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Exact Monte Carlo Simulation of Diffusion and Jump Diffusion Processes with Financial Applications

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The thesis "**Exact Monte Carlo Simulation of Diffusion and Jump Diffusion Processes with Financial Applications**" by **Bruno Casella** is recommended for acceptance by the members of the delegated committee, as stated by the enclosed reports, in partial fulfillment of the requirements for the degree of **Dottore di Ricerca**.

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Alla mia famiglia

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Abstract

Objective of this thesis is to propose some novel techniques for Monte Carlo estimation of classes of diffusion and jump diffusion processes for which finite dimensional distributions are unknown. Given a process Y whose law is not known, we simulate an exact Skeleton at some randomly sampled time points using the techniques (Exact Algorithm) introduced in Beskos et al. (2004). The Skeleton ($\mathcal{S}$) decomposes Y in a sequence of bridge processes. Crucially, we are able to determine the conditional law of Y given the Skeleton, or equivalently the law of the bridges. Knowing the conditional law of the process, we can simulate the target functional ϕ conditional on $\mathcal{S}$ without resorting to any approximation.

We apply this scheme to two challenging Monte Carlo problems of relevant interest in mathematical finance: the barrier problem and the problem of the estimation of the Greeks.

Introduction

Objective of this thesis is to propose some novel techniques for Monte Carlo estimation of classes of diffusion and jump diffusion processes for which finite dimensional distributions are unknown. Typically, in these cases direct Monte Carlo simulation is unfeasible. As a consequence in order to estimate the expected value of a functional ϕ of the process, it is common practice to discretize the process and run Monte Carlo simulation on the discretized process. This clearly introduces a bias in Monte Carlo estimation. Such bias can be reduced only at the cost of a larger computational effort. In comparison the *Exact Monte Carlo method* we propose here is unbiased and efficient. In fact on one hand, our Monte Carlo estimator is free from discretization bias. On the other hand, this achievement is not subjected to any trade off involving the computational time.

The general idea is the following.

Given a process Y whose law is not known, we simulate an exact discrete realization called Skeleton at some randomly sampled time points using the techniques (Exact Algorithm) introduced in Beskos et al. (2004). The Skeleton ($\mathcal{S}$) decomposes Y in a sequence of bridge processes. Crucially, we are able to determine the conditional law of Y given the Skeleton, or equivalently the law of the bridges. Knowing the conditional law of the process, we can simulate the target functional ϕ conditional on $\mathcal{S}$ without resorting to any approximation.

We apply this scheme to two challenging Monte Carlo problems of relevant interest in mathematical finance: the barrier problem and the problem of the estimation of the Greeks.

Part 1 (Chapter 1 to 4) of this Thesis is devoted to the first problem.

Firstly, we consider a one dimensional diffusion process Y with unit diffusion coefficient

$$dY_t = \alpha(Y_t)dt + dW_t; \quad Y_0 = y, \qquad 0 \leq t \leq T \tag{1}$$

Our aim is to estimate the following expected value:

$$\nu = \mathbb{E}^y(\phi(Y)) = \mathbb{E}\left(g(Y_T)\mathbb{I}_{\{\tau<T\}} + h(Y_T)\mathbb{I}_{\{\tau\geq T\}} \mid Y_0 = y\right) \tag{2}$$

where $T > 0$ is a fixed time, τ is the *first exit time* of Y from the set $H := (a, b)$ such that $y \in (a, b)$, $g(\cdot)$ and $h(\cdot)$ are two measurable functions.

It is worth to mention the fact that in general the barrier problem (2) for a generic diffusion X can be reformulated as a barrier problem for a unit diffusion coefficient process Y by applying to X a monotone transformation η, i.e. $Y = \eta(X)$.

In finance the problem of the computation of ν arises in many context, most notably barrier options' pricing and credit risk modeling. However analytic computation is possible just for a small set of simple models (see e.g. Reiner and Rubinstein, 1991, for the Black-Scholes case). On the other hand Monte Carlo strategies based on discretization methods may perform very poorly (Gobet, 2000).

In Chapter 1 we consider a class $\mathcal{D}_1$ of diffusion processes Y (1) defined by a certain set d_1 of conditions on the drift function $\alpha(\cdot)$ and we make use of the *Exact Algorithm 1* (EA1) of Beskos et al. (2004) to generate an exact *Skeleton* ($\mathcal{S}_1$) of the process. The original process is decomposed by the Skeleton in the product of independent bridge processes. For each bridge we are able to determine its distribution law and define an algorithm to simulate its crossing probability, both in the single barrier and in the double barrier case. This allows us to simulate exactly ϕ by conditioning on $\mathcal{S}_1$.

More specifically, the conditional law of the process given the Skeleton $\mathcal{S}_1$ is the law of a product of independent Brownian bridges whose crossing probability and hitting time's density have been extensively investigated in stochastic analysis. Using these results we can simulate exactly the target functional. In particular in the one barrier case we apply the well known Bachelier-Levy theory which provides explicit

formula for the crossing probability of a slope boundary for Brownian motion. In the two barriers case we develop an original procedure to simulate from the two-sided crossing probability. The procedure makes use of Doob's representation of the crossing probability of the Brownian motion as a telescopic sum. Simulation studies show that our estimator is:

- *unbiased*: Euler-based estimators converge to our estimator as the length of the discretization interval decreases
- *efficient*: as the Euler estimator's bias decreases, our estimator overtakes it in terms of efficiency.

The method described in Chapter 1 allows us to construct an unbiased Monte Carlo estimator of ν. Crucially, we don't need knowledge of the transition densities of the process: this makes our method considerably more general than the other unbiased Monte Carlo methods. However restrictions d_1 we impose on Y can be particularly demanding when we deal with models actually used in practice, for example in finance. Therefore it seems a priority to extend our method to a wider class of models.

In Chapter 2 we discuss some ideas which allow to enlarge the class of models to which the Exact Monte Carlo method can be applied.
We show how to construct the Exact Monte Carlo estimator of (2) for two wider classes ($\mathcal{D}_2$ and $\mathcal{D}_3$) of processes. Basically we resort to the scheme set in Chapter 1. However we need to introduce some further results and constructions in order to apply the "old" framework to the "new" models.
Specifically, for $\mathcal{D}_2$ we have made use of the *Exact Algorithm 2* (EA2) of Beskos et al. (2004) for the simulation of the Skeleton. Since the Skeleton ($\mathcal{S}_2$) produced by EA2 has different properties from the Skeleton produced by EA1, the conditional law of the process given $\mathcal{S}_2$ is now the law of a product of independent Bessel bridges. We finally discuss crossing probabilities for Bessel bridges and define an appropriate algorithm to simulate from them in the one barrier and in the two barriers case.
On the contrary, for $\mathcal{D}_3$-processes we just transform the original process $Y \in \mathcal{D}_3$ into a new process $\tilde{Y} \in \mathcal{D}_1$ which leaves the expected value ν unchanged. Therefor, for Monte Carlo estimation of ν, we can apply to $\tilde{Y}$ the framework introduced in Chapter 1.

In Chapter 3, we extend the methodology introduced in Chapter 1 to jump diffusion models. Jump

diffusion processes have become increasingly popular in financial modeling over the last 20 years. It became widely acknowledged that the standard Black and Scholes (1973) assumption of log-normal stock diffusion fails in explaining market behavior. However, dealing with jump diffusion processes, for example in pricing derivatives or modeling credit risk, is mathematically much more challenging than the case of pure diffusions. In fact, except for very basic applications where closed form solutions are available, for most computational problem we have to resort to simulation methods. In particular, for the barrier problem, closed form expressions for the crossing probabilities exist only under very special assumptions. However, resorting to discretization techniques, we observe that if the jump component is independent of the diffusion component, jumps' times and amplitudes are not affected by discretization bias and can be simulated exactly. In this case the discretization error of Monte Carlo estimators is the same as in the case of pure diffusions.

Most critical is the case when the intensity of the marked point process driving the jump component is state dependent. The discretization error reflects on the simulation of the jumps, resulting in larger bias of the Monte Carlo estimator.

In Chapter 3 we present a method to construct unbiased Monte Carlo estimator for the barrier problem when the underlying process Y is a jump diffusion with state dependent intensity. The method is based on a thinning algorithm for the simulation of the jump component and on the Exact Algorithm for the simulation of the diffusion component. The algorithm is adaptive: at each jump epoch the parameters of the Exact Algorithm are updated according to the amplitude of the jumps. Conversely, the thinning probability of a candidate jump depends on the current state of the process.

Compared with Monte Carlo methods based on discretization schemes, our method is exact and efficient. Furthermore, allowing for state-dependent intensity, it is very general. We point out that, in many applications, the assumption that the intensity of the jump component is independent on the state of the process is too restrictive. An example of these situations in the context of credit risk modeling is discussed in Chapter 4.

In Chapter 4 we propose a new jump diffusion approach to credit risk modeling. The firm's value dynamics are described by a jump diffusion process. In particular the diffusion component of the firm's value process is a geometric Brownian motion while the jump component is driven by a marked point process.

Differently from Merton (1976) we let the intensity of the jumps' point process to be state dependent. More specifically, we suppose that the intensity λ of the jumps' arrival process and the mean θ of the jumps depend on the current firm's value. Then, according to the structural approach, the firm defaults when its value falls below a given threshold. Using Monte Carlo techniques developed in Chapter 3, we show how to price a 0 coupon defaultable bond under the assumption that the recovery rate is a fraction of the face value of the bond and depends on the firm's value at the time of default.

The second part of the Thesis (Chapter 5) introduces the idea of Exact Monte Carlo estimation of the Greeks.

In mathematical finance, *Greeks* are the quantities representing the market sensitivities of options or other derivatives. Loosely speaking, Greeks are the partial derivatives of option prices with respect to the parameters of the underlying diffusion process X.

The Greeks are vital tools in risk management. Each represents a specific measure of risk in owning an option. Thus a desirable property of a model of a financial market is the ability to compute the Greeks. Nevertheless closed forms formula for Greeks are very rare. A notable exception is the Black-Sholes case. In general it is clear that, when there are no closed forms expression for prices we have to resort to Monte Carlo methods to estimate both prices and their derivatives. However Monte Carlo simulation for Greeks is not as straightforward as for prices. Prices are genuinely defined as expected values, while Greeks are derivatives of expected values. Therefore in order to perform Monte Carlo simulation we need to define a suitable representation of the Greeks in terms of expectations. There are four approaches to Monte Carlo estimation of Greeks: the *Finite Difference method*, the *Pathwise method*, the *Likelihood Ratio method* and the *Malliavin method*.

In this Thesis, we adopt a Malliavin approach. We propose a new method for Monte Carlo evaluation of Malliavin Greeks. In particular, assuming the Malliavin representation of the Greeks of Fournié et al. (1999), we show how to simulate an unbiased estimator of *delta* and *rho* for a class of diffusion processes X.

The first part of Chapter 5 is descriptive.

We present the challenging problem of the computation of the Greeks and illustrate the traditional Monte Carlo approaches. Discussing the weaknesses of these approaches will lead us to introduce the Malliavin

approach. Malliavin approach relies on the fascinating though complex theory of Malliavin Calculus. It manages to represent the Greeks as expected values of the weighted payoffs. Over competing methods, this representation has the advantage to be exact (unlike Finite Difference method) and general (unlike Pathwise and Likelihood Ratio method). However a major problem is the actual simulation of the Malliavin Monte Carlo estimator.

The second part of the chapter contains our contribution.

We propose a novel method for the Monte Carlo estimation of Malliavin Greeks for the case of *rho* and *delta* when the diffusion X belongs to $\mathcal{D}_1$. Firstly, we simulate the Skeleton $\mathcal{S}_1$. Then given the Skeleton we show how to simulate unbiased estimators of the Greeks. This is not trivial as the Malliavin Greeks are expressed as expected values of the payoff and a stochastic integral (the Malliavin weight). Since direct simulation of a stochastic integral is hardly possible, we have to apply some stochastic calculus to transform the stochastic integral into a simple Riemann integral. We finally construct the unbiased estimators applying to the resulting Riemann integrals suitable simulation techniques.

Part I

BARRIER PROBLEM

Chapter 1

Exact Monte Carlo Simulation

1.1 Introduction

We consider a diffusion process X described by the following SDE:

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t; \quad X_0 = x, \qquad 0 \leq t \leq T \tag{1.1}$$

where $\{W_t : 0 \leq t \leq T\}$ is a standard Brownian Motion and the drift μ and the diffusion coefficient σ are presumed to satisfy the usual conditions that guarantee the existence of a weakly unique global solution of (1.1) (e.g. see Chapter 5 of Oksendal, 1998).

In many applications we are interested in the computation of the expected value, say ν, of a functional of X. This problem arises for example in option pricing where, in frictionless markets, the arbitrage price of derivatives can be expressed as the expected value of the associated payoff, which is usually defined as a functional of the underlying asset process (1.1). In this chapter we will derive and implement efficient and unbiased methods for Monte Carlo evaluation of ν when analytical expressions for the finite dimensional distributions of X are not known and the value of the functional depends on barriers (either a single barrier or double barriers). Generally speaking, let $H := (a, b)$ be an open interval of $\mathbf{R}$ such that $x \in H$; we are interested in the computation of:

$$\nu = \mathbb{E}\left(g(X_T)\mathbb{I}_{\{\tau<T\}} + h(X_T)\mathbb{I}_{\{\tau\geq T\}} \mid X_0 = x\right) \tag{1.2}$$

where $T > 0$ is a fixed time, τ is the *first exit time* of X from the set H, $g(\cdot)$ and $f(\cdot)$ are two measurable functions.

In finance the problem of the computation of (1.2) arises in many contexts. An application to *credit risk* modeling is extensively discussed in Chapter 4.

Another relevant application is related to the pricing of *barrier options*. This is a class of options which give to the owner a different cashflow $(g(X_T),h(X_T))$ according to whether or not H has been exited between 0 and T. Under the assumption of *completeness* of the market, the price of the option at time 0 is the expected value of the discounted payoff given the current price of the asset $X_0 = x$. This leads to the evaluation of (1.2).

Indeed pricing of most popular barrier options can be seen as a special case of (1.2). Suppose that R is the rebate, r is the interest rate, K is the strike price and $(X_T - K)^+$ denote the positive part of $(X_T - K)$.

- *Single Barrier Options*, e.g.

  1. Down and In European (call):

$$\begin{aligned} H &:= (a, +\infty) \\ g(X_T) &= (X_T - K)^+ e^{-rT} \\ h(X_T) &= Re^{-rT} \end{aligned}$$

  2. Down and Out European (call) with no rebate:

$$\begin{aligned} H &:= (a, +\infty) \\ g(X_T) &= 0 \\ h(X_T) &= (X_T - K)^+ e^{-rT} \end{aligned}$$

The definitions above can be modified in an obvious way in order to include *European put* $((K - X_T)^+)$ and *Up and In - Up and Out* options $(H := (-\infty, b))$. We will also show an extension of (1.2) which

allows for the presence of a non null rebate in 2: this requires to introduce a dependence of $g(\cdot)$ on the stopping time τ.

- *Double Barrier Options*, e.g.

  1. Double Barrier European (call) with no rebate:

$$\begin{aligned} g(X_T) &= 0 \\ h(X_T) &= (X_T - K)^+ e^{-rT} \end{aligned}$$

Analytic computation of (1.2) is possible just for a small set of simple models. For example, in the Black-Scholes case, Reiner and Rubinstein (1991) derived explicit formula for the price of one barrier options of European style. However in general cases we have to approximate (1.2) by Monte Carlo methods.
In principle, when using Monte Carlo simulation, many trajectories of X are generated and the value of the functional is evaluated at each sample path. Averaging over all paths provides then an unbiased estimator of ν which converges to the true value as the number of iterations increases.
When the transition densities of X are not known a typical practice is to use some kind of discrete approximation $\tilde{X}$ of the process X. The trajectories of $\tilde{X}$ can be simulated at a discrete set of times. The most popular of these methods is the Euler discretization which manages to approximate (1.1) by means of:

$$\tilde{X}_{i\Delta} = \tilde{X}_{(i-1)\Delta} + \mu(\tilde{X}_{(i-1)\Delta})\Delta + \sigma(\tilde{X}_{(i-1)\Delta})\sqrt{\Delta}\epsilon; \qquad \tilde{X}_0 = x, \quad (i = 1, 2, \ldots, n) \tag{1.3}$$

where $n = T/\Delta$ and ϵ is a standard normal random variable.
Discretization methods introduce a bias into the simulation which tends to 0 as n tends to ∞ (or equivalently $\Delta \to 0$). To reduce the bias to an acceptable level may then require a strong computational investment and even more effort can be necessary to verify that the bias is small enough. In particular, in a Monte Carlo context, the estimator is also affected by a statistical error and it is not easy to isolate the effect of the discretization error in order to assess the performance of the estimator as n increases. A

related problem concerns the optimal allocation of the total computational budget between the number of time steps and the number of simulation trials. Moreover, when dealing with functional involving barriers, the discretization scheme is subjected to a double source of error: one error arising from the actual approximation of (1.1) by its discretized counterpart and the other from the use of a discrete exit time instead of a continuous one. In fact Gobet (2000) proved, under rather general conditions, that in the case of *killed* diffusions the Euler scheme converges weakly at a rate of $1/\sqrt{n}$ which is clearly lower than the usual $1/n$ which applies to the case without barriers.

A widely used technique consists in interpolating the discretized process (1.3) into a continuous Euler scheme by means of Brownian bridges; i.e. for any $i = 1, 2, \ldots, n$ and for any $t \in [t_{(i-1)\Delta}, t_{i\Delta})$ set

$$\tilde{X}_t = \tilde{X}_{(i-1)\Delta} + \mu(\tilde{X}_{(i-1)\Delta})(t - (i-1)\Delta) + \sigma(\tilde{X}_{(i-1)\Delta})(W_t - W_{(i-1)\Delta}) \tag{1.4}$$

The idea here is to produce a realization of the process (1.3) generating $\tilde{X}_\Delta, \tilde{X}_{2\Delta}, \ldots, \tilde{X}_{n\Delta}$; then for any two points $\tilde{X}_{(i-1)\Delta}$ and $\tilde{X}_{i\Delta}$ sample a $[0,1]$-uniformly distributed random variable and compare it with the crossing probability of the corresponding Brownian bridge.

This method improves the rate of convergence of the discrete Euler scheme since it allows for continuous hitting times: in particular Gobet shows that the weak error is now of order n^{-1}. However we point out that, when n is large, the use of (1.4) for Monte Carlo simulation of ν can require the sample of a considerable number of uniform random variables for each Monte Carlo iteration.

As already mentioned our method is instead *exact* and computationally *efficient*. In fact on one hand, the Monte Carlo estimator is free from discretization bias; on the other, this achievement is not subjected to any trade-off involving the computational time.

In constructing the method we will make use of the *Exact Algorithm* of Beskos et al. (2004) to generate an exact *Skeleton* of the process. The original process is decomposed by the Skeleton in the product of independent bridge processes. For each bridge we are able to determine its distribution law and define an algorithm to simulate its crossing probability, both in the one barrier and in the two barriers case. This idea is described in sections 1.3 and 1.4 after some preliminaries in section 1.2. Interesting variants to this general scheme are also possible; we will explore two of them in section 1.5. In section 1.6 we will

show a numerical comparison between our Monte Carlo estimator and the biased Monte Carlo estimators generated by models (1.3) and (1.4).

1.2 Preliminaries

Let $C = C([0,T], \mathbf{R})$ be the set of the continuous functions from $[0,T]$ to $\mathbf{R}$ and $\mathcal{C}$ the usual $\sigma-$algebra generated by the cylinder subsets of C. We denote by $\omega = \{\omega_s : 0 \leq s \leq T\}$ the generic element of C. Let $\mathbb{P}$ denote the probability measure induced by the process X on $(C, \mathcal{C})$ defined by (1.1). Then the following representation is equivalent to (1.2):

$$\nu = \mathbb{E}_{\mathbb{P}}(\phi(\omega)) \mid \omega_0 = x) \tag{1.5}$$

where

$$\phi = \phi(\omega) = g(\omega_T)\mathbb{I}_{\{\tau_H < T\}} + h(\omega_T)\mathbb{I}_{\{\tau_H < T\}^c} \tag{1.6}$$

with τ_H defined by

$$\tau_H := \inf\{t > 0 : \omega_t \notin H\}$$

For convenience of notation, in the sequel of this thesis, whenever we consider an expected value involving a diffusion process with given starting point, we assume that the expectation is conditioned to the starting point. So that, under this convention, the writing $\mathbb{E}_{\mathbb{P}}(\phi(\omega))$ is equivalent to (1.5).
Now consider the function $\eta : \mathbf{R} \to \mathbf{R}$:

$$\eta(u) = \int^u \frac{1}{\sigma(z)} dz$$

By Ito's lemma, the process $Y := \{Y_t = \eta(X_t) : 0 \leq t \leq T\}$ has SDE:

$$dY_t = \alpha(Y_t)dt + dW_t; \quad Y_0 = \eta(x) = y, \qquad 0 \leq t \leq T \tag{1.7}$$

where:

$$\alpha(u) = \frac{\mu\left(\eta^{-1}(u)\right)}{\sigma\left(\eta^{-1}(u)\right)} - \frac{1}{2}\sigma'\left(\eta^{-1}(u)\right)$$

and η^{-1} denotes the inverse transformation.

For reasons that will become clear later, we are going to work with the transformed process Y rather than X. However, denoting $\mathbb{Q}$ the probability measure induced by Y on $(C, \mathcal{C})$, for the (increasing) monotonicity of η:

$$\nu = \mathbb{E}_{\mathbb{P}}\left(\phi(\omega)\right) = \mathbb{E}_{\mathbb{Q}}\left(\xi(\omega)\right)$$

where we define

$$\xi = \xi(\omega) = g(\eta^{-1}(\omega_T))\mathbb{I}_{\left\{\tau_{\eta(H)}<T\right\}}(\omega) + h(\eta^{-1}(\omega_T))\mathbb{I}_{\left\{\tau_{\eta(H)}<T\right\}^c}(\omega) \tag{1.8}$$

with $\eta(H) \equiv (\eta(a), \eta(b))$.

An obvious Monte Carlo estimator of ν is then

$$\hat{\nu} = \frac{1}{N}\sum_{j=1}^{N}\xi^{(j)} \tag{1.9}$$

where $\xi^{(1)}, \xi^{(2)}, \ldots, \xi^{(N)}$ are i.i.d samples from $\mathbb{Q}_\xi$. In general, given any probability measure, say $\mathbb{G}$, on $(C, \mathcal{C})$ and any function $f : C \to \mathbf{R}^n$ we will always denote by $\mathbb{G}_f(\cdot)$ the distribution function of f under $\mathbb{G}$.

To sample ξ we propose a method based on *conditioning*:: first we sample a *Skeleton* of $\omega \sim \mathbb{Q}$ using the Exact Algorithm of Beskos et al. (2004) and then ξ conditional on the Skeleton.
A main feature of our proposal is that in both steps there is no need for the introduction of approximations: the Skeleton as well as the target functional are sampled from their *true* probability measure. As a consequence, unlike the other current methods, the sample of ξ is exact and the resulting Monte Carlo estimator (1.9) is affected only by the Monte Carlo error (decreasing in N).

1.3 The Exact Algorithm 1 (EA1)

1.3.1 Retrospective Rejection Sampling for Diffusions

Given the transformed process $Y = \eta(X)$ defined by (1.7), consider the problem of simulating $\omega \in C$ from $\mathbb{Q}$.
Let $W^y := \{W_t^y : 0 \leq t \leq T\}$ be a Brownian Motion started at y:

$$W^y = (W \mid W_0 = y)$$

and $\mathbb{W}$ the probability measure induced on $(C, \mathcal{C})$ by W^y. Girsanov Theorem provides the Radon-Nikodym derivative of $\mathbb{Q}$ with respect to $\mathbb{W}$:

$$\frac{d\mathbb{Q}}{d\mathbb{W}}(\omega) = e^{\int_0^T \alpha(\omega_t)d\omega_t - \frac{1}{2}\int_0^T \alpha^2(\omega_t)dt}$$

Under the assumption that α is everywhere differentiable (**C1**) we can eliminate the Ito's integral after applying Ito's Lemma to $A(\omega_s)$, where we define:

$$A(u) = \int_{u_0}^{u} \alpha(z)dz; \qquad u \in \mathbf{R} \tag{1.10}$$

be any anti-derivative of $\alpha(\cdot)$. Simple calculations give:

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{W}}(\omega) = \exp\left\{A(\omega_T) - A(y) - \frac{1}{2}\int_0^T \left(\alpha^2(\omega_t) + \alpha'(\omega_t)\right)\mathrm{d}t\right\} \tag{1.11}$$

Since in order to simulate from $\mathbb{Q}$ we want to perform a rejection sampling we need to use as proposal a probability measure $\mathbb{Z}$ such that the likelihood ratio $\frac{d\mathbb{Q}}{d\mathbb{Z}}(\omega)$ is bounded on C.
To achieve this we will use candidate paths ω from a process $\hat{W}$ identical to W^y except for the distribution of its ending point $\hat{W}_T$:

$$\hat{W} = (W^y \mid \hat{W}_T \sim h) \tag{1.12}$$

We call this process *Biased* Brownian motion. In particular we want h to be proportional to:

$$e^{A(u)-\frac{(u-y)^2}{2T}}; \qquad u \in \mathbf{R}$$

which is assumed to be integrable.
To produce a realization $\omega \sim \mathbb{Z}$ at a given collection of time instances is straightforward: it implies sampling a univariate random variable ω_T from h and simulate a Brownian bridge from $(0, y)$ to (T, ω_T) at the required time instances.
The two measures $\mathbb{Z}$ and $\mathbb{W}$ are equivalent and their Radon-Nikodym derivative, as a consequence of Proposition 1 of Beskos et al. (2004), is given by:

$$\frac{d\mathbb{W}}{d\mathbb{Z}}(\omega) = \frac{\mathcal{N}_{y,T}(\omega_T)}{h(\omega_T)} \propto e^{-A(\omega_T)}$$

where $\mathcal{N}_{y,T}$ represents the density of a normal random variable with mean y and variance T. Then:

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{Z}}(\omega) = \frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{W}}(\omega)\,\frac{\mathrm{d}\mathbb{W}}{\mathrm{d}\mathbb{Z}}(\omega) \propto \exp\left\{-\int_0^T \left(\frac{1}{2}\alpha^2(\omega_t) + \frac{1}{2}\alpha'(\omega_t)\right)\mathrm{d}t\right\}$$

Assume now that the function $(\alpha^2 + \alpha')(\cdot)$ is bounded below (**C2**) so that:

$$-\infty < k \leq \inf_{u \in \mathbf{R}} \left\{\frac{1}{2}(\alpha^2 + \alpha')(u)\right\}$$

Thus:

$$\frac{\mathrm{d}\mathbb{Q}}{\mathrm{d}\mathbb{Z}}(\omega) \propto e^{-\int_0^T \varphi(\omega_t)dt} \leq 1, \qquad \mathbb{Z}\ a.s. \tag{1.13}$$

where

$$\varphi(u) = \frac{\alpha^2(u) + \alpha'(u)}{2} - k; \qquad u \in \mathbf{R}$$

The following theorem from Beskos et al. (2004) demonstrates the idea behind the Exact Algorithm.

Theorem 1.1

Let ω be any element of $C([0,T],\mathbf{R})$ and $\mathcal{U}(\omega)$ an upper bound for the mapping $t \mapsto \varphi(\omega_t)$, $t \in [0,T]$. If Π is a homogeneous Poisson process of unit intensity on $[0,T] \times [0,\mathcal{U}(\omega)]$ and $\mathcal{N}$ = number of points of Π found below the graph $\{(t,\varphi(\omega_t))\,;\, t \in [0,T]\}$, then:

$$\mathrm{P}\left[\mathcal{N} = 0 \,|\, \omega\right] = \exp\left\{-\int_0^T \varphi(\omega_t)\mathrm{d}t\right\}$$

Proof:

Conditionally on ω, $\mathcal{N}$ follows a Poisson distribution with mean $\int_0^T \varphi(\omega_t)\mathrm{d}t$.

□

If it was possible to simulate the continuous path $\omega \sim \mathbb{Z}$ then the rejection sampling would be straightforward since the acceptance probability (1.13) can be evaluated simulating a Poisson process of unit intensity on the rectangle $[0,T] \times [0,\mathcal{U}(\omega)]$ and checking if all the points fall above the φ-graph. This is obviously not possible but a simple observation suggests that we can successfully exploit *retrospectivity* inverting the usual rejection sampling procedure: we can sample before the Poisson process Π and then we can simulate the Biased Brownian motion path only at those time instances selected by Π. In this way we need only finite information (which we are able to generate by simulation) in order to decide about the acceptance

of $\omega \sim \mathbb{Z}$.

The main obstacle to our retrospective sampling is to find or simulate $\mathcal{U}(\omega)$ in order to locate the rectangle for the Poisson process. In fact it is clear that in general it depends on the actual sampled path and it can't be fixed *a priori* as in the spirit of the retrospective approach just described. This consideration imposes some further restrictions on the applicability of the algorithm.

1.3.2 The Exact Algorithm 1

Assume that the function φ is bounded above (**C3**) and suppose that $\mathcal{U}$ is any upper bound; in this case

$$\mathcal{U}(\omega) = \mathcal{U}$$

for any $\omega \in C$ so that the upper bound does not depend on the sampled path and the restrospective idea applies fully.

Algorithm 1 Exact Algorithm 1 (EA1)

1. Produce a realization $\{x_1, x_2, \ldots, x_{M-1}\}$ of the Poisson process Π on $[0,T] \times [0,\mathcal{U}]$, where $x_i = (x_{i,1}, x_{i,2})$, $1 \leq i \leq M-1$.

2. Simulate $\omega_T \sim h$ and then the rest of the Skeleton of ω from $\mathbb{Z}$ at the times $x_{i,1}$, $i = 1, \ldots, M-1$.

3. Evaluate $\mathcal{N}$.

4. If $\mathcal{N} = 0$ goto 5, else goto 1.

5. Output the currently constructed Skeleton

Some discussions about the efficiency of this algorithm are given in Beskos et al. (2004).
The output of EA1 is therefore an exact Skeleton of $\omega \sim \mathbb{Q}$; we denote it by

$$\mathcal{S}_1 = \{(t_0, S_0), (t_1, S_1), \ldots, (t_M, S_M)\} \tag{1.14}$$

where $(t_0, S_0) \equiv (0, y)$ and $(t_M, S_M) \equiv (T, \omega_T)$. Then, given the Skeleton, we can continue to construct the accepted path $\omega \sim \mathbb{Q}$ according to the proposal $\mathbb{Z}$ which turns out to be a product of independent Brownian Bridge measures defined between any two points of the Skeleton (Theorem 1.2).
We now summarize the conditions on the process Y which allow the simulation by EA1 and define the class of diffusion processes $\mathcal{D}_1$.

C1 The drift function α is *differentiable*

C2 The function $(\alpha^2 + \alpha')/2$ is *bounded* from below

C3 The function $(\alpha^2 + \alpha')/2$ is *bounded* from above

1.4 Monte Carlo procedure

1.4.1 The main result

Since the Skeleton itself provides a sample of ω_T under $\mathbb{Q}$ ($\omega_T \equiv S_M$), in order to simulate ξ (1.8) all we need is to sample the indicator function

$$I = I(\omega) = \mathbb{I}_{\{\tau_{\eta(H)} < T\}}(\omega)$$

from $\mathbb{Q}_I(\cdot \mid \mathcal{S}_1)$:

$$\mathbb{Q}_I(0 \mid \mathcal{S}_1) = Pr\left[Y_t \in \eta(H),\, 0 \le t \le T \mid \mathcal{S}_1\right] \tag{1.15}$$

To this scope, the following results are crucial:

We first state a simple general lemma that we are going to invoke for the proofs of several results along this thesis (Theorem 1.2, Theorem 2.2 and Theorem 3.1).

Lemma 1.1

Let ω and Φ be two random variables.[1] *Let us denote by $\mathcal{L}_\omega$ the law of ω and define an event A such that $Pr(A \mid \omega, \Phi) \overset{a.s.}{=} \mathcal{P}(\zeta)$ where $\zeta = f(\omega, \Phi)$. Then $\mathcal{L}_{\omega|\zeta,A,\Phi} = \mathcal{L}_{\omega|\zeta,\Phi}$*

Proof:

From Bayes' Theorem $\mathcal{L}_{\omega|\zeta,A,\Phi} = \frac{Pr(A|\omega,\Phi)\mathcal{L}_{\omega|\zeta,\Phi}}{\mathbb{E}(\mathcal{P}(\zeta)|\zeta,\Phi)} = \mathcal{L}_{\omega|\zeta,\Phi}$

□

Let $BB(s,x;t,y) := \{BB_u(s,x;t,y) : s \leq u \leq t\}$ be the Brownian bridge started from x in s and ending at y in t. For any $i = 1, 2, \ldots, M$, we denote by $Y_i \mid \mathcal{S}_1$ the process $\{Y_t;\, t_{i-1} \leq t \leq t_i \mid \mathcal{S}_1\}$.

Theorem 1.2

*Assume that Y satisfies conditions **C1-C3**. Let $\mathcal{S}_1$ be the Skeleton of Y produced by EA1. Then the processes $Y_1 \mid \mathcal{S}_1, Y_2 \mid \mathcal{S}_1, \ldots, Y_M \mid \mathcal{S}_1$ are independent and, for any $i = 1, 2, \ldots, M$:*

$$Y_i \mid \mathcal{S}_1 \overset{d}{=} \{BB_t\left(t_{i-1}, S_{i-1}; t_i, S_i\right);\, t_{i-1} \leq t \leq t_i\}$$

Proof:

Let $\omega \in C([0,T])$ such that $\omega \sim \mathbb{Z}$. Consider the unit intensity marked Poisson process Π on $[0,T] \times [0,U]$:

$$\Pi = (\Phi, \Psi) = \{(x_{i,1}, x_{i,2});\, i = 1, 2, \ldots, M-1\}$$

We define

[1] *With "random variable" here we mean generically a random function with values on a measurable space*

$$\zeta := f(\omega, \Phi) = \left\{ (0, \omega_0), (T, \omega_T), \left\{ \left(x_{i,1}, \omega_{x_{i,1}} \right) \right\}_{i=1,2,\ldots,M-1} \right\}$$

and an event A such that

$$Pr(A \mid \omega, \Phi) = \prod_{i=1}^{M-1} \left[1 - \frac{\varphi\left(\omega_{x_{i,1}}\right)}{\mathcal{U}} \right] = \mathcal{P}(\zeta)$$

Then, for the Girsanov Theorem, it turns out that $Y \stackrel{d}{=} \omega \mid A$. Now, by construction $\mathcal{S}_1 := \zeta \mid A$ and applying Lemma 1.1, since ω and Φ are independent:

$$\mathcal{L}_{Y|\mathcal{S}_1} = \mathcal{L}_{\omega|A,\zeta} = \mathcal{L}_{\omega|A,\zeta,\Phi} = \mathcal{L}_{\omega|\zeta,\Phi}$$

which proves the theorem (by definition of $\mathbb{Z}$).

□

Corollary 1.1

Under the conditions of Theorem 1.2:

$$\mathbb{Q}_I(0 \mid \mathcal{S}_1) = \prod_{i=1}^{M} Pr\left[BB_t(t_{i-1}, S_{i-1}; t_i, S_i) \in \eta(H),\ t_{i-1} \leq t \leq t_i \right] \tag{1.16}$$

Proof:

The result follows immediately from the theorem recalling the definition (1.15) of $\mathbb{Q}_I(\cdot \mid \mathcal{S}_1)$.

□

For simplicity, we introduce the sequence $I_1, I_2, \ldots, I_M$ of independent Bernoulli random variables, defined, for any $i = 1, 2, \ldots, M$, by:

$$Pr\left(I_i = 0\right) = Pr\left[BB_t(t_{i-1}, S_{i-1}; t_i, S_i) \in \eta(H),\ t_{i-1} \leq t \leq t_i\right] \tag{1.17}$$

and the counting random variable $\hat{N}$:

$$\hat{N} = \min_{1 \leq i \leq M} \{i : I_i = 0\} \tag{1.18}$$

where we set $\min\{\emptyset\} = \infty$. Then the Algorithm 2 returns a sample of ξ under $\mathbb{Q}$.
So, when the process $Y = \eta(X)$ belongs to $\mathcal{D}_1$ we are able to determine the Monte Carlo estimator $\hat{\nu}$ applying EA1 provided that we can simulate events of probabilities (1.17). In the following we distinguish two cases: the *single-barrier* case ($H := (-\infty, b)$ or $H := (a, +\infty)$) and the *double-barrier* case ($H := (a, b)$). In fact the simulation from (1.17) involves different technical difficulties in the two cases. In the single barrier case we can compute explicitly (1.17) so that the simulation procedure is straightforward. The double barrier case is definitely more challenging since probability (1.17) comes as a result of an infinite series: to simulate from it we propose an algorithm which exploits the specific structure of the series.

1.4.2 The single barrier problem (P1)

We consider just the upper barrier case since, by symmetry, similar results hold for the lower barrier case ($H := (a, +\infty)$). So suppose $H := (-\infty, b)$. Let's define $\tau_\lambda(s, x; t, y) = \inf\{s \leq u \leq t : BB_u(s, x; t, y) \geq \lambda\}$ under the convention $\inf\{\emptyset\} = \infty$. Then, for any $i = 1, 2, \ldots, M$, (1.17) becomes:

$$\begin{aligned} Pr\left[I_i = 0\right] &= Pr\left[BB_t(t_{i-1}, S_{i-1}; t_i, S_i) \in (-\infty, \eta(b)),\ t_{i-1} \leq t \leq t_i\right] \\ &= Pr\left[\tau_{\eta(b)}\left(t_{i-1}, S_{i-1}; t_i, S_i\right) > t_i\right] \end{aligned}$$

The computation of this probability is known as the *one-sided crossing probability* problem for the Brownian bridge and indicates the problem of determining the probability that a Brownian bridge hits a given

Algorithm 2 Exact Monte Carlo Algorithm

1. Call Algorithm 1 to simulate the Skeleton $\mathcal{S}_1$.

2. If $\exists i \in \{1, \ldots, M\}$ such that $S_i \notin \eta(H)$, go to 5.
   Else go to 3.

3. Sample $\hat{N}$.
   3.1. Set $i = 1$.
   3.2. Sample I_i according to (1.17).
   3.3. If $I_i = 0$ set $\hat{N} = i$ and goto 4.
   3.4. If $i = M$ set $\hat{N} = \infty$ goto 4.
   3.5. Else set $i = i + 1$ and goto 3.2.

4. If $\hat{N} = \infty$ goto 6, else goto 5.

5. Save $\xi = g(\eta^{-1}(S_M))$.

6. Save $\xi = h(\eta^{-1}(S_M))$.

7. Repeat 1-6 a sufficiently large number N of times and output $\hat{\nu} = 1/N \sum_{j=1}^{N} \xi^{(j)}$.

barrier. First we give an explicit formula for it and, after that, we briefly outline the argument that leads to it. For further details, we refer to Lerche (1986).

So, for any $i = 1, 2, \ldots, M$:

$$Pr\left[I_i = 0\right] = \begin{cases} 1 - \exp\left(-2\frac{(\eta(b)-S_i)(\eta(b)-S_{i-1})}{t_i - t_{i-1}}\right) & \text{if} \quad (\eta(b) - S_i)(\eta(b) - S_{i-1}) > 0 \\ 0 & \text{if} \quad (\eta(b) - S_i)(\eta(b) - S_{i-1}) \leq 0 \end{cases} \tag{1.19}$$

We just notice that, given the Skeleton $\mathcal{S}_1$ and the barrier's value b, these probabilities are known. Thus the simulation of the indicator variable given the Skeleton is immediate.

One-sided crossing probability for the Brownian Bridge

Consider the standard Brownian motion $W = \{W_u : u \geq 0\}$ and define the following stopping time $\tau_{\kappa,\rho} = \inf\{u \geq 0 : W_u \geq \kappa + \rho u\}$. Set $\delta = y - x$ and $\Delta = t - s$.
It follows for any $z \in [s, t]$:

$$\begin{aligned} Pr\left[\tau_\lambda(s, x; t, y) > z\right] &= Pr\left[BB_u(s, x; t, y) < \lambda,\, s \leq u \leq z\right] \\ &= Pr\left[BB_u(0, x; t-s, y) < \lambda,\, 0 \leq u \leq z - s\right] \\ &= Pr\left[\frac{u}{\Delta}\delta + \frac{\Delta - u}{\sqrt{\Delta}} W_{\frac{u}{\Delta-u}} < \lambda - x,\, 0 \leq u \leq z - s\right] \\ &= Pr\left[W_{u^*} < \frac{\lambda - x}{\sqrt{\Delta}} + \frac{\lambda - x - \delta}{\sqrt{\Delta}} u^*,\, 0 \leq u^* \leq \frac{z - s}{\Delta - (z - s)}\right] \end{aligned} \tag{1.20}$$

where we have used the usual representation of the Brownian bridge in terms of Brownian motion:

$$BB_u(0; x; \Delta, y) \stackrel{d}{=} x + \frac{u}{\Delta}\delta + \frac{\Delta - u}{\sqrt{\Delta}} W_{\frac{u}{\Delta-u}} \qquad 0 \leq u \leq \Delta$$

It is easy to see that for any $z \in [s, t]$:

$$Pr\left[\tau_\lambda\left(s,x;t,y\right)>z\right]=Pr\left[\tau_{\kappa,\rho}>\frac{z-s}{\Delta-(z-s)}\right] \tag{1.21}$$

where $\kappa=\frac{\lambda-x}{\sqrt{\Delta}}$ and $\rho=\frac{\lambda-y}{\sqrt{\Delta}}$.

It turns out that the probability distribution of $\tau_{\kappa,\rho}$ is absolutely continuous with respect to the Lebesque measure on $[0,+\infty)$ while it has a singular component at $+\infty$. The corresponding density on $[0,+\infty)$ is given by the Bachelier-Levy formula (for a proof. see e.g. Karatzas and Shreve, 1991, Ch. 2):

$$f_{\kappa,\rho}(u)=\frac{\mid\kappa\mid}{u^{3/2}\sqrt{2\pi}}\exp\left\{-\frac{(\rho u+\kappa)^2}{2u}\right\},\quad u>0.$$

We denote by $IG(u;\iota,\theta)$ the inverse gaussian density with parameters $\iota>0$ and $\theta>0$:

$$IG\left(u;\iota,\theta\right)=\sqrt{\frac{\theta}{2\pi u^3}}\exp\left\{-\frac{\theta\left(u-\iota\right)^2}{2\iota^2 u}\right\},\quad u>0 \tag{1.22}$$

It is easy to verify that when $\kappa\rho<0$, the Bachelier-Levy density corresponds to $IG\left(u,-\kappa/\rho,\kappa^2\right)$; on the contrary when $\kappa\rho>0$ it can be written as $\left\{\exp\left(-2\kappa\rho\right)IG\left(u,\kappa/\rho,\kappa^2\right)\right\}$. Therefore it follows:

$$\begin{aligned} Pr\left[\tau_\lambda\left(s,x;t,y\right)>t\right] &= 1-Pr\left[0\leq\tau_{\kappa,\rho}<+\infty\right]= \\ =1-\int_0^\infty f_{\kappa,\rho}(u)du &= \begin{cases} 0 & \text{if}\quad \kappa\rho<0 \\ 1-\exp\left\{-2\kappa\rho\right\} & \text{if}\quad \kappa\rho>0 \end{cases} \end{aligned} \tag{1.23}$$

1.4.3 The double barrier problem (P2)

Suppose $H:=(a,b)$. Then, for any $i=1,2,\ldots,M$, (1.17) becomes:

$$Pr\left[I_i=0\right]=Pr\left[BB_t(t_{i-1},S_{i-1};t_i,S_i)\in(\eta(a),\eta(b)),\ t_{i-1}\leq t\leq t_i\right] \tag{1.24}$$

As we have already stressed, we can not easily derive an explicit expression for this probability. However

we propose an algorithm to sample I_i according to it. Following the same scheme as for the one barrier case, before we present the algorithm and then we provide a theoretical justification.

For any $i = 1, 2, \ldots, M$ consider the two sequences $(\underline{n}_k)_{k=1,2,\ldots}$ and $(\overline{n}_k)_{k=1,2,\ldots}$ defined in the following way:

$$\begin{aligned}\underline{n}_k = \sum_{j=1}^{k} & + \exp\left\{-\tfrac{2}{t_i - t_{i-1}}\left[j\delta + (\eta(b) - S_{i-1})\right]\left[j\delta + (\eta(b) - S_i)\right]\right\} \\ & - \exp\left\{-\tfrac{2j}{t_i - t_{i-1}}\left[j\delta^2 + \delta(\eta(b) - S_i) - \delta(\eta(b) - S_{i-1})\right]\right\} \\ & + \exp\left\{-\tfrac{2}{t_i - t_{i-1}}\left[j\delta - (\eta(a) - S_{i-1})\right]\left[j\delta - (\eta(a) - S_i)\right]\right\} \\ & - \exp\left\{-\tfrac{2j}{t_i - t_{i-1}}\left[j\delta^2 - \delta(\eta(a) - S_i) + \delta(\eta(a) - S_{i-1})\right]\right\} \end{aligned} \tag{1.25}$$

and

$$\begin{aligned}\overline{n}_k = \underline{n}_{k-1} & + \exp\left\{-\tfrac{2}{t_i - t_{i-1}}\left[k\delta + (\eta(b) - S_{i-1})\right]\left[k\delta + (\eta(b) - S_i)\right]\right\} \\ & + \exp\left\{-\tfrac{2}{t_i - t_{i-1}}\left[k\delta - (\eta(a) - S_{i-1})\right]\left[k\delta - (\eta(a) - S_i)\right]\right\} \end{aligned} \tag{1.26}$$

with $\underline{n}_0 = 0$ and $\delta = \eta(b) - \eta(a)$.

The Algorithm 3 then generates a sample of the indicator variables I_i $(i = 1, 2, \ldots, M)$ in the double barrier case.

The algorithm is essentially based on these two limits:

$$\begin{aligned} \underline{n}_k &\uparrow Pr(I_i = 1) \\ \overline{n}_k &\downarrow Pr(I_i = 1) \end{aligned} \tag{1.27}$$

for $k \to +\infty$. A justification of the algorithm together with a formal proof of (1.27) is presented in the

Algorithm 3 Simulation of the indicator variable I_i (P2 CASE)

1. Sample $U \sim \text{Unif}[0,1]$. Set $j = 1$.

2. Evaluate $(\underline{n}_j, \overline{n}_j)$.
   If $U > \overline{n}_j$ go to 4.
   If $U < \underline{n}_j$ go to 3.
   Else set $j = j + 1$ and go to 2.

3. Output $I_i = 1$.

4. Output $I_i = 0$.

last part of this section. Before, we give a result on the random variable N representing the number of times we need to iterate *step* 2 until a decision is taken. This is clearly the crucial variable affecting the efficiency of the algorithm.
Let's denote by $M(\alpha)$ the moment generating function of N:

$$M(\alpha) = \mathbb{E}\left(e^{\alpha N}\right)$$

Proposition 1.1
There exists $\epsilon \in \mathbf{R}$ *such that for any* $\alpha \in (-\epsilon, +\epsilon)$*:*

$$M(\alpha) < \infty$$

Proof:
For the construction of the algorithm, for any $j = 1, 2, \ldots$:

$$\begin{aligned} Pr(N > j) = Pr(U \in (\underline{n}_j, \min\{1, \overline{n}_j\}) &= \min\{1, \overline{n}_j\} - \underline{n}_j \\ &\leq \overline{n}_j - \underline{n}_j \end{aligned}$$

where, for any $i = 1, 2, \ldots, M$, it holds

$$\begin{aligned} \overline{n}_j - \underline{n}_j &= e^{-\frac{2j}{t_i - t_{i-1}}\left[j\delta^2 + \delta(\eta(b) - S_i) - \delta(\eta(b) - S_{i-1})\right]} \\ &+ e^{-\frac{2j}{t_i - t_{i-1}}\left[j\delta^2 - \delta(\eta(a) - S_i) + \delta(\eta(a) - S_{i-1})\right]} \end{aligned}$$

Now:

$$\begin{aligned} M(\alpha) &= \mathbb{E}(e^{\alpha N}) = \sum_{j=0}^{\infty} e^{(j+1)\alpha}\left(Pr(N > j) - Pr(N > j+1)\right) \\ &\leq \sum_{j=0}^{\infty} e^{(j+1)\alpha} Pr(N > j) = e^{\alpha} \sum_{j=0}^{\infty} e^{j\alpha} Pr(N > j) \\ &= e^{\alpha} + e^{\alpha} \sum_{j=1}^{\infty} e^{j\alpha} Pr(N > j) \leq e^{\alpha} + e^{\alpha} \sum_{j=1}^{\infty} e^{j\alpha}(\overline{n}_j - \underline{n}_j) \end{aligned}$$

$M(\alpha)$ is finite for those values of α for which it converges the infinite series:

$$\begin{aligned} \sum_{j=1}^{\infty} e^{j\alpha}(\overline{n}_j - \underline{n}_j) &= \sum_{j=1}^{\infty} e^{-\frac{2j}{t_i - t_{i-1}}\left[j\delta^2 + \delta(\eta(b) - S_i) - \delta(\eta(b) - S_{i-1}) - \frac{t_i - t_{i-1}}{2}\alpha\right]} \\ &+ \sum_{j=1}^{\infty} e^{-\frac{2j}{t_i - t_{i-1}}\left[j\delta^2 - \delta(\eta(a) - S_i) + \delta(\eta(a) - S_{i-1}) - \frac{t_i - t_{i-1}}{2}\alpha\right]} \end{aligned}$$

It is now clear that it exists a neighborhood $(-\epsilon, +\epsilon)$ of 0 such that, if $\alpha \in (-\epsilon, +\epsilon)$ the two series converge; that is $M(\alpha)$ is finite in $(-\epsilon, +\epsilon)$.

□

A main consequence of the proposition is that N has finite moments of every order; that is, for any $k = 1, 2, \ldots$

$$\mathbb{E}(N^k) < \infty$$

Two-sided crossing probability for the Brownian bridge

The problem of determining the *two-sided* crossing probability for the Brownian bridge is more challenging than the one-sided problem for which a straightforward application of Girsanov Theorem provides formula (1.23). In the two barriers case it does not exist a closed-form expression.
This problem has been extensively studied in literature and many different approaches have been proposed (e.g. see Bertoin and Pitman, 1994). Here we follow the classical approach of Doob (1949) and Anderson (1960). For a recent reference, see for example Pötzelberger and Wang (2001).
So, let $BB(s, x; t, y)$ be the usual Brownian bridge and W the standard Brownian motion. Consider two values λ_1 and λ_2 such that $-\infty < \lambda_1 < \lambda_2 < +\infty$ and $x, y \in (\lambda_1, \lambda_2)$. For simplicity, we define:

$$\begin{aligned} \beta_1 &= \frac{\lambda_1 - y}{\sqrt{t-s}} < 0; & \beta_2 &= \frac{\lambda_2 - y}{\sqrt{t-s}} > 0 \\ \alpha_1 &= \frac{\lambda_1 - x}{\sqrt{t-s}} < 0; & \alpha_2 &= \frac{\lambda_2 - x}{\sqrt{t-s}} > 0 \\ a_1(u) &= \beta_1 u + \alpha_1; & a_2(u) &= \beta_2 u + \alpha_2 \end{aligned} \tag{1.28}$$

Now, under the usual convention that $\inf\{\emptyset\} = \infty$, we define the following stopping times:

- $\tau(a_1, a_2) = \inf\{u \geq 0 : W_u \geq a_2(u) \text{ or } W_u \leq a_1(u)\}$
- $\tau_{0,1} = \tau_{0,2} = 0$ and for any $i = 1, 2, \ldots$:

$$\begin{aligned} \tau_{i,1} &= \inf\{u \geq \tau_{i-1,2} : W_u \leq a_1(u)\} \\ \tau_{i,2} &= \inf\{u \geq \tau_{i-1,1} : W_u \geq a_2(u)\} \end{aligned}$$

and the related events:

- $A = \{\tau(a_1, a_2) < \infty\}$
- $A_1 = \{\tau_{1,1} < \tau_{1,2}\}$; $A_2 = \{\tau_{1,2} < \tau_{1,1}\}$
- $\{A_{i,j}\}_{i=1,2;\dots;j=1,2}$: $A_{i,j} = \{\tau_{i,j} < \infty\}$

Notice the following important relationships between these events:

i) $A = A_2 \sqcup A_1 = A_{1,2} \cup A_{1,1}$

ii) $A_2 = \bigsqcup_{i=1}^{+\infty} (A_{2i-1,2} - A_{2i,2})$; $A_1 = \bigsqcup_{i=1}^{+\infty} (A_{2i-1,1} - A_{2i,1})$

iii) for any $i = 1, 2, \cdots$: $A_{i,2} \cap A_{i,1} = A_{i+1,2} \cup A_{i+1,1}$

where with the symbol $\sqcup$ we denote the disjoint union.
Now from (1.20):

$$Pr\left[\lambda_1 < BB_u(s, x; t, y) < \lambda_2,\ s \le u \le t\right] = 1 - Pr(A) \tag{1.29}$$

From Theorem 4.1 of Anderson (1960) we can derive the following expressions:

$$\begin{aligned}
Pr(A_{2j-1,2}) &= e^{-\frac{2}{t-s}[j(\lambda_2-\lambda_1)+(\lambda_2-x)][j(\lambda_2-\lambda_1)+(\lambda_2-y)]} \\
Pr(A_{2j-1,1}) &= e^{-\frac{2}{t-s}[j(\lambda_2-\lambda_1)-(\lambda_1-x)][j(\lambda_2-\lambda_1)-(\lambda_1-y)]} \\
Pr(A_{2j,2}) &= e^{-\frac{2j}{t-s}\left[j(\lambda_2-\lambda_1)^2+(\lambda_2-\lambda_1)(\lambda_2-y)-(\lambda_2-\lambda_1)(\lambda_2-x)\right]} \\
Pr(A_{2j,1}) &= e^{-\frac{2j}{t-s}\left[j(\lambda_2-\lambda_1)^2-(\lambda_2-\lambda_1)(\lambda_1-y)+(\lambda_2-\lambda_1)(\lambda_1-x)\right]}
\end{aligned} \tag{1.30}$$

As a consequence of i) and ii)

$$\begin{aligned}
Pr(A) = \sum_{j=1}^{+\infty} & e^{-\frac{2}{t-s}[j(\lambda_2-\lambda_1)+(\lambda_2-x)][j(\lambda_2-\lambda_1)+(\lambda_2-y)]} - e^{-\frac{2j}{t-s}\left[j(\lambda_2-\lambda_1)^2+(\lambda_2-\lambda_1)(\lambda_2-y)-(\lambda_2-\lambda_1)(\lambda_2-x)\right]} \\
& + e^{-\frac{2}{t-s}[j(\lambda_2-\lambda_1)-(\lambda_1-x)][j(\lambda_2-\lambda_1)-(\lambda_1-y)]} - e^{-\frac{2j}{t-s}\left[j(\lambda_2-\lambda_1)^2-(\lambda_2-\lambda_1)(\lambda_1-y)+(\lambda_2-\lambda_1)(\lambda_1-x)\right]}
\end{aligned}$$

This formula shows the difficulty with computing crossing probabilities in the two barriers case: its computation requires the solution of an infinite series. This is why we propose an iterative algorithm to simulate from it. In the following, we give a probabilistic justification of our method.
Define the two sequences of values $\{\underline{n}_k\}_{k=1,2,...}$ and $\{\overline{n}_k\}_{k=1,2,...}$ such that:

$$\begin{aligned} \underline{n}_k &= \sum_{j=1}^{k} [(Pr(A_{2j-1,2}) - Pr(A_{2j,2})) + (Pr(A_{2j-1,1}) - Pr(A_{2j,1}))] \\ \overline{n}_k &= \underline{n}_{k-1} + [Pr(A_{2k-1,2}) + Pr(A_{2k-1,1})] \end{aligned} \tag{1.31}$$

with $\underline{n}_0 = 0$.
Notice that if we substitute to the probabilities above the corresponding expressions (1.30) we find exactly the sequences (1.25) and (1.26) where the parameters of the Brownian bridge are defined by the points of the Skeleton and the barriers are $\lambda_1 = \eta(a)$ and $\lambda_2 = \eta(b)$. As a consequence, the following proposition is equivalent to (1.27).

Proposition 1.2

$\{\underline{n}_k\}_{k=1,2,...}$ is an increasing sequence of real numbers converging to $Pr(A)$; $\{\overline{n}_k\}_{k=1,2,...}$ is a decreasing sequence of real numbers converging to $Pr(A)$.

Proof:

The first part of the proposition follows directly from i) and ii) just by noticing:

$$\begin{aligned} Pr(A) &= Pr(A_2) + Pr(A_1) \\ &= \sum_{i=1}^{+\infty} (Pr(A_{2i-1,1}) - Pr(A_{2i,1})) + \sum_{i=1}^{+\infty} (Pr(A_{2i-1,2}) - Pr(A_{2i,2})) \\ &= \sum_{j=1}^{+\infty} [(Pr(A_{2j-1,2}) - Pr(A_{2j,2})) + (Pr(A_{2j-1,1}) - Pr(A_{2j,1}))] \end{aligned}$$

For the second part, notice that the sequence $\{\overline{n}_k\}$ can be written in the following way:

$$\begin{aligned}\overline{n}_1 &= Pr(A_{1,2}) + Pr(A_{1,1}) \\ \overline{n}_k &= \overline{n}_{k-1} + \left[Pr(A_{2k-1,2}) + Pr(A_{2k-1,1}) - Pr(A_{2(k-1),2}) - Pr(A_{2(k-1),1})\right]\end{aligned}$$

For any $k = 1, 2, \ldots$, let $q_k = Pr(A_{2k-1,2} \cap A_{2k-1,1})$. From iii) it follows that:

$$q_k - q_{k-1} = Pr(A_{2k-1,2}) + Pr(A_{2k-1,1}) - Pr(A_{2(k-1),2}) - Pr(A_{2(k-1),1}; \qquad (k = 2, 3, \ldots)$$

so that we can rewrite the sequence as:

$$\begin{aligned}\overline{n}_1 &= Pr(A_{1,2}) + Pr(A_{1,1}) = Pr(A_{1,2} \cup A_{1,1}) + Pr(A_{1,2} \cap A_{1,1}) = Pr(A) + q_1 \\ \overline{n}_k &= \overline{n}_{k-1} + (q_k - q_{k-1}) = Pr(A) + q_k\end{aligned}$$

Now, for any $k = 1, 2, \ldots$:

$$Pr(A) + q_k \geq Pr(A)$$

and

$$Pr(A) + q_k \geq Pr(A) + q_{k+1}$$

which follows immediately from iii) observing that $(A_{2k-1,2} \cap A_{2k-1,1}) \supseteq (A_{2k+1,2} \cap A_{2k+1,1})$.

□

1.5 Some remarks on P1

1.5.1 First passage time problem

We propose an extension of **P1** to the case when the function $g(\cdot)$ in (1.6) depends on the first passage time. So we consider a function ϕ' defined as

$$\phi' = \phi'(\omega) = g(\omega_T, \tau_H)\mathbb{I}_{\{\tau_H < T\}} + h(\omega_T)\mathbb{I}_{\{\tau_H < T\}^c} \tag{1.32}$$

with $H := (-\infty, b)$. Then our aim is to construct an estimator of:

$$\nu = \mathbb{E}_{\mathbb{P}}(\phi') = \mathbb{E}_{\mathbb{Q}}(\xi')$$

where ξ' is defined by

$$\xi' = g(\eta^{-1}(\omega_T), \tau_{\eta(H)})\mathbb{I}_{\{\tau_{\eta(H)} < T\}} + h(\eta^{-1}(\omega_T))\mathbb{I}_{\{\tau_{\eta(H)} < T\}^c} \tag{1.33}$$

In the following, for ease of notation, we set $\tau_{\eta(H)} \equiv \tau$.

The interest on this case is motivated by a financial application. In fact suppose we have a *Up and Out* call option of the European type: the option expires when it hits an upper barrier. We begin at $t = 0$ and fix a time to maturity T. If there is no hitting in the interval $[0, T]$ the option gives to the holder a payoff of the European type at T, but if the price hits the barrier it returns a fix positive amount R (*rebate*) at the hitting time. The present value of the rebate (and therefore, the final payoff) depends in this case on the random time τ which affects the discounted factor. We can easily apply (1.32) to this case:

$$\begin{aligned} g(\omega_T, \tau_H) &= \exp\{-r\tau_H\} R \\ h(\omega_T) &= (\omega_T - K)^+ e^{-rT} \end{aligned} \tag{1.34}$$

Analogously, formulation (1.32) covers the case of *Down-Out* option. Notice that in the case of *Up-In* and

Down-in options there is no need to sample the hitting times even in the presence of non null rebate: in fact the option comes to existence after the hitting times and the owner receives either the rebate or the (European) payoff at fixed time instance T.

In order to construct the Monte Carlo estimator of ν, we follow the strategy introduced in section 1.4. We sample a Skeleton of $\omega \sim \mathbb{Q}$ and then, conditionally on the Skeleton, the random variable ξ' is simulated. Now, under the usual assumption that $Y \in \mathcal{D}_1$, $EA1$ provides the Skeleton $\mathcal{S}_1$ of Y and all we need is the simulation of the exit time τ from its conditional distribution given $\mathcal{S}_1$.

To this end we can make use of (1.19) to simulate the random variable $\hat{N}$ (1.18) and then we simulate the hitting times given $\hat{N}$ (and $\mathcal{S}_1$).

If $\hat{N} = +\infty$, $\omega \sim \mathbb{Q}$ does not hit the barrier $\eta(b)$ and the realization of ξ' will be $h(\eta^{-1}(S_M))$. If $\hat{N} \leq M$, for the definition of τ and $\hat{N}$, for any $z \in \left[t_{\hat{N}-1}, t_{\hat{N}}\right]$:

$$\mathbb{Q}_\tau(z \mid \mathcal{S}_1, \hat{N}) = \mathbb{Q}_\tau(z \mid \mathcal{S}_1, t_{\hat{N}-1} \leq \tau \leq t_{\hat{N}})$$

which is equal, as a consequence of Theorem 1.2, to

$$Pr\left[\tau_{\eta(b)}(t_{\hat{N}-1}, S_{\hat{N}-1};\, t_{\hat{N}}, S_{\hat{N}}) \leq z \mid t_{\hat{N}-1} \leq \tau_{\eta(b)}(t_{\hat{N}-1}, S_{\hat{N}-1};\, t_{\hat{N}}, S_{\hat{N}}) \leq t_{\hat{N}}\right] \tag{1.35}$$

We can simulate τ from (1.35) in the following way:

$$\tau \stackrel{d}{=} l^{(\hat{N})}\left(Z_{\hat{N}}\right) \tag{1.36}$$

where $Z_{\hat{N}}$ is a random variable with density function $f_{Z_{\hat{N}}}$ such that, for any $i = 1, 2, \ldots, M$:

$$f_{Z_i}(u) = \begin{cases} IG\left(u; \frac{\eta(b)-S_i}{\eta(b)-S_{i-1}}, \left(\frac{\eta(b)-S_i}{t_i-t_{i-1}}\right)^2\right) & \text{if} \quad (\eta(b)-S_i)(\eta(b)-S_{i-1}) > 0 \\ IG\left(u; -\frac{\eta(b)-S_i}{\eta(b)-S_{i-1}}, \left(\frac{\eta(b)-S_i}{t_i-t_{i-1}}\right)^2\right) & \text{if} \quad (\eta(b)-S_i)(\eta(b)-S_{i-1}) \leq 0 \end{cases} \tag{1.37}$$

and

$$l^{(i)}(u) = t_{i-1} + \frac{(t_i - t_{i-1})\,u}{u+1}; \qquad 0 \le u < \infty \tag{1.38}$$

So to produce a realization from (1.35) we sample an inverse gaussian random variable according to (1.37) apply the transformation (1.38).

Simulating from an inverse gaussian distribution can be done very efficiently (see e.g. Devroye, 1986)

Algorithm 4 Simulation of the first passage time (P1 CASE)

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1. `Call Algorithm 1 to simulate the Skeleton` $\mathcal{S}_1$.

2. `Call Algorithm 2 to simulate` $\hat{N}$.
   `If` $\hat{N} = \infty$ `go to 4.`
   `Else go to 3`

3. `Sample` τ `according to (1.35) and output the result.`

4. `Output` $\tau = \infty$.

First passage time of the Brownian bridge

From (1.20), for any $z \in [s,t]$:

$$Pr\left[\tau_\lambda(s,x;t,y) \le z \mid s \le \tau_\lambda(s,x;t,y) \le t\right] = Pr[\tau_{\kappa,\rho} \le \frac{z-s}{\Delta-(z-s)} \mid 0 \le \tau_{\kappa,\rho} < +\infty]$$

It is now clear that it holds:

$$\tau_\lambda(s,x;t,y) \stackrel{d}{=} l\left(\tau_{\kappa,\rho}\right)$$

where for any $u \in [0,\infty) : l(u) = s + \Delta u/(u+1)$.

So, in order to simulate τ_λ, it is sufficient to simulate $\tau_{\kappa,\rho}$ and apply the transformation above.

Let denote by $f_{\kappa,\rho}(\cdot \mid 0 \leq \tau_{\kappa,\rho} < +\infty)$ the conditional density corresponding to $\tau_{\kappa,\rho}$ with support $[0,+\infty)$. Then:

$$f_{\kappa,\rho}(u \mid 0 \leq \tau_{\kappa,\rho} < +\infty) = \frac{f_{\kappa,\rho}(u)\cdot}{Pr\left[0 \leq \tau_{\kappa,\rho} < +\infty\right]}$$

where the numerator is the Levy-Bachelier density while the probability in the denominator is given by (1.23).

Now the last step follows immediately: for any $u \in [0,+\infty)$

$$f_{\kappa,\rho}(u \mid 0 \leq \tau_{\kappa,\rho} < +\infty) = \begin{cases} IG\left(u, \kappa/\rho, \kappa^2\right) & \text{if} \quad \kappa\rho > 0 \\ IG\left(u, -\kappa/\rho, \kappa^2\right) & \text{if} \quad \kappa\rho \leq 0 \end{cases}$$

1.5.2 A Rao Blackwellized Monte Carlo estimator

For the **P1** case, it is possible to define an alternative estimator $\tilde{\nu}$ characterized by lower variance than $\hat{\nu}$. In particular $\tilde{\nu}$ is a Rao-Blackwellized version of $\hat{\nu}$ (for a general discussion on Rao-Blackwelization see e.g. Casella and Robert, 1996).

Notice that, from the properties of conditional expectation:

$$\nu = \mathbb{E}_{\mathbf{Q}}(\xi) = \mathbb{E}_{\mathbf{Q}}\left(\mathbb{E}_{\mathbf{Q}}(\xi \mid \mathcal{S}_1)\right)$$

So a Monte Carlo estimator alternative to $\hat{\nu}$ is:

$$\tilde{\nu} = \frac{\sum_{j=1}^{N} \mathbb{E}_{\mathbb{Q}}(\xi \mid \mathcal{S}_1^{(j)})}{N} \tag{1.39}$$

where $\mathcal{S}_1^{(1)}, \mathcal{S}_1^{(2)}, \ldots, \mathcal{S}_1^{(N)}$ are N independent copies of the Skeleton.

Like $\hat{\nu}$, $\tilde{\nu}$ is an unbiased estimator of ν, for which asymptotically ($N \to \infty$) the Law of Large Number and the Central Limit Theorem hold. However $\tilde{\nu}$ differs from $\hat{\nu}$ in that, given N,

$$\mathrm{Var}\,(\tilde{\nu}) \leq \mathrm{Var}\,(\hat{\nu}) \tag{1.40}$$

which is a straightforward consequence of *Jensen inequality.*

Inequality (1.40) implies that, given $N \in \mathbb{N}$, under a quadratic loss function, the estimator $\tilde{\nu}$ is more efficient than $\hat{\nu}$ and, clearly, in a Monte Carlo context, the convergence of $\tilde{\nu}$ to the true parameter is faster and the algorithm performs better.

In order to construct (1.39) we have to be able to simulate the random variable $\mathbb{E}_{\mathbb{Q}}(\xi \mid \mathcal{S}_1)$. When the process Y is in $\mathcal{D}_1$, EA1 provides a procedure to produce $\mathcal{S}_1$ and the main difficulty is to compute analytically the conditional expectation. Unfortunately this is not always possible: from Theorem 1.2, it is necessary to determine the expectation of a functional of independent Brownian bridges.

Here we will give an explicit formula for the conditional expectation in the **P1** case.

Using the fact that (under $\mathbb{Q}$) $\omega_T = S_M$, we have:

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}(\xi \mid \mathcal{S}_1) &= \left[h(\eta^{-1}(S_M)) - g(\eta^{-1}(S_M))\right] \mathbb{Q}_I(0 \mid \mathcal{S}_1) + g(\eta^{-1}(S_M)) \\
&= \left[h(\eta^{-1}(S_M)) - g(\eta^{-1}(S_M))\right] \prod_{i=1}^{M} Pr\left[I_i = 0\right] + g(\eta^{-1}(S_M))
\end{aligned}$$

The Rao-Blackwellized Monte Carlo estimator $\tilde{\nu}$ then turns out to be:

$$\begin{aligned}\tilde{\nu} &= \frac{\sum_{j=1}^{N}\mathbb{E}_{\mathbb{Q}}(\xi \mid \mathcal{S}_1^{(j)})}{N} \\ &= \frac{1}{N}\sum_{j=1}^{N}\left[\left(h(\eta^{-1}(S^{(j)}_{M(\mathcal{S}_1^{(j)})})) - g(\eta^{-1}(S^{(j)}_{M(\mathcal{S}_1^{(j)})})\right)\prod_{i=1}^{M\left(\mathcal{S}_1^{(j)}\right)} Pr\left[I_i^{(j)}=0\right] + g(\eta^{-1}(S^{(j)}_{M(\mathcal{S}_1^{(j)})}))\right]\end{aligned}$$

where, for any $j = 1, 2, \ldots, N$, $M(\mathcal{S}_1^{(j)})$ is the number of points of $\mathcal{S}_1^{(j)}$.
For **P1**, given the Skeleton $\mathcal{S}_1^{(j)}$ $(j = 1, 2, \ldots, M)$, $Pr\left[I_i^{(j)} = 0\right]$ $(i = 1, 2, \ldots, M)$ has been explicitly given (1.19) so that the conditional expectation above is available analytically. For **P2** we don't have these probabilities in a closed form and therefore the Rao-Blackwellization fails.

We address now the question whether it is possible to define a Rao-Blackwellized Monte Carlo estimator also for the extension of **P1** we have introduced in section 1.5.1.
In this case in order to construct the Rao-Blackwelized Monte Carlo estimator $\tilde{\nu}$, we need to sample the conditional expectation:

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}(\xi' \mid \mathcal{S}_1) &= \mathbb{E}_{\mathbb{Q}}\left(g(\eta^{-1}(\omega_T), \tau)\mathbb{I}_{\{\tau<T\}} + h(\eta^{-1}(\omega_T))\mathbb{I}_{\{\tau<T\}^c} \mid \mathcal{S}_1\right) \\ &= \mathbb{E}_{\mathbb{Q}}\left(g(\eta^{-1}(S_M), \tau)\mathbb{I}_{\{\tau<T\}} \mid \mathcal{S}_1\right) + h(\eta^{-1}(S_M))\mathbb{E}_{\mathbb{Q}}\left(\mathbb{I}_{\{\tau<T\}^c} \mid \mathcal{S}_1\right)\end{aligned}$$

The first term in the right-hand side can be written as:

$$\sum_{i=1}^{M-1}\left[\prod_{j=1}^{i-1} Pr(I_j = 0)Pr(I_i = 1)\int_{t_{i-1}}^{t_i} g(\eta^{-1}(S_M), t)d\mathbb{Q}_\tau(t \mid \mathcal{S}_1, t_{i-1} \le \tau \le t_i)\right]$$

which is equal, from (1.36), to

$$\sum_{i=1}^{M-1}\left[\prod_{j=1}^{i-1} Pr(I_j = 0)Pr(I_i = 1)\int_0^{+\infty} g(\eta^{-1}(S_M), l^{(i)}(u))f_{Z_i}(u)du\right] \tag{1.41}$$

where $l^{(i)}(\cdot)$ is defined by (1.38), $f_{Z_i}(\cdot)$ is the inverse gaussian density defined by (1.37) and we set $\prod_{j=1}^{0} Pr(I_j = 0) = 1$. The second term is given by:

$$\prod_{j=1}^{M} Pr(I_j = 0) h(\eta^{-1}(S_M))$$

Clearly, the main problem is the computation of the integral in (1.41) for which analytical computation is mostly unfeasible. Our alternative proposal is to estimate the integral using an unbiased estimator. It is straightforward that the new estimator of ν, say ν^*, will present grater variance than the Rao Blackwellized estimator $\tilde{\nu}$.

Concretely, for any $i = 1, 2, \ldots, M$, let

$$s_i(u) = g(\eta^{-1}(S_M), l^{(i)}(u)) f_{Z_i}(u)$$

and let f_i be a density function with support on $[0, +\infty)$ we are able to simulate from. An unbiased estimator of $\int_0^{+\infty} s_i(u)du$ is:

$$T_i = \frac{s_i(X)}{f_i(X)}$$

where X is simulated from f_i. The variance of the estimator depends crucially on the ratio $r(u) = \frac{s_i(u)}{f_i(u)}$ so that, in order to obtain a good Monte Carlo performance of ν^*, we have to choose carefully f_i. Clearly in this case it is not possible to ensure that ν^* is always better than $\hat{\nu}$, as stated in (1.40) for the estimator $\tilde{\nu}$. The easiest available way is to choose $f_i = f_{Z_i}$ (1.37) which corresponds to estimate the integral as $\mathbb{E}\left(g(\eta^{-1}(S_M), l^{(i)}(Z_i))\right)$ by a simple Monte Carlo. However for some $g(\cdot)$ this solution can be very inefficient and the introduction of different importance sampling weights is necessary.

If we consider the payoff defined in (1.34) for the pricing of the Down-Out Call option with rebate, then we have for any $i = 1, 2, \ldots, M$

$$\begin{aligned}\int_0^{+\infty} s_i(u)du &= \int_0^{+\infty} g(\eta^{-1}(S_M), l^{(i)}(u)) f_{Z_i}(u)du \\ &= \int_0^{+\infty} Re^{-rl^{(i)}(u)} f_{Z_i}(u)du \\ &= \int_0^{+\infty} Re^{-r\left(t_{i-1}+\frac{(t_i-t_{i-1})u}{u+1}\right)} f_{Z_i}(u)du = Re^{-rt_{i-1}}\mathbb{E}\left(e^{-r\frac{(t_i-t_{i-1})Z_i}{Z_i+1}}\right)\end{aligned}$$

where R is the rebate and r is the interest rate. Since r is usually very small and $h(u) = \frac{(t_i-t_{i-1})u}{u+1}$ is bounded by 0 and $t_i - t_{i-1}$, the random variable inside the expectation is very closed to 1 and we expect a low variance of the Monte Carlo estimator; in this case it seems there is no need of an importance sampling correction.

1.6 Numerical Example

To test our algorithm we consider the following model:

$$\begin{aligned} dY_t = \sin(Y_t)dt + dW_t \quad & Y_0 = y, \\ & 0 \le t \le T \end{aligned} \tag{1.42}$$

It is easy to verify that Y satisfies conditions $C1 - C3$ for the Exact Algorithm 1. Let us suppose that we want to evaluate:

$$\nu = \mathbb{E}\left(Y_T \mathbb{I}_{\{\tau > T\}}\right) \tag{1.43}$$

where, as usual, $\tau = \inf\{t \ge 0 : Y_t \notin H\}$ with $H = (a, b) \subset \mathbb{R}$ under the assumption that $\inf\{\emptyset\} = +\infty$. This is just a particular case of the general barrier problem (1.5). Since we don't know the law at time T of the killed diffusion it is clear that the explicit computation of (1.43) is not possible and we have to

resort to Monte Carlo methods to estimate ν.

In this simulation study, we investigate the performance of the estimator of ν produced by the Exact Monte Carlo method (hereafter $E1$).

In Fig. 1-4 we check, for different choices of the starting point y and the barriers' values a and b, the convergence of $E1$ plotting the estimated values of ν based on different Monte Carlo sample sizes (up to $2 \cdot 10^6$). In particular Fig. 1 and Fig. 2 show the Monte Carlo estimates in the one-barrier case. The red line corresponds to the Blackwellized version (see section 1.5.2) of $E1$ (black line). Both figures show a strong agreement between the two estimators. Fig 3 and 4 show the convergence of $E1$ in the two barriers case.

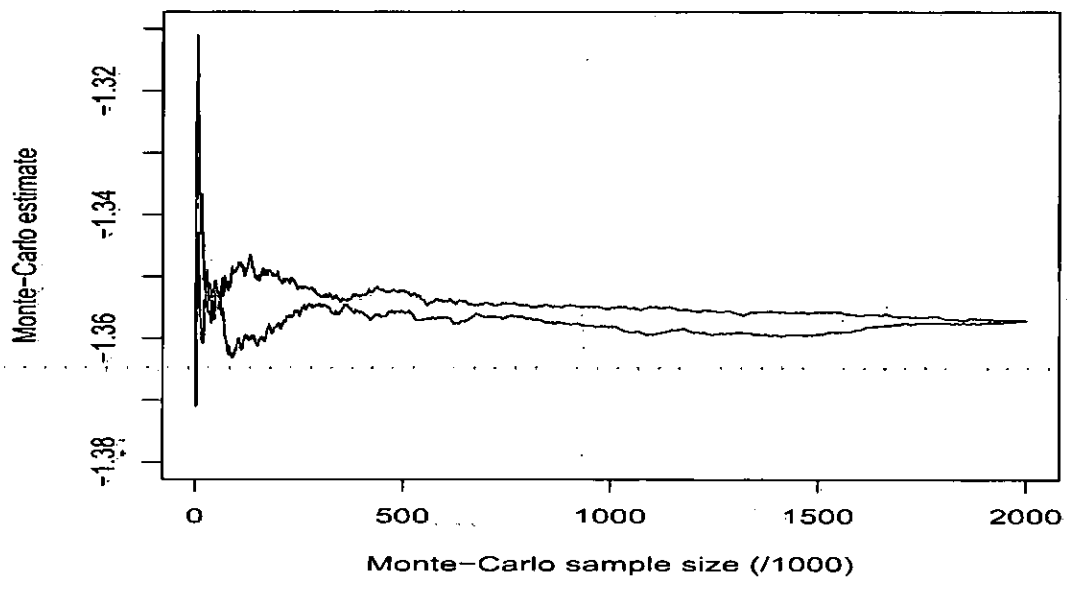

Figure 1.1: Convergence of Monte Carlo estimator: $y = 0$; $b = 3$; $T = 5$

Fig. 5 to 8 propose a comparison between $E1$ and the estimators based on the continuous Euler scheme

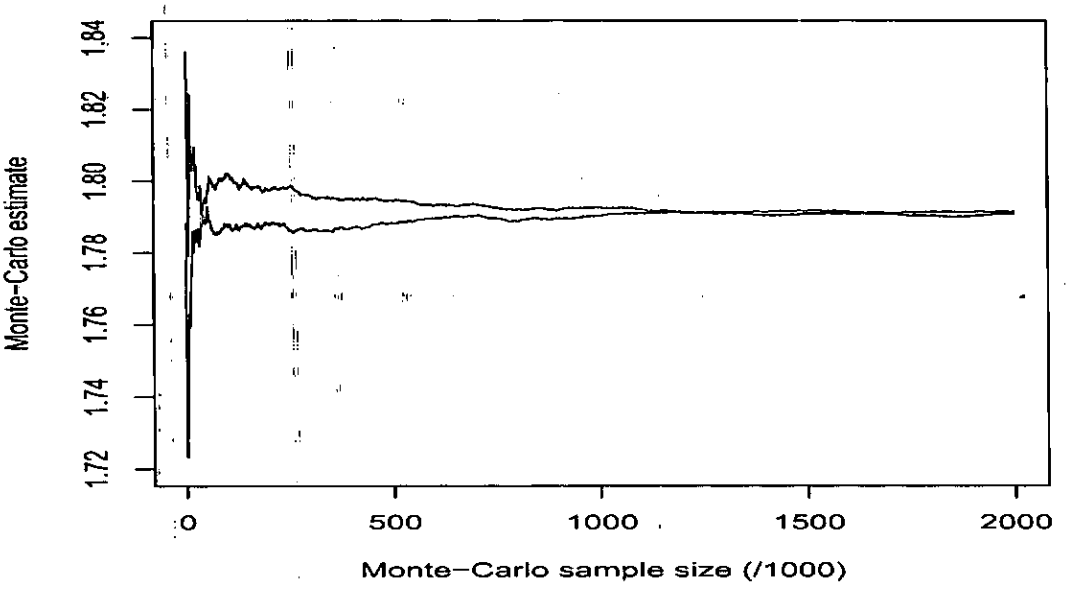

Figure 1.2: Convergence of Monte Carlo estimator: $y = 1.5$; $b = 4.5$; $T = 5$.

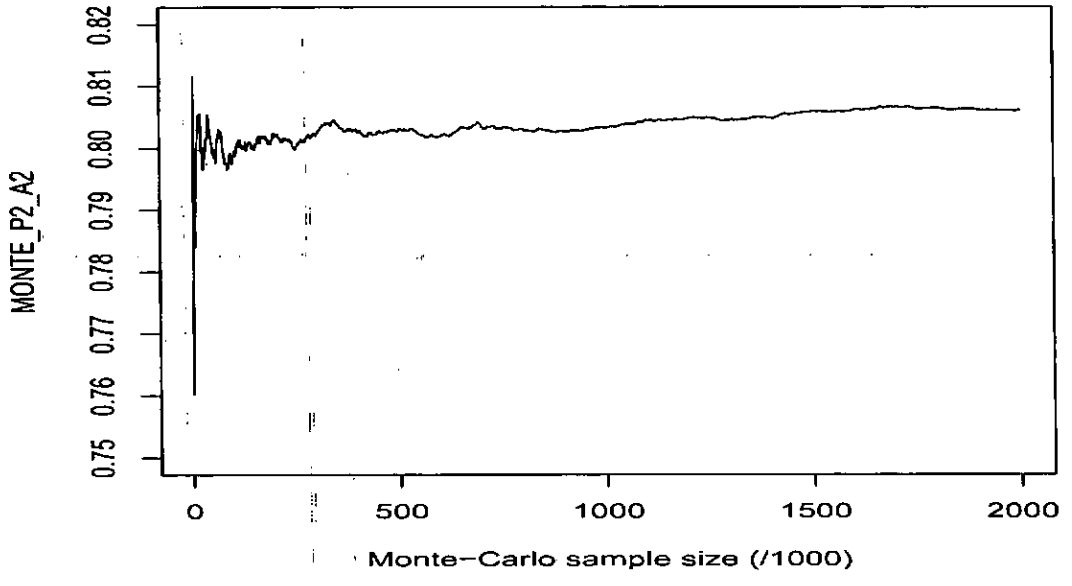

Figure 1.3: Convergence of Monte Carlo estimator: $y = 0$; $a = -3.5$; $b = 4.5$; $T = 5$.

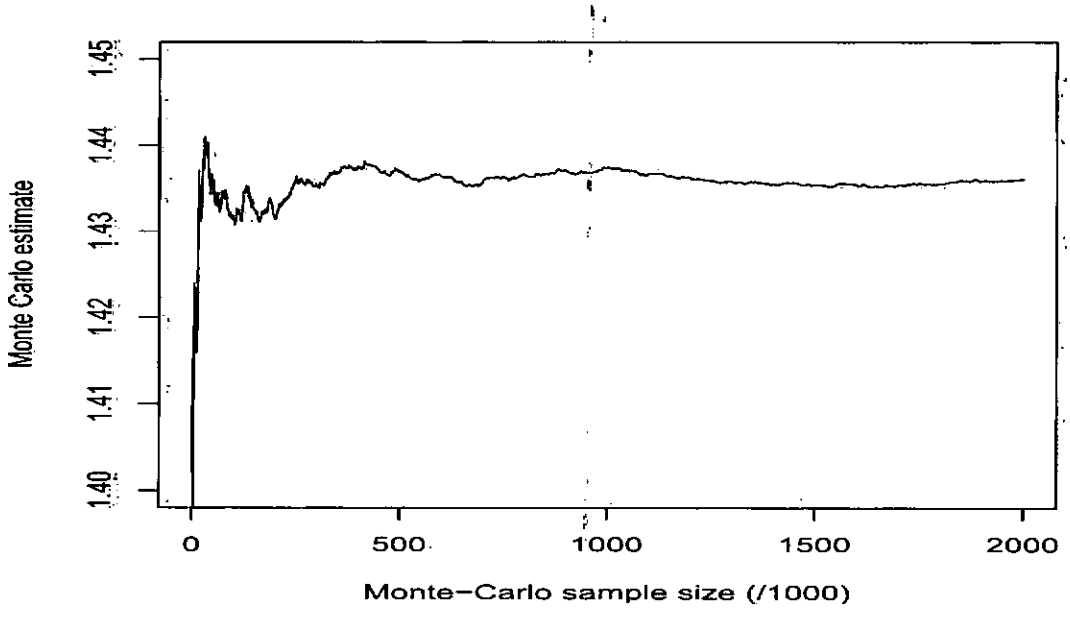

Figure 1.4: Convergence of Monte Carlo estimator: $y = 2$; $a = 1$; $b = 4.5$; $T = 5$.

($E2$) and on the discrete Euler scheme ($E3$). In particular, given a Monte Carlo sample sufficiently large (10^6), for the same sets of parameters of Fig. 1-4, we have computed the estimates of $E1$ (*dotted line*) and the estimates produced by $E2$ and $E3$ for different discretization intervals. Then we have plotted the values of $E2$ (*cross*) and $E3$ (*circle*) versus the number of discretization intervals.

As we expected, the values of $E2$ and $E3$ converge to $E1$ as the number of discretization interval increases. Indeed it was shown by Gobet (2000) that, for killed diffusions, the weak approximation error of Euler schemes decreases to 0 as the number of discretization intervals increases. When the Monte Carlo sample size is large enough, Monte Carlo error is negligible and the estimated values are affected mainly by the discretization error. In this context the distance between the values of $E2$ and $E3$ and the dotted line is a good representation of the (weak) discretization error affecting the Euler schemes and their convergence to the dotted line reflects the theoretical convergence of the corresponding expected values. Furthermore, according to the conclusions of Gobet, we notice that the estimates based on the continuous Euler scheme ($E2$) show a better convergence than the estimates based on the discrete Euler scheme ($E3$).

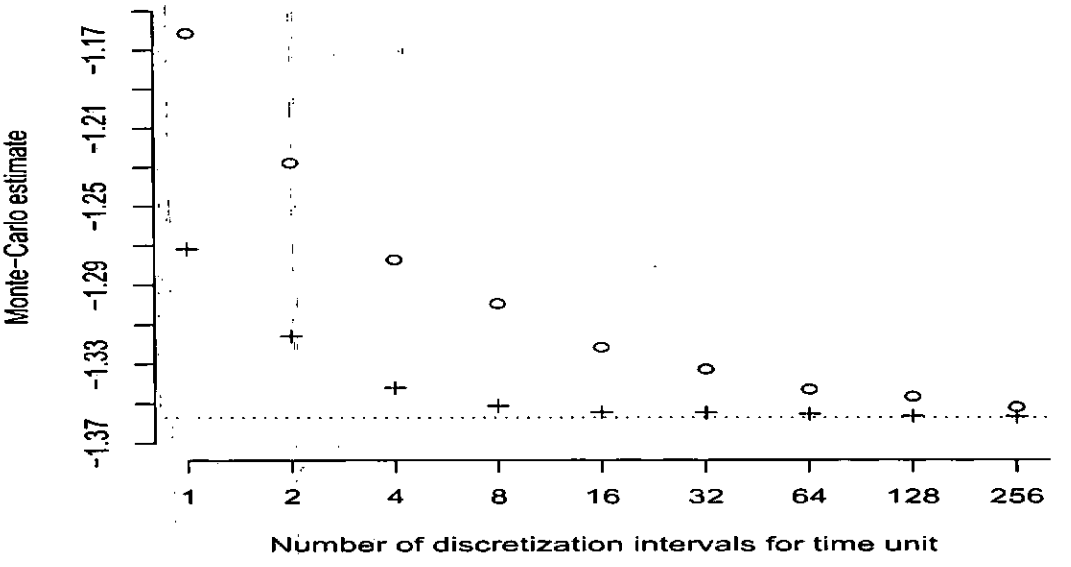

Figure 1.5: Monte Carlo Vs. Euler: $y = 0$; $b = 3$; $T = 5$.

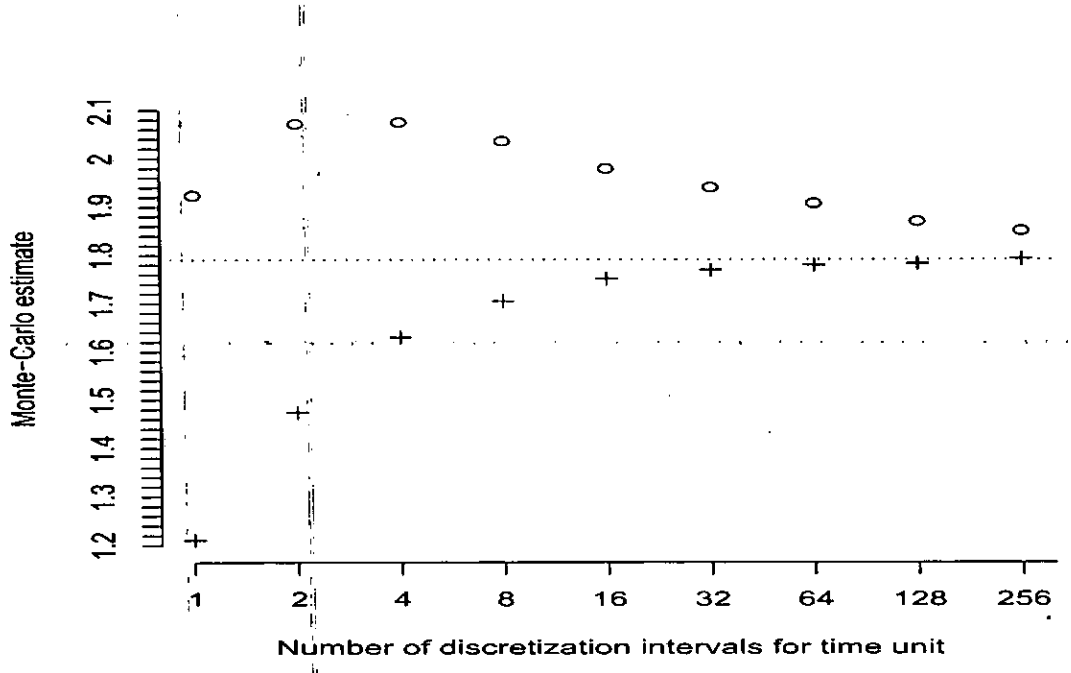

Figure 1.6: Monte Carlo Vs. Euler: $y = 1.5$; $b = 4.5$; $T = 5$.

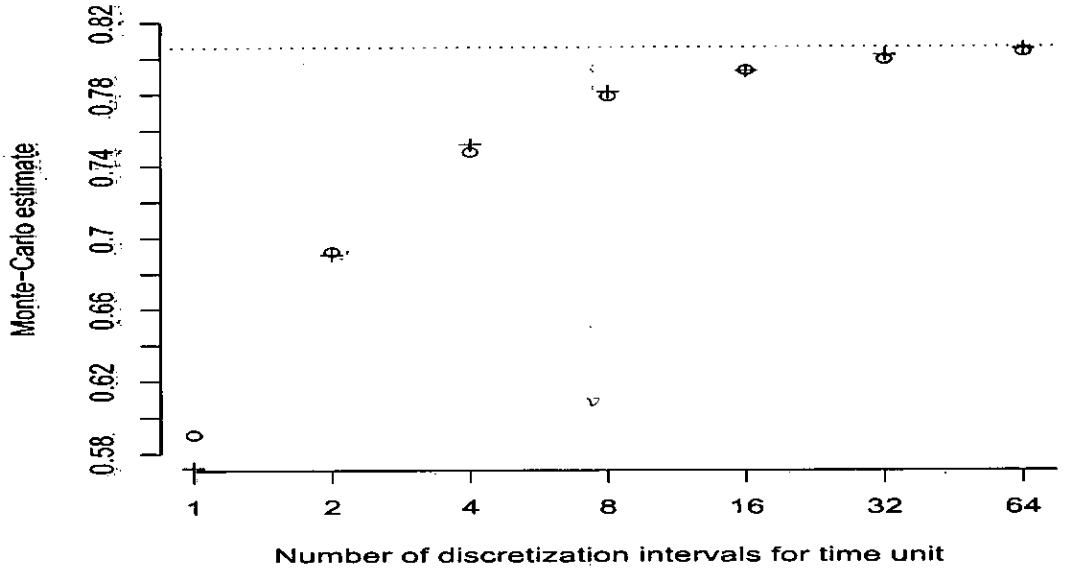

Figure 1.7: Monte Carlo Vs. Euler: $y = 0$; $a = -3.5$; $b = 4.5$; $T = 5$.

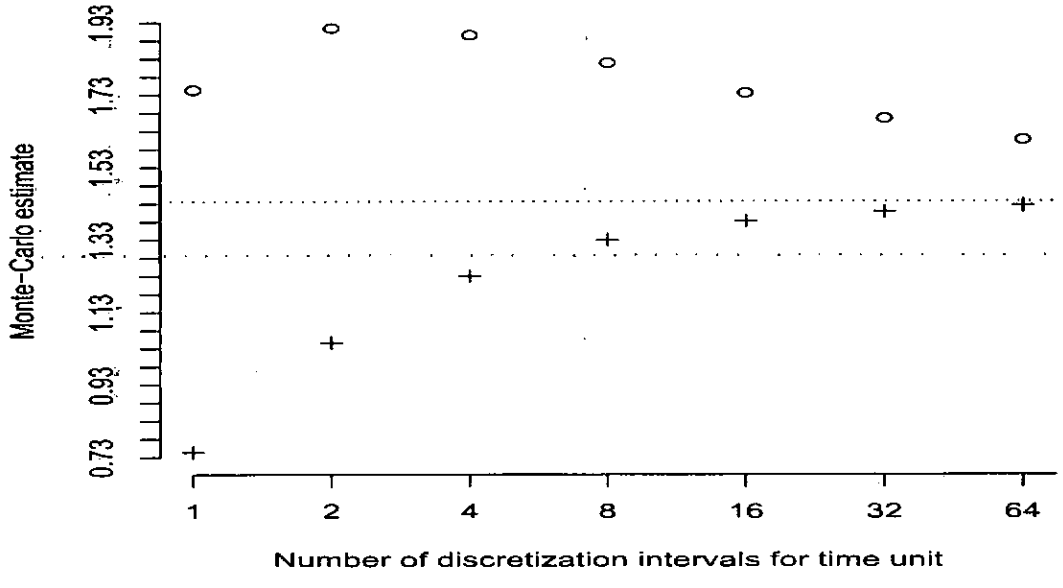

Figure 1.8: Monte Carlo Vs. Euler: $y = 2$; $a = 1$; $b = 4.5$; $T = 5$.

When we compare the performances of $E2$ and $E3$ along the four figures, we observe that both in the one barrier case (Fig. 5 and 6) and in the two barriers case (Fig. 7 and 8) Euler-based estimators behave very poorly when the starting point of the process is different from 0 (Fig. 6 and 8). Moreover in these cases, $E2$ converges to the $E1$ value from below while $E3$ converges from above. The reasons why Euler schemes show such typical behaviors certainly need to be investigated further both from an empirical and a theoretical point of view.

In practical applications, there is an interest in the comparison between the computational times of the Exact Monte Carlo method and the Euler-based methods. However such comparison is not straightforward since the Euler schemes are subjected to a trade-off between computational time and discretization error: i.e, as the estimates of $E2$ and $E3$ converge to the dotted line, the time needed to produce them increases. This is clearly represented in Fig. 5-8 where we have marked with the red color those values of $E2$ and $E3$ whose computational time is greater than the computational time needed for the estimate of $E1$. In these cases our procedure is both more accurate and more efficient. However if we consider just the black points the estimates produced by $E2$ and $E3$ can result strongly biased, in particular when (Fig. 6 and 8) the starting point y of the process is different from 0. We notice that, in the comparison with the Euler schemes, the efficiency of our algorithm decreases in the two barriers case (Fig. 7 and 8): this is a consequence of the fact that in the two barriers case our algorithm is characterized by greater computational complexity (see section 1.4.3). However such effect does not seem to affect dramatically the efficiency of our method in the comparison with the Euler-based methods .

1.7 Conclusion

In this chapter we have developed and implemented a novel Monte Carlo method for the estimation of the expected value of a class of functionals of diffusion processes. In particular we consider functionals involving barriers. In these cases the computation (or approximation) of the expected value is typically challenging. In fact, it is common practice to discretize the underlying diffusion and run Monte Carlo simulation on the discretized process. This clearly introduces a bias in Monte Carlo simulation. Such bias can be reduced only at the cost of a larger computational effort. In comparison, our method is unbiased

and efficient.

The method is essentially based on two steps. In the first step we perform the Exact Algorithm to simulate a Skeleton of the diffusion. In the second step we simulate the target functional conditional on the Skeleton. Theorem 1.2 provides the conditional law of the diffusion given the Skeleton. Crucially this is the law of a product of independent Brownian bridges whose crossing probability and hitting time's density have been extensively investigated in stochastic analysis. Using these results we can simulate exactly the target functional. In particular in the one barrier case we apply the well known Bachelier-Levy theory which provides explicit formula for the crossing probability of a slope boundary for Brownian motion. In the two barriers case we develop an original procedure to simulate from the two-sided crossing probability. The procedure makes use of Doob's representation of the crossing probability of the Brownian motion as a telescopic sum.

The results from the simulation studies confirm our believes. In particular in every case we have considered:

- Euler-based estimators converge to our estimator as the length of the discretization interval decreases (*unbiasedness*)
- as the Euler estimator's bias decreases, our estimator overtakes it in terms of efficiency (*efficiency*).

The theory presented in this chapter inspires many further developments and ideas for future research. One main objective is to extend the methodology to larger classes of diffusion processes as well as to other time-continuous processes of large use, like jump-diffusion processes, Levy processes or stochastic volatility processes. This is the main motivation of the material presented in Chapter 2 and 3.

Another domain of interest is to explore the potentialities of our method in financial applications. An example of such potentialities is given in Chapter 4 where our method is applied to jump diffusion processes with state dependent intensity for modeling credit risk. Another natural area of application is the pricing of barrier and lookback options.

Furthermore, even though the results from our simulation study are fully satisfactory, it is worth to carry out a more comprehensive numerical simulation. In particular the aspects linked to the efficiency of our method in comparison with Euler-based methods need to be deepen. We also mention the fact that a systematic simulation study can be particularly useful also in the context of stochastic numerical analysis to address further investigations on the weak convergence of discretization methods.

Chapter 2

Monte Carlo Simulation for Diffusions with unbounded drift

2.1 Introduction

The method described in Chapter 1 allowed us to construct an unbiased Monte Carlo estimator of ν (1.2) when the process X is a unidimensional diffusion. Crucially, we don't need knowledge of the transition densities of the process: this makes our method considerably more general than the other unbiased Monte Carlo methods. However some restrictions we impose on X can be particularly demanding when we deal with models actually used in practice, for example in finance. Therefore it seems a priority to extend our method to a wider class of models. In this chapter we discuss some ideas which allow to enlarge the class of models to which the Exact Monte Carlo method can be applied.

In section (1.3) we imposed restrictions on the process X by introducing a set of conditions on the drift of its transformation $Y = \eta(X)$:

$$dY_t = \alpha(Y_t)dt + dW_t; \quad Y_0 = y, \qquad 0 \leq t \leq T \tag{2.1}$$

We recall these conditions here:

C_1 The drift function α is *differentiable*

C_2 The function $(\alpha^2 + \alpha')/2$ is *bounded below*

C_3 The function $(\alpha^2 + \alpha')/2$ is *bounded above*

We will denote by d_1 this set of conditions and $\mathcal{D}_1$ the corresponding family of processes Y.
In this chapter we focus only on diffusion processes Y of the type (2.1) (i.e. with unit diffusion coefficient) and consider the problem of the Monte Carlo estimation of:

$$\nu = \mathbb{E}\left(\phi(Y)\right) = \mathbb{E}\left(g(Y_T)\mathbb{I}_{\{\tau>T\}} + f(Y_T)\mathbb{I}_{\{\tau\leq T\}}\right) \tag{2.2}$$

where g and f are measurable real valued functions, τ is the first exit time of the process Y from a set H. The set H will be a generic open interval $(a,b) \subset \mathbf{R}$ such that the first point of the process $Y_0 = y \in (a,b)$. The goal is to apply our Monte Carlo framework for the estimation of (2.2) to families $\mathcal{D}$ of processes Y defined by a set of conditions d such that $d_1 \Rightarrow d$ (equivalently $\mathcal{D}_1 \subseteq \mathcal{D}$). In particular we will try to relax conditions $C2$ and $C3$ since condition $C1$ is generally satisfied. Since in order to apply our method to general diffusions X we have to perform the transformation $Y = \eta(X)$, it is clear that, relaxing conditions on Y, we extend the set of processes X to which the method can be applied. Moreover the restriction of the analysis to (2.2) does not imply any loss of generality as we can reformulate the barrier problem (1.2) for general diffusion X in terms of (2.2) (see section 1.2).
We show how to construct the Exact Monte Carlo estimator of (2.2) for these wider classes of processes. Basically we resort to the scheme set in Chapter 1. However we need to introduce some further results and constructions in order to apply the "old" framework to the "new" models.

2.2 Exact Monte Carlo via Exact Algorithm 2

2.2.1 Preliminaries

In this section we present a first method to generalize $\mathcal{D}_1$. The method is based on the *Exact Algorithm 2* (EA2) of Beskos et al. (2004).

Specifically, we consider a class $\mathcal{D}_2$ of diffusion processes Y (2.1) satisfying the following set of assumptions d_2:

C_1' The drift function α is differentiable

C_2' The function $(\alpha^2 + \alpha')/2$ is *bounded below*

C_3' For all $u \in \mathbb{R}$, $(\alpha^2 + \alpha')/2$ is bounded *either* on $(-\infty, u)$ *or* on $(u, +\infty)$

It is clear that $d_1 \Rightarrow d_2$ so that $\mathcal{D}_1 \subseteq \mathcal{D}_2$. In particular, allowing $\lim_{u\to-\infty}(\alpha^2(u) + \alpha'(u))/2 = \infty$ or $\lim_{u\to+\infty}(\alpha^2(u) + \alpha'(u))/2 = \infty$, we have weakened condition C_3 which is, in practical applications, the most demanding condition in d_1.
In the following we assume that in C_3' the function $(\alpha^2 + \alpha')/2$ is bounded on the intervals $\{(u, +\infty)\}_{u\in\mathbf{R}}$; analogous arguments hold in the symmetric case.
Basically the structure of the algorithm is the same as the one described in Chapter 1. We simulate somehow an exact Skeleton of the process Y and then, conditionally on the Skeleton, our target functional ϕ. Unfortunately in order to sample the Skeleton we can not make use of the Exact Algorithm 1 (section 1.3) since conditions d_2 are more general than d_1. In its place we use the Exact Algorithm 2. Exactly like $EA1$, $EA2$ produces a Skeleton of the process Y: an exact realization of the process at a set of discrete, randomly sampled time points. However the Skeleton produced by EA2 is intrinsically different from the Skeleton ($\mathcal{S}_1$) produced by EA1: it has different structure and gives rise to a different decomposition of the process Y. We will denote it by $\mathcal{S}_2$.

2.2.2 The Exact Algorithm 2

We recall the construction of the rejection sampler of section 1.3.1. We have seen there that, given conditions C_1 and C_2 on the diffusion process Y, the main problem is determining or sampling an upper bound $\mathcal{U}(\omega)$ for the mapping $t \mapsto \varphi(\omega_t)$ where $\omega \sim \mathbb{Z}$, $\mathbb{Z}$ being the measure corresponding to the Biased Brownian motion (1.12). When condition C_3 holds, $\varphi(\cdot)$ is bounded by $\mathcal{U}$ and the problem is easily solved:

$$\mathcal{U}(\omega) = \mathcal{U}$$

The resulting algorithm is then the Exact Algorithm 1 described in section 1.3.2 where the candidate points of the Skeleton are sampled from a Brownian bridge measure.
When on the contrary condition C_3' holds, we can identify an upper bound for $t \mapsto \varphi(\omega_t)$ after decomposing the proposed path ω at its minimum, say $m^* = m^*(\omega)$ and considering:

$$\mathcal{U}(m^*) = \sup\{\varphi(u); u \geq m^*(\omega)\} \tag{2.3}$$

Recalling Theorem 1.1, if the following conditions are satisfied:

1. The law of m^* can be simulated (under $\mathbb{Z}$)

2. The path $\omega \sim \mathbb{Z}$ can be simulated conditionally on its minimum m^* at a discrete set of time points

we can construct an algorithm that generates an exact Skeleton of the process Y.

Algorithm 5 Exact Algorithm 2 (EA2)

1. Initiate a path $\omega \sim \mathbb{Z}$ by drawing $\omega_T \sim h$.

2. Simulate its minimum m^* and the moment when it is achieved t^*.

3. Find an upper bound $\mathcal{U}(m^*)$ for $t \mapsto \varphi(\omega_t)$, $t \in [0, T]$.

4. Produce a realization $\{x_1, x_2, \ldots, x_M\}$, of the Poisson process Π on $[0, T] \times [0, \mathcal{U}(m^*)]$.

5. Simulate the path of $\omega \sim \mathbb{Z}$ at the time instances $\{x_{1,1}, x_{2,1}, \ldots, x_{M,1}\}$ from its conditional distribution given t^* and m^*.

6. Evaluate $\mathcal{N}$.
   If $\mathcal{N} = 0$ go to 7.
   Else go to 1.

7. Output the currently constructed Skeleton of ω: $\mathcal{S}_2 = \mathcal{S}(\omega)$.

We show now how conditions 1 and 2 are satisfied.

1. Distribution of the minimum of a Brownian bridge

The distribution of a minimum of a Brownian bridge is a well known result in stochastic analysis. We apply it for the simulation ot (t^*, m^*).

In fact, given $\omega_T \sim h$, m^* is the minimum of a Brownian bridge started at y in 0 and finishing at ω_T in T and t^* is the moment when the minimum is achieved. Then:

$$(t^*, m^*) \stackrel{d}{=} (t^*, m + y)$$

where, denoting $m^W = \inf\{W_t;\, 0 \le t \le T\}$ and $t^W = \sup\{t \in [0,T] : W_t = m^W\}$,

$$\begin{aligned} Pr\left[m \in db, t^* \in dt\right] &= Pr\left[m^W \in db, t^W \in dt \mid W_T = \omega_T - y\right] \\ &\propto \frac{b(b-(\omega_T - y))}{\sqrt{t^3(T-t)^3}} \exp\left\{-\frac{b^2}{2t} - \frac{(b-(\omega_T-y))^2}{2(T-t)}\right\} \mathrm{d}b\,\mathrm{d}t \end{aligned}$$

with $b \le \min\{0, \omega_T - y\}$ and $t \in [0,T]$. For the derivation of this distribution see for instance Karatzas and Shreve (1991), Ch.2. An efficient way to sample (m, t^*) is presented in Beskos et al. (2004) (Proposition 2).

2. Decomposition of a Brownian bridge at its minimum

In order to sample a realization of $\omega \sim \mathbb{Z}$ at the time instances selected by the Poisson process Π we need a result on the conditional distribution of the Brownian Bridge $BB(0, y; T, \omega_T)$ given its minimum (t^*, m^*).

Let $BES(s_1, x_1; s_2, x_2) = \{BES_u(s_1, x_1; s_2, x_2); s_1 \le u \le s_2\}$ be a 3-dimensional Bessel bridge started at s_1 in x_1 and finishing at s_2 in x_2. For ease of notation we denote $BES(\delta) = BES(0, 0; 1, \delta)$. A complete treatment of definitions and properties related to Bessel processes is given in Revuz and Yor (1994).

We also denote by $BB^c(t^*, m^*) = (BB(0, y; T, \omega_T) \mid t^*, m^*)$ the Brownian bridge starting at y in 0, finishing at ω_T in T and conditioned on obtaining its minimum m^* at time t^*. The following result provides the required characterization of $BB^c(t^*, m^*)$.

Theorem 2.1

The processes $\{BB_s^c(t^*, m^*)\,;\, 0 \le s \le t^*\}$, $\{BB_s^c(t^*, m^*)\,;\, t^* \le s \le T\}$ *are independent with*

$$\{BB_s^c(t^*, m^*)\,;\, 0 \le s \le t^*\} \stackrel{d}{=} \sqrt{t^*}\,\{BES_{(t^*-s)/t^*}(\delta_1)\,;\, 0 \le s \le t^*\} + m^*,$$
$$\{BB_s^c(t^*, m^*)\,;\, t^* \le s \le T\} \stackrel{d}{=} \sqrt{T-t^*}\,\{BES_{(s-t^*)/(T-t^*)}(\delta_2)\,;\, t^* \le s \le T\} + m^*$$

where $\delta_1 = (y - m^*)/\sqrt{t^*}$ *and* $\delta_2 = (\omega_T - m^*)/\sqrt{T-t^*}$.

Proof:

See proof. of Proposition 2 in Asmussen et al. (1995).

□

From the theorem, the Brownian bridge is decomposed by its minimum in two independent Bessel bridges. For simulation purposes, we have rescaled the Bessel processes to obtain bridges of unit length. In fact it is well known (see for instance Bertoin and Pitman, 1994) that:

$$BES_t(\delta) = \sqrt{(\delta \cdot t + BB_{1,t}^{st})^2 + (BB_{2,t}^{st})^2 + (BB_{3,t}^{st})^2}, \quad t \in [0,1]$$

where BB_1^{st}, BB_2^{st}, BB_3^{st} are three independent standard Brownian bridges (bridges of unit length, starting and ending at 0). A standard Brownian bridge can be trivially simulated from a standard Brownian motion by means of:

$$BB_t^{st} \stackrel{d}{=} W_t - tW_1$$

Therefore from a simulation perspective once we have sampled the random vector (t^*, m^*), all we need for the rejection sampling of Algorithm 5 is the simulation of a (marked) Poisson process Π and of six independent Brownian motions evaluated at the set of time points selected by Π. This can be done very efficiently.

2.2.3 Monte Carlo procedure

The Exact Algorithm 2 returns a Skeleton $\mathcal{S}_2$ of the process Y. $\mathcal{S}_2$ is richer than the Skeleton ($\mathcal{S}_1$) produced by EA1. In fact it includes not only a realization of the accepted path ω at the time instances selected by the Poisson process but also the point (t^*, m^*) corresponding to the minimum of ω. In particular, by Theorem 2.1, the point (t^*, m^*) decomposes $\mathcal{S}_2$ in two blocks whose points have been sampled from two different independent processes. It is convenient to choose a representation of $\mathcal{S}_2$ that takes into account this underlying structure:

$$\mathcal{S}_2 = \mathcal{S}_{1,2} \cup \mathcal{S}_{2,2}$$

where, for $j = 1, 2$:

$$\mathcal{S}_{i,j} = \{(t_{i,j}, S_{i,j}) : i = 0, 1, \ldots, M_j\}$$

with $(t_{0,1}, S_{0,1}) \equiv (0, y)$, $(t_{M_2,2}, S_{M_2,2}) \equiv (T, \omega_T)$ and $(t_{M_1,1}, S_{M_1,1}) \equiv (t_{0,2}, S_{0,2}) \equiv (t^*, m^*)$.
Given the Skeleton $\mathcal{S}_2$, all we need in order to simulate the target functional ϕ in 2.2 is the simulation of an event of probability P^* given by:

$$\begin{aligned} P^* &= Pr\left[Y_t \in (a, b),\, 0 \le t \le T \mid \mathcal{S}_2\right] \\ &= Pr\left[Y_t \le b,\, 0 \le t \le T \mid \mathcal{S}_2\right] \mathbb{I}_{\{m^* \ge a\}} \end{aligned}$$

In the one barrier case with lower barrier, b is equal to $+\infty$ and the indicator function gives directly P^* without any further step. In the presence of an upper barrier (either in the one barrier or in the two barriers case) we need to characterize the probability in the right hand side.

For any $i = 1, 2, \ldots, M_j$ and $j = 1, 2$, we denote by $Y_{i,j} \mid \mathcal{S}_2$ the process $\{Y_t : t_{i-1,j} \le t \le t_{i,j} \mid \mathcal{S}_2\}$.

The following result provides the conditional distribution of the process Y given $\mathcal{S}_2$.

Theorem 2.2

The processes $\{Y_{i,j} \mid \mathcal{S}_2\}$ *are independent and, for any* $i = 1, 2, \ldots, M_j$:

$$\begin{aligned} Y_{i,j} \mid \mathcal{S}_2 &\stackrel{d}{=} BES\left(0, S_{i,j} - m^*; \Delta_{i,j}, S_{i-1,j} - m^*\right) + m^*, \qquad j = 1 \\ Y_{i,j} \mid \mathcal{S}_2 &\stackrel{d}{=} BES\left(0, S_{i-1,j} - m^*; \Delta_{i,j}, S_{i,j} - m^*\right) + m^*, \qquad j = 2 \end{aligned}$$

where $\Delta_{i,j} = t_{i,j} - t_{i-1,j}$.

Proof:

Let $\omega \in C\left([0,T]\right)$ such that $\omega \sim \mathbb{Z}$. Consider the unit intensity marked Poisson process Π on $[0,T] \times [0, \mathcal{U}(m^*)]$:

$$\Pi = (\Phi, \Psi) = \{(x_{i,1}, x_{i,2})\,; i = 1, 2, \ldots, M-1\}$$

We define

$$\xi := f(\omega, \Phi) = \left\{(0, \omega_0), (T, \omega_T), (t^*, m^*), \left\{\left(x_{i,1}, \omega_{x_{i,1}}\right)\right\}_{i=1,2,\ldots,M-1}\right\}$$

and an event A such that

$$Pr(A \mid \omega, \Phi) = \prod_{i=1}^{M-1} \left[1 - \frac{\varphi\left(\omega_{x_{i,1}}\right)}{\mathcal{U}(m^*)}\right] = \mathcal{P}(\xi)$$

For the Girsanov Theorem $Y \stackrel{d}{=} \omega \mid A$ and, by construction $\mathcal{S}_2 := \xi \mid A$. Then, applying Lemma 1.1, since, conditionally on the minimum (t^*, m^*), ω and Φ are independent:

$$\mathcal{L}_{Y|\mathcal{S}_2} = \mathcal{L}_{\omega|A,\xi,(t^*,m^*)} = \mathcal{L}_{\omega|A,\xi,(t^*,m^*),\Phi} = \mathcal{L}_{\omega|\xi,(t^*,m^*),\Phi}$$

which proves the theorem (by definition of $\mathbb{Z}$ and Theorem 2.1).

□

As a consequence of the theorem, P^* can be written as the product of *crossing probabilities* of Bessel bridges:

$$P^* = \left\{ \prod_{i=1}^{M_1} p^*_{i,1} \prod_{i=1}^{M_2} p^*_{i,2} \right\} \mathbb{I}_{\{m^* \geq a\}}$$

where

$$\begin{aligned} p^*_{i,1} &= Pr\left[BES_s\left(0, S_{i,j} - m^*; \Delta_{i,j}, S_{i-1,j} - m^*\right) \leq b - m^*, 0 \leq s \leq \Delta_{i,j}\right] \\ p^*_{i,2} &= Pr\left[BES_s\left(0, S_{i-1,j} - m^*; \Delta_{i,j}, S_{i,j} - m^*\right) \leq b - m^*, 0 \leq s \leq \Delta_{i,j}\right] \end{aligned}$$

Crossing probability of a Bessel bridge

In Chapter 1 we have established crossing probabilities (one sided and two sided) of a Brownian bridge exploiting its connections with the Brownian motion. Here we proceed similarly using the relationship between Bessel bridge and Brownian bridge.

In particular we use the fact that a Bessel bridge is distributed as a Brownian bridge conditioned to be positive. This means that we can write:

$$Pr\left[BES_u(s,x;t,y) \leq \lambda, s \leq u \leq t\right] = \frac{Pr\left[BB_u(s,x;t,y) \in (0,\lambda), s \leq u \leq t\right]}{Pr\left[BB_u(s,x;t,y) > 0, s \leq u \leq t\right]} \tag{2.4}$$

for $x, y \in (0, \lambda)$.

The two probabilities in the ratio are familiar.

The one at denominator is the probability that a Brownian bridge does not hit a given barrier (here a lower barrier $\lambda = 0$) and it can be derived explicitly from (1.23):

$$Pr\left[BB_u(s,x;t,y) > 0, s \leq u \leq t\right] = 1 - e^{-2\frac{xy}{t-s}} \tag{2.5}$$

The one at numerator is the probability that a Brownian bridge does not exit a given double barrier $(0, \lambda)$. This probability has been extensively discussed in section 1.4.3: it is given by the infinite sum (1.24) where in (1.30) we set $\lambda_2 = \lambda$ and $\lambda_1 = 0$.

As a consequence, we can write (2.4) as an alternating series and use Proposition 1.2 to construct sequences $\{\overline{n}_k^*\}$ and $\{\underline{n}_k^*\}$ which converge (from above and from below) to (2.4). In particular, given the parameters of the Bessel bridge and the barrier λ,

$$\begin{aligned} \overline{n}_k^* &= \frac{1}{c}\overline{n}_k \\ \underline{n}_k^* &= \frac{1}{c}\underline{n}_k \end{aligned}$$

where c is a constant equal to (2.5) and the sequences $\{\overline{n}_k\}$ and $\{\underline{n}_k\}$ are defined by (1.30) and (1.31).

We can allow the starting point x of the Bessel bridge to be equal 0; in this case we can derive the corresponding crossing probability as the limit of the ratio (2.4) for $x \to 0$.

2.3 Exact Monte Carlo via truncation of the drift

We consider the problem of the Monte Carlo estimation of ν for a family $\mathcal{D}_3$ of diffusion processes Y defined by (2.1) and satisfying the following set of conditions d_3:

C_1'' The drift function α is *differentiable* on the *closure* $\overline{H}$ of H

C_2'' The function $(\alpha^2 + \alpha')/2$ is *bounded* on $\overline{H}$

It is then clear that $d_1 \Rightarrow d_3$.

In this context, by truncating the drift function at the barriers a and b, it is possible to define a process

$\tilde{Y}$:

$$d\tilde{Y}_t = \tilde{\alpha}(\tilde{Y}_t)dt + dW_t; \quad \tilde{Y}_0 = y, \qquad 0 \le t \le T \tag{2.6}$$

such that:

1. $\alpha(u) = \tilde{\alpha}(u)$ for $u \in H$
2. $\tilde{Y} \in \mathcal{D}_1$

It is convenient here to adopt the approach introduced in Chapter 1 thinking of the stochastic processes Y and $\tilde{Y}$ as measures $\mathbb{Q}^{\alpha}$ and $\mathbb{Q}^{\tilde{\alpha}}$ induced on the space $\mathcal{C}$ of the continuous functions $\omega = \{\omega_s : 0 \le s \le T\}$ from $[0,T]$ to $\mathbf{R}$. In this context ϕ in (2.2) is a functional $\mathcal{C} \to \mathbf{R}$:

$$\phi = \phi(\omega) = g(\omega_T)\mathbb{I}_{\{\tau_H < T\}} + h(\omega_T)\mathbb{I}_{\{\tau_H < T\}^c} \tag{2.7}$$

with τ_H defined by

$$\tau_H := \inf\{t > 0 : \omega_t \notin H\}$$

under the usual convention $\inf\{\emptyset\} = +\infty$.

Theorem 2.3

If $\alpha(u) = \tilde{\alpha}(u)$ for any $u \in H$ and the function $\phi : C \to \mathbf{R}$ is measurable with respect to $\mathcal{F}_{T\wedge\tau_H}$ then:

$$\mathbb{E}\left(\phi(Y)\right) = \mathbb{E}(\phi(\tilde{Y}))$$

Proof:

We want to show:

$$\mathbb{E}(\phi(Y)) = \mathbb{E}_{\mathbb{Q}^\alpha}(\phi(\omega)) = \mathbb{E}_{\mathbb{Q}^{\tilde{\alpha}}}(\phi(\omega)) = \mathbb{E}\left(\phi(\tilde{Y})\right)$$

Then:

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}^\alpha}(\phi) &= \mathbb{E}_{\mathbb{Q}^\alpha}(\phi(\omega)) \\
&= \mathbb{E}_{\mathbb{Q}^{\tilde{\alpha}}}\left(\phi(\omega)\frac{d\mathbb{Q}^\alpha}{d\mathbb{Q}^{\tilde{\alpha}}}(\omega)\right) \\
&= \mathbb{E}_{\mathbb{Q}^{\tilde{\alpha}}}(\phi(\omega)M_T(\omega))
\end{aligned}$$

where, by Girsanov Theorem, M_T is the Radon-Nikodym derivative of $\mathbb{Q}^\alpha$ with respect to $\mathbb{Q}^{\tilde{\alpha}}$

$$\begin{aligned}
M_T(\omega) &= \frac{d\mathbb{Q}^\alpha}{d\mathbb{Q}^{\tilde{\alpha}}}(\omega) \\
&= \frac{e^{\int_0^T \alpha(\omega_s)d\omega_s - \frac{1}{2}\int_0^T \alpha^2(\omega_s)ds}}{e^{\int_0^T \tilde{\alpha}(\omega_s)d\omega_s - \frac{1}{2}\int_0^T \tilde{\alpha}^2(\omega_s)ds}}
\end{aligned}$$

and $(M_t, \mathcal{F}_t)_{0\le t\le T}$ is $\mathbb{Q}^{\tilde{\alpha}}$-martingale.

So, by the measurability assumption on ϕ and the martingale property of $\{M_t\}_{0\le t\le T}$ we have:

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}^{\tilde{\alpha}}}(\phi(\omega)M_T(\omega)) &= \mathbb{E}_{\mathbb{Q}^{\tilde{\alpha}}}\left(\mathbb{E}_{\mathbb{Q}^{\tilde{\alpha}}}(\phi(\omega)M_T(\omega) \mid \mathcal{F}_{T\wedge\tau_H})\right) \\
&= \mathbb{E}_{\mathbb{Q}^{\tilde{\alpha}}}\left(\phi(\omega)\mathbb{E}_{\mathbb{Q}^{\tilde{\alpha}}}(M_T(\omega) \mid \mathcal{F}_{T\wedge\tau_H})\right) \\
&= \mathbb{E}_{\mathbb{Q}^{\tilde{\alpha}}}(\phi(\omega)M_{T\wedge\tau_H})
\end{aligned}$$

Since on the interval $[0, T\wedge\tau_H)$ the two drift functions α and $\tilde{\alpha}$ coincide, we have:

$$M_{T\wedge\tau_H} = \frac{e^{\int_0^{T\wedge\tau_H} \alpha(\omega_s)d\omega_s - \frac{1}{2}\int_0^{T\wedge\tau_H} \alpha^2(\omega_s)ds}}{e^{\int_0^{T\wedge\tau_H} \tilde{\alpha}(\omega_s)d\omega_s - \frac{1}{2}\int_0^{T\wedge\tau_H} \tilde{\alpha}^2(\omega_s)ds}} = 1 \qquad a.s.$$

The conclusion then follows.

□

The theorem provides the basis for the extension of the method to processes $Y \in \mathcal{D}_3$. Specifically when $Y \in \mathcal{D}_3$ we can construct a process $\tilde{Y} \in \mathcal{D}_1$ such that

$$\nu = \mathbb{E}(\phi(\tilde{Y})) \tag{2.8}$$

and therefore apply the framework of Chapter 1 for the Monte Carlo estimation of ν.
The main problem is that (2.8) holds only if the functional ϕ (2.7) is $\mathcal{F}_{T\wedge\tau_H}$ measurable. This requires that the function $g(\cdot)$ does not depend on ω_T (equivalently Y_T).
This case covers interesting classes of ϕ-function. For example setting $g \equiv 0$ we obtain the case of the *killed* diffusions:

$$\phi(\omega) = h(\omega_T)\mathbb{I}_{\{\tau_H \geq T\}}(\omega)$$

In finance the computation of ν under this scheme corresponds to the pricing of the Down-Out, Up-Out and Double Barrier Options under the assumption of null rebate $R = 0$. However even if we assume the presence of a positive rebate as in section 1.5.1, $g(\cdot)$ depends only on the stopping time τ_H and the functional ϕ (2.7) is still $\mathcal{F}_{T\wedge\tau_H}$-measurable.

Remark 2.1

The process $\tilde{Y}$ satisfying conditions $1-2$ can be constructed by setting:

$$\tilde{\alpha}(u) = \begin{cases} A + Be^u & for \quad -\infty \leq u \leq \eta(a) \\ \alpha(u) & for \quad \eta(a) < u < \eta(b) \\ C + De^{-u} & for \quad \eta(b) \leq u < +\infty \end{cases}$$

where the parameters A, B, C, D are the solutions of the two systems of equations:

$$\begin{cases} A + Be^{\eta(a)} &= \alpha(\eta(a)) \\ Be^{\eta(a)} &= \alpha'(\eta(a)) \end{cases}$$

and

$$\begin{cases} C + De^{-\eta(b)} &= \alpha(\eta(b)) \\ -De^{-\eta(b)} &= \alpha'(\eta(b)) \end{cases}$$

Remark 2.2

Interestingly this extension is more powerful when we deal with the most problematic situation: the two barriers case. In fact in this case H is a bounded interval and all we need is that the drift α "behaves well" in the compact set $\overline{H}$. This is naturally satisfied in many contexts. In the one barrier case on the contrary we still have to ensure boundedness of $\alpha(\cdot)$ on one side ($(-\infty, b]$ or $[a, +\infty)$).

2.4 Conclusion

We have applied our Monte Carlo method to two new classes of models $\mathcal{D}_2$ and $\mathcal{D}_3$ which include $\mathcal{D}_1$. They allow to enlarge consistently the actual domain of our proposal. To obtain these extensions we have applied to the framework of Chapter 1 some new techniques and constructions.

Specifically, for $\mathcal{D}_2$ we have made use of an algorithm (EA2) for the simulation of the Skeleton which is more general than EA1. Since the Skeleton produced by EA2 ($\mathcal{S}_2$) has different properties from the Skeleton produced by EA1, we had to introduce a new Monte Carlo procedure based on the properties of Bessel bridges (Theorem 2.1 and Theorem 2.2).

In order to obtain $\mathcal{D}_3$ we have transformed the original process $Y \in \mathcal{D}_3$ to a new process $\tilde{Y} \in \mathcal{D}_1$ by a truncation of the drift function which leaves the expected value ν unchanged. From Theorem 2.3, the procedure is justified when the function $g(\cdot)$ in (2.2) does not depend on Y_T. The extent of $\mathcal{D}_3$ depends on the set H: interestingly $\mathcal{D}_3$ is wider in the two barriers case than in the one barrier case.

The techniques we have introduced in this chapter can also be applied simultaneously and in some cases this provides further possibilities of extensions. When $g(\cdot)$ does not depend on Y_T we can use the truncation

of the drift described in section 2.3 to construct a process $\tilde{Y} \in \mathcal{D}_2$ from a process Y (2.1) satisfying the following conditions d_4:

- The drift function α is differentiable on $\overline{H}$
- The function $(\alpha^2 + \alpha')/2$ is *bounded below* on $\overline{H}$
- Given a compact set $U \subseteq \overline{H}$, the function $(\alpha^2 + \alpha')/2$ is bounded above on U

Since α and $\tilde{\alpha}$ coincide on H, Theorem 2.3 holds and we can apply the framework of section 2.2 to the process $\tilde{Y}$ in order to sample a Monte Carlo estimator of ν. Notice that in the two barriers case $\overline{H}$ is itself compact and the new set of conditions are equivalent to d_2: there is no actual extension. On the contrary in the one barrier case the new set of conditions allow us to include the important cases when both $\lim_{u\to+\infty}(\alpha^2 + \alpha')/2 = +\infty$ *and* $\lim_{u\to-\infty}(\alpha^2 + \alpha')/2 = +\infty$.
Summarizing, we show in the tables below which are the minimal sets of conditions on the process (2.1) under which the Exact Monte Carlo method can be applied.

FUNCTIONAL ϕ	**MINIMAL COND.**
g not depending on Y_T	
ONE BARRIER	d_4
TWO BARRIERS	d_3
g depending on Y_T	
ONE BARRIER	d_2
TWO BARRIERS	d_2

Table 2.1: summary of the conditions.

Chapter 3

Exact Monte Carlo Simulation for Jump Diffusions

3.1 Introduction

The aim of this chapter is to extend the methodology introduced in Chapter 1 to jump diffusion models. Jump diffusion processes have become increasingly popular in financial modeling over the last 20 years. It became widely acknowledged that the standard Black and Scholes (1973) assumption of log-normal stock diffusion fails in explaining market behavior. Empirical investigations show that stock prices and other asset prices exhibit more complicated behavior than geometric Brownian motion. In particular heavy tails and volatility smile are well known phenomena at least since the market crash of 1987. Therefor the need of models which capture these dynamics motivated the search for alternatives to the Black Scholes framework. Two proposals have received a large consensus among both academics and practitioners: stochastic volatility models and jump diffusion models.

In jump diffusion approach we add to the (diffusion) dynamics of the process Y a discontinuous component (the *jumps*). By choosing the parameters of the jump process appropriately, the model can generate a rich variety of dynamics which account for fat tailed and volatility smile.

However, dealing with jump diffusion processes, for example in pricing derivatives or modeling credit risk, is mathematically much more challenging than the case of pure diffusions. In fact, except for very basic

applications where closed form solutions are available, for most computational problems we have to resort to simulation methods. In particular, for the barrier problem, closed form expressions for the crossing probabilities exist only under the assumptions that the continuous component is a geometric Brownian motion and the jumps are doubly exponential or exponentially distributed (Kou and Wang, 2003)
Nevertheless barrier problems with jump diffusions arise in finance in many applications, most notably in credit risk modeling (Zhou, 2001) and option pricing (Metwally and Atiya, 2002). Similar problems arise also in other fields like computer vision (Zhu, 1999), adaptive control (Caines and Zhang, 1995) and neurobiology (Giraudo and Sacerdote, 1997).
Metwally and Atiya (2002) propose a Monte Carlo procedure based on Brownian bridge's first passage time density for the case when the underlying process is a Merton (1976) jump diffusion. That is, the continuous component is a geometric Brownian motion, the jump times are selected by a Poisson process and the jumps are i.i.d. lognormally distributed. However this is a particularly simple case since the jump component and the diffusion component are independent and both can be easily simulated.
In more complicated situations, for example when we don't know the transition density of the diffusion component of the process, the only way to go is to discretize the process. If the jump component is independent of the diffusion component, jumps' times and amplitudes are not affected by discretization bias and can be simulated exactly. In this case the discretization error of Monte Carlo estimators is the same as in the case of pure diffusions.
Most critical is the case when the intensity of the marked point process driving the jump component is state dependent. The discretization error reflects on the simulation of the jumps, resulting in larger bias of the Monte Carlo estimator. In a recent paper Glasserman and Merener (2004) develop a simulation procedure to deal with this case. Their method is based on a thinning algorithm with state dependent acceptance probability for the simulation of the jump component and a discretization scheme for the simulation of the diffusion component. Surprisingly, under particular conditions on the coefficient functions of the process, the weak convergence order of this scheme equals the weak convergence order of the discretization scheme for pure diffusions.
In this chapter we present a method to construct unbiased Monte Carlo estimator for the barrier problem (2.2) when the underlying process Y is a jump diffusion with state dependent intensity. The method is based on a thinning algorithm for the simulation of the jump component and on the Exact Algorithm for

the simulation of the diffusion component. The algorithm is adaptive: at each jump epoch the parameters of the Exact Algorithm are updated according to the amplitude of the jumps. Conversely the thinning probability of a candidate jump depends on the current state of the process.
The thinning procedure is in the spirit of Glasserman and Merener (2004). However since the Exact Algorithm returns an exact realization of the process at the candidate jump times, thinning probabilities are exact. Another difference with Glasserman and Merener (2004) is that we consider a functional involving barriers while their analysis is addressed to the case when ϕ depends only on the realization of the process Y at a given time instant, i.e. $\phi = \phi(Y_T)$.
Compared with Monte Carlo methods based on discretization schemes, our method is exact and efficient. Furthermore, allowing for state-dependent intensity, it is very general. We point out that, in many applications, the assumption that the intensity of the jump component is independent on the state of the process is too restrictive. An example in the context of credit risk modeling is proposed in Chapter 4 where the use of a state dependent intensity is motivated by economical arguments. Another application to finance can be found in Glasserman and Merener (2003). Finally, as argued by Glasserman and Merener (2004), it might be the case that, even if jumps are Poisson under the physical measure, they don't preserve this property under the pricing measure since the underlying martingale transformation introduces state dependence in the intensity.

3.2 Exact Monte Carlo for jump diffusions

3.2.1 Preliminaries

Let $Y = \{Y_t;\ 0 \leq t \leq T\}$ be the one-dimensional *jump diffusion* process solving the following SDE:

$$dY_t = \alpha(Y_{t-})dt + dW_t + \int_M g(Y_{t-}, z)m(dz, dt); \quad Y_0 = y, \quad 0 \leq t \leq T \tag{3.1}$$

where $m(dz, dt)$ is a counting random measure on the product space $M \times [0, T]$ with $M \subset \mathbf{R}$ and g is a real valued function on $\mathbf{R} \times M$. Let $q(dz, Y_{t-}, dt)$ define the *intensity measure* of $m(dz, dt)$ of the form:

$$q(dz, Y_{t^-}, dt) = q(z, Y_{t^-}, t)dz\,dt \tag{3.2}$$

In other words, m is the counting measure generated by a *marked point process* on $M \times \mathbf{R}$ driven by the intensity function q. Since there is no possibility of confusion, in the following we will indicate with m both the random measure and the corresponding point process.
We introduce the following assumptions d^* on the process Y (3.1):

C_1^* The drift function α is *differentiable*

C_2^* The function $\varphi = (\alpha^2 + \alpha')/2$ is *bounded* by a constant $\mathcal{U}$

C_3^* The intensity function q is dominated by a function $\hat{q} : M \to \mathbf{R}$ such that:

$$\hat{q}(z) = \hat{\lambda}\hat{f}(z); \quad z \in M \tag{3.3}$$

where $\hat{f}(\cdot)$ is a density function with support M and λ is a positive constant.

We denote by $\mathcal{D}_{jump}$ the class of jump diffusion process Y (3.1) satisfying conditions $C_1^* - C_3^*$.
Let Θ be a realization of m:

$$\Theta = \{(Z_1, \tau_1)(Z_2, \tau_2), \ldots, (Z_{N_T}\tau_{N_T})\}$$

with

$$N_T = \int_0^T \int_M m(dz, dt) \tag{3.4}$$

counting the number of arrivals in $M \times [0, T]$. Furthermore we denote by $Y^c(t_1, t_2, y)$ a diffusion process on the time interval $[t_1, t_2)$ with initial point y whose dynamics is described by the following SDE:

$$dY_t^c = \alpha(Y_t^c)dt + dW_t \tag{3.5}$$

In other words Y^c is the continuous component of Y. Notice that, by conditions C_1^* and C_2^*, for any choice of the triple (t_1, t_2, y), Y^c belongs to $\mathcal{D}_1$.

3.2.2 Jump Exact Algorithm

Following the usual scheme, the first stage for the construction of the Monte Carlo estimator is to define a procedure to simulate a Skeleton of the process Y. It is clear that, due to the jump component in (3.1), we can't use directly EA1.

However when the jump component is distributed as a *marked Poisson* process, the algorithm to generate the Skeleton is a straightforward application of EA1. In fact the arrival times $\tau_1, \tau_2, \ldots, \tau_{N_T}$ represent a realization of a Poisson process of intensity $\lambda(t)$ on $[0, T]$. With each point τ_i we associate a random variable Z (the *mark*) whose density may depend on τ_i but not on the other points τ_j $(j \neq i)$ and the variables Z associated with different arrival times are independent. Then, using representation (3.2):

$$q(z, Y_{t^-}, t) = \lambda(t) f_Z(t, z)$$

where $\lambda(\cdot)$ is a nonnegative function $[0, T] \to \mathbf{R}^+$ and, for any $t \in [0, T]$, $f_Z(t, \cdot)$ is a density function with support M .

In this case we can just simulate a realization Θ of the process m independently on Y. Between any two jump times the process Y follows a pure diffusion Y^c. Therefore, given Θ, we can proceed to simulate a Skeleton of Y updating the parameters of the Exact Algorithm when a jump occurs, according to the amplitude of the jump.

In the general case (3.2) the simulation of the Skeleton is more challenging due to the dependence of the intensity q on the process Y. In particular we are not allowed to sample a realization of the process m on $[0, T] \times M$ before proceeding to the actual simulation of the process Y. The criticism here is that we don't have "a priori" information about the location of the jumps on $[0, T]$.

We circumvent the problem using an *adaptive* simulation procedure which combines the Exact Algorithm for the simulation of the continuous component of Y (3.5) and a *stochastic thinning* algorithm for the simulation of m. The idea is based on the following result.

Let $\hat{m}$ be the Poisson random measure on the product space $M \times [0,1] \times [0,T]$ with intensity given by (3.3). This defines a marked Poisson process with marks $(\hat{Z}, \hat{U})$ such that $\hat{Z}$ and $\hat{U}$ are independently distributed according to

$$\hat{Z} \sim \hat{f}(z); \qquad \hat{U} \sim Unif[0,1]$$

The marks arrive independently at time instances $\hat{\tau}_1, \hat{\tau}_2, \ldots$ generated by a Poisson process of constant intensity λ on $[0,T]$.

We denote by $\hat{\Theta}$ a realization of the process $\hat{m}$; i.e.

$$\hat{\Theta} = \left\{ \left(\hat{Z}_1, \hat{U}_1, \hat{\tau}_1\right) \left(\hat{Z}_2, \hat{U}_2, \hat{\tau}_2\right), \ldots, \left(\hat{Z}_{\hat{N}_T}, \hat{U}_{\hat{N}_T}, \hat{\tau}_{\hat{N}_T}\right) \right\} \tag{3.6}$$

where

$$\hat{N}_T = \int_0^T \int_{M \times [0,1]} \hat{m}(dz, du, dt)$$

Proposition 3.1

Let $\overline{\Theta}$ be the projection of $\hat{\Theta}$ on $[0,T] \times [0,M]$; then

$$\Theta \mid Y, \hat{\Theta} \stackrel{d}{=} \left\{ \left(\hat{Z}_i, \hat{\tau}_i\right) \in \overline{\Theta} : \hat{U}_i \leq R\left(\hat{Z}_i, Y_{\hat{\tau}_i^-}, \hat{\tau}_i\right) \right\} \tag{3.7}$$

where, for any $i = 1, 2, \ldots, \hat{N}_T$:

$$R\left(\hat{Z}_i, Y_{\hat{\tau}_i^-}, \hat{\tau}_i\right) := \frac{q\left(\hat{Z}_i, Y_{\hat{\tau}_i^-}, \hat{\tau}_i\right)}{\hat{\lambda} \hat{f}\left(\hat{Z}_i\right)}$$

Proof:

This is the thinning algorithm with stochastic acceptance probability.

□

According to the proposition, we can make use of a marked Poisson process $\hat{m}$ independent on Y to select a set $\overline{\Theta}$ of *candidate points* of m. Given $\overline{\Theta}$, we can then construct an exact Skeleton of Y applying the Exact Algorithm to simulate the continuous component (3.5) of Y and the thinning formula (3.7) to simulate the jump component. In particular notice that, given $\hat{\Theta}$, for any $i = 1, 2, \ldots, \hat{N}_T$:

- Given $Y_{\hat{\tau}_{i-1}}$, the random variable $Y_{\hat{\tau}_i^-}$ can be simulated exactly using EA1, since on the interval $[\hat{\tau}_{i-1}, \hat{\tau}_i)$ the process Y follows a pure diffusion $Y^c\left(\hat{\tau}_{i-1}, \hat{\tau}_i, Y_{\hat{\tau}_{i-1}}\right) \in \mathcal{D}_1$.
- The random variable $Y_{\hat{\tau}_{i-1}}$ depends on $Y_{\hat{\tau}_{i-1}^-}$ by

$$Y_{\hat{\tau}_{i-1}} = Y_{\hat{\tau}_{i-1}^-} + \hat{Z}_i \mathbb{I}_{\left\{\hat{U}_i \leq R\left(\hat{Z}_{i-1}, Y_{\hat{\tau}_{i-1}^-}, \hat{\tau}_i\right)\right\}}$$

  where $Y_{\hat{\tau}_0} = Y_0 = y$.

Then Algorithm 6 (JEA) returns an exact Skeleton of the process Y of the form

$$\mathcal{S}_{jump} = \mathcal{S}^{(1)} \cup \mathcal{S}^{(2)} \cup \cdots \cup \mathcal{S}^{(\hat{N}_T+1)} \tag{3.8}$$

JEA manages to construct $\mathcal{S}_{jump}$ as a union of Skeletons $\left\{\mathcal{S}^{(j)}\right\}_{j=1,2,\ldots,\hat{N}_T+1}$ produced by EA1. However, differently from $\mathcal{S}_1$, $\mathcal{S}_{jump}$ has instantaneous jumps at the time instances selected by the thinning procedure.

3.2.3 Monte Carlo procedure

Since $\mathcal{S}_{jump}$ contains all the information on the jump component of Y, given $\mathcal{S}_{jump}$, the process Y behaves exactly as a process belonging to $\mathcal{D}_1$. The Skeleton decomposes the process Y in a product of independent Brownian bridges. More formally, we state the following theorem. For any $i = 1, 2, \ldots, M_j$ and

Algorithm 6 Jump Exact Algorithm (JEA)

1. Simulate the marked Poisson process $\hat{m}$.

2. Set $j = 1$.

3. Call EA1 to simulate the Skeleton $\mathcal{S}^{(j)}$ of the process $Y^c(\hat{\tau}_{i-1}, \hat{\tau}_i, Y_{\hat{\tau}_{i-1}})$:
   $\mathcal{S}^{(j)} = \{(t_{0,j}, S_{0,j}), (t_{1,j}, S_{1,j}), \ldots, (t_{M_j}, S_{M_j})\}$. Save $\mathcal{S}^{(j)}$.
   if $j = \tilde{N}_T + 1$ go to 5.
   else go to 4.

4. Simulate the jump event in $\hat{\tau}_j$.
   if $U_j \leq R(\hat{Z}_j, S_{M_j}, \hat{\tau}_j)$:
   - set $Y_{\hat{\tau}_j} = S_{M_j} + \hat{Z}_j$
   - set $j = j + 1$ and go to 3
   else:
   -set $Y_{\hat{\tau}_j} = S_{M_j}$.
   -set $j = j + 1$ and go to 3.

5. Output $\mathcal{S}_{jump} = \mathcal{S}^{(1)} \cup \mathcal{S}^{(2)} \cup \cdots \cup \mathcal{S}^{(\tilde{N}_T+1)}$

$j = 1, 2, \ldots, \hat{N}_T + 1$, let us denote $Y_{i,j} := \{Y_t : t_{i-1,j} \leq t \leq t_{i,j}\}$.

Theorem 3.1

The processes $\{Y_{i,j} \mid \mathcal{S}_{jump}\}_{i,j}$ *are independent and for any* $j = 1, 2, \ldots, \hat{N}_T + 1$:

$$Y_{i,j} \mid \mathcal{S}_{jump} \stackrel{d}{=} BB\left(t_{i-1,j}, S_{i-1,j}; t_{i,j}, S_{i,j}\right); \quad (i = 1, 2, \ldots, M_j) \tag{3.9}$$

Proof:

The result is a consequence of the construction of $\mathcal{S}_{jump}$ by the Jump Exact Algorithm.
In fact $\mathcal{S}_{jump}$ is constructed by simulating sequentially the $EA1$-Skeletons $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_{\hat{N}_T+1}$ (STEP 3 AND 5) whose starting points are selected according to a thinning experiment (STEP 4). Then, by Theorem 1.2, the conditional law of the process Y given the Skeletons (or equivalently given $\mathcal{S}_{jump}$) is the law of a product of independent Brownian bridges.

□

The theorem plays the same role as Theorem 1.2 and Theorem 2.2. It provides the key link between the Exact Algorithm and the Monte Carlo algorithm.
The Monte Carlo procedure is now straightforward and reflects the structure of Algorithm 2. First, we call Algorithm 6 for the simulation of $\mathcal{S}_{jump}$. If any points of the Skeleton have exceeded the barriers H we output directly $\phi = f(S_{M_{\hat{N}_T+1}})$ (the last point of $\mathcal{S}_{jump}$!). Otherwise we proceed to the simulation of ϕ using the results on the crossing probabilities of Brownian bridges stated in Chapter 1, section 1.4. Notice that also the Rao-Blackwellization technique presented in section 1.5 can be applied analogously. In fact, in the next Chapter, we define a Rao-Blackwellized Monte Carlo estimator based on the Jump Exact Algorithm for an application to credit risk modeling.

3.3 Conclusion

In this chapter we have developed a new Monte Carlo procedure for the barrier problem (2.2) when the stochastic process Y is a jump diffusion. The method is based on the Exact Algorithm for the simulation of the continuous component of Y and on a stochastic thinning algorithm for the simulation of the jump

component.

Our proposal presents two advantages over currently used methods based on discretization schemes: the constructed Monte Carlo estimator is unbiased and the simulation algorithm is efficient. Notably, the method applies also to the case when the intensity driving the jump component of the process is (markov) state dependent.

Despite its brevity, this chapter contains several ideas which are worth to be developed in future research work. A first direction of research is to investigate the potential applications of the method in finance. In fact we believe that jump diffusions with state dependent intensity are potentially very useful modeling tools for applied works. In particular they can be used to improve the usual Merton scheme when the assumption of independence between the jump component and the diffusion component is too restrictive. An example of these applications is given in the next chapter. Another direction of research involves to set up the general construction of a rejection sampler for jump diffusions by means of their Girsanov derivative, in the spirit of the $EA1$ rejection sampler illustrated in section 1.3. The rejection sampling step would then combine the Exact Algorithm step and the thinning step. This would lead to more compact algorithm formulation and possible improvements of the efficiency.

Chapter 4

Application to Credit Risk

4.1 Introduction

In this chapter we show an application of the techniques introduced in Chapter 3 to credit risk modeling. In general, models of credit risk are used in finance to address two types of question:

- *Risk management questions:* what is the probability that a firm will default within the next n years? what is the expected loss if this happens? what are our expected losses from all the defaults of our obligors over the next n years?
- *Pricing questions:* what is the equilibrium price of a defaultable bond, a credit derivative or any other security exposed to credit risk? or in other words: how to incorporate credit risk when we price financial instruments?

These two sets of problems are linked: both involve, as basic step, the definition of a mathematical model which describes the dynamics of the *risk of default*. Perhaps, from a methodological point of view, the main difference is that the two analysis are carried out under different measures: the first under the *physical* measure, the second under the *risk-neutral* or pricing measure.
There are two main approaches to modeling default risk: the *structural* approach and the *reduced-form* approach.

1. **Structural approach**

(Merton, 1974; Black and Cox, 1976; Longstaff and Schwarz, 1995)

The evolution of the firm's value is explicitly modeled as a diffusion process. A firm defaults when its value falls below a certain exogenously given threshold level or the level of its debt. In this context the firm never defaults by surprise due to the continuity of diffusion trajectories.

2. **Reduced form approach.**

(Jarrow and Turnbull, 1995; Jarrow et al., 1997; Duffie and Singleton, 1995)

Defaults come by "surprise" as unpredictable Poisson events. There is no explicit link between economic fundamentals and default arrivals. In this context a firm never defaults gradually.

Each of these approaches has its strengths and its drawbacks.

The structural approach gives a satisfactory theoretical insight on the dynamics which has produced a default. However it is quite rigid: it is not able to generate unexpected defaults. As a consequence, credit spreads of corporate bonds go to 0 when time to maturity goes to 0. This is a side effect of the model which is not supported by empirical evidence.

Reduced form approach is more flexible due to its ability to model a wide range of dynamics for credit spreads by an appropriate choice of the intensity function driving the arrival process. As a consequence these models perform better in fitting real data. Their drawback is the poverty of information they provide: they do not account for the actual economic reasons behind a default event. In principle we can circumvent the problem by modeling default events with a Cox process in place of a Poisson process: now the intensity is an exogenous stochastic process which somehow takes into account the economical variables which are relevant to the default process. Nevertheless, we still have to motivate the choice of a particular intensity process and show the link with the default event.

One way to combine structural approach and reduced form approach is to model the firm's value dynamics as a jump diffusion process. In a jump diffusion approach firms can either default gradually when their trajectories actually *hit* the lower barrier or by "surprise" when their trajectories *jump* below the barrier. Although jump diffusions seem a very natural way to overcome the dualism between structural models and reduced form models, not much work has been done in this direction. The most likely explanation is that the introduction of jumps involves an increasing of the computational complexity of the pricing problem. In particular even if we consider the simplest case characterized by a Merton-like jump diffusion process

and a constant threshold level, a closed form solution for the first passage default probability is not known and we have to resort to Monte Carlo simulation to estimate the price of defaultable bonds.

As far as we know Zhou (1997) and Zhou (2001) are the only examples of application of jump diffusion processes in this context. Zhou (2001) [1] assumes that the firm's value is driven by a Merton jump diffusion process and consider the problem of pricing a zero coupon defaultable bond. A solution to the pricing problem is then approximated by Monte Carlo simulation using discretization techniques. Kijima and Suzuki (2001) use a jump-diffusion process with time dependent intensity to model the firm's value but, in order to enable a closed form solution for the pricing problem, they assume that defaults are possible only at maturity time.[2]

In this chapter we propose a new jump diffusion approach to credit risk modeling. In particular the diffusion component of the firm's value process is a geometric Brownian motion while the jump component is driven by a marked point process. Differently from Merton (1976) we let the intensity of the jumps' point process to be state dependent. More specifically, we suppose that the intensity λ of the jumps' arrival process and the mean θ of the jumps depend on the current firm's value. Then, according to the structural approach, the firm defaults when its value falls below a given threshold. Using Monte Carlo techniques developed in Chapter 3, we show how to price a 0 coupon defaultable bond under the assumption that the recovery rate is a fraction of the face value of the bond and depends on the firm's value at the time of default. Our model shares with Zhou (2001) the idea to introduce jumps in the traditional structural framework. However it differs under two respects: 1) from a modeling perspective, our model is more *general*; 2) from a computational perspective our Monte Carlo algorithm for the pricing problem is more *accurate*. In fact

1. We let the intensity of the marked point process driving the jump component to depend on the current firm's value.

2. We construct a (unbiased) Monte Carlo estimator of bonds' prices without appeal to discretization schemes.

[1] Zhou (1997) can be seen seen as a particular case of Zhou (2001)

[2] Somehow related is also the paper of Hilberink and Rogers (2002) which use Levy processes to extend the framework of Leland (1994) and Leland and Toft (1996) on optimal capital structure.

The chapter is organized as follows.

In section 4.2, we review some models of interest proposed in the literature on credit risk. In particular, we provide first a brief account on structural models and reduced form models. For a more complete treatment we recommend Giesecke (2004) or the reference text of Duffie and Singleton (2003). Then we illustrate in detail the jump diffusion approach of Zhou (2001). In section 4.3 we present our proposal. First we set up the general model. Then we describe the Monte Carlo procedure to work out the computational problem. In the final section 4.4 we discuss some features and implications of our proposal both from an economical and from a computational perspective.

4.2 Review of the literature

4.2.1 Structural models

There are two main approaches to modeling default probabilities within the structural models: the Merton (1974) (or *classical*) approach and the Black and Cox (1976) (or *first passage*) approach.

Both approaches assume that a firm defaults when its value V is below a given threshold level K.

The process describing the dynamics of V is a Geometric Brownian motion:

$$dV_t = \mu V_t dt + \sigma V_t dW_t; \quad V_0 = v \tag{4.1}$$

whose solution, by Ito's lemma, is given by:

$$V_t = v \exp\left\{\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t\right\}; \quad t \in [0, +\infty)$$

In Merton's model we assume that a firm can default only at the maturity of the debt (for example a bond). So, assuming that T is the maturity time, at the current time $t = 0$ the default probability is calculated as:

$$\begin{aligned} P_1^m &= Pr\left[V_T \leq K \mid V_0 = v\right] \\ &= \Phi\left(\frac{\log L - mT}{\sigma\sqrt{T}}\right) \end{aligned} \tag{4.2}$$

where $\Phi(\cdot)$ is the standard normal distribution function, $L = \frac{K}{v}$ and $m = \mu - \frac{\sigma^2}{2}$.
In Black and Cox approach default occurs at the first time that the firm's value drops to a sufficiently low default boundary, whether or not at the maturity date of the debt. In this case the probability of default in the time interval $[0, T]$ is given by:

$$\begin{aligned} P_1^{bc} &= Pr\left[\tau \leq T \mid V_0 = v\right] \\ &= Pr\left[\min_{t \leq T}(mt + \sigma W_t) \leq \log L \mid V_0 = v\right] \\ &= \Phi\left(\frac{mT - \ln L}{\sigma\sqrt{T}}\right) - \exp\frac{(2m \ln L)}{\sigma^2}\Phi\left(\frac{mT + \ln L}{\sigma\sqrt{T}}\right) \end{aligned} \tag{4.3}$$

where $\tau = \min\{t \geq 0 : V_t \leq K\}$. This probability can be easily derived from Bachelier-Levy density (see section 1.4.2) exploiting the relationship between standard normal distribution and inverse gaussian distribution. Black and Cox allow also for a time dependent exponential threshold $K = K(t) = K_0 e^{-at}$. In this case

$$P_1^{bc} = Pr\left[\min_{t \leq T}((m-a)t + \sigma W_t) \leq \log L_0 \mid V_0 = v\right]$$

with $L_0 = K_0/v$ and the result follows analogously.
The definition of a suitable probabilistic framework for the default event is the building block for the construction of a pricing model for financial instruments exposed to credit risk.
Let p denote the price at time $t = 0$ of a defaultable zero-coupon bond maturing at time T with face value D. We assume that in case of default the firm stops its activity and the investor does not receive any payment (or *recovery*) back; then

$$p = e^{-rT} D \left(1 - \tilde{P}\right) \tag{4.4}$$

where r is the (constant) interest rate and $\tilde{P}$ is the probability of default under the risk neutral measure. In our context, $\tilde{P}$ is equal to P_1^{bc} (4.3) (or P_1^m (4.2) in the Merton approach) under the risk neutral assumption $\mu = r$.

This is clearly an oversimplified pricing model. However, as long as the default probability is explicitly known, we are still able to determine a closed form expression for p under more realistic pricing schemes. Black and Cox (1976) assume that, in case of default, bondholders receive a recovery equals to the firm's value at the default time $V_\tau = K(\tau) = K_0 e^{-a\tau}$. Longstaff and Schwarz (1995) extend this model to the case of stochastic interest rate using the Vasicek (1977) model for the short rate process r. However if we consider a different process V (not Geometric Brownian motion) or a different time dependent default threshold K (not exponential), the default probability might be intractable and we need simulation to estimate p.

4.2.2 Reduced form models

In reduced form models the fundamental modeling tool is a counting point process $N = \{N_t : t \geq 0\}$. This process determines the arrival of the default event in the time interval $[0, T]$.

Specifically, the time of default τ is modeled as:

$$\tau = \inf \{t \geq 0 : N_t = 1\}$$

Clearly the default probability P depends on the probabilistic structure of the process N:

$$P_2 = Pr[\tau \leq T] = 1 - Pr[N_T = 0]$$

We assume that the dynamics of N are prescribed exogenously directly under the risk-neutral probability measure. We have three cases.

- *Homogeneous Poisson process.*

  N is a Poisson process with constant intensity $\lambda > 0$. The default probability is given by:

$$P_2 = 1 - e^{-\lambda T}$$

- *Inhomogeneous Poisson process.*

  N is a Poisson process with deterministic intensity function $\lambda = \lambda(t)$. Then

$$P_2 = 1 - e^{-\int_0^T \lambda(t)dt}$$

- *Cox process.*

  N is a Cox process with stochastic intensity $\lambda = \{\lambda_t\}_{t\geq 0}$. Conditional on a realization of the stochastic process λ, N is a inhomogeneous Poisson process

$$Pr\left[\tau \leq T \mid (\lambda_t)_{0\leq t\leq T}\right] = 1 - e^{-\int_0^T \lambda_t ds}$$

  so that

$$P_2 = 1 - \mathbb{E}\left(e^{-\int_0^T \lambda_t dt}\right) \tag{4.5}$$

  Intuitively, we first choose a vector $\{S_t\}_{0\leq t\leq T}$ of latent state variables which are interpreted as "risk factors". Then we define a suitable positive function Λ which maps the risk factors into the intensity and set $\lambda_t = \Lambda(S_t)$ for any $t \in [0, T]$.

According to the direction of the recent literature (see e.g. Giesecke, 2004), we focus on the third case. First, we notice that in general a closed form expression for the default probability (4.5) does not exist. However, as shown by Duffie and Kan (1996), under the following assumptions

- the process S driving the intensity process $\lambda = \Lambda(S)$ is a diffusion such that the corresponding drift and diffusion coefficient are *affine* functions of the state variables

- the function $\Lambda(\cdot)$ mapping the state variables S into the intensity is affine

it is possible to determine a closed form expression for (4.5) up to the solution of a system of ordinary differential equations. This framework has been extended by Duffie et al. (2003) to the case when the process S is a jump diffusion under the additional assumption that the intensity of the jump is itself an affine function of S.

We turn now to the pricing problem. To begin, we consider the simple pricing model (4.4). We find immediately that

$$p = De^{-rT}\mathbb{E}\left(e^{-\int_0^T \lambda_t dt}\right)$$

Notice that if the intensity is constant p reduces to $e^{-(r+\lambda)T}$: the value of the defaultable bond is calculated as if the bond was risk-free using a default-adjusted discount rate. The new discount rate is the sum of the riskfree rate r and the intensity λ.

More realistically, we could assume a positive recovery R. The bond's price p becomes

$$p = De^{-rT}\mathbb{E}\left(e^{-\int_0^T \lambda_t dt}\right) + R\mathbb{E}\left(e^{-r\tau}\mathbb{I}_{\{\tau \leq T\}}\right)$$

where we can conveniently rewrite the second expectation in the following way:

$$\begin{aligned}\mathbb{E}\left(e^{-r\tau}\mathbb{I}_{\{\tau \leq T\}}\right) &= \mathbb{E}\left(\mathbb{E}\left(e^{-r\tau}\mathbb{I}_{\{\tau \leq T\}} \mid (S_t)_{t \leq T}\right)\right) \\ &= \mathbb{E}\left(\int_0^T e^{-ru} dPr\left[\tau \leq u \mid (S_t)_{t \leq T}\right]\right)\end{aligned}$$

with

$$dPr\left[\tau \leq u \mid (S_t)_{t \leq T}\right] = \lambda_u e^{-\int_0^u \lambda_s ds} du$$

Finally, we obtain the following simplified expression:

$$\mathbb{E}\left(e^{-r\tau}\mathbb{I}_{\{\tau\leq T\}}\right)=\mathbb{E}\left(\int_0^T e^{-\int_0^u(r+\lambda_s)ds}\lambda_u du\right)$$

This pricing formula can be extended in a straightforward way to the case of stochastic interest rates (see e.g. Lando, 1998).

4.2.3 Zhou's model

We describe now in some details the proposal of Zhou (2001).

Let us consider a model of structural type and assume that the firm's value process V is a jump diffusion. Then, following Merton (1976)

$$dV_t=\mu V_t dt+\sigma V_t dW_t+(e^Z-1)V_t dN_t;\quad V_0=v \tag{4.6}$$

where N is a Poisson process with constant intensity λ, $Z\sim N(\gamma,\beta^2)$ and N, W, Z are independent.

We make the following assumptions on the pricing model:

A1 *Structural form.*

We assume an exponential time dependent threshold function $K=K_t=K_0e^{at}$ such that $K_0<v$. According to the structural approach a default occurs when the firm's value falls to or below the default barrier.

A2 *Recovery rate.*

In case of default the bondholder receives a recovery R given by:

$$R=(1-w(X_\tau))D$$

where D is the face value of the bond and

$$X = \frac{V}{K} = \left\{ \frac{V_t}{K_t} : 0 \leq t \leq T \right\}$$
$$\tau = \inf \{ 0 \leq t \leq T : X_t \leq 1 \}$$

under the usual convention $\inf \{\emptyset\} = +\infty$.

Notice that in traditional first passage models characterized by pure diffusion processes, X_τ is equal to 1 and the recovery is not stochastic (see for example Longstaff and Schwarz, 1995) . Instead, this definition of the recovery rate recalls to some extent reduced form models where the recovery can depend stochastically on the state (at the time of default) of the process S driving the intensity process λ.

A3 *Recovery time.*

The bondholder receives recovery at maturity time T.

A4 *Interest rate.*

The short term risk-free interest rate r is constant over time.

In this framework the price at time $t = 0$ of a zero coupon defaultable bond with maturity time T and face value 1 is given by:

$$p = e^{-rT} - e^{-rT} \mathbb{E} \left(w(X_\tau) \mathbb{I}_{\{\tau \leq T\}} \right) \tag{4.7}$$

where the expectation is taken under the martingale measure.

One feature of jump diffusion pricing models is the existence of many equivalent martingale measures, i.e. many equivalent measures under which the discounted value process is a martingale. This leads us into an *incomplete* market. Each measure corresponds to a specific choice for the price of diffusion risk and the price of market risk. In order to select these two parameters, many different criteria have been proposed in literature: we refer to Pham (1997) and Henderson and Hobson (2003) for a review of some of them. Zhou in particular adopts the approach of Bates (1991) which relies on the assumption of time separable

investors' preferences to derive a price for the jump risk. However since our purpose is to focus on the computational techniques, we can just consider the simplest measure assuming, as in Merton (1976), that the jump risk is unpriced.

In this context, rewriting (4.6) in terms of the process X, applying the martingale transformation and then Ito's formula, we obtain the SDE of the log-process $\ln X$ under the equivalent martingale measure:

$$d\ln X_t = \left(r - a - \sigma^2/2 - \lambda\kappa\right) dt + \sigma dW_t + ZdN_t; \quad \ln X_0 = \ln x = \ln v - \ln k_0 \tag{4.8}$$

where $\kappa = e^{\gamma+\beta^2/2} - 1$.

4.2.4 Zhou's solution

The problem is the computation of (4.7) when the log process of X is defined by (4.8).

The first consideration is that a closed form expression for (4.7) under (4.8) is not known. Therefore we have to apply Monte Carlo simulation to compute p. Monte Carlo estimation of p essentially involves the simulation of the crossing time τ and the simulation of the random variable X_τ. We review the main steps of the simulation procedure proposed by Zhou and then we discuss some relevant features.

1. Divide the time interval $[0, T]$ into n sub-intervals choosing n sufficiently large. Denote $t_i = T \times i/n$
2. Do Monte Carlo simulation by repeating the following sub-procedures for M times ($j = 1, 2, \ldots, M$).
   (a) For each j, generates a series of mutually and serially independent random vectors (y_i, j_i, e_i) for $i = 1, 2, \ldots, n$ according to:

$$\begin{aligned} y_i &\sim N\left((r - a - \sigma^2/2 - \lambda\kappa)T/N, \sigma^2 T/N\right) \\ j_i &\sim N(\gamma, \beta^2) \\ e_i &\sim Bernoulli\,(\lambda T/N) \end{aligned}$$

   (b) Set $\ln X^*_{t_0} = lnx$ and simulate $X^*_{t_i}$ according to:

$$\ln X^*_{t_i} = \ln X^*_{t_{i-1}} + y_i + e_i j_i; \quad i = 1, 2, \ldots, n$$

(c) Find the smallest integer $i \leq n$ such that $\ln X^*_{t_i} \leq 0$. If such an i exists let $W_j = w(X^*_{t_i})$. Otherwise, $W_j = 0$.

3. Let $p^* = e^{-rT}\left(1 - \sum_{j=1}^{M} \frac{W_j}{M}\right)$ be the Monte Carlo approximation of p

We emphasize the fact that Zhou simulation is fully carried out under discrete time setting; i.e. 1) the jump times are discretized; 2) the hitting time is discretized.

As a consequence we use two approximations: one on the jump times and one on the hitting time given the jump times. This clearly introduces in Monte Carlo simulation a bias which goes to 0 as the discretization steps' length goes to 0 (Zhou, 2001, Theorem 1). We observe that the introduction of these approximations seems unnecessary. We can simulate exactly the random variable in (4.7) with much less computational effort just by:

- simulating first the jump times by means of the Poisson process N
- using Brownian bridge results of section 1.5.1 to simulate the hitting event between any two jump times

This is also the approach proposed in a different context by Metwally and Atiya (2002).

4.3 Our proposal

4.3.1 Definition of the model

We assume that, under the physical measure, the process V is defined by the following SDE:

$$dV_t = \mu V_t dt + \sigma V_t dW_t + V_t \int_M g(V_t, z)\, m(dz, dt); \qquad V_0 = v \qquad 0 \leq t \leq T \tag{4.9}$$

where m is a marked point process on $[0,T] \times [0,M]$ with state dependent intensity measure $q(dz, V_t, dt)$. For simplicity we assume that the intensity has the following form:

$$q(dz, V_t, dt) = \lambda(V_t) f(z, V_t) dz dt \tag{4.10}$$

where, for any $v \in \mathbb{R}$, $\lambda(v) > 0$ and $f(\cdot, v)$ is a density function with support M.
We denote by $B = \{B_t : 0 \leq t \leq T\}$ the risk-free asset whose dynamics are described by:

$$dB_t = B_t r(t) dt; \qquad B_0 = 1$$

If we denote by $\mathbb{P}$ the probability measure underlying (4.9), an equivalent martingale measure $\mathbb{Q}$ is a probability measure which is equivalent to $\mathbb{P}$ and such that the discounted process $R = \{R_t : 0 \leq t \leq T\}$

$$R_t = B_t^{-1} V_t = e^{-\int_0^t r(u) du} V_t$$

is a martingale under $\mathbb{Q}$.
Applying Ito's lemma for jump diffusions it is easy to see that a measure $\mathbb{Q}$ is a martingale measure if and only if the dynamics of the process V admit the following representation:

$$dV_t = r(t) V_t dt + dM_t^{\mathbb{Q}} \tag{4.11}$$

where $M^{\mathbb{Q}}$ is a martingale under $\mathbb{Q}$.

Girsanov Theorem (Jump-Diffusions).
Let Γ and Ψ be two previsible processes such that, for any $t \in [0,T]$, $\Psi_t(\cdot)$ is a strictly positive function and for finite t:

$$\int_M \int_0^t \|\Gamma_s\|^2 ds < +\infty; \qquad \int_0^T \lambda(V_t) \int_M |\Psi_t(z)| f(z, V_t) dz dt < +\infty$$

Define the process L by:

$$dL_t = L_t \Gamma_t dW_t + L_t - \int_M (\Psi_t(z) - 1) \{m(dt, dz) - \lambda(V_t) f(z, V_t) dz dt\} \qquad L_0 = 1$$

and suppose that, under $\mathbb{P}$, $\mathbb{E}(L_T) = 1$. Then there exists a probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ with

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_t = L_t$$

such that:

$$\begin{aligned} dW_t &= \Gamma_t dt + dW_t^{\mathbb{Q}} \\ q^{\mathbb{Q}}(z, V_t, t) &= \Psi_t(z) \lambda(V_t) f(z, V_t) \end{aligned}$$

where $W^{\mathbb{Q}}$ is a Brownian motion under $\mathbb{Q}$ and $q^{\mathbb{Q}}$ is the intensity of the process m under $\mathbb{Q}$.

We use the Girsanov theorem to write the SDE of V under $\mathbb{Q}$:

$$\begin{aligned} dV_t &= V_t \mu dt + V_t \sigma (\Gamma_t dt + dW_t) + V_t \lambda(V_t) \left\{ \int_M g(z, V_t) \Psi_t(z) f(z, V_t) dz \right\} dt \\ &+ \int_M g(z, V_t) \{m(dz, dt) - \lambda(V_t) \Psi_t(z) f(z, V_t) dz dt\} \end{aligned}$$

which can be written in the following way:

$$dV_t = V_t \left[\mu + \Gamma_t \sigma_t + \lambda(V_t) \int_M g(z, V_t) \Psi_t(z) f(z, V_t) dz \right] dt + dM_t \tag{4.12}$$

where M is a $\mathbb{Q}$-martingale. Now combining (4.11) and (4.12) we find the equation to select the appropriate Γ and Ψ that make the process R a martingale:

$$r(t) = \mu + \Gamma_t \sigma_t + \lambda(V_t) \int_M g(z, V_t) \Psi_t(z) f(z, V_t) dz \tag{4.13}$$

This equation clarifies what we have already mentioned in section 4.2.3: in presence of jumps the market is not complete since there is not a unique choice of the parameters Γ and Ψ which satisfies the condition above.

In order to avoid nonessential complications we consider the easiest case and, according to Merton's approach, we don't price the jump risk; that is, for any $t \in [0, T]$, $\Psi_t(\cdot) = 1$. We are now ready to write the SDE of the process V under the martingale measure $\mathbb{Q}$ corresponding to our particular choice of Γ and Ψ:

$$\frac{dV_t}{V_t} = \left(r(t) - \lambda(V_t) \int_M g(V_t, z) f(z, V_t) dz \right) dt + \sigma dW_t + \int_M g(V_t, z) m(dz, dt) \tag{4.14}$$

The solution to (4.14) is given by:

$$V_t = v B_t \varepsilon \left(\sigma \cdot W\right)_t \varepsilon \left(g \cdot m_c\right)_t \tag{4.15}$$

where m_c is the $\mathbb{Q}$-compensated point process m and the Dooleans-Dade exponentials are defined as:

$$\varepsilon \left(\sigma \cdot W\right)_t = \exp\left(\sigma W_t - \frac{\sigma^2}{2} t \right)$$

and

$$\varepsilon \left(g \cdot m_c\right)_t = \exp\left(\int_0^t \int_M \ln(1 + g(V_s, z)) m(dz, ds) - \int_0^t \lambda(V_s) \int_M g(V_s, z) f(z, V_s) dz ds \right)$$

From (4.15) we obtain the SDE of the log-process:

$$d\ln V_t = \left(r(t) - \frac{\sigma^2}{2} - \lambda(V_t)\int_M g(V_t,z)f(z,V_t)dz\right)dt + \sigma dW_t + \int_M \ln(1+g(V_t,z))m(dz,dt) \quad (4.16)$$

For concreteness, we make the following assumptions on the model (4.9)-(4.10):

- the random variable g does not depend on V and in particular:

$$g(v,y) = e^y - 1$$

- the density function f is normal and depends on V through its mean; that is:

$$f(z;V_t) \equiv \mathcal{N}_{\theta_t,\beta}(z)$$

where, for any $t \in [0,T]$, $\theta_t = \theta(V_t)$ is the mean of the normal and β^2 is the (constant) variance.

We consider the same pricing model of Zhou characterized by assumptions $A1 - A4$ of section 4.2.3 where the price of a 0 coupon defaultable bond with maturity T at time 0 is defined by (4.7).

4.3.2 Pricing problem

We consider the same pricing model of Zhou (2001) characterized by assumptions **A1 – A4** of section 4.2.3 where the price of a 0 coupon defaultable bond with maturity T at time 0 is defined by (4.7). In particular we reformulate the pricing problem (4.7) in the following equivalent way:

$$p = e^{-rT} + e^{-rT}\mathbb{E}\left(w^*\left(Y_\tau\right)\mathbb{I}_{\{\tau\leq T\}}\right) \quad (4.17)$$

where we have defined $w^*(y) = w(e^{\sigma y})$, $Y_t = h(V_t) = \frac{1}{\sigma}\ln\left(\frac{V_t}{K_t}\right)$ and

$$\tau = \inf\{t \geq 0 : Y_t \leq 0\} \quad (4.18)$$

The SDE of Y can be written as:

$$\begin{cases} dY_t &= \frac{1}{\sigma}\left(r - a - \frac{\sigma^2}{2} - \lambda^*(Y_t)(e^{\theta^*(Y_t)+\frac{\beta^2}{2}} - 1)\right)dt + dW_t + \left(\theta^*(Y_t) + \frac{\epsilon}{\beta}\right)m(dt) \\ Y_0 &= y = \frac{1}{\sigma}(\ln v - \ln K_0) \end{cases} \tag{4.19}$$

where m is a point process on $[0,T]$ with stochastic intensity $\lambda^*(Y_t)$, $\epsilon \sim N(0,1)$ and we have set $\lambda^*(y) = \lambda(h^{-1}(y))$, $\theta^*(y) = \theta\left(h^{-1}(y)\right)$.

4.3.3 Monte Carlo procedure

The goal is to construct a Monte Carlo estimator of (4.17) where the process Y is distributed according to (4.19).

We observe that the problem (4.17) is a typical barrier problem analogous to those considered in Chapter 1-3. Its solution requires to impose the following restrictions on the functions λ^* and θ^*.

1. The function $\lambda^*(y)$ is *differentiable* and *bounded*; i.e. $\exists \lambda_0^* > 0$ such that, for any $y \in \mathbf{R}$, $\lambda^*(y) < \lambda_0^*$
2. The function $\theta^*(y)$ is *differentiable* and *bounded from above*.

Under conditions 1 and 2, it is easy to check that the process Y belongs to $\mathcal{D}_{jump}$ (see section 3.2.1). For these processes, we have defined Algorithm 6 that simulates a Skeleton $\mathcal{S}_{jump}$. By Theorem 3.1, conditionally on $\mathcal{S}_{jump}$, the process is distributed as a product of independent Brownian bridges. Therefore, once we have simulated $\mathcal{S}_{jump}$ calling Algorithm 6, for the construction of the Monte Carlo estimator of (4.17), we can just resort to the theory illustrated in Chapter 1.

At the end the algorithm will be based on the usual two steps: 1) simulation of the Skeleton $\mathcal{S}_{jump}$; 2) construction of the Monte Carlo estimator of (4.17) given $\mathcal{S}_{jump}$.

1. We will briefly recall the main points of the procedure to simulate $\mathcal{S}_{jump}$. For a more detailed description we refer to section 3.2.
   (a) Simulation of the jump component: $\left\{(\tau_1, Z_{\tau_1}), (\tau_2, Z_{\tau_2}), \ldots, \left(\tau_{N_T}, Z_{\tau_{N_T}}\right)\right\}$

i. *Select a set of candidate jump times by a Poisson process* $\left(\hat{\tau}_1, \hat{\tau}_2, \ldots, \hat{\tau}_{\hat{N}_T}\right)$ *of intensity* λ_0^* *on* $[0, T]$.

ii. *Accept or reject* $\hat{\tau}_i$ *according to the criterion prescribed in Proposition 3.1.*

iii. *If a point* $\hat{\tau}_i$ *is accepted, set* $\tau_i = \hat{\tau}_i$ *and simulate the jump* $Z_{\tau_i} = \theta^*(Y_{\tau_i}) + \frac{\epsilon}{\beta}$.

(b) Simulation of the continuous component: $\left\{\mathcal{S}^{(1)}, \mathcal{S}^{(2)}, \ldots, \mathcal{S}^{(N_T+1)}\right\}$

i. *Use EA1 to simulate the Skeleton* $\mathcal{S}^{(i)}$ *of the continuous component between any two candidate jump times*

ii. *When a jump actually occurs update the parameters of the Exact Algorithm accordingly.*

This procedure returns an exact Skeleton $\mathcal{S}_{jump}$ of the process Y which includes the jumps and a discrete realization of the path of the process between any two jump times. It is very convenient to recall here the representation of $\mathcal{S}_{jump}$ introduced in section 3.2:

$$\mathcal{S}_{jump} = \mathcal{S}^{(1)} \cup \mathcal{S}^{(2)} \cup \cdots \cup \mathcal{S}^{(N_T+1)} \tag{4.20}$$

where, for any $i = 1, 2, \ldots, N_T + 1$, $\mathcal{S}^{(i)}$ is the Skeleton generated by the Exact Algorithm between the jump times τ_{i-1} and τ_i:

$$\mathcal{S}^{(i)} = \left\{\left(t_{0,i}, S_{0,i}\right), \left(t_{1,i}, S_{1,i}\right), \ldots, \left(t_{M_i,i}, S_{M_i,i}\right)\right\} \tag{4.21}$$

with $\left(t_{0,i}, S_{0,i}\right) = \left(\tau_{i-1}, Y_{\tau_{i-1}}\right)$ and $\left(t_{M_i,i}, S_{M_i,i}\right) = \left(\tau_i^-, Y_{\tau_i^-}\right)$ under the convention $\tau_0 = 0$ and $\tau_{N_T+1} = T$.

2. Given $\mathcal{S}_{jump}$ we turn to the second step which involves the construction of the actual Monte Carlo estimator for (4.17). Our proposal takes advantage of the *Rao-Blackwelization* technique presented in section 1.5.2 and it is based on the following proposition.

Proposition 4.1

Let S_{jump} defined by (4.20)-(4.21) be the Skeleton of Y. Then:

$$\mathbb{E}\left(w^*(Y_\tau)\mathbb{I}_{\{\tau\leq T\}}\right) = \mathbb{E}\left(\sum_{i=1}^{N_T+1} h_i\left(\mathcal{S}_{jump}\right)\right)$$

with

$$h_i(\mathcal{S}_{jump}) = \left[\prod_{k=1}^{i-1}\left\{\left(\prod_{z=1}^{M_k} p_{z,k}\right)\mathbb{I}_{\{S_{0,k}>0\}}\right\}\right]\left(w^*(S_{0,i})\mathbb{I}_{\{S_{0,i}\leq 0\}} + w^*(0)\mathbb{I}_{\{S_{0,i}>0\}}\left[1-\prod_{z=1}^{M_i} p_{z,i}\right]\right)$$

where, for any $k = 1, 2, \ldots, N_T + 1$

$$p_{z,k} = 1 - \exp\left\{-2\frac{S_{z-1,k}S_{z,k}}{t_{z,k}-t_{z-1,k}}\right\}; \qquad z = 1, 2, \ldots, M_k$$

Proof.

By a property of conditional expectation:

$$\mathbb{E}\left(w^*\left(Y_\tau\right)\mathbb{I}_{\{\tau\leq T\}}\right) = \mathbb{E}\left[\mathbb{E}\left(w^*\left(Y_\tau\right)\mathbb{I}_{\{\tau\leq T\}} \mid \mathcal{S}_{jump}\right)\right]$$

For any $i = 1, 2, \ldots, N_T + 1$:

$$Y_\tau = \begin{cases} S_{0,i} & \text{if} \quad \tau = t_{0,i} \\ 0 & \text{if} \quad t_{0,i} < \tau < t_{0,i+1} \end{cases}$$

As a consequence we can write

$$\mathbb{E}\left(w^*\left(Y_\tau\right)\mathbb{I}_{\{\tau\leq T\}} \mid \mathcal{S}_{jump}\right) = \sum_{i=1}^{N_T+1}\mathbb{E}\left(w^*\left(S_{0,i}\right)\mathbb{I}_{\{\tau=t_{0,i}\}} + w^*(0)\mathbb{I}_{\{t_{0,i}<\tau<t_{0,i+1}\}} \mid \mathcal{S}_{jump}\right) \tag{4.22}$$

For any $i = 1, 2, \ldots, N_T + 1$:

$$\begin{aligned}\mathbb{E}\left(w^*\left(S_{0,i}\right)\mathbb{I}_{\{\tau=t_{0,i}\}} \mid \mathcal{S}_{jump}\right) &= w^*(S_{0,i})\mathbb{E}\left(\mathbb{I}_{\{\tau=t_{0,i}\}} \mid \mathcal{S}_{jump}\right) \\ &= w^*(S_{0,i})\mathbb{I}_{\{S_{0,i}\leq 0\}} Pr\left(Y_t > 0,\ 0 \leq t < t_{0,i} \mid \mathcal{S}_{jump}\right)\end{aligned}$$

and analogously

$$\begin{aligned}\mathbb{E}\left(w^*(0)\mathbb{I}_{\{t_{0,i}<\tau<t_{0,i+1}\}} \mid \mathcal{S}_{jump}\right) &= w^*(0)\left[1 - Pr\left(Y_t > 0,\ t_{0,i} < t < t_{0,i+1} \mid \mathcal{S}_{jump}\right)\right] \\ &\times\ \mathbb{I}_{\{S_{0,i}>0\}} Pr\left(Y_t > 0,\ 0 \leq t < t_{0,i} \mid \mathcal{S}_{jump}\right)\end{aligned}$$

From the Markov property of Y_t, for any $i = 1, 2, \ldots, N_T + 1$:

$$Pr\left(Y_t > 0,\ t_{0,i} < t < t_{0,i+1} \mid \mathcal{S}_{jump}\right) = \prod_{z=1}^{M_i} Pr\left(Y_t > 0,\ t_{z-1,i} < t < t_{z,i} \mid \mathcal{S}^{(i)}\right)$$

and

$$Pr\left(Y_t > 0,\ 0 \leq t < t_{0,i} \mid \mathcal{S}_{jump}\right) = \prod_{k=0}^{i-1} \left\{Pr(Y_t > 0,\ t_{0,k} < t < t_{0,k+1} \mid \mathcal{S}^{(k)})\mathbb{I}_{\{S_{0,k}>0\}}\right\}$$

Since $\mathcal{S}^{(i)}$ is a Skeleton produced by the Exact Algorithm, Theorem 1.2 applies; i.e. for any $z = 1, 2, \ldots, M_i$:

$$\begin{aligned}Pr(Y_t > 0,\ t_{z-1,i} < t < t_{z,i} \mid \mathcal{S}_{jump}) &= Pr\left(BB_t\left(t_{z-1,i}, S_{z-1,i}; t_{z,i}, S_{z,i}\right) > 0,\ t_{z-1,i} < t < t_{z,i}\right) \\ &= 1 - \exp\left\{-2\frac{S_{z-1,i}S_{z,i}}{t_{z,i} - t_{z-1,i}}\right\} \\ &= p_{z,i}\end{aligned}$$

where we have applied expression (1.23) for the one sided crossing probability of the Brownian Bridge. If we substitute in (4.22) and set

$$h_i(\mathcal{S}_{jump}) = \mathbb{E}\left(w^*\left(S_{0,i}\right)\mathbb{I}_{\{\tau=t_{0,i}\}} + w^*(0)\mathbb{I}_{\{t_{0,i}<\tau<t_{0,i+1}\}} \mid \mathcal{S}_{jump}\right)$$

the result follows.

□

As a consequence an unbiased estimator of p (4.17) is

$$\hat{p} = \sum_{i=1}^{N_T+1} h_i(\mathcal{S}_{jump}) \tag{4.23}$$

from which we can construct the Rao-Blackwelized Monte Carlo estimator:

$$\hat{\hat{p}} = \frac{\sum_{j=1}^{N} \hat{p}^{(j)}}{N} \tag{4.24}$$

where $\hat{p}^{(1)}, \hat{p}^{(2)}, \ldots, \hat{p}^{(N)}$ are N independent copies of $\hat{p}$.

This construction applies the Rao-Blackwelization technique introduced in section 1.5.2. The alternative way would prescribe to simulate the Skeleton first and then the random variable in (4.17) conditionally on the Skeleton. However this procedure is more convenient for two reasons. From a computational point of view, we need just the simulation of the Skeleton since the unbiased estimator (4.23) is evaluated as a (deterministic) function of the Skeleton. From a statistical point of view, the constructed Monte Carlo estimator (4.24) has lower variance (given N) since

$$Var\left(w^*\left(Y_\tau\right)\mathbb{I}_{\{\tau\leq T\}}\right) \geq Var\left(\mathbb{E}\left(w^*\left(Y_\tau\right)\mathbb{I}_{\{\tau\leq T\}} \mid \mathcal{S}_{jump}\right)\right)$$

4.4 Conclusion

In this Chapter we have proposed a new framework for modeling default's risk and a Monte Carlo method for unbiased pricing of defaultable bonds.

Our proposal has several nice features.

1. We assume that the firm's value process is a jump diffusion. The idea is to use jump diffusions rather than pure diffusions to describe the evolution of firm's value within a structural approach to credit risk modeling.

   We have observed that the structural approach is more appealing from a modeling perspective since it allows to set a mathematical framework which is based on a model definition of default and which is able to take into account the economic dynamics which lead to the default event; however a serious problem for structural models is the continuity of the process V which does not explain unexpected defaults and generates a rigid term structure of the credit spreads which is not consistent with the empirical evidence.

   Critically, in a jump diffusion context the continuity of the trajectories is broken by the presence of jumps. As a consequence (structural) jump diffusion models capture the advantages of both structural and reduced form models. On one side the structural approach provides a conceptual insights on the economic mechanism of default risk; on the other side jumps give to the model as much flexibility as a reduced form model. An interesting consequence of jump diffusion approach is that the firm's value at default is an endogenous random variable that is not necessarily equal to the default boundary. This means that the model is also able to endogenously generate random variates in recovery rates that are linked to a firm's capital structure and asset value of default.

2. We let both the *mean* of the jumps θ and the *intensity* λ of the jumps' arrival process to be functions of the firm's value V. As a consequence they are *stochastic* and *endogenous*.

   Assuming that the investors have perfect information on the firm's value, the traditional interpretation of jump diffusion models assumes that the continuous component describes the "normal" evolution of the firm while the discontinuous one accounts for unexpected and instantaneous changes

in firm's value determined by the arrival of exogenous shocking events.

We point out the fact that, while the origin of these shocks is actually exogenous, the impact they produce depends on the state of the firm. Loosely speaking, if the firm is healthy it is less exposed to dramatic changes of its value since it will be more able to cushion the impact of exogenous shocks. This clearly reflects both on the intensity of jumps' arrival and on the expected amplitude of jumps.

3. Due to the presence of jumps, the model is able to bound short maturity credit spreads away from 0. Moreover it gives a strong theoretical support to the empirical evidence that bond prices often drop in a surprising manner at or around default time. Duffie and Lando (2001) attribute this phenomenon to incomplete accounting information; that is, around the time of default, substantial accounting information about the issuer will be revealed to the market. Because of a jump in market information , bond prices jump accordingly. In a complete information setting this phenomenon can be explained using our theoretical framework: around the time of default the firm is "weak" and particularly exposed to jumps which result in high volatility of bond prices.

4. To estimate bond prices, we perform a Monte Carlo simulation using the techniques proposed in Chapters 2. The simulation procedure is of interest on its own, as it is new and it can be applied to a variety of problems arising from different contexts. The computational problem we have faced is particularly challenging for two reasons.

   - The intensity of the jump component of the process is state dependent.
   - Under the equivalent martingale measure, also the drift of the (log)-process is state dependent.

   Nevertheless we were able to define a Monte Carlo algorithm which returns an (unbiased) estimator of the prices without appeal to any discretization method. Moreover, due to the Rao-Blackwelization, the simulation of the Monte Carlo estimator is particularly simple since it involves just the simulation of the Skeleton $\mathcal{S}_{jump}$ and the evaluation of a deterministic function of the Skeleton.

Extensions to this framework are also possible.

We can relax condition **A3** and assume, more realistically, that the bondholder receives the recovery when

the default occurs. In this case we have to simulate the default time in order to evaluate the discounted factor in the pricing formula. This can be done by simulating the Skeleton $\mathcal{S}_{jump}$ and then using Bachelier-Levy density to simulate the first passage time of the Brownian bridges (see section 1.5.1). The resulting estimator is still unbiased though not Rao-Blackwellized (see the discussion at the end of section 1.5.2).
We believe that also assumption **A4** can be relaxed, allowing for stochastic interest rates. In this case in order to construct an unbiased estimator of p we should probably resort to more sophisticated simulation techniques such as the *Poisson estimator* (see Wagner, 1988a).
Empirical investigations need to be done in order to fit our model to real data.
In particular a major open problem concerns how to model the dependence between the intensity and the firm's value process. Suitable functional forms for $\lambda(\cdot)$ and $\theta(\cdot)$ should, on the one hand, satisfy conditions 1 and 2 of section 4.3.3. On the other hand, they should be supported by reasonable economic arguments and empirical analysis.

Part II

ESTIMATION OF GREEKS

Chapter 5

Exact Monte Carlo Simulation of Greeks: a Malliavin Approach

5.1 Introduction and brief review of the literature

The aim of this Chapter is to introduce a new approach to Monte Carlo evaluation of the Greeks. We define a Monte Carlo method which is based on the Malliavin representation of the Greeks of Fournié et al. (1999) and can be seen as an unbiased alternative to the traditional Finite Difference method.

In mathematical finance, *Greeks* are the quantities representing the market sensitivities of options or other derivatives. As we have seen, in an equilibrium market, the price of an option is calculated as the (discounted) expected value of its payoff. The payoff is typically a functional of a stochastic (diffusion) process X which describes the dynamics of the underlying asset. In this framework option prices depend on the parameters of X. Loosely speaking, Greeks are the partial derivatives of option prices with respect to the parameters of the process X.

So, suppose that under the martingale measure, the asset process is a diffusion with the following SDE:

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t; \quad X_0 = x, \qquad 0 \leq t \leq T \tag{5.1}$$

and consider an option with simple payoff $\phi(X) = \phi(X_T)$. *Delta* is the derivative of the option price with respect to the initial value $X_0 = x$; that is, denoting by u the price of the option:

$$delta = \frac{\partial u(x)}{\partial x} = \frac{\partial \mathbb{E}(\phi(X_T) \mid X_0 = x)}{\partial x} \tag{5.2}$$

Analogously, we define *gamma* as the second derivative of the price with respect to the initial value, and *rho* and *vega* as the derivative of the price with respect to (resp.) the drift and the diffusion functions of X.

The Greeks are vital tools in risk management. Each represents a specific measure of risk in owning an option. Thus a desirable property of a model of a financial market is the ability to compute the Greeks. Nevertheless closed forms formula for Greeks are very rare. A notable exception is the Black-Scholes case. In general it is clear that, when there are no closed form expressions for prices we have to resort to Monte Carlo methods to estimate both prices and their derivatives. However Monte Carlo simulation for Greeks is not as straightforward as for prices. Prices are genuinely defined as expected values, while Greeks are derivatives of expected values. Therefore in order to perform Monte Carlo simulation we need to define a suitable representation of the Greeks in terms of expectations.

There are four approaches to Monte Carlo estimation of Greeks: the *Finite Difference method*, the *Pathwise method*, the *Likelihood Ratio method* and the *Malliavin method*. In the sequel of this section we give a brief account on the first three methods, while the Malliavin approach will be extensively discussed in section 5.3. For illustrative purposes, here we restrict our description to the computation of *delta* (5.2).

The traditional approach to the estimation of (5.2) is to approximate the derivative by a **Finite Difference**. Indeed for small values of ϵ:

$$delta \approx \frac{u(x+\epsilon) - u(x)}{\epsilon} \tag{5.3}$$

Monte Carlo estimation of the right hand side is then straightforward: we compute a Monte Carlo estimator of $u(x+\epsilon)$ and of $u(x)$ and we calculate the ratio.

This method has the advantage to be simple and generally applicable. The important drawback is that it may perform very poorly. In fact we make use of two approximations, generating two kinds of error: one on the numerical computation of the expectation as in any simulation, and the other on the approximation of the derivative by its Finite Difference. Moreover running independent simulations for $u(x)$ and $u(x+\epsilon)$ the computational time clearly doubles.

Pathwise method, introduced by Broadie and Glasserman (1996), simply suggests the interchange of integration and differentiation in (5.2):

$$delta = \mathbb{E}(\phi'(X_T)Y_T) \tag{5.4}$$

The process $\{Y_t\}_{0\leq t\leq T}$ is called *first variation process* and is defined by

$$Y_t = \frac{\partial X_t}{\partial x} = \lim_{\epsilon\to 0}\frac{X_t^\epsilon - X_t}{\epsilon} \tag{5.5}$$

where we denote by X^ϵ the process X perturbed by a quantity ϵ in the initial state x. Pathwise method can be easily seen as the continuous counterpart of the Finite Difference method since, from

$$\frac{u(x+\epsilon)-u(x)}{\epsilon} = \mathbb{E}\left(\frac{\phi(X_T^\epsilon)-\phi(X_T)}{\epsilon}\right)$$

under suitable conditions on the function ϕ (Dominated Convergence Theorem), we can take the limit for $\epsilon \to 0$ inside the expectation, obtaining (5.4).

This method is just affected by the standard error of Monte Carlo simulation since the derivative is not approximated. However it can be applied only to a small set of options since for the most the payoff function $\phi(\cdot)$ is not smooth enough.

In order to avoid the differentiation of the payoff, Broadie and Glasserman (1996) proposed the **Likelihood Ratio method**. The idea here is to apply the differential operator to the density of the integrand rather than to the random variable. Indeed

$$\begin{aligned} delta &= \frac{\partial u(x)}{\partial x} \\ &= \frac{\partial}{\partial x}\int \phi(z) f_{X_T}(z;x)dz \\ &= \int \phi(z)\frac{\partial f_{X_T}(z;x)}{\partial x}dz = \mathbb{E}\left(\phi(X_T)\frac{\partial \log f(X_T;x)}{\partial x}\right) \end{aligned} \tag{5.6}$$

This method takes advantage of the fact that, differently from payoff functions, density functions are generally smooth in their parameters (consider for example the case of a digital option in a Black-Scholes framework). Moreover representation (5.6) holds for any choice of $\phi(\cdot)$. In other words, Likelihood Ratio method manages to rewrite Greeks as expected values of the product of the payoff itself and a stochastic weight which depends just on the Greek we are considering (and the model, of course!) but not on the payoff of the option:

$$delta = \mathbb{E}(\phi(X_T)WEIGHT^{delta}) \tag{5.7}$$

This formulation of the problem in terms of expected value of the weighted payoff gives to the method a much greater flexibility than the Pathwise method.

As we will see, also Malliavin approach provides a representation of the Greeks in terms of expected values of the weighted payoffs (the *Malliavin weights*). The crucial difference is that in (5.6) in order to compute the Likelihood Ratio we need to have explicit knowledge of the density involved in the integration, while in the Malliavin approach this is not necessary. As a consequence, Malliavin method turns out to be more general than Pathwise and Likelihood Ratio methods. In particular Malliavin approach seems the only alternative to the Finite Difference method when the payoff function is discontinuous or the density

involved in the pricing formula is not known. In fact in these cases, representations (5.4) and (5.6) are not given and unbiased Monte Carlo evaluation of Greeks is possible only by means of the Malliavin way. However in general Monte Carlo simulation of Malliavin Greeks can be extremely complicated. The main problem is that the Malliavin weight is always expressed in terms of a stochastic integral. In principle Monte Carlo estimation would involve the joint simulation of the payoff and the integral. We could clearly resort to discretization methods: however the simulation of a stochastic integral by means of the discretized process might introduce a relevant bias in Monte Carlo estimation.

In this Chapter we present a first step toward a general theory and methodology for unbiased Monte Carlo estimation of Malliavin Greeks. The method is presented for the case of *rho* and *delta* and applies to diffusion processes belonging to $\mathcal{D}_1$. Beyond this, we believe that the framework can be adapted with small changes to the other Greeks and extended to classes of processes more general than $\mathcal{D}_1$.

The Chapter is organized as follows.
In section 5.2 we review basic notions of Malliavin calculus. In section 5.3 we introduce the Malliavin representation of the Greeks. In section 5.4 we present our method for the simulation of the Malliavin Greeks. We end in section 5.5 with general comments and directions for future research.

5.2 Malliavin Calculus

5.2.1 Generalities and intuition

Malliavin Calculus is a young area of research which lies between functional analysis and probability theory. In some sense, it translates some of the concepts familiar to functional analysis, like differentiation in Banach spaces, to a probabilistic framework.
It was firstly introduced by P. Malliavin who was working on giving sufficient conditions under which a random variable has a smooth density with respect to the Lebesgue measure. In the 80's other authors have used Malliavin Calculus, mainly in the field of SPDE. But it is in the last 15 years that Malliavin Calculus has developed impressively due to the potentialities of its application in mathematical finance.
There are two main applications of Malliavin Calculus to finance. The first, dating back to 1991, was

initiated by Karatzas and Ocone who derived explicit formulas for a Δ-hedging strategy in terms of the Malliavin derivative of the payoff function using a generalization of the Clark-Ocone formula (Ocone and Karatzas, 1991). The second one has been firstly proposed by Fournié et al. (1999) who showed, using Malliavin Calculus techniques, that all the Greeks can be represented as expected values of the weighted payoff. Both areas of research have shown to be particularly fruitful, motivating further investigations in many different directions and contributing to the diffusion among practitioners of a theory which has been considered for many years highly theoretical and technical from the mathematical point of view. The accessibility of Malliavin theory to a broader audience has been facilitated by a number of monographs: among others it is worth to mention Oksendal (1997), Bally (2003) and Ikeda and Watanabe (1984) which give a friendly introduction while Malliavin (1997) and Nualart (1995) cover the theory to much greater detail. In particular Nualart (1995) remains the reference text on the subject.

Malliavin calculus, sometimes called *stochastic calculus of variations*, is essentially a theory of differentiation in a Wiener space. Intuitively, suppose that $W = \{W_t : 0 \leq t \leq T\}$ is a Wiener process and consider a random variable $F = f(W)$ such that f is measurable with respect to the σ-algebra generated by W. Loosely speaking, the aim is to differentiate F with respect to W.
Central to the theory of Malliavin calculus is the notion of *Malliavin derivative*. A rigorous definition of Malliavin derivative is given in section 5.2.2 while here we will try to approach the concept from an intuitive point of view. To this scope, we introduce the following *perturbation* of Brownian motion: for any $\epsilon > 0$ and any $s \in [0,T]$ we define a stochastic process $W^{s,\epsilon} = \{W_t^{s,\epsilon} : 0 \leq t \leq T\}$ by:

$$W_t^{s,\epsilon} = W_t + \epsilon \mathbb{I}_{\{t \geq s\}}$$

The Malliavin derivative operator $D : L^2(\Omega) \to L^2(\Omega \times [0,T])$ is then defined by:

$$D_s F(W) = \lim_{\epsilon \to 0} \frac{f(W^{s,\epsilon}) - f(W)}{\epsilon} \tag{5.8}$$

This definition, though not rigorous from a mathematical point of view, is particularly useful since it

provides an intuitive justification of some basic Malliavin derivatives. Important examples are:

$$\begin{aligned} D_s W_t &= \mathbb{I}_{\{t \geq s\}} \\ D_s f(W_t) &= f'(W_t)\mathbb{I}_{\{t \geq s\}} \\ D_s \left(\int_0^T f(t) dW_t \right) &= f(s) \end{aligned} \tag{5.9}$$

which can be obtained from (5.8) applying the usual rules of differentiation.

5.2.2 A formal construction

Let $(\Omega, \mathcal{F}, P)$ be a Wiener Space; so that $\Omega = C_0([0,T])$ is the set of continuous functions on $[0,T]$ starting at 0, $\mathcal{F}$ is the σ-algebra generated by the cylinder subsets of C_0 and P is the usual Wiener measure on $(\Omega, \mathcal{F})$. Let ω be the generic element of Ω: therefore, for any $t \in [0,T]$, $\omega(t) = W_t$. Furthermore let $\{\mathcal{F}_t\}_{\{0 \leq t \leq T\}}$ be the filtration generated by W; that is $\mathcal{F}_t$ is the completion of $\sigma(W_s; s \leq t)$ with the P-null sets.

The fact that we work on a Wiener Space and not on some abstract probability space is crucial: Malliavin Calculus is able to deal with random functions for which the source of randomness is a Wiener process.

The way we are going to "construct" the definition of Malliavin derivative is quite standard in mathematics: first, we define the Malliavin derivative of a set of "simple" random variables and then we extend the definition to a more general class by a limiting argument.

So consider the family C of all random variables $F : \Omega \to \mathbf{R}$ of the form:

$$F(\omega) = f\left(\int_0^T h_1(t) dW_t, \int_0^T h_2(t) dW_t, \ldots, \int_0^T h_n(t) dW_t \right); \quad n \in \mathbb{N}$$

where $h_1, h_2, \ldots, h_n \in L^2([0,T])$ and $f(x_1, x_2, \ldots, x_n)$ is a polynomial in n variables. Such random variables F are called *Wiener Polynomials.*

Definition 5.1

Let $F \in C$. Then the Malliavin derivative of F is a stochastic process $DF := \{D_t F : 0 \leq t \leq T\}$ defined by:

$$D_t F = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} f\left(\int_0^T h_1(t)dW_t, \int_0^T h_2(t)dW_t, \ldots, \int_0^T h_n(t)dW_t\right) h_i(t) \tag{5.10}$$

Definition 5.1 is not surprising: a convenient way to see it is as a consequence of (5.9) after applying the usual rules of differentiation for (deterministic) functions $\mathbf{R}^n \to \mathbf{R}$. Since C is dense in $L^2(\Omega)$ (see e.g. Nualart, 1995) it is tempting to define the Malliavin derivative of a general $F \in L^2(\Omega)$ by a limiting procedure. This is precisely the way we will follow.

Formally we define the following norm $\|\cdot\|_{1,2}$ on C:

$$\begin{aligned} \|F\|_{1,2} &= \|F\|_{L^2(\Omega)} + \|D_t F\|_{L^2([0,T]\times\Omega)} \\ &= \left(\mathbb{E}\left(F^2\right)\right)^{1/2} + \left(\mathbb{E}\left(\int_0^T (D_t F)^2 \, dt\right)\right)^{1/2} \end{aligned}$$

and we define $D_{1,2}$ the *closure* of C under the norm $\|\cdot\|_{1,2}$. That is a random variable $F \in L^2(\Omega)$ belongs to $D_{1,2}$ if it exists a sequence $\{F_n\}_{n=1,2,\ldots} \subset C$ such that:

$$\lim_{n\to\infty} F_n = F \quad \text{in } L^2(\Omega) \tag{5.11}$$

and

$$\{D_t F_n\}_{n=1}^{\infty} \quad \text{is convergent in } L^2([0,T]\times\Omega) \tag{5.12}$$

Furthermore, it is possible to show that the operator $D : L^2(\Omega) \to L^2([0,T]\times\Omega)$ is *closable*: that is, given two sequences $\left\{F_n^{(1)}\right\}, \left\{F_n^{(2)}\right\} \subset C$ satisfying (5.11) and (5.12), it holds

$$\lim_{n\to\infty} D_t F_n^{(1)} = \lim_{n\to\infty} D_t F_n^{(2)} \tag{5.13}$$

We are now ready for a general definition of Malliavin derivative.

Definition 5.2

Let $F \in D_{1,2}$ so that there exists a sequence $\{F_n\} \subset C$ satisfying (5.11) and (5.12). Then the Malliavin derivative of F, $DF := \{D_tF : 0 \leq t \leq T\}$ is defined by:

$$D_tF = \lim_{n\to\infty} D_tF_n \tag{5.14}$$

The definition of the Malliavin derivative of F is now straightforward. It is given as the limit of Malliavin derivatives (5.10). The existence of such limit follows from the definition of $D_{1,2}$ while its uniqueness is ensured by the closability of the operator D. In this context $D_{1,2}$ is then the domain of the Malliavin derivative operator D and from the construction it follows:

$$C \subset D_{1,2} \subset L^2(\Omega) \tag{5.15}$$

Remark 5.1

The stochastic process DF generated by the Malliavin operator is not necessarily $\mathcal{F}_t$-adapted. To see this, the intuitive representation (5.8) is particularly helpful. In fact there it is crucial the dependence of $f(\cdot)$ from the underlying Brownian motion; for instance in (5.9) it is clear that for $s < t$, $D_sf(W_t)$ is not $\mathcal{F}_s$-measurable.

5.2.3 Useful properties of the Malliavin Derivative

We focus here on two properties of the Malliavin derivative which are key to our application to the computation of Greeks.

(Integration by Part)

Let $F \in D_{1,2}$. Then the following integration by part formula holds

$$< DF, u >_{L^2([0,T]\times\Omega)} = < F, u^* >_{L^2(\Omega)}$$

where the adjoint operator of D is the Skorohod integral $\delta : L^2([0,T] \times \Omega) \to L^2(\Omega)$; that is:

$$u^* = \delta(u)$$

A stochastic process $u = \{u_t(\omega) : 0 \leq t \leq T\} \in L^2([0,T] \times \Omega)$ is Skorohod integrable, and we write $u \in Dom(\delta)$, if for any $F \in D_{1,2}$:

$$< DF, u >_{L^2([0,T]\times\Omega)} \leq c \|F\|_{1,2} \tag{5.16}$$

c being a constant independent of u. So the integration by part formula can be written in a more explicit way as:

$$\mathbb{E}\left(\int_0^T D_t F(\omega) u_t(\omega) dt\right) = \mathbb{E}\left(F\delta(u)\right) \tag{5.17}$$

where $F \in D_{1,2}$, $u \in Dom(\delta)$ and the adjoint operators D and δ are respectively the Malliavin derivative and the Skorohod integral.

Remark 5.2

We don't give here a formal definition of Skorohod integral. However it is important to remark that there is a strong connection between the Skorohod integral and the Ito integral. In fact Skorohod integral can be defined in terms of iterated Ito integrals like in Oksendal (1997) where the Ito-Chaos expansion plays a central role. In particular when the process u is F_t-adapted Ito integral and Skorohod integral coincide. In this sense an easy way to see the Skorohod integral is just as a generalization of the Ito integral.

(Malliavin derivative of a SDE)

Let $X := \{X_t : 0 \leq t \leq T\}$ be a diffusion process defined by the SDE:

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t; \quad X_0 = x, \qquad 0 \leq t \leq T \tag{5.18}$$

where μ and σ are supposed to be continuously differentiable functions with bounded derivatives. Then, for any $t \in [0,T]$, $X_t \in D_{1,2}$ and

$$D_s X_t = \frac{Y_t}{Y_s}\sigma(X_s)\mathbb{I}_{\{s\leq t\}}; \quad Y_0 = 1, \quad 0 \leq t \leq T \tag{5.19}$$

where the process $Y := \{Y_t : 0 \leq t \leq T\}$ is the first variation process (5.5) of X

It can be shown that the first variation process satisfies the following SDE:

$$dY_t = \mu'(X_t)Y_t dt + \sigma'(X_t)Y_t dW_t; \quad Y_0 = 1, \quad 0 \leq t \leq T \tag{5.20}$$

Example 5.1

Consider a Geometric Brownian Motion:

$$dX_t = \mu X_t dt + \sigma X_t dW_t; \quad X_0 = x, \tag{5.21}$$

$$0 \leq t \leq T \tag{5.22}$$

Then it is well known

$$X_t = xe^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t}; \quad t \in [0,T]$$

and applying (5.5)

$$Y_t = \frac{X_t}{x}; \quad t \in [0,T] \tag{5.23}$$

Now, from (5.19) it is immediate to derive the Malliavin derivatives for X; given $t \in [0,T]$:

$$D_s X_t = e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma W_t} \sigma \mathbb{I}_{\{t \geq s\}}$$

Before proceeding to the description of how these results can be applied to the computation of the Greeks, it is worth mentioning another very useful property which is familiar to classic differential calculus and it applies analogously in Malliavin calculus.

(Chain Rule)

Let ϕ: $\mathbf{R}^n \to \mathbf{R}$ be a continuously differentiable function with bounded partial derivatives and $F = (F_1, F_2, \ldots, F_n)$ a random vector whose components belong to $D_{1,2}$. Then $\phi(F) \in D_{1,2}$ and:

$$D_t \phi(F) = \sum_{i=1}^{n} \frac{\delta \phi}{\delta x_i}(F) D_t F_i, \qquad 0 \leq t \leq T$$

So, suppose for example that we have, for any $i = 1, 2, \ldots, n$, $F_i = f_i(X_t)$ where $f_i(x)$ is a well behaved function. Then applying the rule of (Malliavin) differentiation of diffusions and the chain rule, we obtain:

$$D_s \phi(F) = \left\{ \sum_{i=1}^{n} \frac{\partial \phi}{\partial x_i} \frac{\partial F_i}{\partial x} \right\} \frac{Y_t}{Y_s} \sigma(X_s) \mathbb{I}_{s \leq t}$$

5.3 Application of Malliavin Calculus to the computation of Greeks

5.3.1 A simple example

We begin with a simple example which is intended to be a friendly introduction to the ideas underlying the material presented in this section.

Here we don't need to resort to Malliavin Calculus arguments since the density involved in the integration is explicitly known. Still the main idea is the same as in the general case: to use an integration by part formula to obtain a smoothed expression for the Greeks where the pay-off function is not differentiated. We consider the Black-Scholes model (5.21) and the *delta* of an option whose (discounted) payoff ϕ depends on the price of the asset at some expiry date $T > 0$. Then applying classic integration by parts formula for real functions:

$$\begin{aligned} delta &= \frac{\partial}{\partial x}\mathbb{E}\left(\phi(X_T) \mid X_0 = x\right) \\ &= \int_{-\infty}^{+\infty} \frac{\partial}{\partial x}\phi\left(xe^{\left(\mu-\frac{\sigma^2}{2}\right)T+\sigma\sqrt{T}y)}\right)\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}dy \\ &= \left[\frac{1}{x\sigma\sqrt{T}}\phi\left(xe^{\left(\mu-\frac{\sigma^2}{2}\right)T+\sigma\sqrt{T}y)}\right)\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}\right]_{-\infty}^{+\infty} \\ &+ \int_{-\infty}^{+\infty} \frac{1}{x\sigma\sqrt{T}}\phi\left(xe^{\left(\mu-\frac{\sigma^2}{2}\right)T+\sigma\sqrt{T}y)}\right)y\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}dy \\ &= \mathbb{E}\left(\frac{W_T}{x\sigma T}\phi(X_T)\right) \end{aligned} \tag{5.24}$$

where we have used the following equality

$$\frac{\partial}{\partial x}\phi\left(xe^{\left(\mu-\frac{\sigma^2}{2}\right)t+\sigma\sqrt{T}y}\right) = \frac{1}{x\sigma\sqrt{T}}\frac{\partial}{\partial y}\phi\left(xe^{\left(\mu-\frac{\sigma^2}{2}\right)t+\sigma\sqrt{T}y}\right) \tag{5.25}$$

So we have first expressed the derivative of the payoff with respect to the initial condition in terms of the derivative of the payoff with respect to the variable we are integrating out. Then we have applied a classic integration by part formula. Somehow, in the general case we will proceed analogously with the only difference that Malliavin Calculus enables us to work with much less information: namely, we don't need an explicit knowledge of the density function.

Representation (5.24) has two main advantages:

1. the differential operator has disappeared from the integrand
2. the weight is not depending on the payoff ϕ

As a consequence of 1, we can estimate the Greeks also in presence of very irregular or discontinuous payoff functions. From 2, the efficiency of the simulation is not affected by the choice of the payoff. On the contrary, the standard way to compute Greeks relies on the payoff function since it takes the Finite Difference approximation of the derivative of the payoff.

Remark 5.3

Representation (5.24) coincides with expression (16) of Broadie and Glasserman (1996) who obtained it directly computing the Likelihood Ratio of X_T.

5.3.2 The general case: mathematical framework

We consider now a general scalar diffusion process X (5.1) and we assume that

- the functions $\mu, \sigma : \mathbf{R} \to \mathbf{R}$ are continuously differentiable with bounded derivatives; this in particular implies that they satisfy Lipschitz conditions for the existence of a unique strong solution of (5.1):

$$\|\mu(u_1) - \mu(u_2)\| + \|\sigma(u_1) - \sigma(u_2)\| \leq K_M \|u_1 - u_2\|; \qquad \|u_1\| \leq M, \ \|u_2\| \leq M \tag{5.26}$$

and

$$\|\mu(u_1)\| + \|\sigma(u_1)\| \leq C(1 + \|u_1\|); \qquad u_1 \in \mathbf{R} \tag{5.27}$$

- the diffusion coefficient is uniformly elliptic; that is $\exists \epsilon > 0$ such that, for any $u \in \mathbf{R}$:

$$|\sigma(u)| \geq \epsilon \tag{5.28}$$

We consider an option with expiry date T whose payoff depends on the values of X at a finite set of times $t_1, t_2 \ldots, t_n$ such that $0 \leq t_1 \leq t_2 \cdots \leq t_n = T$. The payoff function ϕ is a mapping $\mathbf{R}^n \to \mathbf{R}$ and the price of the option at time $t = 0$ is given by:

$$u(x) = \mathbb{E}\left(e^{-rT}\phi\left(X_{t_1}, X_{t_2}, \ldots, X_{t_n}\right) \mid X_0 = x\right)$$

In the following, for ease of notation, we assume $r = 0$. We impose an integrability condition on the payoff:

$$\mathbb{E}\left[\left(\phi\left(X_{t_1}, X_{t_2}, \ldots, X_{t_n}\right) \mid X_0 = x\right)^2\right] < \infty \tag{5.29}$$

Let $\mathcal{T}$ be the set of functions $\phi : \mathbf{R}^n \to \mathbf{R}$ such that (5.29) is satisfied. For ease of notation we denote the vector $(X_{t_1}, X_{t_2}, \ldots, X_{t_n})$ by $\bar{X}$. Our aim is to estimate the sensitivity of u with respect to the parameters of X. We will focus in particular on the *first derivatives*; i.e. *delta*, *rho* and *vega*. We have already introduced *delta* for illustrative purposes. Its definition is straightforward: it is the first derivative of the price with respect to the initial condition. On the contrary *rho* and *vega* need more care. In fact since the drift and the diffusion coefficient are functions of the underlying and not constants, we have to develop a more robust framework than the common sensitivity analysis with respect to a fixed parameter. In this context, the meaning of *rho* and *vega* is to quantify the impact on the price of a small perturbation of either the drift or the diffusion function in a *specified direction*. As a consequence the familiar notion of derivative for functions $\mathbf{R} \to \mathbf{R}$ is replaced here by the notion of *Gateaux derivative* which is defined as the directional derivative (in some specified directions) of a function defined on a general Banach space. Before giving a formal definition for *vega* and *rho* we need to introduce some notation.

Let $\tilde{\mu} : \mathbf{R} \to \mathbf{R}$ and $\tilde{\sigma} : \mathbf{R} \to \mathbf{R}$ be two functions such that, for every $\epsilon \in [-1, +1]$, $\tilde{\mu}(\cdot), (\mu + \epsilon\tilde{\mu})(\cdot), \tilde{\sigma}(\cdot)$ and $(\sigma + \epsilon\tilde{\sigma})(\cdot)$ are continuously differentiable with bounded derivatives and verify Lipschitz conditions. Moreover we assume that $\tilde{\sigma}(\cdot)$ and $(\sigma + \epsilon\tilde{\sigma})(\cdot)$ satisfy the uniform ellipticity condition (5.28); that is for any $u \in \mathbf{R}$ and for any $\epsilon \in [-1, +1]$:

$$\exists \eta > 0 : \left|\left(\sigma + \epsilon\tilde{\sigma}\right)(u)\right| \geq \eta \tag{5.30}$$

For $\tilde{\mu}$ and $\tilde{\sigma}$ given, we define the *perturbed* processes $X^{\epsilon,rho} = \left\{X_t^{\epsilon,rho} : 0 \leq t \leq T\right\}$ and $X^{\epsilon,vega} = \{X_t^{\epsilon,vega} : 0 \leq t \leq T\}$ by:

$$X_t^{\epsilon,rho} = x + \int_0^t \left(\mu(X_s^{\epsilon,rho}) + \epsilon\tilde{\mu}(X_s^{\epsilon,rho})\right) ds + \int_0^t \sigma(X_s^{\epsilon,rho}) dWs; \quad 0 \le t \le T \tag{5.31}$$

and

$$X_t^{\epsilon,vega} = x + \int_0^t \mu(X_s^{\epsilon,vega}) ds + \int_0^t \left(\sigma(X_s^{\epsilon,vega}) + \epsilon\tilde{\sigma}(X_s^{\epsilon,vega})\right) dWs; \quad 0 \le t \le T \tag{5.32}$$

and the corresponding perturbed prices:

$$u^{\epsilon,greek}(x) = \mathbb{E}\left(\phi\left(X_{t_1}^{\epsilon,greek}, X_{t_2}^{\epsilon,greek}, \ldots, X_{t_n}^{\epsilon,greek}\right) \mid X_0 = x\right) \tag{5.33}$$

where either $greek \equiv rho$ or $greek \equiv vega$.

Definition 5.3

We define rho (resp. vega) the Gateaux derivative of the perturbed price function $u^{\epsilon,rho}(x)$ ($u^{\epsilon,vega}(x)$) in the direction given by $\tilde{\mu}(\cdot)(\tilde{\sigma}(\cdot))$; that is:

$$rho \equiv \frac{\partial}{\partial\epsilon} u^{\epsilon,rho}(x)\Big|_{\epsilon=0} \tag{5.34}$$

or, respectively

$$vega \equiv \frac{\partial}{\partial\epsilon} u^{\epsilon,vega}(x)\Big|_{\epsilon=0} \tag{5.35}$$

We also introduce the processes Z^{delta}, Z^{rho} and Z^{vega} defined by

$$Z_t^{greek} = \lim_{\epsilon\to 0} \frac{X_t^{\epsilon,greek} - X_t}{\epsilon}; \quad 0 \le t \le T \tag{5.36}$$

where the processes $X^{\epsilon,rho}$ and $X^{\epsilon,vega}$ are given by (5.31) and (5.32) while $X^{\epsilon,delta}$ is just the diffusion process X perturbed by ϵ in the initial point x. Notice in particular that Z^{delta} is the first variation process of X; that is, using notation of section 1.3, $Y \equiv Z^{delta}$.

Remark 5.4

Generally in finance the Greeks are framed in a Black-Scholes context so that rho and vega are easily defined as the derivative with respect to the expected growth rate and volatility of the log-normal model (see e.g. Hull, 2000, Chapter 14). It is clear that this definition relies on the fact that drift and volatility coefficients are constant and it is inadequate to treat the general case.

5.3.3 The main result

Consider the following representation:

$$greek = \mathbb{E}\left(\phi(\bar{X})\pi^{greek}\right) \tag{5.37}$$

where π^{greek} is any random variable square integrable which satisfies the equality above for any $\phi \in \mathcal{T}$. For each greek we denote by $\mathcal{W}^{greek}$ the corresponding set of such random variables.

Proposition 5.1 (Benhamou (2003))

Let $u = \{u_t : 0 \leq t \leq T\}$ be a Skorohod integrable stochastic process. Then the following statements are equivalent:

1. *$\pi = \delta(u)$ belongs to $\mathcal{W}^{greek}$*
2. *for any $i = 1, 2, \ldots, n$*

$$\mathbb{E}\left[Y_{t_i}\int_0^{t_i}\frac{\sigma(X_t)}{Y_t}u_t dt \mid \bar{X}\right] = \mathbb{E}\left[Z_{t_i}^{greek} \mid \bar{X}\right] \tag{5.38}$$

Sketch of proof:

We propose here a sketch of the proof of $2 \Rightarrow 1$ in order to show how Malliavin calculus techniques introduced in section 1.3.3 play a key role in this context. For a detailed proof of the proposition, see

Benhamou (2003).

We assume for simplicity that the payoff function $\phi \in \mathcal{T}$ is continuously differentiable with bounded partial derivatives. Then, if there exists a Skorohod integrable process u which satisfies (5.38), we have:

$$\begin{aligned}
greek &= \mathbb{E}\left(\sum_{i=1}^{n} \frac{\partial \phi}{\partial x_i}(\bar{X})\, Z_{t_i}^{greek}\right) \\
&= \mathbb{E}\left(\sum_{i=1}^{n} \frac{\partial \phi}{\partial x_i}(\bar{X})\, \mathbb{E}\left(Z_{t_i}^{greek} \mid \bar{X}\right)\right) \\
&= \mathbb{E}\left(\sum_{i=1}^{n} \frac{\partial \phi}{\partial x_i}(\bar{X})\, \mathbb{E}\left(Y_{t_i} \int_0^{t_i} \frac{\sigma(X_t)}{Y_t} u_t dt \mid \bar{X}\right)\right) \\
&= \mathbb{E}\left(\sum_{i=1}^{n} \frac{\partial \phi}{\partial x_i}(\bar{X})\, \mathbb{E}\left(\int_0^{T} D_t X_{t_i} u_t dt \mid \bar{X}\right)\right) \\
&= \mathbb{E}\left(\sum_{i=1}^{n} \frac{\partial \phi}{\partial x_i}(\bar{X}) \int_0^{T} D_t X_{t_i} u_t dt\right) \\
&= \mathbb{E}\left(\int_0^{T} D_t \phi(\bar{X}) u_t dt\right) = \mathbb{E}\left(\phi(\bar{X}) \delta(u)\right)
\end{aligned}$$

The proof of $1 \Rightarrow 2$ is carried out using analogous arguments.

□

Remark 5.5

The differentiability assumptions on the payoff function have allowed us to interchange the expectation and the differential operator and ensured the existence of the partial derivatives. However Proposition 5.1 holds also in the general case (given $\phi \in \mathcal{T}$). Indeed the general result can be proved by a density argument.

Proposition 5.1 states that for each greek there exists a suitable set of Malliavin weights which can be expressed as Skorohod integral of processes u satisfying condition (5.38). A natural question arising in this context is whether it is possible to define an "optimal" weight. Since we are in a Monte Carlo context it seems that the optimal weight must be the one minimizing the variance of the Monte Carlo integrand. That is, π_0^{greek} is the optimal weight for *greek* if, for any given ϕ:

$$\pi_0^{greek} = arg \min_{\pi \in \mathcal{W}^{greek}} \mathcal{V}(\pi) \tag{5.39}$$

where

$$\mathcal{V}(\pi) = \mathbb{E}\left[\left(\phi(\bar{X})\pi - greek\right)^2\right] \tag{5.40}$$

Proposition 5.2 (Benhamou (2003))

π_0^{greek} is the conditional expectation with respect to the random vector $\bar{X} = (X_{t_1}, X_{t_2}, \ldots, X_{t_n})$ of any weight $\pi \in \mathcal{W}^{greek}$; that is, $\forall \pi \in \mathcal{W}^{greek}$:

$$\pi_0^{greek} = \mathbb{E}\left(\pi \mid \bar{X}\right) \tag{5.41}$$

Proof:

For any $\pi \in \mathcal{W}^{greek}$, we have trivially:

$$\begin{aligned} \mathcal{V}(\pi) &= \mathbb{E}\left(\phi^2(\bar{X})\pi^2\right) - \mathbb{E}\left(\phi(\bar{X})\pi_0\right)^2 \\ &= \mathbb{E}\left(\phi^2(\bar{X})(\pi - \pi_0)^2\right) + \mathcal{V}(\pi_0) \end{aligned}$$

and since the argument holds for every $\phi \in \mathcal{T}$ the conclusion follows.

□

Remark 5.6

This proposition indicates that without any further specification on the payoff function, the variance of the integrand in Monte Carlo simulation is bounded below by the variance of π_0. To improve this lower

bound we need to have additional information on the payoff function. Indeed it is worth noticing that the set of the Malliavin weights is defined by conditions that are independent from the payoff function while the global optimal solution in the sense of the Monte Carlo variance depends necessarily on the payoff. In other words, the optimality result of Proposition 5.2 holds only in the subset of the unbiased estimator of 'greek' which admit representation (5.37).

Remark 5.7

Interestingly for each greek the optimal weight coincides with the Likelihood Ratio: this is clear from 5.1. In fact Likelihood Ratio appears there as a proper weight; moreover it is a function of the payoff's state variables and so from Proposition 5.2 is the minimal variance weight. Unfortunately the derivation of the Likelihood Ratio requires explicit knowledge of the density function while Malliavin theory enables us to derive the optimal weight with much less information.

5.3.4 Explicit conditions for the weights

For the construction of Malliavin weight it is then crucial that we are able to use condition (5.38) to select a stochastic process u such that its Skorohod integral is a proper weight. Actually, we can make condition (5.38) more explicit expressing the processes Z^{greek} in terms of the first variation process Y. This is convenient for two reasons. The first reason is that the first variation process is simpler to deal with since we can easily derive its SDE (5.20). The second reason is that the integrand on the left hand side of (5.38) is also expressed in terms of the first variation process due to the link between first variation process and Malliavin derivative of a diffusion.
For the case of *delta* we have already noticed that trivially $Z^{delta} \equiv Y$. For the processes Z^{rho} and Z^{vega} we have the following result.

Proposition 5.3 (Benhamou (2003))

The process Z^{rho} can be expressed in terms of the first variation process by:

$$Z_t^{rho} = Y_t \int_0^t \frac{\tilde{\mu}(X_s)}{Y_s} ds; \quad 0 \leq t \leq T \tag{5.42}$$

Similarly for the process Z^{vega} it holds:

$$Z_t^{vega} = Y_t \int_0^t \frac{\tilde{\sigma}(X_s)}{Y_s} dW_s - Y_t \int_0^t \sigma'(X_s) \frac{\tilde{\sigma}(X_s)}{Y_s} ds; \quad 0 \leq t \leq T \tag{5.43}$$

where $\tilde{\mu}(\cdot)$ and $\tilde{\sigma}(\cdot)$ are the functions which define the direction of perturbation of (resp.) the drift and the diffusion coefficient.

From these results we can reformulate condition (5.38) for each greek:

- *delta*:

$$\mathbb{E}\left[Y_{t_i} \int_0^{t_i} \frac{\sigma(X_t)}{Y_t} u_t dt \mid \bar{X}\right] = \mathbb{E}\left[Y_{t_i} \mid \bar{X}\right] \tag{5.44}$$

- *rho*:

$$\mathbb{E}\left[Y_{t_i} \int_0^{t_i} \frac{\sigma(X_t)}{Y_t} u_t dt \mid \bar{X}\right] = \mathbb{E}\left[Y_{t_i} \int_0^{t_i} \frac{\tilde{\mu}(X_t)}{Y_t} dt \mid \bar{X}\right]; \tag{5.45}$$

- *vega*:

$$\mathbb{E}\left[Y_{t_i} \int_0^{t_i} \frac{\sigma(X_t)}{Y_t} u_t dt \mid \bar{X}\right] = \mathbb{E}\left[Y_{t_i} \int_0^{t_i} \frac{\tilde{\sigma}(X_t)}{Y_t} dW_t - Y_{t_i} \int_0^{t_i} \sigma'(X_t) \frac{\tilde{\sigma}(X_t)}{Y_t} dt \mid \bar{X}\right] \tag{5.46}$$

Remark 5.8

Notice the central role played in this context by the first variation process. While Malliavin calculus provides the integration by part formula which frees the integrand of the differential operator, the first variation process is the necessary link between the Malliavin derivative and the derivative of the process with respect to its parameters. In some sense the first variation process provides the equivalent of formula (5.25) in the general case.

5.3.5 Explicit formula for the weights

From (5.44)- (5.46) we can easily derive some explicit weights for the Greeks. The following correspond to the weights proposed by Fournié et al. (1999).

- *delta*:

$$u_t = a(t)\frac{Y_t}{\sigma(X_t)}, \quad 0 \le t \le T \tag{5.47}$$

where $a = a(t) \in \Gamma_n := \left\{ a \in \mathcal{L}^2[0,T] : \int_0^{t_i} a(t)dt = 1, \forall i = 1,2,\ldots,n \right\}$

- *rho*:

$$u_t = \frac{\tilde{\mu}(X_t)}{\sigma(X_t)}, \quad 0 \le t \le T \tag{5.48}$$

- *vega*:

$$u_t = \frac{\tilde{a}(t)}{\sigma(X_t)} \sum_{i=1}^{n} \left(Z_{t_i}^{vega} - Z_{t_{i-1}}^{vega} \right) \mathbb{I}_{\{t_{i-1} \le t \le t_i\}} \tag{5.49}$$

where $\tilde{a} = \tilde{a}(t) \in \tilde{\Gamma}_n := \left\{ \tilde{a} \in \mathcal{L}^2[0,T] : \int_{t_{i-1}}^{t_i} \tilde{a}(t)dt = 1, \forall i = 1,2,\ldots,n \right\}$

It is easy to verify that these weights satisfy conditions (5.44)- (5.46). For example in the case of *delta*, (5.47) implies that:

$$Y_{t_i} \int_0^{t_i} \frac{\sigma(X_t)}{Y_t} u_t dt = Y_{t_i} \int_0^{t_i} a(t)dt = Y_{t_i} \tag{5.50}$$

so that the integrand in the left hand side and in the right hand side of (5.44) are equal.

Remark 5.9

The solutions of Fournié et al. (1999) are not the only possibilities for the weights. However they are the easiest to verify since they are based on the equality the integrands in (5.38) and do not need any computation involving the conditional expectations. Several other proposals of weights are presented in Benhamou (2003).

5.4 Exact Monte Carlo Simulation of Greeks

5.4.1 Preliminaries

Malliavin Calculus provides a representation of the Greeks in terms of expected values of the weighted payoffs. This representation is very convenient since it is general (differently from Pathwise and Likelihood Ratio method) and exact (differently from Finite Difference method). In particular this approach seems the best one when the payoff function is discontinuous or the density involved in the pricing formula is not known. In fact in these cases, Pathwise method and Likelihood Ratio method are not feasible and unbiased Monte Carlo evaluation of Greeks is possible only by means of the Malliavin approach. Beyond this, it is well known that when the payoff is discontinuous and complex Finite Difference method performs very poorly (see e.g. the simulation study in Benhamou, 2000, Ch.2)

However, from a practical point of view, this nice theoretical framework is useful only if it is actually possible to simulate a Monte Carlo estimator of the Malliavin Greeks. This is particularly challenging since it involves the simulation of a stochastic integrals (the Malliavin weight).

In their seminal paper, Fournié et al. (1999), after constructing the general representations (5.47), (5.48) and (5.49) for *delta*, *rho* and *vega*, test them within a simple Black and Scholes model. In particular they perform Monte Carlo simulation only on those Greeks (*delta* and *vega*) for which the stochastic integral can be easily solved. For example, in the case of *delta*, assuming that X is Geometric Brownian motion (5.21) and the payoff ϕ depends on the value of the underlying at some expiry time $T > 0$, we have from (5.23) and (5.47)

$$u_t = \frac{Y_t}{\sigma(X_t)}\frac{1}{T} = \frac{1}{x\sigma T}$$

where we have chosen $a(t) = \frac{1}{T}$. Then the Malliavin weight is simply

$$\delta(u) = \int_0^T u_t dW_t = \frac{1}{x\sigma T} W_T$$

and we end with the formula:

$$delta = \frac{\partial u(x)}{\partial x} = \mathbb{E}\left(\phi(X_T)\frac{W_T}{\sigma x T}\right)$$

It is worth noticing that the same formula can be obtained in a more direct way applying the integration by part as shown in section 5.3.1 (5.24) or computing the likelihood ratio like in Broadie and Glasserman (1996). In general, for the Black and Scholes model, we can derive suitable representations of the Greeks by the Pathwise method or the Likelihood Ratio method and there is no need to resort to Malliavin theory (see e.g. Broadie and Glasserman, 1996).

However, in not trivial situations, when Pathwise and Likelihood Ratio approach don't provide a suitable representation for the Greeks, the Malliavin approach might fail because we are not able to simulate a Monte Carlo estimator. We could clearly resort to discretization methods: however the simulation of a stochastic integral by means of the discretized process introduces a relevant bias in Monte Carlo estimation.

We request a simulation procedure which translates the general and powerful theory of the Malliavin computation of Greeks into the practice of Monte Carlo simulation. The method should satisfy two minimal conditions. On one side, it should be unbiased (unlike the Finite Difference method); on the other side, it should be general enough to deal with models to which Pathwise method and Likelihood Ratio method can't be applied.

This section is intended to be a first step in this direction.

We consider a diffusion process X defined by

$$dX_t = \mu(X_t)dt + dW_t; \quad X_0 = x, \qquad 0 \le t \le T \tag{5.51}$$

satisfying conditions $C1 - C3$ of section 1.3.2 (i.e. $\mathcal{D}_1$-process) and a payoff function $\phi = \phi(X_T)$. In this context, we show how to simulate an unbiased estimator of *delta* and *rho* with the Malliavin weights (5.47) and (5.48) proposed by Fournié et al. (1999).

5.4.2 Monte Carlo estimation of Rho

Recalling the theory of Malliavin Greeks, *rho* can be represented as the expected value of the payoff weighted by the Skorohod integral of (5.48). Since (5.48) is adapted we can write the Malliavin weight in terms of the Ito integral (see remark 5.2):

$$Rho = \mathbb{E}\left(\phi(X_T)\int_0^T \tilde{\mu}(X_t)dW_t\right) \tag{5.52}$$

The definition of a general procedure for unbiased Monte Carlo simulation of (5.52) is hardly possible. However if the process X satisfies conditions $C1 - C3$, we can successfully apply the Exact Algorithm 1 to our Monte Carlo problem. In particular, from the construction of section 1.3, $EA1$ returns a Skeleton $\mathcal{S}_1$ of X. In this section, for convenience of notation, we represent $\mathcal{S}_1$ in the following way:

$$\mathcal{S}_1 = \{(t_0, S_{t_0}), (t_1, S_{t_1}), \ldots, (t_M, S_{t_M})\} \tag{5.53}$$

where $(t_0, S_{t_0}) \equiv (0, x)$ and $(t_M, S_{t_M}) \equiv (T, X_T)$. Then:

$$\begin{aligned}\mathbb{E}\left(\phi(X_T)\int_0^T \tilde{\mu}(X_t)dW_t\right) &= \mathbb{E}\left(\mathbb{E}\left(\phi(X_T)\int_0^T \tilde{\mu}(X_t)dW_t \mid \mathcal{S}_1\right)\right) \\ &= \mathbb{E}\left(\sum_{i=1}^{M}\mathbb{E}\left(\phi(X_T)\int_{t_{i-1}}^{t_i} \tilde{\mu}(X_t)dW_t \mid \mathcal{S}_1\right)\right)\end{aligned} \tag{5.54}$$

For any $i = 1, 2, \ldots, M$, let us denote:

$$e_i(\mathcal{S}_1) = \mathbb{E}\left(\phi(X_T)\int_{t_{i-1}}^{t_i} \tilde{\mu}(X_t)dW_t \mid \mathcal{S}_1\right)$$

We assume that $\tilde{\mu}(\cdot)$ is everywhere differentiable and we define a function $H : \mathbf{R} \to \mathbf{R}$ by

$$H(u) = \int_0^u \tilde{\mu}(s)ds$$

Proposition 5.4

Let $\mathcal{S}_1$ be the Skeleton (5.53) generated by EA1. For any $i = 1, 2, \ldots, M$

$$e_i(\mathcal{S}_1) = \phi(S_{t_M}) \left[H(S_{t_i}) - H(S_{t_{i-1}}) - \mathbb{E}\left(\int_{t_{i-1}}^{t_i} \left[\frac{1}{2}\tilde{\mu}'(Z_t^{(i)}) + \tilde{\mu}(Z_t^{(i)})\mu(Z_t^{(i)}) \right] dt \right) \right] \tag{5.55}$$

where $Z^{(i)} = \left\{ Z_t^{(i)} = BB_t\left(t_{i-1}, S_{t_{i-1}}; t_i, S_{t_i}\right); t_{i-1} \le t \le t_i \right\}$.

Proof:

For any $i = 1, 2, \ldots, M$:

$$\begin{aligned} e_i(\mathcal{S}_1) &= \mathbb{E}\left(\phi(X_T) \int_{t_{i-1}}^{t_i} \tilde{\mu}(X_t) dW_t \mid \mathcal{S}_1 \right) \\ &= \phi\left(S_{t_M}\right) \mathbb{E}\left(\int_{t_{i-1}}^{t_i} \tilde{\mu}(X_t) dW_t \mid \mathcal{S}_1 \right) \\ &= \phi\left(S_{t_M}\right) \mathbb{E}\left(\int_{t_{i-1}}^{t_i} \tilde{\mu}(X_t) \left(dX_t - \mu(X_t)dt\right) \mid \mathcal{S}_1 \right) \end{aligned} \tag{5.56}$$

By Ito's lemma:

$$dH\left(X_t\right) = \tilde{\mu}(X_t)dX_t + \frac{1}{2}\tilde{\mu}'\left(X_t\right) dt$$

so that

$$\int_{t_{i-1}}^{t_i} \tilde{\mu}(X_t)dX_t = H\left(X_{t_i}\right) - H\left(X_{t_{i-1}}\right) - \frac{1}{2}\int_{t_{i-1}}^{t_i} \tilde{\mu}'\left(X_t\right) dt$$

Substituting in (5.56) gives:

$$\begin{aligned} e_i(\mathcal{S}_1) &= \phi(S_{t_M})\mathbb{E}\left(H\left(X_{t_i}\right) - H\left(X_{t_{i-1}}\right) - \int_{t_{i-1}}^{t_i}\left[\frac{1}{2}\tilde{\mu}'\left(X_t\right) + \tilde{\mu}(X_t)\mu(X_t)\right]dt \mid \mathcal{S}_1\right) \\ &= \phi(S_{t_M})\left[H\left(S_{t_i}\right) - H\left(S_{t_{i-1}}\right) - \mathbb{E}\left(\int_{t_{i-1}}^{t_i}\left[\frac{1}{2}\tilde{\mu}'\left(X_t\right) + \tilde{\mu}(X_t)\mu(X_t)\right]dt \mid \mathcal{S}_1\right)\right] \end{aligned}$$

From Theorem 1.2, for any $i = 1, 2, \ldots, M$:

$$\left\{X_t : t_{i-1} \le t \le t_i\right\} \mid \mathcal{S}_1 \stackrel{d}{=} \left\{BB_t\left(t_{i-1}, S_{t_{i-1}}; t_i, S_{t_i}\right); t_{i-1} \le t \le t_i\right\} := \left\{Z_t^{(i)} : t_{i-1} \le t \le t_i\right\}$$

and the conclusion follows.

□

The advantage of this representation is that we have reformulated the problem of the simulation of the Ito integral of (a function of) a generic diffusion X in terms of the much easier problem of the simulation of a Riemann integral of (a function of) a trajectory of a Brownian bridge.

However, since we are not able to compute analytically the expected value in (5.55), we can't simulate directly $e_i(\mathcal{S}_1)$. We circumvent the problem by simulating an additional uniform random variable to rid the simulation of the integral.

Specifically, let's define the function $h : \mathbf{R} \to \mathbf{R}$:

$$h(u) = \frac{1}{2}\tilde{\mu}'\left(u\right) + \tilde{\mu}(u)\mu(u)$$

Then it is true that

$$\mathbb{E}\left(\int_{t_{i-1}}^{t_i} h\left(Z_t^{(i)}\right)dt\right) = (t_i - t_{i-1})\mathbb{E}\left(h\left(Z_{U_i}^{(i)}\right)\right) \tag{5.57}$$

where the first expectation is w.r.t. the Brownian bridge while the second w.r.t. Brownian bridge and a uniform random variable $U_i \sim Unif\,[t_{i-1}, t_i]$ independent of the Brownian bridge.

We are now able to simulate an unbiased estimator of rho. The construction is an immediate consequence of the argument just outlined.

Given $\mathcal{S}_1$, we simulate an "Enriched" Skeleton

$$\mathcal{S}_1^E = \left\{(t_0, S_{t_0}), (U_1, S_{U_1})\,(t_1, S_{t_1}), \ldots, (t_{M-1}, S_{t_{M-1}}), (U_M, S_{U_M})\,(t_M, S_{t_M})\right\} \tag{5.58}$$

in the following way:

$$\begin{aligned} U_i \mid \mathcal{S}_1 &\sim Unif\,[t_{i-1}, t_i] \\ S_{U_i} \mid \mathcal{S}_1, U_i &\overset{d}{=} S_{t_{i-1}} + \frac{U_i - t_{i-1}}{t_i - t_{i-1}}\left(S_{t_i} - S_{t_{i-1}}\right) + \frac{t_i - U_i}{\sqrt{t_i - t_{i-1}}} W_{\frac{U_i}{t_i - U_i}} \end{aligned}$$

where we have used the representation of the Brownian bridge in terms of Brownian motion.

We can define an unbiased estimator $\hat{rho}$ of rho as a function of $\mathcal{S}_1^E$. In fact from (5.55) and (5.57)

$$\hat{rho} = \sum_{i=1}^{M} e_i^*\left(\mathcal{S}_1^E\right)$$

where for any $i = 1, 2, \ldots, M$

$$e_i^*\left(\mathcal{S}_1^E\right) = \phi(S_{t_M})\left\{H\left(S_{t_i}\right) - H\left(S_{t_{i-1}}\right) - (t_i - t_{i-1})\left[\frac{1}{2}\tilde{\mu}'(S_{U_i}) + \tilde{\mu}(S_{U_i})\mu(S_{U_i})\right]\right\}$$

Finally we construct the Monte Carlo estimator simulating $\hat{rho}$ a sufficiently large number of times and averaging over the simulated values.

5.4.3 Monte Carlo estimation of Delta

We consider now the problem of Monte Carlo estimation of *delta* when the Malliavin weight is defined by the Skorohod integral of (5.47). In particular, we assume $a(t) = \frac{1}{T}$. Then:

$$delta = \frac{1}{T}\mathbb{E}\left(\phi(X_T)\delta(Y)\right) \tag{5.59}$$

where Y is the first variation process of X defined by (5.20) and δ is the Skorohod integral operator. In our case with unit diffusion coefficient (5.51) we have:

$$dY_t = \mu'(X_t)Y_t dt; \qquad Y_0 = 1$$

so that

$$Y_t = e^{\int_0^t \mu'(X_s)ds} \tag{5.60}$$

where μ' is the first derivative of the drift. Since the process Y is adapted, (5.59) becomes:

$$delta = \frac{1}{T}\mathbb{E}\left(\phi(X_T)\int_0^T Y_t dW_t\right) \tag{5.61}$$

Since the process X belongs to $\mathcal{D}_1$ we can simulate a Skeleton $\mathcal{S}_1$. Proceeding as for *rho* we write:

$$\begin{aligned} delta = \frac{1}{T}\mathbb{E}\left(\phi(X_T)\int_0^T Y_t dW_t\right) &= \frac{1}{T}\mathbb{E}\left(\sum_{i=1}^{M}\mathbb{E}\left(\phi(X_T)\int_{t_{i-1}}^{t_i} Y_t dW_t \mid \mathcal{S}_1\right)\right) \\ &= \frac{1}{T}\mathbb{E}\left(\phi(S_{t_M})\sum_{i=1}^{M}\mathbb{E}\left(\int_{t_{i-1}}^{t_i} Y_t dW_t \mid \mathcal{S}_1\right)\right) \end{aligned}$$

We define, for any $i = 1, 2, \ldots, M$:

$$g_i(\mathcal{S}_1) = \mathbb{E}\left(\int_{t_{i-1}}^{t_i} Y_t dW_t \mid \mathcal{S}_1\right)$$

Moreover we introduce the following notation:

$$\epsilon_f(s_1, a; s_2, b) := \mathbb{E}\left(e^{\int_{s_1}^{s_2} f(Z_s)ds}\right) \tag{5.62}$$

where $Z \stackrel{d}{=} BB(s_1, a; s_2, b)$ and f is a function $\mathbf{R} \to \mathbf{R}$. In particular we denote by $\epsilon_f^{(i)}(\mathcal{S}_1) = \epsilon_f(t_{i-1}, S_{t_{i-1}}; t_i, S_{t_i})$.

Proposition 5.5

Let $\mathcal{S}_1$ be the Skeleton produced by EA1. For any $i = 1, 2, \ldots, M$:

$$g_i(\mathcal{S}_1) = \left[\prod_{j=1}^{i-1} \epsilon_{\mu'}^{(j)}(\mathcal{S}_1)\right]\left[S_{t_i}\epsilon_{\mu'}^{(i)}(\mathcal{S}_1) - S_{t_{i-1}} - \mathbb{E}\left(\int_{t_{i-1}}^{t_i} e^{\int_{t_{i-1}}^{t}\mu'(X_s)ds}\left(\mu(X_t) + X_t\mu'(X_t)\right)dt \mid \mathcal{S}_1\right)\right] \tag{5.63}$$

Proof:

$$g_i(\mathcal{S}_1) = \mathbb{E}\left(\int_{t_{i-1}}^{t_i} Y_t\left(dX_t - \mu(X_t)dt\right) \mid \mathcal{S}_1\right)$$

By Ito's Lemma

$$\begin{aligned}\int_{t_{i-1}}^{t_i} Y_t dX_t &= \left[Y_{t_i}X_{t_i} - Y_{t_{i-1}}X_{t_{i-1}} - \int_{t_{i-1}}^{t_i} X_t dY_t\right] \\ &= \left[Y_{t_i}X_{t_i} - Y_{t_{i-1}}X_{t_{i-1}} - \int_{t_{i-1}}^{t_i} X_t\mu'(X_t)Y_t dt\right]\end{aligned}$$

so that

$$g_i(\mathcal{S}_1) = \mathbb{E}\left(Y_{t_i}X_{t_i} - Y_{t_{i-1}}X_{t_{i-1}} - \int_{t_{i-1}}^{t_i} Y_t\left(\mu(X_t) + X_t\mu'(X_t)\right)dt \mid \mathcal{S}_1\right)$$

Now for Theorem 1.2, by the conditional independence of the bridges given the Skeleton:

$$\mathbb{E}(Y_{t_i}X_{t_i} \mid \mathcal{S}_1) = S_{t_i}\mathbb{E}\left(\prod_{j=1}^{i} e^{\int_{t_{j-1}}^{t_j}\mu'(X_s)ds} \mid \mathcal{S}_1\right) = S_{t_i}\left\{\prod_{j=1}^{i-1}\epsilon_{\mu'}^{(j)}(\mathcal{S}_1)\right\}\epsilon_{\mu'}^{(i)}(\mathcal{S}_1) \tag{5.64}$$

By analogous argument

$$\mathbb{E}\left(Y_{t_{i-1}}X_{t_{i-1}} \mid \mathcal{S}_1\right) = S_{t_{i-1}}\left\{\prod_{j=1}^{i-1}\epsilon_{\mu'}^{(j)}(\mathcal{S}_1)\right\} \tag{5.65}$$

and also

$$\mathbb{E}\left(\int_{t_{i-1}}^{t_i} Y_t\left(\mu(X_t) + X_t\mu'(X_t)\right)dt \mid \mathcal{S}_1\right) = \left\{\prod_{j=1}^{i-1}\epsilon_{\mu'}^{(j)}\right\}\mathbb{E}\left(\int_{t_{i-1}}^{t_i} e^{\int_{t_{i-1}}^{t}\mu'(X_s)ds}\left(\mu(X_t) + X_t\mu'(X_t)\right)dt \mid \mathcal{S}_1\right) \tag{5.66}$$

The conclusion follows combining (5.64),(5.65) and (5.66).

□

Recalling the construction of the Enlarged Skeleton $\mathcal{S}_1^E \supset \mathcal{S}_1$ and the uniform sampling technique introduced in the previous section:

$$\mathbb{E}(g_i(\mathcal{S}_1)) = \mathbb{E}(\hat{g}_i(\mathcal{S}_1^E))$$

where

$$\hat{g}_i(\mathcal{S}_1^E) = \left[\prod_{j=1}^{i-1}\epsilon_{\mu'}^{(j)}(\mathcal{S}_1)\right]\left[S_{t_i}\epsilon_{\mu'}^{(i)}(\mathcal{S}_1) - S_{t_{i-1}} - \epsilon_{\mu'}(t_{i-1}, S_{t_{i-1}}; U_i, S_{U_i})\left(\mu(S_{U_i}) + S_{U_i}\mu'(S_{U_i})\right)\right]$$

We can't find explicit expressions for the functions $\epsilon_{\mu'}(\cdot)$ so that we are not able to simulate directly $\hat{g}_i$. The solution is to simulate an unbiased estimator of the conditional expectation (5.62).

We present two methods to construct unbiased estimators for a generic function $e_f(s_1, a; s_2, b)$: the Acceptance method and the Poisson estimator. The two methods have already been used in the field of inference for diffusion processes in Beskos et al. (2005). In particular the Poisson estimator dates back to the works of Wagner (1988b, 1989) who introduced it in the context of statistical physics.
Interestingly, the two estimators are strictly linked; in particular the estimator produced by the Acceptance method turns out to be a particular case of the Poisson estimator (see remark 5.10).

Acceptance Method.
In order to apply this method we need to assume that the function $f(\cdot)$ is bounded:

$$l \leq f(\cdot) \leq l + r$$

Then we can write:

$$\epsilon_f(s_1, a; s_2, b) = \mathbb{E}\left(e^{\int_{s_1}^{s_2} f(Z_u)du}\right) = e^{(l+r)(s_2-s_1)}\mathbb{E}\left(e^{-\int_{s_1}^{s_2} h(Z_u)du}\right)$$

where $h(z) = l + r - f(z)$ is such that $0 \leq h(\cdot) \leq r$.
Define a marked Poisson process Φ of unit intensity on the rectangle $[s_1, s_2] \times [0, r]$. We denote $\Phi = \{\Psi, \Upsilon\}$ with:

$$\begin{aligned} \Psi &= \{\psi_1, \psi_2, \ldots, \psi_k\} \\ \Upsilon &= \{\upsilon_1, \upsilon_2, \ldots, \upsilon_k\} \end{aligned}$$

Let us introduce the following indicator variable:

$$I(Z, \Phi; s_1, a; s_2, b) = \prod_{n=1}^{k} \mathbb{I}_{\{h(Z_{\psi_n}) < \upsilon_n\}}$$

Then we have:

$$\epsilon_f(s_1, a; s_2, b) = e^{(l+r)(t-s)}\mathbb{E}\left(I(Z, \Phi; s_1, a; s_2, b)\right)$$

where the expectation is taken with respect to (Z, Φ). Thus a simple unbiased estimator of $\epsilon_f(s_1, a; s_2, b)$ is just:

$$\hat{\epsilon}_f(Z, \Phi; s_1, a; s_2, b) = e^{(l+r)(s_2-s_1)} \prod_{n=1}^{k} \mathbb{I}_{\{h(Z_{\psi_n})<v_n\}} \tag{5.67}$$

Notice that (5.67) admits a suitable Rao-Blackwelized version. Indeed:

$$\epsilon_f(s_1, a; s_2, b) = e^{(l+r)(s_2-s_1)}\mathbb{E}\left(\mathbb{E}\left(I(Z, \Phi; s_1, a; s_2, b) \mid \Psi, Z\right)\right)$$

The conditional expectation can be written explicitly in the following way:

$$\mathbb{E}\left(I(Z, \Phi; s_1, a; s_2, b) \mid \Psi, Z\right) = \prod_{j=1}^{k}\left[1 - \frac{h\left(Z_{\psi_j}\right)}{r}\right]$$

so that the Rao Blackwellized version of $\hat{\epsilon}_f$ is given by:

$$\hat{\hat{\epsilon}}_f(Z, \Psi; s_1, a, ; s_2, b) = e^{(l+r)(s_2-s_1)} \prod_{j=1}^{k}\left[1 - \frac{h\left(Z_{\psi_j}\right)}{r}\right] \tag{5.68}$$

Notice that, given the parameters of the Brownian bridge Z, the estimators (5.67) and (5.68) can be easily simulated since they depend on the realization of a Poisson process Φ (or Ψ) and of a Brownian bridge at the time points selected by the Poisson process.

Poisson Estimator.

We present now another way to construct an unbiased estimator of $\epsilon_f(s_1, a; s_2, b)$.

First, we observe that for any $c \in \mathbf{R}$, $\lambda > 0$:

$$\begin{aligned} e^{\int_{s_1}^{s_2} f(Z_u)du} &= e^{-c(s_2-s_1)} \sum_{j=0}^{\infty} \frac{1}{j!} \left\{ \lambda(s_2 - s_1) \int_{s_1}^{s_2} \frac{c + f(Z_s)}{\lambda(s_2 - s_1)} ds \right\}^j \\ &= e^{(\lambda - c)(s_2 - s_1)} \mathbb{E}\left[\left\{ \int_{s_1}^{s_2} \frac{c + f(Z_s)}{\lambda \delta} ds \right\}^k \right] \end{aligned}$$

where the expectation is taken with respect to $k \sim Pois(\lambda(s_2 - s_1))$ conditionally on Z. As usual, we can sample k independent uniform random variables $U \sim Unif[s_1, s_2]$ to construct unbiased estimators of the integrals $\int_{s_1}^{s_2}(c + f(Z_s))/(\lambda(s_2 - s_1))ds$. In other terms, letting $\Psi = \{\psi_1, \psi_2, \ldots, \psi_k\}$ be a Poisson process on $[s_1, s_2]$ with intensity λ:

$$e^{\int_{s_1}^{s_2} f(Z_u)du} = e^{(\lambda - c)(s_2 - s_1)} \mathbb{E}\left(\lambda^{-k} \prod_{j=1}^{k} [c - f(Z_{\psi_j})] \right)$$

where the expectation is taken with respect to Ψ.

Now it is clear that an unbiased estimator of $\epsilon_f(s_1, a; s_2, b)$ is given by:

$$\tilde{\epsilon}_f(Z, \Psi; s_1, a; s_2, b) = e^{(\lambda - c)(s_2 - s_1)} \left(\lambda^{-k} \prod_{j=1}^{k} [c - f(Z_{\psi_j})] \right). \tag{5.69}$$

We call this estimator *Poisson estimator*. Again, as in (5.68), this estimator depends only on the realization of a Poisson process Ψ and a Brownian bridge at the time points selected by Ψ. In fact there is a strong connection between (5.68) and (5.69).

Remark 5.10

The estimator (5.68) based on the acceptance method can be seen as a particular case of the Poisson estimator. In fact if in (5.69) we assume $\lambda = r$ and $c = -l$

$$\tilde{\epsilon}_f(Z, \Psi; s_1, a; s_2, b) = e^{(l+r)(s_2-s_1)} \prod_{j=1}^{k} \left[1 - \frac{h(Z_{\psi_j})}{r}\right]$$

where $h(\cdot) = l + r - f(\cdot)$ *and* $\Psi = \{\psi_1, \psi_2, \ldots, \psi_k\}$ *is a Poisson process of rate* r *on* $[s_1, s_2]$.

5.5 Conclusion

In this chapter we have proposed a new method for Monte Carlo evaluation of Greeks. Assuming the Malliavin representation of the Greeks of Fournié et al. (1999), we showed how to simulate an unbiased estimator of *delta* and rho for a general class of diffusion processes X.

The first part of the chapter is descriptive.
We presented the challenging problem of the computation of the Greeks and illustrated the traditional Monte Carlo approaches. Discussing the weaknesses of these approaches led us to introduce the Malliavin approach. Malliavin approach relies on the fascinating though complex theory of Malliavin Calculus. It manages to represent the Greeks as expected values of the weighted payoffs. Over competing methods, this representation has the advantage to be exact (unlike Finite Difference method) and general (unlike Pathwise and Likelihood Ratio method). However a major problem is the actual simulation of the Malliavin Monte Carlo estimator. This involves the joint simulation of the payoff and of a stochastic integral (the Malliavin weight).
The second part of the chapter contains our contribution.
We proposed a novel method for the Monte Carlo estimate of the Malliavin Greeks for the case of rho and *delta* when the diffusion X belongs to $\mathcal{D}_1$. Firstly, we simulate the Skeleton $\mathcal{S}_1$. Then given the Skeleton we showed how to simulate unbiased estimators of the Greeks. Since direct simulation of a stochastic integral is hardly possible, we had to apply some stochastic calculus to transform the stochastic integral into a simple Riemann integral. We finally construct the unbiased estimators applying to the resulting Riemann integrals suitable simulation techniques.

This contribution is meant to be just the first step of an ambitious project. On one hand we have seen that

Malliavin Calculus provides a nice general framework for Monte Carlo treatment of the Greeks. On the other hand we have shown that the Exact Algorithm unveils new and interesting possibilities for the actual simulation of Malliavin Greeks. Main purpose of future research work will be to investigate further the potentialities of the Exact Algorithm and try to exploit them in order to define a general methodology for unbiased Monte Carlo estimation of the Greeks. To this scope, we have to address further investigations in the three main directions.

Firstly, we wish to extend these ideas to the other (first order) derivative, *vega* and also to the second order derivative *gamma*. Even though the Malliavin representation of these Greeks is more complicated, the general scheme does not change: they are given as expected values of the payoff times a (Skorohod) stochastic integral. For this reason we think that this same framework can be applied with relatively small changes to their estimation.

Another direction of research involves the extension of the method to options with different payoffs. It is worth to recall that Malliavin framework of section 5.3 holds when the payoff ϕ can be expressed as a function of the diffusion trajectory at a discrete set of time points. This includes some important option schemes like European options or Discrete Asian options. In our treatment we have considered for simplicity the European case $\phi = \phi(X_T)$. However our method can be extended easily to the general case $\phi = \phi(X_{t_1}, X_{t_2}, \ldots, X_{t_n})$. An interesting question is whether it is possible to define a general framework which includes also more sophisticated path dependent payoff schemes. A major problem is that nice representations (5.47)- (5.49) introduced in section 5.3 do not hold in these cases. Nevertheless it is still possible for some options to derive suitable Malliavin representations for the Greeks. We refer to Benhamou (2000) for the case of Asian options and to Gobet and Kohatsu-Higa (2003) for the case of barrier options. However, especially in the barrier options case, the expressions for the Malliavin weights are much more complicated than (5.47)- (5.49) and consequently it is more difficult to simulate a Monte Carlo estimator.

A third objective is to extend the methodology to classes of diffusions more general than $\mathcal{D}_1$. In Chapter 2 we introduced the classes $\mathcal{D}_2$ and $\mathcal{D}_3$ and showed how to simulate a Skeleton of the processes belonging to these classes. For these processes the construction of the Monte Carlo estimator is the same as for $\mathcal{D}_1$-processes and the extension is straightforward. On the contrary, an extension to the class $\mathcal{D}_{jump}$ introduced in Chapter 3 is more complicated. In fact, the lack of differentiability of the process X due to the

presence of jumps does not allow to apply directly Malliavin Calculus for the derivation of the Malliavin weights (but see El-Khatib and Privault, 2004).

Beyond this, the efficiency of our estimators need to be investigated carefully both with a theoretical and a numerical analysis. In particular the simulation study would involve a numerical comparison between our Monte Carlo estimator and the Monte Carlo estimator constructed with the Finite Difference method. Simulation studies of this type have already been done for the Black-Scholes model in Fournié et al. (1999), Fournié et al. (2001) and Benhamou (2000). However it is not available any simulation study involving other diffusion models due to the difficulty to simulate a Malliavin Monte Carlo estimator. Notably, our simulation procedure allows to set up a simulation study testing the performances of the Malliavin estimator within a general class $\mathcal{D}_1$ of diffusion models.

Conclusion

In this thesis we introduced some novel techniques for Monte Carlo simulation of functionals ϕ of diffusion and jump diffusion processes Y. We proposed in particular a Monte Carlo treatment for two classes of problems of relevant interest in mathematical finance: the barrier problem and the problem of the estimation of the Greeks.

A notable feature of our proposal, is that we are able to simulate ϕ exactly even without explicit knowledge of the transition densities of the process. This allows us to construct unbiased Monte Carlo estimators.

The simulation procedure is based on two main steps. Firstly, we simulate a discrete exact Skeleton of the process using the retrospective rejection sampling introduced by Beskos et al. (2004). Then, conditionally on the Skeleton, we simulate the target functional ϕ. In both steps, there is no need for the introduction of approximations: the Skeleton as well as the target functional are sampled from their "*true*" probability measure. As a consequence, unlike the other current methods, the sample of ϕ is exact and the resulting Monte Carlo estimator is affected only by the Monte Carlo error.

In the first part of the thesis (Chapter 1 to 4) we considered the barrier problem.

In particular in Chapter 1 and 2, we assume that the underlying process Y is a pure diffusion satisfying specific sets of conditions (d_1, d_2, d_3). We showed how to generate a Skeleton for these processes and derived their conditional law given the Skeleton. Finally we discussed the crossing probability of the processes (under the conditional law) and developed suitable algorithms to simulate the crossing event both with one barrier and two barriers.

One main objective of future research work is to extend the methodology to larger classes of diffusion processes as well as to other time-continuous processes of large use, like jump-diffusion processes, Levy

processes or stochastic volatility processes. A first step to this direction is presented in Chapter 3 where the Exact Monte Carlo method was extended to jump-diffusions.

Another domain of interest is to explore the potentialities of our method in financial applications. An example of such potentialities is given in Chapter 4 where it was applied to jump diffusion processes with state dependent intensity for modeling credit risk. Another natural area of application is the pricing of barrier and lookback options.

Furthermore, even though the results from our simulation study are fully satisfactory, it is worth to carry out a more comprehensive numerical simulation. In particular the aspects linked to the efficiency of our method in comparison with Euler-based methods need to be deepen. We also mention the fact that a systematic simulation study can be particularly useful also in the context of stochastic numerical analysis to address further investigations on the weak convergence of discretization methods.

In Chapter 3 we extended the Exact Monte Carlo method to a class $\mathcal{D}_{jump}$ of jump diffusion processes Y. We defined a Jump Exact Algorithm to simulate an exact Skeleton ($\mathcal{S}_{jump}$) of the process Y which incorporates the jumps. The key idea behind the construction of $\mathcal{S}_{jump}$ is to combine the Exact Algorithm for the simulation of the continuous component of Y and a stochastic thinning algorithm for the simulation of the jump component. As in the case of pure diffusions, given $\mathcal{S}_{jump}$, we determined the conditional law of the process and defined a procedure for the simulation of the crossing event.

Clearly top priority of future research is the implementation of this method. Our aim is to test the algorithm on a model of potential interest in financial modeling. To this end, we introduced in Chapter 4, an application to credit risk modeling. Nevertheless, we believe that there are many other applications which worth to be investigated. In fact jump diffusions with state dependent intensity are potentially very useful modeling tools for applied works. In particular they can be used to improve the usual Merton scheme when the assumption of independence between the jump component and the diffusion component is too restrictive.

Another direction of research involves to set up the general construction of a rejection sampler for jump diffusions by means of their Girsanov derivative, in the spirit of the $EA1$ rejection sampler illustrated in section 1.3. The rejection sampling step would then combine the Exact Algorithm step and the thinning step. This would lead to more compact algorithm formulation and possible improvements of the efficiency.

In Chapter 4, we applied the simulation procedure presented in Chapter 3 to a computational problem arising in credit risk. Specifically, within a structural approach to credit risk, we assume that the firm's value is described by a jump diffusion process with state dependent intensity. We show how to price a 0 coupon defaultable bond under the assumptions that the recovery rate is a fraction of the face value of the bond and depends on the firm's value at the time of default (**A2**), the bondholder receives the recovery only at maturity time (**A3**) and the interest rate is constant (**A4**).

Some of these assumptions can be relaxed.

We can certainly relax condition **A3** and assume, more realistically, that the bondholder receives the recovery when the default occurs. In this case we have to simulate the default time in order to evaluate the discounted factor in the pricing formula. This can be done by simulating the Skeleton $\mathcal{S}_{jump}$ and then using Bachelier-Levy density to simulate the first passage time of the Brownian bridges.

We believe that also assumption **A4** can be relaxed, allowing for stochastic interest rates. In this case in order to construct an unbiased estimator of p we should probably resort to more sophisticated simulation techniques such as the *Poisson estimator*.

Empirical investigations need to be done in order to fit our model to real data.

In particular a major open problem concerns how to model the dependence between the intensity and the firm's value process.

In part 2 of the thesis (Chapter 5) we considered the problem of the estimation of the Greeks.

In particular, assuming the Malliavin representation of the Greeks of Fournié et al. (1999), we defined a procedure to simulate an unbiased estimator of *delta* and *rho* for a general class of diffusion processes X.

This contribution is meant to be just the first step of an ambitious project. On one hand we observed that Malliavin Calculus provides a nice general framework for Monte Carlo treatment of the Greeks. On the other hand we showed that the Exact Algorithm unveils new and interesting possibilities for the actual simulation of Malliavin Greeks. Main purpose of future research work will be to investigate further the potentialities of the Exact Algorithm and try to exploit them in order to define a general methodology for unbiased Monte Carlo estimation of the Greeks. To this scope, we have to address further investigations

in the three main directions.

Firstly, we wish to extend these ideas to the other (first order) derivative, *vega* and also to the second order derivative *gamma*. Even though the Malliavin representation of these Greeks is more complicated, the general scheme does not change: they are given as expected values of the payoff times a (Skorohod) stochastic integral. For this reason we think that this same framework can be applied with relatively small changes to their estimation.

Another direction of research involves the extension of the method to options with different payoffs. It is worth to recall that Malliavin framework holds when the payoff ϕ can be expressed as a function of the diffusion trajectory at a discrete set of time points. This includes some important option schemes like European options or Discrete Asian options. In our treatment we have considered for simplicity the European case $\phi = \phi(X_T)$. However our method can be extended easily to the general case $\phi = \phi(X_{t_1}, X_{t_2}, \ldots, X_{t_n})$. An interesting question is whether it is possible to define a general framework which includes also more sophisticated path dependent payoff schemes. A major problem is that Malliavin representations (5.47)-introduced in section 5.3 do not hold in these cases. Nevertheless it is still possible for some options to derive suitable Malliavin representations for the Greeks. We refer to Benhamou (2000) for the case of Asian option and to Gobet and Kohatsu-Higa (2003) for the case of barrier options. However, especially in the barrier options case, expressions for the Malliavin weights are very complicated and consequently it is difficult to simulate a Monte Carlo estimator.

A third objective is to extend the methodology to classes of diffusions more general than $\mathcal{D}_1$. In Chapter 2 we introduced the classes $\mathcal{D}_2$ and $\mathcal{D}_3$ and showed how to simulate a Skeleton of the processes belonging to these classes. For these processes the construction of the Monte Carlo estimator is the same as for $\mathcal{D}_1$-processes and the extension is straightforward. On the contrary, an extension to the class $\mathcal{D}_{jump}$ introduced in Chapter 3 is more complicated. In fact, the lack of differentiability of the process X due to the presence of jumps does not allow to apply directly Malliavin Calculus for the derivation of the Malliavin weights (but see El-Khatib and Privault, 2004).

Beyond this, the efficiency of our estimators need to be investigated carefully both with a theoretical and a numerical analysis. In particular the simulation study would involve a numerical comparison between our Monte Carlo estimator and the Monte Carlo estimator constructed with the Finite Difference method. Simulation studies of this type have already been done for the Black-Scholes model in Fournié et al. (1999),

Fournié et al. (2001) and Benhamou (2000). However it is not available any simulation study involving other diffusion models due to the difficulty to simulate a Malliavin Monte Carlo estimator. Notably, our simulation procedure allows to set up a simulation study testing the performances of the Malliavin estimator within a general class $\mathcal{D}_1$ of diffusion models.

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