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Essays in Network Economics

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Acknowledgements

This is a thesis about networks: how the connections we have to others affect what we think and what we can do. Therefore, it cannot begin but with thanking all the people that made this work possible!

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Last but not least, I am grateful to my family and friends. My parents' love for learning, their taste for details and precision, and constant wish to understand how the world can be made a better place shaped much of what I am as a researcher.

Foreword

The economics of networks is the study of what is the role of connections among agents in determining their behavior and, in turn, how the connections mediate the way individual behavior gives rise to aggregate outcomes. In this thesis, two kinds of connections are studied: the supplier-customer relations among firms, and the network of social contacts. In the first two chapters, I study how the supplier-customer relations among firms affect two things that are crucial in modern economies: market power and the propagation of shocks. The results from the first chapter suggest that competition policy would benefit from taking into account the network structure. In the third, I, with my coauthors, study how the social contacts among groups of people with different perceptions about vaccination affect the diffusion of an epidemic. The results suggest that policies that induce more segregation across groups might make everyone worse off. In the fourth, with my coauthor, we show that a behavioral bias well documented in finance mitigates the information externality in a sequential learning model.

Insofar as it makes sense to try to infer a general conclusion from these works it would be that welfare considerations, and so policy outcomes, are crucially affected by connections among agents. Hence, taking these into account is an important step to refine our ability to understand and regulate the economy.

Contents

Chapter 1

Supply and Demand Function Equilibrium: trade in a network of superstar firms

This paper studies how input-output connections among firms determine the distribution and the welfare impact of market power in a production network. On all links firms compete, symmetrically, in unrestricted supply and demand schedules. As a consequence, firms take strategically into account their position in the network, and have market power on both input and output markets. I show that the fact that firms take into account their position in the network magnifies the welfare impact of oligopolies with respect to the case in which firms ignore their position and take other sectors and markets' prices as given. In input-output models is often assumed that firms have market power only either on inputs or outputs. I show that this kind of assumption adds constraints on the relative market power among firms. Using supply and demand functions firms have market power on both inputs and outputs, in an endogenously determined way. Thus, this model provides a neutral framework to study welfare questions and evaluate competition policy. An equilibrium exists for any network under a technology that yields quadratic profit functions, and I provide an algorithm to compute it. Moreover, horizontal mergers (in absence of synergies) are always harmful for welfare. [1].

1.1 Introduction

Production of goods in modern economies typically features long and interconnected supply chains. Moreover, many sectors are characterized by *superstar firms*, whose size is very large relative to their sector or even the whole economy.[2] How are prices formed in this setting? How is surplus split? How efficient is the process? This paper provides a strategic non-cooperative model of large firms interacting in an input-output network consisting of many specific supply-customer relationships. The model satisfies two requirements:

R1. Symmetric market power : all firms have market power over both input and output goods, and no prices or quantities are taken as given.

R2. Global strategic interactions : firms strategically take into account their position in the network.

To be concrete, consider a competition authority in charge of evaluating merger proposals. Since evaluation takes time and effort, the authority wants to decide on which sectors to focus on.[3] In order to do this, we must be sure not to build into our models assumption that privilege some sectors/firms with respect to others. For example, Section 1.4 shows that in some simple sequential models that have been customarily used ad-hoc differences in the order of moves changes the answer completely. Hence, the importance of requirement R1.

The quantification of the distortions that may arise due to market power has attracted a lot of attention recently, with many scholars arguing that competition is in fact decreasing and market power on the rise.[4] In particular, Baqaee and Farhi (2017b) find that taking the input-output connections into account can dramatically increase the size of the implied misallocation in the economy. This paper shows that, if firms take strategically into account their position in the supply chain, the welfare loss is even larger. Hence, requirement R2 is important to be able to correctly evaluate welfare losses.

The novelty of my approach is to incorporate both requirement R1 and R2. Firms' simultaneously commit to supply and demand functions (a uniform price double auction), a methodology first introduced in Grossman

[2]In the terminology of Autor et al. (2020).

[3]Indeed, this is a common issue: for example, in the USA, legislation requires firms to report merger proposals to the relevant authorities, but only for mergers such that the assets of the firms involved lie above some pre-specified thresholds (see e.g. Wollmann (2019))

[4]De Loecker et al. (2020), Gutiérrez and Philippon (2016), Gutiérrez and Philippon (2017).

(1981) and Klemperer and Meyer (1989). In this context, firms' market power is directly connected with the slope of the supply and demand schedule used, with a mechanism similar to the usual inverse elasticity rule in monopoly pricing. The slopes are endogenous and, by treating input and output goods symmetrically, market power is hence solely determined by network position, competition, and technology. This way, there is no need to introduce asymmetries in the timing of firms' choices and treatment of inputs with respect to outputs. The split of the surplus is also endogenous, and there is no need to introduce parameters connected to bargaining. What allows tractability, in a similar way to most models using supply and demand schedules, are a quadratic functional form for the technology and uncertainty in some (cost) parameters. The quadratic functional form yields linearity of schedules in equilibrium, while uncertainty pins down the best replies uniquely.

Theorems 1, 2 and 3 are the main results of the paper.

Theorem 1 provides an existence result for a Supply and Demand Function Equilibrium in linear strategies in any network. This approach does does not need the assumption that the network is acyclic, as for example the sequential models do.[5] The proof relies on the *strategic complementarity* property of the game: the best reply to a steeper supply curve is a steeper supply curve, where "slopes", being matrices of coefficients, are ordered in the positive semidefinite sense. So in this context, when a firm has larger market power, every other firm has more market power in turn.

Theorem 2 shows that in this setting mergers always increase market power. If, in addition, there is a single aggregate final good, mergers increase the final price. Strategic complementarities are key again: the merged firm will face less competition and so choose a flatter schedule, triggering complementary responses from direct competitors, suppliers and customers, and all firms connected through the network.

Theorem 3 shows that ignoring global strategic considerations (requirement R2) leads to less market power: in a similar model of competition in supply and demand functions, in which firms ignore the rest of the network, market power distortions are smaller. This is because if a firm does not internalize some reactions in the network, this amounts to the firm perceiving a larger elasticity of demand.

The rest of the paper is organized as follows. The next paragraphs describe the related literature. Section 1.2 defines the model in full general-

[5]The relevance of cycles in real production networks is not yet very clear, but on strict terms, they are not acyclic. For example, Tintelnot et al. (2018) estimates that no more than 23% of the links in the Belgian firm-to-firm production network violate acyclicity. This might justify neglecting acyclicity as a first approximation, but is a number distant from zero.

ity and then explains the parametric assumptions needed for the analysis. Section 1.3 illustrates the solution and the existence theorem. Section 1.4 presents some of the benchmark models discussed above and clarifying the differences with my approach. Section 1.5 illustrates the welfare impact of mergers in Theorem 2. Section 1.6 presents the local version of the model and Theorem 3. Section 1.7 explains how it is possible to solve the model numerically. Section 1.8 concludes.

Related literature This paper contributes to three lines of literature: the literature on competition in supply and demand functions, the literature on production networks, and the literature on market power in networked markets.

The use of supply schedules as choice variables in the analysis of oligopoly was introduced in Grossman (1981), and in its modern form by Klemperer and Meyer (1989). These studies feature market power on one side of the market only, as typical in oligopoly models.[6] Vives (2011) studies the case of asymmetric information. Akgün (2004) studies mergers among firms competing in supply functions, without the network dimension, finding that mergers are always welfare-decreasing. My results show that some of the mechanisms extend not only to bilateral trade but to trade in any network. Among the papers that have dealt with the problem of bilateral oligopoly, allowing for market power on both sides of the market, Hendricks and McAfee (2010) is a model of bilateral oligopoly where players compete through choosing a capacity parameter: the elasticity of the demand and supply schedules is a given. My contribution with respect to them is a setting in which the elasticities (slopes) of demand and supply are themselves endogenous. Weretka (2011) attacks the problem constraining the schedules to be linear (instead of getting this as an equilibrium result), thus gaining traction in the analysis for general functional forms.

The use of supply and demand schedules is common also in the finance microstructure literature and in the literature on multi-unit auctions. In finance it was introduced and popularized by Kyle (1989). From a technical point of view, the closest paper to mine is Malamud and Rostek (2017), which studies trade in interconnected financial markets: some of their results are formally similar to the "local" version of the model discussed in Section 1.6. Ausubel et al. (2014) compare uniform price auctions with pay-as-bid auctions and hybrid approaches.

[6] These techniques are customarily used in the study of competition in electricity markets, since Green and Newbery (1992). For a recent contribution, see Delbono and Lambertini (2018).

It is convenient to divide the literature on market power in networks in four parts: sequential models, local competition (sector-level), matching, and bargaining. All differ from my approach, by departing from Requirements 1 and 2. Sequential models of supply chains have been studied in the context of double marginalization by Spengler (1950), in the context of vertical foreclosure by Salinger (1988), Ordover et al. (1990). Recently they have been studied in Hinnosaar (2019) (price setting), Federgruen and Hu (2016) (quantity setting), Kotowski and Leister (2019) (sequential auctions). Carvalho et al. (2020) build a tractable model to identify "bottlenecks" in real production network data. In their terminology bottlenecks are those firms that, if removed, would increase welfare. This mechanism is crucially different from mine, because in their model links have exogenous capacity constraints, and removing a firm removes the capacity constraint. By contrast, in my approach the amount of trade is the result of the balance of market powers, and removing a firm leads always to an increase in market power.

Papers where competition is at the sector level assume that either the markup is an exogenously given *wedge* between prices and marginal costs, such as Baqaee and Farhi (2017b), Huremovic and Vega-Redondo (2016)); or is determined by oligopolistic competition *at the sector level*: Grassi (2017), De Bruyne et al. (2019), Baqaee (2018). My results suggest that care has to be taken in using this models to analyze welfare: limiting strategic interaction at the sector level might make oligopolies less inefficient. Acemoglu and Tahbaz-Salehi (2020) build a model where prices are formed through a link-level bargaining process. This means that relative market power, though affected by the network, will be crucially affected by the choice of bargaining parameters. This means that, e.g., the relative market power across sectors (hence the relative importance of mergers) is crucially affected by these exogenous parameters: in my approach, the split of the surplus is instead endogenous and depends only on the technology parameters and the connections. Example 7 illustrates this point.

Also relevant are models that employ cooperative tools, such as stability and matching. The literature started by Hatfield et al. (2013) and recent contributions are Fleiner et al. (2018) and Fleiner et al. (2019). They consider indivisible goods and firms that are price-takers. Fleiner et al. (2019) studies the model in presence of frictions, that are exogenously given through the utility functions, and not the result of the strategic use of market power.

Some papers study the interconnection of final markets of different firms, without analyzing the input-output dimension. In this category fall Bimpikis et al. (2019), Pellegrino (2019), Chen and Elliott (2019).

My paper is also connected to an older line of literature, called "general oligopolistic competition", studying how to represent a full economy with interconnected trades as a game (for a review see Bonanno (1990)).[7] My contribution is to achieve a fully strategic model of the production side of the economy through the use of competition in supply and demand functions. A recent paper expanding on these themes is Azar and Vives (2018), that develop a model of firms having market power on output and input markets, but without the input-output dimension.

1.2 The Model

In this section I first define the model in full generality, that is without making parametric assumptions on the technology and the consumer utility, to clarify the generality of the setting. In paragraph 1.2.2 I discuss the parametric assumptions needed for the subsequent analysis.

1.2.1 General setting

Firms and Production Network There are N economic sectors, each sector $i = 1, \dots, N$ is populated by a finite number n_i of identical firms. Firms are denoted with greek letters $\alpha = 1, \dots, n_i$. Each sector needs as inputs the goods produced by a subset $\mathcal{N}_i^{in}$ of other sectors, and sells its outputs to a subset of sectors $\mathcal{N}_i^{out}$. I denote the transformation function available to all the firms of sector i as Φ_i. This is a function of the input and output quantities, and also of a stochastic parameter ε_i that will have the role of a technological shock. I denote the joint distribution of $\varepsilon = (\varepsilon_i)_i$ as F.

I denote d_i^{out} the *out-degree* and d_i^{in} as the *in-degree* of sector i. Sectors are connected if one is a customer of the other. E is the set of existing connections, $E \subseteq N \times N$.

Inputs of firms in sector i are $q_{i\alpha,j}, j = 1, \dots d_i^{in}$ and outputs $q_{k,i\alpha}, k = 1, \dots d_i^{out}$. Each of the inputs of sector i has an *input-output weight* ω_{ij}, and the corresponding vector is denoted $\omega_i = (\omega_{i1}, \dots, \omega_{id_i^{in}})$. Firms in each

[7] The closer in spirit is Nikaido (2015), who uses the market clearing conditions to back up quantities as functions of prices, but his method is limited to Leontief technology, and Benassy (1988) which defines an objective demand by means of a fixprice equilibrium, thus not limiting himself to constant returns technology, but as a drawback having to contemplate a rationing rule, and losing a lot in terms of tractability. These methods are the analogous in their setting of the residual demand in 1.3.2. Other important contributions are Dierker and Grodal (1986), Gabszewicz and Vial (1972), and Marschak and Selten (2012).

sector may produce more than 1 good, but different sectors never produce the same good.

Sectors and connections define a weighted directed graph $\mathcal{G} = (N, E)$ which is the *input output network* of this economy. Its adjacency matrix is $\Omega = (\omega_{ij})_{i,j \in N}$. Note that in this model the input-output network is a *sector-level* concept.

Consumers Consumers are a continuum and identical, so that there is a representative consumer.[8] The labor market is assumed competitive, that in particular means firms will have no power over the wage. Hence the wage plays no role, so we are going to assume that the labor is the numeraire good, and normalize it to 1 throughout. Similarly to the firms, I am going to assume that the consumer utility depends on stochastic parameters $\varepsilon_c = (\varepsilon_{i,c})_i$, one for each good consumed: $U(c, L, \varepsilon_{i,c})$. Denote the demand for good i derived by U as $D_{ci}(p_i, \varepsilon_c)$.

Notation I write $i \to j$ to indicate that a good produced by sector i is used by sector j in production (or equivalently that $(i,j) \in E$). I write p_i^{out} for the vector of all prices of ouputs of sector i, and p_i^{in} for the vector of input prices, and $p_i = \begin{pmatrix} p_i^{out} \\ -p_i^{in} \end{pmatrix}$. u_i^{out} denotes a vector of ones of lenght d_i^{out}, while u_i^{in} a corresponding vector of ones of length d_i^{in}, and $\tilde{u}_i = \begin{pmatrix} u_i^{out} \\ -\omega_i \end{pmatrix}$. Similarly, I_i is the identity matrix of size $d_i^{out} + d_i^{in}$, while I_i^{in} and I_i^{out} have respectively size d_i^{in} and d_i^{out}.

Unless specified differently, the inequality $B \geq C$ when B and C are matrices denotes the *positive semidefinite* (Löwner) ordering. That is: $B \geq C$ if and only if $B - C$ is positive semidefinite.

The Game The competition among firms take the shape of a game in which firms compete in supply and demand functions. This means that the players of the game are the firms, and the actions available to each firm α in sector i are vectors of supply and demand functions $(S_{k_1,i\alpha}, \dots, S_{k_{d_i^{out}},i\alpha})$, $(D_{i\alpha,j_1}, \dots, D_{i\alpha,j_{d_i^{in}}})$, $l_{i\alpha,j}(\cdot)$ defined over pairs of output price and realization of sector level stochastic parameter (p_i, ε_i).

The reason to introduce a stochastic parameter is that this type of modeling has a classical multiplicity problem, as illustrated by Figure 1.1. The

[8] In particular, it is assumed that each infinitesimal consumer owns identical shares of all the firms so that we avoid the difficulties uncovered by Dierker and Grodhal: see the Introduction.

solution, both in the oligopoly and in the market microstructure literature, consists in introducing some source of uncertainty, so that all feasible prices can be realized in equilibrium for some realizations of the uncertainty, and this pins down the full demand or supply schedules rather than just a point on them.

Differently from the Klemperer and Meyer (1989) setting, where a stochastic shock to the exogenous demand function is sufficient to pin down unique best replies, in a supply chain, or more generally in a network economy, more prices have to be determined. This means that the uncertainty in demand alone is not able any more to solve the multiplicity problem. In a network setting, we must add a source of uncertainty in every market, that is one *for every price* to be determined. That will be the role of the productivity shock, shifting the amount of good that a firm is willing to buy from its suppliers and simultaneously the quantity that it is willing to sell.

The feasible supply and demand schedules must satisfie:

i) they are nonnegative;

ii) they must satisfy the **technology constraint**, that is for any possible (p_i, ε_i):
$$\Phi_i(S_{i\alpha}(p_i, \varepsilon_i), D_{i\alpha}(p_i, \varepsilon_i), l_{i\alpha}(p_i, \varepsilon_i), \varepsilon_i) = 0 \tag{1.1}$$

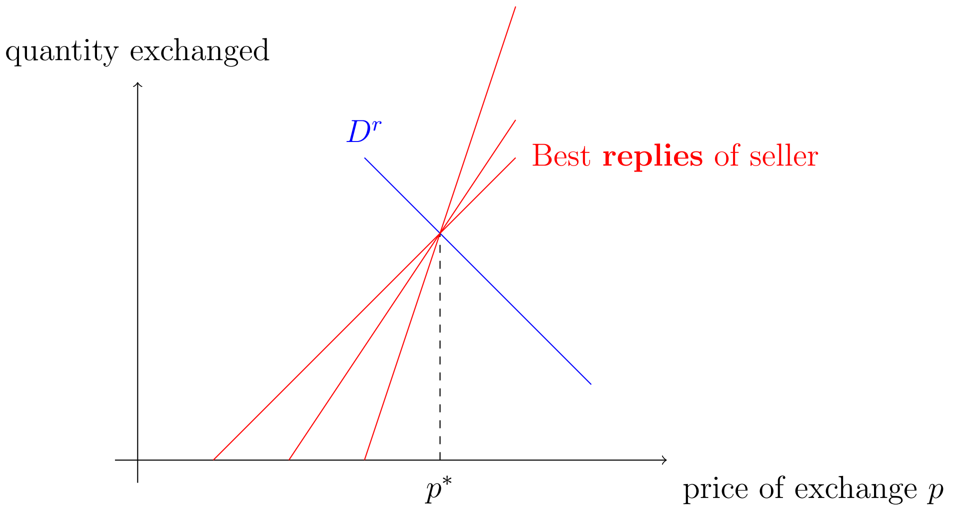
iii) the maps $(S_{i\alpha}, D_{i\alpha})$ must be continuously differentiable and have Jacobian $J_{i,p_i^{out},-p_i^{in}}$ which is everywhere positive semidefinite and has rank at least $d_i - 1$ (the maximum minus 1);[9] note that the differentiation is done with respect to the variables $(p_i^{out}, -p_i^{in})$;

iv) they have a bounded support.

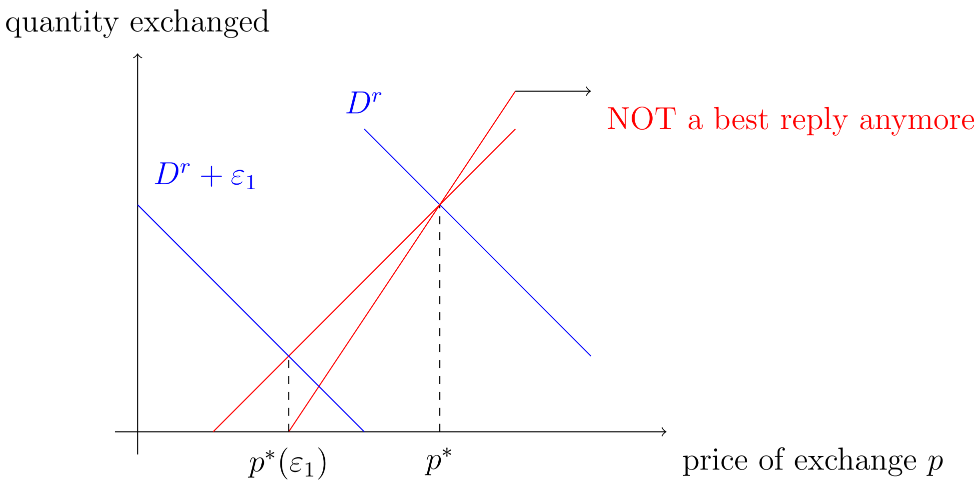
These conditions allow us to define the *realized prices* $p^*(\varepsilon)$ through the market clearing equations. The function p^* is the one implicitly defined by the market clearing equations:

$$\begin{aligned} \sum_\beta D_{j\beta,k}(p_j^{out}, p_j^{in}, \varepsilon_j) &= \sum_\alpha S_{k\alpha}(p_k^{out}, p_k^{in}, \varepsilon_k) \quad \text{if } k \to j \\ D_{ck}(p_{ck}, \varepsilon_{ck}) &= \sum_\beta S_{k\beta}(p_k^{out}, p_k^{in}, \varepsilon_k) \quad \text{if } k \to c \end{aligned} \tag{1.2}$$

[9]The Jacobian might not be positive definite because the technology constraint implies, by the chain rule: $\nabla\Phi_{i,S} JS_i + \nabla\Phi_{i,D} JD_i + \nabla\Phi_{i,l} Jl_i = \mathbf{0}$. Depending on how labor enters the technology this might become a linear constraint on the rows of the Jacobian: it is indeed what happens under the parameterization introduced in 1.2.2, as will be clear in the following.



(a) If optimal price for seller is p, all red lines represents best replies.



(b) Since the parameter ε_1 is stochastic, the seller will adjust its supply function in order to pin down the optimal price for *any* realization of ε_1, thereby destroying the multiplicity.

Figure 1.1: Multiplicity problem and solution in Supply Function Equilibrium.

To show that the regularity conditions indeed imply that the market clearing system can be solved, the crucial step is to show that they translate to regularity conditions on the Jacobian of the function whose zeros define the system above, and then a global form of the implicit function theorem (Gale and Nikaido (1965)) can be applied. This is done in the next Proposition.

Proposition 1. *The market clearing conditions define a function:*

$$p^* : \times_i \mathcal{E}_i \to \mathbb{R}_+^{|E|\times|E|}$$

Note that p^* is also differentiable, but since all the equilibrium analysis and hence the optimizations, will be performed in the linear case, we do not need this property: the goal of this section is simply to define the game.

Now that we built the prices implied by the players's actions, we can define the payoffs. These are the expected profits calculated in the realized prices p^*:

$$\pi_{i\alpha}(S_{i\alpha}, D_{i\alpha}, l_{i\alpha}) = \mathbb{E}\left(\sum_k p^*_{ki} S_{k,i\alpha} - \sum_j p^*_{ij} D_{i\alpha,j} - l_{i\alpha}\right) \tag{1.3}$$

where to avoid clutter I omitted to write each functional variable.

Hence, formally, the game played by firms is: $G = (I, (A_{i\alpha})_{(i,\alpha)\in I}, (\pi_{i\alpha})_{(i,\alpha)\in I}, F)$, where $I = \{(i,\alpha) \mid i = 1, 2,\ \alpha = 1, \dots n_i\}$ denotes the set of firms, and $A_{i\alpha}$ is the set of profiles of supply and demand functions that satisfie the assumptions above.

Example 1. Standard Supply Function Equilibrium

The model by Klemperer and Meyer (1989) can be seen as a special case of this setting, in which there is only one sector and the network $\mathcal{G}$ is empty, as illustrated in Figure 1.2. Their setting is a "partial" equilibrium one, in which the consumers do not supply labor to firms but appear only through a demand function $D(\cdot)$, and firms have a cost function for production $C(\cdot)$, that does not explicitly represent payments to anyone. The strategic environment is precisely the same though: if the setting of this section the transformation function is $\Phi_\alpha(q_\alpha, \ell_\alpha) = C^{-1}(q_\alpha) - \ell_\alpha$, and the consumer utility gives rise to demand D, the game G played by firms is precisely the same as in Klemperer and Meyer (1989).

The welfare of the consumer is $U(C, L)$, where $C(p^*, \varepsilon) = (C_{ci,\alpha}(p_i, \varepsilon_i))_{i,\alpha}$ is the vector of quantities of goods consumed in equilibrium, and $L = \sum_{i,\alpha} l_{i,\alpha}(p^*_i, \varepsilon_i)$ is the total labor used in the economy[10]. The consumers, being atomic, take all prices as given and thus are a non-strategic component

[10] It is not necessary to impose a "labor market clearing" condition because it is redundant with the budget constraint of the consumer, consistently with the decision to normalize the wage to 1.

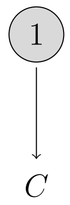


Figure 1.2: The (degenerate) production network of Example 1: there is only 1 Sector whose firm sell to the consumer.

of the model, that enter in the game only through their aggregate demand function.

Supply and Demand Function Equilibrium To compute the predictions of the model I just need to specify the role of the stochastic parameters ε. I will use it as a selection device, as made formal by the next definition.

Definition 1.2.1. A Supply and Demand Function Equilibrium is a profile of prices and quantities of traded goods (p_{ij}, q_{ij}) for all $(ij) \in E$ that realize in a Nash Equilibrium of the game G for $F \xrightarrow{\mathcal{D}} 0$:

$$\begin{aligned} p_{ij} &= p^*_{ij}(0) \\ q_{ij} &= \sum_\alpha D^*_{i\alpha,j}(p^*_{ij}, 0) \quad \forall (i,j) \in E \end{aligned} \tag{1.4}$$

So in practice I am using the stochastic variation to "identify" the equilibrium schedules, but when computing the equilibrium predictions I am considering the case in which the shock vanishes.

1.2.2 Parametric Assumptions

To obtain a tractable solution, I adopt parametric assumptions on the technology. Since firms may produce more than 1 good, I have to express the technology via a *transformation function*. Specifically, assume that the production possibility set of each firm α in sector i be the set of $(q_{k,i\alpha})_k, (q_{i\alpha,j})_j, (l_{i\alpha,kj})_{k,j}$ such that there exists a subdivision $(z_{i\alpha,kj})$ of inputs satisfying $q_{i\alpha j} = \sum_k z_{i\alpha,kj}$, and:

$$q_{k,i\alpha} = \sum_j \omega_{ij} \min\{\bar{\ell}_{i\alpha,kj}, z_{i\alpha,kj}\} \quad k = 1, \ldots, d_i^{out} \tag{1.5}$$

The idea of this functional form is that intermediate inputs $q_{i\alpha,j}$ have to be first allocated to the production of one specific output good: $z_{i\alpha,kj}$ is the amount of input j allocated to the production of the output to be sold to sector k. Moreover, each input needs to be complemented with a specific amount of labor $\bar{\ell}_{i\alpha,kj}$. $\bar{\ell}_{i\alpha,kj}$ represents a measure of "effective labor hours", and is equal to:

$$\bar{\ell}_{i\alpha,kj} = -\varepsilon_i + \sqrt{\varepsilon_i^2 + 2l_{i\alpha,kj}}$$

where $l_{i\alpha,kj}$ is the amount labor hired by the firm to deal with input j in the production of output to be sold to k. ε_i is a sector-level labor productivity shock. It changes the marginal product of labor: a large ε_i means that labor is not very productive.

This functional form[11] turns out to be particularly convenient because at the optimum we must have $-\varepsilon_i + \sqrt{\varepsilon_i^2 + 2l_{i\alpha,kj}} = z_{i\alpha,kj}$, so that $l_{i\alpha,kj} = \varepsilon_i z_{i\alpha,kj} + \frac{1}{2} z_{i\alpha,kj}^2$, and the profit function becomes linear-quadratic:

$$\pi_{i\alpha} = \sum_k p_{ki} q_{k,i\alpha} - \sum_j p_{ij} q_{i\alpha,j} - \varepsilon_i \sum z_{i\alpha,kj} - \frac{1}{2} \sum z_{i\alpha,kj}^2 \tag{1.6}$$

where $\varepsilon_i \sum z_{i\alpha,kj} + \frac{1}{2} \sum z_{i\alpha,kj}^2$ is the cost paid to hire labor. This makes it apparent that ε_i acts reducing the productivity of labor (effective labor hours), and so increasing the amount of labor necessary to achieve the same level of production. This will be crucial in achieving a linear best response.[12]

If a sector uses no intermediate inputs but only labor, the technology is $q_{ki} = \bar{l}_{ki} = -\varepsilon_i + \sqrt{\varepsilon_i^2 + 2l_{i\alpha,k}}$, so that the profit becomes: $\pi_{i\alpha} = \sum_k p_{ki} q_{k,i\alpha} - \varepsilon_i \sum q_{k,i\alpha} - \frac{1}{2} \sum q_{k,i\alpha}^2$.

The analogous assumptions on the utility function of the consumer are that it be quadratic in consumption and (quasi-)linear in disutility of labor L:

$$U((c_i)_i, L) = \sum_i \frac{A_{i,c} + \varepsilon_{i,c}}{B_{c,i}} c_i - \frac{1}{2} \sum_i \frac{1}{B_{c,i}} c_i^2 - L$$

This means that the consumer has demands of the form: $D_{ci} = \max\left\{A_i - B_{c,i}\frac{p_{ci}}{w}, 0\right\}$.

[11] A more classical choice, especially in the macro literature, is the one of a production function belonging to the Constant Elasticity of Substitution class. This does not yield tractable expressions here. Notice, however, that the functional form in 1.5 can be seen as the limit of a *nested* CES:

$$\sum_j \omega_{ij} \min\{\bar{\ell}_{i\alpha,kj}, z_{i\alpha,kj}\} = \lim_{\substack{\sigma\to\infty,\\ \rho\to 0}} \left(\sum_j \omega_{ij} \left(\left(\bar{\ell}_{i\alpha,kj}(\varepsilon_i)^{\frac{\rho}{(\rho-1)}} + z_{i\alpha,kj}^{\frac{\rho}{\rho-1}} \right)^{\frac{\rho-1}{\rho}} \right)^{\frac{\sigma}{\sigma-1}} \right)^{\frac{\sigma-1}{\sigma}}$$

[12] It is the analogous in our setting of the assumption of linear/quadratic cost function, common in standard supply function models (Klemperer and Meyer (1989), Delbono and Lambertini (2018)), and the assumption of gaussian random variables and constant absolute risk aversion in the finance setting (Malamud and Rostek (2017), Kyle (1989)).

Example 2 (Standard Supply and Demand Function equilibrium – parametric). Consider the setting of 1, that is the one sector model. The parametric assumptions in this setting reduce to assuming that the firms have a quadratic cost. Indeed by the same reasoning as above the profit function becomes:

$$\pi_i = pq - \varepsilon_i q - \frac{1}{2}q^2 \tag{1.7}$$

Graphical intuitions Before delving into the formal details, I will give a graphical illustration of the main mechanisms of the model.

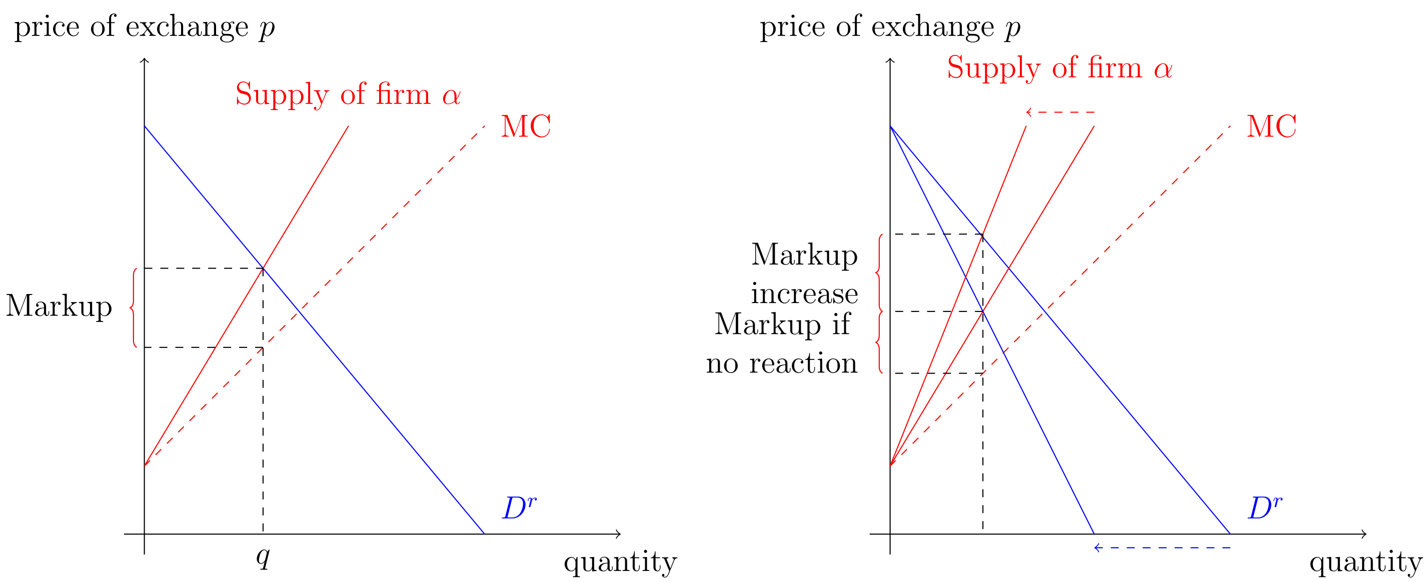


Figure 1.3: Strategic complementarity among demand and supply.

Figure 1.3 illustrates the mechanics behind the strategic complementarity mechanism. In the left panel, it is shown that the supply function (red line) chosen by a firm as a best reply to the residual demand D^r (the blue line) has to have larger slope than the marginal cost curve, which is the supply function chosen by a firm under perfect competition. The gap between the curves is the (absolute) markup charged by the firm. When the residual supply shifts (right panel), firm α is facing a smaller elasticity, so it wants to increase the markup. To do so it must choose a supply function that is steeper.

1.3 Solution and Existence

In the following I will focus on S&D equilibria in symmetric linear schedules.

Definition 1.3.1. A Supply and Demand Function Equilibrium in *symmetric linear schedules* is a profile of functions $\boldsymbol{\sigma} = ((S_{i\alpha})_\alpha, (D_{i\alpha})_\alpha, (l_{i\alpha})_\alpha)_i$ defined on open sets $(\mathcal{P}_{i,\alpha})_{i,\alpha} \times (\mathcal{E}_{i,\alpha})_{i,\alpha}$ such that:

i) $\boldsymbol{\sigma}$ is a Nash Equilibrium of the game G;

ii) (Symmetry) in each sector i firms play the same schedules: $D_{i\alpha} = D_i$, $S_{i\alpha} = S_i$, $l_{i\alpha} = l_i$;

iii) a) (Inactive links) for each i there exists a subset of neighbors $\mathcal{N}_{i,0} \subseteq \mathcal{N}_i$ such that the relative demand or supply function is identically 0; these are called *inactive links*; call the number of active links $d_i^a \leq d_i$, and the prices relative to active links $p_i^a = (p_i^{out,a}, -p_i^{in,a})$;

b) (Linearity) for all i, for all active links $e \in \mathcal{N}_i/\mathcal{N}_{0,i}$ there exist a vector $B_{i,\varepsilon} \in \mathbb{R}^{d_i^a}$ and a matrix $B_i \in \mathbb{R}^{d_i^a \times d_i^a}$ and for all $(p_i, \varepsilon_i) \in \mathcal{E}_i \times \mathcal{P}_i$:

$$\begin{pmatrix} S_i \\ D_i \end{pmatrix} = B_i p_i^a + B_{i,\varepsilon}\varepsilon_i \tag{1.8}$$

and both $S_i > 0$ and $D_i > 0$ hold.

iv) (feasibility) If $p^*(0)$ is the solution of 1.2 for $\varepsilon = 0$, then $p_i^*(0) \in \mathcal{P}_i$ for any i.

Note that i implies that B_i is positive definite for all i, because it is the Jacobian of the schedule with respect to $(p_i^{out}, -p_i^{in})$.

This game is in principle very complex to solve, being defined on an infinite-dimensional space. In practice however, things are simpler, because a standard feature of competition in supply schedules, both in the finance and IO flavors, is that the best reply problem can be transformed from an ex-ante optimization over supply functions in an ex-post optimization over input and output prices, as functions of the realizations of the parameter ε_i. In this way the best reply computation is reduced to an optimization over prices as in a monopoly problem. The crucial complication that the input-output dimension adds to e.g. Malamud and Rostek (2017) is the way the residual demand is computed. In an oligopoly without input-output dimension, as in Example 1, the residual demand is the portion of the final demand that is not met by competitors. In our context this remains true, but in computing it, players have to take into account how a variation in quantity supplied affects the balance of trades, hence the prices, in the rest of the network. Let us first define the residual demand in this setting.

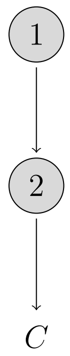


Figure 1.4: A line production network.

Definition 1.3.2 (Residual demand)**.** Given a profile of linear symmetric schedules $(((S_{i\alpha})_\alpha)_i, (D_{i\alpha})_\alpha)_i$, define the *residual demand*, and the *residual supply* of sector i as the amount of demand and supply remaining once all market clearing conditions but those relative to sector i have been solved. Formally:

$$
\begin{aligned}
D^r_{ik}(p^{r,i}_k, p_i, \varepsilon_k, \varepsilon_i) &= \underbrace{n_k D_{ki}(p^{r,i}_k, \varepsilon_k)}_{\text{demand from sector } k} - \underbrace{(n_i - 1) S_{ki}(p^{out}_i, p^{in}_i, \varepsilon_i)}_{\text{supply by competitors}} \\
S^r_{ij}(p^{r,i}_j, p_i, \varepsilon_j, \varepsilon_i) &= \underbrace{n_j S_{ij}(p^{r,i}_j, \varepsilon_j)}_{\text{supply from sector } j} - \underbrace{(n_i - 1) D_i(p_i, \varepsilon_i)}_{\text{demand by competitors}} \quad \forall j, k \in \mathcal{N}_i
\end{aligned}
$$

where $p^{r,i}$ is the *residual price function* of sector i, and is:

i) just the price for all inputs and outputs of i: $p^{r,i}_{ij} = p_{ij}$, $p^{r,i}_{ki} = p_{ki}$;

ii) for all other prices, it is the function of p_i and ε defined by the market clearing conditions 1.2, excluding those relative to the input and output prices of i.

Example 3. (**Line network**)

The easiest setting in which to understand the mechanics of the residual demand is a line network, as illustrated in Figure 1.4.

What is the residual demand (and supply) in this setting? to understand this, consider a firm in sector 1 that needs to compute its best reply to the schedules chosen by all others. (Details can be found in the Proof of Theorem 1). The demand curve faced by a firm in sector 1 is:

$$
\underbrace{n_2 D_2(p^*_2, p_1, \varepsilon_2)}_{\text{Direct demand from sector 2}} - \underbrace{(n_1 - 1) S_1(p_1, \varepsilon_1)}_{\text{Supply of competitors}}
$$

for different choices of a supply function $S_{1\alpha}$, different prices p_1 would realize, as functions of the realizations of ε_2. For the best-responding firm, it is equivalent then to simply choose the price p_1 it would prefer for any given ε_1, and then the function $S_{1\alpha}$ can be backed up from these choices. But naturally also p_2^* is determined in equilibrium, and this has to be taken into account when optimizing. In particular, the market clearing conditions for sector 2:

$$n_2 S_{2\alpha}(p_2, p_1, \varepsilon_2) = D(p_2) + \varepsilon_c$$

define implicitly p_2 as a function of p_1 and the shocks. This allows to internalize in the price setting problem of firm 1 the impact that the variation in p_1 is going to have on p_2, for given supply and demand schedules chosen by other players. The same reasoning holds for the supply function. If we assume that all other players are using *linear* supply and demand schedules $S_1(p_1, \varepsilon_1) = B_1(p_1 - \varepsilon_1)$, $D_2(p_2, p_1, \varepsilon_2) = B_2(p_2 - p_1 - \varepsilon_2)$ we get the following expressions for the residual demands:

$$D_1^r = \frac{n_2 B_2}{B_c + n_2 B_2}\left(A_c + \varepsilon_c - B_c p_1\right) - (n_1 - 1) B_1 (p_1 - \varepsilon_1) \tag{1.9}$$

$$S_2^r = \frac{n_2 B_2}{B_c + n_2 B_2}\left(A_c + \varepsilon_c - B_c p_1\right) - (n_2 - 1) B_2 (p_2 - p_1 - \varepsilon_2) \tag{1.10}$$

$$D_2^r = A_c + \varepsilon_c - B_c p_2 - (n_2 - 1) B_2 (p_2 - p_1 - \varepsilon_2) \tag{1.11}$$

which clarifies how, even if each firms acts "locally" choosing its own input and output prices, actually the problem depends from the parameters of the whole economy.

Figure 1.5 illustrates how the strategic complementarity extends through the residual demand when firms are indirectly connected through the supply chain. Consider the production network depicted in Figure 1.4. The slope (and elasticity) of demand that firms in sector 1 face depends on how a variation in price p_1 implies an adjustment in price p_0. A variation in price p_1 implies a shift in the supply curve of firms in sector 0, as the left panel of Figure 1.5 shows. This implies an upward adjustment of the equilibrium price. The resulting variation in demanded quantity depends on the demand faced by the firms in sector 0, the *downstream market*: the steeper the slope of demand the larger the price adjustment in the downstream market, the smaller the variation in quantity demanded. Hence a large slope of the downstream demand propagates upstream, resulting in a larger slope of demand faced by sector 1. The right panel illustrates that taking into account price adjustment the demand slope perceived is always smaller: this is because all variation is absorbed by quantity, and 0 by the price.

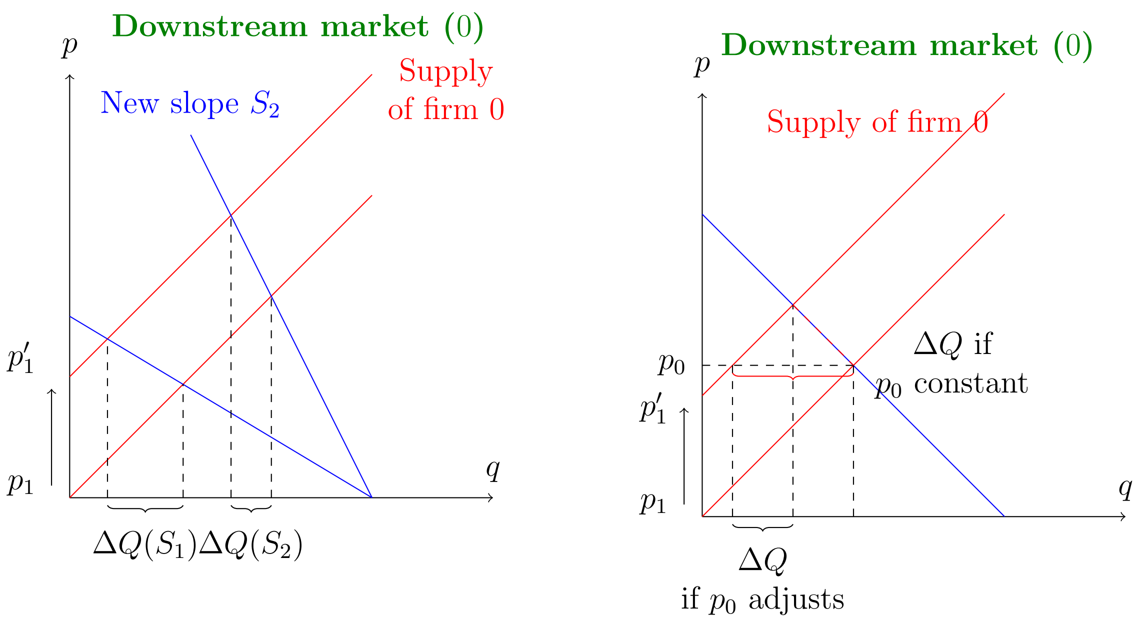


Figure 1.5: Strategic complementarity across the supply chain.

1.3.1 The input-output matrix

Residual demand and supply are the curves against which each firm will be optimizing when choosing its preferred input and output prices. It is natural therefore that they embed the information about relative market power. The key way through which the structure of the economy (i.e. the network) impacts these functions is via the dependence of the prices p^* on the input and output prices of i. To understand this, consider the market clearing equations.

The market clearing equations 1.2 define a system:

$$\begin{aligned} S_{il} &= D_{il} \quad \forall i,l \in N,\ i \to l \\ S_{i,c} &= D_{i,c} \quad \forall i \in N,\ i \to c \end{aligned} \tag{1.12}$$

If all other firms are using symmetric linear schedules with coefficients $(B_i)_i$, then this is a linear system, because all equations are linear in prices. We care about the solution of the system, so the ordering of the equations does not really matter. Let us rewrite the linear supply and demand schedules in a block form as:

$$\begin{pmatrix} S_i \\ D_i \end{pmatrix} = B_i \begin{pmatrix} p_i^{out} \\ -p_i^{in} \end{pmatrix} + \varepsilon_i B_{i,\varepsilon} = \begin{pmatrix} BS_i^{out} & BS_i^{in} \\ BD_i^{out} & BD_i^{in} \end{pmatrix} \begin{pmatrix} p_i^{out} \\ -p_i^{in} \end{pmatrix} + \varepsilon_i B_{i,\varepsilon}$$

(In case the sector employs only labor for production the matrices BD are empty).

So we can rewrite the system 1.12 as:

$$\begin{aligned} & n_l BS_{l,i\cdot}^{out} p_l^{out} - n_l BD_{l,i\cdot}^{in} p_l^{in} - n_i BD_{i,l\cdot}^{out} p_i^{out} + n_i BD_{i,l\cdot}^{in} p_i^{in} = 0 \quad \forall i, l \in N,\, i \to l \\ & n_i BS_{i,c\cdot}^{out} p_i^{out} - n_i BD_{i,c\cdot}^{in} p_i^{in} + B_{i,c} p_{i,c} = A_{i,c} \quad \forall i \in N,\, i \to c \\ & BS_{l,i\cdot}^{out} p_i^{out} - BD_{l,i\cdot}^{in} p_i^{in} \geq 0 \quad \forall i \in N,\, i \to l, \quad p \geq 0 \end{aligned} \tag{1.13}$$

To clarify the structure note that the market clearing equation for link $l \to i$ involves all prices of trades in which sectors l and i are involved.

Definition 1.3.3 (Market clearing coefficient matrix)**.** The *Market clearing coefficient matrix* corresponding to a profile of symmetric linear supply and demand schedules $(S_i, D_i)_i$ is the matrix M of dimension $|L| \times |L|$, where L is the set of active links according to the profile $(S_i, D_i)_i$, such that for all active links the market clearing system 1.12 in matrix form is:

$$Mp^a = \boldsymbol{A} + M_\varepsilon \varepsilon \tag{1.14}$$

$$\tag{1.15}$$

The vector of constants $\boldsymbol{A}$ is zero but for the entries corresponding to links to the consumer (that have value $\boldsymbol{A}_{ci} = A_{ci}$).

This matrix M is the fundamental source of network information in this setting: it is a matrix indexed on the set of *links* of the network (which correspond to prices and equations in 1.2), that has a zero whenever two links do not share a node, and p is a vector that stacks all the prices. To have an example, consider the graph in Figure 1.6 case in which sector 0 has two suppliers: 1 and 2, and 1 itself supplies 2. If the profile of coefficients is $(B_i)_i$, the matrix M (when rows and columns are appropriately ordered) is:

$$\begin{array}{c c} & \begin{array}{cccc} p_{10} & p_{20} & p_{12} & p_{0c} \end{array} \\ \begin{array}{c} (1 \to 0) \\ (2 \to 0) \\ (1 \to 2) \\ (0 \to c) \end{array} & \begin{pmatrix} B_{1,11} + B_{0,22} & B_{0,23} & B_{1,12} & -B_{0,12} \\ B_{0,32} & B_{0,33} + B_{2,11} & -B_{2,12} & -B_{0,13} \\ B_{1,21} & -B_{2,21} & B_{1,22} + B_{2,22} & 0 \\ -B_{0,21} & -B_{0,31} & 0 & B_c + B_{0,11} \end{pmatrix} \end{array}$$

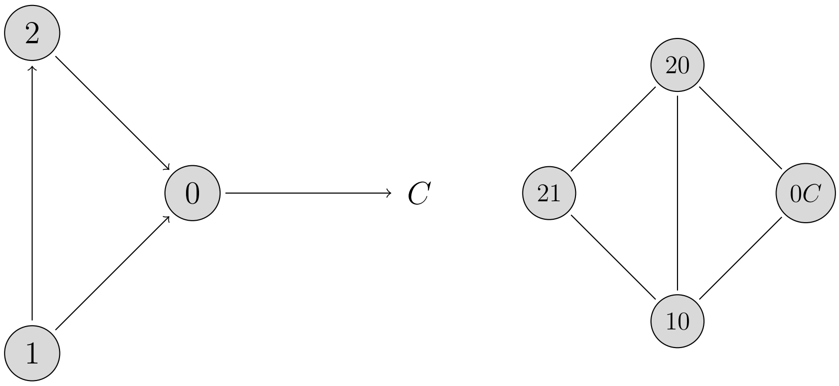


Figure 1.6: (left) A simple production network: c represents the consumer demand, while the other numbers index the sectors. (Right) The *line graph* of the network nearby.

We can see that the only zero is in correspondence of the pair of links $(0, c)$ and $(1, 2)$ which indeed do not share a node.

In network-theoretic language this is the (weighted and signed) adjacency matrix of the *line graph* of the input-output network $\mathcal{G}$. That is the adjacency matrix of the network that has as nodes the link of $\mathcal{G}$ and such that two nodes share a link if and only if the corresponding links in $\mathcal{G}$ have a common sector. Note that this graph is *undirected*, which has the important implication that if all the coefficient matrices B_i are symmetric then also the matrix M is.

To obtain the residual demand, the linear system 1.12 can be partially solved to yield p^*_{-i} – the vector of all the prices of transactions in which sector i is not directly involved – as a function of p_i:

$$p^{r,i}_{-i} = (M_{-i})^{-1}(-M_{C_i}p_i + \boldsymbol{A}_{-i} + M_\varepsilon \varepsilon)$$

where A_{-i} refers to all the rows of matrix A that do not involve links entering or exiting from node i, and M_{C_i} is the i-th column of M. This can be substituted in the supply and demand functions of suppliers and customers of i to yield the expression in the next proposition.

Proposition 2. *If all firms in all sectors $j \neq i$ are using symmetric linear supply and demand schedules with symmetric positive semidefinite coefficients $(B_j)_j$, generically in the values of $(B_j)_j$ there exist a neighborhood of 0 $\mathcal{E}_i$ and a set $\mathcal{P}_i \subset \mathbb{R}^{d^a_i}$ such that the residual supply and demand schedule for active links of sector i is linear and can be written as:*

$$\begin{pmatrix} D^r_i \\ S^r_i \end{pmatrix} = -\tilde{\boldsymbol{A}}_i - ((n_i - 1)B_i + \Lambda_i^{-1})p^a_i - \Lambda_{\varepsilon,i}\varepsilon$$

Moreover, Λ_i is symmetric positive definite and equal to the matrix $[M_i^{-1}]_i$, where:

- *M_i is the matrix obtained by M by setting B_i to 0;*
- *if A is a matrix indexed by edges, $[A]_i$ is the submatrix of A relative to all the links that are either entering or exiting i.*

The coefficient Λ_i can be thought as a (sector level) *price impact*[13]: the slope coefficients of the (inverse) supply and demand schedules, describing what effect on prices firms in sector i can have. It is a measure of market power: the larger the price impact, the larger the rents firms in that sector can extract from the market.

Now we can state the theorem. Define the *perfect competition matrix* for sector i as

$$C_i = \begin{pmatrix} \omega_i'\omega_i I^{out} & u_i^{out}\omega_i' \\ \omega_i(u_{out})_i' & d_i^{out} I^{in} \end{pmatrix}$$

Appendix 1.A.3 shows that this is the matrix of coefficients of demands and supplies chosen by a firm that takes prices as given.

Theorem 1. *1. If there are at least 2 firms per sector, generically in the entries of Ω a non-trivial linear and symmetric Supply and Demand Function equilibrium exists;*

2. The equilibrium coefficients $(B_i)_i$ can be written as

$$B_i = \begin{pmatrix} \tilde{u}_i'\tilde{B}_i\tilde{u}_i & \tilde{u}_i'\tilde{B}_i \\ \tilde{B}_i\tilde{u}_i & \tilde{B}_i \end{pmatrix}$$

for a symmetric positive definite $\tilde{B}_i$ (hence they are positive semi*definite). The equations that characterize them are:*

$$\tilde{B}_i = \left([C_i^{-1}]_{-1} + ((n_i - 1)\tilde{B}_i + \overline{\Lambda}_i)\right)^{-1} \tag{1.16}$$

where $\overline{\Lambda}_i$ is the constrained price impact*:*

$$\overline{\Lambda}_i = [\Lambda_i^{-1}]_{-1} - \frac{1}{\tilde{u}_i'\Lambda_i^{-1}\tilde{u}_i}[\Lambda_i^{-1}\tilde{u}_i\tilde{u}_i'\Lambda_i^{-1}]_{-1}$$

and the equilibrium prices are all strictly positive: $p > 0$.

[13] Using a financial terminology. It is also the reason for the notation: from Kyle (1989) it is common to denote Λ the price impact of traders.

The equilibrium coefficients $(B_i)_i$ can be found by iteration of the best reply map, starting:

- *"from above": the perfect competition matrix C_i;*
- *"from below": any sufficiently small (in 2-norm) initial guess.*

The trivial equilibrium in which every supply and demand function are constantly 0, and so no unilateral deviation yields any profit because there would not be trade anyway, is always present[14]. The condition that there are at least two firms in each sector is sufficient but not necessary, indeed in particular cases without the input output dimension it is sufficient that at least three firms participate in any exchange (Malamud and Rostek (2017)).

Part 3) will be important for the numerical solution of the model, as discussed in Section 1.7.

The *constrained price impact* that appears in equation 1.16 is the matrix that represents the projection on the space of vectors that satisfy the technology constraint $\sum_k q_{ki} = \sum_j \omega_{ij} q_{ij}$. It is thus the necessary adaptation of the concept to an input-output setting: the technology constraint restricts the degrees of freedom that firms have in impacting the market price.

The expression for the best reply highlights the role of the price impact. If $\Lambda = 0$ then $B_i = C_i$ and the outcome is perfect competition. Moreover, we can see that also if $n_i \to \infty$ the model predicts the perfect competition outcome, as it should.

The proof proceeds in two steps:

a) I prove that if a profile of matrices $(B_i^*)_i$ satisfies equation 1.16 on a subnetwork of the original one, and is such that for $\varepsilon = 0$ all implied trades are positive, there exist domains $(\mathcal{E}_i^*, \mathcal{P}_i^*)$ for the linear supply and demand schedules (S_i^*, D_i^*) with coefficients B_i^* such that they are a Nash Equilibrium;

b) I prove that such a profile of matrices exists.

The result can be only stated for generic values of the parameters, and for neighborhoods of $\varepsilon = 0$ because the possibility of corner solutions means that the residual demand in general will only be piecewise linear, and the

[14] This is a feature of the particular technology used, in which labor is a perfect complement to intermediate inputs. In principle if this were not the case a firm producing some final good might find profitable to deviate from the no-trade equilibrium using some labor to sell to the consumers. This would break our assumption on the technology and the linearity of the equilibrium though.

best reply to piecewise linear strategies in this setting might produce a discontinuous schedule (see Anderson and Hu (2008) for an example). To avoid this technical problems, we consider locally defined schedules. In principle it might be the case that precisely at $\varepsilon = 0$ and $p = p^*(0)$ some best reply has a change in slope: but this happens for non generic values of the parameters.

Step *a*) follows the same principles of Klemperer and Meyer (1989) and Kyle (1989): the infinite dimensional optimization problem over supply and demand functions can be reduced to a finite dimensional one of choosing prices taking the stochastic parameters ε as given. The main difference is that we have the input-output dimension, embodied by the residual demand.

Step b) takes advantage of the fact that the best reply equation for coefficient matrices 1.16 is increasing in the coefficients of others with respect to the positive semidefinite ordering, hence a converging sequence can be built. This allows to prove also Part 3). Care must be taken because the positive semidefinite ordering does not have the lattice property, and so the techniques of supermodular games cannot be applied directly. Similar techniques are used in Malamud and Rostek (2017).

All proofs are in the Appendix.

Example 4. Networks with no corner solutions

If the network is a tree such that each sector has just one customer sector, as in Figure 1.7, then it is easy to prove that in equilibrium there is trade on all links. Indeed, by Theorem 1, equilibrium prices are all strictly positive. Then, if i has 0 suppliers, then in equilibrium produces $q_i = B_i p_i > 0$. If sector j has only roots as suppliers, since they all produce strictly positive quantities it follows that $q_j = \sum \omega_{jk} q_k > 0$. Iterating the reasoning we obtain that on all links there is positive trade.

To complete the section, I state two corollaries. The first concerns a partial uniqueness result. Consider sector i, and consider *given* a profile of coefficients of firms in other sectors, that is, consider the sector level price impact Λ_i as given.

Corollary 1.3.1. If we consider the *sector-level* game played just by firms in sector i, this has a unique linear symmetric equilibrium.

The next corollary shows that in an interior equilibrium we do not need to worry about exit of firms: profits are never negative.

Corollary 1.3.2. In equilibrium, if quantities are nonnegative, profits can be expressed as:

$$\pi_i = \left((p_i^{out})^*, -(p_i^{in})^*\right) \left(B_i - \frac{1}{2} V_i' C_i V_i\right) \begin{pmatrix} (p_i^{out})^* \\ -(p_i^{in})^* \end{pmatrix}$$

where $V_i = \tilde{C}_i B_i + \frac{1}{k_i}\tilde{u}_i\tilde{u}_i'\Lambda_i^{-1}(I_i - \tilde{C}_i B_i)$

In particular since $B_i - \frac{1}{2}V_i'C_iV_i$ is positive semidefinite, so profits are always nonnegative in equilibrium.

1.3.2 The role of the network

This section describes how the network of input-output relationships affect the equilibrium of the model. The matrix of coefficients of the market clearing system, M, contains the fundamental network information in this setting. The next Proposition shows that the matrix M has a familiar Leontief form.

Proposition 3. *In equilibrium, the matrix M is positive definite, and has positive diagonal and nonpositive off-diagonal elements. In particular, we can write:*

$$M = D - L$$

where D is a positive diagonal matrix, and L a nonnegative matrix with 0 diagonal elements.

Proposition 3 together with the definition of M imply that L is an adjacency matrix of the line graph $\mathcal{L}(\mathcal{G})$ of $\mathcal{G}$, in the sense that it has a nonzero entry only if the links corresponding to row and column share a node. The weights are endogenous, and depend on the equilibrium profile of demand/supply coefficients.

Inverting the matrix M and collecting the diagonal D on both sides we get:

$$M^{-1} = D^{-1/2}(I - D^{1/2}LD^{1/2})^{-1}D^{-1/2}$$

which shows that M^{-1} is, modulo a normalization, has the familiar form of a *Leontief inverse* matrix. It is standard that entries of matrices of this form constitute a measure of the (weighted) number of *undirected* paths connecting the nodes in the network.

Now with the help of Proposition 2, we can understand how the price impact relates to the network. Indeed, according to Proposition 2, to obtain the price impact of say node 2 first we have to eliminate the links of the line graph connecting input and output links of 2. This is equivalent to building the line graph of the *reduced network* $\mathcal{G}_{-2}$, from which we removed the node 2. Since this is a tree now we have two separate subnetworks. These are illustrated in Figure 1.7 (right). Then, by a reasoning similar to Proposition 3 above, the entries of the matrix Λ_2 count the number of weighted paths between input and outputs of 2. But since in the reduced network input and

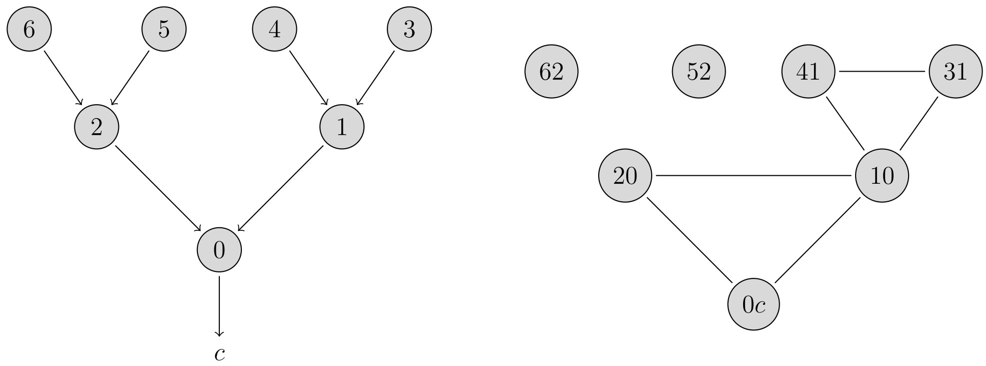


Figure 1.7: Left: A production network shaped as a regular tree. c represents the consumer demand, while the other numbers index the sectors. Right: the reduced line graph with respect to sector 2.

output links are disconnected, the matrix is diagonal, and can be partitioned into:

$$\Lambda_i = \begin{pmatrix} \tilde{D}_i^{-1} & \mathbf{0} \\ \mathbf{0} & \tilde{S}_i^{-1} \end{pmatrix}$$

where $\tilde{D}_i^{-1}$ is the (weighted) number of self loops of the output link in $\mathcal{L}(\mathcal{G})$, and $\tilde{S}_i^{-1}$ is the matrix with on the diagonal the number of self loops of the input links in $\mathcal{L}(\mathcal{G})$.

Figure 1.8 illustrates the network intuition between the decomposition of Λ. It is very similar to the line network: the more upstream the sector is, the larger the portion of the network in which the "self-loops" have to be calculated. Hence the more elastic the demand it is facing. This is because a larger portion of the network is involved in the determination of the demand, and each price variation will distribute on a larger fraction of firms. The intuition is precisely the reverse for the supply coefficients, represented in Figure 1.9

Similar reasonings are at work for other networks, with the difference that in general inputs and outputs are not independent in the reduced network. Consider for example the network in Figure 1.6. What is the price impact of sector 2? In Figure 1.10 is represented the reduced network. Since now input and output links of sector 2 are connected, this means that Λ_2 is not diagonal anymore.

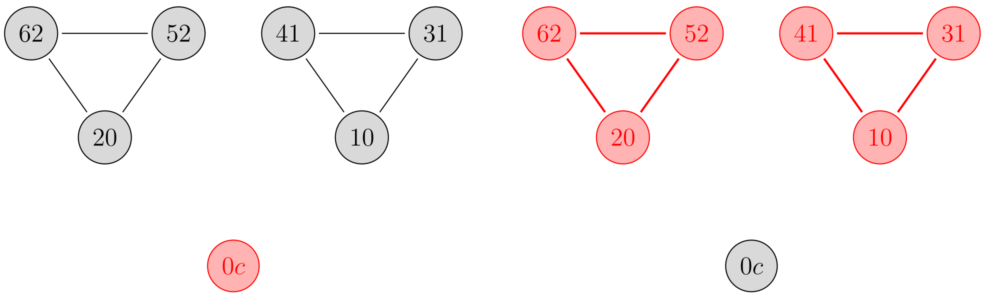


Figure 1.8: The relevant subnetworks of the line graph $\mathcal{L}(\mathcal{G})$ for the calculation of the price impact of sector 2. Left: output, right: inputs.

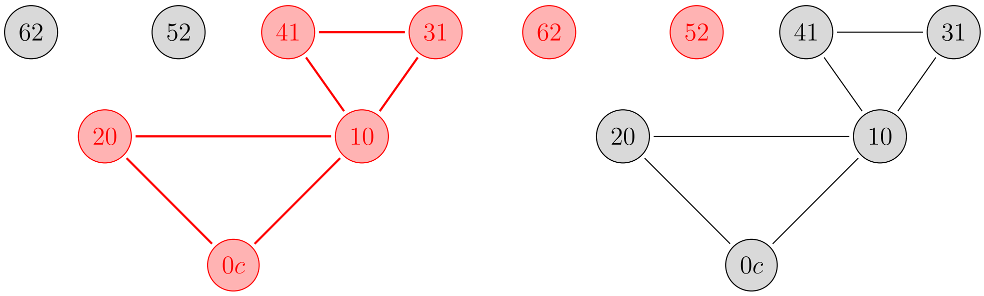


Figure 1.9: The relevant subnetworks for the calculation of the price impact for sector 0. Left: output, right: inputs.

1.4 Benchmarks

Before discussing the results on mergers and local strategic interactions, in this section I am going to describe how some standard models fail to incorporate either Requirement 1 (Symmetric Market Power), or Requirement 2 (Global Strategic Considerations).

1.4.1 Asymmetric market power

The most-clear cut effect is in a line network as depicted in Figure 1.4. Below I will consider more general networks.

Assume that goods in each sector are perfect substitutes, and at each stage of the supply chain firms compete la Cournot, taking as given the input price they face. In our setting this means that firms in sectors 1 and 2 play first, simultaneously, committing to supply a certain quantity. Then firms in sector 0 do the same, taking the price of good 1 and 2 as given. The model can then be solved by backward induction[15].

[15]This is a version of the simplest setting e.g. in Salinger (1988). A similar model, in

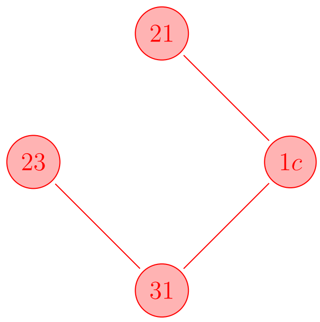


Figure 1.10: The subnetwork of the line graph in Figure 1.6 for the calculation of the price impact.

Call p_0 the inverse demand of the consumer, and assume for simplicity it is concave (this can be sometimes relaxed, as shown below). Assume the technology is linear: $f(q) = Aq$. Capital letters mean sector level quantities, lower case letters are used for firm level quantities.

The markups of firms in sector 0 is equal to the elasticity of the inverse demand, in absolute value. Throughout, I denote elasticities by η:

$$\mu_0 = -\eta_{p_0}$$

What is the markup of upstream sectors? The first order conditions of firms in sector 0 imply that the inverse demand faced by firms in sector 1 is:

$$p_1 = \left(p_0'\left(AQ_1\right)AQ_1 + p_0\left(AQ_1\right)\right)A$$

The markup of firms in sector 1 are then:

$$\begin{aligned}
\mu_1 = -\,\eta_{p_1} &= -\left(\frac{p_0'AQ}{p_0'AQ + p_0}(\eta_{p_0'} + 1) + \frac{p_0}{p_0'AQ + p_0}\eta_{p_0}\right) \\
&= \underbrace{\frac{-p_0'AQ}{p_0'AQ + p_0}}_{>0}(\underbrace{\eta_{p_0'}}_{>0} + 1) + \frac{p_0}{p_0'AQ + p_0}\mu_0 \\
&> \frac{p_0}{p_0'AQ + p_0}\mu_0 > \mu_0
\end{aligned}$$

which puts in evidence that the optimization introduces a force that tends to increase the markup, through the *pass-through*, represented by the term $\frac{p_0}{p_0'f + p_0}$.

The reasoning can be similarly extended to a chain of any lenght.

prices, is Ordover et al. (1990)

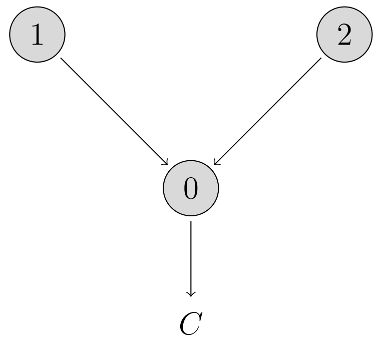


Figure 1.11: A simple supply chain. Firms in Sectors 1 and 2 sell their output to sector 0 firms, which in turn sell to consumers, denoted by C.

More importantly, if there is no compelling physical reason for assuming that firms in sector 1 have precedence over sector 0, an equally reasonable option would be to assume that firms commit to *input* quantities (prices) rather than their output equivalents. An analogous model can then easily be constructed assuming firms in sector 0 decide first, then firms in sector 1. To be more precise, we can compare two different competition structures:

Competition on outputs At $t = 0$ firms in sectors 1 and 2 decide their output quantity; at $t = 1$ firms in sector 0 do the same. Firms in sector 0 face the inverse demand function $p_0(Q_0)$ and all firms take their *input* prices as given.

Competition on inputs At $t = 0$ firms in sector 2 decide their input quantity; at $t = 1$ firms in sector 1 and 2 do the same. Firms in sectors 1 and 2 face the inverse labor supply function $w(L)$ and all firms take their *output* prices as given.

What happens if the network is more general? Consider the case in which 0 has 2 suppliers, as in Figure 1.11. Its production function would then be $f(q_1, q_2)$. Similarly as above we can derive the inverse demand faced by 1:

$$p_1 = (p_0 f_1 + p_0')f_1$$

The markup of firms in sector 1 are then:

$$\begin{aligned}
\mu_1 = &-\left(\eta_{p_1} = \eta_{f_1} + \frac{p_0' f}{p_0' f + p_0}(\eta_{p_0'}\eta_{f,1} + \eta_{f,1}) + \frac{p_0}{p_0' f + p_0}\eta_{p_0}\eta_{f,1}\right) \\
= &\underbrace{-\eta_{f_1}}_{>0} + \underbrace{\frac{-p_0' f}{p_0' f + p_0}}_{>0}(\underbrace{\eta_{p_0'}}_{>0} + 1)\eta_{f,1} + \frac{p_0}{p_0' f + p_0}\mu_0\eta_{f,1} \\
> &\frac{p_0}{p_0' f + p_0}\mu_0\eta_{f,1} > \mu_0\eta_{f,1}
\end{aligned}$$

which puts in evidence that the optimization introduces a force that tends to increase the markup, through the *pass-through*, represented by the terms η_{f_1} and $\frac{p_0}{p_0' f + p_0}$. The opposing force is the substitution effect, which is driven by $\eta_{f,1}$, the output elasticity of good 1. If this is sufficiently close to 1 with respect to the other terms, we have indeed $\mu_1 > \mu_0$. We can sum up the result in a proposition.

We can sum these results up in a proposition.

Proposition 4. *Consider the supply chain illustrated in Figure 11. Assume the consumer has a concave and differentiable demand function, firms have an identical, concave and differentiable constant returns to scale production function f, and there is the same number of firms in each sector. Moreover, assume that at each step of the backward induction the inverse demand remains concave.*

Then:

1. *if the network is a line: under* competition in outputs *firms in sector 1 have larger markup; under* competition in inputs *firms in sector 2 have larger markup.*

2. *if the output elasticity of 0 with respect to input i is close enough to 1, then firms in sector i charge a larger markup than firms in sector 0.*

The conditions are for example satisfied if the technology and utility are quadratic as those used in the main model.

Example 5. (**Markups – Linear-quadratic**)

Assume the firms use the technology introduced in Section 1.2. Analogously, assume that the consumer demand be $Q_0 = A - Bp_0$. Then the markups are:

$$\mu_0 = \frac{1}{B}\frac{Q_0}{p_0 n_0}$$
$$\mu_1 = \frac{1}{B_1}\frac{Q_1}{p_1 n_1}$$

where

$$B_1 = \left(\frac{1}{n_0} + \frac{\omega_{01}^2}{B} \left(1 + \frac{1}{n_0} \right) \right)^{-1}$$

is the *perceived* slope of demand for upstream firms. This represents the pass-through effect: for $\omega_{01} = 1$ it is always smaller than B_1, but for ω_{01} small (corresponding to a situation where 1 is less important in production), the effect can even be reversed. this happens if:

$$B_1 < B_0 \text{ if and only if } B < n_0(1 - \omega_{01}^2) - \omega_{01}^2$$

Now if $\omega_{01} = \omega_{02}$, that is $q_0 = \frac{1}{2}(q_1 + q_2)$. Under this assumption, if $n_0 = n_1 = n_2$, in equilibrium, $Q_0 = Q_1 = Q_2$, and $p_0 > p_1 = p_2$. As a consequence:

$$B_1 < B_0 \Rightarrow \frac{1}{Bp_0} > \frac{1}{B_1 p_1} \Leftrightarrow \frac{Q_0}{Bp_0 n_0} > \frac{Q_1}{B_1 p_1 n_1} \Rightarrow \mu_1 > \mu_0$$

To make things even more concrete, consider a policy maker that is in charge of evaluating merger proposals. She is constrained in her resources, so she wants to know which are the most important mergers she should focus on. This is a concrete issue: for example in USA it is compulsory to report to the Federal Trade Commission only mergers such that the assets of the firms involved lie above some pre-specified thresholds[16]. On which sectors of the economy should she focus? The next example shows that the choice of competition in inputs vs outputs can radically change things.

Example 6. (**Which is the key sector? – Cobb-Douglas**)

Assume the utility of the consumer is $\frac{Q_0^{1-\alpha}}{1-\alpha} - L$, with $\alpha \in (0,1)$[17], and the technology available in sector 0 is Cobb-Douglas: $f(q_1, q_2) = q_1^{\omega_1} q_2^{\omega_2}$, $\omega_1 + \omega_2 \leq 1$. This technology is a classical choice for production network models, it is used (and generalized) among the others by Grassi (2017), Baqaee and Farhi (2017a).

In this case the output elasticity of input 1 is ω_1. The calculation above applies, but notice that here the inverse demand $p_0 = Q_0^{-\alpha}$ is not concave

[16] Thresholds that recently changed: a change that e.g. Wollmann (2019) argues had a large effect on mergers. This is evidence that the costs are substantial, enough to forgo some regulation to reduce them.

[17] The utility would be well defined and concave for $\alpha \in \mathbb{R}_+$, but for $\alpha > 1$ the relative demand function is inelastic, so cannot be used to model oligopoly because it would yield an infinite price.

but convex. The markups are:

$$\mu_0 = \frac{n_0}{n_0 - \alpha} \tag{1.17}$$

$$\mu_1 = \frac{n_1}{n_1 - \frac{\alpha+1}{2}} \tag{1.18}$$

So, since $\alpha < \frac{\alpha+1}{2}$, if $n_0 = n_2$ we get $\mu_1 > \mu_0$.

The (log) welfare impact of mergers is a weighted sum of the log variations in markups:

$$\ln C = -\ln p_0 = -\sum_j L_{ij} \ln \mu_j$$

where L is the Leontief inverse matrix of this economy.

We can see it analytically for "infinitesimal" mergers, that is small variations in the number n_j treated as a continuum parameter[18]:

$$\frac{\partial \ln C}{\partial n_j} = -L_{ij} \frac{1}{\mu_j} \frac{\partial \ln \mu_j}{\partial n_j}$$

and we see that if α is small enough then mergers in sector 1 are more welfare-damaging than mergers in sector 0. This is because the strategic effect is larger the smaller the α. If it is small enough, it dominates the substitution effect caused by the fact that sector 2 produces a substitute good and its presence diminish the possibility of firms in sector 1 to enjoy rents.

Figure 1.12 illustrates numerically that this is true also for non-marginal mergers.

The *competition in inputs* does not yield itself to easy analytical characterizations. Figure 1.13 shows numerically that for small α the sector importance is reversed.

The next example explore a different modeling technique: a bargaining model where the surplus is split according to a parameter δ, and shows that the choice of δ crucially affects relative market power.

Example 7. (**Bargaining**)

In this example I present a simplified variant of the model in Acemoglu and Tahbaz-Salehi (2020), that models the split of surplus between firms in

[18]Farrell and Shapiro (1990a) interpret infinitesimal mergers as changes in asset structure in an oligopoly. Another interpretation can be that of a small change of concentration, in a context where n_j is a reduced form of a measure of concentration in sector j. Farrell and Shapiro (1990b) also use infinitesimal mergers as an analysis tool

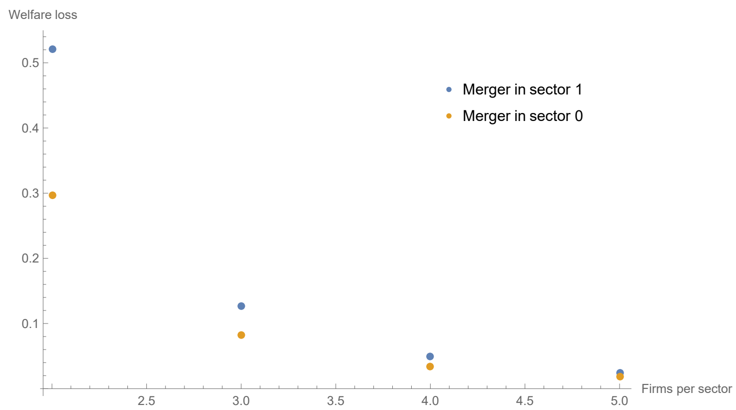


Figure 1.12: Relative welfare loss from a merger of 2 firms in the network of Figure 1.11 under the *competition in outputs* with Cobb-Douglas technology. $\alpha = 1/4$.

an input-output network using a version of Rubinstein repeated offer game. As a result the allocation and the relative market power depend crucially on the parameters δ_{ij} that capture the relative probability of making the first offer. In this example I neglect the exit dimension that they analyze, to simplify the discussion.

There are 2 firms arranged in a line as in Figure 1.4. Each firm produces Q from Q inputs, and the consumer provides inelastically 1 units of labor. Assume for simplicity that the relative bargaining power parameter is the same for all input-output relationships, and is δ. The wage is normalized to 1. The solution of the bargaining problem implies that the prices satisfy (by Equation (5) in the reference):

$$\delta(p_0 - p_1) = (1-\delta)(p_1 - 1)$$
$$A_c - B_c p_0 = 1$$

As shown in the Appendix, solving this equation we find that whether μ_0 is larger than μ_1 depends crucially on δ. For example if $1 < Ac < 2$ and $Bc = 1$ then $\mu_0 > \mu_1$ if and only if $\delta > \overline{\delta}$ for a threshold $\overline{\delta}$.

1.4.2 Local strategic interactions

Most models, mainly in the macroeconomic literature, feature models with *local strategic interactions*. In short, the assumption is that firms internalize the effect of their action on own sector-level variables but not on the other sectors (including suppliers and customers). The purpose of this section is to

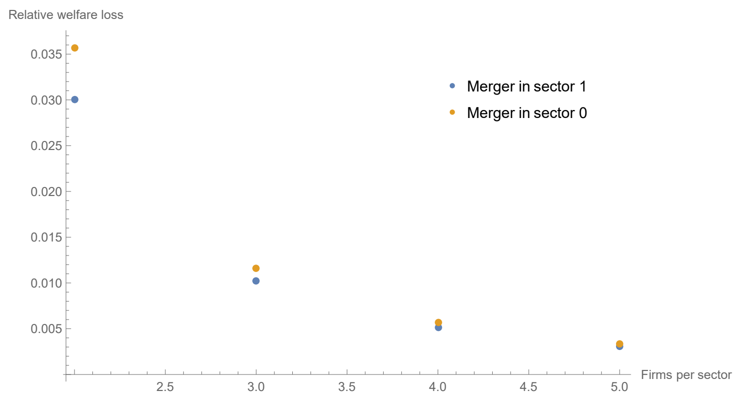


Figure 1.13: Relative welfare loss from a merger of 2 firms in the network of Figure 1.11 under the *competition in inputs* with Cobb-Douglas technology. $\alpha = 1/4$.

show that this assumption can greatly affect the welfare impact of oligopoly power.

The modeling technique relies heavily on parametrical assumptions and to the best of my knowledge there is no clear-cut non-parametric definition, so I present it through an example.

Example 8. (**Local strategic interactions – Cobb-Douglas**)
Assume that the technology available to firms is Cobb-Douglas: $f_i(q_{i1}, \ldots, q_{in}) = \prod q_{ij}^{\omega_{ij}}$, where q_i is the amount of good i bought. In each sector firms are identical and produce perfect substitutes. The *local strategic interaction* assumption works in this way:

1. firm i chooses the bundle of inputs that minimize costs for any given level of output:
$$q_{ij} = \omega_{ij} \frac{f_i MC_i}{p_j}$$

2. the suppliers of i compete committing to output quantities internalizing the inverse demand:
$$p_j = \omega_{ij} \frac{Q_i MC_i}{Q_{ij}} \tag{1.19}$$
where $Q_i MC_i$ is *taken as given.*

This procedure is common in production nework models, it is used among the others by Grassi (2017), Baqaee and Farhi (2017a), Levchenko et al. (2016).

To clarify the difference with the sequential approach inspect equation 1.19: the *perceived* elasticity of demand that the suppliers of i face is 1.

This imposes uniformity across the network, in a radically different way with respect to the sequential approach. Indeed, markups are constant and are:

$$\mu_i = \frac{n_i}{n_i - 1} \tag{1.20}$$

and we can compare with 1.17 to see that they are always smaller. In the sequential economy, taking strategic considerations into account, the original elasticity of demand shrinks as one moves upstream, while here is artificially fixed to 1.

The (log) welfare impact of mergers is a weighted sum of the log variations in markups:

$$\ln C = -\ln p_0 = -\sum_j L_{ij} \ln \mu_j$$

where L is the Leontief inverse matrix of this economy. It immediately follows that in this economy the welfare loss due to market power is *smaller* than in the sequential economy.

Not only: also the welfare impact of mergers is larger. As above, we can study it formally for infinitesimal mergers:

$$\frac{\partial \ln C}{\partial n_j} = -L_{ij} \frac{1}{\mu_j} \frac{\partial \ln \mu_j}{\partial n_j}$$

So the information on the welfare impact of mergers is all contained in the markups. $\frac{\partial \ln \mu_j}{\partial n_j}$ is increasing in the elasticity parameter, so also the welfare impact of a marginal merger is larger in the sequential model.

Moreover, the ratio of the increments under the two models can be arbitrarily large as α gets closer to 0, so the difference is sizable.

Figure 1.14 shows that also a finite (non-marginal) merger has similar properties.

1.5 Market power and mergers

In this section I show how the model provides an answer to the question raised in 1.4, and other examples. As before, a merger in this setting is simply a decrease in the number of firms, n_i. This is because firms are assumed to be identical and to have no capital, hence the merged firm is ex-ante identical to the non-merged firms, but for the fact that there is one firm less in the market now.

First, I show in an example that revenues are not a sufficient statistics for sector importance in this setting.

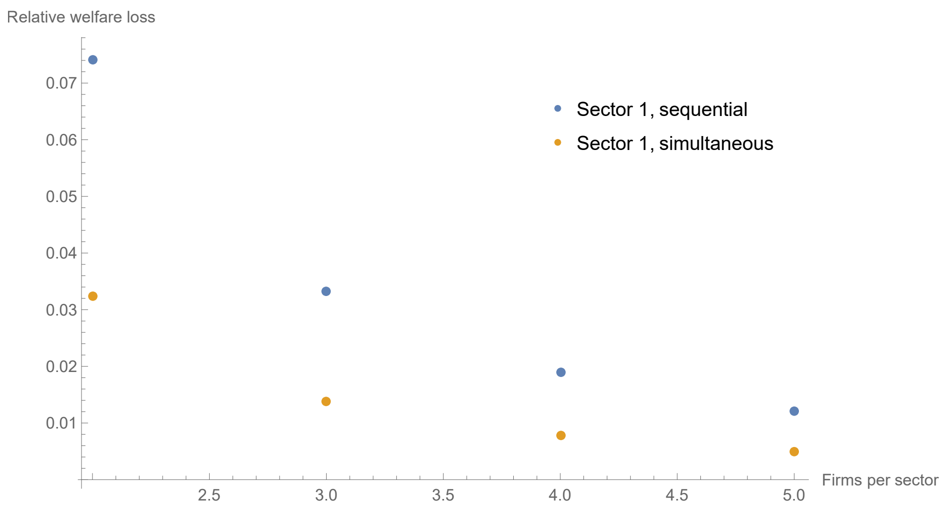


Figure 1.14: Relative welfare losses from a merger of 2 firms in sector 1 in the network of Figure 1.11 under the sequential and the local competition with Cobb-Douglas technology. $\alpha = 1/4$.

Example 9. (**Revenues are not a sufficient statistics**)
Consider a tree oriented differently than in Section 1.4, as in Figure 1.15, consider the case in which the technology is such that $\omega_1 = \omega_2 = \omega_{01} = \omega_{02}$ and all sectors have the same number of firms. In this case parameters are balanced such that *all sectors have the same revenues*. Yet, as in the figure nearby, the welfare loss from mergers is very different in sector 1 and sector 0: it is almost double in sector 0! This shows that a policy maker ignoring the network dimension but focusing only on revenues would choose poorly the sector on which to focus on.

The following Theorem explores the welfare impact of mergers in this setting.

Theorem 2. *Assume a merger does not change the set of active links. Then in the maximal equilibrium it increases all price impacts.*

If there is just one consumer good, any merger decreases the quantity consumed.

Example 10. Tree-Total welfare
Consider a tree network such that each sector has only one customer, as in Figure 1.7, and assume that for each sector all inputs are symmetric, that is $\omega_{ij} = \omega_i$. In this case we can prove that not only the final price increases after a merger, but that also that total welfare decreases. This example is

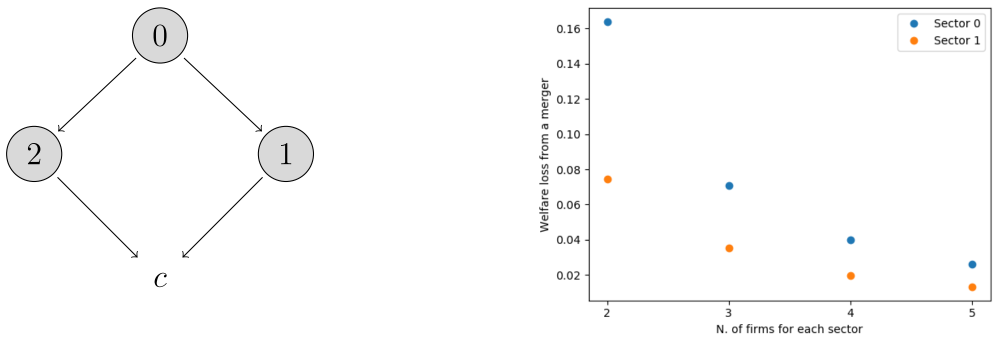


Figure 1.15: On the Left the network considered in the example, on the Right the welfare loss from a merger for different initial numbers of firms.

particularly convenient because the symmetric structure implies that total welfare can be expressed as a function of the consumption of final good only:

$$W = \frac{A_c}{B_c}Q_0 - \frac{1}{2B_c}Q_0^2 - \frac{1}{2}\Omega Q_0^2$$

where Ω is a constant that depends on the degree of each node, the number of firms in each sector, and the input-output coefficients. This expression depends only on market clearing and symmetry, so it is true also under perfect competition. In particular, since there is just one consumer good, we know that Q_0 is maximal under perfect competition. But the expression above is increasing if Q_0 is smaller than the maximum, and from this it follows that total welfare also decreases after a merger.

To explore an example, let us focus on the regular tree of Figure 1.7, and let us assume $\omega_{ij} = \frac{1}{d_i^{in}}$, so that all inputs have the same relative weight in production. Because this choice of technology, this setting allows particularly sharp predictions. This is because, given the symmetry of the problem, all the sectors in the tree will produce the same quantity of output q_i, *no matter the mode of competition*. Hence focusing on this case it is useful can abstract from reallocation and size effects. In the Appendix 1.A.3 I show that in this case under perfect competition profits are identical for all firms.

The results for the S&D equilibrium are numerically calculated in Figure 1.16. It turns out that the equilibrium price impacts are increasing as one moves toward the root of the tree, hence the Corollary above applies in its most useful form. The sector which is the most essential for connecting the whole network is able to extract a larger surplus, and the other are progressively less important the farther upstream one goes.

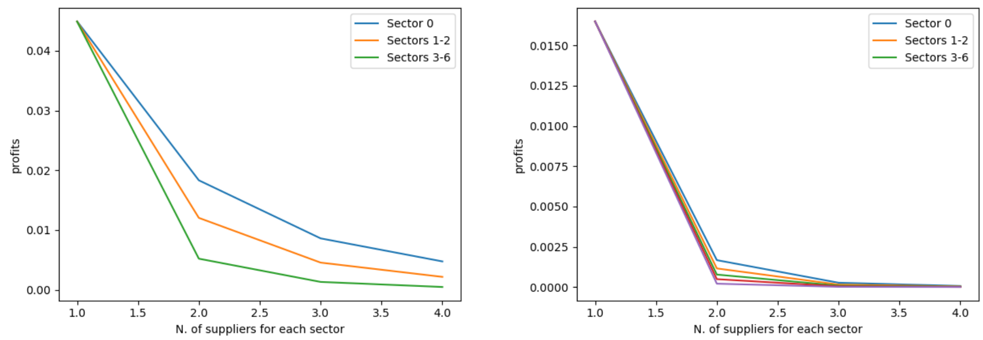


Figure 1.16: Profits for regular trees of height 2 (Left), and 4 (Right), for different numbers of suppliers. Sector 0 is always making the larger profit (except with 1 supplier, which is the case of the line). The number of firms is set to 2 in each sector.

The importance for the regulator follows the same pattern. Figure 1.17 shows that the welfare loss from a merger that brings the number of firms from 2 to 1 is larger in sector 0, and smaller the more we move upstream.

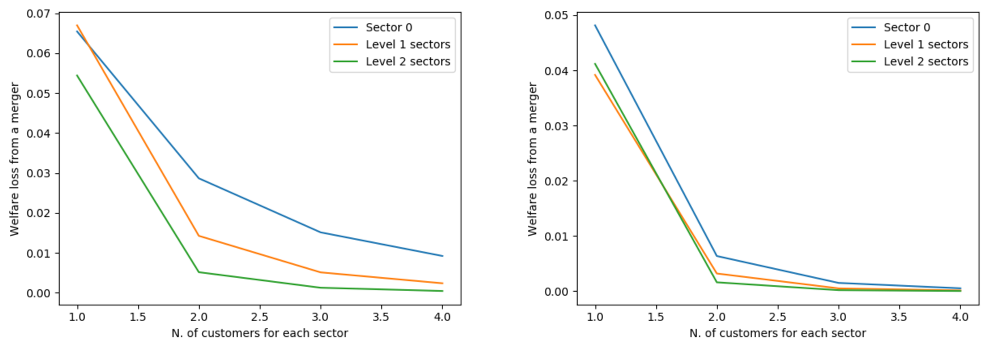


Figure 1.17: Welfare loss from a merger that brings the number of firms from 2 to 1 in different sectors, for different number of suppliers. Left: tree of height 2, Right: tree of height 4.

Example 11. (S&D Equilibrium vs Chain of Oligopolies)
The line network, as in Example 1 is a good setting to gain intuition because sharper results can be obtained. In particular, we can characterize the differences of the sequential competition models with the Supply and Demand Function Equilibrium. Proposition 4 describes how the perceived elasticity of demand tends to decrease as we get closer to the first mover,

and why in that type of sequential model upstream firms tend to have larger market power.

What is the analogous of the markup in the supply and demand function equilibrium setting? to understand this, let us write the problem of the firm in its general form (as in section 1.2.2):

$$\max p_i D_i^r(p_i) - p_{i-1} S_i^r - \frac{1}{2} z_i^2 \tag{1.21}$$

subject to:

$$D_i^r(p_i, p_{i-1}) = z_i \tag{1.22}$$
$$S_i^r(p_i, p_{i-1}) = z_i \tag{1.23}$$

this form is naturally redundant in the case of this simple network. Now define μ_i, the *marginal value of inputs* as the Lagrange multiplier relative to the second constraint, and λ_i, the *marginal value of output*, as the Lagrange multiplier relative to the first constraint. Then we can define simultaneously a *markup* and a *markdown*:

$$M_i = p_i - \lambda_i \quad m_i = \mu_i - p_{i-1} \tag{1.24}$$

which are both zero under perfect competition. The next propostion characterizes their behavior.

Proposition 5. *In a symmetric Supply and Demand Function Equilibrium for the line network, if $n_i = n_j$ for any i, j, then markups are larger the more upstream the sector is, while markdowns are larger the more downstream a sector is.*

This clarifies that the behavior of elasticities in sequential models does not disappear: but here the bilateral nature of the game makes it possible to *both* effects to manifest. How do they balance?

The profit of firms in the symmetric S&D equilibrium can be rewritten as:

$$\pi_i = (M_i + m_i)q_i + \frac{1}{2}q_i^2$$

which makes the intuition transparent: remembering that q_i is constant, the profit in excess of the common component depends on the magnitude of the *sum* of markup and markdown.

Proposition 6. *In a symmetric Supply and Demand Function Equilibrium for the line network, the sector with larger profit is the sector with the smallest number of firms. In particular if $n_i = n_j$ for any i, j, then all sectors have the same profit.*

So, contrary to the sequential competition models, in a S&D equilibrium in a supply chain no one is privileged with respect to others. This follows from the fact that no sector can substitute away from others, they are all essential to produce the consumer good. This allows to shed light on the sequential competition shortcomings: when market power is bilateral one needs to take into account *simultaneously* markup and markdown. When doing so, the paradox disappears and the basic intuition is recovered.

1.6 Global vs Local Strategic Interactions

In this section I explore the question of how the strategic interactions along the supply chains affect welfare. Taking strategically into account what happens in other sectors has in principle ambiguous effects. We could expect more rational agents to be able to extract more surplus, but on the other hand the effect of an increase in price may be larger because it affects all the chain. Moreover, the firm can change the pattern of markups and markdowns charged, shifting market power towards more vulnerable connections, and this may have non trivial distributional effects. Further, firms trade bilaterally, and their reaction makes in principle the problem hard. Strategic complementarities provide a formidable tool to make welfare comparisons.

First, we need to define the equilibrium with short-sighted firms. The idea is to modify Definition 1.2.1 and allow firms to neglect the portion of the network they are not directly connected to.

Definition 1.6.1. A symmetric Local Supply and Demand Function equilibrium is a profile of supply and demand schedules $(S_i, D_i, l_i)_{i \in I}$ such that:

1. the prices and quantities $(p(\varepsilon), q(\varepsilon))$ solve the market clearing conditions when the realization of the shocks is ε;

2. for any firm i, (S_i, D_i, l_i) solves:

$$\max_{(S_{ki})_k, (D_{ij})_j, (z_{i,kj})_{k,j}} \mathbb{E}\left(\sum_k p^*_{ki} S_{ki} - \sum_j p^*_{ij} D_{ij} - \varepsilon_i \sum z_{i\alpha,kj} - \frac{1}{2}\sum_{k,j} z^2_{\alpha,kj}\right) \tag{1.25}$$

subject to:

$$D_{ki}((p_k^{out}, p_k^{in})^*, \varepsilon) = \sum_k z_{i,kj}((p_i^{out}, p_i^{in})^*, \varepsilon_i), \forall i \to k \tag{1.26}$$

$$S_{ij}((p_j^{out}, p_j^{in})^*, \varepsilon) = \sum_j \omega_{ij} z_{i,kj}((p_i^{out}, p_i^{in})^*, \varepsilon_i), \forall j \to i \tag{1.27}$$

$$D_{ki}(p_k^{out}, p_k^{in}, \varepsilon) = S_{ki}((p_i^{out}, p_i^{in})^*, \varepsilon_i), \forall i \to k \tag{1.28}$$

$$S_{ij}(p_k^{out}, p_k^{in}, \varepsilon) = D_{ij}((p_i^{out}, p_i^{in})^*, \varepsilon_i), \forall j \to i \tag{1.29}$$

for given actions chosen by the opponents $(S_j, D_j, l_j)_{j \in I\{i\}}$, and *given* prices of other sectors $(p_j)_{j \notin \mathcal{N}_i}$.

The basic difference with Definition 1.2.1 is that in the firms optimization the *prices of sectors not directly connected with i are taken as given.* Indeed, in the constraints of the optimization there are only the market clearing conditions relative to the links directly connected to i. This is the analogous in this setting of models such as Baqaee (2018), Grassi (2017), Levchenko et al. (2016).

The next theorem explores the welfare implications of this behavioral assumption.

Theorem 3. *In a Local S&D equilibrium, all price impacts are smaller than in the maximal S&D.*

If there is just one consumer good, the quantity consumed is larger.

Example 12. (Welfare effect of Global Strategic Interactions)

Theorem 3 is a qualitative result. In this example I illustrate it quantitatively in the case of a line network, as in Figure 1.4, of length N. As we can see from Figure 1.18, the gap is increasing in the complexity of the network, and sizable: for a line of length 5 the welfare neglecting intersector strategic effects is 25% larger.

Example 13. Welfare effect of mergers – Local vs Global

In this example I show that global strategic interactions are important for the welfare impact of mergers too. I continue to focus on the line network as in the previous example. In this case welfare is simple, because it is just $W = \frac{A_c}{B_c}Q - \frac{1}{2B_c}Q^2 - \frac{N}{2}Q^2$, where N is the number of sectors. Hence the welfare impact of an infinitesimal merger is:

$$\frac{\partial W}{\partial n_i} = \left(\frac{A_c}{B_c} - \left(\frac{1}{B_c} + N\right) Q\right) \frac{\partial Q}{\partial n_i}$$

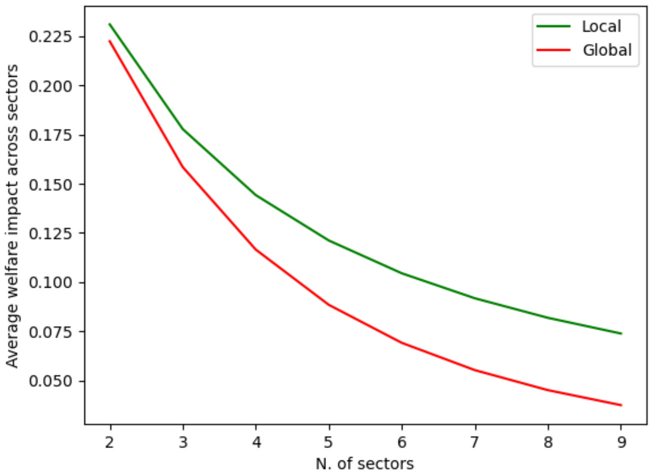


Figure 1.18: Welfare in the global and local versions of the model for different lengths of the line network. A_c and B_c are fixed to 1, $n = 2$.

The first term represents the fact that the closer Q is to the efficient allocation, the smaller the welfare impact is.

Moreover $Q = \frac{A_c}{B_c}\left(\sum_j \frac{1}{n_j B_j}\right)^{-1}$, so that:

$$\frac{\partial Q}{\partial n_i} = \frac{A_c}{B_c}\left(\sum_j \frac{1}{n_j B_j}\right)^{-2} \sum_j \frac{1}{n_j B_j^2}\frac{\partial B_j}{\partial n_i}$$

To understand the mechanics, let us focus on the simplest case: $n_i = n$ for any i. In this case$B_i = B$ for all i, so that $Q = A_c\left(\frac{1}{B_c} + \frac{N}{nB}\right)^{-1} = A_c\frac{nB}{nB+NB_c}$, and:

$$\begin{aligned}\frac{\partial Q}{\partial n_i} &= Q\left(\frac{nBB_c}{nB+NB_c}\right)\sum_j \frac{1}{nB^2}\frac{\partial B_j}{\partial n_i} \\ &= Q\frac{B_c}{(nB+NB_c)}\frac{1}{B}\sum_j \frac{\partial B_j}{\partial n_i}\end{aligned}$$

and in particular, we see that to compare welfare impacts we need to compare the cumulative effect on the coefficients: $\sum_j \frac{\partial B_j}{\partial n_i}$.

To do this, we differentiate the equilibrium conditions to get a fixed point equation for derivatives. For the Global strategic interaction case:

$$\frac{\partial B_j}{\partial n_i} = (1-B)^2\left(\left(\frac{B_c}{(N-1)B_c+nB}\right)^2\left(n\sum_{k\neq j}\frac{\partial B_k}{\partial n_i} + (1-\delta_{ij})B\right) + \delta_{ij}B + (n-1)\frac{\partial B_j}{\partial n_i}\right)$$

while for the Local case:

$$\frac{\partial B_j}{\partial n_i} = (1-B)^2\left(\left(\frac{1}{2}\right)^2\left(n\frac{\partial B_{j-1}}{\partial n_i} + n\frac{\partial B_{j+1}}{\partial n_i} + (\delta_{i,j+1} + \delta_{i,j-1})B\right) + \delta_{ij}B + (n-1)\frac{\partial B_j}{\partial n_i}\right)$$

Comparing the expressions we see that there are 2 distinct effects at play: one is a "crowding out" effect due to the number of sectors: if N is very large, due to the $(N-1)B_c + nB$ factor in the denominator, in the global version derivatives will tend to be smaller. The other is the strategic interaction effect: in the global case an increase in *any* of the other B coefficient reverberates on any other. The following picture illustrates that the strategic interaction effect can prevail in practice.

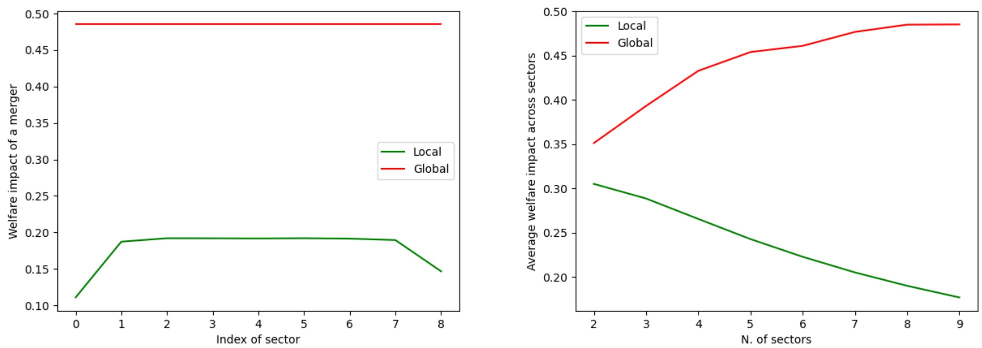


Figure 1.19: Welfare impact of a merger in the line network. On the right the average (relative) impact for different lengths N, on the left the impact differentiated by sectors for $N = 9$. A_c and B_c are fixed to 1, $n = 2$.

1.7 Numerical implementation

The solution by iteration of best reply makes the model numerically tractable for medium sized networks. The main bottleneck is the inversion of the market clearing matrix M, which being a matrix links-by-links, tends to be huge, especially if the network is not very sparse. An application of the Matrix inversion lemma (or Woodbury formula, see Horn and Johnson (2012)) allows to invert the full matrix just once, and then at each step update the inverse by just inverting a small matrix, of size equal to the degree of the involved sector. The gain in this process is especially large when the network is sparse because then the matrices to be inverted are small. The algorithm for solving the model numerically is:

1. initialize all the matrices $B_{i,0}$ as C_i;
2. initialize all relative errors of all nodes to some large number, e.g. 1;
3. start from some node $\hat{i}$. Compute the best reply, inverting the matrix M, and save the inverse.
4. choose the node that has the maximum relative error E_i. Compute its best reply. In doing so, update the inverse of the matrix M using the Matrix Inversion Lemma;
5. Repeat 4 until all E_i are smaller than a threshold (I use 0.01).

In Figure 1.20 I show the computation time to reach the equilibrium for Erdos-Renyi random graphs of 200 nodes, of different densities.

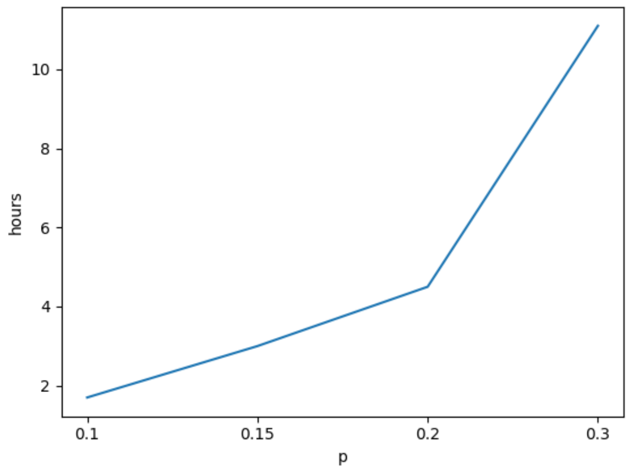


Figure 1.20: Time of model solution for ER random graphs, 200 nodes, average degree $=200p$. Iteration stopped when maximum percentage error $< 0.1\%$

1.8 Conclusion

I build a model of trade among firms as a game in supply and demand function, which allows to study the problem of how the exogenously given network of firm interactions contributes to determine market power. In the case of a tree network, it is possible to connect the endogenous matrix of price impacts to the intuitive notion of Bonacich centrality. I conjecture that the connection is general. Though Bonacich centrality appears often in input-output economics, in this model I show that not only the size of a firm depends on its position, but also its ability to affect prices. The size of a firm (measured e.g. by revenues) will depend on centrality even under perfet competition, as is well known. Here I am introducing another margin: besides begin large, central firms will have more ability to affect market prices. This is a result that can be of interest in the line of research that explores misallocation and its welfare effects.

The results in Section 1.7 show that it is actually possible to use this model in networks of a realistic dimension. A full exploration of the insights that can be obtained from real data is an interesting area to develop further. As I have shown through some examples, the model can in principle be used to assess the market impact of mergers as a function of the position in the network, which might be of interest for antitrust authorities.

Appendix

1.A Proofs of section 1.2

1.A.1 Proof of Proposition 1

The assumption that the consumer demand is zero for very large prices imply that the set of feasible prices is bounded.

Now define the function $F : \mathbb{R}^{|E|\times|E|} \to \mathbb{R}^{|E|\times N}$ (indexed by links) as:

$$F_{ji}(p,\varepsilon) = S_{ji}(p_i,\varepsilon_i) - D_{ji}(p_j,\varepsilon_j) \quad \forall (j,i) \in E \tag{1.30}$$
$$F_{ci}(p,\varepsilon) = S_{ci}(p_i,\varepsilon_i) - D_{ci}(p_c,\varepsilon_c) \tag{1.31}$$

so that the market clearing conditions 1.2 are equivalent to $F(p,\varepsilon) = 0$.

Now I prove that the Jacobian is nonzero, so that the implicit function theorem applies.

Call

$$J_i = \begin{pmatrix} JD_i^{out} & JD_i^{in} \\ JS_i^{out} & JS_i^{in} \end{pmatrix}$$

the blocks of the Jacobian matrix, where "in" and "out" refer to the differentiation variables (prices), and S and D to supply and demand. To prove that JF is positive definite, note that row (il) is composed by:

- $JS_{l,ii} + JD_{i,ll}$ in position (il) (diagonal element);
- $JS_{l,ik}$ in position (kl);
- $-JS_{l,ij}$ in position (lj);
- $-JD_{i,lk}$ in position (ki);
- $JD_{i,lj}$ in position (ij).

Consider $x \in \mathbb{R}^{|E|\times|E|}$ and $x'JFx$. Write as usual x_i for $((x_{ki})_{k,i\to k}, (x_{ij})_{j\,j\to i})$. Inspection of the matrix JF reveals that:

$$x'JFx = \sum_m x_m' \hat{J}_m x_m + x_c' J_c x_c$$

where $\hat{J} = \begin{pmatrix} JS_i^{out} & -JS_i^{in} \\ -JD_i^{out} & JD_i^{in} \end{pmatrix}$, that is again positive semidefinite under our assumptions. Now the expression above is nonnegative because a sum of nonnegative terms. The term $x_c' J_c x_c$ is zero only if x_c is zero. Assume the worst case, that all the Jacobians have rank $d_m - 1$. Call $\tilde{u}_m$ the vector that nullifies $\hat{J}_m$. To prove that $x'Jx$ is positive, we have to prove that for any non zero x at least one of the vectors x_m or x_c is non zero and $x_m \neq \hat{u}_m$. If $x_m = \hat{u}_m$ for all m, then the entries of x_c are different from zero and so the expression is positive. If the entries of x_c are all zero, then there is at least 1 of the m such that x_m has a zero entry, and so $x'_m \hat{J}_m x_m > 0$. So we proved that $x'JFx > 0$ if $x \neq 0$, so JF is positive definite.

1.A.2 Proof of Proposition 2

M is the Jacobian J of Proposition 1 specialized in this linear setting. By the same Proposition, it is invertible and positive definite.

If the supply and demand functions satisfie the conditions of 1, then there exists a set $\mathcal{E}$ such that the price map is defined.

There will be sets $\mathcal{E}_i$ and $\mathcal{P}_i$, such that $\mathcal{E} \subseteq \mathcal{E}_i$ $p_i^*(0) \in \mathcal{P}_i$, such that the partial solution $p_{-i}^*(\varepsilon, p_i)$ is linear. Hence, the residual demand is linear on some set $\mathcal{E}_i \times \mathcal{P}_i$.

Let us calculate it explicitly. we define p_{-i} as the vector of all prices but the prices incident to i. Now we reorder the entries of the matrix M to have in the leading upper left position all the rows that represent equations involving node i, and all the columns relative to prices of input and output of i. Write M_i for the matrix M subject to this reordering. The matrix M can then be partitioned as:

$$M_i = \begin{pmatrix} n_i \tilde{B}_i + B_i^D & M_{R_i} \\ M_{C_i} & M_{-i} \end{pmatrix}$$

where M_{-i} is M from which we cancelled all the rows and columns relative to i, which are M_{R_i} and M_{C_i}, and B_i^D is the matrix with on the diagonal the elements $n_k B_{k,ii}^{out}$ or $n_k B_{k,ii}^{in}$ for all k that are connected to i. Now consider the matrix:

$$\tilde{M}_i = \begin{pmatrix} B_i^D & M'_{C_i} \\ M_{C_i} & M_{-i} \end{pmatrix}$$

The same reasoning proving positive definiteness applied to source nodes shows that at least 2 m are such that $x'_m \hat{B}_m x_m > 0$, so even setting some B_m to zero would not affect invertibility. Hence $\tilde{M}_i$ is still positive definite.

In solving for the objective demand we solve first for p_{-i}:

$$M_{-i} p_{-i} = -M_{C_i} \begin{pmatrix} p_i^{out} \\ p_i^{in} \end{pmatrix} + \boldsymbol{A}_{-i} \Longrightarrow p_{-i} = M_{-i}^{-1}(-M_{C_i} \begin{pmatrix} p_i^{out} \\ p_i^{in} \end{pmatrix} + \boldsymbol{A}_{-i})$$

and then we use it in the expression for objective supplies and demands. The sector level residual demand is, from the market clearing conditions:

$$n_i \begin{pmatrix} S_i \\ D_i \end{pmatrix} = \begin{pmatrix} (n_k D_k)_{k,i\to k} \\ (n_j S_j)_{j,j\to i} \end{pmatrix}$$

Reordering:

$$n_i \begin{pmatrix} -S_i \\ D_i \end{pmatrix} = \begin{pmatrix} -(n_k D_k)_{k,i\to k} \\ (n_j S_j)_{j,j\to i} \end{pmatrix}$$

we can observe that the right hand side corresponds to the market clearing equations 1.2 for inputs and outputs of i after removing the schedules of sector i. That is, the left hand side corresponds to the first i rows of $\tilde{M}_i p$, that is:

$$[\tilde{M}_i p][\textit{first } i \textit{ rows}] = B_i^D \begin{pmatrix} p_i^{out} \\ p_i^{in} \end{pmatrix} + M'_{C_i} p_{-i} =$$

$$(B_i^D - M'_{C_i} M_{-i}^{-1} M_{C_i}) \begin{pmatrix} p_i^{out} \\ p_i^{in} \end{pmatrix} + M'_{C_i} M_{-i}^{-1} \boldsymbol{A}_{-i}$$

and by block matrix inversion can be seen that $(B_i^D - M_{R_i} M_{-i}^{-1} M_{C_i}) = [(\tilde{M}_i)^{-1}]_i^{-1}$, and moreover is positive definite. To obtain the expression for the residual supply and demand schedule, we have to reorder the signs of the blocks of the coefficient matrix. Define: $\Lambda_i^{-1} = P[(\tilde{M}_i)^{-1}]_i^{-1} P$, where:

$$P = \begin{pmatrix} I & \mathbf{0} \\ \mathbf{0} & -I \end{pmatrix}$$

and we obtain:

$$\begin{pmatrix} D_i^r \\ S_i^r \end{pmatrix} = -\Lambda_i^{-1} \begin{pmatrix} p_i^{out} \\ -p_i^{in} \end{pmatrix} + \tilde{\boldsymbol{A}}_i$$

where $\tilde{\boldsymbol{A}}_i = M'_{C_i} M_{-i}^{-1} \boldsymbol{A}_{-i}$

1.A.3 Perfect competition benchmark

If a firm takes prices as given will optimize:

$$\max_{q_{k,i\alpha}, q_{i\alpha,j}, z_{i\alpha,kj}} \sum_k p_{ki} q_{k,i\alpha} - \sum_j p_{ij} q_{i\alpha,j} - \frac{1}{2} \sum z_{i\alpha,kj}^2$$

subject to:

$$q_{k,i\alpha} = \sum_j \omega_{ij} z_{i\alpha,kj}, \quad q_{i\alpha,j} = \sum_k z_{i\alpha,kj}$$

The FOC yield:

$$q_{k,i\alpha} = \sum_j \omega_{ij}^2 p_{ki} - \sum_j \omega_{ij} p_{ij} \tag{1.32}$$

$$q_{i\alpha,j} = \omega_{ij} \sum_k p_{ki} - d_i^{out} p_{ij} \tag{1.33}$$

Or, in matrix form:

$$q = \begin{pmatrix} \omega_i' \omega_i I_i^{out} & u_{out} \omega_i' \\ \omega_i u_{out}' & d_i^{out} I_i^{in} \end{pmatrix} \begin{pmatrix} p_i^{out} \\ -p_i^{in} \end{pmatrix} = C_i \begin{pmatrix} p_i^{out} \\ -p_i^{in} \end{pmatrix}$$

Moreover, the profit is:

$$\pi_i = \frac{1}{2} \sum_{k,j} (\omega_{ij} p_{ki} - p_{ij})^2 = \frac{1}{2} \sum_{k,j} z_{i,kj}^2$$

and we can see that if firms are all producing the same quantity, as in Section 1.4, the profits are the same for all.

1.B Proofs of Section 1.3

1.B.1 Proof of Theorem 1

Step a) - A profile of matrices satisfying 1.16 is a S&D Equilibrium

Rewrite best reply as a finite dimensional optimization Assume all other firms in all other sectors are playing a profile of symmetric linear schedules that for the prices relative to active links have coefficients $(B_j)_j$ which are positive semidefinite. Consider the best reply problem of firm α in sector i. This is:

$$\max_{(S_{ki})_k, (D_{ij})_j, (z_{i,kj})_{k,j}} \mathbb{E} \left(\sum_k p_{ki}^* S_{ki} - \sum_j p_{ij}^* D_{ij} - \varepsilon_i \sum z_{i\alpha,kj} - \frac{1}{2} \sum_{k,j} z_{\alpha,kj}^2 \right)$$

subject to the market clearing conditions 1.2. All the sums run over active links: prices relative to inactive links do not affect the objective function nor the constraints. I already used the fact that at the optimum it must be $l_{i\alpha,kj} = \varepsilon_i z_{i\alpha,kj} + \frac{1}{2} z_{i\alpha,kj}^2$.

Using the residual demand, we can rewrite the optimization as:

$$\max_{(S_{ki})_k, (D_{ij})_j, (z_{i,kj})_{k,j}} \mathbb{E} \left(\sum_k p_{ki}^* S_{ki} - \sum_j p_{ij}^* D_{ij} - \varepsilon_i \sum z_{i,kj} - \frac{1}{2} \sum z_{i,kj}^2 \right)$$

subject to:

$$D^r_{ki}((p_i^{out}, p_i^{in})^*, \varepsilon) = \sum_k z_{i,kj}((p_i^{out}, p_i^{in})^*, \varepsilon_i),\ \forall i \to k \tag{1.34}$$

$$S^r_{ij}((p_i^{out}, p_i^{in})^*, \varepsilon) = \sum_j \omega_{ij} z_{i,kj}((p_i^{out}, p_i^{in})^*, \varepsilon_i),\ \forall j \to i \tag{1.35}$$

$$D^r_{ki}(p_i^{out}, p_i^{in}, \varepsilon) = S_{ki}((p_i^{out}, p_i^{in})^*, \varepsilon_i),\ \forall i \to k \tag{1.36}$$

$$S^r_{ij}(p_i^{out}, p_i^{in}, \varepsilon) = D_{ij}((p_i^{out}, p_i^{in})^*, \varepsilon_i),\ \forall j \to i \tag{1.37}$$

Now assume ε is in the set $\mathcal{E}_i$ where Proposition 2 applies. Then since Λ_i^{-1} is invertible the last two conditions in 1.34 define uniquely a function for the prices of active links $p_i^*(\varepsilon) : \mathcal{E}_i \to \mathbb{R}^{d_i}$. Then we can rewrite the optimization as:

$$\max_{(S_{ki})_k, (D_{ij})_j, (z_{i,kj})_{k,j}, p_i^*} \mathbb{E}\left(\sum_k p^*_{ki} D^r_{ki} - \sum_j p^*_{ij} S^r_{ij} - \varepsilon_i \sum z_{i,kj} - \frac{1}{2} \sum z^2_{i,kj}\right)$$

subject to:

$$D^r_{ki}((p_i^{out}, p_i^{in})^*, \varepsilon) = \sum_k z_{i,kj}((p_i^{out}, p_i^{in})^*, \varepsilon_i),\ \forall i \to k \tag{1.38}$$

$$S^r_{ij}((p_i^{out}, p_i^{in})^*, \varepsilon) = \sum_j \omega_{ij} z_{i,kj}((p_i^{out}, p_i^{in})^*, \varepsilon_i),\ \forall j \to i \tag{1.39}$$

$$\tag{1.40}$$

Now (S, D) do not appear explicitly in the problem any more. For the active links, we can recover them using the information in the pricing function. Indeed, for any x in the range of p_i^*, define:

$$S_{ki}(x, \varepsilon_i) = D^r_{ki}(x, \varepsilon),\ \forall i \to k$$
$$D_{ij}(x, \varepsilon_i) = S^r_{ij}(x, \varepsilon),\ \forall j \to i$$

for some $\varepsilon \in (p_i^*)^{-1}(x)$. By definition of p_i^*, the relation above must be satisfied for all the elements in the counterimage. For all the non-active links, they are both identically zero.

Finally, optimizing with respect to a function of the stochastic variable is equivalent to optimizing ex-post, for any fixed value of ε, as in Klemperer and Meyer (1989). Hence we can write the best reply problem in its final form:

$$\max_{(z_{i,kj})_{k,j},p_i\in\mathcal{P}_i} \sum_k p_{ki}D^r_{ki}(p_i,\varepsilon) - \sum_j p_{ij}S^r_{ij}(p_i,\varepsilon) - \varepsilon_i \sum z_{i,kj} - \frac{1}{2}\sum z^2_{i,kj} \tag{1.41}$$

subject to:

$$D^r_{ki}((p_i,\varepsilon) = \sum_k z_{i,kj},\ \forall i \to k \tag{1.42}$$

$$S^r_{ij}((p_i,\varepsilon) = \sum_j \omega_{ij} z_{i,kj},\ \forall j \to i \tag{1.43}$$

$$z_i \geq 0 \tag{1.44}$$

Optimization The problem 1.41 in the set $\mathcal{E}_i \times \mathcal{P}_i$ it is a simple concave problem, and can be solved by first order conditions. Now I show that the best reply, defined on $\mathcal{E}_i \times \mathcal{P}_i$, is linear and has as coefficient matrix exactly the B^*_i as defined in 1.16.

Call λ_{ki} and μ_{ij} the multipliers for input and output constraints respectively, and $J_i = (n_i - 1)B_i + \Lambda_i^{-1}$ the firm level (inverse) price impact. J is the derivative of the supply and demand schedule, and by Proposition 2 it is positive definite.

The Hessian of the problem is a block diagonal matrix with blocks $-(J_i + J_i')$ and minus the identity (with respect to the zs), so the problem is concave.

The FOCs are:

$$p_{hi}: \quad \sum_k \frac{\partial D^r_{ki}}{\partial p_{hi}}(p_{ki} - \lambda_{ki}) - \sum_j \frac{\partial S^o_{ij}}{\partial p_{hi}}(p_{ij} - \mu_{ij}) + D^r_i = 0$$

$$p_{ih}: \quad \sum_k \frac{\partial D^r_{ki}}{\partial p_{ih}}(p_{ki} - \lambda_{ki}) - \sum_j \frac{\partial S^o_{ij}}{\partial p_{ih}}(p_{ij} - \mu_{ij}) - S^r_i = 0$$

$$z_{i,kj}: \quad -\varepsilon_i - z_{i,kj} + \omega_{ij}\lambda_{ki} - \mu_{ij} + t = 0$$

where $t \geq 0$ is the multiplier relative to the constraint $z \geq 0$.

The first set of equations in matrix form reads:

$$J_i \begin{pmatrix} p_i^{out} - \lambda_i \\ -(p_i^{in} - \mu_i) \end{pmatrix} - \begin{pmatrix} D^r_i \\ S^r_i \end{pmatrix} = 0$$

Since this must be true for any prices, and for any price the market clearing conditions must be satisfied, we can rewrite these as:

$$J \begin{pmatrix} p_i^{out} - \lambda_i \\ -(p_i^{in} - \mu_i) \end{pmatrix} = \begin{pmatrix} S_i \\ D_i \end{pmatrix}$$

Now we can use the constraints to get rid first of the z. To do so, sum the derivatives with respect to z to obtain:

$$D^{obj}_{ki} = \sum_j \omega_{ij} z_{i,kj} = \sum_j \omega^2_{ij}\lambda_{ki} - \sum_j \omega_{ij}\mu_{ij} - \sum_j \omega_{ij}\varepsilon_i + \sum_j \omega_{ij} t_{i,kj}$$

$$S^{obj}_{ij} = \sum_k z_{i,kj} = \omega_{ij}\sum_k \lambda_{ki} - d^{out}_i \mu_{ij} - d^{out}_i \varepsilon_i + \sum_k t_{i,kj}$$

Now notice that these equations have a linear dependence, because $\sum_k \sum_j \omega_{ij} z_{i,kj} = \sum_j \omega_{ij} \sum_k z_{i,kj}$. This is not a problem, because $\sum_k D^{obj}_{ki} = \sum_j \omega_{ij} S^{obj}_{ij}$ is indeed a contraint of the problem, but it means that we need to eliminate one equation to solve for the multipliers. Without loss of generality, I eliminate λ_1. If a is a matrix (vector), a_{-1} will denote the elimination of row and column (element) 1. Call $t_{ij} = \sum_k t_{i,kj}$, $t_{ki} = \sum_j t_{i,kj}$. So we can write the system as:

$$\begin{pmatrix} D^{obj}_{i,-1} \\ S^{obj}_i - \lambda_1\omega_i \end{pmatrix} = \begin{pmatrix} \omega'_i\omega_i I_{-1,out} & u_{-1,out}\omega'_i \\ \omega_i u'_{-1,out} & d^{out}_i I_{in} \end{pmatrix} \begin{pmatrix} \lambda_{i,-1} \\ -\mu_i \end{pmatrix} +$$

$$\begin{pmatrix} t_{ik,-1} \\ t_{ij} \end{pmatrix} - \varepsilon_i \begin{pmatrix} (\omega'_i u^{in}_i) u^{out}_{i,-1} \\ d^{out}_i u^{in}_i \end{pmatrix}$$

Notice that the matrix of the system is $C_{i,-1}$, the $(1,1)-$ minor of the perfect competition matrix.

Solving we get:

$$\begin{pmatrix} \lambda_{i,-1} \\ -\mu_i \end{pmatrix} = C^{-1}_{i,-1}\left[\begin{pmatrix} D^{obj}_{i,-1} \\ S^{obj}_i - \lambda_1\omega_i \end{pmatrix} - \begin{pmatrix} t_{ik,-1} \\ t_{ij} \end{pmatrix} + \varepsilon_i \begin{pmatrix} (\omega'_i u^{in}_i) u^{out}_{i,-1} \\ d^{out}_i u^{in}_i \end{pmatrix}\right]$$

and for the full vector of multipliers:

$$\begin{pmatrix} \lambda_{i1} \\ \lambda_{i,-1} \\ -\mu_i \end{pmatrix} = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & C^{-1}_{i,-1} \end{pmatrix} \left[\begin{pmatrix} 0 \\ D^{obj}_{i,-1} \\ S^{obj}_i \end{pmatrix} - \begin{pmatrix} 0 \\ t_{ik,-1} \\ t_{ij} \end{pmatrix} + \varepsilon_i \begin{pmatrix} 0 \\ (\omega'_i u^{in}_i) u^{out}_{i,-1} \\ d^{out}_i u^{in}_i \end{pmatrix} + \lambda_{1i} \begin{pmatrix} 1 \\ \mathbf{0} \\ -\omega_i \end{pmatrix}\right.$$

Now using the constraint $(u'_{out}, -\omega'_i)\begin{pmatrix} S_i \\ D_i \end{pmatrix} = 0$ we can rewrite:

$$(u'_{out}, -\omega'_i) J_i \begin{pmatrix} p^{out}_i \\ -p^{in}_i \end{pmatrix} = (u'_{out}, -\omega'_i) J_i \begin{pmatrix} \lambda_i \\ -\mu_i \end{pmatrix}$$

and substituting the multipliers we get:

$$\tilde{u}'_i J_i \begin{pmatrix} p^{out}_i \\ -p^{in}_i \end{pmatrix} = \tilde{u}'_i J_i \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & C^{-1}_{i,-1} \end{pmatrix} \left[\begin{pmatrix} 0 \\ D^{obj}_{i,-1} \\ S^{obj}_i \end{pmatrix} - t_{i,-1} + \varepsilon_i \begin{pmatrix} 0 \\ (\omega'_i u^{in}_i) u^{out}_{i,-1} \\ d^{out}_i u^{in}_i \end{pmatrix} + \lambda_{1i} \begin{pmatrix} 1 \\ \mathbf{0} \\ -\omega_i \end{pmatrix}\right.$$

$$-\lambda_{1i}\tilde{u}_i'J_i\begin{pmatrix}1 & \mathbf{0}\\ \mathbf{0} & C_{i,-1}^{-1}\end{pmatrix}\begin{pmatrix}1\\ \mathbf{0}\\ -\omega_i\end{pmatrix}=$$

$$\tilde{u}_i'J_i\left[\begin{pmatrix}1 & \mathbf{0}\\ \mathbf{0} & C_{i,-1}^{-1}\end{pmatrix}\left(\begin{pmatrix}0\\ D_{i,-1}^{obj}\\ S_i^{obj}\end{pmatrix}-t_{i,-1}+\varepsilon_i\begin{pmatrix}0\\ (\omega_i'u_i^{in})u_{i,-1}^{out}\\ d_i^{out}u_i^{in}\end{pmatrix}\right)-\begin{pmatrix}p_i^{out}\\ -p_i^{in}\end{pmatrix}\right]$$

Now the inverse of $C_{i,-1}$ can be calculated through block inversion and Sherman-Morrison formula, and is:

$$C_{i,-1}^{-1}=\begin{pmatrix}\frac{1}{\omega_i'\omega_i}\left(I_{-1,i}^{out}+u_{-1,out}u_{-1,out}'\right) & -\frac{1}{\omega_i'\omega_i}u_{-1,out}\omega_i'\\ -\frac{1}{\omega_i'\omega_i}\omega_iu_{-1,out}' & \frac{1}{d_i^{out}}\left(I_{in}+\frac{d_i^{out}-1}{\omega_i'\omega_i}\omega_i\omega_i'\right)\end{pmatrix}$$

from which we get:

$$\begin{pmatrix}1 & \mathbf{0}\\ \mathbf{0} & C_{i,-1}^{-1}\end{pmatrix}\begin{pmatrix}1\\ \mathbf{0}\\ -\omega_i\end{pmatrix}=\begin{pmatrix}u_{out}\\ -\omega_i\end{pmatrix}$$

and

$$\begin{pmatrix}1 & \mathbf{0}\\ \mathbf{0} & C_{i,-1}^{-1}\end{pmatrix}\begin{pmatrix}0\\ (\omega_i'u_i^{in})u_{i,-1}^{out}\\ d_i^{out}u_i^{in}\end{pmatrix}=\begin{pmatrix}\mathbf{0}\\ u_i^{in}\end{pmatrix}$$

[19]

So that the coefficient of λ_{1i} is $k_i=\tilde{u}_i'J_i\tilde{u}_i>0$. Substituting this into the expression for the multiplier:

$$\begin{pmatrix}\lambda_{i1}\\ \lambda_{i,-1}\\ \mu_i\end{pmatrix}=\begin{pmatrix}1 & \mathbf{0}\\ \mathbf{0} & C_{i,-1}^{-1}\end{pmatrix}\times\left[\left(\begin{pmatrix}0\\ D_{i,-1}^{obj}\\ S_i^{obj}\end{pmatrix}-t_{i,-1}+\varepsilon_i\begin{pmatrix}0\\ (\omega_i'u_i^{in})u_{i,-1}^{out}\\ d_i^{out}u_i^{in}\end{pmatrix}\right)\right.$$

$$\left.-\frac{1}{k}\begin{pmatrix}1\\ \mathbf{0}\\ -\omega_i\end{pmatrix}\tilde{u}_i'J_i\left[\begin{pmatrix}1 & \mathbf{0}\\ \mathbf{0} & C_{i,-1}^{-1}\end{pmatrix}\left(\begin{pmatrix}0\\ D_{i,-1}^{obj}\\ S_i^{obj}\end{pmatrix}-t_{i,-1}+\varepsilon_i\begin{pmatrix}0\\ (\omega_i'u_i^{in})u_{i,-1}^{out}\\ d_i^{out}u_i^{in}\end{pmatrix}\right)-\begin{pmatrix}p_i^{out}\\ -p_i^{in}\end{pmatrix}\right]\right]$$

[19] This can be seen by the explicit calculation of:

$$\begin{pmatrix}1\\ \mathbf{0}\\ -\omega_i\end{pmatrix}=\begin{pmatrix}1 & \mathbf{0}\\ \mathbf{0} & C_{i,-1}\end{pmatrix}\begin{pmatrix}u_{out}\\ -\omega_i\end{pmatrix}$$

so the expression above becomes:

$$\begin{pmatrix} \lambda_{i1} \\ \lambda_{i,-1} \\ \mu_i \end{pmatrix} = \left(I_i - \frac{1}{k_i}\tilde{u}_i\tilde{u}_i' J_i\right)\begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & C_{i,-1}^{-1} \end{pmatrix}\left(\begin{pmatrix} 0 \\ D_{i,-1}^{obj} \\ S_i^{obj} \end{pmatrix} - t_{i,-1}\right) + \varepsilon_i \begin{pmatrix} \mathbf{0} \\ u_i^{in} \end{pmatrix}$$

$$+\frac{1}{k_i}\tilde{u}_i\tilde{u}_i' J_i \begin{pmatrix} p_i^{out} \\ -p_i^{in} \end{pmatrix}$$

So we can finally substitute and get the demand function (after using market clearing to turn objective into supply and demand). So:

$$\begin{pmatrix} S_i \\ D_i \end{pmatrix} = J_i \begin{pmatrix} p_i^{out} \\ -p_i^{in} \end{pmatrix} -$$

$$J_i \left(\left(I_i - \frac{1}{k_i}\tilde{u}_i\tilde{u}_i' J_i\right)\left(\begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & C_{i,-1}^{-1} \end{pmatrix}\left(\begin{pmatrix} 0 \\ D_{i,-1}^{obj} \\ S_i^{obj} \end{pmatrix} - t_{i,-1}\right) + \varepsilon_i \begin{pmatrix} \mathbf{0} \\ u_i^{in} \end{pmatrix}\right) + \frac{1}{k_i}\tilde{u}_i\tilde{u}_i' J_i \begin{pmatrix} p_i^{out} \\ -p_i^{in} \end{pmatrix}\right.$$

Now to re-express everything in terms of supply and demand functions note that:

$$\begin{pmatrix} 0 \\ D_{i,-1}^{obj} \\ S_i^{obj} \end{pmatrix} = \begin{pmatrix} 1 & u_{-1,out}' & -\omega_i' \\ \mathbf{0} & I_{-1,out} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_{in} \end{pmatrix}\begin{pmatrix} D_i^{obj} \\ S_i^{obj} \end{pmatrix}$$

call:

$$\tilde{C}_i = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & C_{i,-1}^{-1} \end{pmatrix}\begin{pmatrix} 1 & u_{-1,out}' & -\omega_i' \\ \mathbf{0} & I_{-1,out} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_{in} \end{pmatrix} = \begin{pmatrix} 1 & \tilde{u}_{i,-1}' \\ \mathbf{0} & C_{i,-1}^{-1} \end{pmatrix}$$

and eventually we get:

$$\begin{pmatrix} S_i \\ D_i \end{pmatrix} = \left(I_i + \left(J_i - \frac{1}{k_i}J_i\tilde{u}_i\tilde{u}_i' J_i\right)\tilde{C}_i\right)^{-1}\left(J_i - \frac{1}{k_i}J_i\tilde{u}_i\tilde{u}_i' J_i\right)\left(\begin{pmatrix} p_i^{out} \\ -p_i^{in} \end{pmatrix} - \varepsilon_i \begin{pmatrix} \mathbf{0} \\ u_i^{in} \end{pmatrix} + t_i\right)$$

where $t_i = \begin{pmatrix} 0 \\ C_{i,-1}^{-1} t_{i,-1} \end{pmatrix}$.

To obtain the expression in the text of the Theorem, notice that $\left(J_i - \frac{1}{k_i}J_i\tilde{u}_i\tilde{u}_i' J_i\right)$ is not invertible because not all equations are independent. Let us solve for the last $d^{in} + d^{out} - 1$ equations:

$$\begin{pmatrix} S_{i,-1} \\ D_i \end{pmatrix} = (I + J_{-R_1}\left(I - \frac{1}{k_i}\tilde{u}\tilde{u}' J\right)_{-C_1}\tilde{C}_i)^{-1}\times$$

$$J_{-R_1}\left(I-\frac{1}{k}\tilde{u}\tilde{u}'J\right)\left(\left(\begin{array}{c} p_i^{out} \\ -p_i^{in} \end{array}\right)+t_i-\varepsilon_i\left(\begin{array}{c} \mathbf{0} \\ u_i^{in} \end{array}\right)\right)$$

Now $J-\frac{1}{k}J\tilde{u}\tilde{u}'J$ is positive semidefinite. To see this, note that $x'(J-\frac{1}{k}J\tilde{u}\tilde{u}'J)x\geq 0$ if and only if $(\tilde{u}'J\tilde{u})(x'Jx)\geq(\tilde{u}'Jx)(x'J\tilde{u})$, which follows from Cauchy Schwartz inequality[20] Moreover canceling the first row and column yields a positive definite matrix, because in that case $1-\frac{1}{\tilde{u}J\tilde{u}'}\tilde{u}_{-1}J_{-1}\tilde{u}'_{-1}>0$. Inverting we get:

$$\left(\begin{array}{c} S_{i,-1} \\ D_i \end{array}\right)=\left(\left(J-\frac{1}{k_i}J\tilde{u}\tilde{u}'J\right)^{-1}_{-1}+C^{-1}_{i,-1}\right)^{-1}\times$$

$$\left(\left(J-\frac{1}{C}J\tilde{u}\tilde{u}'J\right)^{-1}_{-1}J_{-R_1}\left(I-\frac{1}{k}uu'J\right)_{C_1},I_{-1}\right)\left(\left(\begin{array}{c} p_i^{out} \\ -p_i^{in} \end{array}\right)+t_i-\varepsilon_i\left(\begin{array}{c} \mathbf{0} \\ u_i^{in} \end{array}\right)\right)$$

$$=\left(\left(J-\frac{1}{k}J\tilde{u}\tilde{u}'J\right)^{-1}_{-1}+C^{-1}_{i,-1}\right)^{-1}\left(J-\frac{1}{C}J\tilde{u}\tilde{u}'J\right)^{-1}_{-1}J_{-R_1}\left(I-\frac{1}{k}\tilde{u}\tilde{u}'J\right)_{C_1}p_{i,1}^{out}+$$

$$\left(\left(J-\frac{1}{C}J\tilde{u}\tilde{u}'J\right)^{-1}_{-1}+C^{-1}_{i,-1}\right)^{-1}\left(\left(\begin{array}{c} p_{i,-1}^{out} \\ -p_i^{in} \end{array}\right)+t_i-\varepsilon_i\left(\begin{array}{c} \mathbf{0}_{-1} \\ u_i^{in} \end{array}\right)\right)$$

Now the null space of $J-\frac{1}{\tilde{u}'J\tilde{u}}J\tilde{u}\tilde{u}'J$ is parallel to $\tilde{u}$, since $(J-\frac{1}{\tilde{u}'J\tilde{u}}J\tilde{u}\tilde{u}'J)u=J\tilde{u}-J\tilde{u}=0$. Hence we have that

$$J_{-R_1}\left(I-\frac{1}{k}\tilde{u}\tilde{u}'J\right)_{C_1}=-\sum_j J_{-R_1}\left(I-\frac{1}{k}\tilde{u}\tilde{u}'J\right)_{C_j}=-\left(J-\frac{1}{k}J\tilde{u}\tilde{u}'J\right)_{-1}\tilde{u}_{-1}$$

hence we get the final expression for supplies and demands:

$$\left(\begin{array}{c} S_{i,-1} \\ D_i \end{array}\right)=\left(\left(J_i-\frac{1}{k_i}J_i\tilde{u}_i\tilde{u}_i'J_i\right)^{-1}_{-1}+C^{-1}_{i,-1}\right)^{-1}\left(-p_i^{out}\tilde{u}_{-1}+\left(\begin{array}{c} p_{i,-1}^{out} \\ -p_i^{in} \end{array}\right)+t_i-\varepsilon_i\left(\begin{array}{c} \mathbf{0} \\ u_i^{in} \end{array}\right)\right)$$

Finally, note that in equilibrium

$$\tilde{B}_i\tilde{u}_i=(I_i+(J_i-\frac{1}{k_i}J_i\tilde{u}_i\tilde{u}_i'J_i)\tilde{C}_i)^{-1}\left(J_i-\frac{1}{k_i}J_i\tilde{u}_i\tilde{u}_i'J_i\right)\tilde{u}_i=0$$

hence $J_i-\frac{1}{k_i}J_i\tilde{u}_i\tilde{u}_i'J_i=(n_i-1)B_i+\overline{\Lambda}_i$

[20] Which holds even if matrices are not symmetric. To see this:

$$\left(x-\frac{x'Ju}{u'Ju}u\right)'J\left(x-\frac{x'Ju}{u'Ju}u\right)=x'Jx+\left(\frac{x'Ju}{u'Ju}\right)^2u'Ju-\frac{x'Ju}{u'Ju}(x'Ju+u'Jx)$$

$$=x'Jx-\frac{x'Ju}{u'Au}u'Ju$$

which is nonnegative if J is positive semidefinite.

Step b) - A profile of matrices satisfying 1.16 exists

Now I have to show that a non-trivial profile of matrices satisfying 1.16 exists, exhibiting sequences that converge to it.

Increasing best reply Assume B and B' are two profiles of schedules such that $B'_i > B_i$ in the Loewner (positive semidefinite) order for any i. The best reply is a function of B_i and $\overline{\Lambda}_i$ through a double inversion, so is increasing in both. Hence, to prove that the best reply is increasing in the positive semidefinite ordering we have to prove that $\overline{\Lambda}_i$ is increasing in the profile B.

In the notation of Proposition 2, $\hat{B}_i$ is increasing in the Loewner order. Indeed:

$$\hat{B}'_i = \begin{pmatrix} CS_i^{out} & CS_i^{in} \\ CD_i^{out} & CD_i^{in} \end{pmatrix} > \begin{pmatrix} BS_i^{out} & BS_i^{in} \\ BD_i^{out} & BD_i^{in} \end{pmatrix} = \hat{B}_i$$

if and only if

$$\begin{pmatrix} CS_i^{out} - BS_i^{out} & CS_i^{in} - BS_i^{in} \\ CD_i^{out} - BD_i^{out} & CD_i^{in} - BD_i^{in} \end{pmatrix} > 0$$

which is true if and only if

$$\begin{pmatrix} CS_i^{out} - BS_i^{out} & -(CS_i^{in} - BS_i^{in}) \\ -(CD_i^{out} - BD_i^{out}) & CD_i^{in} - BD_i^{in} \end{pmatrix} > 0$$

Since $\hat{B}_i$ is increasing, also the market clearing matrix M is increasing, because remember from 2 that $x'Mx = \sum_m x'_m \hat{B}_m x_m$.

Now $\hat{M}^{-1}$ is decreasing. Canceling rows and columns does not change the Loewner ordering, and so $\Lambda_i^{-1} = (\hat{M}^{-1})_i^{-1}$ is increasing.

Finally, I prove that $\overline{\Lambda}_i = \Lambda_i^{-1} - \frac{1}{k_i}\Lambda_i^{-1}\tilde{u}_i\tilde{u}'_i\Lambda_i^{-1}$ is increasing in Λ_i^{-1}. To see this, assume $J > K$. This is equivalent to $\|KJ^{-1}\|_2 < 1$[21]. Then:

$$\left\| \left(K - \frac{1}{u'Ku}Kuu'K\right)\left(J - \frac{1}{u'Ju}Juu'J\right)^{-1} \right\|_2 =$$

$$\left\| \left(I - \frac{1}{u'Ku}Kuu'\right)KJ^{-1}\left(I - \frac{1}{u'Ju}Juu'\right)^{-1} \right\|_2$$

$$\leq \|\left(I - \frac{1}{u'Ku}Kuu'\right)\|_2 \|_2 KJ^{-1}\|_2\|_2 \left(I - \frac{1}{u'Ju}uu'J\right)^{-1}\|_2$$

[21] Cfr. e.g. Horn and Johnson (2012)

Now $I - \frac{1}{u'Ku}Kuu'$ has one zero eigenvalue and all the others are 1[22], so $\|\left(I - \frac{1}{u'Ku}Kuu'\right)\|_2 = 1$ and similarly $\|\left(I - \frac{1}{u'Ju}uu'J\right)\|_2 = 1$. Finally, $\|KJ^{-1}\|_2 < 1$ by assumption, so it follows that

$$\left\|\left(K - \frac{1}{u'Ku}Kuu'K\right)\left(J - \frac{1}{u'Ju}Juu'J\right)^{-1}\right\|_2 < 1$$

so that $J - \frac{1}{u'Ju}Juu'J > K - \frac{1}{u'Ku}Kuu'K$ as I wanted to show. □

Convergence We are going to need the following lemma.

Lemma 1. If a sequence of *symmetric* matrices B_n is monotone in the positive semidefinite ordering, and bounded in the 2-norm, then it converges.

Proof. Consider the case that the sequence is decreasing, that is $B_n - B_{n+1}$ positive semidefinite. The increasing case is analogous. Assume by contraposition that it does not converge. Then since it is bounded, by compactness there exists a converging subsequence B_{n_k}. Then in particular this sequence is also Cauchy, so:

$$\forall\varepsilon\exists K_0 : \, k_1, k_2 > K_0 \Rightarrow \|B_{n_{k_1}} - B_{n_{k_2}}\|_2 < \varepsilon$$

But then for any $n, m > n_{K_0}$ by the fact that the sequence is decreasing we can find k_1, k_2 such that $B_{n_{k_1}} > B_n > B_m > B_{n_{k_2}}$. Now we can write:

$$B_{n_{k_1}} - B_{n_{k_2}} = B_{n_{k_1}} - B_n + B_n - B_m + B_m - B_{n_{k_2}}$$

and we know that $B_{n_{k_1}} - B_n + B_m - B_{n_{k_2}}$ is positive definite, hence the maximum eigenvalue of the right hand side must be larger than the maximum eigenvalue of $B_n - B_m$. But the maximum eigenvalue is the norm, so $\|B_n - B_m\|_2 \leq \|B_{n_{k_1}} - B_{n_{k_2}}\|_2$ which proves that the whole sequence is Cauchy and so converges.

□

Define:

$$BR_{i,n+1} = ([C_i^{-1}]_{-1} + ((n_i - 1)BR_{i,n} + [\overline{\Lambda}_i]_{-1})^{-1})^{-1}$$

I will prove that the sequnce $(BR_{i,n})_n$ with the proper initial conditions constitute a decreasing sequence in the positive semidefinite ordering. From this, the fact that it is bounded as proven in the existence theorem, and the previous lemma, it follows that they converge.

[22] Summing by the identity matrix results in all eigenvalues being shifted by 1, and $\frac{1}{u'Ku}Kuu'$ has rank 1 with eigenvalue 1, realized by eigenvector Ku.

From above Set $BR_{i,0} = C_{i,-1}$. I prove that $BR_{i,0} - BR_{i,1} = C_{i,-1}(\overline{\Lambda}_i + 2C_{i,-1})^{-1}C_{i,-1}$ and so is positive definite.

$$C_i - BR_{i,1} = C_{i,-1} - ((C_{i,-1} + \overline{\Lambda}_i)^{-1} + C_{i,-1}^{-1})^{-1} = C_{i,-1} - (C_{i,-1} - C_{i,-1}(C_{i,-1} + \overline{\Lambda}_i + C_{i,-1})^{-1}C_{i,-1})$$

$$= C_{i,-1}(2C_{i,-1} + \overline{\Lambda}_i)^{-1}C_{i,-1}$$

where the last but one step is by Woodbury formula. The matrix on the right hand side is positive definite because $(2C_i + \overline{\Lambda}_i)^{-1}$ is.

But then, since the best reply map is increasing when all matrices are symmetric, it follows that $B_{i,n} > B_{i,n+1}$ for each n, so the sequence is decreasing, which is what we wanted to show.

From below Now I prove that if $\tilde{B}_i$ has norm small enough, then $BR_i > \tilde{B}_i$. From this, and the fact that the best reply is increasing will follow convergence from below. Indeed:

$$BR_i > \tilde{B}_i \Leftrightarrow \|\tilde{B}_i BR_i^{-1}\|_2 < 1$$

and

$$\|\tilde{B}_i BR_i^{-1}\|_2 = \|\tilde{B}_i \left(C_{i,-1}^{-1} + (\overline{\Lambda}_i + (n_i - 1)\tilde{B}_i)^{-1} \right)\|_2 =$$

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$$\|\tilde{B}_i C_{i,-1}^{-1} + (\overline{\Lambda}_i \tilde{B}_i^{-1} + (n_i - 1)I)^{-1}\|_2 \leq \|\tilde{B}_i\|_2 \|(C_{i,-1}^{-1}\|_2 + \|(\overline{\Lambda}_i \tilde{B}_i^{-1} + (n_i - 1)I)^{-1}\|_2$$

Moreover:

$$\|(\overline{\Lambda}_i \tilde{B}_i^{-1} + (n_i - 1)I)^{-1}\|_2 = \frac{1}{\lambda_n((\overline{\Lambda}_i \tilde{B}_i^{-1} + (n_i - 1)I))} = \frac{1}{\lambda_n(\overline{\Lambda}_i \tilde{B}_i^{-1}) + (n_i - 1)I} < \frac{1}{n_i - 1}$$

where λ_n is the minimum eigenvalue of $(\overline{\Lambda}_i \tilde{B}_i^{-1} + (n_i - 1)I)$ and it is positive because it is positive definite.

Now if $\|\tilde{B}_i\|_2$ is smaller than $\frac{n_i - 1}{n_i}\|C_i^{-1}\|_2^{-1}$ it follows:

$$\|\tilde{B}_i BR_i^{-1}\|_2 < 1$$

which is what we wanted to show.

There exist a profile of coefficients implying positive trade The previous paragraph prove that a profile of matrix coefficients satisfying 1.16 esits. Now I prove that there exist one that yields positive trade if we limit ourselves to a subset of links - that will be the active links in equilibrium.

Start from the original network $G = (N, E)$. Set $n = 0$ and $L_1 = E$.

1. Find the unconstrained equilibrium profile B^*_n in the network $G_i = (N, L_n)$. Identify the set of links that have negative trade or negative price $E_{n,0}$.
2. set $L_{n+1} = L_n/E_{n,0}$;

The set of links shrink at each step, and when the network is empty there are no negative trades. Hence there must exist an index $\hat{\imath}$ such that for all $i > \hat{\imath}$ $L_i = L_{\hat{\imath}}$. The equilibrium $B^*_{\hat{\imath}}$, augmented with identically zero functions for all excluded links, is an equilibrium of the original game.

Generic Equilibrium existence It remains to prove that the profile of matrices $(B^*_i)_i$ identified above constitute the coefficient matrices of a profile of linear schedules for an open set $\mathcal{P} \times \mathcal{E}$ that contains $(p^*(0), 0)$. To prove this, consider the linear functions defined by $(B^*_i)_i$ and extend them to the whole price space. That is consider:

$$(S_{-1}, D)_i = \tilde{B}_i(-p_1\tilde{u}_{-1} + p_{i,-1} + t_i) + \varepsilon_i B_{\varepsilon,i,-1}$$

where t_i solves the Linear Complementarity problem:

$$\tilde{B}_i(-p_1\tilde{u}_{-1}+p_{i,-1}+t_i)+\varepsilon_i B_{\varepsilon,i,-1} \geq 0 \quad t'_{i,-1}(\tilde{B}_i(-p_1\tilde{u}_{-1}+p_{i,-1}+t_i)+\varepsilon_i B_{\varepsilon,i}) = 0 \quad t_i \geq 0$$

This corresponds to the form of the solution of the Optimization 1.41, where t_i is a function of the Lagrange multipliers on the nonnegativity constraints. Concavity proves that the solution is unique and so non-ambiguous.

Using this form we see that the market clearing conditions can be written as a Linear Complementarity Problem:

$$B_{ij}(p_i + t_i) + \varepsilon_i B_{\varepsilon,i} = B_{ij}(p_j + t_j) + \varepsilon_i B_{\varepsilon,j} \tag{1.45}$$
$$B_i(p_i + t_i) + \varepsilon_i B_\varepsilon \geq 0 \tag{1.46}$$
$$t'_i(B_i(p_i + t_i) + \varepsilon_i B_\varepsilon) = 0 \tag{1.47}$$
$$t_i \geq 0 \tag{1.48}$$

The first set of equations can be rewritten as $M(p + t) = \boldsymbol{A} + M_\varepsilon \varepsilon$ and solved for $p + t$ since M is invertible. So to compute which t variables are not zero it is sufficient to use the complementary slackness condition. Moreover, it is a standard result (Cottle et al. (2009), Proposition 1.4.6) that the solution as a function of ε is piecewise linear.

Now the fact that we can express the residual demand as a linear function for all i relies on the fact that $(0, p^*(0))$ lies in one of the regions where the function is linear and not on one of the boundary regions. Now the boundary regions are identified by a set of equations $F_j((B_i)_i, \varepsilon) = 0$ for some indices j, where the F are analytic functions (see Cottle et al. (2009), Prop. 1.4.6.). This means that the set of profiles of coefficients such that 0 is in one of the boundary regions:

$$\mathcal{B}_F = \{(B_i)_i \mid F_j((B_i)_i, 0) = 0\}$$

is rare. This follows from the fact that if there were an open set in $\mathcal{B}_F$ then since F is analytic it would have be identically zero, which it is not. Moreover $\mathcal{B}_F$ is closed, hence equal to its closure: hence its closure has empty interior, so it is rare.

Now consider the map $\mathcal{O} : (\omega_i)_i \to (B_i^*)$ that maps the values of the parameters to the B_i^* that solve 1.16. I prove that this is one-to-one. To see this, suppose $\mathcal{O}((\omega_i)_i) = \mathcal{O}((\omega_i')_i)$. Then by the construction of 2 we get that $\Lambda_i((\omega_i)_i) = \Lambda_i((\omega_i')_i)$, and by the equation 1.16 we get that the perfect competition matrices must agree too: $(C_i)_i = (C_i')_i$. From this, inspecting the matrix, it follows that $(\omega_i)_i = (\omega_i')_i$. Moreover it is continuous (actually analytic).

Since $\mathcal{O}$ si a homeomorphism the preimage of a rare set is rare, and so we conclude that the property of existence of a linear eqilibrium is generic in $(\omega_i)_i$.

1.B.2 Proof of Corollary 1.3.1

The fixed point equation 1.16 can be rewritten as:

$$(n_i - 1)\tilde{B}_i C_i^{-1} \tilde{B}_i + (\overline{\Lambda}_i C_i^{-1} + (n_i - 2)I)\tilde{B}_i - \overline{\Lambda}_i = 0 \tag{1.49}$$

and premultiplying by C_i^{-1}:

$$(n_i - 1)(C_i^{-1}\tilde{B}_i)^2 + (C_i^{-1}\overline{\Lambda}_i + (n_i - 2)I)(C_i^{-1}\tilde{B}_i) - C_i^{-1}\overline{\Lambda}_i = 0$$

Call $X = C_i^{-1}\tilde{B}_i$, $b = \frac{1}{n_i - 1}(C_i^{-1}\overline{\Lambda}_i + (n_i - 2)I)$ and $c = -\frac{1}{n_i-1}C_i^{-1}\overline{\Lambda}_i$. We can rewrite this as:

$$b^2 - 4c = 4X^2 + 4bX + 4b^2$$

Now note that any solution of 1.49 commutes with b, because taking the transpose of the equation we get that X must solve also $X^2 + Xb + c = 0$, and so $-(X^2 + c) = bX = Xb$. Then the right hand side above is a square, and we have the analogous of the classical quadratic formula:

$$X = \frac{1}{2}\left(-b + \sqrt{b^2 - 4c}\right)$$

and so:

$$B_i = \frac{1}{2} C_i \left(-b + \sqrt{b^2 - 4c} \right)$$

Now $b^2 - 4c$ is the sum of two symmetric positive definite matrices, so is symmetric positive definite, and hence has a unique positive definite square root (Horn and Johnson (2012), Theorem 7.2.6). Hence the equation 1.49 has a unique positive definite solution, so the sector-level symmetric equilibrium is unique. □

1.B.3 Proof of Corollary 1.3.2

I omit the index i because all matrices are relative to sector i.

The quadratic labor cost of the profit is $\sum_{k,j} (\omega_{ij} \lambda_{ki} - \mu_{ij})^2$. This can be written in matrix form as $\begin{pmatrix} \lambda', & -\mu' \end{pmatrix} U'U \begin{pmatrix} \lambda \\ -\mu \end{pmatrix}$ where $U = [I_{out} \otimes \omega_i, u_{out} \otimes I_{in}]$.

Moreover $\begin{pmatrix} \lambda \\ -\mu \end{pmatrix} = V \begin{pmatrix} p_{out} \\ -p_{in} \end{pmatrix}$. and:

$$\pi = p' \left(B - \frac{1}{2} V'U'UV \right) = p' \left(B - \frac{1}{2} V'CV \right)$$

and:

$$V'CV = B \begin{pmatrix} 1 & \mathbf{0} \\ \tilde{u}_{-1} & C_{-1}^{-1} \end{pmatrix} C \begin{pmatrix} 1 & \tilde{u}'_{-1} \\ \mathbf{0} & C_{-1}^{-1} \end{pmatrix} B$$

since $C\tilde{u} = 0$ and $\tilde{u}'C = 0$. Moreover:

$$B \begin{pmatrix} 1 & \mathbf{0} \\ \tilde{u}_{-1} & C_{-1}^{-1} \end{pmatrix} C \begin{pmatrix} 1 & \tilde{u}'_{-1} \\ \mathbf{0} & C_{-1}^{-1} \end{pmatrix} B = \begin{pmatrix} \tilde{u}'_{-1} B_{-1} C_{-1}^{-1} B_{-1} \tilde{u}_{-1} & \tilde{u}'_{-1} B_{-1} C_{-1}^{-1} B_{-1} \\ B_{-1} C_{-1}^{-1} B_{-1} \tilde{u}_{-1} & B_{-1} C_{-1}^{-1} B_{-1} \end{pmatrix}$$

Now $B - V'CV$ has $\tilde{u}$ in the null space, and is positive semidefinite if and only if $B_{-1} - \frac{1}{2} B_{-1} C_{-1}^{-1} B_{-1}$ is. This is true because:

$$\|B_{-1} C_{-1}^{-1} B_{-1} B_{-1}^{-1}\|_2 = \|B_{-1} C_{-1}^{-1}\|_2 < 1$$

because we know that $B_{-1} < C_{-1}$.

1.B.4 Proof of Proposition 3

I am going to prove that, in any equilibrium, B has the following form: it is equal to $P\mathcal{B}P$, where $\mathcal{B}$ is an M-matrix (a positive definite matrix with positive diagonal and nonpositive off-diagonal entries), and:

$$P = \begin{pmatrix} I & \mathbf{0} \\ \mathbf{0} & -I \end{pmatrix}$$

is a matrix that changes signs to the off-diagonal blocks of $\mathcal{B}$.

From the definition of M in 2 it is immediate to see that, if B has the property above, then M is an M-matrix.

First, I prove that the best reply to a profile of coefficients $(B_i)_{i\in I}$ that has the property above has still the property above.

From this, it follows that M is an M-matrix. Then, by 2 we know that $\Lambda^{-1} = PLP$, where L is an M-matrix.[23] Moreover, in equilibrium, since $B\tilde{u} = 0$, we have $\Lambda^{-1}\tilde{u} = -\tilde{\boldsymbol{A}}_i \geq 0$. This is equivalent to $LP\tilde{u} = Lu > 0$, once we define $u = P\tilde{u}$. Then, we have that the matrix:

$$L - \frac{1}{u'Lu} Luu'L$$

is positive semidefinite and still an M-matrix. Then it follows that also $\overline{\Lambda}^{-1}$ has the form $\overline{\Lambda}^{-1} = P\overline{L}P$ for an M-matrix $\overline{L}$, because:

$$\begin{aligned}
\overline{\Lambda}^{-1} =& \Lambda^{-1} - \frac{1}{\tilde{u}'\Lambda^{-1}\tilde{u}} \Lambda^{-1}\tilde{u}\tilde{u}'\Lambda^{-1} \\
=& P(L - \frac{1}{u'Lu} Luu'L)P = P\overline{L}P
\end{aligned}$$

(to get the expression, note that $P^2 = I$.)

Now, also the perfect competition matrix C has the same property: $C = P\mathcal{C}P$. Then, from the best reply equation we get:

$$(I+(\overline{\Lambda}^{-1}+(n-1)B)^{-1}C)^{-1}(\overline{\Lambda}^{-1}+(n-1)B) = P(I+(\overline{L}+(n-1)\mathcal{B})^{-1}\mathcal{C})^{-1}(\overline{L}+(n-1)\mathcal{B})P$$

so that the best reply preserves the property.

To prove that any equilibrium profile has this property, proceed similarly to the proof of Theorem 1, step d). That is define $\tilde{B}_0$ as C, and consider the iteration:

$$\tilde{B}_n = (C_{-1} + ((\overline{\Lambda}_i^{-1})_{-1} + \tilde{B}_{n-1})^{-1})^{-1}$$

Notice that differently from Theorem 1 here $\overline{\Lambda}_i^{-1}$ is kept fixed. By an analogous argument this sequence is increasing and converges to the solution of the best reply equation, which is unique by Corollary 1.3.2. Moreover, each matrix of the sequence has the desired form, hence also the limit has. This is true because weak inequalities are preserved in the limit, and we already know that the limit is positive definite so it must have strictly positive diagonal. Hence it follows that any solution of the best reply equation must have the desired property.

□

[23] This follows because the proof in 2 shows that L is the Schur complement of an M-matrix, which is itself an M-matrix (see Horn et al. (1994)).

1.C Proofs of Section 1.4

1.C.1 Calculations for Example 7

Solving the system we get that:

$$p_0 = -\frac{1 - A_c}{B_c}$$
$$p_1 = -\frac{\delta(1 - A_c) - B_c(1 - \delta)}{B_c}$$

Setting $B_c = 1$ for simplicity we get:

$$\mu_0 - \mu_1 = \frac{(A_c - 2)(\delta((A_c - 2)\delta + 2) - 1)}{(A_c - 1)((A_c - 2)\delta + 1)}$$

and for $A_c = 3$ we get the analytical solution of:

$$\mu_0 > \mu_1 \Leftrightarrow \delta < -1 + \sqrt{2} = \overline{\delta}$$

1.C.2 Proof of Proposition 4

I prove the result for a supply chain (line network) of length K. Denote the production function as f and the inverse demand at stage i of the chain as $P_i(\cdot)$. Assume both are differentiable and concave, $f' > 0$ and $P_i' < 0$. For every step of the chain but the first we have that firms optimize:

$$P_i(Q_i)q_{i\alpha} - p_{i-1}f^{-1}(q_{i\alpha})$$

where $Q_i = \sum_\alpha q_{i\alpha}$. By concavity they do so through the first order conditions[24]:

$$P_i'(Q_i)q_{i\alpha} + P_i(Q_i) - \frac{p_{i-1}}{f'(f^{-1}(q_{i\alpha}))} = 0$$

so in the symmetric equilibrium:

$$P_i'(Q_i)\frac{Q_i}{n_i} + P_i(Q_i) - \frac{p_{i-1}}{f'(f^{-1}\left(\frac{Q_i}{n_i}\right)} = 0$$

[24]The second derivative of the profit function is:

$$P_i''(Q_i)q_{i\alpha} + 2P_i'(Q_i) + \frac{p_{i-1}f''(f^{-1}(q_{i\alpha}))}{f'(f^{-1}(q_{i\alpha}))}\frac{p_{i-1}}{(f'(f^{-1}(q_{i\alpha})))^2}$$

By concavity of P_i and f this is negative.

and the markup is determined by the usual elasticity condition:

$$\frac{p_i - MC_i}{p_i} = -\frac{P_i'(Q_i)\frac{Q_i}{n_i}}{P_i(Q_i)}$$

The equation allows to write directly the inverse demand that sector $i-1$ is facing (using the market clearing $Q_i = n_i f\left(\frac{Q_{i-1}}{n_i}\right)$) [25]:

$$P_{i-1}(Q_{i-1}) = \left[P_i'\left(n_i f\left(\frac{Q_{i-1}}{n_i}\right)\right) f\left(\frac{Q_{i-1}}{n_i}\right) + P_i\left(n_i f\left(\frac{Q_{i-1}}{n_i}\right)\right)\right] f'\left(\frac{Q_{i-1}}{n_i}\right)$$

To compare the elasticities, first calculate the derivative of this:

$$\begin{aligned}P_{i-1}'(Q_{i-1}) = &\left[P_i''\left(n_i f\left(\frac{Q_{i-1}}{n_i}\right)\right) f'\left(\frac{Q_{i-1}}{n_i}\right) f\left(\frac{Q_{i-1}}{n_i}\right)\right.\\ &\left.+\left(1+\frac{1}{n_i}\right) f'\left(\frac{Q_{i-1}}{n_i}\right) P_i'\left(n_i f\left(\frac{Q_{i-1}}{n_i}\right)\right)\right] f'\left(\frac{Q_{i-1}}{n_i}\right) + \\ &\left[P_i'\left(n_i f\left(\frac{Q_{i-1}}{n_i}\right)\right) f\left(\frac{Q_{i-1}}{n_i}\right) + P_i\left(n_i f\left(\frac{Q_{i-1}}{n_i}\right)\right)\right]\frac{1}{n_i} f''\left(\frac{Q_{i-1}}{n_i}\right)\end{aligned}$$

By concavity, the first and last terms are negative, so we conclude:

$$P_{i-1}'(Q_i) < P_i'(f')^2\left(1+\frac{1}{n_i}\right)$$

so

$$\begin{aligned}\frac{P_{i-1}'}{P_{i-1}}\frac{Q_{i-1}}{n_{i-1}} &< \frac{P_i'(f')^2\left(1+\frac{1}{n_i}\right)}{(P_i+P_i'f)f'}\frac{Q_{i-1}}{n_{i-1}}\\ &= \frac{P_i'f'\left(1+\frac{1}{n_i}\right)}{P_i+P_i'f}\frac{n_i}{n_{i-1}}\frac{Q_{i-1}}{n_i}\end{aligned}$$

moreover, we have that:

$$\frac{P_i'f'\left(1+\frac{1}{n_i}\right)}{P_i+P_i'f}\frac{n_i}{n_{i-1}}\frac{Q_{i-1}}{n_i} < \frac{P_i'}{P_i}\frac{Q_i}{n_i}$$

if and only if:

$$P_iP_i'f'\frac{Q_{i-1}}{n_i}\left(1+\frac{1}{n_i}\right)\frac{n_i}{n_{i-1}} < P_iP_i'\frac{Q_i}{n_i} + (P_i')^2\frac{Q_i}{n_i}\frac{Q_{i-1}}{n_i}$$

[25] Differentiating this expression we immediately get that it is decreasing in Q_i.

$$P_i P_i' \left(f' \frac{Q_{i-1}}{n_i} \left(1 + \frac{1}{n_i} \right) \frac{n_i}{n_{i-1}} - \frac{Q_i}{n_i} \right) < (P_i' f)^2$$

Now if n_i and n_{i-1} are sufficiently close the parenthesis is positive by concavity of f (which implies $f' \frac{Q_{i-1}}{n_i} > \frac{Q_i}{n_i}$), hence the inequality is always satisfied. In particular this is true if $n_i = n_{i-1}$. We can conclude that in equilibrium if n_i and n_{i-1} are sufficiently close:

$$\frac{P_i'}{P_i} \frac{Q_i}{n_i} < \frac{P_{i-1}'}{P_{i-1}} \frac{Q_{i-1}}{n_{i-1}}$$

and so firms in sector $i-1$ have larger markup than firms in sector i.

For the case of markdowns, the exact analogous calculations hold, on supply rather than demand functions.

1.C.3 Proof of Propositions 5 and 6

The proofs follow from the following lemmas.

Lemma 2. The profile $\mathcal{B} = ((\frac{D_i S_i}{D_i + S_i})_{i \geq 2}, D_1)$ is a symmetric function of the "sector level" coefficients $(n_i B_i)$. That is $\mathcal{B}((n_i B_i)_i) = \mathcal{B}(n_{\pi(i)} B_{\pi(i)})_i$ where π is any permutation of indices.

Proof. By induction, I prove that $\mathcal{B}$ is equal to:

$$\frac{D_i^r S_i^r}{D_i^r + S_i^r} = \frac{\prod_{k \neq i} n_k B_k B_c}{\prod_{k \neq i} n_k B_k + B_c \sum_{j \neq i} \prod_{k \neq i, k \neq j} n_k B_k} \tag{1.50}$$

$$D_1 = \frac{\prod_{k \neq 1} n_k B_k B_c}{\prod_{k \neq 1} n_k B_k + B_c \sum_{j \neq 1} \prod_{k \neq 1, k \neq j} n_k B_k} \tag{1.51}$$

By induction on the size of the line N. If $N = 2$ it can be checked by calculation. Assume it holds for a line of size $N-1$. To get the corresponding expressions for a line of size N we must substitute B_c with the objective demand of the last but one layer, which is $\frac{n_N B_N B_c}{n_N B_N + B_c}$. If we do it we get that for $i \leq N-1$:

$$\frac{D_i S_i}{D_i + S_i} = \frac{\prod_{k \neq i} n_k B_k \frac{n_N B_N B_c}{n_N B_N + B_c}}{\prod_{k \neq i} n_k B_k + \frac{n_N B_N B_c}{n_N B_N + B_c} \sum_{j \neq i} \prod_{k \neq i, k \neq j} n_k B_k}$$

and reordering and simplifying the denominator we get the expression above. Analogously can be done for D_1. Moreover, always by induction we can find:

$$S_N = \frac{\prod_{k \neq N} n_k B_k}{\prod_{k \neq N} n_k B_k}$$

and $D_N = B_c$, so substituting in the corresponding expression:

$$\frac{D_N S_N}{D_N + S_N} = \frac{\frac{\prod_{k\neq N} n_k B_k}{\prod_{k\neq N} n_k B_k} B_c}{B_c + \frac{\prod_{k\neq N} n_k B_k}{\prod_{k\neq N} n_k B_k}}$$

and simplifying we get the desired result.

□

Lemma 3. In equilibrium $n_i > n_j$ implies $B_i^* > B_j^*$.

Proof. To apply the theory of monotone comparative statics, I will prove that if $n_i \geq n_j$ then $BR_i(x, B_{-i,j}) \geq BR_j(x, B_{-i,j})$, that is the best reply of i dominates the best reply of j conditional on the coefficients of all other sectors.

We have that $BR_i \geq BR_j$ if and only if:

$$\overline{\Lambda}_i^{-1} + (n_i - 1)x \geq \overline{\Lambda}_j^{-1} + (n_j - 1)x$$

In particular, using the characterization of $\overline{\Lambda}_i^{-1}$ above, we have that this is true if and only if:

$$\frac{\mathcal{B} B_c n_j x}{\mathcal{B}(n_j x + b_C) + n_j x \mathcal{F}} + (n_i - 1)x \geq \frac{\mathcal{B} B_c n_i x}{\mathcal{B}(n_i x + b_C) + n_i x \mathcal{F}} + (n_j - 1)x$$

where $\mathcal{B}$ and $\mathcal{F}$ are only functions of the coefficients $B_{-i,j}$ and their respective number of firms. This is true if and only if

$$\frac{\mathcal{B} B_c n_j}{\mathcal{B}(n_j x + b_C) + n_j x \mathcal{F}} - (n_j - 1) \geq \frac{\mathcal{B} B_c n_i}{\mathcal{B}(n_i x + b_C) + n_i x \mathcal{F}} - (n_i - 1)$$

$$\frac{\mathcal{B} B_c n_j - (n_j - 1)(\mathcal{B}(n_j x + B_C) + n_j x \mathcal{F})}{\mathcal{B}(n_j x + B_C) + n_j x \mathcal{F}} \geq \frac{\mathcal{B} B_c n_i - (n_i - 1)(\mathcal{B}(n_j x + B_C) + n_j x \mathcal{F})}{\mathcal{B}(n_i x + B_C) + n_i x \mathcal{F}}$$

$$\frac{\mathcal{B} B_c - (n_j - 1)(\mathcal{B}(n_j x) + n_j x \mathcal{F})}{\mathcal{B}(n_j x + b_C) + n_j x \mathcal{F}} \geq \frac{\mathcal{B} B_c - (n_i - 1)(\mathcal{B}(n_j x) + n_j x \mathcal{F})}{\mathcal{B}(n_i x + B_C) + n_i x \mathcal{F}}$$

which is true if and only if $n_i \geq n_j$ because the function is decreasing.

Then we can conclude that if $n_i \geq n_j$ then $BR_i(x, B_{-i,j}) \geq BR_j(x, B_{-i,j})$, and so, using a result from Lazzati (2013) we can conclude that in equilibrium $B_i^* \geq B_j^*$.

□

Proof of Proposition 5 Calculations reveal that:

$$M_i = p_i - \lambda_i = \frac{S_i^r}{(D_i^r + S_i^r)(1 + B_i) + S_i^r D_i^r}(p_i - p_{i-1})$$

$$= \frac{\frac{S_i^r}{D_i^r + S_i^r}}{(1 + B_i) + \frac{D_i^r S_i^r}{D_i^r + S_i^r}}(p_i - p_{i-1})$$

$$m_i = \mu_i - p_{i-1} = \frac{\frac{D_i^r}{D_i^r + S_i^r}}{(1 + B_i) + \frac{D_i^r S_i^r}{D_i^r + S_i^r}}(p_i - p_{i-1})$$

Now by the previous lemma $B_i = B_j$ for all sectors and so market clearing conditions imply that $p_i - p_{i-1}$ is constant across sectors. Moreover by lemma 2 also $\frac{D_i^r S_i^r}{D_i^r + S_i^r}$. Now inspecting the right hand side of the expressions we see that the markup is decreasing with D_i^r, which is itself decreasing as one goes upstream. Then it follows that the markup is increasing going upstream, and symmetrically for the markdown.

Proof of Proposition 6 We can rewrite the profits as:

$$\pi_i = \frac{1 - \frac{1}{2}B_i}{n_i^2 B_i} c^2$$

where c is the quantity consumed by the consumer. Now by lemma 3 we can conclude.

1.D Proofs of Section 1.5

1.D.1 Proof of Theorem 2

I am going to prove that the best reply function is increasing in the parameters n_i. By monotone comparative statics this implies that in the maximal equilibrium coefficients B are larger, which means that price impacts are smaller.

First, note that M is increasing in any n_i. Indeed, if $n' \geq n$:

$$M(n_i') - M(n_i) = \sum_m x_m'(n_m' - n_m)\hat{B}_m x_m \geq 0$$

Then by the calculations in the proof of 1 it means that the best reply function is increasing in n, which is what we wanted to show.

Now assume that the consumer only buys one "final" good. The vector c has a nonzero entry only in correspondence of the consumer price. This means that $A_c p_c = c'p = c'M^{-1}c$. Since M is increasing in each B_i is also decreasing in each Λ_i, so it follows that the consumer price is increasing in Λ_i. $\square$

1.E Proofs of Section 1.6

1.E.1 Proof of Theorem 3

The best reply matrix for each firm in sector i is:

$$\left([C_i^{-1}]_{-1} + ((n_i - 1)\tilde{B}_i + \overline{B}_i)\right)^{-1} \tag{1.52}$$

where $\overline{B}_i$ is the diagonal matrix that on the diagonal has the coefficient $B_{k,ii}$ for all the neighbors k of i.

Let us define two functions:

$$BR_i(\tilde{B}_{-i}, g) = \left([C_i^{-1}]_{-1} + ((n_i - 1)\tilde{B}_i + \overline{\Lambda}_i)\right)^{-1} \tag{1.53}$$

$$BR_i(\tilde{B}_{-i}, l) = \left([C_i^{-1}]_{-1} + ((n_i - 1)\tilde{B}_i + \overline{B}_i)\right)^{-1} \tag{1.54}$$

$$\tag{1.55}$$

The equilibrium profiles of matrix coefficients in the local or global equilibrium satisfie:

$$\tilde{B}_i^g = BR_i(\tilde{B}_{-i}^g, g) \tag{1.56}$$

$$\tilde{B}_i^l = BR_i(\tilde{B}_{-i}^l, l) \tag{1.57}$$

Now to apply the theory of monotone comparative statics, let us think about the function $BR_i(\cdot, \cdot)$ where the second argument belongs to the space $\{g, l\}$, and consider on this space the ordering such that $g \succ l$. Then the best reply equation is increasing in this parameter. This is because as calculated in 2 $\Lambda_i^{-1} = \overline{B}_i - M_{R_i} M_{-i}^{-1} M'_{R_i}$ and M_{-i}^{-1} is positive definite. Hence $\Lambda_i^{-1} \geq \overline{B}_i$.

By standard arguments now we can conclude that in the maximal equilibrium $\tilde{B}_i^g < \tilde{B}_i^l$

Chapter 2

Dynamic diffusion in production networks

I analyze the dynamics of shocks propagating in a production network, using a variation of the classical Long Jr and Plosser (1983) DSGE. The key feature that allows the dynamic analysis is time to build, which implies that sector purchases and sales react with one period lag to a shock. I study the properties of the transition to the new steady state (or the old, if the shock is temporary). Contrary to the static versions of the model, preference shocks diffuse downstream, similar to productivity shocks. Moreover, even for productivity shocks the ranking of influence of sectors is different, weighing less longer paths, and comovement is smaller. Finally, I provide bounds on the recovery time of the economy hit by a shock.[1]

[1]I wish to thank Fernando Vega-Redondo and Basile Grassi for their comments and advice. I also wish to thank the participants to the 4th NSF Conference on Network Science in Economics at Vanderbilt University, the 7th Annual Workshop on Networks in Economics and Finance at the IMT School for Advanced Studies in Lucca, and seminar participants in Bocconi.

2.1 Introduction

How do shocks to some economic sectors impact the rest of the economy? For many years a widespread view, exemplified by Lucas (1995)' argument, has been that when considering whole economies composed of a large number of agents, idiosyncratic shocks should average out and not have a sizable aggregate impact. Recently, this view has been challenged, noting that the averaging out might not happen if the connections between sectors are sufficiently asymmetric, so that the very well connected sectors will have a sizable impact on aggregate output, as argued in the seminal paper Acemoglu et al. (2012). The understanding of such mechanisms is of crucial importance to understand business cycles and to evaluate and design policies directed to smooth or insure against shocks, such as bailouts or monetary policy.

A growing literature has indeed provided empirical grounding for the importance of idiosyncratic shocks in shaping aggregate outcomes.[2] Yet, most of the analyses have focused on static general equilibrium models or on steady states of the dynamics.[3] While this has certainly allowed many useful insights, production is essentially a dynamic phenomenon, as testified by the sizable literature that studies *time to build* in its own right.[4] It is, therefore, to be expected that the temporal dimension of the propagation of shocks contains many important features that a static analysis would miss. Some are classical questions pertaining to dynamic environments, such as what is the *persistence* of a shock, other are more specific to an input-output level analysis: which sectors are more affected by the shock in the short run rather than the long run? which sectors generate more short than long run impacts on the welfare of the consumers? All these questions simply can't be answered in a static model.[5]

In this work, I want to address these issues, analyzing a model that generates a *dynamic* diffusion, namely a propagation of shocks *over time* as well as over sectors. To generate a dynamic diffusion while keeping analytical tractability, I will follow the original input-output model by Long Jr and Plosser (1983) and in particular assume that the production of any good necessitates 1 period of time. This implies that the reaction of each sector to shocks will be lagged and diffusion will not be instantaneous: a shock to

[2]See section 2.2.

[3]There are exceptions, in particular Pasten et al. (2018), as explained in the literature section.

[4]A classic contribution is Kydland and Prescott (1982), while more recently Meier (2017).

[5]Note that, perhaps not surprisingly, also persistence will depend crucially on network characteristics in this setting.

a sector will trigger a reaction from the immediate neighbors, but in general not from the others. This will generate a dynamic diffusion of the impact, that will take time to spread to the whole economy, allowing us to analyze it in details.

There are two perspectives from which we can analyze such an environment: focusing on the properties of the stationary stochastic process generated by the uninterrupted random disruptions that hit the economy, or analyzing the impact of a single shock and the properties of the *transition* to the (possibly new) steady state - the *impulse response function*.[6]

My results show that the properties of a dynamic diffusion can depart substantially from a static benchmark, even in simple Cobb Douglas environments: productivity shocks propagate exclusively downstream, and an *unexpected* productivity shock has a cumulative welfare impact which is proportional to a *dynamic* version of Bonacich centrality, that takes into account the different value of consumption over time. Moreover, the cumulative impact is equal to the share of sales of the respective sector, its *Domar weight*. This is an analogous of Hulten (1978) theorem, stating that in an efficient economy the first order contribution of a small shock to a sector to aggregate GDP is exactly its sales share. The result, though, is not obvious: here I am considering an *unexpected* shock, which a priori needs not behave as Hulten theorem predicts.

Preference shocks, instead, have a radically different propagation behavior, that can be summarized as such: their *physical* impact propagates downstream, while the *information* impact propagates upstream. By physical impact I mean the impact working through the physical decrease in real output, that through a change in prices causes the customers to vary their purchases and so their production. The information impact is the update in expectations of future demand changes due to autocorrelations in preference shocks over time. The comparative difference in preference shocks with respect to productivity is a by-product of the Cobb-Douglas technology, that implies that productivity shocks do not have nominal effects, and I do not expect it to generalize.

Moreover, thanks to the linear nature of the problem, the dynamics is very close to the iteration of a Markov chain. So we can apply the ergodic theory of Markov chains to provide upper bounds on the time that the economy takes to recover from a negative shock (or to scale down from a positive one), in a spirit similar to Golub and Jackson (2012). These bounds depend

[6]Note that the transition I am referring to is always *along an equilibrium path*, although the analysis of out of equilibrium responses to disruptions is of great interest, and I am addressing it in ongoing work.

crucially on the network characteristics, such as (eigenvector) centrality, the labor share of technology, and community structure.

Finally, I show that the dynamic model generates systematically less comovement, measured as lag 0 autocorrelation. This happens because shocks take time to affect other sectors, so the effect can hit different sectors at lagged times, not generating contemporaneous comovement.

Outline In the next section I present the related literature, then the model and the implied diffusion dynamics. In section 4 I explore the welfare impacts, in section 5 the long run stationary properties of the model. In section 6 I present upper bounds on recovery time of the economy after a shock.

2.2 Related literature

The recent literature on the macroeconomic impact of idyosincratic shocks stems from Acemoglu et al. (2012), which shows that in a competitive equilibrium model with input-output linkages the degree distribution of the network may determine how individual idyosyncratic shocks aggregate into economy-wide fluctuations. In an extended model with entry and exit, Baqaee (2018) shows that individual shocks can be amplified and generate cascading behaviour. Grassi (2017) extends the framework in another direction, analyzing non-atomic firms and how market power interact with the network. Huremovic and Vega-Redondo (2016) analyzes how the impact of price shocks such as taxes depend on network-specific quantities.

All these models cited above belong to the family of general equilibrium models, and the predictions come typically from a comparative statics or a steady state exercise. This means that the properties of the dynamics cannot be analyzed. On quite the opposite side is Contreras and Fagiolo (2014), which analyzes empirically the implications of simple diffusion rules without explicit microfoundations. Levchenko et al. (2016) studies the steady state of an adaptive network, but the decisions of the firms and the adjustment of equilibrium behaviour to shocks are instantaneous. The older contribution by Bak et al. (1993), that study an adaptation of a model coming from statistical physics to an input output environment, to prove that, even if demand shocks are such that aggregate demand is deterministic, aggregate (intermediate) production may not be. Similar to this work, the model is intrinsically dynamic, and the effect relies on the cascade effects that the shocks generate.

There is a handful of papers using a dynamic equilibrium approach. The closest paper conceptually is Pasten et al. (2018), which analyzes analytically how the network affects the response of variables to monetary policy shocks. Cienfuegos (2018) analyzes a similar monetary policy problem, focusing on aggregate variables, and the empirical calibration. Atalay (2017) features a dynamic model, but is concerned with estimating elasticities of substitution in different sectors rather than in characterizing the dynamic properties of diffusion.

On the empirical side, a growing number of papers confirms that the network is an important driver of sizable fluctuations and a crucial propagation mechanism. Notable examples are Carvalho et al. (2016), Barrot and Sauvagnat (2016), Acemoglu et al. (2016), Tintelnot et al. (2018).

Finally, a growing literature is exploring the dynamics of the *evolution* of production networks, or more in general endogenous production networks, different from this work where the network is supposed fixed. Examples are, among others, Levchenko et al. (2016), Carvalho and Voigtländer (2015), Taschereau-Dumouchel (2017).

2.3 Model

The setup is the one in Long Jr and Plosser (1983), but I will depart from it in the case of stochastic preferences. Its ingredients are:

1. Time is infinite and discrete. There are two vector Markov processes A_t and γ_t, which are the sources of stochasticity in the model. For simplicity I assume they have a finite state space $S \subset \mathbb{R}_+^N$. I will denote the history of realizations up to time t as $h^t = ((\gamma_1, A_1), \ldots, (\gamma_t, A_t))$. All the endogenous variables should be indexed by histories. When the context does not strictly require it, I abuse the notation by indexing just with t, as in γ_t.[7]

2. There is one infinitely lived representative consumer maximizing its expected discounted utility. Instantaneous utility is the logarithm of a Cobb-Douglas aggregator $C_t = \prod c_{i,t}^{\gamma_{i,t}}$. The intertemporal utility is a standard discounted sum $U = \sum_t \beta^t \ln C_t = \sum_t \beta^t \sum_i \gamma_{i,t} \ln c_{i,t}$, where $\beta < 1$ is the discount factor. The consumer will maximize the expectation of this intertemporal utility. She has an endowment of 1 unit of labor each period, and she supplies it inelastically.

[7]Note that the process for γ_t has to be such that the normalization $\sum_i \gamma_{i,t} = 1$ is true for all t.

3. There are N sectors, each producing a distinct good, acting as neoclassical firms, that maximize their intertemporal profits $\sum_t \left(p_{i,t}y_{i,t} - \sum_{j=1}^{N} p_{j,t}z_{ij,t} - w_t l_{i,t}\right)$[8] subject to a constant returns Cobb Douglas technology, with the important feature described in the next point.

4. Inputs need to be purchased one period in advance. The specific form of the production function is: $A_{i,t+1}\prod_{j=1}^{N}(z_{ij}^t)^{\alpha\omega_{ij}} l_{i,t}^{1-\alpha}$; the parameters ω_{ij} define a matrix Ω, that defines a directed weighted network which we call the *input-output* network of the economy and represents the strenghts of intersectoral linkages. In particular, due to the Cobb Douglas assumption, ω_{ij} is the share of revenues of sector i spent on input j.

5. The consumer owns the firms, and each period receives or pays the necessary cash flow:

$$f_t = \sum_i f_{i,t} = \sum_i \left(p_{i,t}y_{i,t} - \sum_{j=1}^{N} p_{j,t}z_{ij,t} - w_t l_{i,t}\right)$$

   Despite the Cobb-Douglas assumption, this is not zero, because it is *not* the expected profit, for two reasons: it is a *realized*, not expected quantity, and second it is the sum of earnings today from inputs bought *yesterday*, and expenditure for inputs whose output will be sold *tomorrow*. Hence there is no reason to expect this quantity to be 0;

6. The intertemporal budget constraint of the consumer is:

$$\sum_{h^t}\sum_i p_{i,h^t}c_{i,h^t} \leq \sum_i p_{i,0}\omega_i + \sum_{h^t} w_{h^t}l_{h^t} + \sum_{h^t} f_{h^t}$$

   where $w_t l_{i,t}$ is labor income, f_t is the cash flow she receives from the firms, and ω_i is the endowment of the consumer at period 0. This endowment has to be introduced in order for the model to "kick off", otherwise in the first period there can be no production, but is otherwise unimportant and will not appear in any result.

7. There are forward markets for any contingent commodity.

The equilibrium concept is the standard Arrow-Debreu equilibrium. I report here the definition for further clarity.

[8] Note that the prices that appear in the profit expression are not in real terms, but are intertemporal prices, so they include the interest rate.

Definition 2.3.1 (Equilibrium). An equilibrium of this economy is a vector of prices, consumptions, input demands for each history h^t such that

1. The consumer chooses streams of consumption optimizing its expected utility over its budget constraint, solving:

$$\max \mathbb{E}_0 \sum_t \beta^t \ln C_t = \sum_t \beta^t \sum_i \gamma_{i,t} \ln c_{i,t}$$

   subject to:

$$\sum_{h^t} \sum_i p^*_{i,h^t} c_{i,h^t} \leq \sum_i p^*_{i,0} \omega_i + \sum_{h^t} w_{h^t} l_{h^t} + \sum_{h^t} f_{h^t}$$

   where f_{h^t} is defined above.

2. Firms maximize their expected profits subject to the technology constraint:

$$\max_{(l_{i,t})_{t=0}^{\infty}, (z_{ij,t})_{j=1,t=0}^{n,\infty}} \mathbb{E}_0 \sum_t \left(p_{i,t} A_{i,t+1} \prod_{j=1}^{N} (z^t_{ij})^{\alpha\omega_{ij}} l_{i,t}^{1-\alpha} - \sum_{j=1}^{N} p_{j,t} z_{ij,t} - w_t l_{i,t} \right)$$

3. Prices clear the goods market and the labor market at each history:

$$y_{i,h^t} = c_{i,h^t} + \sum_j z_{ij,h^t} \quad \sum_i l_{i,h^t} = 1 \quad \forall h^t$$

$$\omega_i = c_{i,0} + \sum_j z_{ij,0} \quad \sum_i l_{i,0} = 1$$

2.4 Dynamics

In this section, I report the solutions of the model, respectively for productivity and preference shocks. As in other production network models, *Bonacich centrality* is crucial: we denote it as $d_i(\alpha\beta, \gamma)$, where $d(\alpha\beta, \gamma)$ is the vector such that:

$$d = (I - \alpha\beta\Omega')^{-1}\gamma$$

This also corresponds to what Baqaee and Farhi (2017a) call the *Domar weight*.[9] When the coefficient is clear from the context I will omit the dependence. The next proposition follows Long Jr and Plosser (1983).

[9]They distinguish between *revenue* and *cost* based Domar weights. Since the economy studied here is efficient, the two coincide and there is no ambiguity.

Proposition 7. *If productivity parameters follow a Markov process, while preferences are deterministic, in equilibrium the outputs follow:*

$$\ln y_{i,t+1} = const_i + \ln A_{i,t+1} + \sum_j \alpha\omega_{ij} \ln y_{j,t}$$

The sale shares are constant: $\frac{p_{i,t}y_{i,t}}{GDP_t} = d_i(\alpha\beta, \gamma)$, *where* $GDP_t = \sum_i p_{i,t}c_{i,t}$.

Proof. See Appendix. □

So we can see from the above proposition that productivity shocks diffuse through a very simple linear dynamics. In particular, the process of logarithms of productivity is a filter of the process of the errors, increasing its persistence. For example, if productivity shocks are i.i.d. across time, $\ln y = (I - \alpha\Omega L)(const + \ln A)$, where L is the lag operator. That is, log-output follows a VAR(1).

Moreover, sales share are constant in time and are equal to centralities, as in the static model. Yet, there are significant differences in that the relevant centrality here has as a discount coefficient $\alpha\beta$, as I will argue in section 2.5.1.

Proposition 8. *If preference parameters follow a Markov process, in equilibrium, the dynamics of output follows:*

$$\log y_{i,t+1} = const_i + \alpha \sum_j \omega_{ij} \log y_{j,t} - \alpha \sum_j \omega_{ij} \log(d_{i,t}) +$$

$$\log(\mathbb{E}_t d_{i,t+1})$$

where $d_{i,t} = \gamma_{j,t} + \sum_k \alpha^k \beta^k \sum_h \omega_{hj}^{(k)} \mathbb{E}_t \left[\gamma_h^{t+k}\right]$
The sale shares are: $\frac{p_{i,t}y_{i,t}}{GDP_t} = d_{i,t}$
If γ_t *are i.i.d., then:*

$$d_{i,t} = \Delta\gamma_i + d_i(\alpha\beta, \overline{\gamma})$$

where $\mathbb{E}\gamma_t = \overline{\gamma}$. *Hence the dynamics follows:*

$$\log y_{i,t+1} = const_i + \ln d_i(\alpha\beta, \gamma) + \alpha \sum_j \omega_{ij} \log y_{j,t} - \alpha \sum_j \omega_{ij} \log(\Delta\gamma_{j,t} + d_j(\alpha\beta, \overline{\gamma}))$$

Proof. See Appendix. □

One of the features of static models such as Huremovic and Vega-Redondo (2016) or Acemoglu et al. (2016) is that in a Cobb Douglas environment preference (more in general: demand) shocks diffuse downstream. From the dynamics above we can see that, contrary to the static model, here, despite the Cobb-Douglas assumption, preference shocks diffuse also *downstream.* This happens because when a positive taste shock hits any good j then prices adjust. In particular the price of j increases and so firm i is able to buy less of it, so it will have a (relative) negative impact on its production. There is also a direct effect hitting all firms if shocks are correlated over time: the anticipation of a future higher demand drives the sectors whose demand depend more on the relatively more preferred good to increase their production. Instead, when shocks are i.i.d, the realization of the shock does not give any information on the future, hence the *only* impact is downstream. Summing up: the impact of *realized* preference shocks acts downstream, while the impact of *anticipated* shocks acts both upstream and downstream.

To understand better this behaviour, consider the case in which $\Delta\gamma_{\hat{\imath}} = -\Delta\gamma_{\hat{\jmath}} = \varepsilon$, and all the other components are constant. Then:

$$\begin{aligned}\log y_i^{\hat{t}+1} &= -\alpha\omega_{i\hat{\imath}} \log\left(1 + \frac{\varepsilon}{d_{\hat{\imath}}}\right) - \alpha\omega_{i\hat{\jmath}} \log\left(1 - \frac{\varepsilon}{d_{\hat{\jmath}}}\right) \\ &\sim \alpha\varepsilon\left(\frac{\omega_{\hat{\imath}\hat{\jmath}}}{d_{\hat{\jmath}}} - \frac{\omega_{\hat{\imath}\hat{\imath}}}{d_{\hat{\imath}}}\right)\end{aligned}$$

In this case we can see that the output of sector i increases if the good less preferred because of the shock (and hence costs less) is more important as an input than the good which is more preferred (and so costs more).

2.5 Welfare impact of shocks

In this section, I investigate the welfare impact of a productivity and a preference shock. As anticipated, productivity shocks behave in a way much analogous to the static case, while preference shocks do not. Temporary and permanent shocks behave alike.

2.5.1 Productivity shocks

In this section I investigate productivity shocks.

Definition 2.5.1 (Productivity shocks)**.** In the following, by a **permanent shock** at node $\hat{\imath}$ at time $\hat{t}$ I define an unanticipated change in parameters such that $\ln A_{\hat{\imath},t} \to \ln A'_{\hat{\imath},t}$, for all $t \geq \hat{t}$, and $\ln A'_{i,t} = \ln A_{i,t}$ for all $i \neq \hat{\imath}$

and for all t. By a **temporary shock** I define an unanticipated change in parameters such that $\ln A_{\hat{\imath},\hat{t}} \to \ln A'_{\hat{\imath},\hat{t}}$, and $\ln A'_{i,t} = \ln A_{i,t}$ for all $i \neq \hat{\imath}$ and for all $t \neq \hat{t}$.

Proposition 9. *Consider a* ***permanent shock*** *hitting node $\hat{\imath}$. The consequent impact for the consumer is:*

$$\lim_{\Delta \ln A_{\hat{\imath}} \to 0} \frac{\Delta \ln U}{\Delta \ln A_{\hat{\imath}}} = \beta^{\hat{t}} v_{\hat{\imath}}(\alpha\beta) \tag{2.1}$$

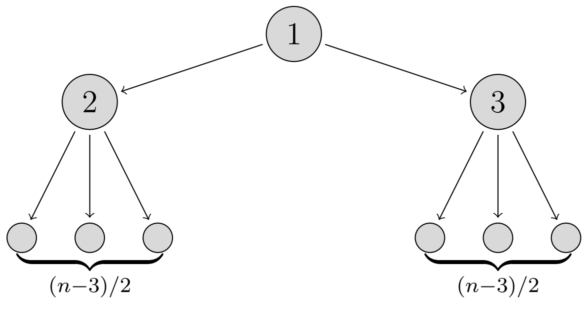
Consider a ***temporary shock*** *hitting node $\hat{\imath}$. The consequent impact for the consumer is:*

$$\lim_{\Delta \ln A_{\hat{\imath}} \to 0} \frac{\Delta \ln U}{\Delta \ln A_{\hat{\imath}}} = \beta^{\hat{t}}(1-\beta) v_{\hat{\imath}}(\alpha\beta) \tag{2.2}$$

This result is an analogous of the well known Hulten Theorem: the impact of the shock in (log) utility is (proportional to) the sales share of the sector hit. Moreover, the dynamics of shocks is linear, so the impact of the realization of the stochastic productivity is identical to a variation in the parameter in a version without uncertainty. This feature depends heavily from the Cobb-Douglas technology assumption, and we do not expect it to be generalizable.

Nevertheless, there are significant differences with Acemoglu et al. (2012): longer paths are more heavily discounted, at a rate $\alpha\beta$ rather than β. This happens because in this model the impact of the shock accrues over time, hence the consumer will discount impacts that are further in the future with its intertemporal discount factor. This can result in changes in the importance of nodes, as in the following example.

Consider the following network, on n (even) nodes:



Centralities:

$$d_1 = \frac{1}{n} + \beta\alpha\frac{2}{n} + \beta^2\alpha^2\frac{n-3}{n} \tag{2.3}$$

$$d_2 = 1/n + \beta\alpha\frac{n-3}{2n} \tag{2.4}$$

If $\beta\alpha > 1/2$ then in the static model a consumer prefers a shock to 2 rather than 1. In the dynamic model instead, the loss in utility are:

$$\Delta U_1 = \varepsilon + \beta\varepsilon^{\frac{2\alpha}{n}} + \beta^2\varepsilon^{\alpha^2\frac{n-3}{n}} \tag{2.5}$$

$$\Delta U_2 = \varepsilon + \beta\varepsilon^{\alpha\frac{n-3}{2n}} \tag{2.6}$$

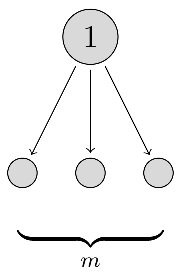
and, e.g. if $n = 6$, $\alpha = 0.6$, $\beta = 0.7$, nodes 2 and 3 are more important than node 1 in the dynamic version.

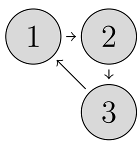
Another feature to be noted is that permanent and temporary shocks behave very much alike. This is due to the fact that permanent shocks converge to a different steady state, but the convergence process to the new steady state is very similar to the convergence back to the old steady state of a temporary shock.

Short run and long run

A possible interpretation of the Hulten-like result above is that, once we know the relative share of revenues, the specific network structure is irrelevant to the impact of the diffusion. In a dynamic setting, though, the same total impact can be achieved in very different ways: there can be shocks whose impact is very strong in the time periods immediately following the realization, but dies out quicly, and there can be shocks whose impact is mild, but diffuses a lot through the network, thereby achieving a high total impact over time nonetheless. The following example is meant to illustrate such behavior.

Example Consider the following two production networks: a circle with n nodes and a star with m leaves. In both cases, consider a temporary shock to node 1.





- a shock on node 1 in the star network exhausts after 1 period: impacts more in the **short run**;

- a shock on node 1 in the circle remains active forever: most important in the **long run**.

In particular, the welfare impacts are: $\frac{1+\alpha\beta m}{m+1}$ and $\frac{1}{n(1-\alpha\beta)}$ in the second. Parameters can be choosed such that the cumulative impacts are the same, despite the two very different structures.

2.5.2 Preference shocks

Definition 2.5.2 (Preference shocks)**.** In the following, by a **permanent shock** at node $\hat{\imath}, \hat{\jmath}$ at time $\hat{t}$ I define an unanticipated change in parameters such that $\gamma'_{\hat{\imath},t} - \gamma_{\hat{\imath},t} = -(\gamma'_{\hat{\jmath},t} - \gamma_{\hat{\jmath},t}) > 0$, for all $t \geq \hat{t}$, and $\gamma'_{i,t} = \gamma_{i,t}$ for all $i \neq \hat{\imath}$ and for all t. By a **temporary shock** I define an unanticipated change in parameters such that $\gamma'_{\hat{\imath},\hat{t}} - \gamma_{\hat{\imath},\hat{t}} = -(\gamma'_{\hat{\jmath},\hat{t}} - \gamma_{\hat{\jmath},\hat{t}}) > 0$, and $\gamma'_{i,t} = \gamma_{i,t}$ for all $i \neq \hat{\imath}$ and for all $t \neq \hat{t}$.

The next proposition describes the welfare impact of preference shocks.

Proposition 10. *Following a permanent shock to γ at time $\hat{t}$, the welfare impact is:*

$$\lim_{\Delta\gamma_{i,\hat{t}}\to 0} \frac{\Delta U}{\Delta\gamma_{\hat{\imath},\hat{t}}} = \beta^{\hat{t}}(\ln c_{\hat{\imath},\hat{t}} - \ln c_{\hat{\jmath},\hat{t}})$$

Following a transitory shock to γ at time $\hat{t}$, the welfare impact is:

$$\lim_{\gamma_{i,\hat{t}}\to 0} \frac{\Delta U}{\Delta\gamma_{\hat{\imath},\hat{t}}} = (1-\beta)\beta^{\hat{t}}(\ln c_{\hat{\imath},\hat{t}} - \ln c_{\hat{\jmath},\hat{t}})$$

Proof. See Appendix. □

Again, we see that transitory and permanent shocks behave in a similar way. And again, shocks with a very similar cumulative impact can differ greatly in the pattern of diffusion. Indeed, in the proof of the proposition we get the following expression:

$$\Delta U^{temp} = \underbrace{-\sum_j (v_j - \gamma_j) \ln\left(\frac{\Delta\gamma_j}{v_j} + 1\right)}_{\substack{\text{diffusion term} \\ \text{as in productivity shocks}}} \text{ periods } t \geq 1$$

$$\underbrace{\Delta \sum \gamma_i \ln \gamma_i + \sum \gamma'_i \ln\left(\frac{\Delta\gamma_i}{v_i} + 1\right)}_{\substack{\text{Direct impact trough revenues} \\ \text{specific to preferences shocks}}} \text{ period } t = \hat{t}$$

so we can see that the cumulative impact is the sum of two terms: one, labeled *diffusion term* above, is the analogous of productivity shocks: the variation in prices creates a chain reaction that affects all reached sectors with the appropriate lag. More interesting is the impact at period $\hat{t}$: this term is due to the adjustment of prices due to produced quantities being pre-determined. This is specific to the preference shock case: the price adjustment in the case of productivity shocks do not impact welfare.

Moreover, again as with productivity shocks, we see that once we know consumption the global impact does not depend on the network anymore. However, if we decompose the impact into a short and a long run impact, we see that centrality is an important modulation factor. Interestingly, the effect of centrality here is the reverse than with productivity shocks: very central nodes will have a small variation in prices, hence have a small utility impact, while nodes with a very low centrality will have a high impact because their prices will be more volatile.

2.6 Time to recovery

One issue of great practical importance about the impact of shocks is how much time does the economy take to absorb it and reach a new steady state (possibly identical to the one it started from). This is what in the following I call *recovery* or *convergence* time. It is a quantity of great interest to policy makers or stakeholders interested in predicting economic variables. One technical issue is that in a smooth equilibrium model as the one I am analyzing, shocks never *totally* die out. Hence we define recovery time as the time the economy takes to arrive ε-close to the steady state, as made precise in the following definition.

Definition 2.6.1 (Recovery time)**.** Given a shock to node k of magnitude $\Delta \ln A_k = 1$ and a bound ε, define the *time to recovery* of node i as the smallest time after which the output of node i differs less than ε from the new steady state. In formulas:

$$CT_{ki}(\varepsilon) = \min\{t : |\ln y_i^{t'} - \ln y_i^{SS}| < \varepsilon, \forall t' \geq t\}$$

Consider a shock, possibly to multiple sectors, satisfying the normalization $\|\ln A\|_2 = 1$. A global convergence time, independent of source and end node is:

$$CT_2(\varepsilon) = \min\{t : \| \ln y^{t'} - \ln y^{SS} \|_2 < \varepsilon, \forall t' \geq t\}$$

The particularly simple dynamics of the model allows to analyze the recovery time in detail. Indeed, since Ω is row stochastic, it can be seen as the transition matrix of a Markov chain, and we can apply the rich theory of mixing times of Markov chains to the task of bounding the recovery time. In order to do this, I maintain throughout the section two assumptions:

Strongly connected network Assume that the production network is strongly connected, meaning that for every pair of nodes i and j there exist a directed path $i_1, \ldots, i_k$ such that $i_1 = i$ and $i_k = j$.

Aperiodic network The minimum common denominator of the length of all cycles is 1.

The first assumption assures that the network cannot be split into separate classes that do not influence each other. If there is a group of sectors that sell output only to themselves a shock hitting one of them (directly or following diffusion) can be analyzed inside the group as a shock on a reduced production network formed just by those sectors, so this is without loss of generality. In particular, rules out sectors that sell only to consumers (i.e. they are not connected to other sectors in the production network). If such a sector exist, a shock to it would just impact consumers and its effect would disappear at the next time period (remember that labor supply is inelastic), so it represents a rather non interesting case.

The second assumption assures that the diffusion of shocks does not feature cycles in such a way that the performance of nodes follows a

In the following, I present two simple results that provide bounds on the convergence time: the first is a global bound that has the advantage of using rather few assumptions, while the second is a sector specific bound, but has a limitation with respect to the first: it requires the Ω matrix to be *reversible*.

Eigenvector centrality The stochastic process for the difference of output from the steady state is defined by the iteration of a Markov chain: $\Delta y^t = \alpha^t \Omega^t \Delta \ln A_0$. Since we assume the matrix to be irreducible and aperiodic the convergence theorem guarantees that the chain converges to the Perron projection of the matrix Ω, that is the matrix with on the rows the leading eigenvalue, which is also the stationary distribution. In formulas:

$$\Omega^t_{ki} \to \pi_i$$

where π is the vector such that $\pi'\Omega = \pi'$. Since here Ω defines a network, π is also the (left) *eigenvector centrality*. The previous discussion shows that in the context of this model eigenvector centrality has the additional interpretation of representing the flow of revenues from nodes that are very far in the production network. Another interpretation can be the vector of revenues that results in the limit as α goes to 1, and so the importance of firms is given by purely network effects.

We note an interesting fact: the time of convergence is connected to eigenvector centrality, while the cumulative impact is connected to Bonacich centrality. These two measures are usually very correlated, but in this context there is an important difference: eigenvector centrality depends only on the *technology* parameters, while Bonacich centrality crucially depends also on the preference parameters of the consumer.

Reversibility An assumption that will be needed for some result in the following is *reversibility*. Consider Ω and π as above. Define $\Omega^*_{ij} = \Omega_{ji}\frac{\pi_j}{\pi_i}$, the *reversibilization* of Ω. A chain is called reversible if $\Omega^* = \Omega$. These concepts are well known in the literature on Markov chains.[10] In this context, following the interpretation of eigenvector centrality as the vector of revenues in an economy where the share of labor goes to zero, reversibility asks that in the same limit the flow of funds from sector i to j be the same as the flow from j to i. This is a rather strong assumption in our context, as it assumes that each link is reciprocal (Ω_{ij} and Ω_{ji} have to be both positive at the same time). Unfortunately the sector-specific result relies on this assumption. That is the reason why I present results for global bounds, which are weaker but do not require reversibility.

2.6.1 Global bound

Adapting corollary 2.14 of Montenegro et al. (2006), we get:

Proposition 11. *Assume Ω is strongly connected and aperiodic. Then:*

$$CT_2(\varepsilon) \leq \max\left\{ \left\lceil \frac{1}{1-\|\Omega^*\|} \ln \frac{1}{\varepsilon \min_i \sqrt{\pi_i}} \right\rceil, \left\lceil \frac{\ln \frac{\varepsilon+1}{\varepsilon}}{\ln 1/\alpha} \right\rceil \right\}$$

where $\Omega^*_{ij} = \Omega_{ji}\frac{\pi_j}{\pi_i}$ is the reversibilization *of* Ω.

The threshold is:

1. decreasing in ε;
2. increasing (weakly) in α;
3. increasing (weakly) in $\|\Omega^*_{ij}\|$.

[10] In the context of Markov chains, this matrix represents the chain that would result if time would go from the future to the past: i.e., the probability of observing first i and then j is the same as the probability of observing first j and then i.

Property 1 is trivial. The second tells us that the more intermediate inputs are important, the longer the recovery time. This is because firms will rely more on produced goods, which are affected by the shock and its propagation, rather than labor (which is not affected by the shock). The third is harder to interpret in general. If Ω is reversible, though, it can be shown that $\|\Omega^*_{ij}\| = \lambda_2$, the second largest eigenvalue of Ω. Under some specific network formation models, in which nodes are partitioned in groups identified by some exogenous characteristics, this has been shown to represent a measure of *homophily*, the tendence of nodes of different groups to be connected together (Golub and Jackson (2012)) or, equivalently, a measure of how strong is the community structure of the network. This is out of this model, but a very interesting empirical as well as theoretical question: is there a community structure in the sectors of an economy? this could happen if for example more productive sectors tended to be comparatively have more exchanges among themselves than with others.

2.6.2 Sector-specific bound

Next, we look for bounds on the convergence time that are sector dependent, to answer the question: which sectors recover first in case of a disruption? which recover later? Unfortunately, since our transition matrix is only sub-stochastic, we can only obtain an upper bound on the convergence time, as the following proposition shows.

Proposition 12 (Sector specific convergence time)**.** *Assume* Ω *is* reversible*, aperiodic and irreducible. Then:*

$$CT_{ki}(\varepsilon) \leq \max \left\{ \left\lceil \frac{\ln\left(\sqrt{\frac{\pi_i}{\varepsilon \pi_k}}\right)}{\ln 1/\lambda_2} \right\rceil, \left\lceil \frac{\ln \frac{\varepsilon + \pi_k}{\varepsilon}}{\ln 1/\alpha} \right\rceil \right\}$$

where λ_2 *is the second largest eigenvalue in absolute value of* Ω *and* π_i *is the eigenvector centrality of node* i*.*

The threshold is:

1. increasing (weakly) in λ_2;
2. (in general) u shaped in π_k;
3. decreasing (weakly) in π_i

The first property is just the adaptation of the result of the previous section to the reversible case, as explained before. The following are more interesting: they suggest that a shock to a more eigenvector central node will take more time to be absorbed, but more central nodes will go back to steady state quicklier than others. To quantify the heterogeneity across nodes, note that variation in eigenvector centrality yields a difference in (the upper bound on) convergence time that is proportional to the second eigenvalue/spectral gap: fix the centrality of the source, if the centrality of the objective is doubled, $\pi_k' = 2\pi_k$, then the convergence time is increased by $CT_{ki}' - CT_{ki} = 1/2 \ln 2 / \ln(1/\lambda_2)$, which can be arbitrarily high if λ_2 is close to 1, or very small if λ_2 is far from 1.

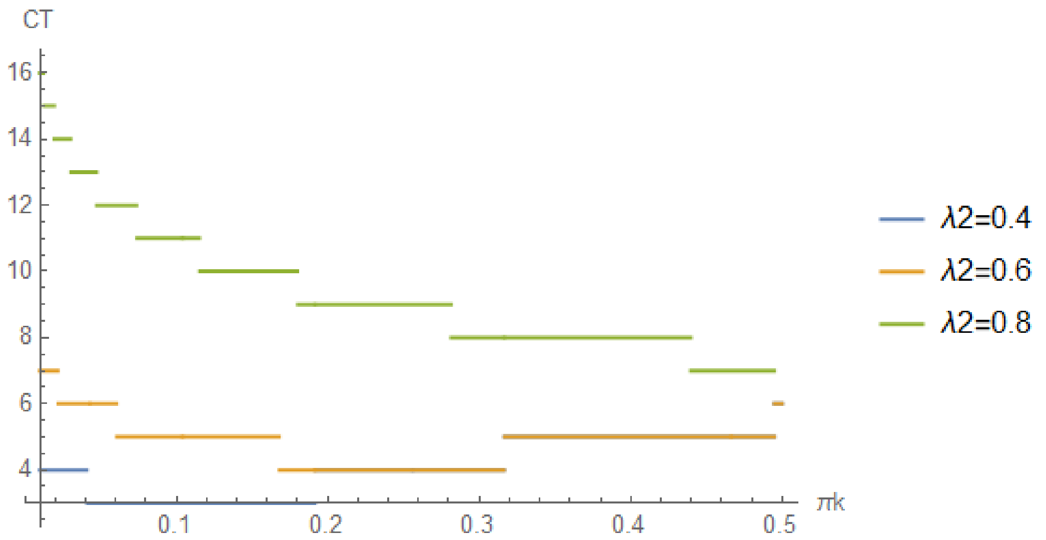


Figure 2.1: The sector-specific upper bound on convergence time as a function of centrality of source node π_k, for fixed centrality of end node π_i.

2.6.3 Heterogeneous primary factor share

The bounds in the previous sections are likely to be strongly driven by α. For this reason in this section we analyze the time of recovery in the case in which the primary factor shares are heterogeneous. The precise meaning of which is the following.

Model with heterogeneous primary factor shares By the model with *heterogeneous primary factor* share, I mean the same model used until now,

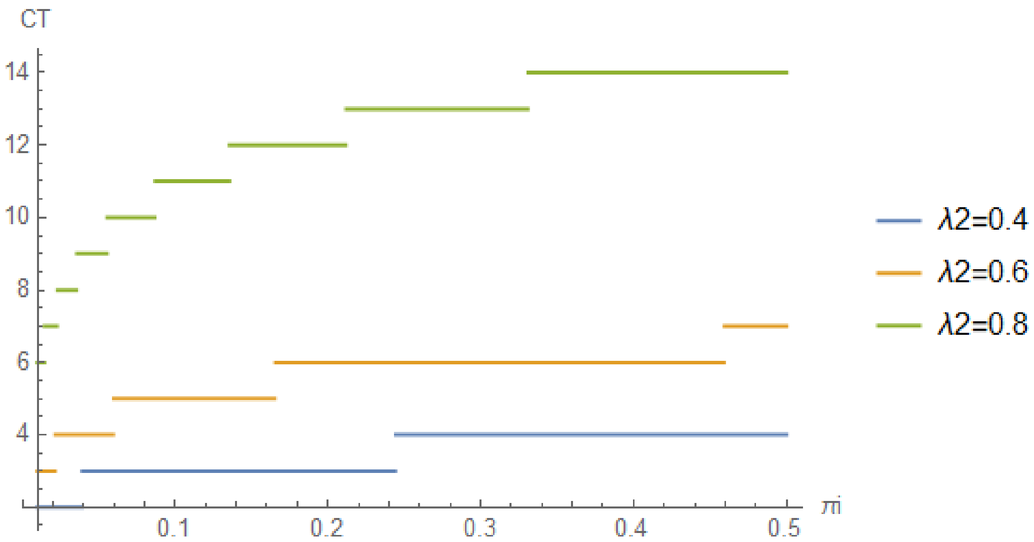


Figure 2.2: The sector-specific upper bound on convergence time as a function of centrality of end node π_i, for fixed centrality of source node π_k.

with one modification, that is the technology is defined as:

$$y_{i,t+1} = A_{i,t+1} \prod_{j=1}^{N} (z_{ij,t})^{\omega_{ij}} l_{i,t}^{1-\alpha_i}$$

where $\sum_j \omega_{ij} = \alpha_i$. That is, the primary factors (labor) in the model are heterogeneous. All expressions and dynamics derived in the special case extend to this case, with the only modification that the matrix $\alpha\Omega$ is replaced by Ω. All the same proofs and propositions go through with obvious modifications. I just report the dynamics of the shock, since is our current object of interest:

$$\ln y_{i,t+1} = \ln A_{i,t+1} + const + \sum_j \omega_{ij} \ln y_{j,t}$$

which implies a deviation from the steady state of:

$$\Delta \ln y_{i,t+1} = \sum_j \omega_{ij} \Delta \ln y_{j,t}$$

for periods following a productivity shock. Hence, the dynamics has a structure very similar to the one studied until now. In this section we exploit the fact that, if Ω is substochastic, nonnegative and irreducible then its eigenvalue maximum in absolute value is real, simple and has a positive left eigenvector (called Perron vector), $\lambda_1 \pi^L = \pi^L \Omega$. Then:

$$P = \lambda_1^{-1} D^{-1} \Omega' D$$

is row stochastic, where $D = diag(\pi_i^L)$ (see proof of the proposition). In this section we differentiate the left and right Perron vectors π^L and π^R because they are different and we will need them both.

Assumption: generalized reversibility The role of reversibility in the substochastic case is played by the condition $\pi_i^R \Omega_{ji} \pi_j^L = \pi_j^R \pi_i^L \Omega_{ij}$. I will call a matrix Ω reversible if it satisfies it. I could derive a global bound without assuming it (see appendix), but since I think the sector-specific bound is more interesting I show the sector specific bound here.

Proposition 13. *Assume Ω is aperiodic, strongly connected and reversible. Then:*

$$CT_{ki}(\varepsilon) \leq \max \left\{ \left\lceil \frac{\ln\left(\frac{1}{\varepsilon}\sqrt{\frac{\pi_i^R \pi_i^L}{\pi_k^R \pi_k^L}}\right)}{\ln \lambda_1/|\lambda_2|} \right\rceil, \left\lceil \frac{\ln\left(\frac{\pi_i^L}{\pi_k^L}\frac{\varepsilon + \pi_k^R \pi_k^L}{\varepsilon}\right)}{\ln 1/\lambda_1} \right\rceil \right\}$$

where λ_1 and λ_2 are respectively the first and second largest eigenvalues in absolute value of Ω, π^L and π^R are the left and right eigenvector centralities.

The intuitions are very similar to the homogeneous case. λ_1 plays the role of α, and is a measure of typical out-degree: it is a classical result that $\sum \alpha_k/N \leq \lambda_1 \leq \max_k \alpha_k$. $|\lambda_2|/\lambda_1$ is still a measure of community structure, normalized by the typical degree.

Centralities role is more complex here. In most terms, we could consider as the "relevant" centrality measure $\pi_i L \pi_i^R$, which ranks nodes according to the fact that they have both out and in-centralities high. This reasoning fails due to the term

2.7 Long run properties

In this section, I assume the (log) productivity shocks are i.i.d. and have mean 0. Then, because of the dynamics described above, (log) sectoral output follows a VAR(1), and analyze the stationary, or long run, properties of the output process, comparing with the static benchmark. To perform this analysis, we need the technical assumption that time starts at $-\infty$. This is not possible in the model analyzed, so we interpret this as a synonym of saying that we analyze the long run behavior of the model.

The aim is to show that the dynamic model generates sistematically less comovement. Let us first see an extreme example.

2.7.1 Example

If

$$\Omega = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

we have a cycle network. For any cycle network (indeed, any network whose adjacency matrix is orthogonal), the covariance in the dynamic model is a multiple of identity, while in the static model: in a connected network all nodes are correlated.

Static **Dynamic**

$$\begin{pmatrix} 1 & \frac{\alpha(1+\alpha+\alpha^2)}{1+\alpha^2+\alpha^4} & \frac{\alpha(1+\alpha+\alpha^2)}{1+\alpha^2+\alpha^4} \\ \frac{\alpha(1+\alpha+\alpha^2)}{1+\alpha^2+\alpha^4} & 1 & \frac{\alpha(1+\alpha+\alpha^2)}{1+\alpha^2+\alpha^4} \\ \frac{\alpha(1+\alpha+\alpha^2)}{1+\alpha^2+\alpha^4} & \frac{\alpha(1+\alpha+\alpha^2)}{1+\alpha^2+\alpha^4} & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The reason is best understood by looking at the elementwise expressions:

$$\mathrm{Cov}_{stat}(i,j) = \sum_k m_{ik} m_{jk}$$

so this covariance between sectors i, j is high when there are sectors k that are very (out)-connected to both i and j, and other sectors that are less connected (there needs to be asymmetry).

The dynamic covariance instead:

$$\mathrm{Cov}_{dyn}(i,j) = \sum_n \alpha^{2n} \sum_k \omega_{ik}^{(n)} \omega_{jk}^{(n)}$$

is high if there are sectors that are connected to both i and j, at *the same distance*. Otherwise shocks to k diffuse in the network but hit i and j at different times, causing less covariation.

We can sum up the results in the following proposition.

Proposition 14. *Assume the* $\ln A_t$ *are a white noise process with covariance matrix* I*. Then for each* i *and* j $\mathrm{Cov}_{dyn}(i,j) \leq \mathrm{Cov}_{stat}(i,j)$.

2.8 Conclusion

In conclusion, if we model shocks as truly stochastic events and look at a model where they gradually spread across the network, different mechanisms for diffusion are at play for preference shocks, while the diffusion of productivity shocks follows the same principles. Moreover, it is possible to derive upper bounds on the recovery time of the economy after a shock, and these apply to both productivity and preference shocks, in both the temporary and permanent case.

2.A Appendix

2.A.1 Proof of Proposition 7

To avoid clutter, I omit the explicit dependence on the history h^t. All variables are to be intended history - dependent.

The firms problems' are essentially static. FOCs:

$$z_{ji}^t : \quad \alpha\omega_{ij} \sum_{h^{t+1}|h^t} p_i^{t+1} y_i^{t+1} = p_j^t z_{ji}^t$$

$$l_i^t : \quad (1-\alpha) \sum_{h^{t+1}|h^t} p_i^{t+1} y_i^{t+1} = w^t l_i^t$$

The transversality is not needed for firms, because it's a sequence of static problems. (Or, equivalently said, the transversality is trivially satisfied.)

Consumer FOCs are:

$$\beta^t \pi(h^t)\gamma_i = \lambda p_i^t c_i^t$$

summing over goods and histories and using the budget constraint we get:

$$\frac{1}{1-\beta} = \lambda \left[\sum_t w^t + \sum_i p_i^0 \omega_i + \sum_{h^t} f_{h^t} \right]$$

Let us write, for brevity, W for $\sum_t w^t$, E for $\sum_i p_i^0 \omega_i$ and f for $\sum_{h^t} f_{h^t}$. From the expression above since in the homogeneous case $U = \lambda(W+E+f)$, we get that in equilibrium $U = \frac{1}{1-\beta}$. Then, substitute the multiplier above into the demand:

$$c_i^t = \beta^t (1-\beta) \frac{\gamma_i}{p_i^t} (W+E+f)\pi(h^t)$$

Goods market clearing yields (for $t > 0$):

$$\alpha \sum_{h^{t+1}|h^t} \sum_{j \in N_i^{in}} \omega_{ji} s_j^{t+1} + \beta^t (1-\beta)\gamma_i (W+E)\pi(h^t) = s_i^t \tag{2.7}$$

Now, consider goods market clearing. We denote for simplicity the revenues $p_{i,t} y_{i,t}$ as $s_{i,t}$.

$$s_i^t = \pi(h^t)\beta^t (1-\beta)(W+E+f)\gamma_i^t + \alpha \sum_{h^{t+1}|h^t} \sum_{j \in N_i^{in}} \omega_{ji} s_j^{t+1}$$

Normalize by the fraction of wealth allocated to time t and call $d_i^t = \frac{s_i^t}{\beta^t \pi(h^t)(1-\beta)(W+E+f)}$ the "per-period" (revenue based) Domar weight:

$$d_i^t = \frac{s_i^t}{\beta^t \pi(h^t)(1-\beta)(W+E+f)} = \gamma_i + \alpha \sum_{h^{t+1}|h^t} \sum_{j \in N_i^{in}} \omega_{ji} \frac{\beta^{t+1}\pi(h^{t+1})(1-\beta)(W+E+f)}{\beta^t \pi(h^t)(1-\beta)(W+E+Pro)}$$

$$
\begin{aligned}
&\times \frac{s_j^{t+1}}{\beta^{t+1}\pi(h^{t+1})(1-\beta)(W+E+f)} \\
&= \gamma_i + \alpha\beta \sum_{h^{t+1}|h^t} \sum_{j \in N_i^{in}} \omega_{ji}\pi(h^{t+1}|h^t) d_j^{t+1}
\end{aligned}
$$

So d follows this difference equation:

$$
d_i^t = \gamma_i + \alpha\beta\mathbb{E}\left[\sum_{j \in N_i^{in}} \omega_{ji} d_j^{t+1} | h^t\right]
$$

iterating forward, since the expectation is bounded (because the state space is finite) and all eigenvalues of Ω are smaller than 1 we obtain that:

$$
d^t = (I - \alpha\beta\Omega')^{-1}\gamma = d^*
$$

are constant over time.

Wage

The amount spent by consumer each period is $\pi(h^t)\beta^t(1-\beta)$, normalizing total wealth $(W+E+f)=1$. This, by market clearing, has to come from the wage and the profit of the firms.

$$
\pi(h^t)\beta^t(1-\beta) = \sum_i \pi_{i,t} + w(h^t) = \sum_i \left(p_{i,t}y_{i,t} - \sum_j p_{j,t}z_{ji,t} - w(h^t)l_{i,t}\right) + w(h^t)
$$

note that profits are not zero because of two reasons: these are the *realized*, not expected, profits, and moreover the profit is computed using contemporaneous values of sales and purchases. These are not related by optimization, since purchases at time t will generate revenues next period, hence there is no reason to think that profits will be zero.

Note also that:

$$
\pi(h^t)\beta^t(1-\beta) = \sum_i \left(p_{i,t}y_{i,t} - \sum_j p_{j,t}z_{ji,t}\right)
$$

so first we see that wage payments and earnings (of course) cancel out. Hence the consumer expenditure comes from the value added on intermediate inputs.

Moreover, using the FOCs:

$$
\pi(h^t)\beta^t(1-\beta) = \pi(h^t)\beta^t(1-\beta)\sum_i \left(d_i - \alpha\sum_j \omega_{ij}\beta d_i - (1-\alpha)\beta d_i\right) + w(h^t)
$$

$$\pi(h^t)\beta^t(1-\beta) = \pi(h^t)\beta^t(1-\beta)\sum_i d_i\,(1-\beta) + w(h^t)$$

hence:

$$w(h^t) = \pi(h^t)\beta^t(1-\beta)\frac{\beta(1-\alpha)}{1-\alpha\beta}$$

and so:

$$f_{i,t} = \pi(h^t)\beta^t(1-\beta)(1-\beta)d_i$$

where we can see that the profit is positive because of the intertemporal dimension: the value of purchases equals the discounted value of revenues tomorrow, which, being discounted, is smaller than the revenues accrued today, even if Cobb-Douglas technology forces everything else to be constant. Moreover, as expected, the profit of each firm is proportional to its dimension, measured by revenues.

Moreover, call $e = \frac{1-\alpha\beta}{\beta(1-\alpha)}$, we get that consumer expenses are: $GDP_{i,t} = \sum p_i c_i = w(h^t)e = \pi(h^t)\beta^t(1-\beta)$, total profit $Pro_t = w(h^t)(e-1)$, and $\sum p_i c_i = \frac{e}{e-1}Pro_t$, so that wage, profit, and GDP are all proportional.

In particular by the calculations above $d_i = \frac{p_{i,t}y_{i,t}}{\pi(h^t)\beta^t(1-\beta)} = \frac{p_{i,t}y_{i,t}}{GDP_t}$

Dynamics

Now, by FOCs input choices are:

$$z_{ji}^{t-1} = \alpha\omega_{ij}\frac{\sum_{h^{t+1}|h^t} d_i^*\beta^{t+1}\pi(h^{t+1})}{p_j^{t-1}} = \alpha\omega_{ij}\frac{d_i^*\beta^{t+1}\pi(h^t)}{p_j^{t-1}}e$$

and in the same way:

$$l_i^{t-1} = (1-\alpha)\frac{\sum_{h^{t+1}|h^t} d_i^*\beta^{t+1}\pi(h^{t+1})}{\beta^t\pi(h^t)} = (1-\alpha)d_i^*\frac{\beta\pi(h^t)}{\pi(h^t)} = (1-\alpha)\beta d_i^* e$$

So, output follows:

$$\ln y_i^{t+1} = \ln A^{t+1} + \sum_j \alpha\omega_{ij}\ln z_{ji}^t + (1-\alpha)\ln l_i^t =$$

$$\ln A^{t+1} + \sum_j \alpha\omega_{ij}\ln\alpha\omega_{ij} - \sum_j \alpha\omega_{ij}\ln p_j^t + \alpha\ln\beta^t\pi(h^t) + \ln\beta + \ln d_i^* + (1-\alpha)\ln(1-\alpha) + \ln e =$$

$$\ln A^{t+1} + \sum_j \alpha\omega_{ij}\ln\alpha\omega_{ij} + (1-\alpha)\ln(1-\alpha) - \sum_j \alpha\omega_{ij}\ln d_j^* + \sum_j \alpha\omega_{ij}\ln y_j^t + \ln\beta + \ln d_i^* + \ln e =$$

and finally:

$$\ln y_i^{t+1} = \ln A_i^{t+1} + C_i + \sum_j \alpha\omega_{ij}\ln y_j^t$$

where $C_i = C_i(\gamma, \alpha, \Omega, \beta) = c_i - \sum_j \alpha\omega_{ij} \ln(d_j^*) + \ln d_i^* + \ln e$, and $c_i = \ln \beta + (1-\alpha)\ln(1-\alpha) + \sum_j \alpha\omega_{ij} \ln(\alpha\omega_{ij})$. Iterating, we can get the relationships between any two outputs at different time periods.

2.A.2 Proof of Proposition 9

We first need a lemma.

Lemma 4. Be $(a_k)_{k\in\mathbb{N}}$ a sequence of nonnegative real numbers, and be ρ and σ real numbers in the interval (0,1). If $\sum_{k=0}^{\infty} a_k$ converges, then:

$$\sum_{k}^{\infty} \rho^k \sum_{n}^{k} \sigma^n a_{k-n} = \frac{1}{1-\rho\sigma} \sum_{k}^{\infty} \rho^k a_k$$

$$\sum_{k}^{\infty} \rho^k \sum_{n}^{k} \sigma^n a_n = \frac{1}{1-\rho} \sum_{k}^{\infty} \rho^k \sigma^k a_k$$

The flow utility of the consumer is, because of homotheticity of the utility:

$$U(c^t) = \sum_i \gamma_i \ln c_i^t = -\ln P^t + \ln w^t$$

hence:

$$U(c^t) = -\sum_i \gamma_i \ln p_i^t + \ln w^t = -\sum_i \gamma_i \ln \frac{p_i^t}{\beta^t \pi(h^t) e} + \ln\left(\frac{w^t}{\beta^t \pi(h^t) e}\right) = -\sum_i \gamma_i \ln \mathcal{P}_i^t$$

Then it is:

$$U^t = \alpha^t \sum_j g_{ji}^t \log \mathcal{P}_j^0 + \sum_{k=0}^{t-1} \alpha^k \sum_j \omega_{ij}^k \Big(-\log A_j^{t-k} - c_j \Big)$$

hence:

$$U = \sum_t \beta^t U^t = \sum_t \beta^t \left(\alpha^t \sum_j g_{ji}^t \log \mathcal{P}_j^0 + \sum_{k=0}^{t-1} \alpha^k \sum_j \omega_{ij}^k \Big(-\log A_j^{t-k} - c_j \Big) \right)$$

Consider a shock at 0. The impact on utility is:

$$\Delta U = \sum_{t \geq \hat{t}} \beta^t \Delta U_t = \sum_t \beta^t \sum_j \gamma_j (c_{j,t}^{\text{shock to } \hat{i}} - c_{j,t}) =$$

$$\sum_{t\geq\hat{t}}\beta^t\sum_j\gamma_j(\ln y_{j,t}^{\text{shock to }\hat{i}}-\ln y_{j,t})=\sum_{t\geq\hat{t}}(\beta\alpha)^t\sum_j\omega_{j\hat{i}}^{(k)}\gamma_j\Delta\ln A_{\hat{i}}$$
$$=\beta^{\hat{t}}\vec{e}_{\hat{i}}'(I-\alpha\beta\Omega')^{-1}\gamma\Delta\ln A_{\hat{i}}$$

because $\ln c_{j,t}^{\text{shock to }\hat{i}}-\ln c_{j,t}=-(\ln p_{j,t}^{\text{shock to }\hat{i}}-\ln p_{j,t})=\ln y_{j,t}^{\text{shock to }\hat{i}}-\ln y_{j,t}$. So

$$\frac{\Delta U/U}{\Delta\ln A_{\hat{i}}}=(1-\beta)\beta^{\hat{t}}v_{\hat{i}}(\alpha\beta)=\frac{p_{\hat{t},\hat{i}}y_{\hat{t},\hat{i}}}{U}$$

$$\lim_{\Delta\ln A_{\hat{i}}\to 0}\frac{\Delta\ln U}{\Delta\ln A_{\hat{i}}}=\lim_{\Delta\ln A_{\hat{i}}\to 0}\frac{\Delta U/U}{\Delta\ln A_{\hat{i}}}=(1-\beta)v_{\hat{i}}(\alpha\beta)$$

Consider now the case of a permanent shock

$$\Delta U=\sum_{t\geq\hat{t}}\beta^t\sum_j\gamma_j(\ln y_{j,t}^{\text{shock to }\hat{i}}-\ln y_{j,t})=\sum_{k=0}(\beta)^k\sum_j\sum_{h=0}^{k}\alpha^h g_{\hat{i}j}^{(h)}\gamma_j\Delta\ln A_{\hat{i}}=$$

$$\frac{1}{1-\beta}\sum_{h=0}(\alpha\beta)^h\omega_{j\hat{i}}^{(h)}\gamma_j\Delta\ln A_{\hat{i}}$$

using the lemma 4.

Remember that utility U is exactly $\frac{1}{1-\beta}$. Hence we get another analogous to Hulten:

$$\frac{\Delta U/U}{\Delta\ln A_{\hat{i}}}=v_{\hat{i}}(\alpha\beta)=p_{\hat{t},\hat{i}}y_{\hat{t},\hat{i}}$$

hence, taking the limit:

$$\lim_{\Delta\ln A_{\hat{i}}\to 0}\frac{\Delta U/U}{\Delta\ln A_{\hat{i}}}=\lim_{\Delta\ln A_{\hat{i}}\to 0}\frac{\Delta\ln U}{\Delta\ln A_{\hat{i}}}=v_{\hat{i}}(\alpha\beta)$$

Proof of Proposition 8

Consumer demand is:

$$c_i^t=\beta^t(1-\beta)\frac{\gamma_{i,t}}{p_i^t}(W+E)\pi(h^t)$$

FOCs:

$$z_{ji}^t:\quad \alpha\omega_{ij}\sum_{h^{t+1}|h^t}p_i^{t+1}y_i^{t+1}=p_j^t z_{ji}^t$$

$$l_i^t:\quad (1-\alpha)\sum_{h^{t+1}|h^t}p_i^{t+1}y_i^{t+1}=w^t l_i^t$$

Hence market clearing:

$$p_i^t y_{i,t} = \beta^t(1-\beta)\gamma_{i,t}(W+E)\pi(h^t) + \alpha \sum_{h^{t+1}|h^t} \sum_j \omega_{ji} s_j^{t+1}$$

Normalize and get:

$$d_i^t = \frac{s_i^t}{\beta^t(1-\beta)\pi(h^t)(W+E)} = \gamma_{i,t} + \alpha \sum_{h^{t+1}|h^t} \sum_{j \in N_i^{in}} \omega_{ji} \frac{\beta^{t+1}\pi(h^{t+1})}{\beta^t \pi(h^t)} \frac{s_j^{t+1}}{\beta^{t+1}\pi(h^{t+1})(1-\beta)(W+E))}$$

$$= \gamma_{i,t} + \alpha\beta \sum_{h^{t+1}|h^t} \sum_j \omega_{ji} \pi(h^{t+1}|h^t) d_j^{t+1}$$

So d follows this difference equation:

$$d_i^t = \gamma_{i,t} + \alpha\beta \mathbb{E}\left[\sum_j \omega_{ji} d_j^{t+1} | h^t\right]$$

and iterating forward and passing to the limit we get:

$$d_i^t = \gamma_{i,t} + \sum_k \alpha^k \beta^k \sum_j \omega_{ji}^{(k)} \mathbb{E}\left[\gamma_j^{t+k} | h^t\right]$$

$$z_{ji,t} = \alpha\beta\omega_{ij} \frac{\mathbb{E}_t d_{i,t+1}}{d_{j,t}} y_{j,t}$$

The wage is

$$w_t = (1-\alpha)\beta \mathbb{E}_t \sum d_i^{t+1} = \frac{(1-\alpha)\beta}{1-\alpha\beta},$$

hence:

$$l_i^t = (1-\alpha)\beta \frac{\mathbb{E}_t d_{i,t+1}}{w_t} = (1-\alpha\beta)\mathbb{E}_t d_{i,t+1}$$

so profits are:

$$Pro_{i,t} = d_{i,t} - \beta \mathbb{E}_t d_{i,t+1}$$

General dynamics:

$$\log y_{i,t+1} = const + \alpha \sum_j \omega_{ij} \log y_{j,t} - \alpha \sum_j \omega_{ij} \log(\gamma_{j,t} + \sum_k \alpha^k \beta^k \sum_{h \in N_j^{in}} g_{jh}^{(k)} \mathbb{E}\left[\gamma_h^{t+k} | h^t\right]) +$$

$$\log(\sum_k \alpha^k \beta^k \sum_{j \in N_i^{in}} g_{ij}^{(k)} \mathbb{E}\left[\gamma_j^{t+1+k} | h^t\right])$$

or

$$\ln y_{i,t+1} = const + \alpha \sum_j \omega_{ij} \log y_{j,t} - \alpha \sum_j \omega_{ij} \ln d_{i,t} + \ln \mathbb{E}_t d_{i,t+1}$$

that for prices (in GDP units) yields:

$$\ln p_{i,t+1} = const + \alpha \sum_j \omega_{ij} \log p_{j,t} + \ln \mathbb{E}_t d_{i,t+1} - \ln d_{i,t+1}$$

i.i.d. case

If γs are i.i.d.:

$$d_i^t = \gamma_{i,t} + \sum_{k=1} \alpha^k \beta^k \sum_{j \in N_i^{in}} g_{ij}^{(k)} \mathbb{E}\left[\gamma\right] = \Delta\gamma_{i,t} + v_i(\alpha\beta, \overline{\gamma})$$

Then plug this into the FOCS:

$$z_{ji,t} = \alpha\omega_{ij} \frac{\sum_{h^{t+1}|h^t} p_i^{t+1} y_i^{t+1}}{p_{j,t}} = \alpha\beta\omega_{ij} \frac{v_i(\alpha\beta, \overline{\gamma})}{\Delta\gamma_{j,t} + v_j(\alpha\beta, \overline{\gamma})} y_{j,t}$$

The wage is the same.

$$l_i^t = (1-\alpha) \frac{\sum_{h^{t+1}|h^t} p_i^{t+1} y_i^{t+1}}{w_t} = (1-\alpha)\beta v_i(\alpha\beta, \overline{\gamma})$$

So the quantity dynamics becomes:

$$\log y_{i,t+1} = c_i + \ln e + \ln v_i + \alpha \sum_j \omega_{ij} \log y_{j,t} - \alpha \sum_j \omega_{ij} \log(\Delta\gamma_{j,t} + v_j)$$

where c_i is defined as above. This captures the idea that when a positive taste shock hits good j then its price increases and so firm i is able to buy less of it, so it will have a (relative) negative impact on its production.

Dynamics of prices:

$$\ln \mathcal{P}_{i,t+1} = \ln(\Delta\gamma_{i,t+1} + v_i) + \alpha \sum_j \omega_{ij} \ln \mathcal{P}_{j,t} - c_i - \ln e - \ln v_i$$

This captures the idea that current prices are directly affected by the realization of γ of the corresponding sector.

profits:

$$Pro_{i,t} = d_{i,t} - \beta\mathbb{E}_t d_{i,t+1} = \Delta\gamma_{i,t} + (1-\beta) v_i(\alpha\beta, \overline{\gamma})$$

2.A.3 Proof of Proposition 10

The expected utility is:

$$U = \sum_{h^t} \beta^t \pi(h^t) \left(\sum_i \gamma_i^t \ln \gamma_i^t - \sum_i \gamma_i^t \ln \mathcal{P}_{i,t} + c_i \right)$$

Now assume to fix ideas that the only stochastic state is 1. All the others are fixed to the average γ. The utility once at 1 has been realized state γ' is:

$$U = \sum_{t>1} \beta^t \pi(h^t) \left(\sum_i \gamma_i \ln \gamma_i - \sum_i \gamma_i \ln \mathcal{P}_{i,t} + c_i \right) + \sum_i \gamma_i' \ln \gamma_i' - \sum_i \gamma_i' \ln \mathcal{P}_{i,1} + c_i$$

Transitory preference shock The impact of a realization γ' at time t is:

$$\Delta U = -\sum_{t>1} \beta^t \pi(h^t) \gamma_i \left(\ln \mathcal{P}'_{i,t} - \ln \mathcal{P}_{i,t} \right) + \sum_i \gamma_i' \ln \gamma_i' - \sum_i \gamma_i \ln \gamma_i - \sum_i \left(\gamma_i' \ln \mathcal{P}'_{i,1} - \gamma_i \ln \mathcal{P}_{i,1} \right)$$

Now:

$$\Delta \ln \mathcal{P}_{i,t} = \alpha^{t-1} \sum_j \omega_{ij} \Delta \ln \mathcal{P}_{j,1}$$

$$\ln \mathcal{P}'_{i,1} = \ln(\Delta\gamma_{i,1} + v_i) + \alpha \sum_j \omega_{ij} \mathcal{P}_{j,0} - c_i - \ln e - \ln v_i = \ln \left(\frac{\Delta \gamma_{i,1}}{v_i} + 1 \right) - c_i - \ln e =$$

$$\ln \left(\frac{\Delta \gamma_{i,1}}{v_i} + 1 \right) + \ln \mathcal{P}_{i,1}$$

so:

$$\Delta \ln \mathcal{P}_{i,1} = \ln \left(\frac{\Delta \gamma_{i,1}}{v_i} + 1 \right)$$

hence:

$$\Delta U_{t>1} = -\sum_t \beta^t \Delta \ln \mathcal{P}_{i,t} = -\sum_t \gamma_i \alpha^{t-1} \beta^{t-1} \sum_j g_{ji}^{(t)} \Delta \ln \mathcal{P}_{j,1} = -\sum_j (v_j - \gamma_j) \ln \left(\frac{\Delta \gamma_j}{v_j} + 1 \right)$$

$$\sim -\sum_j (v_j - \gamma_j) \frac{\Delta \gamma_j}{v_j} = -\sum_j \Delta \gamma_j - \sum_j \gamma_j \Delta \gamma_j = \sum_j \frac{\gamma_j \Delta \gamma_j}{v_j}$$

Instead the terms of the first period can be rewritten as:

$$\Delta \sum \gamma_i (\ln \gamma_i - \ln p_{i,1}) =$$

$$\Delta \sum \gamma_i \ln \gamma_i - \sum_i \left(\gamma_i' \ln \mathcal{P}'_{i,1} - \gamma_i \ln \mathcal{P}_{i,1} \right) = +\Delta \sum \gamma_i \ln \gamma_i - \sum \gamma_i' \Delta \mathcal{P}_i - \sum \Delta \gamma_i \ln \mathcal{P}_i$$

so:

$$\Delta U = -\sum_j (v_j - \gamma_j) \ln\left(\frac{\Delta\gamma_j}{v_j} + 1\right) - \sum_j \gamma_j \ln\left(\frac{\Delta\gamma_j}{v_j} + 1\right) +$$

$$\Delta \sum \gamma_i \ln \gamma_i - \sum \Delta\gamma_i \ln \mathcal{P}_i \sum \Delta\gamma_i \ln\left(\frac{\Delta\gamma_i}{v_i} + 1\right)$$

$$= -\sum_j v_j \ln\left(\frac{\Delta\gamma_j}{v_j} + 1\right) + \Delta \sum \gamma_i \ln \gamma_i - \sum \Delta\gamma_i \ln \mathcal{P}_i + \sum \Delta\gamma_i \ln\left(\frac{\Delta\gamma_i}{v_i} + 1\right)$$

Now note that:

$$\sum \Delta\gamma_i \ln\left(\frac{\Delta\gamma_i}{v_i} + 1\right) \sim \sum \Delta\gamma_i^2 / v_i$$

is second order in the size of the shock, and always at the first order

$$-\sum_j v_j \ln\left(\frac{\Delta\gamma_j}{v_j} + 1\right) \sim -\sum_j \Delta\gamma_j = 0$$

So at the first order:

$$\Delta U = -\sum \Delta\gamma_i \ln \mathcal{P}_i + \Delta \sum \gamma_i \ln \gamma_i$$

$$= -\sum \Delta\gamma_i \ln \mathcal{P}_i + \sum \Delta\gamma_i \ln \gamma_i$$

$$\sum \Delta\gamma_i \ln c_i$$

Permanent preference shock Note that the revenues have a direct impact on the price dynamics only at the moment of the impact. Beyond that, things are equivalent to a productivity shock, of size modified according to centrality. So now:

$$\Delta U = \sum_t \beta^t \left(\Delta \sum \gamma_i \ln \gamma_i - \sum_i \left(\gamma_i' \ln \mathcal{P}_{i,t}' - \gamma_i \ln \mathcal{P}_{i,t} \right) \right)$$

The second part can be rewritten as before $\sum \gamma_i' \Delta \mathcal{P}_{i,t} + \sum \Delta\gamma_i \ln \mathcal{P}_{i,t}$, that is:

$$\sum \gamma_i' \alpha^t \sum_j \omega_{ij}^t \Delta \mathcal{P}_{j,0} + \sum \Delta\gamma_i \alpha^t \sum_j \omega_{ij}^t \ln \mathcal{P}_{j,0} + const$$

Apply lemma 4 to this and get:

$$\frac{1}{1-\beta} \left(\sum_j v_j(\alpha\beta, \gamma') \Delta \ln \mathcal{P}_{j,0} + \sum_j v_j(\alpha\beta, \Delta\gamma) \ln \mathcal{P}_{j,0} \right) + const$$

so in total we get:

$$\Delta U = \frac{1}{1-\beta}\Delta \sum \gamma_i \ln \gamma_i - \frac{1}{1-\beta}\left(\sum_j v_j(\alpha\beta,\gamma')\Delta \ln \mathcal{P}_{j,0} + \sum_j v_j(\alpha\beta,\Delta\gamma)\ln \mathcal{P}_{j,0}\right)$$

$$= \frac{1}{1-\beta}\Delta \sum \gamma_i \ln \gamma_i - \frac{1}{1-\beta}\left(\sum_j v_j(\alpha\beta,\gamma')\ln\left(\frac{v_j(\alpha\beta,\Delta\gamma)}{v_j(\alpha\beta,\gamma)}+1\right) + \sum_j v_j(\alpha\beta,\Delta\gamma)\ln \mathcal{P}_{j,0}\right)$$

Also in this case, at the first order $\sum_j v_j(\alpha\beta,\gamma')\Delta \ln \mathcal{P}_{j,0} = \sum_j v_j(\alpha\beta,\gamma')(\ln v_i(\alpha\beta,\gamma') - \ln v_i(\alpha\beta,\gamma)) \sim \sum_j v_j(\alpha\beta,\gamma')\frac{v_j(\alpha\beta,\Delta\gamma)}{v_j(\alpha\beta,\gamma')} = 0$ (remember that the centrality is a linear combination of the preference parameters).

2.A.4 Proof of Proposition 14

Assume the autocovariance function of $\ln A$ is $\Gamma^k_{\ln A}$, and its autocovariance generating function is $G(z) = \sum \Gamma^k_{\ln A} z^k$. Assume that $\Gamma^k_{\ln A}$ is absolutely summable (it is in our case, as the autocovariance generating function of a white noise is $G(z) = I$).

$\ln y_t$ is a filter of $\ln A_t$ by the filter $(I - \alpha L)^{-1}$, where L is the lag operator. Hence its autocovariance generating function is (Hamilton 10.3):

$$F(z) = (I - \alpha z \Omega)^{-1} G(z) (I - \alpha z^{-1}\Omega')^{-1}$$

From here, we can recover the autocovariance by integrating the spectrum: $\Gamma^k_{\ln y} = \int_{-\pi}^{\pi} F(e^{i\omega})\mathrm{d}\omega$ (recall that $F(e^{i\omega})$ is the spectrum of $\ln y$).

By expanding the three series and taking the Cauchy product (all are absolutely summable) we get:

$$F_{ij}(z) = \sum_n^\infty \sum_k \sum_m \sum_h^n \left(\sum_l^h \alpha^l \omega_{ik}^{(l)} z^l \alpha^{h-l} \omega_{jm}^{(h-l)} z^{l-h}\right) G_{km}^{n-h} z^{n-h}$$

$$F_{ij}(e^{i\omega}) = \sum_n^\infty \sum_k \sum_m \sum_h^n \alpha^h \left(\sum_l^h \omega_{ik}^{(l)} \omega_{jm}^{(h-l)}\right) G_{km}^{n-h} e^{i\omega(2(l-h)+n)}$$

Now since we assume that the autocovariance is absolutely summable then the series is finite and each partial sum is dominated by the total sum, so by the dominated convergence theorem we can exchange series and integral and then we are left with a combination of integrals of $e^{i\omega(2(l-h)+n)}$. These integrals are all zero (integrals of trigonometric functions over multiples of the domain) except the ones for $2(l-h)+n=0$, or $l=h-n/2$, n even, $h \geq n/2$. Hence we can write:

$$\mathrm{Cov}_{dyn}(i,j) = \Gamma^0_{\ln y} = \int_{-\pi}^{\pi} F(e^{i\omega})\mathrm{d}\omega =$$

$$\sum_n^\infty \sum_k \sum_m \sum_h^n \alpha^h \left(\sum_l^h \omega_{ik}^{(l)} \omega_{jm}^{(h-l)} \right) G_{km}^{n-h} \int_{-\pi}^{\pi} e^{i\omega(2(l-h)+n)}\mathrm{d}\omega$$

$$\sum_{n \text{ even}}^\infty \sum_k \sum_m \sum_{h=n/2}^n \alpha^h \left(\omega_{ik}^{(h-n/2)} \omega_{jm}^{(n/2)} \right) G_{km}^{n-h}$$

now we can redefine n as $n/2$, and h as $h-n/2$ to get:

$$\mathrm{Cov}_{dyn}(i,j) = \sum_n^\infty \sum_k \sum_m \sum_h^n \alpha^{h+n} \left(\omega_{ik}^{(h)} \omega_{jm}^{(n)} \right) G_{km}^{n-h}$$

Now we use the assumption that $\ln A$ is a $WN(I)$. In this case the autocovariance function satisfies $G^k = 0$ for any $k \neq 0$, $G^0 = I$. Then the only term surviving in the expression above is the one with $h = n$, hence:

$$\mathrm{Cov}_{dyn}(i,j) = \sum_{n \text{ even}}^\infty \sum_k \alpha^n \left(\omega_{ik}^{(n/2)} \omega_{jk}^{(n/2)} \right)$$

Now compare this to:

$$\mathrm{Cov}_{static}(i,j) = \sum_n^\infty \alpha^n \sum_k \sum_l^n \omega_{ik}^{(l)} \omega_{jk}^{(n-l)}$$

In the expression for the dynamic covariance all the terms are zero, moreover the even terms are $\omega_{ik}^{(n/2)} \omega_{jk}^{(n/2)}$ which is just one addend of the corresponding term in the static expression $\sum_l^n \omega_{ik}^{(l)} \omega_{jk}^{(n-l)}$. Hence, the dynamic is smaller for any i and j.

2.A.5 Proof of Proposition 11

From standard techniques, see e.g. Montenegro et al. (2006) for any irreducible, aperiodic Markov chain with stationary distribution π and transition matrix Ω:

$$CT_2^{Markov}(\varepsilon) \leq \left\lceil \frac{1}{1-\|\Omega^*\|} \ln \frac{1}{\varepsilon \min_i \sqrt{\pi_i}} \right\rceil$$

These result applied to our setting yield:

$$CT_2^{Markov}(\varepsilon) \leq \left\lceil \frac{1}{1-\|\Omega^*\|} \ln \frac{1}{\varepsilon \min_i \sqrt{\pi_i}} \right\rceil$$

here π is the eigenvector centrality.
So, if $t > CT^{Markov}(\varepsilon)$, then:

$$\alpha^t \omega_{ki}^t \leq \alpha^t(\pi_k + \varepsilon)$$

In turn, $\alpha^t(\pi_k + \varepsilon)$ is smaller than ε if and only if $t > \ln \frac{\varepsilon}{\varepsilon+\pi_k} / \ln \alpha$, so that:

$$CT(\varepsilon) = \max\{\left\lceil \frac{\ln\left(\varepsilon\sqrt{\frac{\pi_k}{\pi_i}}\right)}{\ln \lambda_2} \right\rceil, \left\lceil \ln \frac{\varepsilon}{\varepsilon+\pi_k} / \ln \alpha \right\rceil\}$$

2.A.6 Proof of Proposition 12

From Levin and Peres (2017) for any irreducible, aperiodic, reversible Markov chain :

$$CT_{ki}^{Markov}(\varepsilon) \leq \left\lceil \frac{\ln\left(\varepsilon\sqrt{\frac{\pi_k}{\pi_i}}\right)}{\ln \lambda_2} \right\rceil$$

These results applied to our setting yield:

$$CT_{ki}^{Markov}(\varepsilon) \leq \left\lceil \frac{\ln\left(\varepsilon\sqrt{\frac{\pi_k}{\pi_i}}\right)}{\ln \lambda_2} \right\rceil$$

So, if $t > CT_{ki}^{Markov}(\varepsilon)$, then, repeating the reasoning above we get:

$$CT_{ki}(\varepsilon) = \max\{\left\lceil \frac{\ln\left(\varepsilon\sqrt{\frac{\pi_k}{\pi_i}}\right)}{\ln \lambda_2} \right\rceil, \left\lceil \ln \frac{\varepsilon}{\varepsilon+\pi_k} / \ln \alpha \right\rceil\}$$

2.A.7 Proof of Proposition 13

For P defined as in the text:

$$\sum_j P_{ij} = \lambda_1^{-1} \sum_j \Omega_{ji} \frac{\pi_j^L}{\pi_i^L} = \lambda_1^{-1} \frac{\lambda_1 \pi_i^L}{\pi_i^L} = 1$$

Note that the eigenvalues of P are those of Ω divided by λ_1. Our goal is to find t such that $\Omega^t_{ki} < \varepsilon$. But $\Omega^t = \lambda_1^t D P^t D^{-1}$, and the invariant distribution of P is $\pi_i^R \pi_i^L$:

$$\sum_i \pi_i^R \pi_i^L P_{ij} = \lambda_1^{-1} \sum_i \pi_i^R \pi_i^L \Omega_{ji} \frac{\pi_j^L}{\pi_i^L} = \lambda_1^{-1} \sum_i \pi_i^R \Omega_{ji} \pi_j^L = \lambda_1^{-1} \lambda_1 \pi_j^R \pi_j^L = \pi_j^R \pi_j^L$$

Moreover, P is irreducible and aperiodic if and only if Ω is irreducible and aperiodic, since the powers of Ω are positive whenever the ones of P are. Hence $P^t \to \pi^R (\pi^L)^\intercal$.

To proceed further we need P to be reversible, that is:

$$\pi_i^R \pi_i^L \Omega_{ji} \frac{\pi_j^L}{\pi_i^L} = \pi_j^R \pi_j^L \Omega_{ij} \frac{\pi_i^L}{\pi_j^L} \iff \pi_i^R \Omega_{ji} \pi_j^L = \pi_j^R \pi_i^L \Omega_{ij}$$

which is our assumption. So, reasoning as in Proposition 12 we get that for t high enough $P^t_{ik} < \pi_k^R \pi_k^L + \varepsilon$. So if $\lambda_1^t \frac{\pi_k^L}{\pi_i^L} (\pi_k^R \pi_k^L + \varepsilon) < \varepsilon$ then

$$\Omega^t_{ki} = \lambda_1^t \frac{\pi_k^L}{\pi_i^L} P^t_{ik} < \lambda_1^t \frac{\pi_k^L}{\pi_i^L} (\pi_k^R \pi_k^L + \varepsilon) < \varepsilon$$

so get:

$$CT_{ki}(\varepsilon) \leq \max \left\{ \left\lceil \frac{\ln \left(\frac{1}{\varepsilon} \sqrt{\frac{\pi_i^R \pi_i^L}{\pi_k^R \pi_k^L}} \right)}{\ln \lambda_1 / |\lambda_2|} \right\rceil , \left\lceil \frac{\ln \left(\frac{\pi_i^L}{\pi_k^L} \frac{\varepsilon + \pi_k^R \pi_k^L}{\varepsilon} \right)}{\ln 1/\lambda_1} \right\rceil \right\}$$

Chapter 3

Is segregating anti–vaxxers a good idea?

This chapter is joint work with Fabrizio Panebianco[1] *and Paolo Pin*[2]

Some pro-vaccine policies (e.g., compulsory vaccination in public schools) may have the effect of separating those in favor from those against vaccines, inducing a segregating effect. We study an SI-type model, with the possibility of vaccination, where the population is partitioned between pro-vaxxers *and* anti-vaxxers*. We show that, during the outbreak of a disease, segregating people against vaccination from the rest of the population decreases the speed of recovery, may increase the number of cases, and can make the disease endemic. Then, we include endogenous choices based on the trade-off between the cost of vaccinating and the risk of getting infected. We show that the results remain valid under endogenous choices, unless people are very flexible in determining their* pro–vaxxers *or* anti–vaxxers *identity.*[3]

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3.1 Introduction

We model an economy facing the possible outbreak of a disease, for which a vaccine with temporary efficacy is available. This mimics what happens every year for seasonal flu, but it could also be the case in the near future for Covid19.[4]

Even before Covid19, vaccination has been almost unanimously considered the most effective public health intervention by the scientific community (see e.g. Larson et al., 2016 or Trentini et al., 2017). However, in recent years many people either refuse drastically any vaccination scheme or reduce or delay the prescribed vaccination. This phenomenon has become more pronounced in the last decades, especially in Western Europe and in the US,[5] and many public health organizations have issued public calls to researchers to enhance the understanding of the phenomenon and its remedies. Even in the present times of Covid19 epidemic, the opposition to vaccination policies is alive.[6]

The focus of this paper is on the effects of containment measures that aim at reducing contacts between vaccinated and unvaccinated people, and their interaction with vaccination choices of agents and the dynamics of anti–vaxxer movements. During the Covid19 outbreak, governments have implemented very strong and drastic temporary containment and quarantine policies. However, such stringent policies cannot be permanent measures, and in normal times the policy makers are able to implement only milder policies that may segregate people in certain loci of activity. Typical milder measures of this kind, often implemented in recent years, are limitations for attending schools. In order to protect the public, in many countries recent laws forbid enrollment of non vaccinated kids into public schools, and this is believed to have brought to an increase in enrollment in more tolerant

[4] At present, we know that the virus of Covid19 mutates very rapidly (Korber et al., 2020; Pachetti et al., 2020) and that it seems to be seasonal (Carleton and Meng, 2020). Scientists and politicians are considering the possibility that for the next year it could become similar to a seasonal flu that deserves a new vaccine every year: for example, see this report from April 2020. There is also another reason for which vaccination against Covid19 may not be permanent: more recent studies like Seow et al. (2020) have shown that Covid19 antibodies fall rapidly in our body so that it could be the case that people will need to vaccinate regularly (e.g. every 2 years) against the virus.

[5] See Larson et al. (2016) for a general and recent cross country comparison. Most studies are based on the US population: Robison et al. (2012), Smith et al. (2011), Nadeau et al. (2015) and Phadke et al. (2016) are some of the more recent ones. Funk (2017) focuses on measles in various European countries. Rey et al. (2018) analyzes the case of France.

[6] On this, see the recent reports of Johnson et al. (2020), Ball (2020) and Malik et al. (2020).

private schools.[7,8] On an abstract level, this corresponds to a change in the *homophily* of interactions: the policy has a segregating effect, incentivizing people with anti-vaccination beliefs to interact more together.

First, we consider a mechanical model in which vaccination choices are exogenous. The policy parameter that the policy-maker can tune is the segregation between two groups of people: those that are against vaccination and all the others, which we call for simplicity *anti-vaxxers* and *pro-vaxxers* (or just vaxxers), respectively. The two groups differ in the judgment about the real cost of vaccination, which is deemed higher by anti-vaxxers. This can be thought of as a psychological cost, a sheer mistake, or any phenomenon that may lead to a difference in perceived cost: we remain agnostic on the cause of it: our aim is to study its consequences. We think of this difference in the perceived cost as a basic cultural trait, which, as the literature on cultural transmission, affects the *preferences* (or beliefs) of agents, who are still free to choose to vaccinate or not, based on a heterogeneous component of the cost. This means that an anti-vaxxer in our model may still vaccinate, if the heterogeneous component of the cost is small enough. Through this, we mean to capture not the extremists that would never take a vaccine, but the much more general phenomenon of *vaccine hesitancy*, which is much more widespread and, so, potentially much more dangerous (Trentini et al., 2017).

We model the segregation policy as a parameter $h \in [0, 1]$, which is the percentage of contacts that people cannot have with the other group (because, for example, their kids are not in the same schools, or they cannot meet in the same job and leisure places). We think h as a number that is far from one (which would be the case of total segregation). This partial segregation policy is implemented before the epidemic actually takes place. We show that the policy may backfire: more segregation may cause the disease to die out more slowly and cause more infection in the whole population, or even more infection among *vaxxers*. In particular, our results suggest care both to a social planner concerned with total infection in the population *and* to a social planner concerned only with infection among the vaxxers. The choice between the two approaches depends on the attitude toward society we want to model, and in particular on the specific interpretation of the difference in

[7] This phenomenon is documented for California by Silverman and Yang (2019). Recent evidence shows that similar trends happened in Italy and have been considered a cause of the measles outbreak in Manhattan in April 2019.

[8] As another example, in light of the policies enforced during the Covid19 crisis, many companies and other public and private organizations have applied *rotation schemes* to limit physical interaction between people (on this, see the recent work by Ely et al., 2020): it is admissible that a policy maker may want to include unvaccinated people all in the same group.

perceived cost, e.g., as a pure bias that the social planner should consider as such, or as a form of real psychological cost that we may want to factor in the welfare computation.[9] As a consequence of these considerations, we remain agnostic on a general welfare criterion and explore instead the physical outcome of the amount of infection, which in such an environment is likely to be a prominent, if not the only, element of any welfare analysis.

The reason why an increase in h may generate more infection is that homophily protects the group with less infected because it decreases the contacts and so the diffusion of the disease across groups. Which group has a larger infection rate will in turn depend on initial conditions and on the difference in vaccination rates between the two groups. If the total number of agents initially infected is the same across the two groups [10] then homophily has no effect on total infection. Hence, a planner that cares only about the infection among vaxxers has no clear choice: she will desire an increase of h (e.g., in case of an outbreak among anti-vaxxers, but would have opposite preferences in case of an outbreak among vaxxers).

If, instead, the two groups differ in the number of infected agents, the effect on total infection depends on the interplay of initial conditions and vaccinations. If the less vaccinated group happens to have more infections (because suffered a larger share of the initial outbreak), we know homophily further increases infections in such group. The crucial observation is that it increases infections at a disproportionally larger rate than when the more vaccinated group has more infections. As a result, if the outbreak is among anti-vaxxers, total infection in the population increases with h, while if the outbreak is among vaxxers it decreases.

In the second part of the paper, we endogenize the vaccination choices of people. Vaccination choices are taken before the disease spreads out. We view this as a classical trade-off between the perceived cost of vaccinating and the expected cost of getting sick. In the model, the difference between anti–vaxxers and pro–vaxxers is only in the perceived costs of vaccination. We show that even if we endogenize these choices, the qualitative predictions of the mechanical model are still valid: a policy that increases segregation is counterproductive.

Finally, we endogenize the choice of agents on whether to be anti–vaxxer or pro–vaxxer. This choice is modeled as the result of social pressure, with the transmission of a cultural trait. There is a well-documented fact about vaccine hesitancy that seems hard to reconcile with strategic models: the geo-

[9] These are complex issues at the forefront of research in behavioral economics, see Bernheim (2009).

[10] This can happen, e.g., if the initial seeds are unequally distributed, and initially more vaxxers are infected, see Section 3.3.

graphical and social clustering of vaccine hesitancy. Various studies, reviewed e.g. by Dubé and MacDonald (2016), find that people are more likely to have positive attitudes toward vaccination if their family or peers have. This is particularly evident in the case of specific religious confessions that hold anti-vaccination prescriptions and tend to be very correlated with social contacts and geographical clustering. These studies, though observational and making no attempt to assess causal mechanisms, present evidence at odds with the strategic model: if the main reason not to vaccinate is free riding, people should be *less* likely to vaccinate if close to many vaccinated people, and not vice versa. In addition, Lieu et al. (2015) show that vaccine-hesitant people are more likely to communicate together than with other people. Edge et al. (2019) document that vaccination patterns in a network of social contacts of physicians in Manchester hospitals are correlated with being close in the network. It has also been shown that, in many cases, providing more information does not make vaccine-hesitant people change their minds (on this, see Nyhan et al., 2013, 2014 and Nyhan and Reifler, 2015). However, people do change their minds about vaccination schemes, but they do so under psychological rules that look irrational, as documented recently by Brewer et al. (2017), for example. In a review of the literature, Yaqub et al. (2014) finds that lack of knowledge is cited less than distrust in public authorities as a reason to be vaccine-hesitant. This is true both among the general public and professionals: in a study of French physicians, Verger et al. (2015) finds that only 50% of the interviewed trusted public health authorities. They both find a correlation between vaccine hesitancy and the use or practice of alternative medicine.

When we fully endogenize the choices of agents (both membership to groups and vaccination choices), we find that the predictions of the simple mechanical model remain valid only if the groups of the society are rigid enough, and people do not change their minds easily about vaccines. If instead people are more prone to move between the anti–vaxxers and pro–vaxxers groups, then segregation policies can have positive effects. The simple intuition for this is that when anti–vaxxers are forced to interact more together, they internalize the higher risk of getting infected and, as a result, they are more prone to become pro–vaxxers.

We contribute to three lines of literature, related to three steps of our analysis highlighted above: the analysis of the effects of segregation in epidemiological models, the economics literature on vaccination and its equilibrium effects, and the literature on diffusion of social norms and transmission of cultural traits.

The medical and biological literature using SI-type models is wide, and a review of it is beyond our scope. We limit ourselves to note that recently some papers have considered dynamic processes with superficial similarity to ours. Jackson and López-Pintado (2013) and Izquierdo et al. (2018) are the first, to our knowledge, to study how homophily affects diffusion. Pananos et al. (2017) analyze critical transitions in the dynamics of a three equation model including epidemic and infection.

The literature on strategic immunization has analyzed models where groups are given and the focus is the immunization choice, as in Galeotti and Rogers (2013), or both the immunization and the level of interaction are endogenous, as in Goyal and Vigier (2015). At an abstract level, the difference with respect to our setting is that we endogenize the group partition through the diffusion of social norms.[11]

The economics of social norms and transmission of cultural traits is a lively field, surveyed by Bisin and Verdier (2011). Common to this literature is the use of simple, often non-strategic, dynamic models of evolution of preferences. We adopt this framework, finding it useful despite the differences we discuss later. A paper close to ours is Panebianco and Verdier (2017), that considers how social networks affect cultural transmission in a SI-type model, with a more concrete network specification through degree distributions.

The paper is organized as follows. Section 3.2 describes the mechanical model and shows its results. Sections 3.3 and 3.4 introduce respectively endogenous choices and endogenous group membership, deriving our analytical results for these cases. We conclude in Section 3.5. In the appendices we consider extensions of the model (Appendices 3.A, 3.B and 3.C) and we prove our results (Appendix 3.D).

3.2 Mechanical model

We consider a simple SIS model with vaccination and with two groups of agents, analogous to the setup in Galeotti and Rogers (2013). To understand the main forces at play, we start by taking all the decisions of the agents as exogenous, and we focus on the infection dynamics. In the following sections, we endogenize the choices of the players.

Our society is composed of a continuum of agents of mass 1, which is partitioned into two groups of agents. To begin with, in this section this partition is exogenous. Agents in each group are characterized by their attitude

[11] There is also a recent literature in applied physics that studies models where the diffusion is simultaneous for the disease and for the vaccination choices. On this, see the review of Wang et al. (2015), and the more recent analysis of Alvarez-Zuzek et al. (2017) and Velásquez-Rojas and Vazquez (2017).

towards vaccination. In details, following a popular terminology, we label the two groups with a, for *anti-vaxxers*, and with v, for *vaxxers*. Thus, the set of the two groups is $G := \{a, v\}$, with $g \in G$ being the generic group. Let $q^a \in [0,1]$ denote the fraction of *anti-vaxxers* in the society, and $q^v = 1 - q^a$ the fraction of *vaxxers*. To ease the notation, when this does not create ambiguity, we write q for q^a.

People in the two groups meet each other with an *homophilous* bias. We model this by assuming that an agent of any of the two groups has a probability h to meet someone from her own group and a probability $1 - h$ to meet someone else randomly drawn from the whole society.[12] This implies that anti-vaxxers meet each others at a rate of $\tilde{q}^a := h + (1-h)q^a$, while vaxxers meet each others at a rate of $\tilde{q}^v := h + (1-h)q^v = h + (1-h)(1-q^a)$. Note that h is the same for both groups, but if $q^a \neq q^v$ and $h > 0$, then $\tilde{q}^a \neq \tilde{q}^v$.

For each $g \in G$, let $x^g \in [0,1]$ denote the fraction of agents in group g that are vaccinated against our generic disease. It is natural to assume, without loss of generality, that $x^a < x^v$, and by now this is actually the only difference characterizing the two groups. Let μ be the recovery rate of the disease, while its infectiveness is normalized to 1.

3.2.1 The dynamical system

Setting the evolution of the epidemic in continuous time, we study the fraction of infected people in each group. When this does not generate ambiguity, we drop time indexes from the variables. For each $i \in G$, let ρ^i be the share of infected agents in group i. Since vaccinated agents cannot get infected, we have $\rho^a \in [0, 1 - x^a]$ and $\rho^v \in [0, 1 - x^v]$, respectively.

The differential equations of the system are given by:

$$\begin{aligned} \dot{\rho}^a &= \big(1 - \rho^a - x^a\big)\Big(\tilde{q}^a\rho^a + (1-\tilde{q}^a)\rho^v\Big) - \rho^a\mu; \\ \dot{\rho}^v &= \big(1 - \rho^v - x^v\big)\Big(\tilde{q}^v\rho^v + (1-\tilde{q}^v)\rho^a\Big) - \rho^v\mu. \end{aligned} \tag{3.1}$$

For each $g \in G$, $\big(1 - \rho^g - x^g\big) \in [0,1]$ represents the set of agents who are neither vaccinated, nor infected, and thus susceptible of being infected by other infected agents. Moreover, the share of infected agents met by vaxxers

[12] h is the *imbreeding homophily* index, as defined in Coleman (1958), Marsden (1987), McPherson et al. (2001) and Currarini et al. (2009). It can be interpreted in several ways, as an outcome of choices or opportunities. As we assume that h can be affected by policies, we can interpret it as the amount of time in which agents are kept segregated by group, while in the remaining time they meet uniformly at random.

and anti-vaxxers is given by $\Big(\tilde{q}^a\rho^a + (1-\tilde{q}^a)\rho^v\Big)$ and by $\Big(\tilde{q}^v\rho^v + (1-\tilde{q}^v)\rho^a\Big)$, respectively. Finally, $\rho_a\mu$ and $\rho_v\mu$ are the recovered agents in each group.

Result 1 (Homophily and endemic disease)**.** The system (3.1) always admits a trivial steady state: $(\rho_1^a, \rho_1^v) := (0,0)$. For each h, there exists a $\hat{\mu}(h) > 0$ such that (i) if $\mu < \hat{\mu}(h)$, $(0,0)$ is unstable, whereas (ii) if $\mu > \hat{\mu}(h)$, $(0,0)$ is stable.[13]

This result is obtained in the standard way, by setting to zero the two right–hand side parts of the system in (3.1) and solving for ρ^a and ρ^v. The formal passages are in Appendix 3.D, as those of the other results that follow.

In the remaining of the paper, we focus on the case in which $\mu > \hat{\mu}(h)$, because it is consistent with diseases that are not endemic but show themselves in episodic or seasonal waves. For those diseases, society lays for most of its time in a steady state where no one is infected. However, exogenous shocks increase the number of infected people temporarily. Eventually, the disease dies out, as it happens, for example, for the seasonal outbreaks of flu.

Note that $\hat{\mu}(h)$ is increasing in h, so that we can highlight a first important role for h in the comparative statics. If h increases, it is possible that a disease that was not endemic, because $\mu > \hat{\mu}(h)$, becomes so because $\hat{\mu}(h)$ increases with h, and the sign of the inequality is reversed. Indeed, higher homophily counterbalances the negative effect that the recovery rate μ has on the epidemic outbreak.

3.2.2 Cumulative Infection

The main focus of our interest is to see what is the welfare loss due to the epidemic, and how this depends on the policy parameter h. In our simple setting, the welfare loss is measured by the total number of infected people over time, that is **cumulative infection**. For analytical tractability, we will approximate the dynamics of outbreaks with the linearized version of the dynamics $\hat{\rho}$, that satisfies:

$$\dot{\hat{\rho}}_t = \mathbf{J}\begin{pmatrix} \hat{\rho}_t^a \\ \hat{\rho}_t^v \end{pmatrix}, \qquad \hat{\rho}_0 = \begin{pmatrix} \rho_0^a \\ \rho_0^v \end{pmatrix} \tag{3.2}$$

where $\mathbf{J}$ is the Jacobian matrix of (3.1) calculated in the $(0,0)$ steady state, and $(\rho_0^a, \rho_0^v)'$ is the initial magnitude of the outbreak. We can think of it as

[13] Note that $\hat{\mu}(h) := \frac{1}{2}(T+\Delta) \in [0,1]$, where $T := \tilde{q}^a(1-x^a) + \tilde{q}^v(1-x^v)$ and $\Delta := \sqrt{T^2 - 4h(1-x^a)(1-x^v)}$. Δ is always positive and it is increasing in q. Moreover $\hat{\mu}(h) \in [0,1]$ and its value is $1 - x^v + q(x^v - x^a)$ for $h = 0$ and $1 - x^a$ for $h \to 1$.

the number of infected agents at the beginning of a particular outbreak or as the expected value of the initial number of infected agents according to some probability distribution.

The cumulative infection in the two groups and in the overall population is:

$$\begin{cases} CI^a := \int_0^\infty \hat{\rho}^a(t)dt \\ CI^v := \int_0^\infty \hat{\rho}^v(t)dt \\ CI := q^a CI^a + (1-q^a)CI^v \end{cases} \tag{3.3}$$

Note that, since q^a is fixed, CI takes into account both the number of infected agents of each group at each period and also the length of the outbreak. In the range of parameters for which $(0,0)$ is stable, all the integrals are finite, so here we do not add discounting, for simplicity. We will explore the implications of introducing time preferences in Section 3.2.3 below. The expressions can be found in Lemma 5 in the Appendix 3.D. To understand the mechanics that regulates the share of agents that get infected during the outbreak, let us consider three different types of initial conditions: The epidemic starts (i) among vaxxers ($\rho_0^v > 0$ and $\rho_0^a = 0$), (ii) among anti-vaxxers ($\rho_0^v = 0$ and $\rho_0^a > 0$), and (iii) in both groups symmetrically ($\rho_0^v = \rho_0^a > 0$).

First, we analyze which group has more infected agents throughout the epidemic.

Result 2 (Who is better off?)**.** The cumulative number of infected agents is such that $CI^a \geq CI^v$ if and only if:

$$\rho_0^v(1-x^a) - \rho_0^a(1-x^v) + \mu(\rho_0^a - \rho_0^v) \geq 0 \tag{3.4}$$

The result simply follows from comparing the explicit expressions for CI^a and CI^v (we derive it in Lemma 5 in the Appendix 3.D). Inequality (3.4) underlines the roles of the parameters in determining the welfare of the groups. The left–hand side is increasing in x^v and decreasing in x^a: the gap in vaccinations tends to penalize the less vaccinated group. Since the cumulative infection is an intertemporal measure, the initial conditions also concur in determining which group is better off: the difference is increasing in ρ_0^a and decreasing in ρ_0^v.[14] μ regulates the importance of this effect in the discrepancy of initial conditions: the larger μ, the shorter the epidemic, the larger the importance of the initial conditions. In particular, we have:

[14]Because the stability assumptions imply $-1+x^v+\mu>0$ and $-1+x^a+\mu>0$.

i) if the outbreak starts among vaxxers, vaxxers have a larger cumulative infection;

ii) if the outbreak starts among anti–vaxxers, anti–vaxxers have a larger cumulative infection;

iii) if the outbreak starts symmetrically in both groups, the group with less vaccinated (anti–vaxxers, under our assumptions) has the largest cumulative infection.

In particular, the evaluation of what group is better off in terms of infections is *independent* of homophily. However, the levels of contagion do depend on homophily, as the following result shows. It is obtained applying definitions from (3.3) and taking derivatives.

Result 3 (Effect of h and q^a)**.**

a) CI and CI^a are increasing (decreasing) in h if and only if $CI^a > CI^v$ ($CI^a < CI^v$); CI^v is decreasing (increasing) in h if and only if $CI^a > CI^v$;

b) CI, CI^a and CI^v are increasing (decreasing) in q if and only if $CI^a > CI^v$ ($CI^a < CI^v$).

In particular, the marginal effects of h and q^a for different outbreak types are those reported in Table 3.1.

	If the outbreak is among···		
	vaxxers	anti–vaxxers	symmetric: $\rho_0^a = \rho_0^v$
the effect of h is:	$\frac{\partial CI^a}{\partial h} < 0$, $\frac{\partial CI^v}{\partial h} > 0$, $\frac{\partial CI}{\partial h} < 0$	$\frac{\partial CI^a}{\partial h} > 0$, $\frac{\partial CI^v}{\partial h} < 0$, $\frac{\partial CI}{\partial h} > 0$	$\frac{\partial CI^a}{\partial h} > 0$, $\frac{\partial CI^v}{\partial h} < 0$, $\frac{\partial CI}{\partial h} > 0$
the effect of q^a is:	$\frac{\partial CI^a}{\partial q^a} < 0$, $\frac{\partial CI^v}{\partial q^a} < 0$, $\frac{\partial CI}{\partial q^a} < 0$	$\frac{\partial CI^a}{\partial q^a} > 0$, $\frac{\partial CI^v}{\partial q^a} > 0$, $\frac{\partial CI}{\partial q^a} > 0$	$\frac{\partial CI^a}{\partial q^a} > 0$, $\frac{\partial CI^v}{\partial q^a} > 0$, $\frac{\partial CI}{\partial q^a} > 0$

Table 3.1: Marginal effects of h and q^a on CI^a, CI^v, and CI, when there is an outbreak among vaxxers, anti–vaxxers, or symmetrically in both groups.

The previous results show how initial conditions and parameters contribute to determining the effect of an increase in h. As anticipated in the introduction, if the initial parameters are such that $CI^a = CI^v$, then both the total infection, CI, and the group level ones, CI^a and CI^v, do not depend on homophily. If instead, the initial parameters are such that $CI^a \neq CI^v$, then homophily hurts the group with more infected, because it causes the infection to spread to more members of the group and less outside. Table 3.1 helps us understand the behavior in prototypical cases and analyze whether a policy that increases h has the desired effect.

To better understand the mechanics, let us first focus on the effects of homophily (first row of Table 3.1). First note that, if the outbreak happens just in one of the two groups, homophily protects the group that is not infected ex-ante. So, intuitively, the outbreak has the strongest effect in terms of infected agents in the group in which the outbreak has taken place. The effect of homophily on the overall CI is however ambiguous and depends on the initial condition.

Consider first the case in which the outbreak takes place among vaxxers. Then, at the beginning, the infection takes over among the group with the highest vaccination rate, since $x^v > x^a$. The higher the homophily h, the more vaxxers interact with each other, and thus the more the infection remains within the group that is more protected against it. For this reason, the higher h, the less the CI. For the opposite reason, if the outbreak takes place in the anti-vaxxers group, homophily makes infection stay more in the less protected group, and CI increases.

So the crucial message is that a policy having the effect of increasing h cannot be considered unanimously beneficial neither from a planner concerned with total infection, nor from a planner concerned with just infection among vaxxers.

To understand the role of q^a on the CI (second row of Table 3.1), recall that a higher q^a means a higher share of agents less protected against the disease. Consider first the case in which the outbreak takes place in the vaxxers group. Then, a higher q^a means that the number of infected agents, which are in the v group, is lower. Thus, all CI measures are decreasing in q^a. For the opposite reasoning, all CI measures are increasing in q^a if the outbreak takes place in the anti-vaxxers group. If the outbreak is symmetric, then the two forces mix. However, if q^a increases, the share of agents who are not protected against the disease increases, and thus CI measures increase.

3.2.3 Time preferences

In this section we explore the implications of the degree of impatience of the planner on the evaluation of the impact of homophily. Time preferences can be crucial for the planner. As we have seen, for example, in the Covid19 epidemic, the planner, given a CI, may prefer not to have all infected agents soon because of some capacity constraints of the health system.

For example, Figure 3.1 shows the time evolution of the infection of both groups and the overall society in case of an outbreak among the vaxxers. In

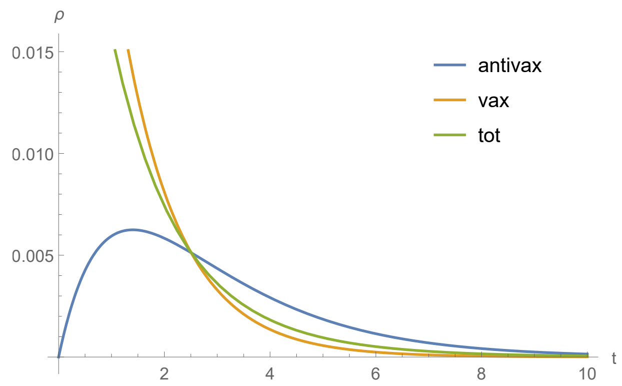

Figure 3.1: CI as a function of time in case the outbreak starts among vaxxers ($\rho_0^a = 0$). Here $\rho_0^v = 0.1$, $x_a = 0.3$, $x_v = 0.9$, $q = 0.3$, $h = 0.5$, $\mu = 1$.

this case, since the outbreak starts among the vaxxers, it is among this group that infection is higher initially. In contrast, eventually infection becomes larger among the anti–vaxxers, due to the lower vaccination levels. The effects on cumulative infection depend on how the planner trades off today and tomorrow infections: the more the planner is patient, the more the infection among anti–vaxxers becomes prominent.

Moreover, since in our setting the impact of segregation policies depends on the relative amount of infected agents in the two groups, as specified in Result 3, in our context the time preference is also crucial for the evaluation of the impact of homophily on the total cumulative infection.

Thus, we first define the *discounted* cumulative infection:

$$\begin{cases} CI^a := \int_0^\infty e^{-\beta t}\rho^a(t)dt \\ CI^v := \int_0^\infty e^{-\beta t}\rho^v(t)dt \\ CI := q^a CI^a + (1-q^a)CI^v \end{cases} \tag{3.5}$$

where $\beta > 0$ is the discount rate. Analytically, things turn out to be very simple, due to the exponential nature of the solutions, as the following observation lays out.

Result 4. Discounted cumulative infections are equivalent to cumulative infections in a model with recovery rate $\mu' = \mu + \beta$.

This is not too surprising: μ is a measure of how fast the epidemic dies out, and β is a measure of how fast the welfare loss dies out. The previous result carries on even when, as we do in the following sections, choices on vaccination and on types are made endogenous.

The impact can be made more precise if we stick to exogenous choices, as it is done below.

Result 5. In the model with discounting:
$CI^a \geq CI^v$ if and only if $-\rho_0^a(1-x^v)+\rho_0^v(1-x^a)+(\mu+\beta)(\rho_0^a-\rho_0^v)\geq 0$

The proof is immediate from the previous result and from Result 3.4. In details:

1. An increase in the degree of impatience β makes initial conditions more important in the welfare evaluation. For example, without time preferences, we may have that $\rho_0^a < \rho_0^v$ but $CI^a > CI^v$, because the difference in vaccinated agents dominates the difference in the initial outbreak. However, if time preferences are introduced, or β gets larger, a planner may evaluate that $CI^a < CI^v$ because she is putting more weight on the earlier moments of the epidemic.

2. An increase in the degree of impatience β can change the impact of homophily, as illustrated in Figure 3.2. To understand this point, given a population share q, there exists a β such that homophily does not impact the CI (with time preferences). In this CI, groups get infected at different rates over time. As we change β, the planner gives more weight to the group getting infected earlier. As we have seen above, homophily plays a role in this process, keeping the infection more into each group. In Figure 3.2, we consider the case in which $q = .3$, so that there are more vaxxers than anti-vaxxers, and vaxxers are also more vaccinated. Thus, the more the planner is impatient, the more she is satisfied by the fact that most agents (vaxxers) are less infected when homophily increases.

3.2.4 Convergence time

In this section, we show that homophily slows down the diffusion dynamics, as has already been studied in a different context by Golub and Jackson (2012). We consider as a measure of convergence time the magnitude of the leading eigenvalue, which in this case is the one with the smallest absolute value. This is because the solution of our linear system is a linear combination of exponential terms whose coefficients are the eigenvalues (which are negative

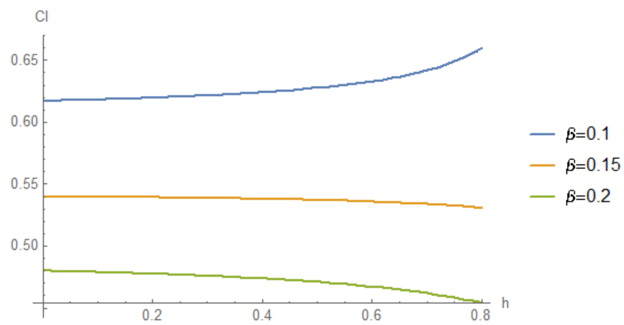

Figure 3.2: Cumulative infection as a function of homophily for different values of time preference. Here $\mu = 0.7$, $x^a = 0.2$, $x^v = 0.9$, $q = 0.3$.

by stability). Hence, when t is large, the dominant term is the one containing the eigenvalue which has smallest absolute value.[15]

Result 6. Consider a perturbation around the stable steady state $(0, 0)$. The time of convergence (as measured by the leading eigenvalue) back to $(0, 0)$ is increasing in h.

This result shows that homophily, by making the society more segregated, makes the convergence to the zero infection benchmark slower once an outbreak occurs. This is obtained by analyzing the eigenvalues of the Jacobian matrix, computed in the steady state. All results are obtained analytically (see Appendix 3.D), and the resulting eigenvalues are decreasing in absolute value in h.

If we look at the effects of other parameters, we have that the eigenvalues are increasing in absolute value in both x^a and x^v. This is because a larger number of vaccinated agents means a smaller space for infection to diffuse. Finally, since $x^a < x^v$, then the smallest eigenvalue is decreasing (in absolute value) in q^a, while the largest eigenvalue is increasing. Since the long-run

[15] We should be careful, though, because this is true non–generically outside of the *eigendirection* of the *second* eigenvector. Indeed, in our case the eigenvectors are:

$$\boldsymbol{e}_1 = \left(-\frac{(1 - x^v)\,\tilde{q}^a + (x^a - 1)\,\tilde{q}^a + \Delta}{2\,(1 - x^v)\,(1 - \tilde{q}^a)}, 1 \right)$$

and

$$\boldsymbol{e}_2 = \left(\frac{-\,(1 - x^v)\,\tilde{q}^a - (x^a - 1)\,\tilde{q}^a + \Delta}{2\,(1 - x^v)\,(1 - \tilde{q}^a)}, 1 \right) \quad .$$

So, we can see that the first eigendirection does not intersect the first quadrant, while the second does. Hence, we should remember that the first eigenvalue is a measure of the speed of convergence only generically, outside of the eigendirection identified above.

dynamics (i.e. asymptotic convergence) depends on the smallest eigenvalue, this means that the dynamics is *asymptotically slower* the larger the fraction of the population with less vaccinated agents.

Results 3.1, 3 and 6 in this section provide clear implications that should be taken into account when considering policies that affect the level of homophily h in the society. Any increase in segregation between vaxxers and anti–vaxxers may induce the disease to become endemic. Additionally, a larger h, if there is a temporary outbreak, will slow down the recovery time, and in some cases (i.e. when the outbreak does not start only among va,xxers), it may increase the cumulative infection caused by the disease.

When applied to the real world, the results of this section can be seen as first-order effects because, in general, a policy that changes h may have effects also on x^a, x^v, and q^a. Indeed, in the following sections, we endogenize the shares of vaccinated agents and the shares of vaxxers and anti–vaxxers in the society. The outcomes then depend also on the second order effects: the impact of homophily on the endogenous variables.

3.3 Vaccination choices

In this section we start introducing elements of endogeneity. First of all we consider vaccination choices, to make the shares x^a and x^v endogenous. To begin with, we still consider the partition of our society in anti-vaxxers and vaxxers, q^a and q^v, as exogenously fixed (we will relax this assumption in the next section). In our approach, people take vaccination decisions ex-ante, before an epidemic actually takes place, and cannot update their decision during the diffusion. This mimics well diseases, like seasonal flu, for which the vaccine takes a few days before it is effective, and the disease spreads rapidly among the population.

We model the behavior of agents who consider the trade-off between paying some fixed cost for vaccinating or incurring the risk of getting infected, and thus paying with some probability a cost associated with health. We need to set some assumptions about vaccination costs and agents' perception of the risk of being infected. Now that x^a and x^v are endogenous, it is the difference in the perception of costs that characterizes the difference between the two groups.[16]

[16]See, for example, Bricker and Justice (2019) and Greenberg et al. (2019) for a recent analysis of the anti–vaxxers arguments: Those are mostly based on conspiracy theories that attribute hidden costs to the vaccination practice and not so much on minimizing the effects of getting infected. Our model would not change dramatically if we attribute the difference in perception on the costs of becoming sick (see equations (3.8) and (3.9)), but

Vaccination costs Vaxxers and anti-vaxxers have different perceptions about vaccination costs. For vaxxers we assume that vaccination costs are $c^v \sim U[0,1]$, while for anti-vaxxers $c^a \sim U[d, 1+d]$, with $d > 0$. This is to say that anti-vaxxers perceive a higher cost of getting a vaccine than vaxxers do.

Risk of infection We assume that agents think about the risk of infection as proportional to the fraction of unvaccinated people that they meet. This is reasonable, because they form a belief before the actual outbreak occurs (e.g., agents decide to vaccinate against flu a few months before winter). Agents multiply this fraction of unvaccinated people by a factor $k > 0$, that represents the perceived damage from the disease, which is the same for the two groups. Let σ^v be the share of unvaccinated people met by vaxxers, then

$$\sigma^v = \tilde{q}^v(1-x^v) + (1-\tilde{q}^v)(1-x^a) \quad , \tag{3.6}$$

so that vaxxers perceive the risk of infection to be $k\sigma^v$. Similarly

$$\sigma^a = \tilde{q}^a(1-x^a) + (1-\tilde{q}^a)(1-x^v) \quad , \tag{3.7}$$

so that anti–vaxxers perceive the risk of infection to be $k\sigma^a$. Now, only those for which costs are lower than perceived risk decide to vaccinate. So:

$$x_v^* = \min\{k\sigma_v, 1\} \quad , \tag{3.8}$$

whereas, for an anti–vaxxer, we have

$$x_a^* = \max\{0, \min\{k\sigma_a - d, 1\}\} \quad . \tag{3.9}$$

The two solutions are both interior, for any value of the other parameters, whenever $d < \min\left\{\frac{1}{k^2}, \frac{k}{k+1}\right\}$, and we call this the *interiority condition*, which is analyzed in depth in Appendix 3.A. We use interiority condition as a maintaned assumption for the remainder of the paper. In this case, equations (3.6)–(3.9) imply a system that provides

$$\begin{aligned} x^a &= 1 - \frac{1+dq^a}{1+k} - \frac{d(1-q^a)}{1+hk} \\ x^v &= 1 - \frac{1+dq^a}{1+k} + \frac{dq^a}{1+hk} \quad . \end{aligned} \tag{3.10}$$

we stick to the first interpretation because it makes the computations cleaner.

In what follows, we focus on the interior solutions. In Appendix 3.B we analyze numerically the case in which $x^v = 0$, and we see that the qualitative results are analogous to those that we present here.

First of all, we note that (i) $x^v > x^a$ - since vaxxers perceive a lower vaccination costs than anti-vaxxers; (ii) x^a is increasing in h whereas x^v is decreasing in h - since a higher homophily makes vaxxers more in contact with agents who are less susceptible than anti-vaxxers and, as a consequence, $(x^v - x^a)$ is decreasing in h; (iii) x^a and x^v are increasing in q^a - since the higher the share of anti-vaxxers, the more agents are in touch with other subjects at risk of infection; (iv) the total number of vaccinated people is $q^a x^a + (1 - q^a) x^v = \frac{k - dq^a}{1+k}$, it is independent of h, but decreasing in q^a - this is due to a Simpson paradoxical effect: both groups vaccinate more, but since anti-vaxxers increase, in the aggregate vaccination decreases.

We can also examine cumulative infection rates in a neighborhood of the stable steady state $(0, 0)$. We limit ourselves to the symmetric initial condition $\rho_0^a = \rho_0^v$, that can be compared with Result 3, looking at the third column of Table 3.1. Also, analytical tractability is obtained only for values of h that are *small*, as would be the effect of a policy that limits contacts between vaxxers and anti-vaxxers only in a few of the daily activities (e.g. only in schools).

Proposition 15. *Consider a perturbation around* $(0, 0)$*, such that* $\rho_0^a = \rho_0^v > 0$*. Then, there exists* $\bar{h} > 0$ *such that, if* $h < \bar{h}$*:*

(i) $\partial x^a / \partial h = -\partial x^v / \partial h$*;*

(ii) CI *is increasing in* h*;*

(iii) CI *is increasing in* q^a *(but the marginal effect is lower than in the exogenous case of Result 3).*

This proposition analyzes what happens for an outbreak that is symmetric in the two groups when homophily is *low* enough. We have already seen above that the effect of homophily is opposite for x^a and x^v. Here we find that the magnitude of the effects is the same for both groups and that the higher the homophily, the higher CI. Thus, homophily policy does not seem to be a good policy to be implemented in these cases. At the same time, the more the anti-vaxxers, the more the number of infected agents.

To complete our analysis of endogenous choices, in the next section we also endogenize the partition between the two groups.

3.4 Endogenous groups

In the previous section we have illustrated the trade-off faced by agents between two different costs: the act of vaccinating, and the risk of being infected, which is based on the fraction of unvaccinated agents they expect to meet, σ_v and σ_a. We now consider how the shares of vaxxers and anti-vaxxers change with time, that is how q is determined. In the real world, this decision does not seem to be updated frequently, and can be considered as fixed during a single flu season. So, in the model, we assume that this decision is taken before actual vaccination choices, which are in turn taken before the epidemic eventually starts. Our aim here is to offer a simple and flexible theory of the diffusion of opinions to be integrated into our main epidemic model. As explained in the Introduction, the empirical observations that important drivers of vaccination opinions are peer effects and cultural pressure leads us to discard purely rational models, where the decision of not vaccinating descends from purely strategic considerations. Given the complex pattern of psychological effects at play, we opt for a simple reduced form model capturing the main trade-offs. In particular, we are going to assume the diffusion of traits in the population to be driven by *expected advantages*: the payoff advantage that individuals in each group estimate to have with respect to individuals in the other group. This is made precise by the next definition.

Definition 3.4.1 (Expected advantage)**.** Consider an individual in group a. Define ΔU^a as the *Expected advantage* individual a estimates to have with respect to individuals in group v. Specifically:

$$\Delta U^a = U^{a\to a} - U^{a\to v} \tag{3.11}$$

$$U^{a\to a} = -\mathbb{E}_c\left[(c+d)\mathbb{1}_{k\sigma^a - d < c} + k\sigma^a \mathbb{1}_{k\sigma^a - d \geq c}\right] \tag{3.12}$$

$$U^{a\to v} = -\mathbb{E}_c\left[(c+d)\mathbb{1}_{k\sigma^v < c} + k\sigma^v \mathbb{1}_{k\sigma^v \geq c}\right] \tag{3.13}$$

where $U^{a\to a}$ is the payoff of individuals with trait a evaluated by an individual with trait a, while $U^{a\to v}$ is the payoff of individuals with trait v evaluated by individuals with trait a. ΔU^v is defined analogously, and can be found in Appendix 3.C.2.

Agents in each group perceive a differential in expected utilities from being of their own group and being of the other group. Note that, apart from the bias d, agents correctly evaluate all other quantities, including the risks from the disease of the two groups, $k\sigma^v$ and $k\sigma^a$. Indeed, even if both groups evaluate the choice of the other group as suboptimal, this perceived difference can be negative for *anti–vaxxers*, because they understand that *vaxxers* have less chances of getting infected.

To clarify Definition 3.4.1, consider Figure 3.3. The black line is the disutility of agents in groups a, as a function of the cost c, *as perceived by agents in group* a. The shape of this line mirrors the fact that an agent in group a undertakes vaccination only if her costs are in the $[0, k\sigma_a - d]$ interval, in which a agents incur in a disutility $c + d$. If $c > k\sigma_a - d$, a agents do not vaccinate, and the disutility is the risk of infection, which is $k\sigma_a$. The grey area below this curve is then $U^{a\to a}$. Consider now the red line. This represents the disutiliy of agents in group v *as perceived by agents in group* a. In particular, agents in group v have a different perception of costs with respect to agents in group a, and so take different choices. In particular, they vaccinate in the $[0, k\sigma_v]$ interval, while if they are in the $[k\sigma_v, 1]$ interval they do not vaccinate and incur a risk of infection. Note, however, that this is the evaluation from the perspective of agents in group a, and thus the cost of vaccination is $c + d$ instead of c. Hence $U^{a\to v}$ is the area below the red curve. The difference ΔU^a is given by the red area minus the blue area. $U^{v\to v}$ and $U^{v\to v}$ are computed accordingly. The details of the calculation and the corresponding figure are in Appendices 3.C.1 and 3.C.2.

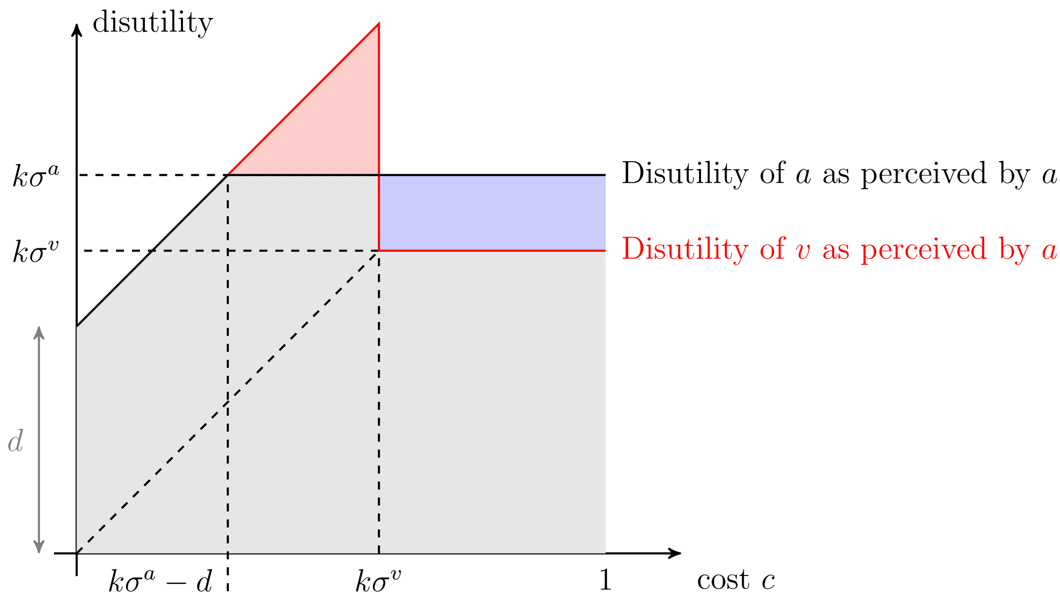

Figure 3.3: Composition of ΔU^a. The graph represents the disutility incurred by an individual as a function of its cost c. ΔU^a is the red area minus the blue area.

To ease notation, let $q = q^a$. Then, we make the following assumption over the population dynamics

Assumption 1. Given an $\alpha \in \mathbb{R}$, the level of q increases when $q^\alpha \Delta U^a > (1-q)^\alpha \Delta U^v$ and it decreases when $q^\alpha \Delta U^a < (1-q)^\alpha \Delta U^v$.

Clearly, the implication of the previous assumption is that the resting points of the dynamics are such that $q^\alpha \Delta U^a = (1-q)^\alpha \Delta U^v$, but stability has to be addressed. The simplest example of dynamics satisfying Assumption 1 is:

$$\dot{q} = q(1-q)[q^\alpha \Delta U^a - (1-q)^\alpha \Delta U^v] \ \ ,$$

but we allow also for any non linear generalization.

The dynamics obtained from Assumption 1 generalizes the standard workhorse model in cultural transmission, the one by Bisin and Verdier (2001), in two ways: (*i*) endogenizing the socialization payoffs, as from Definition 3.4.1, and (*ii*) introducing a parameter α regulating the *stickiness* agents have in changing their identity *via* social learning. Indeed, at the limit $\alpha \to \infty$, $\dot{q} = 0$ and types are fixed. Note also that α regulates the strength of *cultural substitution*, a phenomenon often observed in cultural transmission settings: the tendency of members of minorities to preserve their culture by exerting larger effort to spread their trait.[17] Thus, we are able to encompass different types of social dynamics. (*i*) If $\alpha = 0$ this is a standard replicator dynamics (see e.g. Weibull, 1997). (*ii*) If $\alpha < 0$, the model displays cultural substitution, as most standard cultural transmission models. Moreover, the more α is negative, the more there is substitution. In particular, if $\alpha = -1$ the dynamics has the same steady state and stability properties as the dynamics of Bisin and Verdier (2001).[18] (*iii*) If $\alpha > 0$, the model displays cultural complementarity, so that the smaller the minority, the less the effort exerted, and the less the minority survives. Note that cultural complementarity is increasing in α.

The environment of social influence is not only shaped by physical contacts and it is not the same of the epidemic diffusion of the actual disease (because in the real world many contacts are online and are channeled by social media). Hence, any policy on h can have a limited effect on it, because for us h is a restriction on the physical meeting opportunities. As a consequence, h does not appear explicitly in Assumption 1.

If $\Delta U^a = 0$, naturally there will be no anti–vaxxers. This will happen if for example the bias d is very high, or homophily is very high, so that the

[17] See Bisin and Verdier (2011).

[18] To be precise, the model by Bisin and Verdier (2001) refers to *intergenerational* transmission. In the Appendix 3.C we show how a similar equation can be recovered in a context of *intragenerational* cultural transmission

increased infection risk from being an anti-vaxxer (the blue area in Figure 3.3) is so large that no one wants to be an anti-vaxxer. This is of course an uninteresting case, so from now on we are going to assume the following:

Assumption 2 (**Interiority conditions**)**.** x_a, x_v, and ΔU^a are interior (details in terms of exogenous parameters are in the Appendix 3.A).

The following result shows that only the case $\alpha < 0$ is of some interest for the analysis, because in the other cases the population become all of one type, with $q^* = 0$ or $q^* = 1$.

Proposition 16. *Under the interiority conditions: (i) If $\alpha \geq 0$ there are no interior stable steady states of the dynamics for q. (ii) If $\alpha < 0$, there exists a $q^* \in (0, 1)$ such that $q^{*\alpha}\Delta U^a = (1 - q^*)^\alpha \Delta U^v$. Moreover, there is always an interior stable steady state and there exists a threshold $\overline{h}$ such that, for $h < \overline{h}$, the steady state is unique and stable.*

Again, the proof of this result is obtained with standard methods, applying the implicit function theorem to the condition from Assumption 1, looking at results for $h \to 0$, and using the continuity of the system to prove that results hold in an interval $[0, \bar{h})$ for h. The equilibrium level of q can be computed analytically only in the case $\alpha = -1$ (which is the case in which the dynamic is equivalent to Bisin and Verdier, 2001) and for $h = 0$, since the differences in payoffs across groups become null, and $q^* = \frac{1}{2}$.

We are now interested in the effect of an increase in homophily on q^*. Figure 3.4 shows, on the basis of numerical examples with $\alpha = -\frac{1}{2}$, $\alpha = -1$, and $\alpha = -3$, that homophily has a negative effect on q^* and that this result seems to extend to any $\alpha < 0$.

We can actually prove it analytically for *small* values of h.

Proposition 17. *Under the interiority conditions, and if $\alpha < 0$ there exists a threshold $\overline{h}$ such that, for $h < \overline{h}$, q^* is decreasing in h in the unique and stable steady state.*

The intuition is that a larger h magnifies the negative effects of being anti-vaxxers in terms of infection, relatively to vaxxers. This is internalized in the cultural dynamics, *via* the ΔUs. This long run effect of h on anti-vaxxers share is one of the few positive effects of segregating policies.
As we have done in the preliminary model with exogenous choices, we can analyze the effects of homophily on the cumulative infection, when the initial perturbation is symmetric across both groups (see Result 3, summarized in the third column of Table 3.1). We find that the effects depend on the magnitude of α, the parameter regulating how agents are rigid/prone towards social influence.

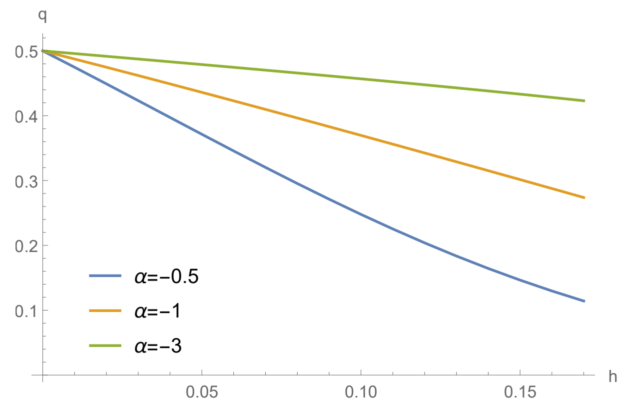

Figure 3.4: q as a function of h. $d = 0.5$, $k = 1$, $\mu = 1$. The range of h is restricted as prescribed by the interiority conditions 3.A

Proposition 18. *Consider the model with endogenous q, $\alpha < 0$ and interiority conditions. Consider an outbreak affecting both groups symmetrically, starting from the unique stable steady state and $h = 0$. Then, there exists a threshold $\overline{\alpha}$ such that:*

- *if $\alpha < \overline{\alpha}$, CI is increasing in h* $\left(\frac{\mathrm{d}CI}{\mathrm{d}h}\big|_{h=0} > 0 \right)$*;*
- *if $\alpha > \overline{\alpha}$, CI is decreasing in h* $\left(\frac{\mathrm{d}CI}{\mathrm{d}h}\big|_{h=0} < 0 \right)$.

With this proposition we consider the effects of the *introduction* of some form of segregation policy, taking also into account the cultural dynamics. This means that if α is *large* in magnitude (the first bullet point, since α is negative) then the society is rigid in its opinions, and the effects are qualitatively the same that we would have if types and vaccination choices were fixed (Result 3). If instead α is *small* in magnitude (the second bullet point), then the reaction of q^* to a policy change of h is large, and this reverts the effect: cumulative infection is decreasing in homophily. In this last case, the effects of a policy based on partial segregation will be the desired ones. In this respect, how agents are subjected to social influence can revert the effects of a policy based on homophily. Figure 3.5 shows this effect for two values of $\alpha < 0$. These are also compared with what would happen, with the same parameters, under the assumptions of Result 3 (all choices are exogenous) and Proposition 15 (only vaccination choices are endogenous, but groups are fixed). The figure shows that, only when α is negative and small in absolute value, the cumulative infections decreases in homophily. In all the other cases a policy based on homophily can increase the cumulative infection at various degrees.

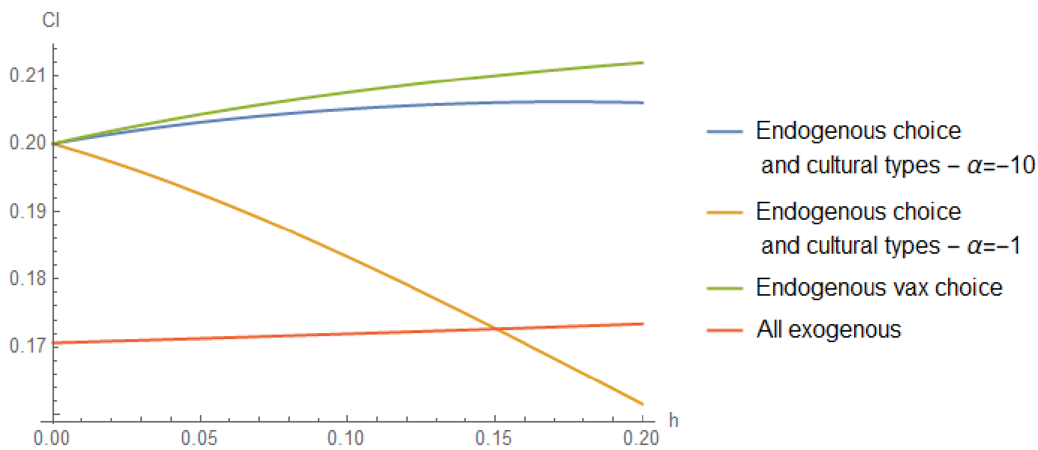

Figure 3.5: Cumulative infection in the three models. Whenever exogenous, q, x^a and x^v are set using the median value of $h = 0.1$. The other parameters are set at $k = 2$, $d = 0.5$, $\mu = 1$, $\rho_0^a = \rho_0^v = 0.1$.

The intuition for the different marginal effects of h on cumulative infection seems to lie on the marginal effects on the speed of the dynamics, via the first eigenvalue (see Result 6), as Figure 3.6 illustrates: the cases in which cumulative infection increases with h are those in which the leading eigenvalue is decreasing in magnitude, and vice versa.

Note that Result 4 about the discount rates of a policy maker are still valid with endogenous choices, as we discussed in Section 3.2.3.

3.5 Conclusion

The problem of vaccine skepticism is a complex one, that requires analysis from multiple angles: psychological, medical, social. In this paper, we propose an analysis of the trade-offs faced by a policy maker interested in minimizing infection in a world with vaxxers and anti–vaxxers, having available a policy inducing some degree of segregation, or homophily, h. The key observation is that reducing contact with anti–vaxxers may be counterproductive both from the perspective of vaxxers and the society as a whole because it slows down the dynamics of the disease to its steady state, if there is an outbreak. Homophily may actually increase the duration of the outbreaks, and depending on the time preferences of the planner this might crucially change the impact of the policy. Further, if cultural types are endogenous, the intensity of cultural substitution is key in determining the impact of the policy. Our results suggest that the study of policy responses to the spread

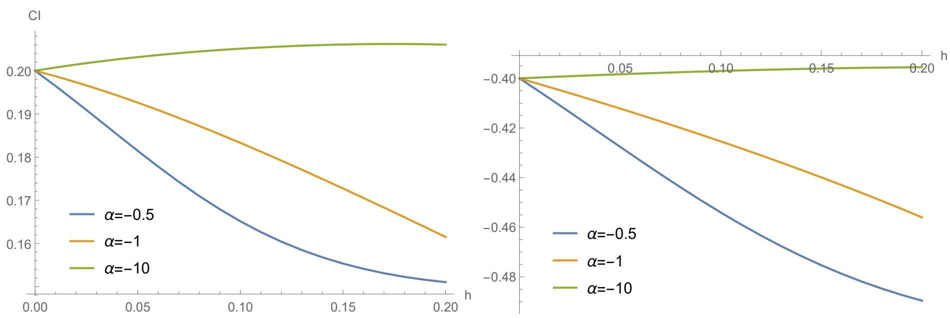

Figure 3.6: **Left panel**: Cumulative infection as function of homophily in the interior equilibrium. The other parameters are set at $k = 2$, $d = 0.5$, $\mu = 1$, $\rho_0^a = \rho_0^v = 0.1$. **Right panel**: corresponding leading eigenvalue of dynamical system as a function of h.

of vaccine-hesitant sentiment would benefit from trying to pin down more precisely the intensity of these mechanisms.

Appendices

3.A Interiority conditions

In Section 3.3 we have included endogenous choices about vaccinations, using two parameters, d and k, for the beliefs about the expected costs of vaccination and of becoming infected. We assume *interiority conditions*, which are essentially the conditions for which $\Delta U^a > 0$ (from (3.11)). The conditions under which ΔV^a is positive is $\frac{2hk(hk+1)}{k+1} < d$. For this to be compatible with x^a and x^v being interior, we need $d < \min\{\frac{1}{k}, \frac{1}{k(1+k)}\}$, hence we need also $2h(1 + hk) < \frac{1}{k^2}$. So in addition to k high enough we also need h small enough. Figure 3.7 shows the regions in the h–d plane for which the interiority conditions are satisfied, depending on the value of k.

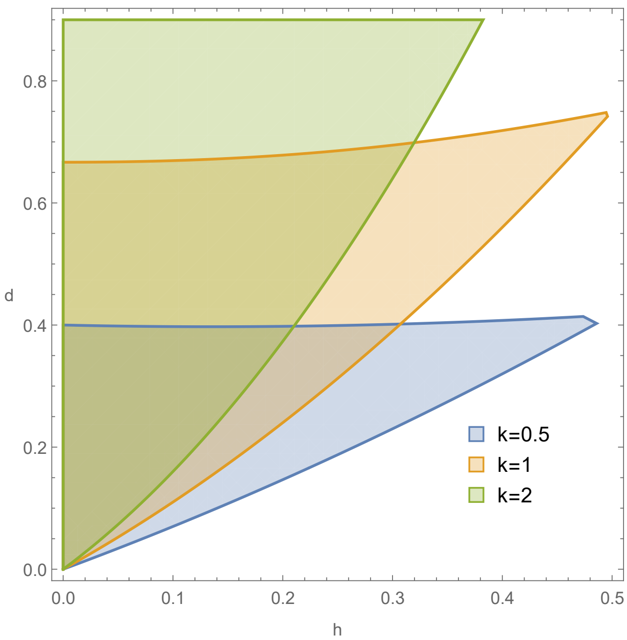

Figure 3.7: Region of parameters where all endogenous variables are interior. μ is fixed to 1.

3.B Analysis of corner solution $x_a = 0$

In this section we explore the case in which the interiority conditions are not satisfied, and d is large enough so that the unique equilibrium in the vaccination game is:

$$x^a = 0 \ , \tag{3.14}$$

$$x^v = \frac{k}{(h-1)kq + k + 1} \ , \tag{3.15}$$

provided $d < 1/k$ so that $x^v \neq 0$.[19] Indeed, if both $x^a = 0$ and $x^v = 0$, then everything is exogenous and we are back to the mechanical model.

Note that x^v maintains the properties we expect: it is decreasing in h and increasing in q, as can be directly seen from the expression above.

Cumulative infection

Consider a symmetric initial conditions. Since x^v is decreasing in h, it means that now when h increases we have the direct effect on CI which is increasing, plus a decrease in vaccination, which further increases the effect on CI. The effect of x^a, which is of the opposite sign, disappears in the computations, so we have exactly the same behavior as in the interior solution, and, moreover, in this case the negative effect of h on CI is even stronger. The derivatives are:

$$\frac{\partial CI}{\partial h} = -\frac{k^2(q-1)q\rho_0^v}{2\mu(k(h(\mu-1)q + \mu - \mu q) + \mu - 1)^2} \tag{3.16}$$

$$\frac{\partial CI}{\partial q^a} = \frac{k\rho_0^v((h-1)kq + k + 1)}{2(k(h(\mu-1)q + \mu - \mu q) + \mu - 1)^2} \tag{3.17}$$

$$\frac{\partial CI}{\partial x_a} = -\frac{q\rho_0^v((h-1)kq + k + 1)^2}{2(k(h(\mu-1)q + \mu - \mu q) + \mu - 1)^2} \tag{3.18}$$

$$\frac{\partial CI}{\partial x_v} = \frac{(q-1)\rho_0^v((h-1)kq + k + 1)^2}{2(k(h(\mu-1)q + \mu - \mu q) + \mu - 1)^2} \tag{3.19}$$

[19] This is possible if $\frac{hk^2+k}{hkq-kq+k+1} < d < \frac{1}{k}$ and either $k < 1$ or $\left(1 < k < \frac{1}{2}\left(1+\sqrt{5}\right) \wedge 0 < q < \frac{-k^2+k+1}{k} \wedge 0 < h < \frac{-k^2-kq+k+1}{k^3-kq}\right)$

Endogenous cultural types

The socialization payoffs are:

$$\Delta U^a = -k\sigma^a + dk\sigma^v + \frac{1}{2}k^2\sigma^{2v} + k\sigma^v(1-k\sigma^v) \tag{3.20}$$

$$= \frac{k(2(d-hk)((h-1)kq+k+1)-k)}{2((h-1)kq+k+1)^2} \tag{3.21}$$

$$\Delta U^v = k\sigma^a - \frac{1}{2}k^2\sigma^{2v} - k\sigma^v(1-k\sigma^v) \tag{3.22}$$

$$= \frac{k^2(2h((h-1)kq+k+1)+1)}{2((h-1)kq+k+1)^2} \tag{3.23}$$

We can apply the intermediate value theorem provided ΔU^a does not become negative. $\Delta U^a(q=0) = -\frac{k(2(k+1)(hk-d)+k)}{2(k+1)^2}$. If $h < \frac{1}{2}$, this is positive under the condition $\frac{hk^2+k}{hkq-kq+k+1} < d$. So for h small enough we get an interior solution for $\alpha < 0$.

It is not possible to obtain analytical results in the case of q endogenous. Nevertheless, numerical simulations reveal a picture very similar to the one described in the case of interior solution, in the main text. Specifically, the magnitude of α is crucial to determine the effect of an increase in homophily, as illustrated in Figure 3.8. Again, the main mechanism through which homophily acts is via the increased length of the outbreak, as measured by the leading eigenvalue, as shown in Figure 3.9.

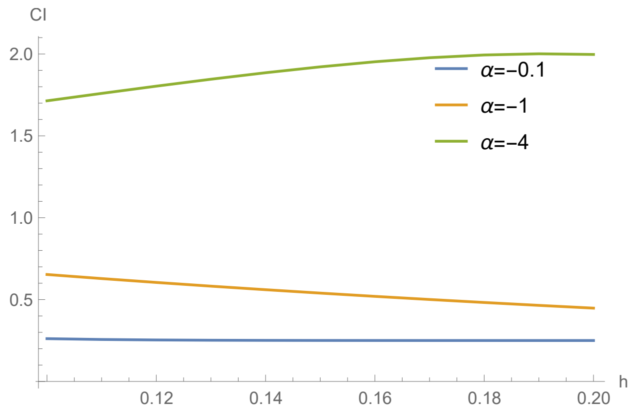

Figure 3.8: Cumulative infection as function of homophily in the equilibrium where $x^a = 0$. The other parameters are set at $k = 1$, $d = 0.6$, $\mu = 0.7$, $\rho_0^a = \rho_0^v = 0.1$.

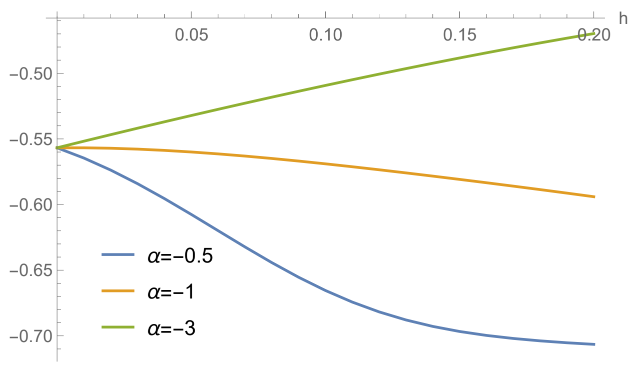

Figure 3.9: Eigenvalues in the equilibrium where $x_a = 0$. The other parameters are set at $k = 1$, $d = 0.6$, $\mu = 0.7$, $\rho_0^a = \rho_0^v = 0.1$

3.C Endogenous groups

3.C.1 A simple model of intragenerational cultural transmission

In this section we illustrate how equation (1) with $\alpha = -1$ can arise from a simple adaptation of the Bisin and Verdier (2001) model to an intragenerational context.

At each time period, each agent meets another agent selected randomly. When they meet, they are assigned two roles: the *influencer* and the *target*. The incentive for the influencer is based only on *other–regarding* preferences, for two reasons: it is consistent with some survey evidence (Kümpel et al. 2015, Walsh et al. 2004), and in this economy every agent has negligible impact on the spread of the disease, so socialization effort cannot be driven by the desire to minimize the probability of infection, or similar motivations. The timing of the model is as follows.

- Before the matching, agents choose a **proselitism effort** level τ_t^a, τ_t^v;
- When 2 agents meet, if they share the same cultural trait nothing happens. Otherwise, one is selected at random with probability $\frac{1}{2}$ to exert the effort and try to have the other change cultural trait.

The fraction of cultural types evolves according to:

$$q_{t+1}^a = q_t^a P_t^{aa} + (1 - q_t^a) P_t^{va}, \tag{3.24}$$

where the transition rate P_t^{aa} is the probabilities that an agent a is matched with another agent who, next period, results to be of type a and P_t^{va} is the probabilities that an agent v is matched with another agent who, next period, results to be of type a. These probabilities are determined by efforts according to the following rules:

$$P_t^{aa} = \tilde{q}_t^a + (1 - \tilde{q}_t^a)\frac{1}{2} + (1 - \tilde{q}_t^a)\frac{1}{2}(1 - \tau_t^v), \tag{3.25}$$

$$P_t^{va} = \frac{1}{2}(1 - \tilde{q}_t^a)\tau_t^v, \tag{3.26}$$

(P_t^{vv} and P_t^{av} are defined similarly) which yield the following discrete time dynamics:

$$\Delta q_t^a = q_t^a (1 - q_t^a)(1 - h)\Delta\tau_t, \tag{3.27}$$

where $\Delta\tau_t := \tau_t^a - \tau_t^v$.

Effort has a psychological cost, which, as in Bisin and Verdier (2001), we assume quadratic. Hence, agents at the beginning of each period (before the matching happens) solve the following problem:

$$\max_{\tau_t^a} \underbrace{-\frac{(\tau_t^a)^2}{2}}_{\text{cost of effort}} + \underbrace{q_t^a U_t^{a\to a} + (1-q_t^a)\frac{1}{2}(\tau_t^a U_t^{a\to a} + (1-\tau_t^a)U_t^{a\to v})}_{\text{expected social payoff}}, \tag{3.28}$$

which yields as a solution:

$$\tau_t^a = (1-q_t^a)\underbrace{(U_t^{a\to a} - U_t^{a\to v})}_{\text{"cultural intolerance"}}, \tag{3.29}$$

$$\tau_t^v = (1-q_t^v)(U_t^{v\to v} - U_t^{v\to a}). \tag{3.30}$$

Hence, the dynamics implied by our assumptions is:

$$\Delta q_t^a = q_t^a(1-q_t^a)((1-q_t^a)\Delta U^a - q_t^v \Delta U^v). \tag{3.31}$$

The steady state of this dynamics is determined by the equation:

$$(1-q_t^a)\Delta U^a = q_t^v \Delta U^v \tag{3.32}$$

which is precisely the steady state implied by (1) when $\alpha = -1$.

3.C.2 Socialization payoffs

We have:

$$U^{v\to v} = -\int_0^{k\cdot\sigma_v} c\ dc - \int_{k\cdot\sigma_v}^1 (k\cdot\sigma_v)\ dc \quad ; \tag{3.33}$$

$$U^{a\to a} = -\int_0^{k\cdot\sigma_a - d} (c+d)\ dc - \int_{k\cdot\sigma_a - d}^1 (k\cdot\sigma_a)\ dc \quad ; \tag{3.34}$$

$$U^{v\to a} = -\int_0^{k\cdot\sigma_a - d} c\ dc - \int_{k\cdot\sigma_a - d}^1 (k\cdot\sigma_a)\ dc \quad ; \tag{3.35}$$

$$U^{a\to v} = -\int_0^{k\cdot\sigma_v} (c+d)\ dc - \int_{k\cdot\sigma_v}^1 (k\cdot\sigma_v)\ dc \quad . \tag{3.36}$$

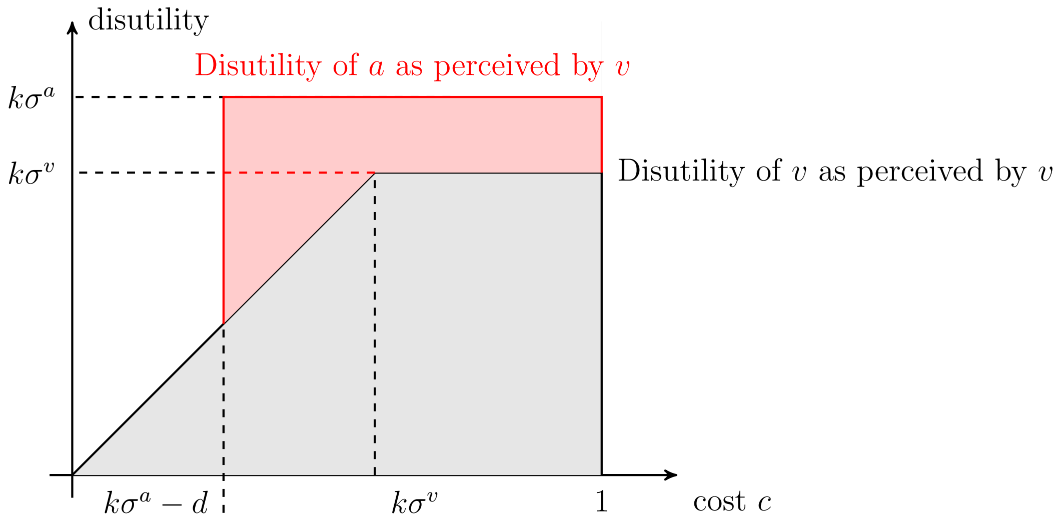

Figure 3.10: Composition of ΔU^v. The area below the black line is $U^{v\to v}$, the are below the red line is $U^{v\to a}$, ΔU^v is the red area.

Integration and the use of 3.10 yields:

$$\Delta U^a = \frac{1}{2}(x_v - x_a)^2 - (d - (x_v - x_a))(1 - x_v) \tag{3.37}$$

$$= \frac{d\left(d\left(-2(h-1)hk^2q + k + 1\right) - 2hk(hk+1)\right)}{2(k+1)(hk+1)^2} \tag{3.38}$$

$$\Delta U^v = \frac{1}{2}(x_v - x_a)^2 + (d - (x_v - x_a))(1 - x_a) \tag{3.39}$$

$$= \frac{d(d(2hk((h-1)kq + k + 1) + k + 1) + 2hk(hk+1))}{2(k+1)(hk+1)^2} \tag{3.40}$$

Figure 3.10 shows the composition of $\Delta U^v = U^{v\to v} - U^{v\to a}$. It is the analogous of Figure 3.4.1 for ΔU^a.

3.D Proofs

Proof or Result 1

Proof. To analyze stability, we need to identify the values of parameters for which the Jacobian matrix of the system is negative definite when calculated in $(0,0)$. The matrix is:

$$\mathbf{J} = \begin{pmatrix} (1-x_a)\,\tilde{q}^a - \mu & (x_a - 1)\,(\tilde{q}^a - 1) \\ (x_v - 1)\,(\tilde{q}^v - 1) & (1-x_v)\,\tilde{q}^v - \mu \end{pmatrix}$$

We can directly compute the eigenvalues, which are:

$$\begin{aligned} e_1 &= \hat{\mu} - \mu \\ e_2 &= \hat{\mu} - \mu - \Delta. \end{aligned}$$

where $\hat{\mu} := \frac{1}{2}(T + \Delta) \in [0,1]$, $T := \tilde{q}^a(1-x^a) + \tilde{q}^v(1-x^v)$, and $\Delta := \sqrt{T^2 - 4h(1-x^a)(1-x^v)}$.

The eigenvalues are real and distinct because, given $(x+y)^2 > 4xy$ whenever $x \neq y$, we get

$$\Delta^2 = T^2 - 4h(1-x^a)(1-x^v) \geq 4\tilde{q}^a(1-x^a)\tilde{q}^v(1-x^v) - 4h(1-x^a)(1-x^v)$$

Now $\tilde{q}^a\tilde{q}^v = h^2 + h(1-h) + (1-h)^2 q(1-q) \geq h$, so we conclude $\Delta^2 > 0$.

Since eigenvalues are all distinct, the matrix is diagonalizable, and it is negative definite whenever the eigenvalues are negative. Inspecting the expression, this happens whenever $\mu > \hat{\mu}$.

□

Cumulative infection

Lemma 5. Let (ρ_0^a, ρ_0^v) be the infected share for each group at the outbreak. Then in the linearized approximation around the (0,0) steady state:

$$CI^a = \frac{2\left[\rho_0^a\left(\mu - (1-x^v)\tilde{q}^v\right) + \rho_0^v\left(1-x^a\right)\left(1-\tilde{q}^a\right)\right]}{(T - 2\mu - \Delta)(T - 2\mu + \Delta)}; \tag{3.41}$$

$$CI^v = \frac{2\left[\rho_0^a\left(1-x^v\right)\left(1-\tilde{q}^v\right) + \rho_0^v\left(\mu - (1-x^a)\tilde{q}^a\right)\right]}{(T - 2\mu - \Delta)(T - 2\mu + \Delta)}; \tag{3.42}$$

$$CI = \frac{2\left[\rho_0^a\left(\mu + (1-x^v)(1-2\tilde{q}^v)\right) + \rho_0^v\left(\mu + (1-x^a)(1-2\tilde{q}^a)\right)\right]}{(T - 2\mu - \Delta)(T - 2\mu + \Delta)}. \tag{3.43}$$

Proof. The linearized dynamics is:

$$\dot{d\rho}(t) = M d\rho(t)$$
$$d\rho(0) = \rho_0$$

where $\rho_0 = (\rho_0^a, \rho_0^v)$, and:

$$M_{11} = \frac{1}{\Delta} e^{\frac{1}{2}t(T-2\mu)} \left(\sinh\left(\frac{\Delta t}{2}\right) \left(-x^a \tilde{q}^a + \tilde{q}^a - \mu + \frac{1}{2}(2\mu - T) \right) + \frac{1}{2}\Delta \cosh\left(\frac{\Delta t}{2}\right) \right)$$
$$M_{12} = \frac{1}{\Delta} (1 - x^a)(1 - \tilde{q}^a) \sinh\left(\frac{\Delta t}{2}\right) e^{\frac{1}{2}t(T-2\mu)}$$
$$M_{21} = \frac{1}{\Delta} (1 - x^v)(1 - \tilde{q}^v) \sinh\left(\frac{\Delta t}{2}\right) e^{\frac{1}{2}t(T-2\mu)}$$
$$M_{22} = \frac{1}{\Delta} e^{\frac{1}{2}t(T-2\mu)} \left(\sinh\left(\frac{\Delta t}{2}\right) \left(-x^v \tilde{q}^v + \tilde{q}^v - \mu + \frac{1}{2}(2\mu - T) \right) + \frac{1}{2}\Delta \cosh\left(\frac{\Delta t}{2}\right) \right)$$

The cumulative infection in time in the two groups can be calculated analytically by integration, since it is just a sum of exponential terms. Integration yield, for CI^v:

$$\begin{aligned}
CI^v &= \int_0^\infty d\rho^v(t) dt \\
&= \frac{2\left(\rho_0^a (1 - x^v)(1 - \tilde{q}^v) + \rho_0^v (\mu - (1 - x^a)\tilde{q}^a)\right)}{(-\Delta - 2\mu + T)(\Delta - 2\mu + T)} + \\
&\lim_{t->\infty} e^{\frac{1}{2}t(T-2\mu)} \left(2\Delta \cosh\left(\frac{\Delta t}{2}\right) \left(\rho_0^a (x_v - 1)(\tilde{q}^v - 1) + \rho_0^v ((1 - x_v)\tilde{q}^v + \mu - T) \right) + \right. \\
&\left. \sinh\left(\frac{\Delta t}{2}\right) \left(\rho_0^v \left((T - 2\mu)(2(x_v - 1)\tilde{q}^v + T) + \Delta^2 \right) - 2\rho_0^a (T - 2\mu)(x_v - 1)(\tilde{q}^v - 1) \right) \right)
\end{aligned}$$

and the limit is zero if $\mu > \hat{\mu}$ because the leading term is $Exp\left(\frac{1}{2}t(T - 2\mu) + \frac{\Delta}{2}\right) = \hat{\mu} - \mu$. An analogous reasoning for CI^a yields:

$$CI^a = \int_0^\infty d\rho^a(t) dt = \frac{2\left(\rho_0^a (\mu - (1 - x^v)\tilde{q}^v) + \rho_0^v (1 - x^a)(1 - \tilde{q}^a)\right)}{(-\Delta - 2\mu + T)(\Delta - 2\mu + T)} \tag{3.44}$$
$$CI^v = \int_0^\infty d\rho^v(t) dt = \frac{2\left(\rho_0^a (1 - x^v)(1 - \tilde{q}^v) + \rho_0^v (\mu - (1 - x^a)\tilde{q}^a)\right)}{(-\Delta - 2\mu + T)(\Delta - 2\mu + T)} \tag{3.45}$$

The total CI in the population is $CI = q^a CI^a + (1 - q^a) CI^v$

$$CI = \frac{2}{(-\Delta - 2\mu + T)(\Delta - 2\mu + T)} \left(q^a \left(\rho_0^a (\mu - (1 - x^v)\tilde{q}^v) + \rho_0^v (1 - x^a)(1 - \tilde{q}^a) \right) + \right.$$
$$\left. (1 - q^a)\left({\rho_0}^a (1 - x^v)(1 - \tilde{q}^v) + {\rho_0}^v (\mu - (1 - x^a)\tilde{q}^a) \right) \right)$$

$$= \rho_0^a \frac{2\left(q^a\left(\mu-(1-x^v)\tilde{q}^v\right)+(1-q^a)\left(1-x^v\right)\left(1-\tilde{q}^v\right)\right)}{(-\Delta-2\mu+T)(\Delta-2\mu+T)}+$$
$$\rho_0^v \frac{2(q^a\left(1-x^a\right)\left(1-\tilde{q}^a\right)+\left(1-q^a\right)\left(\mu-(1-x^a)\tilde{q}^a\right))}{(-\Delta-2\mu+T)(\Delta-2\mu+T)}$$

□

Proofs for Result 3

First, note that $\mu > \hat{\mu}$ implies:

$$\begin{aligned}
\mu >& 1-x_a > h(1-x_a)\\
\mu >& 1-x_v > h(1-x_v)\\
\mu >& \frac{h(1-x_a)}{1-(1-h)q}\\
\mu >& \frac{h(1-x_v)}{1-hq}
\end{aligned}$$

The expressions of the derivatives are:

$$\begin{aligned}
\frac{\partial CI^a}{\partial h} &= \frac{(q-1)\left(x_a-1\right)\left(\mu+x_v-1\right)\left(\rho_0^a\left(\mu+x_v-1\right)-\rho_0^v\left(x_a+\mu-1\right)\right)}{2\left(h\mu\left(-qx_a+x_a+qx_v-1\right)+h\left(x_a-1\right)\left(x_v-1\right)+\mu\left(q\left(x_a-x_v\right)+\mu+x_v-1\right)\right)^2}\\
\frac{\partial CI^a}{\partial q^a} &= \frac{(h-1)\left(x_a-1\right)\left(h\left(x_v-1\right)+\mu\right)\left(\rho_0^a\left(\mu+x_v-1\right)-\rho_0^v\left(x_a+\mu-1\right)\right)}{2\left(h\mu\left(-qx_a+x_a+qx_v-1\right)+h\left(x_a-1\right)\left(x_v-1\right)+\mu\left(q\left(x_a-x_v\right)+\mu+x_v-1\right)\right)^2}\\
\frac{\partial CI^a}{\partial x^a} &= \frac{\left((h-1)q\left(x_v-1\right)+\mu+x_v-1\right)\left(\mu(h(q-1)-q)\left(\rho_0^a-\rho_0^v\right)-h\rho_0^a\left(x_v-1\right)-\mu\rho_0^v\right)}{2\left(h\mu\left(-qx_a+x_a+qx_v-1\right)+h\left(x_a-1\right)\left(x_v-1\right)+\mu\left(q\left(x_a-x_v\right)+\mu+x_v-1\right)\right)^2}\\
\frac{\partial CI^a}{\partial x^v} &= \frac{(h-1)(q-1)\left(x_a-1\right)\left(\mu\left((h-1)q\left(\rho_0^v-\rho_0^a\right)+\rho_0^v\right)+h\left(x_a-1\right)\rho_0^v\right)}{2\left(h\mu\left(-qx_a+x_a+qx_v-1\right)+h\left(x_a-1\right)\left(x_v-1\right)+\mu\left(q\left(x_a-x_v\right)+\mu+x_v-1\right)\right)^2}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial CI^v}{\partial h} &= \frac{q\left(x_v-1\right)\left(x_a+\mu-1\right)\left(\rho_0^a\left(\mu+x_v-1\right)-\rho_0^v\left(x_a+\mu-1\right)\right)}{2\left(h\mu\left(-qx_a+x_a+qx_v-1\right)+h\left(x_a-1\right)\left(x_v-1\right)+\mu\left(q\left(x_a-x_v\right)+\mu+x_v-1\right)\right)^2}\\
\frac{\partial CI^v}{\partial q^a} &= \frac{(h-1)\left(x_v-1\right)\left(h\left(x_a-1\right)+\mu\right)\left(\rho_0^a\left(\mu+x_v-1\right)-\rho_0^v\left(x_a+\mu-1\right)\right)}{2\left(h\mu\left(-qx_a+x_a+qx_v-1\right)+h\left(x_a-1\right)\left(x_v-1\right)+\mu\left(q\left(x_a-x_v\right)+\mu+x_v-1\right)\right)^2}\\
\frac{\partial CI^v}{\partial x_a} &= -\frac{(h-1)q\left(x_v-1\right)\left(h\rho_0^a\left(\mu+\mu(-q)+x_v-1\right)+\mu q\rho_0^a+(h-1)\mu(q-1)\rho_0^v\right)}{2\left(h\mu\left(-qx_a+x_a+qx_v-1\right)+h\left(x_a-1\right)\left(x_v-1\right)+\mu\left(q\left(x_a-x_v\right)+\mu+x_v-1\right)\right)^2}\\
\frac{\partial CI^v}{\partial x_v} &= \frac{\left(h(q-1)\left(x_a-1\right)+q\left(-x_a\right)-\mu+q\right)\left(\mu\left((h-1)q\left(\rho_0^v-\rho_0^a\right)+\rho_0^v\right)+h\left(x_a-1\right)\rho_0^v\right)}{2\left(h\mu\left(-qx_a+x_a+qx_v-1\right)+h\left(x_a-1\right)\left(x_v-1\right)+\mu\left(q\left(x_a-x_v\right)+\mu+x_v-1\right)\right)^2}
\end{aligned}$$

and combining them, we get:

$$\frac{\partial CI}{\partial h} = \frac{\mu(q-1)q\left(x_a-x_v\right)\left(\rho_0^a\left(\mu+x_v-1\right)-\rho_0^v\left(x_a+\mu-1\right)\right)}{2\left(h\mu\left(-qx_a+x_a+qx_v-1\right)+h\left(x_a-1\right)\left(x_v-1\right)+\mu\left(q\left(x_a-x_v\right)+\mu+x_v-1\right)\right)^2}$$
$$\frac{\partial CI}{\partial q^a} = \frac{(h-1)\left(\rho_0^a\left(\mu+x_v-1\right)-\rho_0^v\left(x_a+\mu-1\right)\right)\left(h\left(x_a-1\right)\left(x_v-1\right)+\mu\left(q\left(x_a-x_v\right)+x_v-1\right)\right)}{2\left(h\mu\left(-qx_a+x_a+qx_v-1\right)+h\left(x_a-1\right)\left(x_v-1\right)+\mu\left(q\left(x_a-x_v\right)+\mu+x_v-1\right)\right)^2}$$
$$\frac{\partial CI}{\partial x_a} = -\frac{q\left(h\left(x_v-1\right)+\mu\right)\left(h\rho_0^a\left(\mu+\mu(-q)+x_v-1\right)+\mu q\rho_0^a+(h-1)\mu(q-1)\rho_0^v\right)}{2\left(h\mu\left(-qx_a+x_a+qx_v-1\right)+h\left(x_a-1\right)\left(x_v-1\right)+\mu\left(q\left(x_a-x_v\right)+\mu+x_v-1\right)\right)^2}$$
$$\frac{\partial CI}{\partial x_v} = \frac{(q-1)\left(h\left(x_a-1\right)+\mu\right)\left(\mu\left((h-1)q\left(\rho_0^v-\rho_0^a\right)+\rho_0^v\right)+h\left(x_a-1\right)\rho_0^v\right)}{2\left(h\mu\left(-qx_a+x_a+qx_v-1\right)+h\left(x_a-1\right)\left(x_v-1\right)+\mu\left(q\left(x_a-x_v\right)+\mu+x_v-1\right)\right)^2}$$

Note that all the denominators are positive, so to control the sign from now on we focus on the numerators. In particular, recognising the numerators of the first two as precisely the terms arising in 3.4 we can conclude that CI is increasing in h if and only if $CI^a > CI^v$ and CI is increasing in q if and only if $CI^a > CI^v$.

If initial conditions are symmetric:

$$\frac{\partial CI^a}{\partial h} > 0 \iff -(q-1)\rho_0^a\left(x_a-1\right)\left(x_a-x_v\right)\left(\mu+x_v-1\right) > 0$$
$$\frac{\partial CI^a}{\partial q^a} > 0 \iff -(h-1)\rho_0^a\left(x_a-1\right)\left(x_a-x_v\right)\left(h\left(x_v-1\right)+\mu\right) > 0$$
$$\frac{\partial CI^a}{\partial x^a} > 0 \iff -\rho_0^a\left(h\left(x_v-1\right)+\mu\right)\left(\mu-(1-h)(1-q)\left(1-x_v\right)\right) > 0$$
$$\frac{\partial CI^a}{\partial x^v} > 0 \iff (h-1)(q-1)\rho_0^a\left(x_a-1\right)\left(h\left(x_a-1\right)+\mu\right) > 0$$

Now, using the first four inequalities presented above, we can conclude that $\frac{\partial CI^a}{\partial h} > 0$, $\frac{\partial CI^a}{\partial q^a} > 0$, $\frac{\partial CI^a}{\partial x^a} < 0$ and $\frac{\partial CI^a}{\partial x^v} < 0$. Similarly, if $\rho_0^a = 0$:

$$\frac{\partial CI^a}{\partial h} > 0 \iff -(q-1)\left(x_a-1\right)\rho_0^v\left(x_a+\mu-1\right)\left(\mu+x_v-1\right) > 0$$
$$\frac{\partial CI^a}{\partial q^a} > 0 \iff -(h-1)\left(x_a-1\right)\rho_0^v\left(x_a+\mu-1\right)\left(h\left(x_v-1\right)+\mu\right) > 0$$
$$\frac{\partial CI^a}{\partial x^a} > 0 \iff -(1-h)(1-q)\rho_0^v\left(\mu-(1-q)(1-h)\left(1-x_v\right)\right) > 0$$
$$\frac{\partial CI^a}{\partial x^v} > 0 \iff (h-1)(q-1)\left(x_a-1\right)\rho_0^v\left(h\left(x_a-1\right)+\mu\left((h-1)q+1\right)\right) > 0$$

and we conclude that $\frac{\partial CI^a}{\partial h} < 0$, $\frac{\partial CI^a}{\partial q^a} < 0$, $\frac{\partial CI^a}{\partial x^a} < 0$ and $\frac{\partial CI^a}{\partial x^v} < 0$.

If $\rho_0^v = 0$:

$$\begin{aligned}
\frac{\partial CI^a}{\partial h} > 0 &\Longleftrightarrow (q-1)\rho_0^a\,(x_a-1)\,(\mu+x_v-1)^{\,2} > 0 \\
\frac{\partial CI^a}{\partial q^a} > 0 &\Longleftrightarrow (h-1)\rho_0^a\,(x_a-1)\,(\mu+x_v-1)\,(h\,(x_v-1)+\mu) > 0 \\
\frac{\partial CI^a}{\partial x^a} > 0 &\Longleftrightarrow -\rho_0^a\,(\mu-(1-q)(1-h)(1-x_v))\,(h\,(\mu-(1-x_v))+\mu(1-h)q) > 0 \\
\frac{\partial CI^a}{\partial x^v} > 0 &\Longleftrightarrow -(h-1)^2\mu(q-1)q\rho_0^a\,(x_a-1) > 0
\end{aligned}$$

and we conclude that $\frac{\partial CI^a}{\partial h} > 0$, $\frac{\partial CI^a}{\partial q^a} > 0$, $\frac{\partial CI^a}{\partial x^a} < 0$ and $\frac{\partial CI^a}{\partial x^v} < 0$.

The other cases are analogous.

Proof of Result 4

The linearized dynamics is (from Lemma 5):

$$\begin{aligned}
\dot{\rho}^a =& \frac{1}{\Delta}e^{\frac{1}{2}t(T-2\mu)}\left(\sinh\left(\frac{\Delta t}{2}\right)\left(-x^a\tilde{q}^a+\tilde{q}^a-\frac{1}{2}T\right)+\frac{1}{2}\Delta\cosh\left(\frac{\Delta t}{2}\right)\right)\rho_0^a \\
&+\frac{1}{\Delta}\left(1-x^a\right)\left(1-\tilde{q}^a\right)\sinh\left(\frac{\Delta t}{2}\right)e^{\frac{1}{2}t(T-2\mu)}\rho_0^v \\
\dot{\rho}^v =& \frac{1}{\Delta}\left(1-x^v\right)\left(1-\tilde{q}^v\right)\sinh\left(\frac{\Delta t}{2}\right)e^{\frac{1}{2}t(T-2\mu)}\rho_0^a \\
&+\frac{1}{\Delta}e^{\frac{1}{2}t(T-2\mu)}\left(\sinh\left(\frac{\Delta t}{2}\right)\left(-x^v\tilde{q}^v+\tilde{q}^v-\frac{1}{2}T\right)+\frac{1}{2}\Delta\cosh\left(\frac{\Delta t}{2}\right)\right)\rho_0^v
\end{aligned}$$

In particular, it depends on μ just through the exponential term $e^{\frac{1}{2}t(T-2\mu)}$. So we can rewrite it as:

$$\begin{aligned}
\dot{\rho}^a &= e^{\frac{1}{2}t(T-2\mu)}\mathcal{A}(t) \\
\dot{\rho}^v &= e^{\frac{1}{2}t(T-2\mu)}\mathcal{V}(t)
\end{aligned}$$

where $\mathcal{A}(t)$ and $\mathcal{V}(t)$ do not depend on μ. Now the discounted cumulative infection for anti–vaxxers is equal to:

$$CI^a = \int_0^\infty e^{-\beta t}e^{\frac{1}{2}t(T-2\mu)}\mathcal{A}(t)dt = \int_0^\infty e^{\frac{1}{2}t(T-2(\mu+\beta))}\mathcal{A}(t)dt$$

which is precisely the expression for the non discounted cumulative infection in a model where the recovery rate is $\mu' = \mu + \beta$.

Proof of Result 6

From the proof of Result 1, the eigenvalues are:

$$\begin{aligned} e_1 &= \hat{\mu} - \mu \\ e_2 &= \hat{\mu} - \mu - \Delta. \end{aligned}$$

Moreover, they are both decreasing in absolute value as h increases (this is easy to see for e_1, given that $\hat{\mu}$ is positive and increases in h, but it holds also for e_2).

Proof of Proposition 15

We have:

$$\left(\frac{dCI}{dq}\right)_{|h=0} = \left(\frac{\partial CI}{\partial q}\right)_{|h=0} + \left(\frac{\partial CI}{\partial x^a}\frac{\partial x^a}{\partial q} + \frac{\partial CI}{\partial x^v}\frac{\partial x^v}{\partial q}\right)_{|h=0}$$

The first term can be obtained setting $h = 0$ in the expressions from the proof of Result 3:

$$\left(\frac{\partial CI}{\partial q}\right)_{|h=0} = \frac{(k+1)\left(\rho_0^a\left(\mu + x^v - 1\right) - \rho_0^v\left(x^a + \mu - 1\right)\right)}{2(k+1)\left(q\left(x^a - x^v\right) + \mu + x^v - 1\right)^2}$$

The correction term instead is:

$$\left(\frac{\partial CI}{\partial x^a}\frac{\partial x^a}{\partial q} + \frac{\partial CI}{\partial x^v}\frac{\partial x^v}{\partial q}\right)_{|h=0} = \frac{dk\left((q-1)\rho_0^v - q\rho_0^a\right)}{2(k+1)\left(q\left(x^a - x^v\right) + \mu + x^v - 1\right)^2}$$

which is negative: so endogenizing the vaccination choices always yields a smaller effect of a change in q. Moreover, if the initial conditions are symmetric the numerator of the derivative becomes:

$$(k+1)(x^v - x^a) - dk = (k+1)d - dk = d > 0$$

so CI is still increasing in q, but at a lower rate.

Similarly, the derivative with respect to h is:

$$\left(\frac{dCI}{dh}\right)_{|h=0} = \left(\frac{\partial CI}{\partial h}\right)_{|h=0} + \left(\frac{\partial CI}{\partial x^a}\frac{\partial x^a}{\partial h} + \frac{\partial CI}{\partial x^v}\frac{\partial x^v}{\partial h}\right)_{|h=0}$$

and the correction term is null:

$$\left(\frac{\partial CI}{\partial x^a}\frac{\partial x^a}{\partial h} + \frac{\partial CI}{\partial x^v}\frac{\partial x^v}{\partial h}\right)_{|h=0} =$$

$$-dkq\frac{(q-1)\left(\rho_0^v - q\left(\rho_0^v - \rho_0^a\right)\right)}{2\left(q\left(x^a - x^v\right) + \mu + x^v - 1\right)^2} - dk(q-1)\frac{q\left(\mu q\rho_0^a - \mu(q-1)\rho_0^v\right)}{2\mu\left(q\left(x^a - x^v\right) + \mu + x^v - 1\right)^2} = 0$$

so the derivative is exactly the same as in Result 3.

Proof of Proposition 16

Consider the function $F(q) = q^\alpha \Delta U^a - (1-q)^\alpha \Delta U^v$. Both ΔU^a and ΔU^v are bounded from above and bounded away from 0, so when $q \to 0$ the negative term remains bounded while $q^\alpha \to \infty$ (because $\alpha < 0$). The reverse happens when $q \to 1$. By the intermediate value theorem, there exist a solution $q^* \in (0, 1)$.

Concerning stability, we can calculate the derivative of the function F:

$$\frac{\mathrm{d}F}{\mathrm{d}q} = \frac{d}{2(k+1)(hk+1)^2} \times$$

$$\Big(aq^{a-1}\left(d\left(-2(h-1)hk^2q + k + 1\right) - 2hk(hk+1)\right) - 2d(h-1)hk^2\left(q^a + (1-q)^a\right)$$
$$+a(1-q)^{a-1}\left(2d(h-1)hk^2q + d(k+1)(2hk+1) + 2hk(hk+1)\right)\Big)$$

If $q \to 0$, $\frac{\mathrm{d}F}{\mathrm{d}q} \to -\infty$, whereas if $q \to 1$ $\frac{\mathrm{d}F}{\mathrm{d}q} \to +\infty$, so that, by continuity, there must be a stable steady state. If if $h \to 0$, $\frac{\mathrm{d}F}{\mathrm{d}q} \to \alpha d^2 2^{1-\alpha} < 0$, so for h in a neighborhood of 0 the steady state is unique and stable.

Proof of Proposition 17

For $h = 0$ we have that $q = \frac{1}{2}$. We can compute the derivative using the implicit function theorem. The first derivative is above. The second is

$$\frac{\mathrm{d}F}{\mathrm{d}h} = \frac{d}{2(k+1)(hk+1)^2} \times$$

$$dk\left((1-q)^a(hk(d(-(k+2)q + k + 1) - 1) + dkq - 1)\right.$$
$$\left.-q^a(d(kq(h(k+2) - 1) + k + 1) + hk + 1)\right)$$

so that:

$$\left.\frac{\mathrm{d}q}{\mathrm{d}h}\right|_{h=0} = -\frac{\frac{\mathrm{d}F}{\mathrm{d}h}}{\frac{\mathrm{d}F}{\mathrm{d}q}} = \frac{2k + dk}{\alpha(2d + 2dk)}$$

and we can see that q is always decreasing with homophily, but with a different level of intensity according to the magnitude of α.

Proof of Proposition 18

Using the implicit function theorem, we can analyze the behavior of cumulative infection for h close to 0:

$$\left.\frac{\mathrm{d}CI}{\mathrm{d}h}\right|_{h=0} = \frac{(k+1)}{4(dk - 2(k+1)\mu + 2)^2}\left(\frac{4(d+2)k(d\rho_0^v k + (k+1)\mu(\rho_0^a - \rho_0^v) - \rho_0^a + \rho_0^v)}{ad(k+1)}\right.$$

$$+\frac{dk\left(\rho_0^a\left(dk^2+2(k+1)\mu-2\right)+\rho_0^v(dk(k+2)-2(k+1)\mu+2)\right)}{\mu}\Bigg)$$

With a symmetric initial condition we get:

$$\frac{\mathrm{d}CI}{\mathrm{d}h}\bigg|_{h=0}=\frac{\rho_0^a k^2\left(ad^2(k+1)^2+2(d+2)\mu\right)}{2a\mu(dk-2(k+1)\mu+2)^2}$$

which is positive if $\alpha < \frac{-2d\mu-4\mu}{d^2k^2+2d^2k+d^2}$ and negative otherwise.

Chapter 4

Learning, Over-reaction and the Wisdom of the Crowd

This chapter is joint work with Daniele D'Arienzo[1]

We study the classical sequential social learning problem in a setting where agents depart from the standard Bayesian updating rule. We consider the case of over-reacting - as well under-reacting - individual posterior beliefs, two well known biases in beliefs updating (Benjamin (2019)). Agent posterior beliefs over-react (or under-react) to the current information according to how much it is surprising relative to past information. We study the interplay of distorted posterior beliefs and social learning. We find that in a context with fine grained signals the biases do not impact on the eventual learning, while in a context with coarse signals, such as in the cascades setting of Banerjee (1992), over-reaction can make it easier for agents to learn, because past actions of others become more informative, hence a moderate level of over-reaction is socially optimal.

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4.1 Introduction

Departures from the standard Bayesian updating rule are well documented using both experiments and survey data (e.g. Benjamin (2019)). For example, in financial markets, the typical departure is that of over-optimistic beliefs, when good news are observed (e.g. Bordalo et al. (2018)). However, much of the literature focuses on decisions taken by agents in isolation, abstracting away from another fundamental economic fact: interaction. On the other hand, the study of interaction in the formation of expectations and learning has been widely studied (Golub and Sadler (2017)), both under Bayesian behavior (Banerjee (1992), Acemoglu et al. (2011), Bala and Goyal (1998)) and under simple mechanical updating rules, such as averaging neighbors beliefs (e.g. Golub and Jackson (2010)). The literature on social networks has focused on how the network structure of interactions facilitates or forbids reaching consensus and learning. Recently, departures from the Bayesian paradigm has been investigated in the context of social learning. For example, Molavi et al. (2018) consider the case of beliefs updating with imperfect recall.

Here we bridge the gap by considering a simple model of sequential learning where agents - due to information processing limitations - depart from full Bayesian rationality. Agents exhibit under/over-reaction to signals. Our main contribution is to characterize the information externalities caused by departures from Bayesian rationality. Specifically, we find that over-reaction to news entails a positive externality, which partially heals the informational cascade phenomenon, thereby increasing social informational efficiency. This is surprising because one may expect individual biases to be socially inefficient.

We first introduce a model of non Bayesian updating, which features two specific observed biases: *over-reaction* or *under-reaction* to information. While the origin of the two mechanisms is thought to be different in nature [2] our model includes both cases. In the case of over-reaction, our model is a learning analog of the diagnostic expectation model of Bordalo et al. (2018), which is a model of extrapolative predictions. We then apply the model to study the learning problem of an isolated agent. We find that in the long-run limit, the agent learns the true state of the world as in the Bayesian case. We show that however the mean square loss of biased agents - a measure of individ-

[2]Under-reaction to information can be rationalized by costly information acquisition. On the contrary, over-reaction to information may be grounded in the Tversky and Kahneman (1974) representativeness heuristic.

ual inefficiency - is not symmetric: under-reaction to information leads to greater losses than over-reaction to information. In both cases, however, it is sub-optimal to have biased expectations.
What happens instead when agents interact? We focus on the stylized case of a sequential decision task, as in the cascade literature started by Banerjee (1992) and Bikhchandani et al. (1992). At each time step, an agent is born and she has to take a binary action (e.g. buy or sell a financial asset) which corresponds to guess the true state of nature, which is binary as well (e.g. the fundamental value of the asset). Her information set consists of a private signal and past actions of other agents (e.g. past buy/sell orders). In the Bayesian setting, this framework leads to the phenomenon of *informational cascades*: when the actions of previous agents are aligned enough, then future private signals become irrelevant and each future agent is stuck in a specific action. This may result in a cascade of wrong guesses. This is so because the mapping between private signals and the history of actions (which is what is observed by future agents) is highly non injective. Much of the information in the economy remain unexploited. We consider our model of non Bayesian updating and we find that over-reaction helps injecting more information into the history of actions, which is then exploited by future agents. We find that there exists a unique socially optimal level of over-reaction, which maximizes the probability of learning the true state of the world.
This insight also clarify that departures from Bayesian rationality - overreaction in particular - may be seen from an evolutionary perspective as optimal with respect to the objective of social informational efficiency.

4.2 A non Bayesian learning model

Consider an agent that has to *learn* the state of the world ω, on which she has a prior belief, with density $p_0(\omega)$. The agent observes a signal X, whose likelihood is known to be $l(X|\omega)$. The Bayesian updating operator takes as inputs the prior density, the likelihood function and the observed signal and it prescribes to move beliefs about ω from $p_0(\omega)$ to the Bayesian posterior:

$$\mathcal{BU}(l, p_0)(\omega) := \frac{l(X|\omega)p_0(\omega)}{\int l(X|\omega')p_0(\omega')d\omega'}. \tag{4.1}$$

We propose the following distorted updating rule:

$$\mathcal{BU}^{\theta}(l, p_0)(\omega) = \frac{l(X|\omega)^{1+\theta}p_0(\omega)}{\int l(X|\omega')^{1+\theta}p_0(\omega')d\omega'}. \tag{4.2}$$

The scalar parameter $\theta > -1$ controls the departure from the Bayesian case. When $\theta > 0$ the model delivers *over-reaction* to information and it is a learning analog of the diagnostic expectation model of Bordalo et al. (2018). To see this point, consider the case $\theta > 0$ and rewrite expression (4.2) as:

$$\mathcal{BU}^{\theta}(l, p_0)(\omega) = \frac{1}{Z}\mathcal{BU}(l, p_0)(\omega)\left(\frac{\mathcal{BU}(l, p_0)(\omega)}{p_0(\omega)}\right)^{\theta}, \tag{4.3}$$

where Z is a normalization constant. The previous formula says that states ω which are more likely under $\mathcal{BU}(l, p_0)(\omega)$ than under $p_0(\omega)$, i.e. *representative* states, are over-weighted. On the contrary, states ω which are less likely under $\mathcal{BU}(l, p_0)(\omega)$ than under $p_0(\omega)$, are under-weighted. Thus, we say that for $\theta > 0$ posterior beliefs over-react to information. On the contrary, for $-1 < \theta < 0$, posterior beliefs under-react to information. When facing multiple data, $X_1, \ldots, X_t$, agents can update beliefs sequentially: in at each step, the prior belief is the precious step distorted posterior belief. Alternatively agents could update their beliefs only once, after observing the string $X_1, \ldots, X_t$. Define the distorted updating given t observations as:

$$\mathcal{BU}_t^{\theta}(l, p_0)(\omega) = \frac{l(X_1, \ldots X_t|\omega)^{1+\theta} p_0(\omega)}{\int l(X_1, \ldots X_t|\omega')^{1+\theta} p_0(\omega') d\omega'}.$$

Then, the following consistency result shows that it is irrelevant which of the two strategy is implemented.

Theorem 1. For $k \in \{1, \ldots, t-1\}$:

$$\mathcal{BU}_t^{\theta}(l, p_0)(\omega) = \mathcal{BU}_{t-k}^{\theta}(l, \mathcal{BU}_k^{\theta}(l, p_0))(\omega)$$

Does learning take place? Expression (4.3) suggests the the Bayesian and the diagnostic distribution are connecting by a continuous transformation, which preserve convergence.

Theorem 2. Call ω^* the true value of ω.
Learning occurs under Bayesian updating, i.e. $\mathcal{BU}_t(l, p_0) \xrightarrow{d} \delta_{\omega^*}$ as $t \to \infty$ if and only if it occurs under distorted beliefs, i.e. $\mathcal{BU}_t^{\theta}(l, p_0) \xrightarrow{d} \delta_{\omega^*}$ as $t \to \infty$.

We now characterize the loss from using distorted posterior beliefs. Assume that the goal of the agent is to minimize the sum of future discounted losses:

$$\sum_{t^0}^{\infty} \beta^t \mathbb{E}_t(\omega - \mathbb{E}_t^\theta \omega)^2.$$

Then, the following results characterizes the losses occurring with distorted beliefs.

Theorem 3. Under the updating model (4.2), the (cumulative) mean square error reads:

$$-\log(1-\beta) + \sum_{t=1}^{\infty} \beta^t (\mathbb{E}_t\omega - \mathbb{E}_t^\theta \omega)^2.$$

As expected, distorted beliefs are, in general, sub-optimal since $(\mathbb{E}_t\omega - \mathbb{E}_t^\theta\omega)^2 > 0$. Thus, in a world with distorted beliefs, an isolated agents eventually learn (or does not) if the only if the Bayesian agent does. We now move to a concrete example to gain more intuition.

4.2.1 Learning the mean from Gaussian i.d.d. draws

Suppose that an agent observes *iid* realizations of $X \sim \mathcal{N}(\mu^{true}, \sigma^2)$. She knows the variance σ^2 and she has to learn the mean. Given $X \sim \mathcal{N}(\mu^{true}, \sigma^2)$ and prior $p_0(\mu) \sim \mathcal{N}(\mu_0, \sigma_0^2)$, and t observations $X_1, \ldots, X_t$, we have:

$$p_t(\mu) := \mathcal{BU}_t(l, p_0) \sim \mathcal{N}\left(\left(\frac{1}{\sigma_0^2} + \frac{t}{\sigma^2}\right)^{-1}\left(\frac{\mu_0}{\sigma_0^2} + \frac{\sum_{i=1}^t X_i}{\sigma^2}\right), \left(\frac{1}{\sigma_0^2} + \frac{t}{\sigma^2}\right)^{-1}\right).$$

The distorted posterior distribution is:

$$p_t^\theta(\mu) = \mathcal{BU}_t^\theta(l, p_0) = \mathcal{N}\left(\left(\frac{1}{\sigma_0^2} + \frac{t(1+\theta)}{\sigma^2}\right)^{-1}\left(\frac{\mu_0}{\sigma_0^2} + \frac{(1+\theta)\sum_{i=1}^t X_i}{\sigma^2}\right), \left(\frac{1}{\sigma_0^2} + \frac{t(1+\theta)}{\sigma^2}\right)^{-1}\right),$$

since the only effect of the distortion is to modify the variance of the likelihood function from σ^2 to $\frac{\sigma^2}{1+\theta}$. Thus, the variance of the posterior diagnostic distribution is:

$$\mathbb{V}[\mu_t^\theta] = \left(\frac{1}{\sigma_0^2} + \frac{(\theta+1)t}{\sigma^2}\right)^{-1} \sim \frac{\sigma^2}{(\theta+1)t},$$

which is smaller then the variance of the Bayesian posterior. Also, for large t, convergence to the truth is guaranteed by theorem (2). What about the mean square error? As shown in Appendix, in the Gaussian case the loss reads:

$$(\mathbb{E}_t^\theta \omega - \mathbb{E}_t \omega)^2 \sim \frac{1}{t^2} \frac{\theta^2}{(\theta+1)^2} \left((\mu_0 - \omega)^2 + \sigma_t^2 \right).$$

The bias term depends on: the initial prior, the variance of the posterior under the bayesian updating, and the term $\frac{\theta^2}{(\theta+1)^2}$, which is asymmetric: under-reaction makes it increase very fast above 1, while even for strong over-reaction it is always smaller than 1. Hence it seems that underreaction is much worse in terms of learning than overreaction. Inspection of the calculations reveal that the numerator comes from the error in prediction, while the denominator from the precision. So the error tends to increase with bias, while the precision is increasing in overreaction. With one agent, though, the numerator always prevails, so that the optimal θ is 0. We now introduce social interactions.

4.3 Sequential learning and efficiency

In this section, we apply our behavioral model of learning to a simple social learning environment, to show that overreaction can be more socially efficient than bayesian updating.

Let us consider the simplest model of informational cascades, analogous to Banerjee (1992). There is a binary state of the world $\omega \in \{0,1\}$ which agents have to guess. Formally, agents are infinite and indexed with natural numbers $i = 1, 2, \ldots$. They act sequentially, first observing a private signal s_i and then choosing a public action a_i. Both actions and signals are binary $a_i, s_i \in \{0,1\}$ and we assume that $Pr(s_i = 1|\omega = 1) = q > \frac{1}{2}$ (i.e. signals are informative). Symmetrically (for convenience) let $Pr(s_i = 0|\omega = 0) = q > \frac{1}{2}$. Agents have a common prior $Pr(\omega = 1) = p$.

Agent i information set includes all actions of past agents $(a_1, \ldots, a_{i-1})$ and his own private signal s_i. Each agent has a utility v_1 from choosing action $a_i = 1$ if the correct state is $\omega = 1$ and v_0 from choosing the correct action when the state is $\omega = 0$, and they want to maximize their expected payoff. This means that, e.g., Mr 1, when observing signal s_1 will form the following posterior:

$$Pr^\theta(\omega = 1|s_1 = 1) = Pr(\omega = 1|s_1 = 1) \left(\frac{Pr(\omega = 1|s_1 = 1)}{Pr(\omega = 1)} \right)^\theta \frac{1}{Z(\theta, s_1 = 1)} = \frac{p}{p + (1-p) \left(\frac{1-}{q} \right.}$$

and he will choose action $a_1 = 1$ if and only if:

$$v_1 Pr^\theta(\omega = 1|s_1 = 1) > v_0 Pr^\theta(\omega = 0|s_1 = 1),$$

which is equivalent to:

$$Pr^{\theta}(\omega = 1|s_1) > \frac{v_0}{v_0 + v_1}.$$

In the following, we will be interested in regions in the parameter space, so we do not need to specify tie-breaking rules. We define $\tau = \frac{v_0}{v_0+v_1}$.

We defined over and under-reaction relative to the estimation of the parameter done with past information. When considering interacting agents, we have two sources of information: private signals and actions of others. In the following, we are treating actions of others and the private signal in a symmetric way, as past information. This means that Mr 2 will compute his posterior as:

$$Pr^{\theta}(\omega = 1|s_1, a_1, s_2) = Pr(\omega = 1|s_1, a_1, s_2)\left(\frac{Pr(\omega = 1|s_1, a_1, s_2)}{Pr(\omega = 1, s_1, a_1)}\right)^{\theta} \frac{1}{Z(\theta, s_1, a_1, s_2)}$$

To understand what are the implications of the distortion for cascades and learning, let us start with the following definition.

Definition 4.3.1. The *Informational efficient region* (IE) is the set of parameters given by the union of:

$$\theta + 1 \geq \frac{\ln\left(\frac{p}{1-p}\left(\frac{1}{\tau} - 1\right)\right)}{\ln\frac{1-q}{q}} \qquad p \geq \tau$$

and

$$\theta + 1 \leq \frac{\ln\left(\frac{p}{1-p}\left(\frac{1}{\tau} - 1\right)\right)}{\ln\frac{q}{1-q}} \qquad p < \tau$$

The Informational efficient region is the region of parameters such that Mr 1 plays $a_1 = s_1$ and therefore "communicate" his private signal to future agents. Outside the efficient region, agent 1 instead chooses the action consistent with his prior regardless the signal. This is crucial in characterizing the behavior of the model. The following proposition describes such behavior.

Proposition 19. *If the parameters are in the Informational efficient region, then:*

If $p \geq \tau$ *If the first signal is 1, there is a cascade on 1. If the first two signals are* $(0,0)$ *there is a cascade on 0. If the first two signals are* $(0,1)$*, then the third agent faces the same problem of agent 1. The probability of learning is* $Pr(a_\infty = \omega) = \frac{pq+(1-p)q^2}{1-q(1-q)}$.

If $p < \tau$ *There is a cascade on 1 if the first signals are* $(1,1)$*, there is a cascade on 0 if the first signal is 0. If the first two signals are* $(1,0)$*, then the third agent faces the same problem of agent 1. The probability of learning is* $Pr(a_\infty = \omega) = \frac{pq^2+(1-p)q}{1-q(1-q)}$.

If the parameters are outside of the Informational efficient region, then:

1. *If* $\tau > p$*, then all agents play 0 with probability 1 and the probability of learning is* $1 - p$.
2. *If* $\tau \leq p$*, then then all agents play 1 with probability 1, and the probability of learning is* p.

By the form of the results, we can already see that a larger θ creates more room for learning, by enlarging the Informational efficient region. In the following, we make this argument formal. For simplicity in this section we fix $\tau = \frac{1}{2}$.

A way to quantify the size of the parameter space is to think of the parameters p and q as drawn before the process starts. from a distribution μ, with full support on $(0,1) \times (\frac{1}{2}, 1)$. Denote $a_\infty = \lim_{t\to\infty} a_t$. Consider regions as $R_1 = IE \cup \{p > \tau\}$, $R_2 = IE \cup \{p \leq \tau\}$, $N_1 = \overline{IE} \cup \{p > \tau\}$, and $N_2 = \overline{IE} \cup \{p \leq \tau\}$. Then the ex-ante probability of learning the correct state of the world is:

$$
\begin{aligned}
&Pr(a_\infty = \omega) = \\
&= \int \left(pI_{N_1} + (1-p)I_{N_2} + \frac{pq + (1-p)q^2}{1 - q(1-q)} I_{R_1} + \frac{pq^2 + (1-p)q}{1-q(1-q)} I_{R_2} \right) \mathrm{d}\mu,
\end{aligned}
$$

where $I(\cdot)$ represents the indicator function.

Let us define a level of θ *ex-ante efficient* if it achieves the maximum of this probability. The following figures illustrate the situation. In figure 4.1 we draw the region where Mr1 playing $a_1 = s_1$ is *socially* efficient, and the region where (Bayesian) Mr1 playing $a_1 = s_1$ is *individually* efficient (i.e. optimal). As is clear from the figure, a Bayesian updating rule does not maximize the learning probability: there is a region where it would be socially efficient that Mr1 plays $a_1 = s_1$, but a Bayesian agent, since he does not internalize the information externality on other agents, does not. In figure 4.2, we plot instead the informational efficient regions for different values of the parameter θ: we can see that there are moderate values of overreaction that increases the probability of learning. In the following proposition, we show that there is a value of θ that actually achieve ex-ante efficiency.

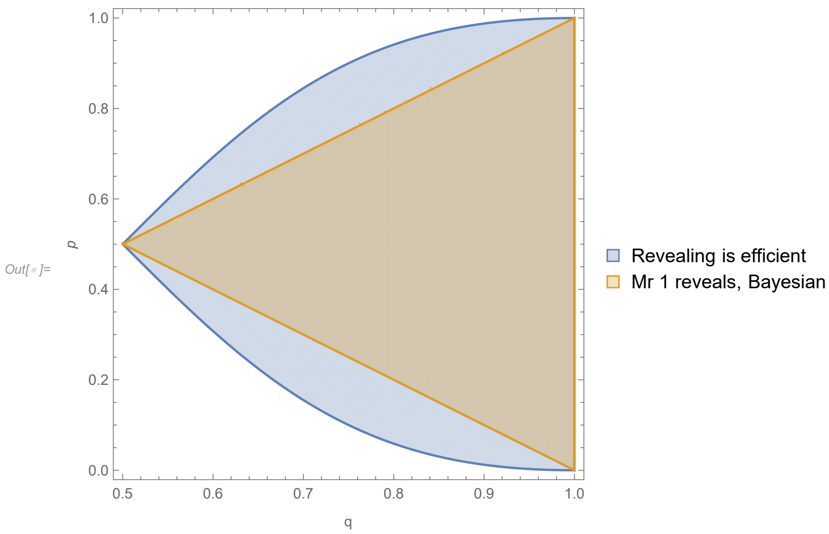

Figure 4.1: The areas of the parameter space where revealing is efficient, and the area where a Bayesian agent reveals. The figure shows that the Bayesian updating fails to be socially efficient: there is a region where agent 1 does not reveal but it would be socially optimal to do so.

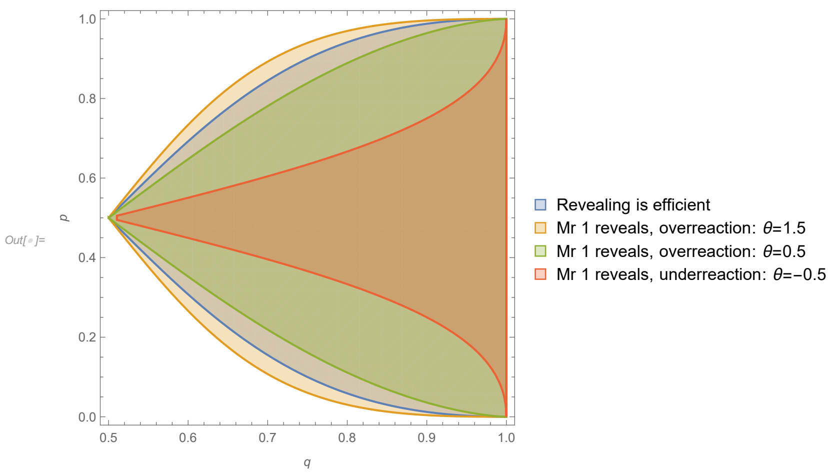

Figure 4.2: The areas of the parameter space where revealing is efficient, and the area where agent with different degree of distortion reveal. It is clear that underreaction is always worse than Bayesian, while a moderate overreaction can be socially better than Bayesian.

Theorem 4. If the parameters p, q are drawn from a distribution μ with full support on $(0,1) \times (\frac{1}{2}, 1)$, the distorted updating with $\theta = 1$ is ex-ante efficient.

Given the importance of the result, we report the proof here in the main text.

Proof. Let us focus on the subset of the parameter space where $\{p > \tau\}$, the reasoning for $p < \tau$ is analogous. The the ex-ante probability of learning is:

$$\int \left(pI \left(\theta + 1 \leq \frac{\ln\left(\frac{p}{1-p}\right)}{\ln \frac{q}{1-q}} \right) + \frac{pq + (1-p)q^2}{1 - q(1-q)} I \left(\theta + 1 > \frac{\ln\left(\frac{p}{1-p}\right)}{\ln \frac{q}{1-q}} \right) \right) \mathrm{d}\mu.$$

In the Informational efficient region IE, the probability of learning is $\frac{pq+(1-p)q^2}{1-q(1-q)}$, while outside is just p. The probability of learning is higher inside the Revelation region if and only if:

$$p < \frac{pq + (1-p)q^2}{1 - q(1-q)},$$

which is equivalent to:

$$p < \frac{q^2}{(1-q)^2 + q^2}.$$

Now we can rewrite the condition defining the IE region:

$$p < \frac{q^{\theta+1}}{(1-q)^{\theta+1} + q^{\theta+1}}.$$

Depending on θ, $\frac{q^2}{(1-q)^2+q^2}$ can be larger or smaller than $\frac{q^{\theta+1}}{(1-q)^{\theta+1}+q^{\theta+1}}$, with equality for $q = \frac{1}{2}$ $q = 1$, and for all q if $\theta = 1$.

If $\theta < 1$, it means that there is a region outside the IE region (with positive mass, because of the full support assumption on μ) with probability of learning p, strictly smaller than the corresponding probability if it belonged to the IE, hence by increasing θ the probability of learning would increase. If $\theta > 1$, on the contrary, there is a region that belongs to IE where the probability of learning is smaller than p. Hence, the maximum is achieved for $\theta = 1$.

□

4.A Proofs

4.A.1 Theorem (1)

Proof.

$$\begin{aligned}
&\mathcal{BU}_t(l, p_0)(\omega) = \frac{l(X_1, \ldots, X_t|\omega)p_0(\omega)}{\int d\omega' l(X_1, \ldots, X_t|\omega')p_0(\omega')} \\
&= \frac{l(X_\tau, \ldots, X_t|\omega, X_1, \ldots, X_{\tau-1})l(X_1, \ldots, X_{\tau-1}|\omega')p_0(\omega)}{\int l(X_\tau, \ldots, X_t|\omega', X_1, \ldots, X_{\tau-1})l(X_1, \ldots, X_{\tau-1}|\omega')p_0(\omega')d\omega'} \times \frac{\int l(X_1, \ldots, X_{\tau-1}|\omega')p_0(\omega')d\omega'}{\int l(X_1, \ldots, X_{\tau-1}|\omega')p_0(\omega')d\omega'} \\
&= \frac{l(X_\tau, \ldots, X_t|\omega, X_1, \ldots, X_\tau)\mathcal{BU}_\tau(l, p_0)(\omega)}{\int l(X_\tau, \ldots, X_t|\omega', X_1, \ldots, X_\tau)\mathcal{BU}_\tau(l, p_0)(\omega')d\omega'} \\
&= \mathcal{BU}_{t-\tau}(l, \mathcal{BU}_\tau(l, p_0)(\omega))(\omega)
\end{aligned}$$

Remaining close to the diagnostic expectations literature, we can assume that the reference group for representativeness be the information collected until period t. The diagnostic Bayesian operator is defined for $t = 1$ as:

$$\mathcal{BU}_1^\theta(l, p_0)(\omega) = \frac{1}{\int \mathcal{BU}_1(l, p_0)\left(\frac{\mathcal{BU}_1(l,p_0)(\omega')}{p_0(\omega')}\right)^\theta d\omega'} \mathcal{BU}_1(l, p_0)\left(\frac{\mathcal{BU}_1(l, p_0)(\omega)}{p_0(\omega)}\right)^\theta.$$

For $t > 1$ it is defined as:

$$l, p_0)(\omega) = \frac{1}{\int \mathcal{BU}_t(l, \mathcal{BU}_{t-1}^\theta(l, p_0))(\omega')\left(\frac{\mathcal{BU}_t(l,\mathcal{BU}_{t-1}^\theta(l,p_0))(\omega')}{\mathcal{BU}_{t-1}^\theta(l,p_0)(\omega')}\right)^\theta d\omega'} \mathcal{BU}_t(l, \mathcal{BU}_{t-1}^\theta(l, p_0))(\omega)\left(\frac{\mathcal{BU}_t(l, \mathcal{BU}_{t-1}^\theta(l, p_0))(\omega)}{\mathcal{BU}_{t-1}^\theta(l, p_0)(\omega)}\right)^\theta.$$

Note that this can be rewritten as:

$$\begin{aligned}
&\mathcal{BU}_t^\theta(l, p_0)(\omega) \propto \mathcal{BU}_t(l, \mathcal{BU}_{t-1}^\theta(l, p_0))(\omega)\left(\frac{\mathcal{BU}_t(l, \mathcal{BU}_{t-1}^\theta(l, p_0))(\omega)}{\mathcal{BU}_{t-1}^\theta(l, p_0)(\omega)}\right)^\theta \\
&\propto \mathcal{BU}_t(l, \mathcal{BU}_{t-1}^\theta(l, p_0))(\omega)\left(\frac{l(X_t|X_1, \ldots, X_{t-1}, \omega)^\theta \mathcal{BU}_{t-1}^\theta(l, p_0)(\omega)}{\mathcal{BU}_{t-1}^\theta(l, p_0)(\omega)}\right)^\theta \\
&\propto l(X_t|X_1, \ldots, X_{t-1}, \omega)^{1+\theta}\mathcal{BU}_{t-1}^\theta(l, p_0)(\omega) \\
&\propto \mathcal{BU}_t(l^{1+\theta}, p_0)(\omega).
\end{aligned}$$

It turns out that this is a model of over-inference. The prior is corrected processed by the diagnostic operator, while the likelihood is over-weighted.

Note that for any $1 < \tau < t$:

$$\mathcal{BU}^{\theta}_{t-\tau}(l, \mathcal{BU}^{\theta}_{\tau}(l, p_0)(\omega))(\omega) = \mathcal{BU}_{t-\tau}(l^{1+\theta}, \mathcal{BU}_{\tau}(l^{1+\theta}, p_0)(\omega))(\omega) = \mathcal{BU}(l^{1+\theta}, p_0)(\omega) = \mathcal{BU}^{\theta}_{t}(l, p_0)(\omega).$$

This says that sequential updating or "one shot" updating are equivalent.
□

4.A.2 Theorem (3)

Proof. One way to state the optimality of the Bayesian updating is to say that the Bayesian posterior mean μ_t is the best predictor according to the quadratic loss function, $(\omega - \mu_t)^2$. This means that if an agent uses a different predictor, say $\mu^{\theta}_t = \mathbb{E}^{\theta}_t \omega$, then this is not the correct posterior mean, hence the expected utility of such an agent is:

$$-\mathbb{E}_t(\omega - \mathbb{E}^{\theta}_t \omega)^2 \leq -\mathbb{E}_t(\omega - \mathbb{E}_t \omega)^2.$$

This is a reduced form reasoning, in that it uses just the conditional expectations. If, as it is used in standard learning exercises, we assume that agents myopically optimize their quadratic utility at each time period (or any utility which has the conditional expectation of X_{t+1} as the optimal point), then we get that their intertemporal utility is in expectation smaller at every period:

$$-\mathbb{E}_0 \sum_t \beta^t (\omega - \mathbb{E}^{\theta}_t \omega)^2 \leq -\mathbb{E}_0 \sum_t \beta^t (\omega - \mathbb{E}_t \omega)^2.$$

We can understand better this discrepancy. First of all, applying the law of iterated expectations we can show that:

$$-\mathbb{E}_0 \sum_t \beta^t (\omega - \mathbb{E}^{\theta}_t \omega)^2 = -\mathbb{E}_0 \sum_t \beta^t \mathbb{E}_t (\omega - \mathbb{E}^{\theta}_t \omega)^2.$$

Consider the time t term:

$$(\omega - \mathbb{E}^{\theta}_t \omega)^2 = (\omega - \mathbb{E}_t \omega)^2 + (\mathbb{E}^{\theta}_t \omega - \mathbb{E}_t \omega)^2 + 2(\omega - \mathbb{E}_t \omega)(\mathbb{E}^{\theta}_t \omega - \mathbb{E}_t \omega)$$

and in expectation:

$$\mathbb{E}_t(\omega - \mathbb{E}^{\theta}_t \omega)^2 = \mathbb{V}_t \omega + \mathbb{E}_t(\mathbb{E}^{\theta}_t \omega - \mathbb{E}^{P}_t \omega)^2,$$

which shows that the disutility of an error has two components: the (im)precision of the rational posterior plus the discrepancy of the diagnostic from the bayesian posterior. □

MSE for Example 1.

$$\mathbb{E}_t^\theta \omega - \mathbb{E}_t \omega = \frac{\theta(t\mu_0 - \sum^t X_s)}{(t+1)(t(\theta+1)+1)}$$

and so, if t large:

$$(\mathbb{E}_t^\theta \omega - \mathbb{E}_t \omega)^2 = \frac{\theta^2}{(t(\theta+1)+1)^2}\mathbb{E}_t\left(\frac{t\mu_0 - \sum^t X_s}{t+1}\right)^2 \sim \frac{\theta^2}{(t^2(\theta+1))^2}\mathbb{E}_t\left(\mu_0 - \frac{\sum^t X_s}{t}\right)^2.$$

Moreover:

$$\mathbb{E}_t\left(\mu_0 - \frac{\sum^t X_s}{t}\right)^2 = \mathbb{E}_t\left(\mu_0 - \omega\right)^2 + \mathbb{E}_t\left(\omega - \frac{\sum^t X_s}{t}\right)^2.$$

4.A.3 Proof of Proposition 19

Mr 1 will play action $a_1 = 1$ after observing signal $s_1 = 1$ if and only if his posterior is higher than τ, that is:

$$\frac{pq^{1+\theta}}{pq^{1+\theta} + (1-p)(1-q)^{1+\theta}} \geq \tau$$

$$\iff 1 + \frac{1-p}{p}\left(\frac{1-q}{q}\right)^{1+\theta} \leq \frac{1}{\tau}$$

$$\iff (1+\theta)\log\left(\frac{1-q}{q}\right) \leq \log\left(\frac{1}{\tau} - 1\right)\frac{p}{1-p}$$

$$\iff \theta + 1 \geq \frac{\ln\left(\frac{p}{1-p}\left(\frac{1}{\tau} - 1\right)\right)}{\ln\frac{1-q}{q}} = \frac{-\ln\left(\frac{p}{1-p}\right) + \ln\left(\frac{\tau}{1-\tau}\right)}{\ln\frac{q}{1-q}}.$$

In the last line we used $q > \frac{1}{2}$ to change the inequality sign. This condition is always true if $p \geq \tau$, given that $\theta > -1$. Its interpretation is that if Mr 1 is ex ante indifferent or in favor of alternative 1, then if he observes signal $s_1 = 1$, he plays $a_1 = 1$ for any value of θ. On the contrary if $p < \tau$, meaning that the agent is ex ante in favor of alternative 0, when he observes signal $s_1 = 1$ he might or might not play action $a_1 = 1$, depending on the parameter values. The condition above says that the bigger θ is, the bigger the set of parameters under which Mr1 revises the prior and plays action $a_1 = 1$.
Similarly, after seeing $s_1 = 0$, Mr 1 will play action $a_1 = 1$ if and only if:

$$\frac{p(1-q)^{1+\theta}}{p(1-q)^{1+\theta} + (1-p)q^{1+\theta}} \geq \tau$$

$$\iff \theta + 1 \leq \frac{\ln\left(\frac{p}{1-p}\left(\frac{1}{\tau}-1\right)\right)}{\ln\frac{q}{1-q}} = \frac{\ln\left(\frac{p}{1-p}\right) - \ln\left(\frac{\tau}{1-\tau}\right)}{\ln\frac{q}{1-q}}.$$

This is never the case never if $p < \tau$, which means that if the agent is in favor of alternative 0 and then he sees the signal $s_1 = 0$, he never revises his opinion. On the contrary, depending on θ, the opposite case may be true. Call the above condition 2. Note that the space of parameter such that condition two is violated increases with θ.
Summing up: if the agent sees $s_1 = 1$, then he plays $a_1 = 1$ the if either $\tau \leq p$ or $\tau > p$ and condition 1 is satisfied. If the agent sees $s_1 = 0$, then he plays $a_1 = 0$ if $\tau > p$ or $\tau \leq p$ and condition 2 is violated.

The behavior of Mr 2 will depend on which conditions are satisfied. Consider the informationally efficient region IE (for agent 1) defined as:

$$IE = \{(p,\tau) \in [0,1] \times \mathbb{R}_+ | \, (\tau \leq p \text{ and condition 2 is true}) \, \text{or} \, (\tau > p \text{ and condition 1 is true })\}.$$

If the parameters lie inside IE, then Mr 2 can perfectly infer Mr1 signal by observing his action, since $a_1 = s_1$. Thus Mr 2 effectively observes two signals.
Consider first the case $p > \tau$, namely the prior is in favor of 1. In this case if $s_1 = 1$ than Mr2 will do his Bayesian updating, leading him to play action $a_2 = 1$ regardless of his signal. Similarly for subsequent agents: a cascade therefore starts on state 1 in this case. If instead $a_1 = s_1 = 0$, then if $s_2 = 0$ Mr2 will do his Bayesian updating, leading him to play action $a_2 = 0$ regardless of his signal. Similarly for subsequent agents: a cascade therefore starts on state 0 in this case. Finally if $a_1 = s_1 = 0$ and $s_2 = 1$, then Mr2 will do his Bayesian updating, leading him to play action $a_2 = 1$. However in this case a cascade does not start immediately, as Mr 3 faces the same problem of Mr1. Here the intuition is straightforward: opposite signals s_1 and s_2 cancel out, and therefore $Mr2$ only relies on his prior belief. The case $p < \tau$ is symmetric.
Resuming the dynamics is characterized as follows:

- if $p > \tau$ then the probability of learning is:

$$Pr(a_\infty = \omega) = pPr(a_\infty = 1) + (1-p)Pr(a_\infty = 0) = \left(pq + (1-p)q^2\right)\left(\sum_{i=0}^{\infty}(q(1-q))^i\right);$$

- if $p < \tau$ then the probability of learning is:

$$Pr(a_\infty = \omega) = pPr(a_\infty = 1) + (1-p)Pr(a_\infty = 0) = \left(pq^2 + (1-p)q\right)\left(\sum_{i=0}^{\infty}(q(1-q))^i\right);$$

Resuming, if $s_1 = s_2$, then $a_2 = s_2$; if instead $s_1 \neq s_2$ then M2 2 will stick to his prior belief. In the former case Mr 3 will also play $a_3 = s_2$ regardless of his signal (if $s_3 = s_2 = s_1$ this is true since $IE_1 \subseteq IE_2 \subseteq IE_3$; if $s_3 \neq s_2 = s_1$ then Mr3 problem is the same problem of Mr1, therefore $a_3 = a_3$); in the latter case Mr3 problem is the same problem of Mr 1, therefore $a_3 = s_3$. Therefore, if the parameters lie IE_1, then i

plays 1 if $p \geq \tau$, and 0 viceversa. This means that if $p \geq \tau$ and the first agent revealed his signal to be 1, then the second agent will always play 1, and this will not be informative for Mr 3, which will act as if he observed only the signal of the first agent. On the contrary, if the first agent revealed his signal to be 0 and still $p \geq \tau$, the second agent reveals his signal, and Mr 3 updates consequently. Hence, if the conditions on θ for Mr 1 are satisfied, the first agent reveals and the second follows if observes the same, and if observes a different signal it depends on the prior, as should be. If the conditions are satisfied for the first but not the second agent, it means that the first agent actually does not reveal information, hence the second agent actually behaves as the first, and it means that he will not reveal anything either, and we have the applicable cascade (because all subsequent agents will follow).

f Mr 1 plays 1 regardless of the signal s_1 observed, then Mr 2 has no updating to do, and will act as if she were the first of the line. This happens with probability $1 - q$.

Hence, if agents are all homogeneous, there is a trivial cascade on 1, and the probability of learning is p.

Mr 3 if he observes 2 identical zeros will ignore his private signals, and we have a cascade on 0 (the conditions on θ are trivially satisfied). If he observes 3 different signals, will follow the most frequent. Anyway, the first 2 signals are sufficient to determine which cascade we have.

4.B Additional Material

Large deviations

In the calculations above only the variance and the error matter. If we consider general concave utility functions $u(-(a-\omega)^2)$ (or general "risk aversion"), instead we have that the term t of the sum is:

$$\mathbb{E}_t^P u\left(-(\mathbb{E}_t^{P'}\omega-\omega)^2\right) = \mathbb{E}_t^P u\left(-\left((\omega-\mathbb{E}_t^P\omega)^2 + (\mathbb{E}_t^{P'}\omega-\mathbb{E}_t^P\omega)^2 + 2(\omega-\mathbb{E}_t^P\omega)(\mathbb{E}_t^{P'}\omega-\mathbb{E}_t^P\omega)\right.\right.$$

$$\leq u\left(-\mathbb{E}_t^P\left((\omega-\mathbb{E}_t^P\omega)^2 + (\mathbb{E}_t^{P'}\omega-\mathbb{E}_t^P\omega)^2 + 2(\omega-\mathbb{E}_t^P\omega)(\mathbb{E}_t^{P'}\omega-\mathbb{E}_t^P\omega)\right)\right) = u\left(-Var_t^P\omega - \mathbb{E}\right.$$

so now the expression obtained above in this case are just useful as upper bounds on the utility. The correct utility involves the term $\mathbb{E}_t^{P'}\omega-\mathbb{E}_t^P\omega$. We know that as n becomes large, by the large deviations principle $P(|\sum Y_n - \mu_0| > a) \sim e^{-\frac{a^2}{2}}$ if the Y are standard normal i.i.d. Hence:

$$P(|\mathbb{E}_t^{P'}\omega-\mathbb{E}_t^P\omega|) = P(\frac{\theta(t\mu_0 - \sum^t X_s)}{(t+1)(t(\theta+1)+1)} > a) =$$

$$P(|\mu_0 - \frac{\sum^t X_s}{t}| > a\frac{(t+1)(t(\theta+1)+1)}{t\theta}) \sim e^{-\frac{1}{2\sigma^2}\left(a\frac{(\theta+1)}{\theta}\right)^2 t^2}$$

so the variance of the distribution of large deviations is proportional to $\frac{\theta^2}{(\theta+1)^2}$, the same term as before, with same intuitions: underreaction leads to much worse large deviations.

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