An Explicit Representation for Disappointment Aversion and Other Betweenness Preferences*

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Abstract

One of the most well-known models of non-expected utility is Gul (1991)'s model of Disappointment Aversion. This model, however, is defined implicitly, as the solution to a functional equation; its explicit utility representation is unknown, which may limit its applicability. We show that an explicit representation can be easily constructed, using solely the components of the implicit one. We also provide a more general result: an explicit representation for preferences in the Betweenness class that also satisfy Negative Certainty Independence (Dillenberger, 2010) or its counterpart. We show how our approach gives a simple way to behaviorally identify the parameters of the representation and to study the consequences of disappointment aversion in a variety of applications.

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KEYWORDS: Betweenness, Cautious Expected Utility, Disappointment Aversion, Non-Expected Utility, Utility Representation, Strategic Rationality.

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1 Introduction

One of the most well-known models of non-expected utility preferences is Gul (1991)'s model of Disappointment Aversion (henceforth DA). Its popularity is related both to the intuitive nature of the model, where the value of each outcome is determined relatively to an endogenously-defined "expected" payoff, capturing reference dependence; and because it generalizes expected utility by adding only one parameter. Despite its appeal, there is one limitation to its applicability: the value of each lottery is the solution of an equation that changes with the lottery—a so-called implicit representation. The (explicit) utility representation is instead unknown. This may be a concern if one wishes to apply this model, for example, to carry out monotone comparative statics exercises. The same concern applies to the broader class of Betweenness preferences, studied in Dekel (1986) and Chew (1989) and to which the DA model belongs; for such preferences only an implicit representation is known, while the explicit one is still elusive.[1]

The goal of this paper is to address these issues. First, we provide an explicit representation for DA preferences, showing that it can be easily obtained using solely the components of its implicit one. Second, we generalize this result: we provide an explicit representation for Betweenness preferences that satisfy either Negative Certainty Independence (Dillenberger, 2010; Cerreia-Vioglio et al., 2015), or its positive version, Positive Certainty Independence. Third, we show how our approach may be useful to identify parameters of the model and in comparative statics exercises.

Let p be a lottery over monetary outcomes. Its value according to the DA model is the unique v that solves

$$v = \mathbb{E}_p(k_v) \tag{1}$$

where k_v is given by

$$k_v(x) = \begin{cases} u(x) & u(x) \leq v \\ \frac{u(x)+\beta v}{1+\beta} & u(x) > v \end{cases} .$$

Here u is a utility function over money and $\beta \in (-1, \infty)$ is the coefficient of either disappointment aversion ($\beta > 0$) or elation seeking ($\beta < 0$). Note that this is an implicit equation, as the value v appears on both sides of Equation (1). In this model the value v is similar to expected utility, except that the individual gives an additional weight β to disappointing outcomes—those with a utility lower than the value of the lottery itself.[2] The DA model is thus a model of endogenous reference dependence: possible payoffs generate disappointment (or elation) depending on how their utilities

[1] This is the case not only for the broad class, but also for most of its special cases. A notable exception is Chew and MacCrimmon (1979a,b)'s model of weighted-utility.

[2] To see this, note that the value of a simple lottery p can be equivalently defined as the unique v

compare to an endogenously-determined value—the utility of the lottery.[3] When $\beta > 0$, the disappointing outcomes receive greater weight, whereas the opposite is true for $\beta < 0$, justifying the terms disappointment aversion/elation seeking. If $\beta = 0$, the model reduces to expected utility.

In Section 3 we show that these preferences admit the following explicit representation. When $\beta > 0$, the case of disappointment aversion, preferences are represented by

$$V(p) = \min_{v} k_v^{-1} \left(\mathbb{E}_p \left(k_v \right) \right),$$

while when $\beta \in (-1, 0)$, the case of elation seeking, they are represented by

$$V(p) = \max_{v} k_v^{-1} \left(\mathbb{E}_p \left(k_v \right) \right).$$

This means that one can easily construct an explicit representation for preferences in this class using solely the components of the implicit representation in Equation (1)—taking the min or the max of the certainty equivalents computed using each of the possible utilities involved.

After formally stating the result above for DA preferences (Theorem 3), we discuss an explicit representation for general Betweenness preferences that also satisfy Negative Certainty Independence (Theorem 4), or its counterpart (Positive Certainty Independence). The previous results are corollaries of this more general theorem. Again, the explicit representation is the minimum (resp., maximum) of the certainty equivalents using the functions (called local utilities) present in Dekel (1986)'s implicit representation.

There are at least two benefits of having an explicit representation. The first is conceptual: it may help capturing the mental process adopted by the agent. While highly idealized, one can imagine a cautious decision process that involves the max min criterion. It is perhaps less plausible to take the solution of an implicit equation as a descriptive decision making procedure. That said, this argument is not behavioral, but relies on going beyond the standard 'as if' approach.[4]

The second and possibly main advantage of an explicit representation is practical: it facilitates the application of these models by allowing for a better understanding

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that solves

$$v = \frac{\sum_{\{x:u(x)>v\}} u(x)p(x) + (1+\beta) \sum_{\{x:u(x)\leq v\}} u(x)p(x)}{1 + \beta \sum_{\{x:u(x)\leq v\}} p(x)}.$$

[3]We should stress that this is conceptually and behaviorally distinct from other models of endogenous reference dependence under risk, e.g., Köszegi and Rabin (2006, 2007). For example, the DA model satisfies Betweenness, while both models above do not. See Masatlioglu and Raymond (2016) for further discussion on the implications of these alternative models.

[4]A related argument appears in Chapter 17 of Gilboa (2009) and in Dekel and Lipman (2010).

of optimization problems with these preferences. This is particularly relevant because DA preferences, while continuous, are not even Gateaux differentiable (Safra and Segal, 2009). Therefore, one cannot apply standard differential methods or Machina (1982)'s local utility approach and its extensions to the Gateaux case.

In Section 4 we start demonstrating the practical benefit of our approach, by showing that it can be useful to identify the parameters of DA preferences from behavior: while it is known that both u and β are unique (the former up to positive affine transformations), using the original implicit representation it is not easy to see how to easily pin-point them from behavioral data. Our approach, instead, offers a way to do so directly, identifying u independently of β.

With this identification in hand, in Section 5 we discuss optimization and applications. Using Sion's minmax theorem in conjunction with our characterizations, we show that the solution to a maximization problem over a convex and compact set of options with DA preferences must coincide with the solution of the original problem under expected utility, but with the agent being more risk averse. The key point is not that disappointment aversion increases risk aversion, but rather that the solution under disappointment aversion must also be a solution under *expected utility*, with a more concave utility. That is, solving the problem under disappointment aversion is tantamount to solving it with expected utility with higher risk aversion. As the latter is typically easier to solve, this may greatly simplify the problem when it comes to comparative statics.

We illustrate the usefulness of this result with a variety of applications. We show that, in a standard portfolio problem, a disappointment averse agent invests less in the risky asset compared to an expected utility agent with the same utility over outcomes; that the set of justifiable actions in general, and rationalizable strategies in a game in particular, is smaller if we make agents elation seeking; and that in a simple problem of Bayesian Persuasion, making the sender of information disappointment averse reduces the revealed information.

2 Preliminaries

Consider a nontrivial compact interval $[w, b] \subseteq \mathbb{R}$ of monetary prizes. Let Δ be the set of lotteries (Borel probability measures) over $[w, b]$, endowed with the topology of weak convergence. We denote by x, y, z generic elements of $[w, b]$; by p, q, r generic elements of Δ; and by $\delta_x \in \Delta$ the degenerate lottery (Dirac measure at x) that gives the prize $x \in [w, b]$ with certainty. We denote by $C([w, b])$ the space of continuous functions on $[w, b]$ and endow it with the topology induced by the supnorm. The set $\mathcal{U}_{\text{nor}} \subseteq C([w, b])$ is the collection of all strictly increasing and continuous functions

$v:[w,b]\to\mathbb{R}$ such that $v(w)=0$ and $v(b)=1$. Given $p\in\Delta$ and a strictly increasing $v\in C([w,b])$, we define $c(p,v)=v^{-1}(\mathbb{E}_p(v))$. Lastly, $\succcurlyeq_{FSD}$ denotes the First Order Stochastic Dominance relation, that is, $p\succcurlyeq_{FSD} q$ means $\mathbb{E}_p(v)\geq\mathbb{E}_q(v)$ for all $v\in\mathcal{U}_{\text{nor}}$.

The primitive of our analysis is a binary relation $\succcurlyeq$ over Δ. The symmetric and asymmetric parts of $\succcurlyeq$ are denoted by $\sim$ and, respectively, $\succ$. A certainty equivalent of a lottery $p\in\Delta$ is a prize $x_p\in[w,b]$ such that $\delta_{x_p}\sim p$. Throughout the paper, we focus on binary relations $\succcurlyeq$ that satisfy the following three standard assumptions.

A 1 (Weak Order) *The relation $\succcurlyeq$ is complete and transitive.*

A 2 (Continuity) *For each $q\in\Delta$, the sets $\{p\in\Delta : p\succcurlyeq q\}$ and $\{p\in\Delta : q\succcurlyeq p\}$ are closed.*

A 3 (Strict First Order Stochastic Dominance) *For each $p,q\in\Delta$*

$$p\succ_{FSD} q \implies p\succ q.$$

Betweenness Preferences. We study binary relations that satisfy the following assumption:

A 4 (Betweenness) *For each $p,q\in\Delta$ and $\lambda\in[0,1]$*

$$p\sim q \implies p\sim\lambda p+(1-\lambda)q\sim q.$$

Betweenness implies neutrality toward mixing: if satisfied, then the agent has no preference for, or aversion to, mixing between indifferent lotteries. Binary relations satisfying this property were studied by Dekel (1986) and Chew (1989).

We say that a binary relation is a *Betweenness preference* if and only if it satisfies Weak Order, Continuity, Strict First Order Stochastic Dominance, and Betweenness. Dekel (1986) proves a version of the following result:[5]

Theorem 1 (Dekel, 1986) *If $\succcurlyeq$ is a Betweenness preference, then there exists a function $k:[w,b]\times[0,1]\to\mathbb{R}$ such that:*

[5] Dekel's original result deals with a general set of consequences and considers a weaker form of monotonicity. At the same time, it uses a stronger form of Betweenness. Given these differences, we prove Theorem 1 in Appendix A. For convenience, we focus on the normalized representation (that is, k satisfies the condition in point 3). In Remark 4 in Appendix A we comment on how to use our results for unnormalized representations. Also observe that even though $k(\cdot,0)$ and $k(\cdot,1)$ are not assumed to be continuous, they are implicitly assumed to be integrable, given Equation (2) in the representation.

1. $x \mapsto k(x,t)$ *is strictly increasing and continuous on* $[w,b]$ *for all* $t \in (0,1)$,

2. $t \mapsto k(x,t)$ *is continuous on* $(0,1)$ *for all* $x \in [w,b]$,

3. $k(w,t) = 0$ *and* $k(b,t) = 1$ *for all* $t \in [0,1]$,

4. $\succcurlyeq$ *can be represented by a continuous utility function which strictly preserves first order stochastic dominance,* $\hat{V} : \Delta \to [0,1]$, *where for each* $p \in \Delta$, $\hat{V}(p)$ *is the unique number in* $[0,1]$ *such that*

$$\int_{[w,b]} k\left(x, \hat{V}(p)\right) dp = \hat{V}(p). \tag{2}$$

Fixing t, the function $k(\cdot,t)$ is called the *local utility at* t. The function k thus summarizes the collection of local utilities, one for each $t \in [0,1]$. While the theorem above provides a representation for Betweenness preferences, it does not provide an explicit one: indeed, $\hat{V}$ is the solution to Equation (2), thus a fixed point of a functional equation.

An important class of Betweenness preferences is the one generated by Gul (1991)'s model of Disappointment Aversion (DA). These preferences admit a continuous utility function $\tilde{V} : \Delta \to \mathbb{R}$ such that, for each $p \in \Delta$, $\tilde{V}(p)$ is the unique number that solves

$$\int_{[w,b]} \tilde{k}\left(x, \tilde{V}(p)\right) dp = \tilde{V}(p) \tag{3}$$

where $\tilde{k} : [w,b] \times \operatorname{Im} u \to \mathbb{R}$ is defined by

$$\tilde{k}(x,s) = \begin{cases} u(x) & \text{if } u(x) \leq s \\ \frac{u(x)+\beta s}{1+\beta} & \text{if } u(x) > s \end{cases} \qquad \forall x \in [w,b], \forall s \in \operatorname{Im} u; \tag{4}$$

here u is a strictly increasing continuous utility function from $[w,b]$ to $\mathbb{R}$ and $\beta \in (-1,\infty)$.[6] We discussed its interpretation in the Introduction. We say that a binary relation is a *DA preference* with parameters (u,β) if and only if it admits a utility function $\tilde{V}$ which satisfies (3).

Negative Certainty Independence. As noted by Dillenberger (2010), a DA preference with $\beta > 0$ satisfies the following axiom.

A 5 (Negative Certainty Independence) *For each* $p, q \in \Delta$, $x \in [w,b]$, *and* $\lambda \in [0,1]$

$$p \succcurlyeq \delta_x \implies \lambda p + (1-\lambda) q \succcurlyeq \lambda \delta_x + (1-\lambda) q. \tag{5}$$

[6]A careful inspection of (4) also suggests that two types of normalizations are due to link the implicit representation of Gul (1991) to the one of Dekel (1986) as in Theorem 1. In proving our results below, we also address these minor technical points (see Remark 4).

Negative Certainty Independence, initially suggested by Dillenberger (2010), is meant to capture the certainty effect. It states that if the sure outcome x is not enough to compensate the agent for the risky prospect p, then mixing it with any other lottery, thus eliminating its certainty appeal, will not result in the mixture of δ_x being more attractive than the corresponding mixture of p. The opposite condition, termed Positive Certainty Independence, simply inverts the role of p and δ_x in (5).

We say that a binary relation is a *Cautious Expected Utility preference* if and only if it satisfies Weak Order, Continuity, Strict First Order Stochastic Dominance, and Negative Certainty Independence. Cerreia-Vioglio et al. (2015) prove the following:[7]

Theorem 2 (Cerreia-Vioglio, Dillenberger, Ortoleva, 2015) *If $\succcurlyeq$ is a Cautious Expected Utility preference, then there exists $\mathcal{W} \subseteq \mathcal{U}_{\mathrm{nor}}$ such that $V : \Delta \to \mathbb{R}$, defined by*

$$V(p) = \inf_{v \in \mathcal{W}} c(p, v) \qquad \forall p \in \Delta, \tag{6}$$

is a continuous utility representation of $\succcurlyeq$.

3 Explicit representations

We start by providing an explicit representation of DA preferences.

Theorem 3 *Let $\succcurlyeq$ be a DA preference with parameters (u, β) and $\mathcal{W}_{\mathrm{da}} = \left\{\tilde{k}(\cdot, z)\right\}_{z \in \operatorname{Im} u}$. The following statements are true:*

1. *If $\beta > 0$, then $V : \Delta \to \mathbb{R}$, defined by*

$$V(p) = \min_{v \in \mathcal{W}_{\mathrm{da}}} c(p, v) \qquad \forall p \in \Delta, \tag{7}$$

is a continuous utility representation of $\succcurlyeq$.

2. *If $\beta = 0$, then $V : \Delta \to \mathbb{R}$, defined by*

$$V(p) = c(p, u) \qquad \forall p \in \Delta, \tag{8}$$

is a continuous utility representation of $\succcurlyeq$.

3. *If $\beta < 0$, then $V : \Delta \to \mathbb{R}$, defined by*

$$V(p) = \max_{v \in \mathcal{W}_{\mathrm{da}}} c(p, v) \qquad \forall p \in \Delta,$$

is a continuous utility representation of $\succcurlyeq$.

[7] More precisely, Cerreia-Vioglio et al. (2015) state the result below as an equivalence using a weaker form of monotonicity. However, for ease of comparison with Theorem 1, we provide it using Strict First Order Stochastic Dominance. In Remark 3 in Appendix A, we also discuss the uniqueness features of the representation. See also Cerreia-Vioglio et al. (2015, Theorem 2).

In the case of disappointment aversion ($\beta > 0$), our utility representation is the smallest of the certainty equivalents obtained using the local utilities. In the opposite case of elation seeking ($\beta < 0$), it is instead the largest. Thus, the difference between the two behaviors is not only in the way in which disappointing/elating outcomes are weighted, but also in how they are aggregated—using the min or the max.

As discussed above, when $\beta > 0$, Gul's model satisfies Negative Certainty Independence. We thus know that it must admit a Cautious Expected Utility representation. However, from previous results we do not know which utilities are used. The content of Theorem 3 is to show that this involves *precisely* the local utilities used in the implicit representation. Thus, the explicit representation can be derived directly from the implicit one. When $\beta < 0$, the model does *not* satisfy Negative Certainty Independence, but its counterpart Positive Certainty Independence (Artstein-Avidan and Dillenberger, 2015). In this case the individual is elation seeking, and violates expected utility in a way opposite to the certainty effect.

A General Result. We now generalize Theorem 3 showing that any Betweenness preference that satisfies Negative Certainty Independence also admits an *explicit* representation of the Cautious Expected Utility form, where the utilities in $\mathcal{W}$ are the local ones obtained in Theorem 1, that is, $\mathcal{W}_{\text{bet}} = \{k(\cdot, t)\}_{t\in(0,1)}$.

Theorem 4 *Let $\succcurlyeq$ be a Betweenness preference. The following statements are equivalent:*

(i) $\succcurlyeq$ satisfies Negative Certainty Independence;

(ii) The functional $V : \Delta \to \mathbb{R}$, defined by

$$V(p) = \min_{v\in\mathcal{W}_{\text{bet}}} c(p, v) \qquad \forall p \in \Delta, \tag{9}$$

is a continuous utility representation of $\succcurlyeq$. In particular, for each $p \in \Delta \setminus \{\delta_w, \delta_b\}$ the function $v_p = k\left(\cdot, \hat{V}(p)\right)$ is such that

$$v_p \in \operatorname{argmin}_{v\in\mathcal{W}_{\text{bet}}} c(p, v). \tag{10}$$

Like in the case of the previous result, the contribution of Theorem 4 does not lie in showing that these preferences admit an explicit representation of the Cautious Expected Utility class—this was already known (it follows from Theorem 2). As before, the contribution lies in showing that the utilities involved are exactly the local utilities identified in Theorem 1 and included in $\mathcal{W}_{\text{bet}}$. Here too, the explicit representation can be derived directly from the implicit one. In addition, Equation (10) shows that the local utility giving the implicit representation of Dekel (1986) is also the one achieving the minimum in representation (9).

Remark 1 A specular version of this theorem also holds for Positive Certainty Independence. In particular, by keeping the same premises, Theorem 4 takes a similar form with (i) and (ii) replaced by:

(i)' $\succcurlyeq$ satisfies Positive Certainty Independence;

(ii)' The functional $V : \Delta \to \mathbb{R}$, defined by

$$V(p) = \max_{v \in \mathcal{W}_{\text{bet}}} c(p, v) \qquad \forall p \in \Delta, \tag{11}$$

is a continuous utility representation of $\succcurlyeq$. In particular, for each $p \in \Delta \backslash \{\delta_w, \delta_b\}$ the function $v_p = k\left(\cdot, \hat{V}(p)\right)$ is such that

$$v_p \in \text{argmax}_{v \in \mathcal{W}_{\text{bet}}} c(p, v). \tag{12}$$

$\triangledown$

While Theorem 4 provides an explicit characterization for Betweenness preferences that satisfy Negative Certainty Independence, a natural question is how to check if a given Betweenness preference satisfies Negative Certainty Independence. In Appendix B we show how this could be easily done using only the properties of the local utilities. This result allows us to derive an example of a Betweenness preference that satisfies Negative Certainty Independence, but does not belong to the class of DA preferences.

Example 1 Consider a Betweenness preference with local utilities $k : [0,1] \times [0,1] \to \mathbb{R}$ defined as

$$k(x,t) = \begin{cases} x & \text{if } x \leq t \\ x(x-t)+t & \text{if } x > t \end{cases} \qquad \forall x \in [0,1], \forall t \in [0,1].$$

This is a special case of Chew (1985)'s model of semi-implicit weighted utility, where $[w, b]$ is set to be equal to $[0,1]$. It retains the idea of disappointment aversion, but allows the weight, $x - t$, to depend on x. In Appendix B we show that this preference relation satisfies Negative Certainty Independence and therefore admits an explicit representation as in Theorem 4 with the utilities above. $\triangledown$

4 Behavioral identification of parameters

Even though in the DA model both the utility function u and the disappointment aversion parameter β are unique (the former up to positive affine transformations), we are not aware of simple ways of identifying them from behavior. We now use our approach to derive an identification technique, as well as a behavioral notion of comparative disappointment aversion.

The expected utility core. In what follows it will be useful to recall the notion of expected utility core of a preference relation $\succcurlyeq$, i.e., the subrelation $\succcurlyeq'$ defined as

$$p \succcurlyeq' q \iff \lambda p + (1-\lambda) r \succcurlyeq \lambda q + (1-\lambda) r \quad \forall \lambda \in (0,1], \forall r \in \Delta.$$

That is, $p \succcurlyeq' q$ if not only p is preferred to q, but this ranking is preserved even if both are mixed with another lottery. Under expected utility, we have $\succcurlyeq = \succcurlyeq'$; however, in general the two may not coincide and, in this case, $\succcurlyeq'$ is incomplete. This notion is central in proving our main results above and in the study of Cautious Expected Utility preferences.[8] We now show that it is also useful for the identification of the parameters characterizing a DA preference.

Behavioral identification of parameters. We begin by recalling how identification works under expected utility. Consider an expected utility agent with a continuous and strictly increasing von Neumann-Morgenstern utility u. If we normalize $u(b) = 1$ and $u(w) = 0$, it is known that for each $x \in [w, b]$ we can identify $u(x)$ by

$$u(x) = \max\{\lambda \in [0,1] : \delta_x \succcurlyeq \lambda \delta_b + (1-\lambda)\delta_w\}.$$

That is, when u is normalized, the utility of x is equal to the highest of the weights λ that, if given to the best prize, keeps δ_x preferred to $\lambda \delta_b + (1-\lambda)\delta_w$. Because under expected utility $\succcurlyeq$ coincides with $\succcurlyeq'$, this can be equivalently stated as

$$u(x) = \max\{\lambda \in [0,1] : \delta_x \succcurlyeq' \lambda \delta_b + (1-\lambda)\delta_w\}.$$

We now show that for a DA preference with parameters (u, β), where $\succcurlyeq'$ and $\succcurlyeq$ cease to coincide whenever $\beta \neq 0$, it is this latter formula that allows us to identify the utility u *independently* from β. Before stating our result, recall that two von Neumann-Morgenstern functions $u, \bar{u} \in C([w, b])$ are cardinally equivalent if and only if $u = a\bar{u} + b$ where $a > 0$ and $b \in \mathbb{R}$. It is easy to see that if (u, β) represents a DA preference $\succcurlyeq$, so does $(\bar{u}, \beta)$, provided u and $\bar{u}$ are cardinally equivalent. Recall that given a lottery p and a preference relation $\succcurlyeq$, we denote by x_p the certainty equivalent of p.

Proposition 1 *Let $\succcurlyeq$ be a DA preference with parameters (u, β). The following statements are true:*

1. *If $\beta \geq 0$, then u is cardinally equivalent to*

$$\bar{u}(x) = \max\{\lambda \in [0,1] : \delta_x \succcurlyeq' \lambda \delta_b + (1-\lambda)\delta_w\} \qquad \forall x \in [w, b]$$

[8] See also Remark 3 and Propositions 5-7 in Appendix A. Under our assumptions, $\succcurlyeq'$ is the largest subrelation of $\succcurlyeq$ that satisfies Independence. See Cerreia-Vioglio (2009); Cerreia-Vioglio et al. (2015, 2017).

and if we define $\bar{p} = \frac{1}{2}\delta_b + \frac{1}{2}\delta_w$*, then the exact value of* β *is*

$$\beta = \frac{1}{\bar{u}(x_{\bar{p}})} - 2 = \frac{1}{\max\{\lambda \in [0,1] : \delta_{x_{\bar{p}}} \succcurlyeq' \lambda\delta_b + (1-\lambda)\delta_w\}} - 2.$$

2. *If* $\beta \leq 0$*, then* u *is cardinally equivalent to*

$$\bar{u}(x) = \min\{\lambda \in [0,1] : \lambda\delta_b + (1-\lambda)\delta_w \succcurlyeq' \delta_x\} \qquad \forall x \in [w,b]$$

and if we define $\bar{p} = \frac{1}{2}\delta_b + \frac{1}{2}\delta_w$*, then the exact value of* β *is*

$$\beta = \frac{1}{\bar{u}(x_{\bar{p}})} - 2 = \frac{1}{\min\{\lambda \in [0,1] : \lambda\delta_b + (1-\lambda)\delta_w \succcurlyeq' \delta_{x_{\bar{p}}}\}} - 2.$$

To distinguish behaviorally between the two cases above, i.e., $\beta \geq 0$ (resp., $\beta \leq 0$), recall that by Theorem 3 they correspond to Negative (resp., Positive) Certainty Independence.

Moreover, in both cases above the value of β can be determined using any binary lottery between the best and the worst outcomes—not just $\bar{p}$. This is because, for a DA preference with parameters (u, β), for each $\lambda \in [0,1]$

$$u(x_{\lambda\delta_b+(1-\lambda)\delta_w}) = \tilde{V}(\lambda\delta_b + (1-\lambda)\delta_w) = \frac{\lambda u(b) + (1+\beta)(1-\lambda)u(w)}{1+\beta(1-\lambda)}.$$

Since u is cardinally equivalent to $\bar{u}$, if we define $\hat{p} = \lambda\delta_b + (1-\lambda)\delta_w$, then $\bar{u}(x_{\hat{p}}) = \frac{\lambda}{1+\beta(1-\lambda)}$, which means $\beta = \frac{1}{1-\lambda}[\frac{\lambda}{\bar{u}(x_{\hat{p}})} - 1]$.

Comparative Disappointment Aversion. The result above also allows us to derive simple comparative statics, showing how to behaviorally identify when one agent is more disappointment averse than the other, i.e., has higher β fixing the utility function u.

Proposition 2 *Let* $\succcurlyeq_1$ *and* $\succcurlyeq_2$ *be two DA preferences with parameters* (u_1, β_1) *and* (u_2, β_2)*, with* $\beta_1, \beta_2 \geq 0$*. The following statements are equivalent:*

(i) u_1 *is cardinally equivalent to* u_2 *and* $\beta_1 \geq \beta_2$*;*

(ii) For each $p \in \Delta$ *and* $x \in [w,b]$

$$\delta_x \succcurlyeq_1' p \iff \delta_x \succcurlyeq_2' p \quad \textit{and} \quad \delta_x \succcurlyeq_2 p \implies \delta_x \succcurlyeq_1 p.$$

In words, an agent is more disappointment averse than the other, if the two agents have the same risk attitudes in terms of the expected utility core, but the first agent is more risk averse than the second in terms of final choices.

Now that we understand the behavioral meaning of comparing two agents with identical u solely in terms of their coefficient β, we can proceed to illustrate how our explicit representations can be used to provide comparative statics results in applications.

5 Optimization and applications

5.1 Explicit representations and optimization

Typically in economic models, agents need to pick the best action from a convex and compact set of alternatives. However, explicitly solving such problems with Betweenness preferences or even just obtaining qualitative predictions, can be nontrivial: as pointed out in the introduction, standard differential methods cannot be used, as DA preferences are not even Gateaux differentiable. We now show how our explicit representation results might help.

Consider a Betweenness preference $\succcurlyeq$ that satisfies Negative Certainty Independence—for DA preferences, this corresponds to the typical case of $\beta \geq 0$. By Theorem 4, $\succcurlyeq$ is represented by

$$V(p) = \min_{v \in \mathcal{W}_{\text{bet}}} c(p, v) \qquad \forall p \in \Delta \tag{13}$$

where $\mathcal{W}_{\text{bet}} = \{k(\cdot, t)\}_{t \in (0,1)}$. We further assume:
(a) $k(\cdot, t)$ is strictly increasing on $[w, b]$ for all $t \in [0, 1]$;
(b) k is jointly continuous on $[w, b] \times [0, 1]$.

Note that both assumptions are satisfied by DA preferences as well as the preferences in Example 1 (see also Remark 4 in Appendix A).

Proposition 3 *Let $\succcurlyeq$ be a Betweenness preference that satisfies Negative Certainty Independence and such that k satisfies (a) and (b). If $A \subseteq \Delta$ is convex and compact, then*

$$\max_{p \in A} \min_{v \in \bar{\text{co}}(\mathcal{W}_{\text{bet}})} c(p, v) = \min_{v \in \bar{\text{co}}(\mathcal{W}_{\text{bet}})} \max_{p \in A} c(p, v).$$

In particular, if $p^* \in A$ is such that $V(p^*) \geq V(p)$ for all $p \in A$, then there exists $\hat{v} \in \bar{\text{co}}(\mathcal{W}_{\text{bet}})$ such that $\mathbb{E}_{p^*}(\hat{v}) \geq \mathbb{E}_p(\hat{v})$ for all $p \in A$.

Proposition 3 shows that any alternative that maximizes the original preference in A is also a maximizer of an *expected utility* preference with von Neumann-Morgenstern utility $\hat{v}$ which is a convex combination of the utilities in $\mathcal{W}_{\text{bet}}$. Note that this result was derived exploiting the explicit representation in (13). This result holds because the map $(p, v) \mapsto c(p, v)$ satisfies all the conditions of Sion's minmax theorem and so do the sets A and $\bar{\text{co}}(\mathcal{W}_{\text{bet}})$, leading to the equality in Proposition 3 and the existence of a saddle point.

To see the usefulness of this result, consider a maximization problem of the form

$$\max V(p) \text{ subject to } p \in A \tag{14}$$

where $V : \Delta \to \mathbb{R}$ is continuous and represents the agent's preferences. For simplicity, assume also it admits a unique maximizer. As it is often the case, assume that V has been first considered to be an expected utility functional with a strictly increasing and continuous von Neumann-Morgenstern function u, and call p^*_{EU} the solution of (14). Now suppose that we are interested in knowing what happens when agents are instead disappointment averse ($\beta > 0$), and call p^*_{DA} the solution for this case. How do the predictions of the model change? Proposition 3 simplifies answering this question. It shows that p^*_{DA} must also be the solution of the same optimization problem, but for an expected utility agent with von Neumann-Morgenstern function $\hat{v} \in \bar{\mathrm{co}}\,(\mathcal{W}_{\mathrm{bet}})$. It follows from the shape of the functions in $\mathcal{W}_{\mathrm{bet}}$ that each $v \in \bar{\mathrm{co}}\,(\mathcal{W}_{\mathrm{bet}})$ is more concave than u, which is easy to see because our characterization theorems give a precise functional form to these functions. Therefore, the solution to the problem with a DA preference must coincide with the solution of the original problem under expected utility, but with the agent being more risk averse. This means that monotone comparative statics exercises in terms of introducing disappointment aversion are equivalent to ones under expected utility in terms of concavity of u. As the latter are typically easier to solve, or at least use more familiar techniques, this may simplify the problem. The conceptual contribution here does not lie in showing that introducing disappointment aversion increases risk aversion—this is already well-known (Gul, 1991). Rather, it lies in showing that the solution under disappointment aversion must also be a solution under *expected utility* with a more risk averse agent.

Remark 2 In the optimization problem above, alternatives are lotteries and agents' preferences are over Δ. A different common formalization of the problem in (14) is instead the following: the agent chooses an action from a set A, a state of the world from a state space S realizes, and the decision maker receives a monetary outcome specified by the function $g : A \times S \to \mathbb{R}$. Suppose A and S are separable metric spaces where the latter is endowed with the Borel σ-algebra. Also assume that g is measurable with respect to s and bounded, i.e., $\mathrm{Im}\, g \subseteq [w, b]$. The decision maker has a Borel probability measure μ over S and preferences over Δ, represented by the function $V : \Delta \to \mathbb{R}$. Each action a thus induces a probability measure over $[w, b]$, say $p_{a,\mu} \in \Delta$, where $p_{a,\mu}$ is the distribution of $g(a, \cdot) : S \to \mathbb{R}$ under μ. That is, $p_{a,\mu}(B) = \mu(\{s \in S : g(a, s) \in B\})$ for all Borel sets B of $[w, b]$.

We assume that the agent is probabilistically sophisticated: given a belief μ on S, she values each action solely in terms of its distribution over payoffs, that is, if $a, b \in A$ are such that $p_{a,\mu} = p_{b,\mu}$, then the agent is indifferent between a and b. Problem (14) then becomes

$$\max V(p_{a,\mu}) \text{ subject to } a \in A. \tag{15}$$

In Appendix A, we show that a result similar to Proposition 3 also holds in this setting with an identical conceptual interpretation (cf. Propositions 8 and 9). $\triangledown$

5.2 Applications

We now present a list of applications of our previous results. The common feature is that they almost all follow rather immediately from a combination of our comparative statics and known results in the literature. This is because we have proved that introducing disappointment aversion yields the same monotone comparative statics of increasing risk aversion within expected utility.

Portfolio choice. Consider a standard two assets portfolio problem a la Arrow (noting we are in the framework of Remark 2). The risky asset $r : S \to [0, \infty)$ is a simple nonconstant measurable random variable which pays a return $r(s)$ if state s realizes. The risk free asset is instead a constant random variable $r_f : S \to [0, \infty)$. To avoid trivial cases, assume that $\mathbb{E}_\mu(r) > r_f$.

A decision maker has wealth $W > 0$ which she has to allocate between r and r_f. Denote by a and $W - a$ the amounts invested, respectively, in the risky and the risk-free assets. Assume short-selling and borrowing are not allowed, which implies $a \in [0, W] = A$. Assume also that $ar(s) + (W - a) r_f(s) \in [w, b]$ for all $a \in A$ and for all $s \in S$. Denote by a^*_{EU} the optimal allocation of a decision maker with expected utility preferences ($\beta = 0$) and a von Neumann-Morgenstern utility u such that $u' > 0$ and $u'' < 0$. Denote by a^*_{DA} the optimal allocation of a disappointment averse agent ($\beta > 0$) with the same u. It is easy to show that both solutions are unique.

Corollary 1 $a^*_{\text{DA}} \le a^*_{\text{EU}}$.

Recall that, under expected utility, increasing risk aversion reduces the absolute investment in the risky asset. Since we have shown that the same comparative statics holds if we make the agent disappointment averse, the result that disappointment averse decision makers invest less in the risky asset follows.

Justifiability and rationalizability. A decision maker chooses an action $a \in A$ without knowing which state of the world $s \in S$ will realize and receives a monetary payoff $g(a, s)$ (again, we are in the framework of Remark 2). Denote by $\Delta(S)$ the set of Borel probability measures on S.

The set of *justifiable* actions is the set of actions that are optimal for *some* belief μ. That is,

$$\mathcal{J}_V = \{a \in A : \exists \mu \in \Delta(S) \text{ s.t. } V(p_{a,\mu}) \ge V(p_{b,\mu}) \quad \forall b \in A\}.$$

This set and the corresponding operator are commonly used in game theory to derive rationalizable strategies: at the initial iteration, A is the set of actions of player i and

$S = A_{-i}$ is the profile of the opponents' actions. We are interested in: How is the set of justifiable actions affected by disappointment aversion? Denote by $\mathcal{J}_{\text{DA}}$ and $\mathcal{J}_{\text{EU}}$, respectively, the set of justifiable actions of a DA agent with parameters (u, β) and that of an expected utility agent with utility u.

Corollary 2 *Let $g : A \times S \to \mathbb{R}$ be jointly continuous. If $\beta < 0$ and S is compact, then $\mathcal{J}_{\text{DA}} \subseteq \mathcal{J}_{\text{EU}}$.*

In words, making the agent *elation seeking* shrinks the set of justifiable actions. To see why, recall that it is known that increasing risk aversion increases the set of justifiable actions (Battigalli et al., 2016; Weinstein, 2016); then the same comparative statics holds here, as moving from $\beta = 0$ to negative β *decreases* risk aversion.

The result above does not cover the case $\beta > 0$. To see why, note that the set of justifiable actions is a union of argmax sets ($\mathcal{J}_V = \cup_{\mu \in \Delta(S)} \operatorname{argmax}_{a \in A} V(p_{a,\mu})$), while our results for monotone comparative statics are in terms of argmax sets. By our maxmin results with DA preferences, we have that if either $\beta > 0$ or $\beta < 0$, then $\mathcal{J}_V \subseteq \cup_{v \in \mathcal{W}_{\text{da}}} \mathcal{J}_v$. If $\beta < 0$, then each v is less risk averse than u and $\mathcal{J}_v \subseteq \mathcal{J}_u$ for all v, yielding $\mathcal{J}_V \subseteq \mathcal{J}_u$. But if $\beta > 0$, then each v is more risk averse than u and $\mathcal{J}_u \subseteq \mathcal{J}_v$ for all v, and the same conclusion cannot be derived.

Bayesian persuasion. We now consider the implications of adding disappointment aversion to a problem of Bayesian persuasion. We focus on the special case studied in Dworczak and Martini (2019, Section VI.A). Sender and Receiver share a full-support prior r with continuous cdf and expectation $\bar{\mu}$ on a state of the world $x \in [0,1]$. Sender observes the state before Receiver's choice and chooses a disclosure policy, which she commits to before observing the state. Receiver chooses action $e \in [0, \bar{e}]$ where $\bar{e} \leq 1$. Following Dworczak and Martini (2019), we focus on the case in which both Sender and Receiver's utility, u_{S} and u_{R} respectively, depend only on Receiver's posterior mean y and action e:

$$u_{\text{S}}(e, y) = (1-\sigma)ey \qquad u_{\text{R}}(y, e) = \sigma ye - e^{\alpha} \text{ where } 0 < \sigma < 1 \text{ and } \alpha > 1.$$

In equilibrium, Sender's utility can be rewritten as

$$u(y) = (1-\sigma)e^{*}(y)y \qquad \forall y \in [0,1] \tag{16}$$

where $e^{*}(y)$ is Receiver's optimal action.[9] As standard in this literature, Sender's problem amounts to choosing a distribution over Receiver's posteriors; and since here

[9]For example, these would be the utilities in which Sender was selling a project of value xe to Receiver for a price $(1-\sigma)ye$, with ye being the value of the project for Receiver at the time of purchase and σ reflecting bargaining power.

only their mean matter, this simplifies to choosing a distribution over posterior means. Note that fully revealing the state corresponds to generating a distribution of posterior means equal to r; while revealing less information implies a disclosure policy p of which r is a mean preserving spread—denote this by $p \geq_{\text{MPS}} r$.[10] Sender's problem thus corresponds to choosing a distribution from the set $A = \{p \in \Delta : p \geq_{\text{MPS}} r\}$.

We now compare two cases: when Sender follows expected utility with utility as in (16); and when Sender is disappointment averse with u as in (16) and $\beta > 0$. Denote by p^*_{EU} and p^*_{DA} the optimal disclosure policy in each of the two cases (p^*_{EU} is the solution computed by Dworczak and Martini 2019; see also the proof of Proposition 4). To make the problem interesting, we assume $0 < \bar{\mu} < \bar{y} < 1$, where $\bar{y}$ is the smallest posterior mean for which Receiver chooses the maximum action $\bar{e}$.[11]

Proposition 4 $p^*_{\text{DA}} \geq_{\text{MPS}} p^*_{\text{EU}}$.

If Sender is disappointment averse, then it is optimal to reveal less information than the case of expected utility. Intuitively, the more information is transmitted, the more variation is in Receiver's action. Disappointment aversion increases Sender's desire to avoid this volatility and thus to communicate less information.

[10]Formally, for each $p, q \in \Delta$, we write $p \geq_{\text{MPS}} q$ if and only if $\mathbb{E}_p(v) \geq \mathbb{E}_q(v)$ for all real-valued, continuous, and concave functions v on $[0, 1]$. In other words, $\geq_{\text{MPS}}$ is the concave order over Δ.

[11]If, instead, we had $\bar{\mu} \geq \bar{y}$, then an optimal strategy for both types of Senders would be to simply reveal nothing, that is, $\delta_{\bar{\mu}}$.

Appendix

In Appendix A, we provide the proofs of all the results in the paper. In Appendix B, we give a condition that allows to check whether or not a given Betweenness preference satisfies Negative Certainty Independence. This condition is instrumental in proving the statement contained in Example 1.

A Proofs

Roadmap. We begin by proving Theorem 1, a mildly modified version of Dekel's representation result. We proceed by recalling the notion of expected utility core and its role for Cautious Expected Utility preferences (Formula (19) and Remark 3). Using it, we characterize the expected utility core of Betweenness preferences (Propositions 5-7). These allow us to prove our most general representation result, Theorem 4 (resp., Remark 1). We then focus our attention to DA preferences. First, we discuss the issue of renormalization of Gul's locals (Remark 4); with this in mind, we prove Theorem 3 as a special case of Theorem 4. We then move to the proofs of our identification results (Propositions 1 and 2). Finally, we prove our maxmin result (Proposition 3) which allows us to discuss some monotone comparative statics applications.

Proof of Theorem 1. Compared to (Dekel, 1986, Proposition 2), we only need to prove that the following form of Betweenness holds

$$
\begin{aligned}
& p \succcurlyeq q \implies p \succcurlyeq \lambda p + (1-\lambda) q \succcurlyeq q \quad \forall \lambda \in (0,1) \\
& \quad \textit{and} \\
& p \succ q \implies p \succ \lambda p + (1-\lambda) q \succ q \quad \forall \lambda \in (0,1) .
\end{aligned}
$$

The proof of the first implication is routine (see, e.g., the techniques in (Cerreia-Vioglio et al., 2011, Lemma 56)). As for the second, suppose $p \succ q$. By the first implication, we have that $p \succcurlyeq \lambda p + (1-\lambda) q \succcurlyeq q$ for all $\lambda \in (0,1)$. By contradiction, assume that there exists $\bar{\lambda} \in (0,1)$ such that $p \sim \bar{\lambda} p + \left(1-\bar{\lambda}\right) q$. We have two cases:

1. $p = \delta_b$. Since $p \succ q$, we obtain that $\delta_b = p \neq q$, yielding that $p \succ_{FSD} \bar{\lambda} p + \left(1-\bar{\lambda}\right) q$. Since $\succcurlyeq$ satisfies Strict First Order Stochastic Dominance, we can conclude that $p \succ \bar{\lambda} p + \left(1-\bar{\lambda}\right) q$, a contradiction.

2. $p \neq \delta_b$. Since $\succcurlyeq$ satisfies Betweenness, it follows that

$$
1 \geq \lambda \geq \bar{\lambda} \Rightarrow \lambda p + (1-\lambda) q \sim p. \tag{17}
$$

Since $\succcurlyeq$ satisfies Strict First Order Stochastic Dominance, we have that $\gamma p + (1-\gamma) \delta_b \succ p$ for all $\gamma \in (0,1)$. By (17) and since $\succcurlyeq$ satisfies Strict First Order

Stochastic Dominance, we can conclude that

$$1 \geq \lambda \geq \bar{\lambda} \implies \lambda \left(\gamma p + (1-\gamma)\, \delta_b\right) + (1-\lambda)\, q \succ p \quad \forall \gamma \in (0,1).$$

Next, we are going to define an ancillary object $r_{\eta,\gamma} = \eta\left(\gamma p + (1-\gamma)\delta_b\right) + (1-\eta) q$ for all $\eta, \gamma \in (0,1)$. Note that for each $\eta, \gamma \in (0,1)$ and for each $\lambda \in \left(\bar{\lambda}, 1\right) \subseteq (0,1)$, we have that

$$\begin{aligned}
&\lambda p + (1-\lambda)\, r_{\eta,\gamma} = \\
&= \lambda p + (1-\lambda)\left[\eta\left(\gamma p + (1-\gamma)\delta_b\right) + (1-\eta)\, q\right] \\
&= \left(\lambda + (1-\lambda)\,\eta\gamma\right) p \\
&+ \left(1-\lambda-(1-\lambda)\,\eta\gamma\right)\left[\frac{(1-\lambda)\,\eta\,(1-\gamma)}{(1-\lambda-(1-\lambda)\,\eta\gamma)}\delta_b + \frac{(1-\lambda)\,(1-\eta)}{(1-\lambda-(1-\lambda)\,\eta\gamma)} q\right].
\end{aligned}$$

Since $\gamma p + (1-\gamma)\delta_b \succ p \succ q$ for all $\gamma \in (0,1)$ and $\succcurlyeq$ satisfies Continuity, for each $\gamma \in (0,1)$ there exists $\bar{\eta}_\gamma \in (0,1)$ such that $r_{\bar{\eta}_\gamma,\gamma} = \bar{\eta}_\gamma\left(\gamma p + (1-\gamma)\delta_b\right) + (1-\bar{\eta}_\gamma) q \sim p$. Since $\succcurlyeq$ satisfies Betweenness, $\lambda p + (1-\lambda)\, r_{\bar{\eta}_\gamma,\gamma} \sim p$ for all $\lambda \in \left(\bar{\lambda}, 1\right)$ and for all $\gamma \in (0,1)$. Fix a generic $\gamma \in (0,1)$. Choose $\lambda \in \left(\bar{\lambda}, 1\right)$ close enough to 1, so that $\hat{\lambda} = \lambda + (1-\lambda)\,\bar{\eta}_\gamma\gamma \in \left(\bar{\lambda}, 1\right)$. Note that

$$\hat{r} \overset{def}{=} \frac{(1-\lambda)\,\bar{\eta}_\gamma\,(1-\gamma)}{(1-\lambda-(1-\lambda)\,\bar{\eta}_\gamma\gamma)}\delta_b + \frac{(1-\lambda)\,(1-\bar{\eta}_\gamma)}{(1-\lambda-(1-\lambda)\,\bar{\eta}_\gamma\gamma)} q \succ_{FSD} q.$$

By the characterization of $\lambda p + (1-\lambda)\, r_{\bar{\eta}_\gamma,\gamma}$, we can also conclude that

$$\left(\lambda + (1-\lambda)\,\bar{\eta}_\gamma\gamma\right) p + \left(1-\lambda-(1-\lambda)\,\bar{\eta}_\gamma\gamma\right) \hat{r} \sim p. \tag{18}$$

By (17) and (18), we can conclude that $\hat{\lambda} \in \left(\bar{\lambda}, 1\right)$,

$$\hat{\lambda} p + \left(1-\hat{\lambda}\right) \hat{r} \sim p \sim \hat{\lambda} p + \left(1-\hat{\lambda}\right) q \text{ and } \hat{\lambda} p + \left(1-\hat{\lambda}\right) \hat{r} \succ_{FSD} \hat{\lambda} p + \left(1-\hat{\lambda}\right) q.$$

Since $\succcurlyeq$ satisfies Strict First Order Stochastic Dominance, it follows that $\hat{\lambda} p + \left(1-\hat{\lambda}\right) \hat{r} \succ \hat{\lambda} p + \left(1-\hat{\lambda}\right) q$, a contradiction. A similar proof yields that $\lambda p + (1-\lambda)\, q \succ q$ for all $\lambda \in (0,1)$. ■

We recall the definition of expected utility core of $\succcurlyeq$, i.e., the subrelation $\succcurlyeq'$ defined as:[12]

$$p \succcurlyeq' q \iff \lambda p + (1-\lambda)\, r \succcurlyeq \lambda q + (1-\lambda)\, r \quad \forall \lambda \in (0,1], \forall r \in \Delta. \tag{19}$$

This notion is useful for three reasons. First, as Remark 3 below shows, in order to find a (canonical) representation of a Cautious Expected Utility preference, it is

[12]Under Axioms A 1-2, one can show that $\succcurlyeq'$ satisfies all the assumptions of expected utility with possibly the exception of completeness. See Cerreia-Vioglio (2009); Cerreia-Vioglio et al. (2015, 2017).

sufficient to find an expected multi-utility representation of $\succcurlyeq'$. This is instrumental in proving Theorem 4 (cf. Proposition 7). Second, $\succcurlyeq'$ allows to retrieve the parameters characterizing a DA preference from behavioral data (cf. Propositions 1 and 2). Third, as shown by Cerreia-Vioglio et al. (2017), in general $\succcurlyeq'$ summarizes the risk attitudes of the decision maker irrespective of whether or not $\succcurlyeq$ satisfies Negative Certainty Independence. In particular, $\succcurlyeq$ is averse to Mean Preserving Spreads if and only if $\succcurlyeq'$ is, which is equivalent to have all the utilities representing the latter being concave.

Remark 3 In addition to what is stated in Theorem 2, it can also be shown that the following are true:

1. There exists a set $\mathcal{W} \subseteq \mathcal{U}_{\text{nor}}$ such that

$$p \succcurlyeq' q \iff \mathbb{E}_p(v) \geq \mathbb{E}_q(v) \qquad \forall v \in \mathcal{W} \tag{20}$$

   and $V : \Delta \to \mathbb{R}$ defined as in Equation (6) is a continuous utility representation of $\succcurlyeq$.

2. If $\mathcal{W} \subseteq \mathcal{U}_{\text{nor}}$ satisfies (20), then it satisfies (6).

3. The set $\mathcal{W} \subseteq \mathcal{U}_{\text{nor}}$ can be chosen to be

$$\mathcal{W}_{\max-\text{nor}} = \{v \in \mathcal{U}_{\text{nor}} : p \succcurlyeq' q \implies \mathbb{E}_p(v) \geq \mathbb{E}_q(v)\}.$$

4. If $\mathcal{W} \subseteq \mathcal{U}_{\text{nor}}$ satisfies (20), then

$$\mathcal{W} \subseteq \mathcal{W}_{\max-\text{nor}} \text{ as well as } \bar{\text{co}}(\mathcal{W}) = \text{cl}(\mathcal{W}_{\max-\text{nor}}).$$

   In particular, this latter result, paired with point 2, allows us to discuss uniqueness of the representation in (6). In fact, a canonical representation $\mathcal{W}$, that is a set in $\mathcal{U}_{\text{nor}}$ that also represents $\succcurlyeq'$, represents $\succcurlyeq$ and is unique up to the closed convex hull. $\triangledown$

We next prove a few results pertaining to the expected utility core of a Betweenness preference. These results rely on some of the techniques developed in Cerreia-Vioglio et al. (2017). We start with a definition and an observation. Define $K : \Delta \times [0,1] \to \mathbb{R}$ by

$$K(r,t) = \int_{[w,b]} k(x,t)\, dr \qquad \forall r \in \Delta, \forall t \in [0,1].$$

It is immediate to see that K is affine wrt the first component. Note that for each $r \in \Delta$ and for each $t \in [0,1]$

$$\begin{aligned}
K\left(r, \hat{V}(r)\right) &= \int_{[w,b]} k\left(x, \hat{V}(r)\right) dr = \hat{V}(r) = \hat{V}(r)\, k(b,t) + \left(1 - \hat{V}(r)\right) k(w,t) \\
&= \int_{[w,b]} k(x,t)\, d\left(\hat{V}(r)\,\delta_b + \left(1 - \hat{V}(r)\right)\delta_w\right) \\
&= K\left(\hat{V}(r)\,\delta_b + \left(1 - \hat{V}(r)\right)\delta_w, t\right).
\end{aligned}$$

Finally, we have that for each $p \in \Delta$ the number $\hat{V}(p) \in [0,1]$ is the unique number such that

$$\hat{V}(p) = K\left(p, \hat{V}(p)\right).$$

Proposition 5 *Let $\succcurlyeq$ be a Betweenness preference. If $K(p,t) \geq K(q,t)$ for all $t \in (0,1)$, then $p \succcurlyeq q$.*

Proof. Consider $p, q \in \Delta$. By contradiction, assume that $K(p,t) \geq K(q,t)$ for all $t \in (0,1)$ and $q \succ p$. We have two cases: either $q = \delta_b$ or $q \neq \delta_b$. In the first case, note that $1 \geq K(p,t) \geq K(q,t) = 1$ for all $t \in (0,1)$, that is, $K(p,t) = 1$ for all $t \in (0,1)$. Since each $k(\cdot,t)$ is strictly increasing and normalized, we can conclude that $p = \delta_b = q$, a contradiction with $q \succ p$. In the second case, we have that $\hat{V}(q) \in (0,1)$. On the one hand, since $\succcurlyeq$ admits a representation a la Dekel, note that

$$\hat{V}(q) = K\left(q, \hat{V}(q)\right) \leq K\left(p, \hat{V}(q)\right). \tag{21}$$

On the other hand, by working hypothesis, we have $q \succ p$ which implies that $\hat{V}(q) > \hat{V}(p)$. It follows that

$$\begin{aligned}
\hat{V}(q) > \hat{V}(p) &= K\left(\hat{V}(p)\,\delta_b + \left(1 - \hat{V}(p)\right)\delta_w, \hat{V}(q)\right) \\
&= K\left(\hat{V}(p)\,\delta_b + \left(1 - \hat{V}(p)\right)\delta_w, \hat{V}(p)\right) = \hat{V}(p) = K\left(p, \hat{V}(p)\right).
\end{aligned}$$

In particular, it follows that

$$K\left(\hat{V}(p)\,\delta_b + \left(1 - \hat{V}(p)\right)\delta_w, \hat{V}(p)\right) = \hat{V}(p) = K\left(p, \hat{V}(p)\right) \tag{22}$$

and

$$\hat{V}(q) > K\left(\hat{V}(p)\,\delta_b + \left(1 - \hat{V}(p)\right)\delta_w, \hat{V}(q)\right). \tag{23}$$

Define $r = \hat{V}(p)\,\delta_b + \left(1 - \hat{V}(p)\right)\delta_w$. By (21) and (23) and since K is affine wrt the first component, it follows that there exists $\lambda \in (0,1]$ such that

$$K\left(\lambda p + (1-\lambda) r, \hat{V}(q)\right) = \hat{V}(q),$$

proving that $\lambda p+(1-\lambda) r \sim q$. By (22), we have that $r \sim p$. Since $\succcurlyeq$ is a Betweenness preference, this yields that $p \sim \lambda p+(1-\lambda) r \sim r$. We can conclude that $q \succ p \sim \lambda p+(1-\lambda) r \sim q$, a contradiction. ∎

Proposition 6 *Let $\succcurlyeq$ be a Betweenness preference. If $p \succcurlyeq' q$, then $K(p,t) \geq K(q,t)$ for all $t \in (0,1)$.*

Proof. Consider $p, q \in \Delta$. By contradiction, assume that $p \succcurlyeq' q$ and that there exists $\bar{t} \in (0,1)$ such that $K(p,\bar{t}) < K(q,\bar{t})$. Then, there exist $\lambda \in (0,1]$ and $y \in [w,b]$ such that $\hat{V}(\lambda p+(1-\lambda)\delta_y) = \bar{t}$.[13] It follows that

$$\begin{aligned}\bar{t} &= K(\lambda p+(1-\lambda)\delta_y, \bar{t}) = \lambda K(p,\bar{t}) + (1-\lambda) K(\delta_y, \bar{t}) \\ &< \lambda K(q,\bar{t}) + (1-\lambda) K(\delta_y,\bar{t}) = K(\lambda q+(1-\lambda)\delta_y, \bar{t}).\end{aligned}$$

Define $r_1 = \lambda p + (1-\lambda)\delta_y$ and $r_2 = \lambda q+(1-\lambda)\delta_y$ so that $\bar{t} = \hat{V}(r_1)$. In particular, we obtain that

$$\hat{V}(r_1) < K\left(r_2, \hat{V}(r_1)\right). \tag{24}$$

Since $p \succcurlyeq' q$ and $\succcurlyeq'$ satisfies Independence, it follows that $r_1 \succcurlyeq' r_2$. Since $\succcurlyeq'$ is a subrelation of $\succcurlyeq$, this implies that $r_1 \succcurlyeq r_2$, that is, $\hat{V}(r_1) \geq \hat{V}(r_2)$. Define $r_3 = \hat{V}(r_2)\delta_b + \left(1-\hat{V}(r_2)\right)\delta_w$. On the one hand, it is immediate to see that $r_2 \sim r_3$. On the other hand, by (24), we obtain that

$$K\left(r_3, \hat{V}(r_1)\right) = \hat{V}(r_2) \leq \hat{V}(r_1) < K\left(r_2, \hat{V}(r_1)\right).$$

Since K is affine wrt the first component, there exists $\gamma \in [0,1)$ such that

$$K\left(\gamma r_2 + (1-\gamma) r_3, \hat{V}(r_1)\right) = \hat{V}(r_1),$$

yielding that $\gamma r_2 + (1-\gamma) r_3 \sim r_1$. Since $\succcurlyeq$ satisfies Betweenness and $r_2 \sim r_3$, this yields that

$$r_2 \sim \gamma r_2 + (1-\gamma) r_3 \sim r_1.$$

We can then conclude that $\hat{V}(r_2) = \hat{V}(r_1)$, that is, $\hat{V}(r_1) = \hat{V}(r_2) = K\left(r_2, \hat{V}(r_2)\right) = K\left(r_2, \hat{V}(r_1)\right)$, a contradiction with (24). ∎

Proposition 7 *If $\succcurlyeq$ is a Betweenness preference, then*

$$p \succcurlyeq' q \iff \mathbb{E}_p(v) \geq \mathbb{E}_q(v) \qquad \forall v \in \mathcal{W}_{\text{bet}}.$$

Moreover, the set $\mathcal{W}_{\text{bet}}$ is either a singleton or infinite.

[13] If $\hat{V}(p) \geq \bar{t} > 0 = \hat{V}(\delta_w)$, then $y = w$ and if $\hat{V}(p) < \bar{t} < 1 = \hat{V}(\delta_b)$, then $y = b$. The existence of λ is then granted by the continuity of $\hat{V}$.

Proof. Define $\succsim''$ by

$$p \succsim'' q \iff \mathbb{E}_p(v) \geq \mathbb{E}_q(v) \qquad \forall v \in \mathcal{W}_{\text{bet}}.$$

By Proposition 6, we have that if $p \succsim' q$, then $K(p,t) \geq K(q,t)$ for all $t \in (0,1)$, that is, $p \succsim'' q$. By Proposition 5, if $p \succsim'' q$, that is $K(p,t) \geq K(q,t)$ for all $t \in (0,1)$, then $p \succsim q$. By (Cerreia-Vioglio et al., 2017, Lemma 1 and Footnote 10), we can conclude that $p \succsim'' q$ implies $p \succsim' q$, proving that $\succsim''$ coincides with $\succsim'$. Finally, assume that $\mathcal{W}_{\text{bet}}$ is not a singleton. It follows that there exist $t_1, t_2 \in (0,1)$ and $\bar{x} \in (w,b)$ such that $k(\bar{x}, t_1) \neq k(\bar{x}, t_2)$. Without loss of generality, assume that $k(\bar{x}, t_1) < k(\bar{x}, t_2)$. By contradiction, assume that $|\mathcal{W}_{\text{bet}}| \in \mathbb{N}$. By the intermediate value theorem and since $k(\bar{x}, \cdot)$ is continuous on $(0,1)$, it follows that

$$\{k(\bar{x}, t)\}_{t \in (0,1)} \supseteq [k(\bar{x}, t_1), k(\bar{x}, t_2)].$$

Since $k(\bar{x}, t_1) < k(\bar{x}, t_2)$, it follows that $\left|\{k(\bar{x}, t)\}_{t \in (0,1)}\right| = \infty$, a contradiction with $\left|\{k(\bar{x}, t)\}_{t \in (0,1)}\right| \leq |\mathcal{W}_{\text{bet}}| \in \mathbb{N}$. ■

We now prove Theorem 4.

Proof of Theorem 4. (ii) implies (i). By (Cerreia-Vioglio et al., 2015, Theorem 1), the statement trivially follows.

(i) implies (ii). Since $\succsim$ is a Betweenness preference, it satisfies Weak Order, Continuity, and Strict First Order Stochastic Dominance. By Theorem 2 and Remark 3, and since $\mathcal{W}_{\text{bet}} = \{k(\cdot, t)\}_{t \in (0,1)}$ represents $\succsim'$, it follows that $\mathcal{W}$ in (6) can be chosen to be $\mathcal{W}_{\text{bet}}$. This yields (6) and, in particular, (9) with inf in place of min. Note that for each $v \in \mathcal{W}_{\text{bet}}$ we have that $V(\delta_w) = w = c(\delta_w, v)$ and $V(\delta_b) = b = c(\delta_b, v)$. Thus the inf is attained for δ_w and δ_b. The proof below yields that the inf is attained at each $p \in \Delta$, proving (9).

We next prove (10). Consider $p \in \Delta \setminus \{\delta_w, \delta_b\}$. Since $\succsim$ satisfies Strict First Order Stochastic Dominance, we have that $\hat{V}(p) \in (0,1)$ and it is the unique number in $[0,1]$ such that

$$\int_{[w,b]} k\left(x, \hat{V}(p)\right) dp = \hat{V}(p). \tag{25}$$

Define $v_p = k\left(\cdot, \hat{V}(p)\right) \in \mathcal{W}_{\text{bet}}$. Define $\bar{x} \in [w,b]$ to be such that $\bar{x} = c(p, v_p)$. Note that

$$v_p(\bar{x}) = v_p(c(p, v_p)) = v_p\left(v_p^{-1}\left(\int_{[w,b]} k\left(x, \hat{V}(p)\right) dp\right)\right) = \int_{[w,b]} k\left(x, \hat{V}(p)\right) dp.$$

By (25), it follows that

$$\int_{[w,b]} k\left(x, \hat{V}(p)\right) d\delta_{\bar{x}} = v_p(\bar{x}) = \hat{V}(p).$$

Since $\succsim$ is a Betweenness preference, we can conclude that $\hat{V}(\delta_{\bar{x}}) = \hat{V}(p)$, that is, $\delta_{\bar{x}} \sim p$ and so $\bar{x} = x_p$.[14] This yields that

$$V(p) = x_p = \bar{x} = c(p, v_p),$$

proving that the inf is attained at v_p. ■

Proof of Remark 1. (i)' implies (ii)'. Since $\succsim$ satisfies Weak Order, Continuity, and Strict First Order Stochastic Dominance, there exists a continuous utility function $V : \Delta \to \mathbb{R}$ such that $V(\delta_x) = x$ for all $x \in [w, b]$. By Proposition 7, the expected utility core $\succsim'$ of $\succsim$ admits an expected multi-utility representation with set $\mathcal{W}_{\text{bet}}$. Define $\bar{V} : \Delta \to \mathbb{R}$ by $\bar{V}(p) = \sup_{v \in \mathcal{W}_{\text{bet}}} c(p, v)$ for all $p \in \Delta$. We next show that $V = \bar{V}$. Fix $p \in \Delta$. Note that $\bar{V}(p), V(p) \in [w, b]$. By construction, it is immediate to see that

$$c\left(\delta_{\bar{V}(p)}, v'\right) = \bar{V}(p) = \sup_{v \in \mathcal{W}_{\text{bet}}} c(p, v) \geq c(p, v') \quad \forall v' \in \mathcal{W}_{\text{bet}}.$$

By Proposition 7, this yields that $\delta_{\bar{V}(p)} \succsim' p$. Since $\succsim'$ is a subrelation of $\succsim$, we can conclude that $\delta_{\bar{V}(p)} \succsim p$, that is, $\bar{V}(p) = V\left(\delta_{\bar{V}(p)}\right) \geq V(p)$, proving that $\bar{V} \geq V$. By contradiction, assume that $\bar{V} \not\leq V$. It would follow that there exists $p \in \Delta$ such that $\bar{V}(p) > V(p)$. This would imply that there exists $y \in [w, b]$ such that $\bar{V}(p) > y > V(p)$. We could conclude that $\delta_y \succsim p$. By Proposition 7 and since $\succsim$ satisfies Positive Certainty Independence, we could conclude that $\delta_y \succsim' p$, that is, $y = c(\delta_y, v) \geq c(p, v)$ for all $v \in \mathcal{W}_{\text{bet}}$, yielding that $\bar{V}(p) > y \geq \sup_{v \in \mathcal{W}_{\text{bet}}} c(p, v) = \bar{V}(p)$, a contradiction. This proves that

$$V(p) = \sup_{v \in \mathcal{W}_{\text{bet}}} c(p, v) \qquad \forall p \in \Delta.$$

In order to prove that (12) and (11) hold, the same arguments used in proving (10) and (9) apply.

(ii)' implies (i)'. Consider $x \in [w, b]$ and $p, q \in \Delta$ as well as $\lambda \in [0, 1]$. Note that

$$\begin{aligned}
\delta_x \succsim p &\implies \sup_{v \in \mathcal{W}_{\text{bet}}} c(\delta_x, v) \geq \sup_{v \in \mathcal{W}_{\text{bet}}} c(p, v) \implies x \geq \sup_{v \in \mathcal{W}_{\text{bet}}} c(p, v) \\
&\implies c(\delta_x, v) = x \geq c(p, v) \quad \forall v \in \mathcal{W}_{\text{bet}} \implies \delta_x \succsim' p \\
&\implies \lambda \delta_x + (1 - \lambda) q \succsim \lambda p + (1 - \lambda) q,
\end{aligned}$$

proving that $\succsim$ satisfies Positive Certainty Independence. ■

[14] Recall that a Betweenness preference is a binary relation on Δ which satisfies Weak Order, Continuity, Strict First Order Stochastic Dominance, and Betweenness. In this case, given $p \in \Delta$, x_p is the unique number such that $\delta_{x_p} \sim p$.

Remark 4 As already observed, a careful inspection of Gul's model formulation (see (4)) suggests that two types of normalizations are due to link the implicit Gul's representation to Dekel's one. In order to do so, we define few objects: $\alpha : \operatorname{Im} u \to \mathbb{R}$, $\gamma : \operatorname{Im} u \to \mathbb{R}$, $g : [0,1] \to \mathbb{R}$, $\hat{V} : \Delta \to \mathbb{R}$ and $k : [w,b] \times [0,1] \to \mathbb{R}$. We set

$$\alpha(s) = \frac{1}{\tilde{k}(b,s) - \tilde{k}(w,s)} \text{ and } \gamma(s) = \frac{-\tilde{k}(w,s)}{\tilde{k}(b,s) - \tilde{k}(w,s)} \qquad \forall s \in \operatorname{Im} u.$$

We also set

$$g(\lambda) = \tilde{V}(\lambda \delta_b + (1-\lambda)\delta_w) \qquad \forall \lambda \in [0,1]$$

and, since g is strictly increasing, continuous, and $\operatorname{Im} g = \operatorname{Im} u = \operatorname{Im} \tilde{V}$,

$$\hat{V}(p) = g^{-1}\left(\tilde{V}(p)\right) \qquad \forall p \in \Delta.$$

Finally, we set

$$k(x,t) = \alpha(g(t))\,\tilde{k}(x, g(t)) + \gamma(g(t)) \qquad \forall x \in [w,b], \forall t \in [0,1].$$

It is easy to check that k and $\hat{V}$ satisfy all the assumptions of Theorem 1.[15] Since

$$k(\cdot,t) = \alpha(g(t))\,\tilde{k}(\cdot, g(t)) + \gamma(g(t)) \qquad \forall t \in [0,1]$$

and $g : [0,1] \to \operatorname{Im} u$ is strictly increasing, continuous, and onto, we have that for each $t \in [0,1]$ there exists an element $z \in \operatorname{Im} u$ such that $k(\cdot,t)$ is a positive affine transformation of $\tilde{k}(\cdot,z)$. Similarly, for each $z \in \operatorname{Im} u$ there exists an element $t \in [0,1]$ such that $\tilde{k}(\cdot,z)$ is a positive affine transformation of $k(\cdot,t)$. Recall that $\mathcal{W}_{\text{da}} = \left\{\tilde{k}(\cdot,z)\right\}_{z \in \operatorname{Im} u}$. In particular, this implies that $\inf_{v \in \mathcal{W}_{\text{bet}}} c(p,v) = \min_{v \in \mathcal{W}_{\text{da}}} c(p,v)$ for all $p \in \Delta$ as well as $\sup_{v \in \mathcal{W}_{\text{bet}}} c(p,v) = \max_{v \in \mathcal{W}_{\text{da}}} c(p,v)$. ∇

Proof of Theorem 3. 1 and 2. By Dillenberger (2010) and Artstein-Avidan and Dillenberger (2015) and since $\beta \geq 0$, it follows that $\succcurlyeq$ satisfies Negative Certainty Independence. By Theorem 4 and Remark 4, it follows that if $\beta \geq 0$, then $V : \Delta \to \mathbb{R}$, defined by

$$V(p) = \min_{v \in \mathcal{W}_{\text{bet}}} c(p,v) = \min_{v \in \mathcal{W}_{\text{da}}} c(p,v) \qquad \forall p \in \Delta$$

is a continuous utility representation of $\succcurlyeq$ where $\mathcal{W}_{\text{bet}} = \{k(\cdot,t)\}_{t \in (0,1)}$. Thus, if $\beta > 0$, then (7) follows. If $\beta = 0$, then $\mathcal{W}_{\text{da}} = \{u\}$ and (8) follows.

3. By Dillenberger (2010) and Artstein-Avidan and Dillenberger (2015) and since $\beta \in (-1,0)$, it follows that $\succcurlyeq$ satisfies Positive Certainty Independence. By Remarks 1 and 4, it follows that if $\beta \in (-1,0)$, then $V : \Delta \to \mathbb{R}$, defined by

$$V(p) = \max_{v \in \mathcal{W}_{\text{bet}}} c(p,v) = \max_{v \in \mathcal{W}_{\text{da}}} c(p,v) \qquad \forall p \in \Delta$$

[15] Indeed, points 1 and 2 are satisfied on $[0,1]$ and not just $(0,1)$.

is a continuous utility representation of $\succcurlyeq$ where $\mathcal{W}_{\text{bet}} = \{k(\cdot, t)\}_{t\in(0,1)}$. ■

We next prove our results on the behavioral identification of parameters for DA preferences.

Proof of Proposition 1. Before starting, define by $\bar{u}$ the positive and affine transformation of u such that $\bar{u}(w) = 0$ and $\bar{u}(b) = 1$. Recall that $(\bar{u}, \beta)$ also represents $\succcurlyeq$. We denote by $\mathcal{W}_{\text{da}}$ the corresponding set of local utilities given by $\tilde{k}$ defined as in (4) and computed with $(\bar{u}, \beta)$. By Proposition 7 and Remark 4, recall that $\succcurlyeq'$ is such that

$$p \succcurlyeq' q \iff \mathbb{E}_p(v) \geq \mathbb{E}_q(v) \quad \forall v \in \mathcal{W}_{\text{da}} \iff c(p, v) \geq c(q, v) \quad \forall v \in \mathcal{W}_{\text{da}}. \tag{26}$$

We list four facts that are going to be useful:

1. For each $v \in \mathcal{W}_{\text{da}}$ there exists a strictly increasing and continuous function $f_v :$ $\operatorname{Im}\bar{u} \to \mathbb{R}$ such that $v = f_v \circ \bar{u}$.

2. For each $v \in \mathcal{W}_{\text{da}}$, if $\beta \geq 0$ (resp., $\beta \leq 0$), then f_v is concave (resp., convex).

3. $\bar{u} \in \mathcal{W}_{\text{da}}$.

4. Since $\succcurlyeq$ is a DA preference, recall that for each $x \in [w, b]$ and for each $\lambda \in [0, 1]$

$$\tilde{V}(\delta_x) = \bar{u}(x) \text{ and } \tilde{V}(\lambda\delta_b + (1-\lambda)\delta_w) = \frac{\lambda\bar{u}(b) + (1+\beta)(1-\lambda)\bar{u}(w)}{1 + \beta(1-\lambda)}.$$

If $\bar{p} = \frac{1}{2}\delta_b + \frac{1}{2}\delta_w$, then

$$\bar{u}(x_{\bar{p}}) = \frac{1}{2}\frac{1}{1+\beta/2} \implies \beta = \frac{1}{\bar{u}(x_{\bar{p}})} - 2.$$

1. By points 1 and 2 and since $\beta \geq 0$, we have that if $p \in \Delta$ and $x \in [w, b]$

$$x \geq c(p, \bar{u}) \implies x \geq c(p, v) \quad \forall v \in \mathcal{W}_{\text{da}}.$$

By point 3, we have that

$$x \geq c(p, v) \quad \forall v \in \mathcal{W}_{\text{da}} \implies x \geq c(p, \bar{u}),$$

yielding that

$$x \geq c(p, \bar{u}) \iff x \geq c(p, v) \quad \forall v \in \mathcal{W}_{\text{da}}. \tag{27}$$

By (26) and (27) and since $\bar{u}(w) = 0 = \bar{u}(b) - 1$, we can conclude that for each $x \in [w, b]$

$$\begin{aligned}\max\{\lambda \in [0,1] : \delta_x \succcurlyeq' \lambda\delta_b + (1-\lambda)\delta_w\} &= \max\{\lambda \in [0,1] : x \geq c(\lambda\delta_b + (1-\lambda)\delta_w, \bar{u})\} \\ &= \max\left\{\lambda \in [0,1] : x \geq \bar{u}^{-1}(\lambda)\right\} = \bar{u}(x).\end{aligned}$$

By point 4, we conclude that

$$\beta = \frac{1}{\bar{u}\left(x_{\bar{p}}\right)} - 2 = \frac{1}{\max\left\{\lambda \in [0,1] : \delta_{x_{\bar{p}}} \succsim' \lambda\delta_b + (1-\lambda)\,\delta_w\right\}} - 2,$$

proving the statement.

2. By points 1 and 2 and since $\beta \leq 0$, we have that if $p \in \Delta$ and $x \in [w,b]$

$$c\left(p,\bar{u}\right) \geq x \implies c\left(p,v\right) \geq x \quad \forall v \in \mathcal{W}_{\mathrm{da}}.$$

By point 3, we have that

$$c\left(p,v\right) \geq x \quad \forall v \in \mathcal{W}_{\mathrm{da}} \implies c\left(p,\bar{u}\right) \geq x,$$

yielding that

$$c\left(p,\bar{u}\right) \geq x \iff c\left(p,v\right) \geq x \quad \forall v \in \mathcal{W}_{\mathrm{da}}. \tag{28}$$

By (26) and (28) and since $\bar{u}\left(w\right) = 0 = \bar{u}\left(b\right) - 1$, we can conclude that for each $x \in [w,b]$

$$\begin{aligned}\min\left\{\lambda \in [0,1] : \lambda\delta_b + (1-\lambda)\,\delta_w \succsim' \delta_x\right\} &= \min\left\{\lambda \in [0,1] : c\left(\lambda\delta_b + (1-\lambda)\,\delta_w, \bar{u}\right) \geq x\right\} \\ &= \min\left\{\lambda \in [0,1] : \bar{u}^{-1}\left(\lambda\right) \geq x\right\} = \bar{u}\left(x\right).\end{aligned}$$

By point 4, we conclude that

$$\beta = \frac{1}{\bar{u}\left(x_{\bar{p}}\right)} - 2 = \frac{1}{\min\left\{\lambda \in [0,1] : \lambda\delta_b + (1-\lambda)\,\delta_w \succsim' \delta_{x_{\bar{p}}}\right\}} - 2,$$

proving the statement. ■

Proof of Proposition 2. We adopt the same normalizations and objects of Proposition 1. Since we have two preferences we index the corresponding objects by 1 and 2. Before starting, recall that $V_i : \Delta \to \mathbb{R}$, defined by

$$V_i\left(p\right) = \min_{v \in \mathcal{W}_{\mathrm{da}}^i} c\left(p,v\right) \qquad \forall p \in \Delta,$$

represents $\succsim_i$ for $i \in \{1,2\}$.

(i) implies (ii). Since $\bar{u}_1$ and $\bar{u}_2$ are normalized and u_1 is cardinally equivalent to u_2, we have that $\bar{u}_1 = \bar{u}_2$. Define $\bar{u} = \bar{u}_1 = \bar{u}_2$. By (26) and (27), we have that for each $x \in [w,b]$ and for each $p \in \Delta$

$$\begin{aligned}\delta_x \succsim_1' p &\iff x \geq c\left(p,v\right) \quad \forall v \in \mathcal{W}_{\mathrm{da}}^1 \iff x \geq c\left(p,\bar{u}\right) \\ &\iff x \geq c\left(p,v\right) \quad \forall v \in \mathcal{W}_{\mathrm{da}}^2 \iff \delta_x \succsim_2' p.\end{aligned}$$

Since $\beta_1 \geq \beta_2 \geq 0$, for each $v \in \mathcal{W}^2_{\mathrm{da}}$ there exists $\tilde{v} \in \mathcal{W}^1_{\mathrm{da}}$ which is a strictly increasing, continuous, and concave transformation of v. This implies that

$$\min_{v \in \mathcal{W}^2_{\mathrm{da}}} c(p, v) \geq \min_{v \in \mathcal{W}^1_{\mathrm{da}}} c(p, v) \qquad \forall p \in \Delta.$$

Since the functional on the left-hand (resp., right-hand) side represents $\succsim_2$ (resp., $\succsim_1$), it follows that for each $x \in [w, b]$ and for each $p \in \Delta$

$$\delta_x \succsim_2 p \implies \delta_x \succsim_1 p,$$

proving the implication.

(ii) implies (i). By (26) and (27), we have that for each $x \in [w, b]$ and for each $p \in \Delta$

$$\begin{aligned} x \geq c(p, \bar{u}_1) &\iff x \geq c(p, v) \quad \forall v \in \mathcal{W}^1_{\mathrm{da}} \\ &\iff \delta_x \succsim'_1 p \iff \delta_x \succsim'_2 p \\ &\iff x \geq c(p, v) \quad \forall v \in \mathcal{W}^2_{\mathrm{da}} \iff x \geq c(p, \bar{u}_2), \end{aligned}$$

yielding that $\bar{u}_1$ is an affine transformation of $\bar{u}_2$. Since $\bar{u}_1$ and $\bar{u}_2$ are normalized, it follows that $\bar{u}_1 = \bar{u}_2$, in particular, proving that u_1 and u_2 are cardinally equivalent. Define $\bar{p} = \frac{1}{2}\delta_b + \frac{1}{2}\delta_w$. Let $x_{i,\bar{p}}$ be the certainty equivalent of $\succsim_i$. Since $\bar{u}_1$ is equal to $\bar{u}_2$, let $\bar{u} = \bar{u}_1 = \bar{u}_2$. By Proposition 1 and its proof, we have that

$$\beta_1 = \frac{1}{\bar{u}(x_{1,\bar{p}})} - 2 \text{ and } \beta_2 = \frac{1}{\bar{u}(x_{2,\bar{p}})} - 2.$$

Since $\succsim_1$ is more risk averse than $\succsim_2$, we have that $x_{2,\bar{p}} \geq x_{1,\bar{p}}$, that is, $\beta_1 \geq \beta_2$. ■

Proof of Proposition 3. Define $\mathcal{W} = \bar{\mathrm{co}}(\mathcal{W}_{\mathrm{bet}})$. Given the assumptions, $\mathcal{W}$ is convex and compact and (13) holds with $\mathcal{W}$ in place of $\mathcal{W}_{\mathrm{bet}}$.[16] First, note that the

[16] Since $k : [w, b] \times [0, 1] \to \mathbb{R}$ is jointly continuous and $k(x, t) \in [0, 1]$ for all $x \in [w, b]$ and all $t \in [0, 1]$, it follows that $\mathcal{W}_{\mathrm{bet}} = \{k(\cdot, t)\}_{t \in (0,1)}$ is a (uniformly) bounded and equicontinuous family of functions in $C([w, b])$. This implies that the convex hull of $\mathcal{W}_{\mathrm{bet}}$, $\mathrm{co}(\mathcal{W}_{\mathrm{bet}})$, is bounded and equicontinuous. Since closure in supnorm preserves boundedness and equicontinuity, we can conclude that $\bar{\mathrm{co}}(\mathcal{W}_{\mathrm{bet}}) = \mathrm{cl}(\mathrm{co}(\mathcal{W}_{\mathrm{bet}}))$ is a bounded and equicontinuous family of functions of $C([w, b])$. By Arzela-Ascoli Theorem, $\bar{\mathrm{co}}(\mathcal{W}_{\mathrm{bet}})$ is compact. Finally, we are left to show that $\bar{\mathrm{co}}(\mathcal{W}_{\mathrm{bet}})$ is a subset of $\mathcal{U}_{\mathrm{nor}}$. Clearly, each $v \in \bar{\mathrm{co}}(\mathcal{W}_{\mathrm{bet}})$ is continuous and such that $v(w) = 0$ and $v(b) = 1$. Thus, we only need to prove that v is strictly increasing. Let $x, y \in [w, b]$ be such that $x > y$. Since k is strictly increasing in the first argument and continuous in the second, we have that there exists $\bar{t} \in [0, 1]$ and $\varepsilon > 0$ such that

$$\inf_{v \in \mathcal{W}_{\mathrm{bet}}} [v(x) - v(y)] \geq \min_{t \in [0,1]} [k(x, t) - k(y, t)] = k(x, \bar{t}) - k(y, \bar{t}) \geq \varepsilon > 0.$$

It is immediate to show that $\inf_{v \in \mathrm{co}(\mathcal{W}_{\mathrm{bet}})} [v(x) - v(y)] \geq \varepsilon$, yielding that $\inf_{v \in \bar{\mathrm{co}}(\mathcal{W}_{\mathrm{bet}})} [v(x) - v(y)] \geq \varepsilon$. This implies that $v(x) > v(y)$ for all $v \in \bar{\mathrm{co}}(\mathcal{W}_{\mathrm{bet}})$. Since x and y were arbitrarily chosen, this shows that each element of $\bar{\mathrm{co}}(\mathcal{W}_{\mathrm{bet}})$ is strictly increasing.

map $c : \Delta \times \mathcal{W} \to [w, b]$, defined by

$$c(p, v) = v^{-1}(\mathbb{E}_p(v)) \qquad \forall (p, v) \in \Delta \times \mathcal{W},$$

is quasiconcave and upper semicontinuous in the first argument and quasiconvex and lower semicontinuous in the second argument. By Sion's minimax theorem and since A is a convex and compact set of Δ, this implies that

$$\max_{p \in A} \min_{v \in \mathcal{W}} c(p, v) = \min_{v \in \mathcal{W}} \max_{p \in A} c(p, v).$$

Let $\hat{v} \in \mathcal{W}$ be such that $\max_{p \in A} c(p, \hat{v}) \leq \max_{p \in A} c(p, v)$ for all $v \in \mathcal{W}$. Note that

$$\begin{aligned} \max_{p \in A} \min_{v \in \mathcal{W}} c(p, v) &= \max_{p \in A} V(p) = V(p^*) = \min_{v \in \mathcal{W}} c(p^*, v) \leq c(p^*, \hat{v}) \\ &\leq \max_{p \in A} c(p, \hat{v}) \leq \min_{v \in \mathcal{W}} \max_{p \in A} c(p, v). \end{aligned}$$

Since $\max_{p \in A} \min_{v \in \mathcal{W}} c(p, v) = \min_{v \in \mathcal{W}} \max_{p \in A} c(p, v)$, this yields that

$$c(p^*, \hat{v}) = \max_{p \in A} c(p, \hat{v}),$$

proving the statement. ■

We next move to the applications contained in Section 5.2. We begin by offering, in a framework action-space, two versions of Proposition 3. The first, for $\beta > 0$, is key to prove our result on portfolio choice. The second, for $\beta < 0$, is needed to prove our result on justifiability. We then discuss our Bayesian Persuasion application. This latter one is a direct consequence of Proposition 3.

Proposition 8 *Consider the framework of Remark 2 and the problem in (15). If*

1. $\succcurlyeq$ *is a DA preference with parameters* (u, β) *where* $\beta > 0$*;*

2. A *is a compact and convex subset of a separable normed vector space;*

3. $a \mapsto u(g(a, s))$ *is continuous and concave for all* $s \in S$*;*

4. $a^* \in A$ *is such that* $V(p_{a^*,\mu}) \geq V(p_{a,\mu})$ *for all* $a \in A$*;*

then, there exists $\hat{v} \in \bar{\mathrm{co}}(\mathcal{W}_{\mathrm{bet}})$ *such that* $\mathbb{E}_{p_{a^*,\mu}}(\hat{v}) \geq \mathbb{E}_{p_{a,\mu}}(\hat{v})$ *for all* $a \in A$.

Proof. Given Theorem 3 and Remark 4, define $\mathcal{W} = \bar{\mathrm{co}}(\mathcal{W}_{\mathrm{bet}})$. Given the assumptions, $\mathcal{W}$ is convex and compact and (13) holds with $\mathcal{W}$ in place of $\mathcal{W}_{\mathrm{bet}}$ (cf. the proof of Proposition 3). Recall that for each $v \in \mathcal{W}$ there exists a strictly increasing, continuous, and concave function $f_v : \operatorname{Im} u \to [0,1]$ such that $v = f_v \circ u$. By the Change of Variable Theorem, this implies that

$$\begin{aligned} c\left(p_{a,\mu}, v\right) &= v^{-1}\left(\mathbb{E}_{p_{a,\mu}}(v)\right) = v^{-1}\left(\int_{[w,b]} f_v\left(u\left(x\right)\right) dp_{a,\mu}\right) \\ &= v^{-1}\left(\left(\int_S f_v\left(u\left(g\left(a,s\right)\right)\right) d\mu\right)\right) \quad \forall a \in A, \forall v \in \mathcal{W}. \end{aligned}$$

Since $a \mapsto u\left(g\left(a,s\right)\right)$ is concave for all $s \in S$, it follows that $a \mapsto f_v\left(u\left(g\left(a,s\right)\right)\right)$ is concave for all $s \in S$. This implies that $a \mapsto c\left(p_{a,\mu}, v\right)$ is quasiconcave on A for all $v \in \mathcal{W}$. It is immediate to verify that $a \mapsto c\left(p_{a,\mu}, v\right)$ is upper semicontinuous on A for all $v \in \mathcal{W}$. By standard arguments, it also follows that $v \mapsto c\left(p_{a,\mu}, v\right)$ is quasiconvex and lower semicontinuous on $\mathcal{W}$ for all $a \in A$. By Sion's minimax theorem, this implies that

$$\max_{a \in A} V\left(p_{a,\mu}\right) = \max_{a \in A} \min_{v \in \mathcal{W}} c\left(p_{a,\mu}, v\right) = \min_{v \in \mathcal{W}} \max_{a \in A} c\left(p_{a,\mu}, v\right).$$

Let $\hat{v} \in \mathcal{W}$ be such that $\max_{a \in A} c\left(p_{a,\mu}, \hat{v}\right) \le \max_{a \in A} c\left(p_{a,\mu}, v\right)$ for all $v \in \mathcal{W}$. Note that

$$\begin{aligned} \max_{a \in A} \min_{v \in \mathcal{W}} c\left(p_{a,\mu}, v\right) &= \max_{a \in A} V\left(p_{a,\mu}\right) = V\left(p_{a^*,\mu}\right) = \min_{v \in \mathcal{W}} c\left(p_{a^*,\mu}, v\right) \le c\left(p_{a^*,\mu}, \hat{v}\right) \\ &\le \max_{a \in A} c\left(p_{a,\mu}, \hat{v}\right) \le \min_{v \in \mathcal{W}} \max_{a \in A} c\left(p_{a,\mu}, v\right). \end{aligned}$$

Since $\max_{a \in A} \min_{v \in \mathcal{W}} c\left(p_{a,\mu}, v\right) = \min_{v \in \mathcal{W}} \max_{a \in A} c\left(p_{a,\mu}, v\right)$, this yields that

$$c\left(p_{a^*,\mu}, \hat{v}\right) = \max_{a \in A} c\left(p_{a,\mu}, \hat{v}\right),$$

proving the statement. ■

Proof of Corollary 1. First, observe that $A = [0, W]$ and $g\left(a,s\right) = ar\left(s\right) + \left(W - a\right) r_f\left(s\right)$ for all $a \in A$ and for all $s \in S$. Clearly, A is a compact and convex subset of a separable normed vector space. Since r is measurable and r_f is constant, $s \mapsto g\left(a,s\right)$ is measurable for all $a \in A$. Finally, by construction, $a \mapsto g\left(a,s\right)$ is continuous and affine for all $s \in S$. Since $u'' < 0$, u is concave and, in particular, $a \mapsto u\left(g\left(a,s\right)\right)$ is continuous and concave for all $s \in S$. By Proposition 8, there exists $\hat{v} \in \bar{\text{co}}\left(\mathcal{W}_{\text{bet}}\right)$ such that $\mathbb{E}_{p_{a^*_{\text{DA}},\mu}}\left(\hat{v}\right) \ge \mathbb{E}_{p_{a,\mu}}\left(\hat{v}\right)$ for all $a \in A$. In other words, a^*_{DA} is the solution of the portfolio problem for a decision maker who is expected utility and more risk averse than u. It is well-known that this implies that $a^*_{\text{DA}} \le a^*_{\text{EU}}$. ■

Proposition 9 *Consider the framework of Remark 2 and the problem in (15). If*

1. $\succsim$ *is a DA preference with parameters* $\left(u, \beta\right)$ *where* $\beta < 0$*;*
2. $a^* \in A$ *is such that* $V\left(p_{a^*,\mu}\right) \ge V\left(p_{a,\mu}\right)$ *for all* $a \in A$*;*

then, there exists $\hat{v} \in \mathcal{W}_{\text{bet}}$ *such that* $\mathbb{E}_{p_{a^*,\mu}}(\hat{v}) \geq \mathbb{E}_{p_{a,\mu}}(\hat{v})$ *for all* $a \in A$*.*

Proof. By Theorem 3 and Remark 4, we have that $V(p_{a^*,\mu}) = c(p_{a^*,\mu}, \hat{v})$ for some $\hat{v} \in \mathcal{W}_{\text{bet}}$. By Theorem 3 and assumption, we have that

$$c(p_{a^*,\mu}, \hat{v}) = V(p_{a^*,\mu}) \geq V(p_{a,\mu}) = \max_{v \in \mathcal{W}_{\text{bet}}} c(p_{a,\mu}, v) \geq c(p_{a,\mu}, \hat{v}) \qquad \forall a \in A,$$

proving the statement. ■

Proof of Corollary 2. Given a strictly increasing and continuous von Neumann-Morgenstern function $v : [w, b] \to \mathbb{R}$, we denote by $\mathcal{J}_v$ the set of justifiable actions for the expected utility preference generated by v. If $a \in \mathcal{J}_{\text{DA}}$, then there exists $\mu \in \Delta(S)$ such that $V(p_{a,\mu}) \geq V(p_{b,\mu})$ for all $b \in A$ where V is the functional in point 3 of Theorem 3 (see also Remark 4). By Proposition 9, we have that there exists $\hat{v} \in \mathcal{W}_{\text{bet}}$ such that $c(p_{a,\mu}, \hat{v}) \geq c(p_{b,\mu}, \hat{v})$ for all $b \in A$, that is, $a \in \mathcal{J}_{\hat{v}}$. Since $\beta < 0$, we have that u is more risk averse than $\hat{v}$. By Battigalli et al. (2016), this implies that $\mathcal{J}_{\hat{v}} \subseteq \mathcal{J}_u = \mathcal{J}_{\text{EU}}$. We can conclude that $a \in \mathcal{J}_u = \mathcal{J}_{\text{EU}}$. Since a was arbitrarily chosen, the statement follows. ■

Proof of Proposition 4. Let $[w, b] = [0, 1]$. Denote by $\mathcal{B}$ the Borel σ-algebra of $[0, 1]$. Given $p \in \Delta$, we denote by $e(p)$ the expectation of p. Denote by $\succcurlyeq$ Sender's DA preference with parameters (u, β) where u is as in (16) and $\beta > 0$. Assume $\succcurlyeq$ is represented by $V : \Delta \to \mathbb{R}$ as in Theorem 3. The problem of Sender is

$$\max V(p) \text{ subject to } p \in A \tag{29}$$

where $A = \{p \in \Delta : p \geq_{\text{MPS}} r\}$. Before starting, we need some notation. Given an element $p \in \Delta$, we denote by $F_p : [0, 1] \to [0, 1]$ the cumulative distribution of p, that is, $F_p(t) = p([0, t])$ for all $t \in [0, 1]$. Standard calculations show that in (16) we have $e^*(y) = \min\left\{\left(\frac{\sigma y}{\alpha}\right)^{\frac{1}{\alpha - 1}}, \bar{e}\right\}$ for all $y \in [0, 1]$ and $\bar{y} = (\alpha/\sigma)\, \bar{e}^{\alpha - 1}$. Dworczak and Martini (2019, Proposition 3) prove that if $e(r) = \bar{\mu} < \bar{y}$, then a solution of (29) for the expected utility agent with u as in (16) is p^*_{EU} with cumulative distribution $F_{\text{EU}} : [0, 1] \to [0, 1]$ defined by

$$F_{\text{EU}}(x) = \begin{cases} F_r(x) & x < x^\star \\ F_r(x^\star) & x^\star \leq x < \bar{y} \\ 1 & x \geq \bar{y} \end{cases} \qquad \forall x \in [0, 1]$$

where $x^\star$ is such that the probability s, defined by $s(B) = r(B \cap [x^\star, 1]) / r([x^\star, 1])$ for all $B \in \mathcal{B}$, satisfies $e(s) = \bar{y}$. Define $F_{\text{DA}} = F_{p^*_{\text{DA}}}$ where p^*_{DA} is a solution to (29). Finally, define $\mathcal{U}_{\text{conc}}$ to be the set of all functions $v : [0, 1] \to \mathbb{R}$ which are continuous and concave. We can begin.

Since A is compact and V is continuous, let p^*_{DA} be a solution of (29). By Proposition 3 and since A is convex, there exists $\hat{v} \in \bar{\mathrm{co}}\left(\mathcal{W}_{\mathrm{bet}}\right)$ such that

$$\mathbb{E}_{p^*_{\mathrm{DA}}}\left(\hat{v}\right) \geq \mathbb{E}_{p}\left(\hat{v}\right) \qquad \forall p \in A. \tag{30}$$

By contradiction, assume that $p^*_{\mathrm{DA}} \not\geq_{\mathrm{MPS}} p^*_{\mathrm{EU}}$.[17] Define $\Phi_{\mathrm{DA}}, \Phi_{\mathrm{EU}} : [0,1] \to \mathbb{R}$ by

$$\Phi_{\mathrm{DA}}\left(t\right) = \int_0^t F_{\mathrm{DA}}\left(x\right) dx \text{ and } \Phi_{\mathrm{EU}}\left(t\right) = \int_0^t F_{\mathrm{EU}}\left(x\right) dx \quad \forall t \in [0,1].$$

It is immediate to see that Φ_{DA} and Φ_{EU} are continuous functions. Since, by construction, $p^*_{\mathrm{EU}}, p^*_{\mathrm{DA}} \geq_{\mathrm{MPS}} r$, we necessarily have that $\Phi_{\mathrm{DA}}\left(0\right) = \Phi_{\mathrm{EU}}\left(0\right)$ as well as

$$\Phi_{\mathrm{DA}}\left(1\right) = \int_0^1 F_{\mathrm{DA}}\left(x\right) dx = \int_0^1 F_r\left(x\right) dx = \int_0^1 F_{\mathrm{EU}}\left(x\right) dx = \Phi_{\mathrm{EU}}\left(1\right). \tag{31}$$

Since $p^*_{\mathrm{DA}} \not\geq_{\mathrm{MPS}} p^*_{\mathrm{EU}}$, this yields that there exists $\bar{t} \in (0,1)$ such that

$$\Phi_{\mathrm{DA}}\left(\bar{t}\right) = \int_0^{\bar{t}} F_{\mathrm{DA}}\left(x\right) dx > \int_0^{\bar{t}} F_{\mathrm{EU}}\left(x\right) dx = \Phi_{\mathrm{EU}}\left(\bar{t}\right). \tag{32}$$

Since $p^*_{\mathrm{DA}} \geq_{\mathrm{MPS}} r$ and $F_{\mathrm{EU}}\left(x\right) = F_r\left(x\right)$ for all $x \in [0, x^\star]$, we can conclude that

$$\Phi_{\mathrm{DA}}\left(t\right) = \int_0^t F_{\mathrm{DA}}\left(x\right) dx \leq \int_0^t F_r\left(x\right) dx = \int_0^t F_{\mathrm{EU}}\left(x\right) dx = \Phi_{\mathrm{EU}}\left(t\right) \quad \forall t \in [0, x^\star].$$

This implies that $\bar{t}$ in (32) must belong to $(x^\star, 1)$. Since Φ_{DA} and Φ_{EU} are continuous as well as $\Phi_{\mathrm{DA}}\left(x^\star\right) \leq \Phi_{\mathrm{EU}}\left(x^\star\right)$ and $\Phi_{\mathrm{DA}}\left(\bar{t}\right) > \Phi_{\mathrm{EU}}\left(\bar{t}\right)$ for some $\bar{t} \in (x^\star, 1)$, we have that

$$\hat{t} = \min\left\{t \in [x^\star, 1] : \Phi_{\mathrm{DA}}\left(t\right) = \Phi_{\mathrm{EU}}\left(t\right)\right\}$$

is well defined. Note that $\bar{t} > \hat{t} \geq x^\star$ and $\hat{t} \leq \bar{y}$.[18,19] By (31), we have that

$$\int_0^{\hat{t}} F_{\mathrm{DA}}\left(x\right) dx + \int_{\hat{t}}^1 F_{\mathrm{DA}}\left(x\right) dx = \Phi_{\mathrm{DA}}\left(1\right) = \Phi_{\mathrm{EU}}\left(1\right) = \int_0^{\hat{t}} F_{\mathrm{EU}}\left(x\right) dx + \int_{\hat{t}}^1 F_{\mathrm{EU}}\left(x\right) dx,$$

[17] Recall that, equivalently, $p \geq_{\mathrm{MPS}} q$ if and only if $\int_0^t F_p\left(x\right) dx \leq \int_0^t F_q\left(x\right) dx$ for all $t \in [0,1]$ and $\int_0^1 F_p\left(x\right) dx = \int_0^1 F_q\left(x\right) dx$.

[18] Otherwise, $\bar{t} \leq \hat{t}$. We have two cases:

1. $\bar{t} = \hat{t}$. In this case, we would have that

$$0 = \Phi_{\mathrm{DA}}\left(\hat{t}\right) - \Phi_{\mathrm{EU}}\left(\hat{t}\right) = \Phi_{\mathrm{DA}}\left(\bar{t}\right) - \Phi_{\mathrm{EU}}\left(\bar{t}\right) > 0,$$

a contradiction.

2. $\bar{t} < \hat{t}$. In this case, we would have that

$$\Phi_{\mathrm{DA}}\left(x^\star\right) - \Phi_{\mathrm{EU}}\left(x^\star\right) \leq 0 < \Phi_{\mathrm{DA}}\left(\bar{t}\right) - \Phi_{\mathrm{EU}}\left(\bar{t}\right).$$

Since $\Phi_{\mathrm{DA}} - \Phi_{\mathrm{EU}}$ is a continuous function on $[x^\star, \bar{t}]$, this would imply that there exists $\tilde{t} \in [x^\star, \bar{t}] \subseteq [x^\star, 1]$ such that

$$\Phi_{\mathrm{DA}}\left(\tilde{t}\right) - \Phi_{\mathrm{EU}}\left(\tilde{t}\right) = 0,$$

that is, $\tilde{t} \leq \bar{t} < \hat{t} \leq \tilde{t}$, a contradiction.

[19] By contradiction, assume that $\hat{t} > \bar{y}$. Since F_{EU} is such that $F_{\mathrm{EU}}\left(x\right) = 1 \geq F_{\mathrm{DA}}\left(x\right)$ for all

yielding that

$$\int_{\hat{t}}^{1} F_{\mathrm{DA}}(x)\,dx = \int_{\hat{t}}^{1} F_{\mathrm{EU}}(x)\,dx.$$

Next, we show that $\bar{t}$ can be chosen to be $\bar{y}$. First, by (32) and since $\Phi_{\mathrm{DA}}(\hat{t}) = \Phi_{\mathrm{EU}}(\hat{t})$, observe that

$$\begin{aligned}\Phi_{\mathrm{DA}}(\hat{t}) + \int_{\hat{t}}^{\bar{t}} F_{\mathrm{DA}}(x)\,dx &= \int_{0}^{\hat{t}} F_{\mathrm{DA}}(x)\,dx + \int_{\hat{t}}^{\bar{t}} F_{\mathrm{DA}}(x)\,dx = \int_{0}^{\bar{t}} F_{\mathrm{DA}}(x)\,dx \\ &> \int_{0}^{\bar{t}} F_{\mathrm{EU}}(x)\,dx = \int_{0}^{\hat{t}} F_{\mathrm{EU}}(x)\,dx + \int_{\hat{t}}^{\bar{t}} F_{\mathrm{EU}}(x)\,dx \\ &= \Phi_{\mathrm{EU}}(\hat{t}) + \int_{\hat{t}}^{\bar{t}} F_{\mathrm{EU}}(x)\,dx,\end{aligned}$$

yielding that

$$\int_{\hat{t}}^{\bar{t}} F_{\mathrm{DA}}(x)\,dx > \int_{\hat{t}}^{\bar{t}} F_{\mathrm{EU}}(x)\,dx. \tag{33}$$

Since F_{EU} is such that $F_{\mathrm{EU}}(x) = 1 \geq F_{\mathrm{DA}}(x)$ for all $x \in [\bar{y}, 1]$, note that if $\bar{t} \geq \bar{y}$, then

$$\begin{aligned}0 &< \int_{\hat{t}}^{\bar{t}} [F_{\mathrm{DA}}(x) - F_{\mathrm{EU}}(x)]\,dx = \int_{\hat{t}}^{\bar{y}} [F_{\mathrm{DA}}(x) - F_{\mathrm{EU}}(x)]\,dx + \int_{\bar{y}}^{\bar{t}} [F_{\mathrm{DA}}(x) - F_{\mathrm{EU}}(x)]\,dx \\ &\leq \int_{\hat{t}}^{\bar{y}} [F_{\mathrm{DA}}(x) - F_{\mathrm{EU}}(x)]\,dx.\end{aligned}$$

This implies that $\Phi_{\mathrm{DA}}(\bar{y}) > \Phi_{\mathrm{EU}}(\bar{y})$. Since $F_{\mathrm{EU}}(x) = F_{\mathrm{EU}}(x^{\star})$ for all $x \in [x^{\star}, \bar{y})$, viceversa, if $\bar{t} < \bar{y}$, then it follows that $F_{\mathrm{DA}}(\bar{t}) > F_{\mathrm{EU}}(\bar{t})$. It follows that $F_{\mathrm{DA}}(x) \geq F_{\mathrm{DA}}(\bar{t}) > F_{\mathrm{EU}}(\bar{t}) = F_{\mathrm{EU}}(x^{\star}) = F_{\mathrm{EU}}(x)$ for all $x \in [\bar{t}, \bar{y}) \subseteq [x^{\star}, \bar{y})$, yielding that

$$\int_{\hat{t}}^{\bar{y}} [F_{\mathrm{DA}}(x) - F_{\mathrm{EU}}(x)]\,dx = \int_{\hat{t}}^{\bar{t}} [F_{\mathrm{DA}}(x) - F_{\mathrm{EU}}(x)]\,dx + \int_{\bar{t}}^{\bar{y}} [F_{\mathrm{DA}}(x) - F_{\mathrm{EU}}(x)]\,dx > 0.$$

This implies that $\Phi_{\mathrm{DA}}(\bar{y}) > \Phi_{\mathrm{EU}}(\bar{y})$. It follows that

$$\int_{\hat{t}}^{\bar{y}} F_{\mathrm{DA}}(x)\,dx + \int_{\bar{y}}^{1} F_{\mathrm{DA}}(x)\,dx = \int_{\hat{t}}^{1} F_{\mathrm{DA}}(x)\,dx = \int_{\hat{t}}^{1} F_{\mathrm{EU}}(x)\,dx = \int_{\hat{t}}^{\bar{y}} F_{\mathrm{EU}}(x)\,dx + \int_{\bar{y}}^{1} F_{\mathrm{EU}}(x)\,dx.$$

$x \in [\bar{y}, 1]$. This would imply that

$$\begin{aligned}\Phi_{\mathrm{EU}}(\bar{t}) &= \int_{0}^{\bar{t}} F_{\mathrm{EU}}(x)\,dx = \int_{0}^{\hat{t}} F_{\mathrm{EU}}(x)\,dx + \int_{\hat{t}}^{\bar{t}} F_{\mathrm{EU}}(x)\,dx = \Phi_{\mathrm{EU}}(\hat{t}) + \int_{\hat{t}}^{\bar{t}} F_{\mathrm{EU}}(x)\,dx \\ &= \Phi_{\mathrm{DA}}(\hat{t}) + \int_{\hat{t}}^{\bar{t}} F_{\mathrm{EU}}(x)\,dx \geq \Phi_{\mathrm{DA}}(\hat{t}) + \int_{\hat{t}}^{\bar{t}} F_{\mathrm{DA}}(x)\,dx \\ &= \int_{0}^{\hat{t}} F_{\mathrm{DA}}(x)\,dx + \int_{\hat{t}}^{\bar{t}} F_{\mathrm{DA}}(x)\,dx = \int_{0}^{\bar{t}} F_{\mathrm{DA}}(x)\,dx = \Phi_{\mathrm{DA}}(\bar{t}),\end{aligned}$$

a contradiction with (32).

By (33) and since $\bar{t}$ can be chosen to be $\bar{y}$, this yields that

$$0 < \int_{\hat{t}}^{\bar{y}} F_{\mathrm{DA}}(x)\, dx - \int_{\hat{t}}^{\bar{y}} F_{\mathrm{EU}}(x)\, dx = \int_{\bar{y}}^{1} F_{\mathrm{EU}}(x)\, dx - \int_{\bar{y}}^{1} F_{\mathrm{DA}}(x)\, dx.$$

Since $\int_{\bar{y}}^{1} F_{\mathrm{EU}}(x)\, dx - \int_{\bar{y}}^{1} F_{\mathrm{DA}}(x)\, dx > 0$ and $F_{\mathrm{EU}}(x) = 1 \geq F_{\mathrm{DA}}(x)$ for all $x \in [\bar{y}, 1]$, we have that $F_{\mathrm{DA}}(\bar{y}) < 1$ and, in particular, $p^*_{\mathrm{DA}}((\bar{y}, 1]) > 0$. Since $e(p^*_{\mathrm{DA}}) = e(r) < \bar{y}$, note also that $1 > p^*_{\mathrm{DA}}([\bar{y}, 1]) \geq p^*_{\mathrm{DA}}((\bar{y}, 1]) > 0$. With this in mind, define $q_{\mathrm{DA}}(B) = p^*_{\mathrm{DA}}(B \cap [0, \bar{y})) / p^*_{\mathrm{DA}}([0, \bar{y}))$ and $r_{\mathrm{DA}}(B) = p^*_{\mathrm{DA}}(B \cap [\bar{y}, 1]) / p^*_{\mathrm{DA}}([\bar{y}, 1])$ for all $B \in \mathcal{B}$. Since $p^*_{\mathrm{DA}}([\bar{y}, 1]) \in (0, 1)$, q_{DA} and r_{DA} are well defined elements of Δ. Let also λ be $p^*_{\mathrm{DA}}([0, \bar{y})) = 1 - p^*_{\mathrm{DA}}([\bar{y}, 1]) \in (0, 1)$. Note that $p^*_{\mathrm{DA}} = \lambda q_{\mathrm{DA}} + (1 - \lambda) r_{\mathrm{DA}}$ as well as $e(q_{\mathrm{DA}}) \in [0, \bar{y})$ and $e(r_{\mathrm{DA}}) = y \in [\bar{y}, 1]$. We now have two cases:

* $y = \bar{y}$. It would follow that $e(r_{\mathrm{DA}}) = y = \bar{y}$, yielding that $r_{\mathrm{DA}} = \delta_{\bar{y}}$. This would imply that $p^*_{\mathrm{DA}} = \lambda q_{\mathrm{DA}} + (1 - \lambda) r_{\mathrm{DA}} = \lambda q_{\mathrm{DA}} + (1 - \lambda) \delta_{\bar{y}}$. In particular, we would have that $F_{\mathrm{DA}}(\bar{y}) = 1$, a contradiction.

** $y > \bar{y}$. Since $\lambda \in (0, 1)$ and $e(q_{\mathrm{DA}}) < \bar{y} < e(r_{\mathrm{DA}})$, consider $\varepsilon, \tau \in (0, \min\{\lambda, 1 - \lambda\})$ such that $\frac{\tau}{\tau+\varepsilon} e(q_{\mathrm{DA}}) + \frac{\varepsilon}{\tau+\varepsilon} e(r_{\mathrm{DA}}) = \bar{y}$. It follows that $(\lambda - \tau), (1 - \lambda - \varepsilon) \in (0, 1)$. Consider the probability

$$\hat{p}_{\mathrm{DA}} = (\lambda - \tau) q_{\mathrm{DA}} + (\tau + \varepsilon) \delta_{\bar{y}} + (1 - \lambda - \varepsilon) r_{\mathrm{DA}}.$$

Since $e\left(\frac{\tau}{\tau+\varepsilon} q_{\mathrm{DA}} + \frac{\varepsilon}{\tau+\varepsilon} r_{\mathrm{DA}}\right) = \bar{y}$, note that

$$\mathbb{E}_{\delta_{\bar{y}}}(v) \geq \mathbb{E}_{\frac{\tau}{\tau+\varepsilon} q_{\mathrm{DA}} + \frac{\varepsilon}{\tau+\varepsilon} r_{\mathrm{DA}}}(v) = \frac{\tau}{\tau + \varepsilon} \mathbb{E}_{q_{\mathrm{DA}}}(v) + \frac{\varepsilon}{\tau + \varepsilon} \mathbb{E}_{r_{\mathrm{DA}}}(v) \quad \forall v \in \mathcal{U}_{\mathrm{conc}}.$$

This implies that

$$\begin{aligned}
\mathbb{E}_{\hat{p}_{\mathrm{DA}}}(v) &= \mathbb{E}_{(\lambda-\tau) q_{\mathrm{DA}} + (\tau+\varepsilon)\delta_{\bar{y}} + (1-\lambda-\varepsilon) r_{\mathrm{DA}}}(v) \\
&= (\lambda - \tau) \mathbb{E}_{q_{\mathrm{DA}}}(v) + (\tau + \varepsilon) \mathbb{E}_{\delta_{\bar{y}}}(v) + (1 - \lambda - \varepsilon) \mathbb{E}_{r_{\mathrm{DA}}}(v) \\
&= \lambda \mathbb{E}_{q_{\mathrm{DA}}}(v) + (1 - \lambda) \mathbb{E}_{r_{\mathrm{DA}}}(v) \\
&+ (\tau + \varepsilon) \left(\mathbb{E}_{\delta_{\bar{y}}}(v) - \frac{\tau}{\tau + \varepsilon} \mathbb{E}_{q_{\mathrm{DA}}}(v) - \frac{\varepsilon}{\tau + \varepsilon} \mathbb{E}_{r_{\mathrm{DA}}}(v) \right) \\
&\geq \lambda \mathbb{E}_{q_{\mathrm{DA}}}(v) + (1 - \lambda) \mathbb{E}_{r_{\mathrm{DA}}}(v) = \mathbb{E}_{\lambda q_{\mathrm{DA}} + (1-\lambda) r_{\mathrm{DA}}}(v) = \mathbb{E}_{p^*_{\mathrm{DA}}}(v) \quad \forall v \in \mathcal{U}_{\mathrm{conc}}
\end{aligned}$$

that is, $\hat{p}_{\mathrm{DA}} \geq_{\mathrm{MPS}} p^*_{\mathrm{DA}} \geq_{\mathrm{MPS}} r$ and, in particular, $\hat{p}_{\mathrm{DA}} \in A$. Finally, note that the affine function $\ell : [0, 1] \to \mathbb{R}$, defined by $\ell(x) = (1 - \sigma) \bar{e} x$ for all $x \in [0, 1]$, is such that

$$\ell(x) > u(x) \quad \forall x \in (0, \bar{y}) \text{ and } \ell(x) = u(x) \quad \forall x \in [\bar{y}, 1] \cup \{0\}.$$

Note also that $\operatorname{supp} q_{\mathrm{DA}} \neq \{0\}$.[20] This implies that

$$\begin{aligned} u(\bar{y}) = \ell(\bar{y}) = \ell\left(e\left(\frac{\tau}{\tau+\varepsilon} q_{\mathrm{DA}} + \frac{\varepsilon}{\tau+\varepsilon} r_{\mathrm{DA}}\right)\right) &= \mathbb{E}_{\frac{\tau}{\tau+\varepsilon} q_{\mathrm{DA}} + \frac{\varepsilon}{\tau+\varepsilon} r_{\mathrm{DA}}}(\ell) \\ = \frac{\tau}{\tau+\varepsilon}\mathbb{E}_{q_{\mathrm{DA}}}(\ell) + \frac{\varepsilon}{\tau+\varepsilon}\mathbb{E}_{r_{\mathrm{DA}}}(\ell) &> \frac{\tau}{\tau+\varepsilon}\mathbb{E}_{q_{\mathrm{DA}}}(u) + \frac{\varepsilon}{\tau+\varepsilon}\mathbb{E}_{r_{\mathrm{DA}}}(u) \\ &= \mathbb{E}_{\frac{\tau}{\tau+\varepsilon} q_{\mathrm{DA}} + \frac{\varepsilon}{\tau+\varepsilon} r_{\mathrm{DA}}}(u). \end{aligned}$$

Since $\hat{v}$ is a continuous, strictly increasing, and concave transformation of u, we have that

$$\mathbb{E}_{\delta_{\bar{y}}}(\hat{v}) = \hat{v}(\bar{y}) > \mathbb{E}_{\frac{\tau}{\tau+\varepsilon} q_{\mathrm{DA}} + \frac{\varepsilon}{\tau+\varepsilon} r_{\mathrm{DA}}}(\hat{v}) = \frac{\tau}{\tau+\varepsilon}\mathbb{E}_{q_{\mathrm{DA}}}(\hat{v}) + \frac{\varepsilon}{\tau+\varepsilon}\mathbb{E}_{r_{\mathrm{DA}}}(\hat{v}).$$

We can conclude that

$$\begin{aligned} \mathbb{E}_{\hat{p}_{\mathrm{DA}}}(\hat{v}) &= \mathbb{E}_{(\lambda-\tau) q_{\mathrm{DA}} + (\tau+\varepsilon)\delta_{\bar{y}} + (1-\lambda-\varepsilon) r_{\mathrm{DA}}}(\hat{v}) \\ &= (\lambda-\tau)\mathbb{E}_{q_{\mathrm{DA}}}(\hat{v}) + (\tau+\varepsilon)\mathbb{E}_{\delta_{\bar{y}}}(\hat{v}) + (1-\lambda-\varepsilon)\mathbb{E}_{r_{\mathrm{DA}}}(\hat{v}) \\ &= \lambda\mathbb{E}_{q_{\mathrm{DA}}}(\hat{v}) + (1-\lambda)\mathbb{E}_{r_{\mathrm{DA}}}(\hat{v}) \\ &\quad + (\tau+\varepsilon)\left(\mathbb{E}_{\delta_{\bar{y}}}(\hat{v}) - \frac{\tau}{\tau+\varepsilon}\mathbb{E}_{q_{\mathrm{DA}}}(\hat{v}) - \frac{\varepsilon}{\tau+\varepsilon}\mathbb{E}_{r_{\mathrm{DA}}}(\hat{v})\right) \\ &> \lambda\mathbb{E}_{q_{\mathrm{DA}}}(\hat{v}) + (1-\lambda)\mathbb{E}_{r_{\mathrm{DA}}}(\hat{v}) = \mathbb{E}_{\lambda q_{\mathrm{DA}} + (1-\lambda) r_{\mathrm{DA}}}(\hat{v}) = \mathbb{E}_{p^*_{\mathrm{DA}}}(\hat{v}), \end{aligned}$$

a contradiction with (30).

Cases * and ** prove the statement. ■

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[20] If $\operatorname{supp} q_{\mathrm{DA}} = \{0\}$, there would exist $k \in (0,1)$ such that $F_{\mathrm{DA}}(x) = k$ for all $x \in [0,\bar{y})$. In this case, note that $F_r(0) \geq F_{\mathrm{DA}}(0)$. Otherwise, since F_r is continuous, if $F_r(0) < F_{\mathrm{DA}}(0)$, then there would exist $\bar{x} \in [0,1]$ such that $F_r(x) < F_{\mathrm{DA}}(0)$ for all $x \in [0,\bar{x}]$. It would follow that $F_r(x) < F_{\mathrm{DA}}(0) \leq F_{\mathrm{DA}}(x)$ for all $x \in [0,\bar{x}]$, yielding that

$$\int_0^{\bar{x}} F_r(x)\,dx < \int_0^{\bar{x}} F_{\mathrm{DA}}(x)\,dx,$$

a contradiction with $p^*_{\mathrm{DA}} \geq_{\mathrm{MPS}} r$. Since $F_{\mathrm{EU}}(x) = F_r(x)$ for all $x \in [0,x^\star]$ and $F_r(0) \geq F_{\mathrm{DA}}(0)$, it follows that $F_{\mathrm{EU}}(x) = F_r(x) \geq F_r(0) \geq F_{\mathrm{DA}}(0) = F_{\mathrm{DA}}(x)$ for all $x \in [0,x^\star]$. Moreover, $F_{\mathrm{EU}}(x) = F_{\mathrm{EU}}(x^\star) \geq F_{\mathrm{DA}}(0) = F_{\mathrm{DA}}(x)$ for all $x \in [x^\star,\bar{y})$. Finally, since $F_{\mathrm{EU}}(x) = 1 \geq F_{\mathrm{DA}}(x)$ for all $x \in [\bar{y},1]$, we can conclude that $F_{\mathrm{EU}}(x) \geq F_{\mathrm{DA}}(x)$ for all $x \in [0,1]$. This would yield that

$$\int_0^{\bar{t}} F_{\mathrm{DA}}(x)\,dx \leq \int_0^{\bar{t}} F_{\mathrm{EU}}(x)\,dx,$$

a contradiction with (32).

B When do Betweenness preferences satisfy NCI?

While Theorem 4 provides an explicit characterization for Betweenness preferences that satisfy Negative Certainty Independence, a natural question is how to actually check if a given Betweenness preference does or does not satisfy Negative Certainty Independence. The next result, paired with the next remark, provides a rather simple tool to answer this question, solely in terms of the functional form of k. We conclude the appendix by illustrating its use, by proving the results in Example 1.

In stating these results, the following notation will be useful: given $f : [0,1] \to [0,1]$, we say that f is *convex (resp., concave) at* $t \in (0,1)$ if and only if for each $n \in \mathbb{N}$, $\{t_i\}_{i=1}^n \subseteq [0,1]$, and $\{\lambda_i\}_{i=1}^n \subseteq [0,1]$ such that $\sum_{i=1}^n \lambda_i = 1$

$$t = \sum_{i=1}^n \lambda_i t_i \implies f(t) \leq \sum_{i=1}^n \lambda_i f(t_i) \text{ (resp., } \geq \text{)}.$$

For each $s, t \in (0,1)$, define $f_{s,t}$ to be the transformation from $k(\cdot, t)$ to $k(\cdot, s)$, that is, $f_{s,t} : [0,1] \to [0,1]$ is such that $k(x,s) = f_{s,t}(k(x,t))$ for all $x \in [w,b]$. Note that $f_{s,t}$ must exist since $k(\cdot,t)$ and $k(\cdot,s)$ are strictly increasing and continuous. Moreover, $f_{s,t}$ is strictly increasing, continuous, and such that $f_{s,t}(0) = 0$ and $f_{s,t}(1) = 1$.

Proposition 10 *Let $\succcurlyeq$ be a Betweenness preference. The following statements are equivalent:*

(i) For each $t \in (0,1)$ and for each $s \in (0,1)$ the function $f_{s,t}$ is convex (resp., concave) at t;

(ii) $\succcurlyeq$ satisfies Negative (resp., Positive) Certainty Independence.

Proposition 10 characterizes Negative Certainty Independence within the class of Betweenness preferences, just in terms of the parameters of their representation. In fact, testing Negative (resp., Positive) Certainty Independence amounts to checking if for each $t \in (0,1)$ the transformations $f_{s,t}$ are convex (resp., concave) at t for all $s \in (0,1)$. This is a handy tool since $f_{s,t} = k(\cdot,s) \circ k^{-1}(\cdot,t)$ and is thus computable. Moreover, checking convexity and concavity at t is rather simple in light of Remark 5 below.

Proof of Proposition 10. Before starting, define $V : \Delta \to \mathbb{R}$ by

$$V(p) = \inf_{v \in \mathcal{W}_{\text{bet}}} c(p,v) \text{ (resp., } = \sup_{v \in \mathcal{W}_{\text{bet}}} c(p,v)\text{)} \qquad \forall p \in \Delta.$$

Define $v_t = k(\cdot,t)$ for all $t \in [0,1]$. Denote also by Δ_0 the subset of Δ of all simple lotteries (convex linear combinations of Dirac measures), that is, $\Delta_0 = \text{co}\left(\{\delta_x\}_{x \in [w,b]}\right)$.

Claim: *If $s,t \in (0,1)$ and $f_{s,t}$ is convex (resp., concave) at t, then for each $p \in \Delta_0$*

$$\mathbb{E}_p(v_t) = t \implies c(p, v_t) \leq c(p, v_s) \text{ (resp., } \geq \text{)}.$$

Proof of the Claim. Let $p \in \Delta_0$ and $\mathbb{E}_p(v_t) = t$. If $p = \delta_x$, then the statement is trivially true, since $c(p, v_s) = x = c(p, v_t)$. Otherwise, we have that there exist $n \in \mathbb{N} \setminus \{1\}$, $\{x_i\}_{i=1}^n \subseteq [w,b]$, and $\{\lambda_i\}_{i=1}^n \subseteq [0,1]$ such that $\sum_{i=1}^n \lambda_i = 1$ and $\sum_{i=1}^n \lambda_i \delta_{x_i} = p$. Define $t_i = v_t(x_i) \in [0,1]$ for all $i \in \{1,...,n\}$. Since $\mathbb{E}_p(v_t) = t$, this implies that $\sum_{i=1}^n \lambda_i t_i = \sum_{i=1}^n \lambda_i v_t(x_i) = \mathbb{E}_p(v_t) = t$. Since $f_{s,t}$ is convex (resp., concave) at t, it follows that

$$f_{s,t}(\mathbb{E}_p(v_t)) = f_{s,t}(t) \leq \sum_{i=1}^n \lambda_i f_{s,t}(t_i) = \sum_{i=1}^n \lambda_i f_{s,t}(v_t(x_i)) = \sum_{i=1}^n \lambda_i v_s(x_i) = \mathbb{E}_p(v_s) \text{ (resp., } \geq \text{)}.$$

Since $v_s = f_{s,t} \circ v_t$, we have that $f_{s,t} = v_s \circ v_t^{-1}$. This implies that $c(p, v_t) \leq c(p, v_s)$ (resp., $\geq$). $\square$

(i) implies (ii). Let $p \in \Delta \setminus \{\delta_w, \delta_b\}$. Since $\succcurlyeq$ satisfies Strict First Order Stochastic Dominance, we can conclude that $\hat{V}(p) \in (0,1)$ and it is the unique number in $[0,1]$ such that

$$\int_{[w,b]} k\left(x, \hat{V}(p)\right) dp = \hat{V}(p).$$

Define $t = \hat{V}(p)$ and consider v_t. Let also s be an element of $(0,1)$ and consider v_s as well as $f_{s,t}$. Since Δ_0 is dense in Δ and $\succcurlyeq$ satisfies Weak Order, Continuity, and Strict First Order Stochastic Dominance, we have that there exists a sequence $\{p_n\}_{n \in \mathbb{N}} \subseteq \Delta_0$ such that $p_n \sim p$ for all $n \in \mathbb{N}$ and $p_n \to p$. The condition $p_n \sim p$ yields that $\mathbb{E}_{p_n}(v_t) = t$ for all $n \in \mathbb{N}$. By the previous claim and since $\{p_n\}_{n \in \mathbb{N}} \subseteq \Delta_0$ and $f_{s,t}$ is convex (resp., concave) at t, this implies that $c(p_n, v_t) \leq c(p_n, v_s)$ (resp., $\geq$) for all $n \in \mathbb{N}$. By passing to the limit and since $s \in (0,1)$ was arbitrarily chosen, we obtain that

$$c(p, v_t) \leq c(p, v_s) \text{ (resp., } \geq \text{)} \qquad \forall s \in (0,1).$$

We can conclude that

$$V(p) = \min_{s \in (0,1)} c(p, v_s) = \min_{v \in \mathcal{W}_{\text{bet}}} c(p, v) = c(p, v_t)$$
$$\text{(resp., } V(p) = \max_{s \in (0,1)} c(p, v_s) = \max_{v \in \mathcal{W}_{\text{bet}}} c(p, v) = c(p, v_t)).$$

By using the same technique in the proof of (i) implies (ii) in Theorem 4, we have that $\bar{x} = c(p, v_t)$ is such that $p \sim \delta_{\bar{x}}$,[21] that is, $\bar{x} = x_p$. Since $p \in \Delta \setminus \{\delta_w, \delta_b\}$ was arbitrarily

[21] Recall that $p \in \Delta \setminus \{\delta_w, \delta_b\}$ and $\hat{V}(p) \in (0,1)$ where the latter is the unique number in $[0,1]$ such

chosen, we obtain that $V(p) = x_p$ for all $p \in \Delta$.[22] This implies that V is a utility representation of $\succcurlyeq$. Since $\succcurlyeq$ satisfies Continuity and $V(\delta_x) = x$ for all $x \in [w,b]$, it is immediate to see that V is continuous. By Theorem 4 (resp., Remark 1), this implies that $\succcurlyeq$ satisfies Negative Certainty Independence (resp., Positive Certainty Independence).

(ii) implies (i). By Theorem 4 (resp., Remark 1), we have that $V : \Delta \to \mathbb{R}$, defined by

$$V(p) = \min_{v \in \mathcal{W}_{\text{bet}}} c(p,v) = \min_{s \in (0,1)} c(p, v_s) \qquad \forall p \in \Delta$$
$$(\text{resp., } V(p) = \max_{v \in \mathcal{W}_{\text{bet}}} c(p,v) = \max_{s \in (0,1)} c(p, v_s) \qquad \forall p \in \Delta),$$

is a continuous utility representation of $\succcurlyeq$. By contradiction, assume that there exist $t \in (0,1)$ and $s' \in (0,1)$ such that $f_{s',t}$ is not convex (resp., concave) at t. It follows that there exist $n \in \mathbb{N} \setminus \{1\}$, $\{t_i\}_{i=1}^n \subseteq [0,1]$, and $\{\lambda_i\}_{i=1}^n \subseteq [0,1]$ such that $\sum_{i=1}^n \lambda_i = 1$ and $t = \sum_{i=1}^n \lambda_i t_i$ as well as $f_{s',t}(t) > \sum_{i=1}^n \lambda_i f_{s',t}(t_i)$ (resp., $<$). Consider $\{x_i\}_{i=1}^n$ such that $v_t(x_i) = t_i$. Define $p \in \Delta_0$ to be such that $p = \sum_{i=1}^n \lambda_i \delta_{x_i}$. It follows that $\mathbb{E}_p(v_t) = \sum_{i=1}^n \lambda_i v_t(x_i) = \sum_{i=1}^n \lambda_i t_i = t$. Since $\succcurlyeq$ is a Betweenness preference, this implies that $p \sim \delta_{\bar{x}}$ where $\bar{x} = c(p, v_t)$. In particular, this implies that $x_p = \bar{x}$. At the same time, we also have that

$$f_{s',t}(\mathbb{E}_p(v_t)) = f_{s',t}(t) > \sum_{i=1}^n \lambda_i f_{s',t}(t_i) = \sum_{i=1}^n \lambda_i f_{s',t}(v_t(x_i)) = \sum_{i=1}^n \lambda_i v_{s'}(x_i) = \mathbb{E}_p(v_{s'}) \ (\text{resp., } <).$$

Since $f_{s',t} = v_{s'} \circ v_t^{-1}$, we can conclude that

$$\min_{s \in (0,1)} c(p, v_s) = V(p) = x_p = c(p, v_t) > c(p, v_{s'}) \geq \min_{s \in (0,1)} c(p, v_s)$$
$$(\text{resp., } \max_{s \in (0,1)} c(p, v_s) = V(p) = x_p = c(p, v_t) < c(p, v_{s'}) \leq \max_{s \in (0,1)} c(p, v_s)),$$

a contradiction. ■

that

$$\int_{[w,b]} k\left(x, \hat{V}(p)\right) dp = \hat{V}(p).$$

Recall also that $v_t = k\left(\cdot, \hat{V}(p)\right) \in \mathcal{W}_{\text{bet}}$ and $t = \hat{V}(p)$. Define $\bar{x} \in [w,b]$ to be such that $\bar{x} = c(p, v_t)$. Note that

$$v_t(\bar{x}) = v_t(c(p, v_t)) = v_t\left(v_t^{-1}\left(\int_{[w,b]} k\left(x, \hat{V}(p)\right) dp\right)\right) = \int_{[w,b]} k\left(x, \hat{V}(p)\right) dp.$$

It follows that

$$\int_{[w,b]} k\left(x, \hat{V}(p)\right) d\delta_{\bar{x}} = v_t(\bar{x}) = \hat{V}(p).$$

Since $\succcurlyeq$ is a Betweenness preference, we can conclude that $\hat{V}(\delta_{\bar{x}}) = \hat{V}(p)$, that is, $\delta_{\bar{x}} \sim p$ and so $\bar{x} = x_p$.

[22] Clearly, $V(\delta_x) = x$ if either $x = w$ or $x = b$.

Remark 5 Convexity at t is implied by the following sufficient condition: the subdifferential of $f_{s,t}$ is nonempty at t, $\partial f_{s,t}(t) \neq \emptyset$. This takes a simple geometric interpretation, as it amounts to saying that the graph of $f_{s,t}$ is supported by a line at the point $(t, f_{s,t}(t))$, that is, there exists a function $g : [0,1] \to \mathbb{R}$ such that $g(t') = mt' + l$ for all $t' \in [0,1]$, where $l, m \in \mathbb{R}$ and $f_{s,t}(t) = g(t)$ as well as $g(t') \leq f_{s,t}(t')$ for all $t' \in [0,1]$. $\triangledown$

Proof. Denote $f_{s,t}$ simply by f. Let $t \in (0,1)$. Assume that $f : [0,1] \to [0,1]$ is such that $\partial f(t) \neq \emptyset$. By assumption, it follows that there exists $m \in \mathbb{R}$ such that

$$f(t') - f(t) \geq m(t' - t) \qquad \forall t' \in [0,1].$$

Define $g : [0,1] \to \mathbb{R}$ by $g(t') = mt' + l$ for all $t' \in [0,1]$ where $l = f(t) - mt$. Note that

$$f(t) = g(t) \text{ and } g(t') \leq f(t') \quad \forall t' \in [0,1].$$

Next consider $n \in \mathbb{N}$, $\{t_i\}_{i=1}^n \subseteq [0,1]$, and $\{\lambda_i\}_{i=1}^n \subseteq [0,1]$ such that $\sum_{i=1}^n \lambda_i = 1$ and $\sum_{i=1}^n \lambda_i t_i = t$. It follows that

$$f(t) = g(t) = g\left(\sum_{i=1}^n \lambda_i t_i\right) = \sum_{i=1}^n \lambda_i g(t_i) \leq \sum_{i=1}^n \lambda_i f(t_i),$$

proving convexity at t. ■

We conclude by putting into use the results above, showing that the Betweenness preference in Example 1 satisfies Negative Certainty Independence.

Proof of Example 1. For each $t \in [0,1]$ define $v_t(x) = k(x,t)$ for all $x \in [0,1]$. Given $s, t \in (0,1)$, we need to show that $f = v_s \circ v_t^{-1}$ is convex at t. Before starting, observe that $v_t^{-1} : [0,1] \to \mathbb{R}$

$$v_t^{-1}(x) = \begin{cases} x & \text{if } x \leq t \\ \frac{t + \sqrt{t^2 + 4(x-t)}}{2} & \text{if } x > t \end{cases} \qquad \forall x \in [0,1].$$

Clearly, if $s = t$, then $f = v_s \circ v_t^{-1}$ is the identity on $[0,1]$ and it is convex at t. We then have two cases:

1. $t > s$. In this case, we have that for each $x \in [0,1]$

$$f(x) = v_s\left(v_t^{-1}(x)\right) = \begin{cases} x & \text{if } x \leq s \\ x^2 - sx + s & \text{if } s < x \leq t \\ \left(\frac{t+\sqrt{t^2+4(x-t)}}{2}\right)^2 - s\left(\frac{t+\sqrt{t^2+4(x-t)}}{2}\right) + s & \text{if } x > t \end{cases}.$$

   Consider $g : [0,1] \to \mathbb{R}$ to be such that $g(x) = m(x-t) + f(t)$ and $m = \max\{2t - s, 1\}$. We have three cases:

(a) $0 \le t' \le s$. Note that

$$g(0) = f(t) - mt \le f(t) - t = t^2 - st + s - t = (t-1)(t-s) < 0.$$

We can conclude that

$$\begin{aligned} g(t') &= m(t'-t) + f(t) \le f(t) + t' - t \\ &= t' + f(t) - t \le t' = f(t'). \end{aligned}$$

(b) $s < t' \le t$. Define $h : [0,1] \to \mathbb{R}$ by $h(x) = x^2 - sx + s$ for all $x \in [0,1]$. Note that $h(t) = f(t)$ and $h'(t) = 2t - s \le m$, yielding $h'(t)(t'-t) \ge m(t'-t)$ for all $t' \le t$. Since h is convex, we have that

$$f(t') = h(t') \ge h'(t)(t'-t) + h(t) \ge m(t'-t) + f(t) = g(t') \qquad \forall t' \in (s,t].$$

(c) $t' > t$. Define $\tilde{h} : [t,1] \to \mathbb{R}$ by $\tilde{h}(x) = \left(\frac{t+\sqrt{t^2+4(x-t)}}{2}\right)^2 - s\left(\frac{t+\sqrt{t^2+4(x-t)}}{2}\right) + s$ for all $x \in [t,1]$. It follows that $\tilde{h}$ is concave. Note that $\tilde{h}(t) = f(t) = g(t)$. Since $\tilde{h}$ is concave and g is affine, it is enough to verify that $\tilde{h}(1) \ge g(1)$ to prove that $f(t') = \tilde{h}(t') \ge g(t')$ for all $t' \in [t,1]$. Since $t \in (0,1)$ and $\tilde{h}(1) = 1$, observe that if $m = 2t - s$, then

$$\begin{aligned} g(1) &= m(1-t) + f(t) = (2t-s)(1-t) + t^2 - st + s \\ &= 2t - 2t^2 - s + st + t^2 - st + s \\ &= 2t - t^2 = t + t(1-t) \le 1 = \tilde{h}(1). \end{aligned}$$

Since $0 < s < t < 1$, if $m = 1$, then

$$\begin{aligned} g(1) - \tilde{h}(1) &= g(1) - 1 = 1 - t + f(t) - 1 = 1 - t + t^2 - st + s - 1 \\ &= -t + t^2 - st + s = t(t-1) + s(1-t) \\ &= (t-s)(t-1) < 0, \end{aligned}$$

proving that $g(1) < \tilde{h}(1)$.

Subpoints a–c just showed that the subdifferential of f is nonempty at t and, in particular, f is convex at t.

2. $t < s$. In this case, we have that for each $x \in [0,1]$

$$f(x) = v_s\left(v_t^{-1}(x)\right) = \begin{cases} x & \text{if } x \le t \\ \frac{t+\sqrt{t^2+4(x-t)}}{2} & \text{if } t < x \le \bar{s} \\ \left(\frac{t+\sqrt{t^2+4(x-t)}}{2}\right)^2 - s\left(\frac{t+\sqrt{t^2+4(x-t)}}{2}\right) + s & \text{if } x > \bar{s} \end{cases}$$

where $\bar{s}$ is such that $\frac{t+\sqrt{t^2+4(\bar{s}-t)}}{2} = s$.[23] Consider $g : [0,1] \to \mathbb{R}$ to be such that $g(x) = x$. We have three cases:

(a) $0 \leq t' \leq t$. Clearly, we have that $f(t') \geq g(t')$.

(b) $t < t' \leq \bar{s}$. Define $h : [t, \bar{s}] \to \mathbb{R}$ by $h(x) = \frac{t+\sqrt{t^2+4(x-t)}}{2}$ for all $x \in [t, \bar{s}]$. Since h is concave and g is affine, if we verify that $h(t) \geq g(t)$ and $h(\bar{s}) \geq g(\bar{s})$, then $f(t') = h(t') \geq g(t')$ for all $t' \in [t, \bar{s}]$. Note that $h(t) = t = g(t)$. On the other hand, we have that

$$\begin{aligned} h(\bar{s}) &= \frac{t+\sqrt{t^2+4(\bar{s}-t)}}{2} \geq \frac{t+\sqrt{t^2+4\bar{s}(\bar{s}-t)}}{2} \\ &= \frac{t+\sqrt{t^2+4\bar{s}^2-4\bar{s}t}}{2} = \frac{t+\sqrt{(2\bar{s}-t)^2}}{2} = \bar{s} = g(\bar{s}). \end{aligned}$$

(c) $t' > \bar{s}$. Define $\tilde{h} : [\bar{s}, 1] \to \mathbb{R}$ by $\tilde{h}(x) = \left(\frac{t+\sqrt{t^2+4(x-t)}}{2}\right)^2 - s\left(\frac{t+\sqrt{t^2+4(x-t)}}{2}\right) + s$ for all $x \in [\bar{s}, 1]$. Since $\tilde{h}$ is convex, $\tilde{h}(1) = 1$, and $\tilde{h}'(1) = \frac{2-s}{2-t} \in (0,1)$, we have that

$$\begin{aligned} \tilde{h}(t') &\geq \tilde{h}'(1)(t'-1) + \tilde{h}(1) \geq 1(t'-1) + \tilde{h}(1) \\ &= t' - 1 + 1 = t' = g(t') \qquad \forall t' \in [\bar{s}, 1]. \end{aligned}$$

Subpoints a–c just showed that the subdifferential of f is nonempty at t and, in particular, f is convex at t. ■

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[23]Since $\frac{t+\sqrt{t^2+4(t-t)}}{2} = t < s < 1 = \frac{t+\sqrt{t^2+4(1-t)}}{2}$ and the map $x \mapsto \frac{t+\sqrt{t^2+4(x-t)}}{2}$ is strictly increasing and continuous on $[t,1]$, we have that $\bar{s}$ exists and $\bar{s} > t$.

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