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Abstract

This thesis comprises three chapters that examine strategic behavior under asymmetric information in auctions and bilateral trade.
Chapter 1 studies bidding behavior in concurrent auctions for heterogeneous goods when bidders can participate in only one auction. The analysis highlights how endogenous market entry generates a stochastic number of competitors and a self-selection effect that shapes bidding strategies, which collectively determine the equilibrium market-entry pattern and bidding behavior upon entry. Under independent private values, bidders always bid for their favorite good, while with affiliated values, bidders may enter a less-preferred market to avoid competition.
Chapters 2 and 3 both investigate bilateral trade environments with two-sided private information, but with different economic motivations. Chapter 2, which is a joint work with Nenad Kos, analyzes expert markets in which an informed expert sells services to a consumer who privately knows the difficulty (and value) of his problem. Prices serve as signals of expertise. The chapter characterizes the equilibria and evaluates welfare across different outcomes. It shows that the expert can sometimes increase her profit by grouping her types to segment demand more effectively, while full separation yields the highest welfare by revealing information and lowering prices.
Chapter 3 studies a product market in which a seller privately knows a horizontal product attribute, while the consumer privately knows his taste. The seller chooses whether to disclose this attribute before setting a price. The analysis uncovers how disclosure incentives depend on the transport cost specifications and characterizes when partial pooling or full disclosure emerges in equilibrium. It also shows that mandatory disclosure can harm both the seller and the consumer when the market is not fully covered.
Together, these chapters demonstrate how asymmetric information shapes market entry, price signaling, and disclosure decisions, and provide new insights into how private information interacts with market structure in both auction and bilateral-trade environments.

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Contents

Chapter 1

Concurrent Auctions with Heterogeneous Objects

Abstract

This paper examines bidding behavior in concurrent auctions of two heterogeneous goods, where each bidder can participate in only one auction. We consider both first-price and second-price auction formats. Each bidder has a private valuation of each good, thus having a two-dimensional type. After observing their types, bidders simultaneously make (i) market entry decisions and (ii) bidding plans. We show that endogenous market entry leads to an unknown number of bidders in the auction markets, and introduces a self-selection effect that shapes bidding strategies. We define a monotone entry equilibrium as an equilibrium where each bidder enters the higher-value market. Under independent information, a monotone entry equilibrium always exists in both first-price and second-price auctions. When the private values are correlated, bidders infer the potential competition in markets and enter strategically to avoid competition. This is demonstrated in affiliated private values second-price auctions: bidders may deviate to bid on the less-preferred good to avoid intense competition when the valuations are highly correlated across bidders.

1.1 Introduction

There is a large body of research on bidder behavior in auctions across a wide range of settings, including different auction formats, information structures, and risk preferences.[1] However, an important dimension often overlooked lies outside the auction itself: bidders often need to make market entry decisions before participating in an auction. Such decisions arise naturally in many institutional contexts. For example, several Chinese municipalities have historically permitted each developer to bid on only one land parcel in a given auction round, preventing simultaneous bidding across parcels and generating precisely the type of single-auction entry constraint modeled in this paper.[2] Similarly, in U.S. Forest Service timber auctions, binding capacity and financial-exposure limits restrict firms from bidding on multiple timber sales at once, inducing a strategic choice over which auction to enter.[3][4]

Beyond institutional restrictions, bidders often face practical limits that shape their entry across auctions. Budget constraints may make participation in multiple high-stakes auctions financially infeasible, particularly when large upfront deposits are required. Time limitations and limited attention further restrict a bidder's ability to engage in all auctions of interest. Auctions may occur simultaneously in different physical locations, and even in online environments, such as eBay, bidders must still choose which auctions to actively

[1]See Milgrom and Weber (1982), Matthews (1987), Krishna (2009), and Milgrom (2004) for foundational work on bidder behavior across auction formats, information structures, and risk preferences.

[2]For example, land auction rules in cities such as Shanghai, Shenzhen, and Guangzhou have periodically included "one bidder–one parcel" restrictions, in which a developer may submit a bid for only one plot in a given auction event. Evidence includes municipal land bureau announcements and translated policy summaries such as: Guangzhou Land Resources and Urban Planning Committee, "Transfer Rules for State-Owned Land Use Rights" (various years); Shenzhen Public Resource Exchange Center, "Land Use Rights Transfer Procedures" (e.g., 2012–2015).

[3]U.S. Forest Service timber auctions impose strict financial and operational exposure limits under the "Timber Sale Contract" framework. Firms must satisfy bonding requirements, logging-capacity constraints, and maximum uncompleted-purchase limits (FSM 2400—Timber Management; 36 CFR §223). For example, 36 CFR §223.101 ("Bidding methods") and §223.48 ("Limitations on Purchaser Eligibility") describe how insufficient capacity or excessive existing obligations can disqualify a firm from bidding on additional sales. Official documentation is available from the U.S. Forest Service: `https://www.fs.usda.gov/about-agency/regulations-policies`.

[4]A closely related example appears in Italian local public real-estate auctions. For instance, Aler Milano (the public housing authority of the Lombardy Region) regularly conducts auctions in which multiple apartments are offered simultaneously, but each participant is allowed to submit only one sealed bid across all properties in the session. See `https://alermipianovendite.it/asta-alloggi/`.

follow within a large and diverse set of ongoing listings.

In light of these considerations, market entry becomes a critical and strategic decision that precedes bidding. Strategic entry, in turn, alters bidders' beliefs and bidding behavior—a topic that has received limited attention in the literature. Moreover, in many real-world auction environments, the goods offered across markets are heterogeneous rather than identical, so bidders' individual preferences play a central role in their entry decisions.

We study how bidders make market-entry decisions when faced with multiple concurrent auctions offering heterogeneous goods, and how these entry choices subsequently influence bidding behavior and strategic inference. To address these questions, the paper departs from the standard assumption of a single auction market and instead examines bidding behavior within a concurrent-auctions framework.[5]

The general framework for concurrent auctions is as follows. There are N bidders competing for two heterogeneous goods, each sold in a separate auction market. The auctions are concurrent. Bidders can bid on only one of the goods. Hence, bidders must make market entry decisions prior to bidding. Each bidder draws a private signal/valuation for each good, resulting in a two-dimensional private type. After observing their types, all bidders simultaneously (i) make market entry decisions and (ii) submit bids. The Bayesian Nash equilibrium is a double-fixed point of entry decision and bidding (upon entry). We consider both first-price and second-price auctions in this framework. A bidder's payoff equals her valuation of the good minus the payment if she wins, and zero otherwise.

Before previewing the main results, we define a monotone entry equilibrium: each bidder enters the market corresponding to her higher private signal. We show in Proposition 1.1 that, in both first-price and second-price auctions, a monotone entry equilibrium exists when private information is independent across bidders. Intuitively, when information

[5]Concurrent auctions are fundamentally different from simultaneous auctions. In a simultaneous auction, bidders submit bids on a range of items at the same time, and items are allocated separately to the highest bidders. There is no restriction on bidders' market entry in a simultaneous auction, whereas market selection is a crucial feature of concurrent auctions.

only reveals bidders' own valuations and contains no inference about others' types, each market appears equally competitive from the bidder's perspective, so she always enters the market where her valuation is higher. Proposition 1.1 therefore allows us to abstract from the entry stage under independent information.

We then illustrate in Section 1.4.1, using independent private-value (IPV) first-price auctions, that endogenous entry in concurrent auctions generates (i) an uncertain number of bidders in each market and (ii) a self-selection effect that alters bidding behavior. Proposition 1.2 formalizes this result, showing that bidders bid as if they believe the number of rivals follows $B\left(N-1, \frac{1}{2}\right)$, and that each rival's valuation is drawn from a distribution with a more convex cumulative distribution function (CDF) than in the standard single-auction case. Revenue equivalence continues to hold under independent information, because the two markets behave as identical, independent auctions with stochastic numbers of bidders.

When information is affiliated, the structure of entry changes fundamentally. A bidder who receives a high signal in one market infers that others are likely to have high signals as well, and therefore anticipates stronger competition in that market. By focusing on private-value second-price auctions, we abstract from bidding behavior conditional on entry and isolate the strategic forces shaping market choice. Proposition 1.3 characterizes the symmetric equilibrium in the binary-signal environment and shows that monotone entry is more likely when affiliation is weaker. Bidders face a clear trade-off: a high signal raises their valuation in a market, but it also predicts more competition there. As affiliation strengthens, the desire to avoid competition becomes more important, pushing some bidders toward lower-signal markets. Extensions to general discrete types and a specific continuous-type family are provided in the appendix.

This paper makes three main contributions. First, it characterizes equilibrium entry in concurrent auctions and shows that under independent private values and ex ante symmetric markets, every symmetric equilibrium features monotone entry, giving a clean benchmark where entry depends only on a bidder's own information. Second, it identifies two forces that shape outcomes in concurrent auctions—a stochastic number of bidders and the self-selection of types—and shows how they govern both entry and bidding across formats. Third, it demonstrates how affiliation can overturn monotone entry, fully characterizing the symmetric equilibrium with binary signals and deriving sharp comparative statics linking the strength of affiliation and the value ratio to mixed entry behavior.

Table 1 summarizes the settings examined in the paper. The classification is useful for decomposing the equilibrium analysis. In particular, the structure of second-price private-value auctions, together with the independent-information assumption, allows us to abstract from either the entry decision or the bidding stage when characterizing the equilibrium.

	Independent private values	Affiliated private values
Second-price	entry: monotone entry	entry: analysis in section 1.4.2
	bid: truthful bidding	bid: truthful bidding
First-price	entry: monotone entry	analysis in section 1.4.2
	bid: analysis in section 1.4.1	

Table 1.1: Summary of the environments considered in the paper.

The rest of the paper is organized as follows. Section 1.2 reviews the related literature. Section 1.3 introduces the general framework. Section 1.4 presents results on symmetric equilibrium. Specifically, we first analyze auctions under independent information in subsection 1.4.1, and then extend the analysis to affiliated information in subsection 1.4.2. Section 1.5 examines a type of asymmetric equilibrium. Finally, Section 1.6 concludes.

1.2 Related Literature

Auctions with endogenous entry This model relates to the literature on auctions with endogenous entry, where most studies introduce an entry or participation cost for bidders. For instance, Engelbrecht-Wiggans (1993), Levin and Smith (1994), Ye (2004), Hausch and Li (1993), and Murto and Välimäki (2015) assume bidders make market entry decisions before types realize. In contrast, Chakraborty and Kosmopoulou (2001), Menezes and Monteiro (2000) assume bidders first observe their private type, then decide whether to pay to enter the market. The fundamental difference between these two lines of literature lies in the presence of the self-selection effect in the latter. Our concurrent-auction setup shares this self-selection feature but differs in that we do not impose an explicit entry cost. Instead, bidders compare expected payoffs across the two markets and make entry decisions accordingly. By fixing one auction market, we can equivalently interpret that each bidder has an opportunity cost of entering the market, which is the expected payoff in the other auction market. Thus, our model can be viewed as imposing a type-dependent entry cost for each market.

Unknown number of bidders Several studies have examined the implications of an unknown number of bidders in auctions. The mainstream auction literature, however, typically assumes that the number of bidders is common knowledge among participants and derives equilibrium bidding strategies and revenue implications under this assumption.[6] Nonetheless, it is arguably more realistic to relax the common knowledge assumption regarding the number of participants or the market size. Levin and Smith (1994) consider a model with endogenous entry, in which bidders decide whether to enter the auction before their private types are realized. In symmetric equilibrium, each bidder enters with the same probability, leading to a binomial distribution over the number of participants in the auction. Our framework shares this feature of a stochastic number of bidders but incorporates an additional self-selection effect, as bidders choose between multiple markets based on their private information. We introduce a variant of their model as a benchmark in Subsection 1.4.1 for comparison.

Market selection The majority of the auction literature has focused on single mar-

[6] See Milgrom and Weber (1982), Krishna (2009), Klemperer (2004), and Milgrom (2004) for standard treatments of auction models with a known number of bidders.

kets, with limited research on multiple markets. While some papers study simultaneous auctions (e.g., Szentes (2007) and Peters and Severinov (2006)), where bidders place multiple bids in different auctions at the same time, our work is more related to the following papers that assume buyers need to select market prior to entry. McAfee (1993), Burguet and Sákovics (1999) and Delnoij and De Jaegher (2020) study competing auctions where each buyer can only participate in one auction. These models typically feature identical goods and strategic differentiation through auction formats. In contrast, we analyze bidding in two concurrent auctions with heterogeneous goods, where market differences stem from bidders' preferences rather than auction formats. Our focus is on how private information and heterogeneous tastes shape market entry decisions, not seller competition.

Relatedly, Landi et al. (2023) examine competing auctions with heterogeneous goods and independent private values. Sellers conduct second-price auctions and compete by setting reserve prices. We differ in two key ways: first, we allow for affiliated values among bidders, enabling the study of how informational linkages influence equilibrium bidding behavior; second, we include first-price auctions, offering a broader perspective on bidding strategies across auction formats.

1.3 Framework

There are $N \geq 2$ risk-neutral bidders who simultaneously submit bids for two heterogeneous goods, indexed by $k = 1, 2$. The markets for the two goods are separate, and are referred to as market 1 and market 2, respectively. We will use *market* k and *good* k interchangeably for the rest of the text. For each good k, bidder i, $i = 1, \ldots, N$, receives one private signal $s_k^i \in [\underline{s}, \overline{s}] \equiv S \subset \mathbb{R}_+$, which represents her private valuation of good k. Hence, bidder i has a two-dimensional type $s^i = (s_1^i, s_2^i) \in S^2$. Let $s_k = \left(s_k^1, s_k^2, \ldots, s_k^N\right)$ denote the vector of signal realizations in market k, and let $s = (s_1, s_2)$ denote the overall signal realizations. We assume that the markets are independent, i.e., $F\left(s\right) = F_1\left(s_1\right) \times F_2\left(s_2\right)$, and that the markets are symmetric, so $F_1 = F_2$.

Upon observing their private signals, each bidder i submits a pair of bids $b^i = (b^i_1, b^i_2) \in \mathbb{R}^2_+$, where b^i_k denotes bidder i's bid for good $k = 1, 2$. We impose the constraint $b^i_1 b^i_2 = 0$, meaning that each bidder may submit a positive bid for at most one of the two goods. We say that bidder i *enters market* k or *bids on good* k if $b^i_k > 0$ (so $b^i_{-k} = 0$). This reflects that bidders must decide which market to enter prior to submitting a bid. In other words, each bidder has two decisions to make at the same time: (i) *entry decision* k such that $b^i_k \geq 0$ and $b^i_{-k} = 0$ and (ii) *bid* b^i_k if $b^i_k > 0$. In each auction market k, the bidder who submits the highest bid wins the good. In the event of a tie, the good is awarded with equal probability among the highest bidders. If bidder i wins good k, her payoff equals her valuation v^i_k minus the payment, where the payment is b^i_k in a first-price auction and $\max_{j \neq i} b^j_k$ in a second-price auction; the payoff is zero otherwise. A (pure) bidding strategy[7] $\beta^i_k(s^i)$ maps S^2 into pairs of bids (b^i_1, b^i_2). A bidding plan $\hat{\beta}^i : S^2 \to \mathbb{R}_+$ is defined as $\hat{\beta}^i(s^i) := \max_{k=1,2} \beta^i_k(s^i)$, so that a bidding plan always refers to the actual bid placed in the chosen market.

1.4 Symmetric Equilibrium

In this section, we adopt the notion of Bayesian equilibrium and analyze the symmetric equilibrium bidding strategy under various auction environments. Subsection 1.4.1 examines the independent private values setting in both first-price and second-price concurrent auctions, highlighting the direct impact of concurrent auctions on equilibrium bidding behavior. In subsection 1.4.2, we relax the independent information assumption and explore how affiliated information shapes equilibrium entry decisions.

To solve for the equilibrium, we decompose the problem into two subproblems: (i) fixing entry decisions to determine optimal bidding plans, and (ii) fixing bidding plans to determine optimal entry decisions. The equilibrium corresponds to a double fixed point of *entry decision* and *bidding plan.* This decomposition allows us to begin by considering

[7] Although it is called bidding strategy, it indicates both entry decision and the actual bid to be placed in the corresponding auction market, after observing two private signals about the goods.

a symmetric equilibrium with an intuitive entry rule. A natural benchmark is for each bidder to enter the market corresponding to their higher private signal. We refer to an equilibrium in which every bidder selects the good associated with the higher signal as a *monotone entry equilibrium.*

Definition 1.1. *A monotone entry equilibrium is an equilibrium in which, for every bidder* $i = 1, \ldots, N$ *and for each good* $k = 1, 2$, $s_k^i > s_{-k}^i$ *implies* $b_k^i > 0$ *and* $b_{-k}^i = 0$.

One important remark is that there is no restriction on entry decisions when $s_k^i = s_{-k}^i$. Hence, it is possible to have mixed entry decisions in a monotone entry equilibrium: bidders can randomize between the two markets when they observe the same private signals about two goods. Note that bidders' entry decisions are non-trivial: one may have an incentive to bid on the good with a low signal if she anticipates intense competition in the market where a higher private signal is observed.

1.4.1 Independent Private Values

We begin by analyzing concurrent auctions under independent private values. Independence across bidders implies that a bidder's private signals do not reveal any information about rivals' valuations or their entry decisions. Combined with the assumption of ex-ante symmetry across markets, bidders perceive the two auction markets as equally competitive before signals are realized. This benchmark isolates the role of private information in shaping market selection and allows a clean characterization of equilibrium entry behavior.

The following proposition establishes that independence eliminates all inference about rivals and therefore leads to monotone entry in any symmetric equilibrium.

Proposition 1.1. *In both first-price and second-price concurrent auctions, if for* $k = 1, 2$ *and for all* i *and* j, $\boldsymbol{s}_k^i$ *and* $\boldsymbol{s}_k^j$ *are independent, then any symmetric equilibrium is a monotone entry equilibrium. Furthermore, in equilibrium, for all* $i = 1, \ldots, N$ *and for* $k = 1, 2$, $\beta_k^i(s^i)$ *is increasing in* s_k^i *and constant in* s_{-k}^i *when* $s_k^i > s_{-k}^i$.

Independent information across bidders implies that a bidder's private signal carries no information about others' signals or about the number of competitors she will face in either market. Under ex-ante symmetry, both markets therefore have the same expected level of competitiveness before signals are observed. The only difference between entering market 1 or 2 comes from the bidder's own valuations. Because the expected payoff from entering market k is strictly increasing in s_k, a bidder strictly prefers the market in which she receives her higher signal, and every symmetric equilibrium must exhibit monotone entry.[8]

The second part of Proposition 1.1 characterizes bidding behavior under independence. Conditional on entering market k, bidder i's bid depends solely on her valuation for that market, s^i_k; the other signal s^i_{-k} is irrelevant for beliefs about competitors. Thus, entry decisions and bidding strategies separate cleanly under independent information.

IPV Second-Price Auctions: A Stochastic Number of Bidders and Self-Selection

We begin with second-price auctions, where bidding truthfully is weakly dominant.[9] This environment provides a transparent benchmark for understanding how endogenous entry shapes behavior under independent information. Because truthful bidding is optimal and Proposition 1.1 ensures monotone entry, the equilibrium is straightforward: each bidder enters the market for which she has the higher valuation and then bids her true value once she enters.

Endogenous entry generates two key effects. The first is a stochastic number of bidders. Under monotone entry, each bidder selects either market with probability $\frac{1}{2}$, so the number of entrants in market k, denoted M_k, follows a binomial distribution $M_K \sim B(N, \frac{1}{2})$. The second effect is self-selection: only bidders with $s^i_k > s^i_{-k}$ enter market k. Conditional on

[8]The analysis relies on the assumption of symmetric markets. For example, if the value distribution in market 1, F_1, first-order stochastically dominates that in market 2, F_2, bidders might expect more intense competition in market 1, leading them to tilt toward market 2 when the values of the two goods are similar. Monotone entry is not guaranteed even under independent information across bidders.

[9]Truthful bidding is weakly dominant in private-value second-price auctions. Although multiple equilibria may exist, the analysis focuses on the truthful-bidding equilibrium.

entry, a bidder's valuation in market k, therefore has cumulative distribution

$$\begin{aligned}
F(x|s_k^i > s_{-k}^i) &= \Pr\left(s_k^i < x | s_k^i > s_{-k}^i\right) \\
&= \Pr\left(s_k^i > s_{-k}^i\right)^{-1} \cdot \Pr\left(s_{-k}^i < s_k^i < x\right) \\
&= \left(\frac{1}{2}\right)^{-1} \int_{\underline{s}}^{x} \left[F(x) - F\left(s_{-k}^i\right)\right] dF\left(s_{-k}^i\right) \\
&= 2\left[F(x) F\left(s_{-k}^i\right) - \frac{1}{2} F\left(s_{-k}^i\right)^2\right]_{\underline{s}}^{x} \\
&= F(x)^2,
\end{aligned}$$

which is strictly more convex than the prior distribution F, meaning that the types are more concentrated on the higher values. Entrants are thus positively selected, with types more concentrated on higher values. These two forces fully determine the composition of entrants in each market.

Revenue in a second-price concurrent auction follows directly from these features. Conditional on $M_k = m > 1$, the object is awarded to the bidder with the highest valuation, and the payment equals the second-highest valuation among the m entrants. Thus, the expected revenue in market k is

$$\mathbb{E}[R_k] = \sum_{m=2}^{N} \binom{N}{m} \left(\frac{1}{2}\right)^N \mathbb{E}\left[V_{(2)}^{(m)}\right],$$

where $V_{(2)}^{(m)}$ denotes the second order statistic from m i.i.d. draws of s_k. Since the two markets are symmetric, the total expected revenue is simply $2\mathbb{E}[R_k]$. This benchmark highlights that under truthful bidding, endogenous entry affects revenue only through the induced distribution of market sizes and entrant types.

IPV First-Price Auctions: Bidding Behavior

We now turn to first-price auctions under independent private values. Proposition 1.1 implies that equilibrium entry is monotone: each bidder enters the market for which she receives the higher signal. Unlike in second-price auctions, the two forces generated by endogenous entry, a stochastic number of bidders and the self-selection effect, directly shape bidding behavior upon entry.

As in Levin and Smith (1994), bidders face uncertainty about the number of rivals they will compete against. In our environment, this uncertainty is compounded by the fact that bidders self-select into different markets. The conditional beliefs about rivals' types upon entry differ from the unconditional one, resulting in different bidding behavior.

Define $H_k := \left\{j : s_k^j > s_{-k}^j\right\}$ as the set of players that receive higher signals in market k than market $-k$. Since F is continuous, the set $\left\{j : s_k^j = s_{-k}^j\right\}$ is of zero measure and we omit it.

Proposition 1.2. *If $\boldsymbol{s}_k^i$ and $\boldsymbol{s}_k^j$ are independent for all i and j, then revenue-equivalence holds. Furthermore, in first-price auctions, upon entering market k, each bidder bids*

$$b_k^i = \mathbb{E}\left[\max_{j \in H_k \setminus \{i\}} s_k^j \,\middle|\, s_k^i = \max_{j \in H_k} s_k^j\right],$$

with the following conditional beliefs:

1. *the number of rivals in market k follows $B\left(N-1, 1/2\right)$;*

2. *each rival's private valuation for good k follows F^2.*

This result shows that the standard logic of first-price auctions is preserved: bidders shade their bids toward the expected second-highest value conditional on winning. The key differences lie in bidders' conditional beliefs about rival participation and type distributions. Bidders treat the number of competitors as random, and they expect competitors' valuations to follow the positively selected distribution F^2 rather than the prior F. These adjustments arise directly from the two forces identified earlier: the random number of bidders and the self-selection effect. Unlike in second-price auctions, these forces feed back into bidding behavior once a bidder enters a market.

The equilibrium parallels the framework of Levin and Smith (1994), in which a stochastic number of bidders emerges due to endogenous entry. Our setting adds a second layer of selection through market choice, producing a smooth shift in the type distribution rather than an explicit truncation. The expected payoff from the alternative market functions as an endogenous, type-dependent outside option, effectively acting as an implicit entry cost.

Despite these additional layers, revenue equivalence continues to hold under independent information. Under monotone entry, each market is equivalent to a single auction with a stochastic number of bidders. In both first- and second-price formats, the good is allocated to the agent with the highest valuation. In second-price auctions, the winner pays the second-highest valuation; in first-price auctions, the optimal bid equals the expected second-highest valuation. The two formats therefore generate the same expected revenue.[10]

1.4.2 Affiliated Private Values Auctions

We now relax the assumption of independent private information and examine the implications of affiliated information for equilibrium bidding behavior. We begin with the analysis of second-price auctions and then extend the discussion to first-price auctions.

APV Second-Price Auctions: Entry Decisions

Since truthful bidding remains optimal in second-price auctions, our analysis focuses on entry decisions under affiliated information. We examine equilibrium entry behavior in this setting by considering both discrete and continuous signal structures.

Binary signals Consider the binary-signal environment where each bidder's valuation

[10] Levin and Smith (1994) also showed revenue equivalence between first- and second-price auctions under a stochastic number of bidders. Under monotone entry rule, the concurrent auctions can be interpreted as two independent and identical markets with stochastic number of bidders. Their result thus applies.

in market k takes one of two values,$s_k^i \in \{s_H, s_L\}$ with $s_H > s_L$. Let $\theta_k \in \{H, L\}$ denote the state of market k, and let $\omega \in \{H, L\}^2$ denote the pair of market states. The signals satisfy the monotone likelihood ratio property:

$$\Pr\left(s_k^i = s_H | \theta_k = H\right) = \Pr\left(s_k^i = s_L | \theta_k = L\right) = \rho \in \left[\frac{1}{2}, 1\right].$$

The correlation between any two bidders' signals is $\text{Corr}(s_i, s_j) = (2\rho - 1)^2$, hence ρ captures the degree of affiliation across bidders' values. A higher ρ implies stronger positive correlation across bidders' signals, making it more likely that they draw similar valuations for a given market.

In a symmetric equilibrium, player i (i) randomizes between the two markets when $s_1^i = s_2^i$; (ii) enters market k with probability $h \in [0, 1]$ when $s_k^i > s_{-k}^i$; and (iii) bids her true value upon entry. The case $h = 1$ corresponds to a monotone entry equilibrium.

Proposition 1.3. *In the affiliated private values second-price auctions with binary signals, there exists a unique symmetric equilibrium σ^* such that for all $i = 1, \ldots, N$, for $k = 1, 2$, for any $\rho \in \left[\frac{1}{2}, 1\right]$, there exists a unique $h \in \left(\frac{1}{2}, 1\right]$ such that player i's strategy is*

$$\sigma^{*i}\left(\left(s_1^i, 0\right)\right) = \begin{cases} \frac{1}{2} & \text{if } s_1^i = s_2^i \\ h & \text{if } s_1^i > s_2^i \\ 1 - h & \text{if } s_1^i < s_2^i \end{cases} \quad \text{and} \quad \sigma^{*i}\left(\left(0, s_2^i\right)\right) = 1 - \sigma^{*i}\left(\left(s_1^i, 0\right)\right).$$

Note that h must be strictly greater than $\frac{1}{2}$ in any symmetric equilibrium. First, $h = \frac{1}{2}$ cannot occur. To see this, suppose by contradiction that $h = \frac{1}{2}$. This would imply that all bidders randomize uniformly between the two markets, regardless of the signals they observe. Consequently, each bidder would expect the number of rivals in each market to follow the same binomial distribution, $B\left(N - 1, \frac{1}{2}\right)$. However, if a bidder observes different valuations for the two goods, she strictly prefers the market where her valuation is higher, and hence would deviate from uniform randomization. This violates equilibrium. Similarly, if $h < \frac{1}{2}$, the low-value market would attract more competitors, making it strictly less profitable to enter. Thus no bidder would enter the low-value market, contradicting the equilibrium. Therefore, any symmetric equilibrium must satisfy $h > \frac{1}{2}$.

In the symmetric equilibrium, the value of h is uniquely determined by the correlation parameter ρ and the value ratio $\frac{s_H}{s_L}$. First, h is strictly decreasing in ρ. As signals become more correlated, high-valuation bidders expect stronger competition in the high-signal market. This pushes them toward the low-value market to avoid congestion. There exists a threshold $\overline{\rho}$ such that when $\rho < \overline{\rho}$, we have monotone entry. Second, h is strictly increasing in the value ratio $\frac{s_H}{s_L}$. Intuitively, a larger valuation gain from receiving s_H reduces the incentive to migrate to the low-value market to avoid competition.

This illustrates the trade-off bidders face when making entry decisions: A higher signal increases a bidder's expected valuation in that market (the value effect), but under affiliation it also implies stronger expected competition in that market (the competition effect). A monotone entry equilibrium arises when the value effect dominates: that is, when the relative valuation gain $\frac{s_H}{s_L}$ is sufficiently large to offset the increased likelihood that rivals also enter the high signal market.

The logic extends beyond the binary-signal environment. General discrete and continuous affiliated signal structures lead to analogous patterns in which monotone entry survives only when affiliation is sufficiently weak. The formal generalizations are presented in Appendix A.

APV First-Price Auctions

Under affiliated information in first-price auctions, the equilibrium analysis becomes significantly more involved, as both entry and bidding decisions depend on the information conveyed by the bidder's full signal vector.

We begin by considering a monotone entry equilibrium. Unlike the independent-information case, a bidder's signal for the other market, s^i_{-k}, now affects her optimal bid in market k. Intuitively, s^i_{-k} conveys information about competitors' signals in both markets; thus it affects the bidder's belief about the expected intensity of competition in market k. Thus, the bid for good k, $\beta_k : S^2 \to \mathbb{R}_+$, maps a two-dimensional type to a positive number.

Given a bid b in market k, bidder i's probability of winning is

$$G(b) = \sum_{\omega \in \{H,L\}^2} \Pr\left(\omega | s^i\right) \sum_{m=0}^{n-1} \Bigg[\underbrace{\binom{n-1}{m} \Pr\left(s^j_{-k} > s^j_k \mid \omega\right)^{n-1-m} \Pr\left(s^j_k > s^j_{-k} \mid \omega\right)^m}_{\substack{\text{Conditional probability of} \\ \text{having } m \text{ competitors in market } k}}$$
$$\Pr\left(b > \beta_k\left(s^j\right) \mid \omega, \underbrace{s^j_k > s^j_{-k}}_{\text{self-selection}}\right)^m \Bigg],$$

where the inner sum accounts for the probability that exactly m rivals enter market k (self-selection) and the final term is the probability that bidder i's bid exceeds their bids.

The optimal bidding b solves

$$\max_b \quad \left(s^i_k - b\right) G\left(b\right).$$

A key difficulty in characterizing the equilibrium is that the bidding function β_k depends on two signals, which generally prevents invertibility. Multiple signal pairs (s^i_k, s^i_{-k}) to recover valuations cannot be applied. The underlying economic reason is that a bidder's optimal bid reflects not only her valuation for market k but also her inference about expected competition, for which s^i_{-k} is informative.

Although a closed-form bidding strategy is unavailable, a natural hypothesis emerges from the structure of the problem. One would expect the optimal bid in market k to be increasing in s^i_k, since a higher valuation raises willingness to pay, and decreasing in s^i_{-k}, since a higher signal in the other market suggests weaker competition in market k and therefore induces more shading. This conjectured monotonicity reflects the strategic tension introduced by affiliated information: bidders must weigh their valuation against the expected competitiveness implied by their full signal vector.

Finally, I conjecture that revenue equivalence should continue to hold conditional on entry, provided that first- and second-price auctions induce the same entry pattern. Once entry decisions are fixed, each market reduces to a private-value auction with (i) a stochastic number of bidders and (ii) a type distribution shaped by self-selection. Conditional on entry, these primitives are identical across formats, so both auction types implement the same allocation and induce the same expected second-highest valuation. This environment satisfies the standard conditions under which revenue equivalence applies. Thus, my hypothesis is that any revenue differences between the two formats must arise from differences in entry, not from the bidding stage.

1.5 Robustness to Collusion

The preceding analysis assumes that bidders select markets independently. In practice, however, bidders may attempt to coordinate their entry decisions to reduce competition, effectively engaging in collusive market division. Many forms of such collusion are conceivable—bidders could, for instance, jointly agree to enter one market with higher probability. Here we focus on the simplest and most credible form of collusion: bidders commit to enter different markets before observing their private information. This requires no information sharing or conditional strategies and is therefore the most likely type of coordinated behavior.

To illustrate, consider a second-price concurrent auction with two bidders. In principle, the bidders could follow a convention in which bidder 1 always enters market 1 and bidder 2 always enters market 2. Such coordination eliminates competition entirely, allowing both bidders to obtain goods at zero cost. The key question is whether an equilibrium of this form can arise more generally.

Definition 1.2. *A coordination equilibrium is an equilibrium such that for all bidder* $i = 1, \ldots, N$*, either one of the following is true:*

1. $\beta_2^i(s^i) = 0$ *for all* $s^i \in S^2$

2. $\beta_1^i(s^i) = 0$ *for all* $s^i \in S^2$.

Equivalently, bidders can be partitioned into two groups C_1 and C_2, where $C_k := \{i \in I : \beta_2^{-k}(s^i) = 0 \text{ for all } s^i \in S^2\}$, so that all bidders in C_k always enter market k.

Proposition 1.4. *In first-price concurrent auctions, there does not exist a coordination equilibrium; in second-price concurrent auctions (with truthful bidding), a coordination equilibrium exists only when* $N = 2$.

This result shows that coordinated market separation is sustainable only under very restrictive conditions. When $N \geq 3$, at least one market must contain more than one bidder under any fixed assignment of bidders to markets. In that market, say market 1, a bidder with type $(\underline{s}, s')$ with $s' > \underline{s}$ would strictly prefer to deviate to market 2 to obtain positive surplus. This deviation breaks any attempted coordination.

Thus, collusive market division is highly fragile. Except for the knife-edge case of second-price auctions with exactly two bidders, bidders cannot sustain collusion through fixed market assignments.

1.6 Conclusion

By departing from the single-auction assumption, we have uncovered important implications for the bidding behavior through the analysis of the concurrent-auctions model. In our model, bidding strategies involve two components: (i) the market entry decision and (ii) the bidding plan. We examine how different information structures and auction formats shape both elements of the equilibrium strategy.

The information assumption plays a critical role in shaping the equilibrium bidding strategy. Under independent private values (IPV), a monotone entry equilibrium always exists. Since bidders cannot infer rivals' information, their entry decisions are driven solely by their own valuations—leading them to enter the market where they expect the highest payoff. Even in this setting, the concurrent-auction framework introduces new dynamics: endogenous entry leads to uncertainty in the number of bidders and creates a self-selection effect.

When we relax the assumption of independence and allow for affiliated private values, the entry decision becomes more strategic. A bidder's private signal now conveys information about the competitiveness of each market. High signals suggest high competition, creating a trade-off between entering the market for the preferred good and avoiding competition. If the incentive to avoid competition dominates, a monotone entry equilibrium may fail to exist. Thus, a necessary condition for monotone entry is that the degree of affiliation across bidders' signals is sufficiently low.

This paper contributes to the auction literature by extending analysis beyond a single market and revealing strategic interactions that are not captured in standard models. This framework also raises new questions about revenue implications. For example, adding a bidder may be less effective than in standard auctions, as it intensifies competition and can distort entry decisions. Conversely, a well-chosen reserve price may reduce inefficient (non-monotone) entry and enhance revenue. These possibilities call into question the classic result of Bulow and Klemperer (1996), which favors attracting an additional bidder over setting a reserve price. Understanding these revenue effects remains an open and important avenue for future research.

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Appendix

Appendix A: Extensions

General result under discrete signals

General discrete signals Consider a discrete type setting where each bidder i receives a signal $s_k^i \in \{s_1, s_2, \ldots, s_m\}$, with $s_1 < s_2 < \cdots < s_m$. In each auction market k, there are two equally likely unobservable states, $\theta_k \in \{H, L\}$. Conditional on the state θ_k, bidder i's private value for good k is independently drawn from a probability mass function P_{θ_k}. The monotone likelihood ratio property is satisfied such that $\Pr\left(s_k^i = s_l | \theta_k = H\right) / \Pr\left(s_k^i = s_l | \theta_k = L\right)$ is increasing in l. The introduction of states induces affiliation among bidders' private values within a market, but the markets remain independent.

Since bidders' private values are positively correlated, a bidder who receives a high private value for good k will infer that her rivals are also more likely to have high private values. As a result, bidders can use their private information to form beliefs about the relative competitiveness of the two markets, making the market entry decision non-trivial. The following proposition characterizes the necessary and sufficient conditions for the existence of a monotone entry equilibrium.

Proposition 1.5. *In the affiliated private values second-price auctions with m signals, a monotone entry equilibrium exists if and only if for some functions $\{\kappa_l\}_{l=1,\ldots,m}$,*[11] *the*

[11] See Appendix B for the complete characterization of the functions.

following inequalities hold:

$$\frac{s_2}{s_1} \geq \kappa_2(\Gamma)$$
$$\frac{s_l}{s_1} \geq \kappa_h\left(\frac{s_2}{s_1}, \frac{s_3}{s_1}, \ldots, \frac{s_{l-1}}{s_1}; \Gamma\right) \qquad \textit{for all } l = 3, \ldots, m$$

where Γ *is the information structure.*

The set of inequalities is characterized by the incentive compatibility constraints that ensure no bidder has an incentive to deviate to the lower-value market. Proposition 1.5 states that when s_l (normalized by s_1) is sufficiently high for all $l = 1, \ldots, m$, a monotone entry equilibrium exists. Intuitively, when the value of the higher-value good is sufficiently high, bidders prefer to remain in the higher-value market despite potentially greater competition, rather than switch to the lower-value market to avoid rivals.

Affiliated private values with continuous signals

Continuous signals We consider a class of continuous distributions where the probability density functions $f_{\theta_k} : [0,1] \to \mathbb{R}_+$, conditional on the state of market k is $\theta_k \in \{H, L\}$, are given by

$$f_H(s) = (1+\alpha)\, s^{\alpha} \qquad \text{and} \qquad f_L(s) = (1+\alpha)(1-s)^{\alpha}$$

for some $\alpha > 0$. The corresponding cumulative distribution functions are

$$F_H(s) = s^{1+\alpha} \qquad \text{and} \qquad F_L(s) = 1 - (1-s)^{1+\alpha}.$$

Conditional on the market state $\theta_k \in \{H, L\}$, players' private values are drawn independently from F_{θ_k}. The parameter α captures the correlation of private values across bidders, so it can be viewed as the continuous counterpart of ρ. When α goes to zero, the private valuations are uncorrelated; when α goes to infinity, the private valuations are perfectly correlated.

As we proceed with private value second-price auctions, bidding the true value upon entry remains weakly dominant. In the following, we visualize a (breakdown or an existence of) monotone entry equilibrium under this environment.

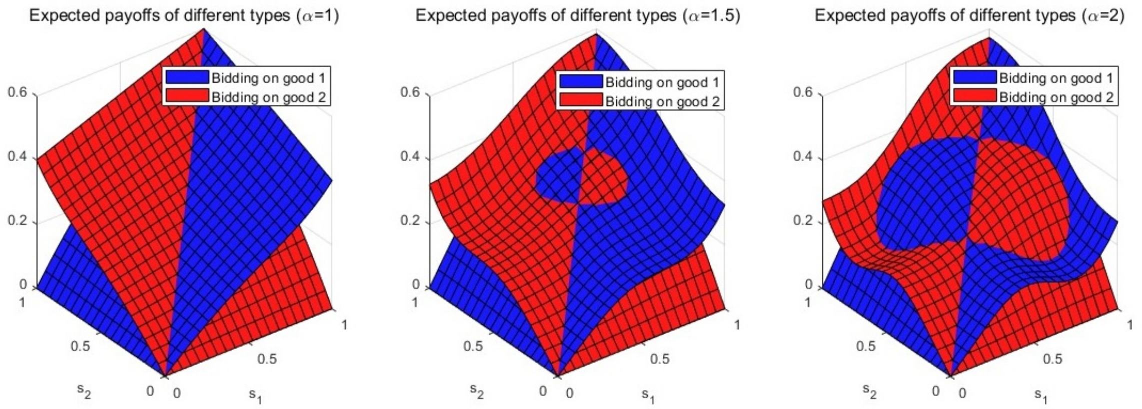

Figure 1.1: Expected payoff of player 1, conditional on player 2 following monotone entry and truthful bidding.

Figure 1 shows player 1's expected payoffs of entering two markets under truthful bidding, given that player 2 follows (i) monotone entry and (ii) truthful bidding. The visualization is conducted under $\alpha \in \{1, 1.5, 2\}$. When the expected payoff of bidding on good 1 (blue) is higher than that of good 2, player 1 should optimally enter market 1, vice versa. Panel 1 demonstrates a monotone entry equilibrium: whenever player 1's valuation of good 1 is greater than good 2 ($s_1 > s_2$), he would always follow the high signal and enter market 1.

Panel 2 and 3 illustrate a breakdown of monotone entry equilibrium. For instance, when player 1's type is $s = (0.6, 0.4)$, the expected payoff of bidding on good 2 is higher. Even though player 1 has a higher valuation for good 1, she has an incentive to switch to the low-value market (market 2). Thus, there does not exist a monotone entry equilibrium under $\alpha \in \{1.5, 2\}$. Furthermore, the deviation region[12] is located at the middle,

[12]In this simulation, the deviation region indicates the set of player 1's types such that it is optimal for her to enter the lower-value market, given that player 2 always enters her higher-value market and bids truthfully.

where the valuations of the two goods are close. This reconfirms the intuition from the last subsection: when the divergence of values of goods is small, the gain from avoiding competition is greater than the loss in valuation when switching market, resulting in a deviation from monotone entry. Building on this observation, we formally establish a necessary condition for the existence of a monotone entry equilibrium.

Proposition 1.6. *In a two-player affiliated private values second-price auction, suppose each signal* $s_k \in [0,1]$ *has conditional densities*

$$f_H(s) = (1+\alpha)s^{\alpha}, \qquad f_L(s) = (1+\alpha)(1-s)^{\alpha},$$

there exists a threshold $\overline{\alpha}$ *such that if* $\alpha > \overline{\alpha}$*, there does not exist any monotone entry equilibrium.*

A monotone equilibrium under affiliated information cannot be sustained when the affiliation is strong, since the incentive to avoid competition is amplified by a high correlation of signals across players. To illustrate, consider the limit case where α goes to infinity. The distributions of signals converge to the following degenerate distributions:

$$F_H(s) = \begin{cases} 1 & \text{if } s \geq 1 \\ 0 & \text{if } s < 1 \end{cases} \qquad \text{and} \qquad F_L(s) = 1 \qquad \text{for all } s \geq 0.$$

The state in each market k is perfectly revealed and players' private signals/valuations are perfectly correlated. Suppose $(\theta_1, \theta_2) = (H, L)$, and so $(s_1^1, s_2^1) = (s_1^2, s_2^2) = (1, 0)$. If all rivals follow monotone entry, the bidder has no incentive to do the same because there is extreme competition in market 1.

Appendix B: Proofs

Proof of Proposition 1.1.

Fix bidder i with type $s^i = (s^i_1, s^i_2)$ and consider market $k \in \{1,2\}$. In a symmetric equilibrium with i.i.d. signals, the distribution of opponents' bids is the same in both markets, and conditional on entering market k, bidder i faces a winning probability $G_k(b)$ and an expected payment $P_k(b)$ that do not depend on s^i_k.

Bidder i therefore chooses a bid to solve

$$V_k(s^i_k) = \max_{b \geq 0} \Pi_k(b; s^i_k), \qquad \Pi_k(b; s^i_k) = s^i_k\, G_k(b) - P_k(b).$$

For any fixed b, the expected payoff $\Pi_k(b; s^i_k)$ is strictly increasing in s^i_k; hence the maximized value $V_k(s^i_k)$ is strictly increasing in s^i_k.

By symmetry of markets and bidders,

$$V_1(\cdot) = V_2(\cdot) \equiv V(\cdot).$$

Thus bidder i prefers market 1 over market 2 iff

$$V(s^i_1) > V(s^i_2),$$

which, by strict monotonicity of V, occurs exactly when $s^i_1 > s^i_2$.

Therefore, in any symmetric equilibrium, each bidder enters the market for which she receives the higher signal; entry is monotone.

Q.E.D.

Proof of Proposition 1.2.

Solving the standard ODE for first-price auctions yields the symmetric bidding function

$$\beta_k^*(s) = \frac{\int_{\underline{s}}^{s} \sum_{n=1}^{N-1} \binom{N-1}{n} nF(x)^{n-1} f(x)\, x\, dx}{\sum_{n=0}^{N-1} \binom{N-1}{n} F(s)^n}.$$

The expected payment in market k under first-price auctions is

$$\begin{aligned}
&\mathbb{E}_{\text{FPA}}\left[\beta_k^*(s_k^i)\mathbf{1}\{s_k^i = \max_{j\in M_k} s_k^j\}\right] \\
&= \left(\frac{1}{2}\right)^{N-1} \sum_{n=0}^{N-1} \binom{N-1}{n} F(s)^{2k} \cdot \frac{2\int_0^s \sum_{n=1}^{N-1} \binom{N-1}{n} nF(x)^{2n-1} f(x)\, x dx}{\sum_{n=0}^{N-1} \binom{N-1}{n} F(s)^{2n}} \\
&= \left(\frac{1}{2}\right)^{n-2} \int_0^s \sum_{n=1}^{N-1} \binom{N-1}{n} nF(x)^{2n-1} f(x)\, x dx.
\end{aligned}$$

Under second-price auctions, conditioning on n rivals in the market, the distribution of the highest rival signal is F^{2n} with density $2nf(x)F(x)^{2n-1}$. Hence the expected payment is

$$\mathbb{E}_{\text{SPA}}\left[\max_{j\neq i} s_k^j \cdot \mathbf{1}\{s_k^i = \max_{j\in M_k} s_k^j\}\right] = \left(\tfrac{1}{2}\right)^{N-2} \int_0^s \sum_{n=1}^{N-1} \binom{N-1}{n} nF(x)^{2n-1} f(x)\, x\, dx.$$

The expressions coincide, so revenue equivalence holds. *Q.E.D.*

Proof of Proposition 1.3.

We will guess and verify the following equilibrium:

1. enters each market with equal probability if and only if $s_1^i = s_2^i$;
2. enters market k with probability h and market $-k$ with probability $1-h$ if and only if $s_k^i > s_{-k}^i$; and
3. bids truthfully after entering market k, i.e. $b_k^i = s_k^i$ and $b_{-k}^i = 0$.

Denote $\omega \in \{H, L\}^2$ the state of the world. Bidders who observe the same signals in both markets randomize because of symmetric markets assumption. We consider the bidders who observe different signals in two markets. Without loss of generality, assume player i's type is $s^i = (s_H, s_L)$. If he chooses to bid on good 1, first consider the winning probability. The probability that j submits the low bid s_L in market 1 is

$$\Pr(b_1^j = s_L \mid \omega) = \begin{cases} (1-h)\rho(1-\rho) + \frac{1}{2}(1-\rho)^2, & \omega = HH, \\ (1-h)(1-\rho)^2 + \frac{1}{2}\rho(1-\rho), & \omega = HL, \\ (1-h)\rho^2 + \frac{1}{2}\rho(1-\rho), & \omega = LH, \\ (1-h)\rho(1-\rho) + \frac{1}{2}\rho^2, & \omega = LL. \end{cases}$$

The probability that player j submits a low bid on good 1 (conditional on the state ω) is

$$\begin{aligned} \Pr\big(b_1^j = s_L \mid \omega\big) &= \Pr(s_j = (s_L, s_H) \mid \omega) \cdot (1-h) + \Pr(s_j = (s_L, s_L) \mid \omega) \cdot \frac{1}{2} \\ &= \begin{cases} (1-h)\,\rho(1-\rho) + \frac{1}{2}(1-\rho)^2, & \omega = HH, \\ (1-h)\,(1-\rho)^2 + \frac{1}{2}\rho(1-\rho), & \omega = HL, \\ (1-h)\,\rho^2 + \frac{1}{2}\rho(1-\rho), & \omega = LH, \\ (1-h)\,\rho(1-\rho) + \frac{1}{2}\rho^2, & \omega = LL. \end{cases} \end{aligned}$$

Winning probability (excluding tie) is

$$
\begin{aligned}
&\Pr\left(s_1^i = s_H > b_1^j \quad \forall j \neq i | s^i\right) \\
&= \sum_{\omega} \Pr\left(\omega | s^i\right) \left[\Pr\left(b_1^j = 0 | \omega\right) + \Pr\left(b_1^j = s_L | \omega\right)\right]^{n-1} \\
&= \rho\left(1-\rho\right) \left[\frac{1}{2} + \left(1-h\right)\rho\left(1-\rho\right) + \frac{1}{2}\left(1-\rho\right)^2\right]^{n-1} \\
&\quad + \rho^2 \left[\rho + h - 2h\rho + \left(1-h\right)\left(1-\rho\right)^2 + \frac{1}{2}\rho\left(1-\rho\right)\right]^{n-1} \\
&\quad + \left(1-\rho\right)^2 \left[-\rho - h + 2h\rho + 1 + \left(1-h\right)\rho^2 + \frac{1}{2}\rho\left(1-\rho\right)\right]^{n-1} \\
&\quad + \rho\left(1-\rho\right) \left[\frac{1}{2} + \left(1-h\right)\rho\left(1-\rho\right) + \frac{1}{2}\rho^2\right]^{n-1} \\
&= \rho\left(1-\rho\right) \left[\rho^2 \left(h - \frac{1}{2}\right) - h\rho + 1\right]^{n-1} \\
&\quad + \rho^2 \left[\rho^2 \left(\frac{1}{2} - h\right) - \frac{1}{2}\rho + 1\right]^{n-1} \\
&\quad + \left(1-\rho\right)^2 \left[\rho^2 \left(\frac{1}{2} - h\right) + \rho\left(2h - \frac{1}{2}\right) - h + 1\right]^{n-1} \\
&\quad + \rho\left(1-\rho\right) \left[\rho^2 \left(h - \frac{1}{2}\right) + \rho\left(1-h\right) + \frac{1}{2}\right]^{n-1}.
\end{aligned}
$$

The expected payoff from entering market 1 is

$$
\begin{aligned}
E_1 = s_H \cdot \Pr\big(b_1^j = 0 \ \forall j \neq i \,\big|\, s^i = (s_H, s_L)\big) \\
+ (s_H - s_L)\Big[\Pr\big(s_H > b_1^j \ \forall j \neq i \mid s^i\big) - \Pr\big(b_1^j = 0 \ \forall j \neq i \,\big|\, s^i = (s_H, s_L)\big)\Big].
\end{aligned}
$$

The expected payoff from entering market 2 is

$$
E_2 = s_L \cdot \Pr\big(b_2^j = 0 \ \forall j \neq i \,\big|\, s^i = (s_H, s_L)\big).
$$

We next show that an equilibrium with parameter h exists and is unique. Fix any player i and suppose all other players $j \neq i$. We compare player i's expected payoffs from entering the two markets.

First, we show that the E_1 and E_2 are strictly decreasing and increasing in h, respectively.

$$\begin{aligned}
&\frac{\partial E_1}{\partial h} \\
&= s_H (n-1) \left(-(2\rho-1)^2\right) \left(2\rho^2 - 2\rho + 1 - (2\rho-1)^2 h\right)^{n-2} \\
&\quad + (s_H - s_L) \sum_{k=1}^{n-1} \binom{n-1}{k-1} (n-k) \left(-4\rho^2 (1-\rho)^2\right) \left(2\rho^4 - 4\rho^3 + \rho^2 + \rho - 4\rho^2 (1-\rho)^2 h\right)^{n-k-1} \left(2\rho^2 - 2\rho + 1 - (2\rho-1)^2 h\right)^{k-1} \\
&\quad + (s_H - s_L) \sum_{k=1}^{n-1} \binom{n-1}{k-1} (k-1) \left(-(2\rho-1)^2\right) \left(2\rho^4 - 4\rho^3 + \rho^2 + \rho - 4\rho^2 (1-\rho)^2 h\right)^{n-k} \left(2\rho^2 - 2\rho + 1 - (2\rho-1)^2 h\right)^{k-2} \\
&= -s_H (n-1) (2\rho-1)^2 \left((2\rho-1)^2 (1-h) + 2\rho (1-\rho)\right)^{n-2} \\
&\quad - (s_H - s_L) \sum_{k=1}^{n-1} \binom{n-1}{k-1} (n-k)\, 4\rho^2 (1-\rho)^2 \left(2\rho^4 - 4\rho^3 + \rho^2 + \rho - 4\rho^2 (1-\rho)^2 h\right)^{n-k-1} \left((2\rho-1)^2 (1-h) + 2\rho (1-\rho)\right)^{k-1} \\
&\quad - (s_H - s_L) \sum_{k=1}^{n-1} \binom{n-1}{k-1} (k-1) (2\rho-1)^2 \left(2\rho^4 - 4\rho^3 + \rho^2 + \rho - 4\rho^2 (1-\rho)^2 h\right)^{n-k} \left((2\rho-1)^2 (1-h) + 2\rho (1-\rho)\right)^{k-2} \\
&< 0
\end{aligned}$$

$$\frac{\partial E_2}{\partial h} = s_L (n-1) (2\rho-1)^2 \left(2\rho (1-\rho) + (2\rho-1)^2 h\right) > 0$$

Second, we show that when $h = 0$, $E_1 > E_2$. When $h = 0$,

$$\begin{aligned}
E_1 = {} & s_H \left(\rho^2 + (1-\rho)^2\right)^{n-1} \\
& + (s_H - s_L) \sum_{k=1}^{n-1} \binom{n-1}{k-1} \left(4\rho^2 (1-\rho)^2 + \rho (1-\rho) \left(\rho^2 + (1-\rho)^2\right)\right)^{n-k} \left(\rho^2 + (1-\rho)^2\right)^{k-1};
\end{aligned}$$

and

$$E_2 = s_L (2\rho (1-\rho))^{n-1} < s_H \left(\rho^2 + (1-\rho)^2\right)^{n-1} < E_1$$

Finally, consider the boundary case $h = 1$. At $h = 1$, two cases arise:

- If $E_1 > E_2$ at $h = 1$, then by monotonicity $E_1 > E_2$ for every $h \in [0, 1]$, and the unique equilibrium is the monotone entry equilibrium $h^* = 1$.

- If $E_1 < E_2$ at $h = 1$, then continuity and strict monotonicity imply, by the Intermediate Value Theorem, a unique solution

$$h^* \in (0,1) \quad \text{satisfying } E_1 = E_2.$$

We proved the existence and uniqueness of h^*. Now we show that $h^* > \frac{1}{2}$ in equilibrium. When $h = \frac{1}{2}$, $E_1 > E_2$:

$$\begin{aligned} E_1 = {} & s_H \left(\rho^2 + (1-\rho)^2 - (2\rho-1)^2 \frac{1}{2}\right)^{n-1} \\ & + (s_H - s_L) \sum_{k=1}^{n-1} \binom{n-1}{k-1} \left(4\rho^2 (1-\rho)^2 + \rho(1-\rho)\left(\rho^2 + (1-\rho)^2\right) - 2\rho(1-\rho)^2\right)^{n-k} \\ & \left(\rho^2 + (1-\rho)^2 - \frac{1}{2}(2\rho-1)^2\right)^{k-1}; \end{aligned}$$

and

$$E_2 = s_L \left(2\rho(1-\rho) + (2\rho-1)^2 \frac{1}{2}\right)^{n-1} < s_H \left(\rho^2 + (1-\rho)^2 - (2\rho-1)^2 \frac{1}{2}\right)^{n-1} < E_1$$

Hence, either $E_1 = E_2$ for a unique $h \in \left(\frac{1}{2}, 1\right]$ or $E_1 > E_2$ for all $h \in \left(\frac{1}{2}, 1\right]$, imply $h^* \in \left(\frac{1}{2}, 1\right]$ or $h^* = 1$ respectively. Next, we find the threshold r such that if $\frac{s_H}{s_L} > r$, there is the pure monotone entry equilibrium ($h = 1$). Take $h = 1$, we have

$$\begin{aligned} E_1 = {} & s_H \left(\rho^2 + (1-\rho)^2 - (2\rho-1)^2\right)^{n-1} \\ & + (s_H - s_L) \sum_{k=1}^{n-1} \binom{n-1}{k-1} \left(4\rho^2 (1-\rho)^2 + \rho(1-\rho)\left(\rho^2 + (1-\rho)^2\right) - 4\rho^2(1-\rho)^2\right)^{n-k} \\ & \left(\rho^2 + (1-\rho)^2 - (2\rho-1)^2\right)^{k-1} \\ = {} & s_H (2\rho(1-\rho))^{n-1} + (s_H - s_L) \sum_{k=1}^{n-1} \binom{n-1}{k-1} \left(\rho(1-\rho)\left(\rho^2 + (1-\rho)^2\right)\right)^{n-k} (2\rho(1-\rho))^{k-1}; \end{aligned}$$

and

$$E_2 = s_L \left(2\rho(1-\rho) + (2\rho-1)^2\right)^{n-1} = s_L \left(\rho^2 + (1-\rho)^2\right)^{n-1}.$$

Put $E_1 = E_2$, we find the threshold

$$r(\rho, n) = \frac{s_H}{s_L} = \frac{\left(\rho^2 + (1-\rho)^2\right)^{n-1} + \sum_{k=1}^{n-1} \binom{n-1}{k-1} \left(\rho(1-\rho)\left(\rho^2 + (1-\rho)^2\right)\right)^{n-k} (2\rho(1-\rho))^{k-1}}{(2\rho(1-\rho))^{n-1} + \sum_{k=1}^{n-1} \binom{n-1}{k-1} \left(\rho(1-\rho)\left(\rho^2 + (1-\rho)^2\right)\right)^{n-k} (2\rho(1-\rho))^{k-1}}.$$

If $\frac{s_H}{s_L} > r(\rho, n)$, then $E_1 > E_2$ at $h = 1$, and the unique equilibrium is the pure monotone entry equilibrium. *Q.E.D.*

Proof of Proposition 1.4. Define

$$C_k := \{i : \beta_2^{-k}(s^i) = 0 \text{ for all } s^i\}.$$

If $N > 2$, then $|C_1| > 1$ or $|C_2| > 1$. WLOG, suppose $|C_1| > 1$ and consider $i \in C_1$. Since i bids 0 in market 2 for all types, a bidder with type $s_1^i = \underline{s}$ receives zero surplus in market 1, and can profitably deviate to market 2. Thus such a configuration cannot be an equilibrium. *Q.E.D.*

Proof of Proposition 1.5.

Denote $\omega \in \{H, L\}^2$ the state of the world. Suppose there is a monotone entry equilibrium such that player i enters market k if $s_k^i > s_{-k}^i$ and randomizes between two markets with equal probability if $s_k^i = s_{-k}^i$. Probability of player j entering market 2 (conditional on state) is

$$
\begin{aligned}
&\Pr\left(b_1^j = 0|\omega\right) \\
=\ &\Pr\left(s_1^j < s_2^j|\omega\right) + \frac{1}{2}\Pr\left(s_1^j = s_2^j|\omega\right) \\
=\ &\sum_{l'=2}^{m}\sum_{l=1}^{l'-1}\Pr\left(s_1^j = s_l|\omega\right)\cdot\Pr\left(s_2^j = s_{l'}|\omega\right) + \frac{1}{2}\sum_{l=1}^{m}\Pr\left(s_1^j = s_l|\omega\right)\cdot\Pr\left(s_2^j = s_l|\omega\right).
\end{aligned}
$$

Probability of player j bidding s_l in market 1 (conditional on state) is

$$
\begin{aligned}
&\Pr\left(b_1^j = s_l|\omega\right) \\
=\ &\Pr\left(s_1^j = s_l, s_2^j < s_l|\omega\right) + \frac{1}{2}\Pr\left(s_1^j = s_2^j = s_l|\omega\right) \\
=\ &\Pr\left(s_1^j = s_l|\omega\right)\Pr\left(s_2^j < s_l|\omega\right) + \frac{1}{2}\Pr\left(s_1^j = s_l|\omega\right)\Pr\left(s_2^j = s_l|\omega\right) \\
=\ &\Pr\left(s_1^j = s_l|\omega\right)\left[\sum_{l'=1}^{l-1}\Pr\left(s_2^j = s_{l'}|\omega\right) + \frac{1}{2}\Pr\left(s_2^j = s_l|\omega\right)\right].
\end{aligned}
$$

First, consider (s_2, s_1). The incentive constraint reads:

$$
\sum_{\omega}\Pr\left(\omega|s^i = (s_2, s_1)\right)\left\{\overbrace{s_2\left[Pr\left(b_1^j = 0|\omega\right) + \Pr\left(b_1^j = s_1|\omega\right)\right]^{N-1}}^{WP_1|_\omega(\text{lose when tie})} - \overbrace{s_1\left[\Pr\left(b_1^j \leq s_1|\omega\right)^{N-1} - \Pr\left(b_1^j = 0|\omega\right)^{N-1}\right]}^{EP_1|_\omega}\right\}
$$

$$
> s_1\sum_{\omega}\Pr\left(\omega|s^i = (s_2, s_1)\right)\underbrace{\left[1 - Pr\left(b_1^j = 0|\omega\right)\right]^{N-1}}_{WP_2|_\omega(\text{lose when tie})}
$$

$$
\begin{aligned}
&s_2\sum_{\omega}\Pr\left(\omega|s^i = (s_2, s_1)\right)\left[Pr\left(b_1^j = 0|\omega\right) + \Pr\left(b_1^j = s_1|\omega\right)\right]^{N-1} \\
&-s_1\sum_{\omega}\Pr\left(\omega|s^i = (s_2, s_1)\right)\left[\Pr\left(b_1^j \leq s_1|\omega\right)^{N-1} - \Pr\left(b_1^j = 0|\omega\right)^{N-1} + \left[1 - Pr\left(b_1^j = 0|\omega\right)\right]^{N-1}\right] > 0
\end{aligned}
$$

$$
\Rightarrow \quad \frac{s_2}{s_1} > \frac{\sum_{\omega}\Pr\left(\omega|s^i = (s_2, s_1)\right)\left[\Pr\left(b_1^j \leq s_1|\omega\right)^{N-1} - \Pr\left(b_1^j = 0|\omega\right)^{N-1} + \left[1 - Pr\left(b_1^j = 0|\omega\right)\right]^{N-1}\right]}{\underbrace{\sum_{\omega}\Pr\left(\omega|s^i = (s_2, s_1)\right)\left[Pr\left(b_1^j = 0|\omega\right) + \Pr\left(b_1^j = s_1|\omega\right)\right]^{N-1}}_{WP_1}}.
$$

Then, consider a general case (s_h, s_l) where $h > l$. The incentive constraint reads:

$$\sum_\omega \Pr\left(\omega|s^i=(s_h,s_l)\right)\left\{s_h \overbrace{\left[\Pr\left(b_1^j=0|\omega\right)+\sum_{l'=1}^{h-1}\Pr\left(b_1^j=s_{l'}|\omega\right)\right]^{N-1}}^{WP_1|_\omega} \overbrace{-\sum_{l'=1}^{h-1}s_{l'}\left[\Pr\left(b_1^j\leq s_{l'}|\omega\right)^{N-1}-\Pr\left(b_1^j<s_{l'}|\omega\right)^{N-1}\right]}^{EP_1|_\omega}\right\}$$
$$>\sum_\omega \Pr\left(\omega|s^i=(s_h,s_l)\right)\left\{s_l \underbrace{\left[1-Pr\left(b_1^j=0|\omega\right)+\sum_{l'=1}^{l-1}\Pr\left(b_2^j=s_{l'}|\omega\right)\right]^{N-1}}_{WP_2|_\omega} \underbrace{-\sum_{l'=1}^{l-1}s_{l'}\left[\Pr\left(b_2^j\leq s_{l'}|\omega\right)^{N-1}-\Pr\left(b_2^j<s_{l'}|\omega\right)^{N-1}\right]}_{EP_2|_\omega}\right\}$$

$$\Rightarrow \quad \frac{s_h}{s_1} > \frac{\begin{array}{c}\frac{s_l}{s_1}\overbrace{\sum_\omega \Pr\left(\omega|s^i=(s_h,s_l)\right)\left[1-Pr\left(b_1^j=0|\omega\right)+\sum_{l'=1}^{l-1}\Pr\left(b_2^j=s_{l'}|\omega\right)\right]^{N-1}}^{WP_2} \\ \underbrace{-\sum_\omega\sum_{l'=1}^{l-1}\frac{s_{l'}}{s_1}\Pr\left(b_1^j=s_{l'}|\omega\right)\Pr\left(\omega|s^i=(s_h,s_l)\right)}_{EP_2}+\underbrace{\sum_\omega\sum_{l'=1}^{h-1}\frac{s_{l'}}{s_1}\Pr\left(b_1^j=s_{l'}|\omega\right)\Pr\left(\omega|s^i=(s_h,s_l)\right)}_{EP_1}\end{array}}{\underbrace{\sum_\omega \Pr\left(\omega|s^i=(s_h,s_l)\right)\left[\Pr\left(b_1^j=0|\omega\right)+\sum_{l'=1}^{h-1}\Pr\left(b_1^j=s_{l'}|\omega\right)\right]^{N-1}}_{WP_1}}.$$

At last, by induction, we can write the IC constraint for each pair of values $(s_1, s_2) \in \{S\}^2$. There will be $m(m-1)/2$ inequalities in total. These inequalities are the necessary and sufficient conditions for the existence of a monotone entry equilibrium.

We define the following functions to complete the characterization. Let

$$\kappa_2(\Gamma) := \frac{\sum_\omega \Pr\left(\omega|s^i=(s_2,s_1)\right)\left[\Pr\left(b_1^j\leq s_1|\omega\right)^{N-1}-\Pr\left(b_1^j=0|\omega\right)^{N-1}+\left[1-Pr\left(b_1^j=0|\omega\right)\right]^{N-1}\right]}{\sum_\omega \Pr\left(\omega|s^i=(s_2,s_1)\right)\left[Pr\left(b_1^j=0|\omega\right)+\Pr\left(b_1^j=s_1|\omega\right)\right]^{N-1}}$$

$$\kappa_{hl} := \frac{\begin{array}{c}\frac{s_l}{s_1}\sum_\omega \Pr\left(\omega|s^i=(s_h,s_l)\right)\left[1-Pr\left(b_1^j=0|\omega\right)+\sum_{l'=1}^{l-1}\Pr\left(b_2^j=s_{l'}|\omega\right)\right]^{N-1} \\ -\sum_\omega\sum_{l'=1}^{l-1}\frac{s_{l'}}{s_1}\Pr\left(b_1^j=s_{l'}|\omega\right)\Pr\left(\omega|s^i=(s_h,s_l)\right)+\sum_\omega\sum_{l'=1}^{h-1}\frac{s_{l'}}{s_1}\Pr\left(b_1^j=s_{l'}|\omega\right)\Pr\left(\omega|s^i=(s_h,s_l)\right)\end{array}}{\sum_\omega \Pr\left(\omega|s^i=(s_h,s_l)\right)\left[\Pr\left(b_1^j=0|\omega\right)+\sum_{l'=1}^{h-1}\Pr\left(b_1^j=s_{l'}|\omega\right)\right]^{N-1}}$$

$$\kappa_h\left(\frac{s_2}{s_1},\frac{s_3}{s_1},\ldots,\frac{s_{l-1}}{s_1};\Gamma\right) := \max_l \kappa_{hl}.$$

Q.E.D.

Proof of Proposition 1.6.

Suppose the state ω is known by the players. Assume player j follows monotone entry and bids truthfully. Winning probability of entering market 1 is

$$\begin{aligned} & WP_1\left(s^i\right)|_\omega \\ = & \Pr\left(s_2^j > s_1^j|\omega\right) + \Pr\left(s_1^i > s_1^j > s_2^j|\omega\right) \\ = & F_1\left(s_1\right)F_2\left(s_1\right) + \int_{s_1}^1 F_1\left(x\right)f_2\left(x\right)dx. \end{aligned}$$

The marginal change in WP w.r.t. s_1 is

$$\frac{\partial WP_1\left(s^i\right)|_\omega}{\partial s_1} = f_1\left(s_1\right)F_2\left(s_1\right) + F_1\left(s_1\right)f_2\left(s_1\right) - F_1\left(s_1\right)f_2\left(s_1\right) = f_1\left(s_1\right)F_2\left(s_1\right).$$

Let $\mathcal{F}\left(s\right) := \int_0^s F\left(x\right)dx$ be the integral of the CDF F. Expected payment in market 1 is

$$\begin{aligned} & EP_1\left(s^i\right)|_\omega \\ = & \Pr\left(s_1^i > s_1^j > s_2^j|\omega\right)\mathbb{E}\left[s_1^j|s_1^i > s_1^j > s_2^j, \omega\right] \\ = & \int_0^{s_1}\int_s^{s_1} xdF_1\left(x\right)dF_2\left(s\right) \\ = & s_1F_1\left(s_1\right)F_2\left(s_1\right) - \int_0^{s_1} xF_1\left(x\right)f_2\left(x\right)dx - \int_0^{s_1} F_1\left(x\right)F_2\left(x\right)dx. \end{aligned}$$

The marginal change in EP w.r.t. s_1 is

$$\begin{aligned} & \frac{\partial EP_1\left(s^i\right)|_\omega}{\partial s_1} \\ = & F_1\left(s_1\right)F_2\left(s_1\right) + s_1f_1\left(s_1\right)F_2\left(s_1\right) + s_1F_1\left(s_1\right)f_2\left(s_1\right) - s_1F_1\left(s_1\right)f_2\left(s_1\right) - F_1\left(s_1\right)F_2\left(s_1\right) \\ = & s_1f_1\left(s_1\right)F_2\left(s_1\right). \end{aligned}$$

The expected payoff in market 1 is

$$E_1\left(s^i\right)|_\omega := WP_1\left(s^i\right)|_\omega \cdot s_1 - EP_1\left(s^i\right)|_\omega$$
$$= s_1 \int_{s_1}^{1} F_1(x)\, dF_2(x) + \int_0^{s_1} xF_1(x)\, dF_2(x) + \int_0^{s_1} F_1(x)\, F_2(x)\, dx.$$

The marginal change in expected payoff w.r.t. s_1 is

$$\frac{\partial E_1(s^i)|_\omega}{\partial s_1} = \frac{\partial WP_1(s^i)|_\omega}{\partial s_1} \cdot s_1 + WP_1\left(s^i\right)|_\omega - \frac{\partial EP_1(s^i)|_\omega}{\partial s_1}$$
$$= s_1 f_1(s_1)\, F_2(s_1) + WP_1\left(s^i\right)|_\omega - s_1 f_1(s_1)\, F_2(s_1)$$
$$= WP_1\left(s^i\right)|_\omega$$
$$= F_1(s_1)\, F_2(s_1) + \int_{s_1}^{1} F_1(x)\, f_2(x)\, dx.$$

Now consider the case that the players do not know the state. The expected payoff of entering market 1 is

$$E_1 = \sum_\omega \Pr\left(\omega|s^i\right)\left(WP|_\omega \cdot s_1 - EP|_\omega\right).$$

The marginal change in E_1 w.r.t. s_1 is

$$\frac{\partial E_1}{\partial s_1} = \sum_\omega \left[\frac{\partial \Pr(\omega|s^i)}{\partial s_1} E_1|_\omega + \Pr\left(\omega|s^i\right) \frac{\partial E_1|_w}{\partial s_1}\right].$$

When $s_1 = s_2 = s$, the expected payoffs from two markets are the same, i.e. $E_1 = E_2$. In a monotone entry equilibrium, it must be that $\frac{\partial E_1}{\partial s_1}|_{s_1=s_2=s} > \frac{\partial E_2}{\partial s_1}|_{s_1=s_2=s}$. For the following analysis, I always look at the diagonal ($s_1 = s_2 = s$), hence $|_{s_1=s_2=s}$ is omitted whenever there is no confusion to simplify notation.

$$
\begin{aligned}
& \left.\frac{\partial E_1}{\partial s_1}\right|_{s_1=s_2=s} - \left.\frac{\partial E_2}{\partial s_1}\right|_{s_1=s_2=s} \\
= & \left.\frac{\partial E_1}{\partial s_1}\right|_{s_1=s_2=s} - \left.\frac{\partial E_1}{\partial s_2}\right|_{s_1=s_2=s} \\
= & \sum_{\omega} \left[\left(\left.\frac{\partial \Pr(\omega|s^i)}{\partial s_1}\right|_{s_1=s_2=s} - \left.\frac{\partial \Pr(\omega|s^i)}{\partial s_2}\right|_{s_1=s_2=s} \right) E_1|_{\omega} + \Pr\left(\omega|s^i\right) \left(\overbrace{\frac{\partial E_1|_w}{\partial s_1}}^{WP|_{\omega}} - \overbrace{\frac{\partial E_1|_w}{\partial s_2}}^{=0} \right) \right] \\
= & \underbrace{\frac{f_H'(s) f_L(s) - f_H(s) f_L'(s)}{\left(f_H(s) + f_L(s)\right)^2}}_{>0} \left(\underbrace{E_1|_{HL} - E_1|_{LH}}_{<0} \right) + WP
\end{aligned}
$$

$$
\begin{aligned}
& E_1|_{LH} \\
= & \left(F_H(s) F_L(s) + \int_s^1 F_L(x) f_H(x)\, dx \right) \cdot s - s F_H(s) F_L(s) \\
& + \int_0^s x F_L(x) f_H(x)\, dx + \int_0^s F_L(x) F_H(x)\, dx \\
= & \; s \int_s^1 F_L(x) f_H(x)\, dx + \int_0^s x F_L(x) f_H(x)\, dx + \int_0^s F_L(x) F_H(x)\, dx
\end{aligned}
$$

$$
\begin{aligned}
& E_1|_{LH} - E_1|_{HL} \\
= & \; s \left(\int_s^1 F_L(x) f_H(x)\, dx - \int_s^1 F_H(x) f_L(x)\, dx \right) + \int_0^s x F_L(x) f_H(x)\, dx - \int_0^s x F_H(x) f_L(x)\, dx \\
= & \; s \left(\int_s^1 \underbrace{\left[F_L(x) f_H(x) - F_H(x) f_L(x)\right]}_{>0 \quad (MLRP)} dx \right) + \int_0^s x \underbrace{\left[F_L(x) f_H(x) - F_H(x) f_L(x)\right]}_{>0 \quad (MLRP)} dx \quad > 0
\end{aligned}
$$

Take the following distribution of values:

$$\begin{aligned} f_H(s) &= (1+\alpha) s^{\alpha} \\ f_L(s) &= (1+\alpha)(1-s)^{\alpha} \\ F_H &= s^{1+\alpha} \\ F_L &= 1-(1-s)^{1+\alpha} \end{aligned}$$

Consider the type $s_1 = s_2 = 0.5$. We proceed to show the followings.

1. $\frac{\partial E_1}{\partial s_1}\big|_{s_1=s_2=0.5} - \frac{\partial E_2}{\partial s_1}\big|_{s_1=s_2=0.5}$ is decreasing in α.
2. There exists $\overline{\alpha}$ such that $\frac{\partial E_1}{\partial s_1}\big|_{s_1=s_2=0.5} - \frac{\partial E_2}{\partial s_1}\big|_{s_1=s_2=0.5} < 0$.
3. For all $\alpha > \overline{\alpha}$, $\frac{\partial E_1}{\partial s_1}\big|_{s_1=s_2=0.5} - \frac{\partial E_2}{\partial s_1}\big|_{s_1=s_2=0.5} < 0$, thus there does not exist any monotone entry equilibrium.

The difference in conditional expected payoff (HL and LH)

$$\begin{aligned} & E_1|_{LH} - E_1|_{HL} \\ &= 0.5\left(\int_{0.5}^{1} (1+\alpha) x^{\alpha} \left[1-(1-x)^{\alpha}\right] dx\right) + \int_{0}^{0.5} (1+\alpha) x^{1+\alpha} \left[1-(1-x)^{\alpha}\right] dx \end{aligned}$$

Winning probability of entering market 1 is

$$\begin{aligned} & WP_1\left(s^i\right)\big|_{s_1=s_2=0.5} \\ &= \sum_{\omega} \Pr\left(\omega | s_1 = s_2 = 0.5\right) \left[F_{\omega 1}(0.5) F_{\omega 2}(0.5) + \int_{0.5}^{1} F_{\omega 1}(x) f_{\omega 2}(x) dx\right] \\ &= \frac{1}{4}\left[1 + (1+\alpha)\int_{0.5}^{1} \left(x^{\alpha} + (1-x)^{\alpha}\right)\left(1 + x^{1+\alpha} - (1-x)^{1+\alpha}\right) dx\right] \\ &= \frac{5}{8}. \end{aligned}$$

Since

$$\begin{aligned}
&\frac{\partial}{\partial \alpha}\left(\left.\frac{\partial E_1}{\partial s_1}\right|_{s_1=s_2=0.5}-\left.\frac{\partial E_2}{\partial s_1}\right|_{s_1=s_2=0.5}\right)\\
&=\frac{\partial}{\partial \alpha}\left(\frac{f_H'(s)\,f_L(s)-f_H(s)\,f_L'(s)}{\left(f_H(s)+f_L(s)\right)^2}\left(E_1|_{HL}-E_1|_{LH}\right)+WP\right)\\
&=\underbrace{\left(E_1|_{HL}-E_1|_{LH}\right)}_{<0}\cdot\underbrace{\frac{\partial}{\partial \alpha}\left(\frac{f_H'(s)\,f_L(s)-f_H(s)\,f_L'(s)}{\left(f_H(s)+f_L(s)\right)^2}\right)}_{>0}\\
&\quad+\underbrace{\frac{f_H'(s)\,f_L(s)-f_H(s)\,f_L'(s)}{\left(f_H(s)+f_L(s)\right)^2}}_{>0}\cdot\frac{\partial}{\partial \alpha}\underbrace{\left(E_1|_{HL}-E_1|_{LH}\right)}_{<0}+\underbrace{\frac{\partial}{\partial \alpha}WP}_{=0}\\
&<0,
\end{aligned}$$

it follows that

$$\left.\frac{\partial E_1}{\partial s_1}\right|_{s_1=s_2=0.5}-\left.\frac{\partial E_2}{\partial s_1}\right|_{s_1=s_2=0.5}$$

is decreasing in α.

Since at $\alpha = 1.5$ this difference is already negative, it must remain negative for all $\alpha > 1.5$. Hence, for any $\alpha > 1.5$, the condition required for monotone entry fails, and no monotone entry equilibrium exists.

Q.E.D.

Chapter 2

Expert Service Provided by an Informed Expert

Abstract

An expert is privately informed about her skill level, while a consumer faces a problem whose difficulty is known only to him. The problem is solved if and only if the expert's expertise exceeds the problem's difficulty. Knowing her own type, the expert sets a price, and the consumer then chooses whether to purchase the service. When the expert has only two possible types, a pooling equilibrium always maximizes her ex-ante profit. However, with more than two types, the expert can sometimes earn higher profits by optimally grouping her types to better segment demand. The welfare-optimal equilibrium is always fully separating: all the expert information is revealed and the prices are low.

2.1 Introduction

In many expert service markets, the expert's ability is unclear. Even when consumers fully understand their own needs, they remain uncertain whether the expert is capable of addressing their specific problems. For example, patients recognize the value of an accurate diagnosis but cannot easily assess a physician's true capability. Similar dynamics arise in financial advising, legal consulting, and technical support, where consumers know their valuations conditional on success, but the expert's competence remains private information. These markets also share a feature central to our analysis: the marginal cost of delivering expert advice is typically small, making posted prices a natural way for experts to convey information about their capability.

We introduce a bilateral trade model in which the expert is privately informed about her level of expertise. A consumer, facing a problem, considers purchasing the expert's service. The consumer's problem is solved if and only if the expert's expertise exceeds the problem's difficulty level. After the expert's type is realized, she makes a take-it-or-leave-it price offer. The consumer observes the posted price and then decides whether to buy the service. If the problem is solved, the consumer receives the value of resolution minus the price; otherwise, the consumer incurs a loss equal to the price paid. The expert incurs no cost in providing the service.

We study Perfect Bayesian equilibrium in pure strategies. The posted price serves as a signal of the expert's type, and the consumer updates his belief after observing it. Any pricing strategy induces a partition of the expert's type space into pools, with all types in a given pool posting the same price. A key observation is that, in equilibrium, all types must earn the same expected profit: if a type received a strictly lower profit, she could costlessly mimic a more profitable type by posting that type's price. Our first result shows that any partition of types can be supported in equilibrium. For any proposed pooling structure, one can construct a pricing strategy under which all on-path prices deliver the same profit, and pessimistic off-path beliefs ensure that no deviation is profitable.

A key insight of this paper is that fully pooling equilibria may not yield the highest profit for the expert, challenging the conventional view that an informed seller benefits most by withholding private information. The intuition is that a proper partition of types allows for better market segmentation and more effective consumer targeting. This motivates the following questions: When is full pooling optimal for the expert? Which type partition generates the highest equilibrium profit?

Finally, we compare welfare across the fully separating and fully pooling equilibria. A fully separating equilibrium yields the lowest profit for the expert, but generates higher welfare relative to a pooling equilibrium. The low expert profit arises from the fact that all types must earn the same payoff in equilibrium, which is bounded by the payoff of the lowest-type expert. This makes the fully separating outcome the least favorable from the expert's perspective. However, expertise is perfectly revealed through prices in a separating equilibrium, and the resulting price level is lower. This ensures that all trades are efficient and facilitates a greater volume of trade. As a result, total welfare improves. Transitioning from a pooling to a separating equilibrium reallocates surplus from the expert to the consumer, enhancing efficiency at the cost of expert rents.

Our paper enriches the theory of markets for expert advice, relevant to consulting, repair services, and medical diagnostics, by showing how hidden information about capability can create inefficiencies even in the absence of moral hazard. First, we demonstrate substantial equilibrium multiplicity: any partition of types can be sustained through appropriate off path beliefs. Second, we characterize the expert optimal equilibrium and show that, unlike in standard signaling models, prices decrease with expertise in any separating equilibrium because all types must earn the same profit. Third, we show that full pooling can be strictly suboptimal for the expert. Partial pooling allows the expert to segment demand and serve high-valuation consumer more effectively. Taken together, these results provide a new perspective that connects signaling models with bilateral trade environments and contributes to the broader literature on informed principals.

The remainder of the paper is structured as follows. Section 2.2 reviews the related literature. Section 2.3 presents the model. Section 2.4 analyzes the benchmark case with known expertise. Section 2.5 examines the equilibrium with unknown expertise. Section 3.6 concludes.

2.2 Related Literature

The literature on informed sellers, originating with Akerlof (1970), has extensively examined how sellers reveal or signal product quality before setting a price. In particular, our framework relates closely to models where price itself serves as a signaling device. Wolinsky (1983), Milgrom and Roberts (1986), and Bagwell and Riordan (1991) explore how informed sellers signal product quality through pricing. Milgrom and Roberts (1986) includes both price and advertising as signals for the product quality; while Bagwell and Riordan (1991) consider a dynamic pricing problem.[1] Nonetheless, they all confirm the existence of a separating equilibrium where a high price signals a high quality. By contrast, we show that a welfare-dominant separating equilibrium features decreasing price. That is, a low price signals a high quality.

This paper relates to the literature on informed principal models, where the principal holds private information relevant to contracting. Seminal contributions include Myerson (1983), Maskin and Tirole (1990), and Maskin and Tirole (1992). Our framework is particularly close to Maskin and Tirole (1992), where the agent's payoff depends directly on the principal's type. They show that when the principal's type affects outcomes and preferences, equilibrium contracts may fail to separate types due to incentive constraints on the informed party. Nishimura (2022) extends this to bilateral trade, showing that implementable mechanisms must deliver type-independent payoffs for the seller, restricting separation and highlighting the importance of pooling.

Most of the literature models expert services as a credence good, where consumer lacks prior information about their valuation of the good, while the seller is informed. The information asymmetry in credence goods markets is typically one-sided. Dulleck and Kerschbamer (2006) and Balafoutas and Kerschbamer (2020) provide comprehensive surveys of such markets. To complement existing studies, this paper adopts a different modeling approach. We assume that consumer knows exactly what his problem is and the

[1] The takeover-signaling framework by Ekmekci and Kos (2014) also studies price as a signal in a bilateral-trade environment with an informed principal.

potential payoff from having it solved. However, he is uncertain about the expert's ability to solve the problem. We embed Garicano (2000)'s model of expertise into a bilateral trade setting: a problem is only solved if the expert's ability exceeds the complexity of the problem. At the same time, the expert has no prior knowledge of the consumer's specific problem. We therefore study a bilateral trade setting with two-sided asymmetric information.

In addition to the information structure, this model features a distinctive utility function. In most of the bilateral trade and industrial organization literature, the consumer derives utility from receiving the good, with their utility depending on the product's quality and their individual preferences. These preferences are typically modeled through vertical differentiation, horizontal differentiation (à la Hotelling (1929)), or a combination of both, and the utility function is usually continuous in these dimensions. In contrast, our study focuses on a service provision context where the consumer only gains utility if the product type exceeds his own type. This creates a discrete jump in utility at a critical threshold.

2.3 Model

An expert (she) offers her services to a consumer. Her expertise level, $e \in [\underline{e}, 1]$, is privately known, where $\underline{e} \in (0,1)$. A consumer (he) has a problem of difficulty (and value[2]) $v \in [0,1]$, drawn from distribution $F_b \in \Delta[0,1]$. The consumer's prior belief about the expertise e is captured by $F_s \in \Delta\left[\underline{e}, 1\right]$. Both F_b and F_s are common knowledge.

After the expertise level is realized, the expert posts a price $p \in [0,1]$. Denote the pricing strategy by $P : [\underline{e}, 1] \to [0,1]$. Note that $P(\cdot)$ also serves as a signaling device. The consumer updates his belief about the expert's type after observing the price. The consumer buys the service if and only if his expected utility is non-negative. The consumer's

[2] We assume that solving a more difficult problem yields a higher value to the consumer. For simplicity, we take the value to be equal to the level of difficulty.

payoff from buying the service from the expert with type e at price p is

$$u(v, e, p) = v \cdot \mathbf{1}_{e \geq v} - p.$$

Suppose the consumer's posterior belief about the expert's type is G. Then, a type-v consumer's expected value of the service is $EV(v|G) = v(1 - G(v))$. A type-v consumer buys if $EV(v|G) \geq p$. The expert has no production cost[3], and thus obtains profit

$$\pi(p) = p \Pr(p \leq EV(v|G)).$$

Note that the expert's profit does not directly depend on her expertise e, but rather on the consumer's belief about her expertise G.

The timing is as follows. In the first stage, the expert draws her type $e \sim F_s$ and posts a price $p = P(e)$. In the second stage, the consumer observes the price p and makes buying decisions.

2.4 Benchmark: Known Expertise

We first analyze the benchmark case in which the expert's type is common knowledge. Then the posted price does not convey any information, but simply maximizes the expert's profit.

Since the expertise is commonly known, the consumer will only buy when (1) they know that the service is useful for them, i.e. $v \leq e$, and (2) the price is lower than the value of the expert service, i.e. $p \leq v$. Hence, a type-v consumer buys the service if and only if $p \leq v \leq e$. The expert solves

$$\max_{p \in [0,1]} \quad p(F_b(e) - F_b(p)).$$

[3]This normalization is natural for expert services such as medical advice, legal consultation, or technical troubleshooting, where the marginal cost of providing the service is negligible relative to the informational component.

Lemma 2.1. *When the expertise e is commonly known, the optimal price $P(e)$ and the profit $\pi^*(e)$ increase in e.*

Proof. See appendix.

Lemma 2.1 confirms that a more capable expert charges a higher price and makes a higher profit when her expertise level is commonly known. The intuition is simple: a higher-skilled expert can solve more problems and thus faces higher demand. However, this result relies crucially on the absence of any signaling motive. In the next section, we show that the relationship reverses when the expert's type is private.

2.5 Unknown Expertise

With the expert's type e being her private information, the posted price also carries a signaling effect. Taking the expert's pricing strategy as given, the consumer infers some information about the expertise from the observed price.

We focus on Perfect Bayesian equilibria in pure strategies.

Definition 2.1. *A Perfect Bayesian equilibrium consists of expert's pricing strategy P, the consumer's buying strategy and the consumer's belief system. For any type e expert, the pricing strategy $P(\cdot)$ maximizes her profit given the consumer's buying strategy; the consumer's buying strategy maximizes his expected payoff given $P(\cdot)$ and his belief about the expert's type. The consumer's belief about the expert's type is updated according to the expert's strategy and Baye's rule whenever possible.*

To begin, we introduce the following Lemma.

Lemma 2.2. *Any on-equilibrium-path price p yields the same profit.*

Since any type of expert can freely announce a price that is posted by some other types of expert, and the profit does not directly depend on the expertise, any on-equilibrium-path price must yield the same profit. Otherwise, if some price were to yield strictly lower profit, the types posting that price would have an incentive to deviate to the more profitable one.

In equilibrium, the pricing strategy $P(e)$ implicitly induces a partition of the type space, where all types posting the same price are grouped into a pool. For instance, in a fully pooling equilibrium, the partition consists of a single element, $\{[\underline{e}, 1]\}$, since all types post the same price. As another example, in a partial pooling equilibrium where types $e \in [\underline{e}, 0.5]$ post price p_1 and types $e \in (0.5, 1]$ post price p_2, the pricing strategy induces the partition $\{[\underline{e}, 0.5], (0.5, 1]\}$.

Importantly, multiple equilibria may exist even for a fixed equilibrium partition. As long as the condition stated in Lemma 2.2 is satisfied, types within each pool can coordinate on any common price. The following example illustrates this multiplicity of equilibria under the same partition structure.

Example 2.1. *Suppose $v \sim U[0, 1]$ and $e \in \{0.8, 1\}$, with each level of expertise equally likely. Below are two equilibria with the same partition $\big\{\{0.8\}, \{1\}\big\}$.*

1. *The expert posts a price of 0.4 when $e = 0.8$ and posts a price of 0.2 when $e = 1$. After observing $p = 0.4$, the consumer perfectly infers that the expert is of type 0.8, and buys when $v \in [0.4, 0.8]$. After observing $p = 0.2$, the consumer perfectly infers that the expert is of type 1, and buys when $v \in [0.2, 1]$. Both types of expert earn the same equilibrium profit of 0.16.*

2. *The expert posts a price of 0.4 when $e = 0.8$ and posts a price of 0.8 when $e = 1$. After observing $p = 0.4$, the consumer perfectly infers that the expert is of type 0.8, and buys when $v \in [0.4, 0.8]$. After observing $p = 0.8$, the consumer perfectly infers that the expert is of type 1, and buys when $v \in [0.8, 1]$. Both types of expert earn the same equilibrium profit of 0.16.*

Note that both equilibria can be supported by pessimistic off-equilibrium beliefs that assign probability 1 to type 0.8. Under these beliefs, no type has an incentive to deviate.

Theorem 2.1. *Any partition of types can be sustained in equilibrium with a pricing strategy $P(e)$ satisfying $\pi(P(e), \mu(P(e))) = \pi^*$, together with off-equilibrium beliefs that assign probability 1 to the lowest type $\underline{e}$.*

Proof. Consider a partition of the type space into $n \in \mathbb{N}$ pools. For each pool, let $\hat{\pi}_i$denote the maximal profit that any type in pool i could earn if that pool were isolated, that is, if no other types could mimic it. Let $\pi^* = \min_{i=1,\ldots,n} \hat{\pi}_i$. Each pool can choose a price that yields exactly π^*. Types in pools with $\hat{\pi}_i > \pi^*$ simply burn surplus by charging a suboptimal price. Thus all types earn the same profit π^*, so no type can profitably deviate to any other on-path price.

Next, impose pessimistic off-equilibrium beliefs assigning posterior probability 1 to the lowest type $\underline{e}$ at any off-path price. These beliefs make all off-path deviations unprofitable, since the deviation payoff equals the lowest-type payoff, which is exactly π^*. Thus no off-path deviation is profitable. *Q.E.D.*

Any equilibrium partition can be constructed by identifying a pricing strategy that satisfies the condition in Lemma 2.2.

As demonstrated in Example 2.1, even conditional on a given equilibrium partition, multiple equilibria may still arise. To address this multiplicity, we introduce the following refinement of the equilibrium concept.

Definition 2.2. *An expert-optimal equilibrium maximizes the expert's profit among all equilibria that induce the same partition.*

For the remainder of the paper, our analysis focuses exclusively on expert-optimal equilibria. For brevity, we omit the term "expert-optimal" when referring to such equilibria. The definition implies the following property.

Lemma 2.3. *In an expert-optimal equilibrium that features an equilibrium partition with $n \in \mathbb{N}$ pools, the equilibrium profit is given by $\pi^* \equiv \min_{i=1,\ldots,n} \hat{\pi}_i$, where $\hat{\pi}_i$ is the highest attainable profit within pool i.*

For each pool i, $\hat{\pi}_i$ is the highest attainable profit for the types in that pool—that is, the maximal profit they could earn if they did not need to worry about other types mimicking them. Any lower profit level is also attainable since the profit function is continuous in price, and zero profit is trivially achievable by setting the price to zero. By Lemma 2.2, all pools must earn the same profit in equilibrium. Therefore, the equilibrium profit is bounded above by the minimum of $\hat{\pi}_i$. The highest equilibrium profit is then equal to $\min_{i=1,\ldots,n} \hat{\pi}_i$. This result is reminiscent of the "curse of high types" in the classical informed-principal model of Maskin and Tirole (1992), in which the high type must burn money to keep the low-type incentive constraint binding; otherwise, the low-type expert would mimic the high type.

The following example demonstrates how an expert-optimal equilibrium profit is determined.

Example 2.2. *Suppose $v \sim U[0,1]$ and $e \in \{0.8, 1\}$, with each level of expertise equally likely. We begin by fixing the equilibrium partition as $\big\{\{0.8\},\{1\}\big\}$; that is, we focus on fully separating equilibria.*

Now we compute the optimal profits of the two types of expert.

Type 0.8 expert solves

$$\max_p \quad p(0.8-p),$$

which gives $p_1 = 0.4$ and $\hat{\pi}_1 = 0.16$.

Type 1 expert solves

$$\max_p \quad p(1-p),$$

which gives $p_2 = 0.5$ and $\hat{\pi}_2 = 0.25$. Hence, the equilibrium profit is $\pi^* = \min\{\hat{\pi}_1, \hat{\pi}_2\} = 0.16$.

In the equilibrium, type 0.8 expert can act optimally while type 1 expert must burn *surplus to reveal her type. Otherwise, type 0.8 expert would have an incentive to mimic type 1.*

Fixing type 1 expert's profit at $\pi^* = 0.16$*, one of the expert-optimal equilibria is such that type 0.8 expert posts a price of 0.4 and type 1 expert posts a price of* 0.8*. Both types earn the same equilibrium profit of 0.16. The consumer believes that (i) the expert is type 1 when* $p = 0.8$*, and (ii) the expert is type 0.8 otherwise. Hence, we have successfully constructed an expert-optimal equilibrium for a given partition, which also illustrates Theorem 2.1.*

Remark. *In this example, all equilibria are expert-optimal. That is, the type 0.8 expert always optimizes. The lowest type (*$e = 0.8$*) does not pool with any other type, and there is no off-equilibrium belief that can deter her from deviating to her optimal price.*

2.5.1 Fully Separating Equilibrium

By Theorem 2.1, there always exists a fully separating equilibrium in which each type of expert charges a distinct price, thereby perfectly revealing her type. However, such an equilibrium is also the least favorable to the expert.

Proposition 2.1. *Among all expert-optimal equilibria, the fully separating equilibrium yields the lowest equilibrium profit.*

Proof. By Lemma 2.3, the equilibrium profit in any expert-optimal equilibrium is given by the lowest optimal profit fixing the pool: $\pi^* \equiv \min_i \{\hat{\pi}_i\}_{i=1,\ldots,n}$. In an expert-optimal fully separating equilibrium, the equilibrium profit is given by the optimal profit of type $\underline{e}$ expert, which is the lowest optimal profit among all pools under any partition. To see this, let $\delta_{\underline{e}}$ denote the Dirac measure at point $\underline{e}$. Any distribution $F \in \Delta[\underline{e}, 1]$ first-order stochastic dominates $\delta_{\underline{e}}$. The optimal profit under any distribution F is higher than that under $\delta_{\underline{e}}$. It follows that the optimal profit of type $\underline{e}$ expert is the lowest. *Q.E.D*

The result hinges on the fact that the equilibrium price is pinned down by the lowest type's optimal profit. In fact, the argument extends directly: any equilibrium that involves separating the lowest type yields the lowest equilibrium profit among all equilibria.

By Lemma 2.3, the expert-optimal equilibrium profit in a fully separating equilibrium is determined by the lowest type's optimal profit. Hence, the equilibrium profit is given by

$$\pi^* = \hat{\pi}\left(\underline{e}\right) = \max_{p\in[0,1]} \quad p(F_b(\underline{e}) - F_b(p)).$$

In the equilibrium, Lemma 2.2 requires that the pricing strategy $P(e)$ must yield the same profit across all types:

$$P(e)\left[F_b(e) - F_b(P(e))\right] = \pi^*, \qquad \forall e \in \left[\underline{e}, 1\right]. \tag{2.1}$$

However, for a given profit level π^* and expert's type e, there could be multiple prices that satisfy (2.1). As a result, expert-optimal separating equilibrium is not unique. To address this multiplicity, we introduce a stronger notion of equilibrium that also accounts for consumer surplus. See the following illustrative example.

Example 2.3. *Suppose $v \sim U[0,1]$ and $e \sim U\left[0.5, 1\right]$. To find an expert-optimal separating equilibrium, we first compute the lowest type ($e = 0.5$)'s optimal profit when her type is commonly known.*

Solving

$$\max_{p\in[0,1]} \quad p(0.5 - p),$$

we have $P(0.5) = 0.25$ and $\pi = 0.0625$.

Now we know that the expert-optimal separating equilibrium profit is $\pi^* = 0.0625$. For any type $e > 0.5$, a pricing strategy $P(e)$ constitutes an expert-optimal equilibrium as long *as equation (2.1) is satisfied:*

$$P(e)\left[e - P(e)\right] = 0.0625, \qquad \forall e \in (0.5, 1]\,.$$

We then have $P(e) = \frac{e}{2} \pm \frac{\sqrt{e^2-0.25}}{2}$*. For each type, there are two potential prices that can sustain the equilibrium. Since we have a continuum of types, there is an infinite number of equilibria even if we only focus on the expert-optimal ones. Among these equilibria, the following pricing strategy constitutes an equilibrium that yields the highest consumer surplus (and welfare):*

$$P(e) = \frac{e}{2} - \frac{\sqrt{e^2 - 0.25}}{2}, \qquad \forall e \in [0.5, 1]\,.$$

That is, the expert always chooses the lower price at every expertise level. We refer to this as a welfare dominant equilibrium, meaning an expert optimal equilibrium that maximizes consumer surplus among all equilibria that yield the same profit to the expert. In a welfare dominant equilibrium, the expert always charges $P(e) = \min_{k=1,\ldots,K} p_k$*, where each* p_k *solves*

$$p\left[F_b(e) - F_b(p)\right] = \pi^*, \qquad k = 1, \ldots, K.$$

Throughout the rest of our analysis, we restrict our focus to welfare-dominant equilibria.

An important remark about Example 2.3 is that $P(e)$ is strictly decreasing in e. In other words, a more capable expert charges a low price in the equilibrium. It might be counterintuitive at first glance, but the following Lemma states that this is a general property of welfare-dominant separating equilibria.

Lemma 2.4. *If* $2f_b(p) + pf_b'(p) > 0$*, then in a welfare-dominant separating equilibrium, the equilibrium price* $P(e)$ *is decreasing in* e*.*

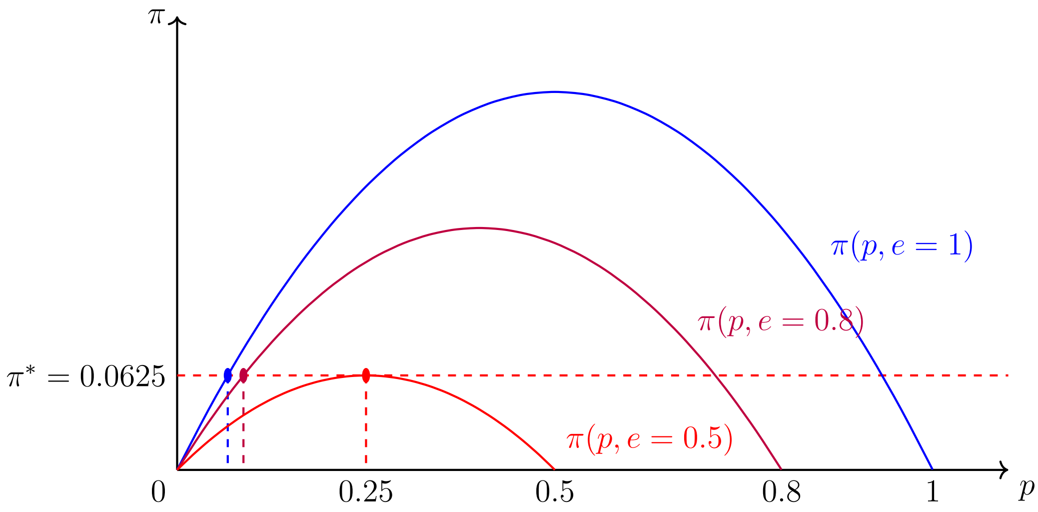

Figure 2.1: Profit functions under fully separating in Example 2.3.

Proof. The condition $2f_b(p) + pf_b'(p) > 0$ is a sufficient condition for the profit function $\pi(p,e) = p\,(F_b(e) - F_b(p))$ to be single-peaked and concave in p. For any fixed profit level π^*, the equation $\pi(p,e) = \pi^*$ has at most two solutions, which we denote by $p' < p''$. In a welfare-dominant separating equilibrium, the expert sets a lower price $P(e) = p'$ to make a profit π^*. Since p' lies to the left of the peak, the slope is positive:

$$\left.\frac{\partial \pi}{\partial p}\right|_{p=p'} = F_b(e) - F_b(p') - p' f_b(p') > 0.$$

By the implicit function theorem applied to $P(e)\,(F_b(e) - F_b(P(e))) = \pi^*$, we have

$$\frac{dP(e)}{de} = -\frac{P(e) f_b(e)}{F_b(e) - F_b(P(e)) - P(e) f_b(P(e))} < 0.$$

Q.E.D.

Lemma 2.4 stands in sharp contrast to Lemma 2.1. Even though the consumer perfectly infers the expert's type in a fully separating equilibrium, the equilibrium price decreases with expertise. The key reason is that all types must earn the same profit in any signaling equilibrium, as established in Lemma 2.2. Higher-type experts face higher demand, so to reduce their profit to the common level they must charge lower prices. The low price is therefore not a signal of low quality but a consequence of the equal-profit constraint.

Fig. 2.1 illustrates the pricing structure and profit functions across types under full separation in Example 2.3. The figure shows the profit as a function of price, given three different expertise levels, $e \in \{0.5, 0.8, 1\}$. As the expert's type increases, the demand for her service is higher, leading to the overall increase in the profit. By Lemma 2.4, all types of expert earn the same level of profit in the equilibrium. To reach a profit of $\pi^* = 0.0625$, any expert with $e > 0.5$ can choose from two prices. In the welfare-dominant equilibrium, all types of expert posts the lower of the two. As the type increases, the profit function expands and a lower price is needed to achieve the equilibrium profit π^*.

Remark. There also exists a separating equilibrium in which the equilibrium price $P(e)$ increases with e, which may appear more realistic. The key point is that all types must earn the same equilibrium profit. Higher types therefore need to charge an inefficiently high price and reduce demand in order to burn surplus and prevent lower types from mimicking them.

2.5.2 Fully Pooling Equilibrium

In a fully pooling equilibrium, all expert's types post the same price. Since the price does not convey any information, the consumer forms belief based on the prior distribution of types.

By Theorem 2.1, there always exists a fully pooling equilibrium with price p that generates a profit no less than $\hat{\pi}(\underline{e})$. Consider a punishing belief such that whenever an off-equilibrium price $p' \neq p$ is observed, the consumer believes that the expert is of type $\underline{e}$.[4] Under this belief, any deviation from the equilibrium price is unprofitable for the expert.

[4] Forward-induction refinements (including the Intuitive Criterion, D1, D2, and Divinity) do not eliminate any equilibria because all types have identical deviation payoffs at any off-path price, so no deviation is more attractive to one type than another. The pessimistic beliefs that sustain these equilibria are therefore robust to forward-induction reasoning.

Take the equilibrium price p as given, a type-v consumer buys the service if and only if her expected value from the purchase is no less than the price, i.e., $EV(v) = v \Pr(e \geq v) = v(1 - F_s(v)) \geq p$. The equilibrium profit is then given by $\pi = p \Pr(v(1 - F_s(v)) \geq p)$.

To start, consider the case in which the expert's type is binary, $e \in \{e_L, e_H\}$, with $0 < e_L < e_H < 1$. Let $\lambda \equiv \Pr(e = e_H) \in (0, 1)$ denote the probability that the expert is of the high type.

Proposition 2.2. *When the expert's type is binary, the pooling equilibrium yields the highest profit among all equilibria.*[5]

Proof. Let $e \in \{e_L, e_H\}$ with $0 \leq e_L < e_H \leq 1$ and $\Pr(e = e_H) = \lambda \in [0, 1]$. Let $\mu(p)$ be consumer's posterior probability that the expert is type e_H after observing posted price p. Then the expert's equilibrium profit is determined solely by the price and consumer's belief, $\pi = \pi(p; \mu(p))$. Note that in a fully pooling equilibrium, the equilibrium profit is optimized under the prior: $\pi_{\text{pooling}}(\lambda) = \max_p \pi(p; \lambda)$.

Fix any equilibrium that induces a distribution over posted prices. Pick any p in the support. Either $\mu(p) = \lambda$ for every p in the support, or there exists $\tilde{p}$ in the support with $\mu(\tilde{p}) < \lambda$. In the first case, $\pi = \pi(p; \lambda) \leq \pi_{\text{pooling}}(\lambda)$ trivially. In the second case, we have $\pi = \pi(\tilde{p}; \mu(\tilde{p})) \leq \pi_{\text{pooling}}(\mu(\tilde{p})) \leq \pi_{\text{pooling}}(\lambda)$. The last inequality holds because $\pi(p; q)$ (and thus $\pi_{\text{pooling}}(q)$) is weakly increasing in the posterior q, i.e. demand is weakly increasing in μ. *Q.E.D.*

This result hinges on the assumption that beliefs about the expert's type can be ordered based on the value of μ. When the expert's type space includes more than two values, such a ranking of beliefs is no longer possible. Indeed, a fully pooling equilibrium may not yield the highest equilibrium profit among all equilibria when we consider a general distributions of expertise. To illustrate this point, consider the following counterexample.

[5]The proposition remains valid when mixed-strategy equilibria are considered.

Example 2.4. *Suppose* $v \sim U[0,1]$ *and let* δ_x *denote the Dirac measure at point* x*. Suppose the distribution of expert's type is* $0.1\delta_0 + 0.8\delta_{0.5} + 0.1\delta_1$*.*

In an expert-optimal pooling equilibrium, the expert sets a uniform price of 0.225 regardless of her type and makes a profit of $\pi_{pooling} = 0.05625$*. The consumer buys when* $v \in [0.25, 0.5]$*. In particular, a type-*v *consumer with* $v > 0.5$ *does not expect his problem to be solved with sufficiently high probability to make purchasing at that price sensible.*

Now consider a partial pooling equilibrium in which (1) types 0 and 1 pool together, and (2) type 0.5 separates. That is, the equilibrium partition is $\big\{\{0,1\},\{0.5\}\big\}$*. We begin by computing the optimal profits for each of the two pools.*

1. *The posterior distribution of pool (1) is* $0.5\delta_0 + 0.5\delta_1$*. The pool optimally posts a price of 0.25 and earns a profit of* $\hat{\pi}_1 = 0.125$*. The consumer buys when* $v \in [0.5, 1]$*.*

2. *The posterior distribution of pool (2) is* $\delta_{0.5}$*. Type 0.5 expert optimally posts a price of 0.25 and earns a profit of* $\hat{\pi}_2 = 0.0625$*. The consumer buys when* $v \in [0.25, 0.5]$*.*

The equilibrium profit in the partial pooling case is $\pi_{partial\ pooling} = \min\{\hat{\pi}_1, \hat{\pi}_2\} = 0.0625 > \pi_{pooling}$*. This shows that a fully pooling equilibrium does not always yield the highest equilibrium profit for the expert.*

Fig. 2.2 shows the expected value of the service as a function of the consumer's type v, which also corresponds to type-v consumer's willingness to pay. We can focus on the comparison between full pooling and pool 2, because pool 2 has a lower optimal profit and thus determines the equilibrium profit under this partial pooling. In pool 2, a consumer with $v \leq 0.5$ exhibits a higher willingness to pay for the expert's service, as he now knows with certainty that his problem can be solved. Hence, even though the market coverage is the same under full pooling and under pool 2, the expert is able to charge a higher price and achieve a higher equilibrium profit in the partial pooling equilibrium.

Under pooling, a consumer with a high valuation ($v > 0.5$) is never served, which can be considered a "waste" of potential demand. By separating the high-type expert from

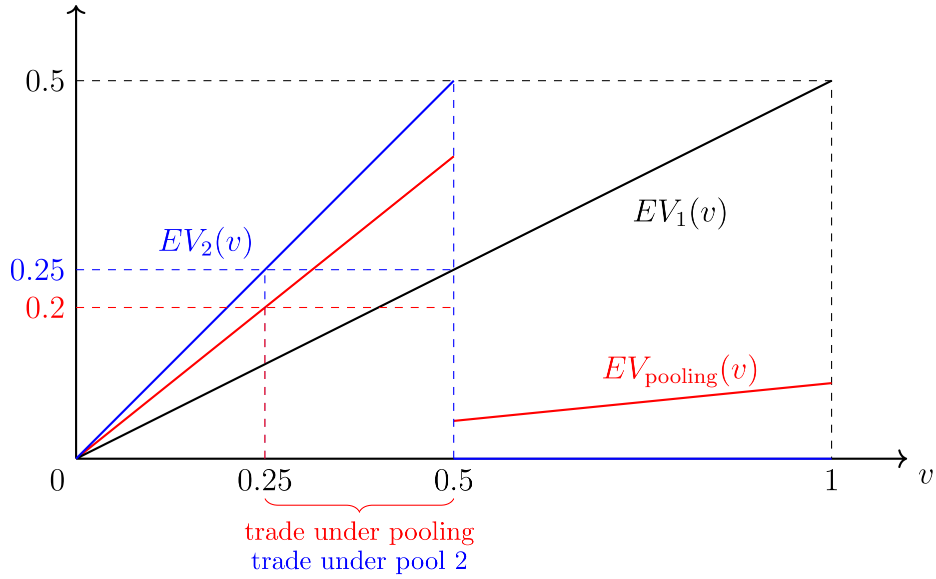

Figure 2.2: Expected value functions in Example 2.4.

the intermediate type, a high-valuation consumer is now served with positive probability. Intuitively, distinguishing the high-type expert allows the market to target different subsets of the consumer's types more effectively, enabling the expert to exploit demand more efficiently than under full pooling.

It also raises two important questions: When is pooling equilibrium (not) optimal? Is there always a way for the expert to separate types and earn a higher profit compared to setting a uniform price?

2.5.3 Welfare

Finally, we discuss welfare under different equilibria. We compare the two extrema: fully separating equilibrium and fully pooling equilibrium.

We impose a regularity condition on the distribution of expertise.

Assumption 2.1. *The hazard rate* $\frac{f_s(e)}{1-F_s(e)}$ *is increasing.*[6]

Proposition 2.3. *Under Assumption 2.1, both total welfare and the consumer surplus*

[6]Expertise distributions in many professional service markets are typically unimodal and right-skewed, making an increasing hazard rate empirically plausible. Technically, this assumption ensures that the expected-value function is single-peaked, which simplifies the welfare analysis, but it does not affect the main equilibrium results. See the appendix for a proof.

are higher under a fully separating equilibrium than under a fully pooling equilibrium.

Proof. The welfare under full separation is

$$\int_{\underline{e}}^{\overline{e}} \int_{P(e)}^{e} v dF_b(v) dF_s(e),$$

because a type-e expert posts price $P(e)$ and the consumer buys when $v \in [P(e), e]$.

Under full pooling at price p, the consumer buys when $v \in [\underline{v}(p), \overline{v}(p)]$, where $\underline{v}(p) < \overline{v}(p)$ are the two roots of $v(1 - F_s(v)) = p$. (Assumption 1 gives single-peakedness of $v(1 - F_s(v))$, hence two roots for p in the relevant range). Thus, the welfare under full pooling is

$$\int_{\underline{e}}^{\overline{e}} \int_{\underline{v}(p)}^{\min\{\overline{v}(p), e\}} v dF_b(v) dF_s(e).$$

First, it is immediate that $e \geq \min\{\overline{v}(p), e\}$. Next, by Lemma 2.4, $P(e) < P(\underline{e})$. It follows that $P(e) < P(\underline{e}) < p_{\text{pooling}} < \underline{v}(p)$. Therefore, the total welfare is higher under full separating.

Furthermore, from Proposition 2.1, the expert earns less profit under separation. Therefore, the consumer surplus must be strictly higher under separation, as total welfare increases but expert surplus falls. *Q.E.D.*

In a fully separating equilibrium, expertise information is fully revealed, leading to efficient matching: only the consumers whose problems the expert can solve are served. Moreover, equilibrium prices in a fully separating equilibrium are always lower than those in a fully pooling equilibrium. This results in increased welfare and a significant redistribution of surplus from the expert to the consumer.

2.6 Discussion

We have shown that any partition of types can be sustained in equilibrium, resulting in a multiplicity of equilibria. Hence, we focus on the equilibria that maximize either profit or welfare. Among these equilibria, the fully separating equilibrium features decreasing price and yields the lowest profit.

Moving from the fully separating to the fully pooling equilibrium, it is surprising that pooling can be suboptimal for the expert under certain parameter configurations. An intriguing direction for future research is to explore how the expert might optimally partition her type space to increase her ex ante profit. Our hypothesis is that when the distribution of expertise has relatively low density among high types, the expert could improve her expected payoff by segmenting her types strategically and targeting different market segments ex post.

Finally, our welfare analysis shows that the fully separating equilibrium generates higher social welfare than full pooling. The consumer benefits from lower prices and perfect information about expertise, which substantially improves his welfare. Welfare is redistributed from the expert to the consumer. However, due to the multiplicity of equilibria, the policy implications of mandatory disclosure are not straightforward. Mandating disclosure of expertise replicates the case of known expertise. While this improves the consumer surplus relative to full pooling, it does not reach welfare level of full separation.

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Appendix

Regularity condition: Increasing hazard rate

Lemma 2.5. *If the hazard rate* $\frac{f_s(e)}{1-F_s(e)}$ *is increasing, then* $EV(v)$ *is single-peaked and the set of buyer types is convex.*

Proof. Put $\frac{d}{dv}EV(v) = 0$.

$$\frac{d}{dv}EV(v) = -vf_s(v) + 1 - F_s(v) = 0 \quad \Leftrightarrow \quad v = \frac{1 - F_s(v)}{f_s(v)}.$$

Since the hazard rate $\frac{f_s(e)}{1-F_s(e)}$ is increasing, R.H.S. is decreasing. L.H.S. and R.H.S intersect only once. We have a unique critical point.

Since $EV(v)$ is quasiconcave, the set of buyer types $\{v : p \leq EV(v)\}$ is convex. *Q.E.D.*

Proof of Lemma 2.1.

Since $P(e)$ is optimal, it follows that

$$P(e)\left[F(e) - F(P(e))\right] \geq p'\left[F(e) - F(p')\right] \quad \forall p' \leq P(e)$$

Suppose $e' > e$, for all $p' \leq P(e)$,

$$
\begin{aligned}
&\pi\left(P(e), e'\right) - \pi\left(p', e'\right) \\
&= P(e)\left[F(e') - F(P(e))\right] - p'\left[F(e') - F(p')\right] \\
&= P(e)\left[F(e) - F(P(e))\right] - p'\left[F(e) - F(p')\right] + \left[P(e) - p'\right]\left[F\left(e'\right) - F(e)\right] \\
&\geq \left[P(e) - p'\right]\left[F\left(e'\right) - F(e)\right] \\
&\geq 0
\end{aligned}
$$

Hence, we have $P(e') \geq P(e)$.

We then show that the optimal profit $\pi^*(e) = \pi\big(P(e), e\big)$ also increases in e. Take e and e' such that $e' > e$. It follows that

$$
\begin{aligned}
\pi^*(e') - \pi^*(e) &= p(e')\big[F_b(e') - F_b(p(e'))\big] - P(e)\big[F_b(e) - F_b(P(e))\big] \\
&> p(e')\big[F_b(e') - F_b(p(e'))\big] - P(e)\big[F_b(e') - F_b(P(e))\big] \\
&\geq p(e')\big[F_b(e') - F_b(p(e'))\big] - p(e')\big[F_b(e') - F_b(p(e'))\big] \quad \text{(by optimality of } p\text{)} \\
&= 0.
\end{aligned}
$$

Q.E.D.

Chapter 3

When to Disclose Horizontal Product Attributes?

Abstract

A seller offers a horizontally differentiated product to a consumer on a Hotelling line. The seller privately knows her product's location, and the consumer privately knows his taste. After observing her type, the seller chooses whether to disclose her location and then sets a price; the consumer decides whether to buy based on the price and the disclosed information. Under convex transport cost, no fully pooling equilibrium, where all seller types hide information, can exist, because the mean type in any pooling set always prefers to disclose. When the market is fully covered, an unravelling equilibrium maximizes consumer surplus but minimizes the seller's profit. Conversely, if the market is not covered, a pooling equilibrium may strictly benefit both sides by lowing price and expanding trade. Therefore, mandatory disclosure policies can potentially harm both parties.

3.1 Introduction

Advertising can make consumers aware of a product's existence and can also convey valuable information about its features. In practice, however, firms often omit important horizontal attributes from their advertisements, leaving many details to be discovered only through product experience. Because sellers frequently hold private information about these attributes, they may choose strategically whether to reveal or conceal it. This raises several fundamental questions: When does a seller have an incentive to disclose product information? How do market characteristics, such as the distribution of consumer tastes, shape these incentives? And under what conditions can advertising regulation improve consumer welfare?

Much of the existing literature relies on uniform consumer distributions and specific transport cost specifications, making it unclear which results reflect fundamental economic forces and which depend on these assumptions. By allowing for arbitrary distributions and a general transport cost function, this paper moves beyond these restrictions and isolates the underlying determinants of disclosure incentives. This broader framework shows how market heterogeneity and mismatch costs shape equilibrium behavior and welfare.

Selective disclosure of horizontal characteristics is common in many markets, including electronics, online platforms, and food products. Even when product information is verifiable, firms often reveal only partial details. Understanding when firms voluntarily reveal such characteristics, and whether mandatory disclosure benefits consumers, requires analyzing the trade-offs between location signaling and increased mismatch uncertainty. The framework developed here sheds light on these forces and on the welfare consequences of disclosure policies.

To address these questions, we consider a Hotelling model in which a monopolist sells to a consumer with a privately known taste.[1] The seller privately knows the horizontal attribute of her product and chooses whether to disclose it before setting a price. The consumer then decides whether to buy based on the disclosed information and his preference. We study Perfect Bayesian Equilibria (PBE) in pure strategies.

We first show that the specification of the transport cost determines the seller's pooling

[1] Kim and Kos (2025b) and Kim and Kos (2025a) analyze related settings in which a seller faces a continuum of consumers along the Hotelling line, focusing on product design, pricing, and their effects on consumer surplus and profits.

incentive, and thus the existence of certain equilibria. Hiding information shifts the perceived product location and increases mismatch uncertainty. The former effect is beneficial for sellers in disadvantaged locations, while the latter depends critically on the transport cost. When the cost function is convex, consumers are averse to mismatch uncertainty and their willingness to pay decreases. As a result, any pooling set must be disconnected, and a fully pooling equilibrium cannot exist. Convexity of the transport cost is thus the fundamental economic force underlying the impossibility of full pooling.

The second part of the analysis characterizes the equilibrium pooling set by comparing disclosure and non-disclosure profits. All non-disclosing types earn the same profit, so the shape of the disclosure profit function determines which types optimally reveal their information. For example, when demand for the product is sufficiently strong that the market is always covered, the disclosure profit function is single-peaked, and only edge types pool in equilibrium.

Finally, we discuss the welfare implications of different equilibria. When the demand is strong enough that the market is always covered, the unravelling equilibrium yields the highest consumer surplus but the lowest seller profit. However, when market coverage is incomplete under unravelling, a pooling equilibrium may generate higher payoffs for both the seller and consumers. The intuition is that non-disclosure leads the seller to charge a lower price, thereby expanding market coverage. Consumers benefit from both the lower price and the increased trade volume. This overturns the common presumption that mandatory disclosure always benefits consumers.

Related literature There is a vast amount of studies on advertising and information provision; Bagwell (2007) offers a comprehensive overview. The papers most closely related to the present study are those that examine disclosure of horizontal attributes in Hotelling-type environments. Sun (2011) considers a seller privately informed about both product location and quality and shows that equilibrium involves partial disclosure, with central types revealing and edge types hiding. Celik (2014) extends this framework by allowing for partial, verifiable disclosure of intervals, leading to pooling among types equidistant from the center. Both analyses rely on uniform consumer distributions, symmetric seller types and specific transport cost assumptions, which play an important role in shaping the equilibrium structure.

The paper also connects to the broader disclosure and unraveling literature. Classic results by Grossman (1981) and Milgrom (1981) show that with verifiable information, full disclosure generally occurs. Departures from unraveling typically require frictions such as disclosure costs or discrete type spaces. In contrast, in horizontally differentiated markets, mismatch uncertainty induced by convex transport cost provides a novel and frictionless reason for partial pooling, linking the model to this broader literature.

My analysis diverges from the prior horizontal disclosure literature in three main respects. First, it allows for arbitrary consumer and seller distributions and a general transport cost function. This generalization uncovers economic forces that were obscured by the symmetry and linearity imposed in earlier models. In particular, I show that convex transport cost is the fundamental driver of disconnected pooling sets, a result that holds irrespective of distributional assumptions. Second, the analysis delivers a general impossibility result: under convex transport cost, a fully pooling equilibrium cannot exist because the mean type in any pooling set always strictly prefers disclosure. In earlier work, full pooling does not arise in equilibrium but is not ruled out by a general argument. Third, it examines welfare in settings where market coverage is endogenous. When the separating equilibrium fails to cover the market, pooling may expand trade and increase welfare for both the seller and consumers. This welfare reversal does not appear in existing models, which focus on environments with full market coverage where unravelling is always welfare-improving.

The rest of the paper is organized as follows. Section 3.2 presents the model. Section 3.3 analyzes how transport cost specification affects equilibrium. Section 3.4 discusses equilibrium properties. Section 3.5 studies welfare implications. Section 3.6 discusses.

3.2 Model

We consider a Hotelling model with one seller (she) and one consumer (he), both located on the interval $[0, 1]$. The seller's type $l \in [0, 1]$ represents the product's horizontal attribute (or location) and is privately observed by the seller. The consumer has a privately known taste $\theta \in [0, 1]$. The consumer's taste is drawn from a commonly known distribution $F_b \in \Delta[0, 1]$, and the seller's type is drawn from a commonly known prior distribution $F_s \in \Delta[0, 1]$. Both F_b and F_s are common knowledge.

After observing her type, the seller chooses a disclosure message $m(l) \in \{l, n\}$, where $m(l) = l$ corresponds to full disclosure and $m(l) = n$ corresponds to hiding all information.[2] Disclosure is verifiable: if the seller discloses, the message must be correct, and any non-disclosure message reveals nothing. The seller then sets a price $p \geq 0$. The consumer observes both the message and the price and decides whether to buy the product.

The consumer's payoff from purchasing a product of type l at price p is

$$u(\theta, l, p) = V - C(|\theta - l|) - p,$$

where $V > 0$ is the intrinsic valuation of the product, and $C(\cdot)$ is the transport cost capturing mismatch disutility. We assume that C is strictly increasing, $C(0) = 0$, and $C(1) < V$, ensuring that trade is always efficient.[3] For brevity, write $c_\theta(l) := C(|\theta - l|)$.

If the seller hides her type ($m = n$), the consumer forms a posterior belief about l based on the prior F_s. His expected utility is $\mathbb{E}_{F_s}\left[u(\theta, l, p) \middle| m\right]$. The consumer buys the product if and only if this expected utility is non-negative.

The seller can produce at zero cost, and thus obtains profit

$$\pi(m) = p \Pr\left(p \leq \mathbb{E}_{F_s}\left[u(\theta, l, p) \middle| m\right]\right).$$

The timing is as follows. In the first stage, the seller draws her type $l \sim F_s$, sends

[2] Sun (2011) studies this "all-or-nothing" disclosure structure.

[3] The assumption $C(1) < V$ rules out the possibility that extreme mismatch makes trade inefficient.

a disclosure message $m(l) \in \{l, n\}$ and posts a price $p \geq 0$. In the second stage, the consumer observes the message m and the price p and makes a buying decision.

We study Perfect Bayesian Equilibria in pure strategies. A *fully separating equilibrium* is an equilibrium in which all seller types disclose their location. Any equilibrium in which some types hide information is referred to as a *(partial) pooling equilibrium*. Define the *equilibrium pool* as

$$N := \{\, l \in [0,1] : m(l) = n \,\},$$

the set of seller types that conceal their location in equilibrium.

3.3 Transport Costs vs. Pooling Incentives

To begin with, note that there always exists an unravelling equilibrium, which can be supported by a punishing off-path belief.[4] In the subsequent discussion, we focus on (partial) pooling equilibria, which are more relevant for studying the seller's disclosure incentives.

Pooling affects the seller's profit through two distinct mechanisms. First, hiding shifts the consumer's perceived product location toward the posterior mean. This effect benefits seller types that are disadvantaged relative to this mean. For example, an edge-type seller typically faces weaker demand than a centrally located seller; by pooling, she can shift the perceived location toward the center and attract more buyers. Second, pooling increases mismatch uncertainty. The consumer internalizes this uncertainty when evaluating expected transport cost, and the magnitude of this effect depends on the curvature of the transport cost function. These two forces jointly determine which types find pooling attractive and therefore shape the equilibrium pooling set.

The next example illustrates these two effects in a tractable setting.

[4] This also holds in Sun (2011) and Koessler and Renault (2012). Since non-disclosure is off-equilibrium-path, the consumer' conditional belief can be arbitrary. Under disclosure, there exists a seller type that gets the minimum profit among all types, which can be regarded as occupying the worst location. Consequently, a separating equilibrium can be supported by the consumer' belief that any deviation from disclosure implies that the seller is at the worst location.

Example 3.1. *Consider a quadratic transport cost $C(d) = d^2$. Non-disclosure affects a type-θ consumer's willingness to pay through two channels:*

1. *changing the perceived location from l to $\mu_N = \mathbb{E}_N[l]$;*

2. *increasing uncertainty (variance) of location $\sigma_N^2 = \mathbb{E}_N[l^2] - \mu_N^2$.*

A type-θ consumer's expected transport cost of buying the product without disclosure is

$$\mathbb{E}_N[c_\theta(l)] = \mathbb{E}_N\left[(\theta - l)^2\right] = (\theta - \mu_N)^2 + \sigma_N^2 = c_\theta(\mu_N) + \sigma_N^2,$$

which is as if buying a type-μ_N product with an additional uncertainty cost.

A convex transport cost implies that consumers dislike uncertainty when buying the product. Thus, any increase in uncertainty strictly reduces their willingness to pay. In this case, the second effect of pooling lowers the seller's profit. We then have the following lemma.

Lemma 3.1. *Under strictly convex transport cost $C(\cdot)$, the seller type located at the expected location of the equilibrium pool discloses her location in equilibrium, i.e., $\tilde{l} \equiv \mathbb{E}_N[l] \notin N$.*

Proof. Since $C(\cdot)$ is strictly convex, $c_\theta(\cdot)$ is strictly convex. Then by Jensen's inequality,

$$WTP(N, \theta) = V - \mathbb{E}_N[c_\theta(l)] < V - c_\theta(\mathbb{E}_N[l]) = V - c_\theta\left(\tilde{l}\right) = WTP\left(\tilde{l}, \theta\right)$$

Since for all consumers, the willingness to pay strictly increases under disclosure, type-$\tilde{l}$ seller always discloses. *Q.E.D.*

Lemma 1 captures a key economic insight. The seller type located at the expected value of the pooling distribution gains nothing from shifting perceived location but is strictly harmed by the added mismatch uncertainty generated by pooling. Consequently, this type always reveals her location. Since the mean type must lie outside the pooling set, any pooling set must be disconnected.[5] This immediately rules out the possibility of full pooling.

Corollary 3.1. *Under convex transport cost $C(\cdot)$, there does not exist a fully pooling equilibrium.*

Earlier models do not establish this impossibility result; in those settings, full pooling simply fails to arise in equilibrium under their particular assumptions. The result here holds for any distributions F_s and F_b, thereby generalizing the patterns observed in prior work. Convexity of $C(\cdot)$ alone ensures that the mean-type strictly prefers disclosure, ruling out full pooling for fundamental economic reasons.

Define the equilibrium profit function $\pi^* : [0,1] \cup \{n\} \to \mathbb{R}_+$, where $\pi^*(l)$ is the type-l seller's equilibrium profit under disclosure and $\pi^*(n)$ is the equilibrium profit under non-disclosure. The next proposition provides a general ranking of these profits under convex transport cost

Proposition 3.1. *In any partial pooling equilibrium with pooling set $N \subseteq [0,1]$, for any $l \in N$, $\pi^*(\mu_N) \geq \pi^*(n) \geq \pi^*(l)$ under convex transport cost, where $\mu_N = \mathbb{E}_N[l] \in [0,1]$ is the expected location of the pooling distribution.*

This ordering highlights how the two forces behind pooling incentives operate. The gap $\pi^*(\mu_N) - \pi^*(l)$ represents the benefit a type-l seller obtains from shifting the consumer's perceived product location to the mean of the pooling set; this captures the location-shifting motive for hiding information. In contrast, the gap $\pi^*(\mu_N) - \pi^*(n)$ reflects the loss generated by additional mismatch uncertainty when the seller chooses not to disclose. Together, these differences summarize the trade-off each type faces between improving perceived location and increasing the consumer's expected transport cost, and thus determine which types are willing to remain in the pooling set and which strictly prefer to reveal their locations.

[5] Sun (2011) and Celik (2014) both find disconnected pooling sets, but only under uniform consumer distributions and specific cost functions. Here, convexity of the transport cost is shown to be the fundamental force behind the result.

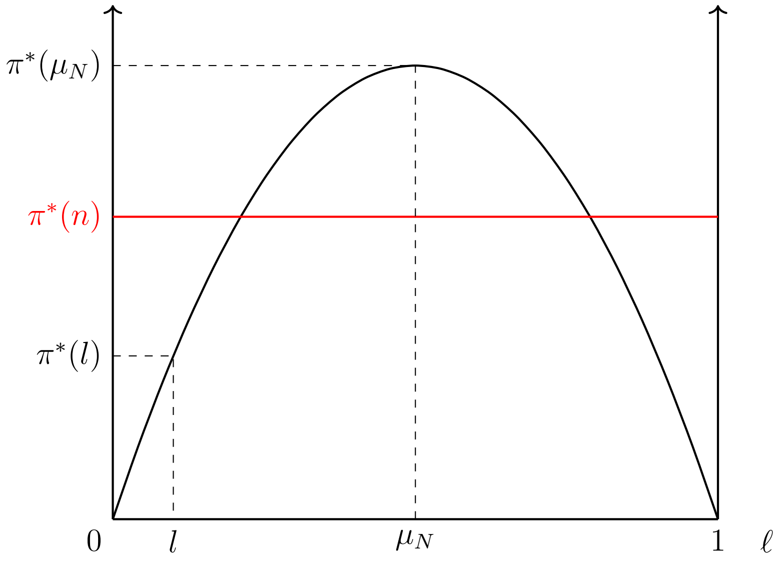

Figure 3.1: A visualization of Proposition 1.

3.4 Equilibrium Pooling

To determine the equilibrium pool, a natural approach is to compare the equilibrium profits under disclosure and non-disclosure. Intuitively, seller types located in disadvantaged regions have stronger incentives to hide information. We begin by analyzing the equilibrium profit under non-disclosure.

Lemma 3.2. *All non-disclosing seller types make the same equilibrium profit.*

Proof. Suppose not. There exist prices p' and p'' posted by non-disclosing seller types that leads to profit levels π' and π'', respectively. Suppose $\pi'' > \pi'$. Then a non-disclosing seller type that makes a lower profit π' can deviate to post π'' and earn $\pi'' > \pi'$. *Q.E.D.*

Lemma 3.2 implies that the non-disclosure profit function $\pi^*(n)$ is constant in l. Therefore, the shape of the disclosure profit function is critical in determining the structure of the equilibrium pool. By comparing profits under full disclosure and non-disclosure, we can identify which seller types optimally choose to reveal their information.

Figure 3.3 illustrates how the shape of the disclosure profit function determines the structure of the equilibrium pool. The red line represents the constant non-disclosure profit, and the black curve represents the disclosure profit. The left panel shows a single-peaked disclosure profit function: any partial pooling equilibrium must then feature pooling of edge types. The right panel displays a less standard case where the disclosure profit function is neither convex nor concave; here, the equilibrium pool may be a connected set.

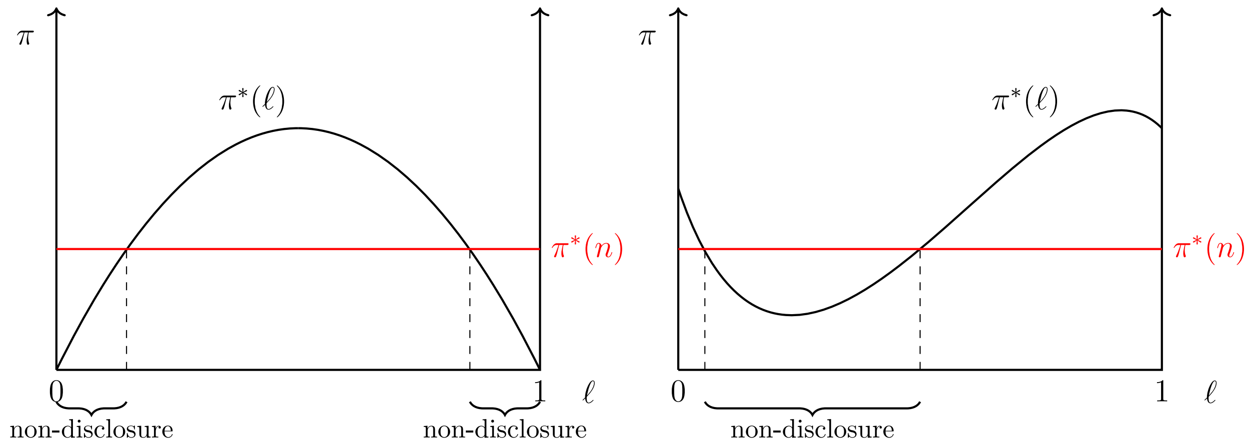

Figure 3.2: Two examples of partial equilibria.

3.4.1 Full Market Coverage

Next, we consider a simplified environment in which V is sufficiently large, such that the seller always finds it optimal to serve the entire market, under both disclosure and non-disclosure. Under this condition, the partial pooling equilibrium always takes a consistent form regardless of the specifications of the distributions and the transport cost.

Proposition 3.2. *When V is sufficiently large, there always exists a partial pooling equilibrium with $N = [0, x] \cup [1-x, 1]$ for some $x \in \left[0, \frac{1}{2}\right]$.*

Proof. Since it is optimal for any seller $l \in [0, 1]$ to serve the whole market, the seller l optimally charges $p^*(l) = \min\left\{V - c(|\theta - l|), V - c(|\theta - (1-l)|)\right\}$, so that even the edge-type consumer is willing to buy the product. The seller earns $\pi^*(l) = p^*(l)$ under disclosure. The profit $\pi^* p^*(l)$ is symmetric, i.e., $\pi^*(l) = \pi^*(1-l)$, and single-peaked at

$l = 0.5$. We have shown that $\pi^*(n)$ is constant in l. The intersections of $\pi^*(l)$ and $\pi^*(n)$ must also be symmetric around $l = 0.5$. *Q.E.D.*

Since V is large, all seller types prefer to cover the market. To do so, the seller sets a price that leaves zero surplus for the most distant consumer. The further a seller is from the center, the lower the price she must charge. Hence, the disclosure profit function is single-peaked and symmetric around 0.5. As a result, edge-type sellers have a stronger incentive to pool. Moreover, it can be shown that the pooling profit strictly exceeds the disclosure profit for sellers of types 0 and 1. This ensures the existence of the partial pooling equilibrium described above.

Proposition 3.3. *When V is sufficiently large, there always exists a full pooling equilibrium under symmetric F_s and concave transport cost.*

Proof. See appendix.

In a fully covered market, the seller's optimal price is determined by the willingness to pay of the most distant consumer. Under concave transport cost, these consumers exhibit behavior analogous to risk-loving agents. By pooling, the seller increases their expected willingness to pay, and thus all types prefer non-disclosure. This highlights the pivotal role that transport cost specification plays in shaping equilibrium outcomes.

3.4.2 Perfect Knowledge of the Consumer's Taste

Finally, we establish a non-existence result for pooling equilibria under an extreme specification of the consumer taste distribution. In particular, we consider the case in which the seller has perfect knowledge of the consumer's taste,[6] that is, $\theta \sim \delta_s$, where δ_s is the Dirac measure at $s \in [0, 1]$.

Proposition 3.4. *If $\theta \sim \delta_s$ for some $s \in [0,1] \setminus \left\{\frac{1}{2}\right\}$, there is no pooling equilibrium. If $\theta \sim \delta_{\frac{1}{2}}$, only type-0 and type-1 seller hide information in any partial pooling equilibrium.*

[6] Because the model is equivalent to one with a continuum of consumers whose tastes follow the distribution F_b, this extreme case can be interpreted as a market in which all consumers have identical tastes.

Proof. See appendix.

Since the consumer is located at a single point s, that location becomes the unique point of absolute advantage. Because the consumer's willingness to pay is strictly decreasing in the distance to the seller, any seller type located closer to s than the pooling mean strictly prefers to deviate and disclose. The only potential pooling set that can survive is the pair of edge types when the consumer is located exactly at the center: in that case, the two edge types are symmetrically the farthest from the consumer and therefore earn the lowest disclosure profit. For these types, pooling does not change payoffs, so they have no incentive to deviate. Outside this knife-edge case, the presence of a unique best location destroys all pooling incentives. Thus, the unique equilibrium outcome is full separation.

This outcome is consistent with the classic unravelling results of Grossman (1981) and Milgrom (1981). When the seller has perfect knowledge of the consumer's taste, the environment effectively collapses to one of vertical differentiation: the distance between the product and the consumer's ideal point can be interpreted as a quality shortfall. In such settings, higher "quality" types always have an incentive to distinguish themselves from lower ones, leading to full disclosure.

3.5 Welfare Analysis

We now examine the welfare effects of disclosure policies. The key distinction is whether the market is fully covered. With full coverage, total surplus is fixed, and mandatory disclosure simply redistributes surplus from the seller to the consumer by lowering price. When the market is not fully covered, however, disclosure can raise prices and reduce trade, harming both sides.

Proposition 3.5. *When V is sufficiently high, the unravelling equilibrium gives the highest consumer surplus and the lowest profit among all equilibria.*

Proof. Since the market is always covered and $V > C(1)$, there will always be trade and trade is efficient. The total social surplus, $\int_0^1 \int_0^1 [V - c_\theta(l)] dF_b(\theta) dF_s(l)$, is the same across all equilibria, given specifications of V, c_θ, F_b and F_s. Then the difference in welfare distribution among equilibria lies in the equilibrium prices, which are redistribution of surplus between consumers and the seller. Therefore, we only need to show that the unravelling equilibrium price is weakly lower than any other (partial) pooling equilibria.

Consider an arbitrary partial pooling equilibrium. We have $\max_l \pi(l) \geq \pi(n) \geq \min_l \pi(l)$. Since the market is covered, the profits equal the corresponding prices, i.e., $\pi(l) = p(l)$ and $\pi(n) = p(n)$. The equilibrium price posted by type-l seller is $p^*(l) = \max\{p(l), p(n)\}$. Hence, the unravelling equilibrium price is weakly lower than any partial pooling equilibrium price, i.e., $p(l) \leq p^*(l)$. *Q.E.D.*

This yields a clear policy implication: when demand is strong enough that the seller optimally covers the whole market, mandatory disclosure (which induces unravelling) increases consumer surplus and reduces seller profit without changing total surplus. By contrast, when the market is not fully covered, both consumers and sellers may prefer a pooling equilibrium. In such cases, mandatory disclosure leads to the (Pareto-dominated) unravelling outcome and should not be imposed.

Example 3.2. *Consider $\theta \sim U[0,1]$ and $l \in \{0,1\}$ with equal probabilities. Consumers have a quadratic transport cost $C(d) = d^2$ and valuation $V = 1$.*

Under disclosure, both types of seller post $p(0) = p(1) = \frac{2}{3}$ and earn a disclosure profit $\pi(0) = \pi(1) = \frac{2}{3}\sqrt{\frac{1}{3}} \approx 0.3849$. Only $\theta \in \left[0, \frac{1}{\sqrt{3}}\right]$ consumers buy type 0 product; only $\theta \in \left[1 - \frac{1}{\sqrt{3}}, 1\right]$ consumers buy type 1 product. The corresponding consumer surplus is

$$CS_{separating} = TSS_{separating} - \pi(0) = \int_0^{\frac{1}{\sqrt{3}}} \left(1 - \theta^2\right) d\theta - \frac{2}{3}\sqrt{\frac{1}{3}} \approx 0.1283$$

Under pooling, the seller charges $p(n) = \frac{1}{2}$ and earn a pooling profit $\pi(n) = \frac{1}{2} > \pi(0) = \pi(1)$. All consumers buy the product. The corresponding consumer surplus is

$$CS_{pooling} = TSS_{pooling} - \pi(n) = \int_0^1 \left(1 - \theta^2\right) d\theta - \frac{1}{2} = \frac{1}{6} > CS_{separating}$$

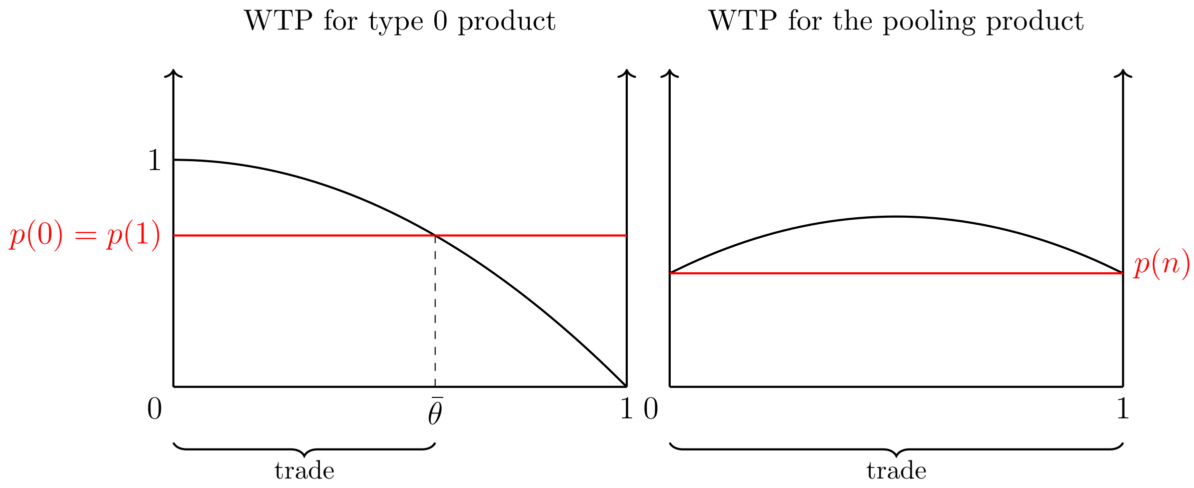

Figure 3.3: type-θ consumer's willingness to pay in Example 3.2.

Pooling increases welfare because it lowers the price and expands trade. The separating equilibrium leaves part of the market unserved, whereas pooling serves all consumers. This expansion in coverage is the key force behind the welfare improvement.

3.6 Discussion

We have shown that the specification of transport costs plays a crucial role in determining equilibrium properties. For example, a fully pooling equilibrium does not exist under convex transport costs. Conversely, with concave transport costs, the seller may exploit this by hiding information, thereby creating matching uncertainty that can benefit some risk-loving consumers. Beyond equilibrium characterization under different conditions, the welfare analysis provides important policy implications, as highlighted in the previous section.

It would be worthwhile to investigate the general conditions under which a pooling equilibrium yields better welfare outcomes than a separating equilibrium. My hypothesis is that when the market is not covered in the separating equilibrium, there is potential for welfare improvement through a pooling equilibrium. As illustrated in Example 3.2, pooling equilibrium lowers the equilibrium price and expands trade, thereby opening the possibility of increased consumer welfare. However, as shown in Proposition 3.5, when demand is strong from the outset, pooling equilibrium cannot further enhance trade, and consumers do not benefit relative to a separating equilibrium.

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Appendix

Proof of Proposition 3.3.

Consider a full pooling equilibrium. We only need to show the middle type-$l = 0.5$ seller does not have an incentive to deviate and disclose.

Type 0.5 seller's profit under disclosure is

$$\pi^* (0.5) = V - \max \{c_0 (0.5), c_1 (0.5)\} = V - C (0.5).$$

The profit under no disclosure is

$$\pi^* (n) = V - \max \{\mathbb{E} [c_0 (l)], \mathbb{E} [c_1 (l)]\} = V - \max \{\mathbb{E} [C (l)], \mathbb{E} [C (1 - l)]\}.$$

WLOG, assume $\mathbb{E} [C (l)] \geq \mathbb{E} [C (1 - l)]$. By Jensen's inequality, since C is concave, it follows that $\mathbb{E} [C (l)] \leq C (\mathbb{E} [l]) = C (0.5)$. Hence, $\pi^* (n) \geq \pi^* (0.5)$ in a complete pooling equilibrium. Since the profit under disclosure is single-peaked at $l = 0.5$ under sufficiently large V, we have proved that all types of seller would not deviate from the full pooling equilibrium. *Q.E.D.*

Proof of Proposition 3.4.

We first prove for $s \in \{0, 1\}$. WLOG, consider $s = 0$. Consider any pool $N \subseteq [0, 1]$. The consumers' willingness to pay is $V - \mathbb{E}_N [c (l)] < V - \min_{l \in N} c (l)$. The smallest type in the pool must deviate to earn a higher profit.

Next, we prove for $s \in (0, 1) \setminus \left\{\frac{1}{2}\right\}$. First, in the pool, the seller type with the shortest distance to the consumer always wants to deviate and disclose location. Under disclosure, type-l firm earns $V - c\left(\left|l-\theta\right|\right)$. Under no disclosure, the profit is $V - \mathbb{E}_N\left[c\left(\left|l-\theta\right|\right)\right] \leq V - \min_{l' \in N} c\left(\left|l'-\theta\right|\right)$. If $\mathbb{E}_N\left[c\left(\left|l-\theta\right|\right)\right] > \min_{l' \in N} c\left(\left|l'-\theta\right|\right)$, then $\tilde{l} = \arg\min_{l' \in N} c\left(\left|l'-\theta\right|\right)$ have a strict incentive to disclose and N cannot be sustained in equilibrium. Suppose $\mathbb{E}_N\left[c\left(\left|l-\theta\right|\right)\right] = \min_{l' \in N} c\left(\left|l'-\theta\right|\right)$. Then for all $l \in N$, $c\left(\left|l-\theta\right|\right) = c\left(\left|\tilde{l}-\theta\right|\right)$. Since c is strictly increasing, we have $\left|l-\theta\right| = \left|\tilde{l}-\theta\right|$. Thus, $l = \tilde{l}$ or $2\theta - \tilde{l}$. In words, the pool can only consist of two points that are equidistant from θ.

Second, given the two-point pool, the seller type that discloses information but locates furthur away from the consumer than the pool would want to deviate and hide information. Hence, the two-point pool cannot sustain. *Q.E.D.*