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Part I

Ambiguity, Prudence and Optimal Portfolio

Abstract

Recent experimental evidence suggests that ambiguity prudence plays an important role in decision-making. In this work, I investigate the contribution of ambiguity prudence when an investor evaluates portfolio compositions according to the smooth ambiguity criterion. In chapter 1, I derive a third-order approximation of the certainty equivalent that disentangles prudence effects among beliefs and behavioural attitudes. In chapter 2, I analyse two relevant portfolio problems to show that, when the certainty equivalents implied by models under evaluation are significantly different, ambiguity prudence might induce sizeable non-linearities in the optimal stock allocations. These effects emerge because of the investor's willingness to hedge against downside model uncertainty.

0.1 Introduction

In the first two chapters of this thesis, I investigate the impact of ambiguity prudence on the optimal stock allocation for an investor who evaluates portfolio compositions according to the smooth model under ambiguity approach of Klibanoff et al. (2005). The choice of the smooth preferences is motivated by the possibility to span the whole set of ambiguity attitudes extending many popular decision criteria. However, the non-linear composition of the utility indexes, combined with the two integration layers, constitutes a cost in terms of interpretability.

In chapter 1, I overcome this problem deriving a third-order approximation of the certainty equivalent that allows to study ambiguity prudence in its full generality, without imposing restrictive assumptions over preferences or probability distributions. Theorem 1.2.1 shows that ambiguity prudence induces an uncertainty premium that depends on two statistical moments: the third central moment of the expected return and the covariance among models between the conditional expected return and risk variance. I exploit the third-order approximation to calibrate, consistently with the qualitative evidence collected by Boiney (1993), the value of the preference parameter that governs attitudes towards model uncertainty.

In chapter 2, I analyse the relevance of ambiguity prudence for two portfolio problems in which an investor has to optimally allocate her wealth between a risk-free bond and an ambiguous stock. In the first setting, a *buy-and-hold* investor believes that excess-returns are i.i.d. with known variance and unknown equity premium, over which she has Skew-normal beliefs. I derive the implicit equation of the optimal stock allocation quantifying the relevance of ambiguity prudence and comparing its contribution with the one induced by risk prudence. Results confirm that, for a smooth investor, both risk and ambiguity prudence induce sizeable variations from the robust mean-variance solution of Maccheroni et al. (2013) but the characteristics of the two effects are different from each other. In particular, the relevance of risk prudence is large at short horizons but, as the maturity of the investment increases, the optimal allocation converges rapidly and monotonically to the one implied by the robust mean-variance solution. On the other hand, the contribution of ambiguity prudence is weaker at very short horizons but, because of the faster growth of model uncertainty, its relevance increases rapidly and it exhibits a high persistence across maturities. However, the severe penalization of model uncertainty, due to ambiguity aversion, causes an overall reduction of the stock allocation; thus, at very long horizons, the relevance of ambiguity prudence is mitigated by the limited exposure of the portfolio to sources of uncertainty. These joint effects imply that the contribution of third-order terms induce an inverted U-shape variation from the robust mean-variance solution as a function of the maturity of the investment.

In the second setting, I analyse the problem of an investor who is ambiguous about the time-varying expected excess-return. In a similar way to Chen et al. (2014), I assume that the investor has concerns about excess-returns predictability and she evaluates two possible models: in the first one, excess-returns are i.i.d. while, in the second one, they contain a predictable component.[1] I consider a one-period problem to show that, if the

[1] The predictability of excess-returns has been the subject of an important academic debate among

certainty equivalents implied by models under evaluation are significantly different and the subjective beliefs over the two models are asymmetric, then ambiguity prudence puts non-linearities into the optimal stock allocation. These effects are due to the investor's willingness to hedge her exposure against downside model uncertainty and, regardless of the maturity of an investment, they have a sizeable impact on the optimal portfolio composition.

Finally, I relate the portfolio implications of ambiguity prudence to some empirical evidence that emerges from the robust asset pricing literature. When the smooth preferences coincide with the multiplier preferences of Hansen and Sargent (2001), ambiguity attitudes are embedded in the endogenous distortion of beliefs. If the investor ignores the process governing the growth of aggregate consumption and, as in Hansen and Sargent (2010), she evaluates the i.i.d. growth model and the long-run risk model of Bansal and Yaron (2004), then ambiguity prudence puts non-linearities in the marginal evaluation of the stochastic discount factor. These non-linearities have been shown to provide a relevant contribution to explain some empirical evidence of aggregate financial markets, e.g. the time-varying market uncertainty premium or the excess and counter-cyclical volatility of prices.

To the best of my knowledge, this is the first work in the literature that studies financial implications of attitudes beyond ambiguity aversion and I motivate the research question by three main facts. First, recent experimental evidence, e.g. Baillon et al. (2018), supports the idea that individuals exhibit a strong preference towards prudent choices both under risk and model uncertainty. Second, the interest to investigate asymmetries over the expected return is motivated by the recent work of Martin (2017), in which the author proposes a novel proxy for the equity premium and he shows that, at different investment horizons, the expected excess-return is positively skewed. More generally, several works seem to suggest that higher-order moments of the models' distribution might help to simultaneously explain puzzling links between macroeconomics and aggregate financial

researchers, e.g. Welch and Goyal (2008) or Campbell and Thompson (2007).

markets.[2] Third, ambiguity prudence is a natural complement to risk prudence, whose relevance has been extensively studied in the literature. In fact, as risk prudence is necessary to explain precautionary savings motives, ambiguity prudence might capture economic properties that go beyond ambiguity aversion. This intuition is supported by the work of Guerdjikova and Sciubba (2015), who show how ambiguity prudence explains the survival of ambiguity averse agents in long-run market equilibria.

Related Literature

Since the seminal work of Markowitz (1952), researchers have underlined the importance of future returns distributions for optimal portfolio allocations. Traditionally, this problem has been studied within the rational expectations hypothesis that assumes common knowledge over the objective probability distribution of returns. However, several surveys have shown how investors tend to differ in their assessments about future returns and any consensus about the characteristics of the underlying distribution is difficult to reach. Following the insight of Knight (1921), recent studies have underlined how in a decision process, a sophisticated decision-maker (hereafter DM) has to account for different sources of uncertainty.[3] The first one, typically considered in economic models, is denoted as "Risk" (or uncertainty "within the model") and reflects the stochastic nature of phenomena. A second source, denoted "Model Uncertainty" or "Ambiguity", reflects the limits of DM to quantify the uncertainty related to potential outcomes. Not being able to detect the true mechanism behind some given observations, she evaluates a set of possible models (or data generating processes) and she attaches them a degree of belief (or subjective probability). Finally, models under evaluation might just approximate the true mechanism, creating misspecification issues.

The contribution of model uncertainty in affecting optimal portfolio allocations has

[2]The two most important puzzles are the equity premium puzzle of Mehra and Prescott (1985) and Weil (1989), or the volatility puzzle of Shiller (1981).

[3]See Marinacci (2015), Gilboa and Marinacci (2016) and Hansen et al. (2016).

been initially underlined by important seminal papers, e.g. Klein and Bawa (1976), Bawa et al. (1979), Barberis (2000), Xia (2001) or Kan and Zhou (2007). However, these works consider a Bayesian investor for which risk and ambiguity might be compounded in a single probability measure and the reduced distribution is denoted mean-measure (also barycenter or predictive distribution) of returns. This approach is inconsistent with many experimental results that confirm the validity of Ellsberg (1961) paradox. In the last decades, the flourishing of new decision criteria, e.g. Schmeidler and Gilboa (2004) or Maccheroni et al. (2006), has introduced additional sophistication in the analysis, giving the possibility to disentangle behavioural attitudes among types of uncertainty. When investors exhibit non-neutral ambiguity attitudes, then the certainty equivalent of the portfolio accounts for *ad-hoc* corrections due to ambiguity effects.

In this work, given the key relevance of non-neutral ambiguity attitudes, I relate more closely to the literature that studies the portfolio implications of ambiguity aversion, e.g. Chen and Epstein (2002), Uppal and Wang (2003), Epstein and Wang (2004), Maenhout (2004), Garlappi et al. (2006), and in particular to those works that adopt a smooth representation of preferences, e.g. Gollier (2011), Maccheroni et al. (2013) and Chen et al. (2014). However, these works differ from mine since they focus on financial implications of ambiguity aversion.

Finally, I relate to the asset pricing literature that studies the market price of model uncertainty, e.g. Hansen et al. (2006) or Hansen (2007). The empirical evidence provided by several works, e.g. by Hansen and Sargent (2010) or Ju and Miao (2012), seems to suggest that higher-order ambiguity effects have an important contribution to explain empirical regularities of financial markets, e.g. the time-varying uncertainty premium or the excess volatility of stock prices. More recently, Hansen and Sargent (2019) analyse the macroeconomic implications of model uncertainty when a representative investor is ambiguous about the persistence of the shocks hitting the fundamentals of the economy.

Chapter 1

Approximating Smooth Preferences

1.1 Preferences

In order to study the contribution of ambiguity prudence to the optimal portfolio composition, I assume preferences are represented by the smooth model of decision making under ambiguity in which the certainty equivalent of an ambiguous variable X is given by[1]

$$C(X) = v^{-1}\left(\int_{\Omega} v \circ u^{-1}\left(\int_{\mathbb{X}} u(x)\, K(\omega, dx)\right)\mu(d\omega)\right). \qquad (1.1)$$

Such preferences are characterized by two probabilistic objects: the space of possible realizations $(\mathbb{X}, \mathfrak{X})$ and an abstract set of models which I denote by $(\Omega, \mathcal{F})$.[2] A "model" completely specifies the distribution of all the observable variables and the subjective probability μ over $(\Omega, \mathcal{F})$ reflects the degree of belief of the investors towards different data generating processes. Conditioning on a given $\omega \in \Omega$, $K(\omega, \cdot)$ defines a probability measure over $(\mathbb{X}, \mathfrak{X})$ so that, for each pair $(\omega_1, A) \in \Omega \times \mathfrak{X}$, $K(\omega_1, A)$ denotes the probability that model ω_1 attaches over the event A. Since I do not want to impose *a priori* any constraint over uncertainty, I assume $K : \Omega \times \mathfrak{X} \to [0, 1]$ to be a Random Probability

[1] These preferences have been axiomatized by Klibanoff et al. (2005).

[2] I assume $(\mathbb{X}, \mathfrak{X})$ to be a Polish space, e.g. an Euclidean space. This assumption guarantees the existence of a regular version of the conditional (to the model) probability.

Measure.[3]

A key feature of smooth preferences is the possibility to span the whole sets of risk and ambiguity attitudes.[4] In particular, attitudes towards model uncertainty are characterized using an *ad hoc* second order utility index and this favours the adoption of the (properly adapted) machinery developed within risk theory. To see this, notice that $C(X)$ is a composition of two certainty equivalents: for each model $\omega \in \Omega$, $c(\omega, X) = u^{-1}\left(\int_{\mathbb{X}} u(x) K(\omega, dx)\right)$ denotes the certainty equivalent associated to the data generating process $K(\omega, \cdot)$. Subsequently, each $c(\omega, X)$ is evaluated through v and averaged with a weight given by μ, $C(X) = v^{-1}\left(\int_{\Omega} v(c(\omega, X)) \mu(d\omega)\right)$.

The key tenet of Subjective Expected Utility (or Bayesian) framework requires that a DM evaluates lotteries under the mean-measure probability distribution, i.e. $u = v$, and this mechanically implies she has neutral attitudes towards ambiguity. However, starting from Ellsberg (1961), many experiments demonstrate that individuals dislike situations in which the true probabilistic model is unknown. This behavioural attitude has been defined ambiguity aversion and a formal definition has been given by Schmeidler and Gilboa (2004) in terms preferences over Anscombe et al. (1963) acts. More recently, Berger (2014) has introduced in the literature the idea of prudence towards ambiguity. The insight behind this behavioural attitude is that a DM might be afraid to face model uncertainty when she believes to be most likely to suffer.

In order to highlight the difference between ambiguity aversion and prudence, I consider a framework that is similar to the one analysed by Eeckhoudt and Schlesinger (2006), but with some important differences. In particular, risk attitudes are defined in terms of preferences over pairs of simple lotteries that differ in the aggregation or disaggrega-

[3]For the formal definition of Random Probability Measure see Appendix. In parametric settings, elements of Ω might be indexed by a finite dimensional vector of parameters that characterizes the data generating process.

[4]For example, the classical Bayesian setting is retrieved when u and v have the same shape, i.e. $\phi(x) = id(x)$, while the Multiple-Priors approach is obtained in the limit case as $A_{\phi}(x) \to +\infty$.

tion of sure-losses or mean-preserving spread at the outcome level. On the other hand, ambiguity attitudes are defined in terms of preferences over sets of models that aggregate/disaggregate probability losses or mean-preserving spread of outcomes' probabilities. To help the intuition, suppose a DM has to rank the following lotteries

$$\mathbf{L}_1 = \left(\frac{1}{2}, x - k_1; \frac{1}{2}, x - k_2\right) \quad \mathbf{L}_2 = \left(\frac{1}{2}, x; \frac{1}{2}, x - k_1 - k_2\right),$$

whose outcomes are monetary values. If, for all $k_1, k_2 > 0$, a DM prefers the lottery that disaggregates the sure losses across the two equally-likely states of the world instead of the lottery that aggregates them, i.e. $\mathbf{L}_1 \succsim \mathbf{L}_2$, then she displays risk aversion. Similarly, given any two outcomes $x \succsim y$ and a subjective probability over models under evaluation

$$\mathbf{L}_1^c = \left(\frac{1}{2}, M_1; \frac{1}{2}, M_2\right) \quad \mathbf{L}_2^c = \left(\frac{1}{2}, M_3; \frac{1}{2}, M_4\right),$$

where

$$M_1 = ((p - q_1), x; (1 - p + q_1), y), \quad M_2 = ((p - q_2), x; (1 - p + q_2), y),$$

$$M_3 = (p, x; (1 - p), y), \quad M_4 = ((p - q_1 - q_2), x; (1 - p + q_1 + q_2), y),$$

a DM is ambiguity averse if $\mathbf{L}_1^c \succsim \mathbf{L}_2^c$ for every $q_1, q_2 > 0$.[5] The intuition is that an ambiguity averse agent dislikes situations in which the favourable outcome x is, in relative terms, either very likely or very unlikely. Therefore, she dislikes $\mathbf{L}_2^c$ that aggregates the probability losses within the same model M_3.

Recently experimental evidence, e.g. Baillon et al. (2018), suggests that DMs exhibit also attitudes beyond ambiguity aversion and, in particular, ambiguity prudence. Suppose that in $\mathbf{L}_1, \mathbf{L}_2$, the sure monetary loss k_2 is replaced by an uninsurable zero-mean risk ε

$$\mathbf{L}_3 = \left(\frac{1}{2}, x - k_1; \frac{1}{2}, x + \varepsilon\right) \quad \mathbf{L}_4 = \left(\frac{1}{2}, x; \frac{1}{2}, x - k_1 + \varepsilon\right).$$

If, for all $k_1 > 0$ and ε s.t. $\mathbb{E}[\varepsilon] = 0$, a DM prefers to disaggregate a sure loss from an additional source of risk, i.e. $\mathbf{L}_3 \succsim \mathbf{L}_4$, then she is risk prudent. Likewise, if in $\mathbf{L}_1^c, \mathbf{L}_2^c$ the

[5] Here, it is implicitly required that $p, (p - q_1), (p - q_2), (p - q_1 - q_2) \in (0, 1)$. Intuitively, the probability losses q_1, q_2 are the counterparts of sure money losses under risk.

probability loss q_2 over the favourable outcome x is replaced by a mean-preserving spread of p, $\tilde{\varepsilon} = \left(\frac{1}{2}, \tilde{p}; \frac{1}{2}, -\tilde{p}\right)$, I obtain

$$\mathbf{L}_3^c = \left(\frac{1}{2}, M_5; \frac{1}{2}, M_6\right) \quad \mathbf{L}_4^c = \left(\frac{1}{2}, M_7; \frac{1}{2}, M_8\right),$$

where

$$M_5 = \left(\left(p - q_1\right), x; \left(1 - p + q_1\right), y\right) \quad M_6 = \left(\left(p + \tilde{\varepsilon}\right), x; \left(1 - p - \tilde{\varepsilon}\right), y\right)$$

$$M_7 = \left(p, x; \left(1 - p\right), y\right) \quad M_8 = \left(\left(p - q_1 + \tilde{\varepsilon}\right), x; \left(1 - p + q_1 - \tilde{\varepsilon}\right), y\right),$$

and $p, \left(p - q_1\right), \left(p + \tilde{\varepsilon}\right), \left(p - q_1 + \tilde{\varepsilon}\right) \in (0, 1)$.

Definition. *A DM is ambiguity prudent if* $\mathbf{L}_3^c \succsim \mathbf{L}_4^c$ *for all* $x \succsim y$.

The intuition underlying the above definition is that, given a set of models, an ambiguity prudent DM prefers to disentangle a probability loss over the most favourable outcome from an additional source of ambiguity. Rephrasing, a DM is afraid to face uncertainty in those models in which she is most likely to suffer and prefers lotteries that prevent such possibility. Figure 1.1 summarizes the risk and ambiguity attitudes discussed so far.

Finally, a good feature of the smooth model is that attitudes towards uncertainty might be easily characterized in terms of mathematical properties of utility indexes u and $\phi := v \circ u^{-1}$.[6] Klibanoff et al. (2005) have shown that a DM is ambiguity averse iff $\phi'' < 0$, while Baillon (2017) has shown that a DM is ambiguity prudent iff $\phi''' > 0$. If the utility indexes are monotone and increasing and $u'' < 0$, then

$$\phi''(x) < 0 \iff A_v(x) - A_u(x) > 0, \tag{1.2}$$

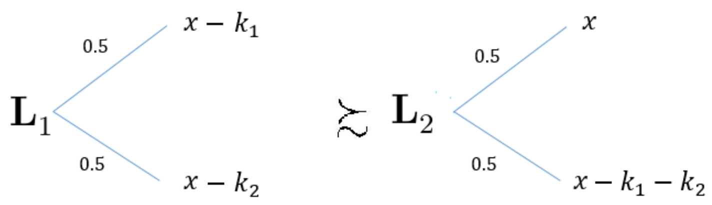
$$\phi'''(x) > 0 \iff A_v(x) P_v(x) - A_u(x) P_u(x) - 3A_u(x)\left(A_v(x) - A_u(x)\right) > 0, \tag{1.3}$$

where $A_f(x) := -f''(x)/f'(x)$ and $P_f(x) := -f'''(x)/f''(x)$, $f \in \{u, v\}$, are the standard Arrow-Pratt measures of the degree of absolute risk/model uncertainty aversion and prudence.

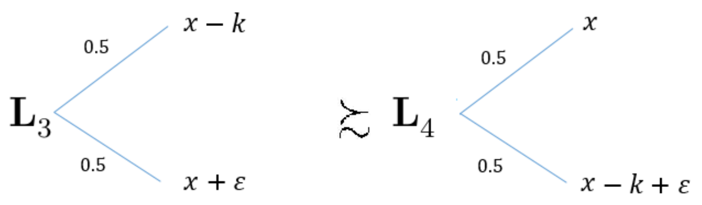
[6] The function $v : c(\Omega, \mathbb{X}) \to \mathbb{R}$ captures attitudes towards model uncertainty *stricto sensu*.

Figure 1.1: Risk and ambiguity attitudes

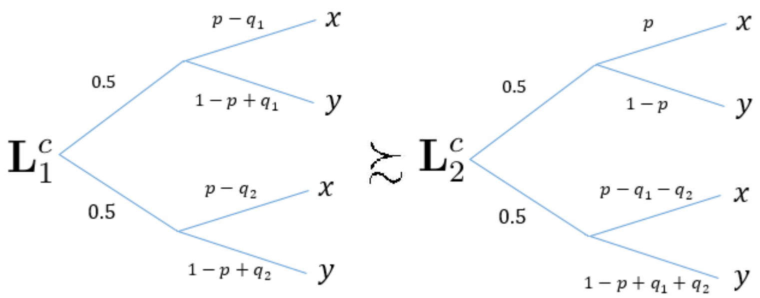
(a) Risk-aversion



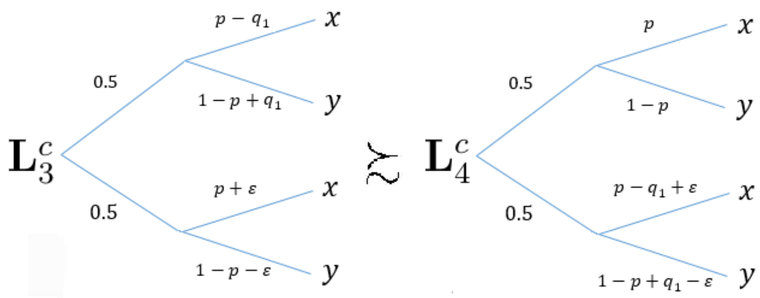
(b) Risk-prudence



(c) Ambiguity-aversion



(d) Ambiguity-prudence



1.2 Analytical treatment

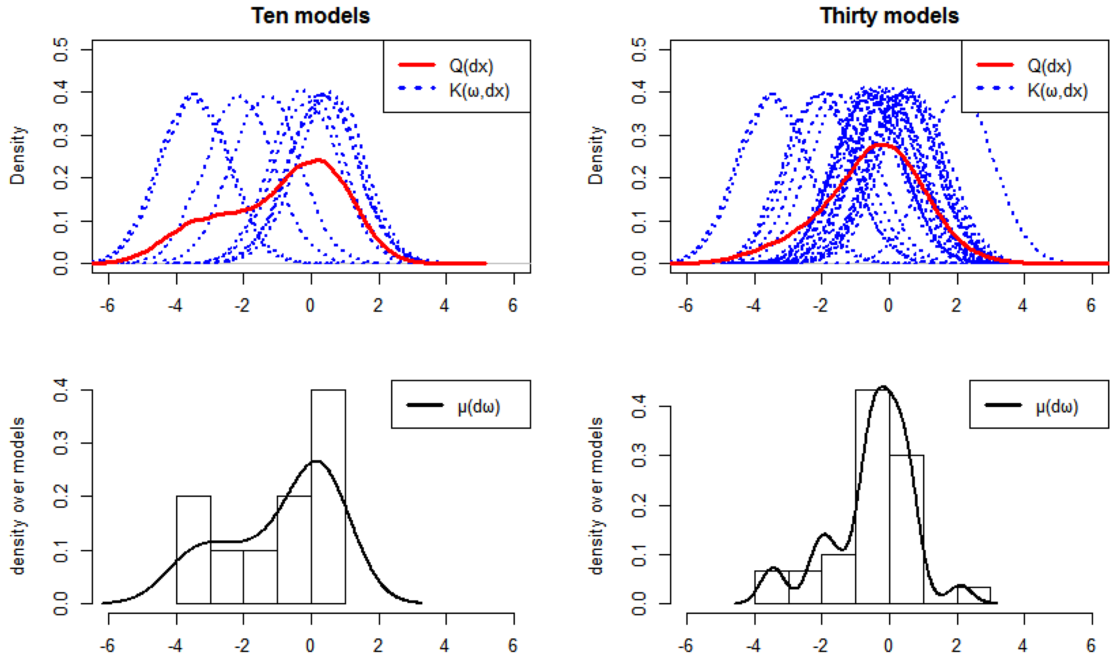
Although the signs of the derivatives of ϕ provide information about the type of ambiguity attitudes, the proper contribution of each behavioural feature to the optimal choice is nested with the distribution of beliefs. In order to disentangle the overall effect between ambiguity aversion and prudence, the non-linear composition of (1.1), combined with the two integration's layers, constitutes a price in terms of tractability and interpretability. A natural way to overcome the problem is to use a linear approximation and, similar to Maccheroni et al. (2013), I propose a Maclaurin expansion that introduces third-order effects in the analysis. This choice allows to study ambiguity prudence in its full generality, i.e. without imposing restrictive assumptions over preferences or probability distributions.

Before proceeding, I define Q to be the mean-measure distribution of X, such that for any event $A \in \mathfrak{X}$,

$$Q(A) := \int_{\Omega} \int_{\mathbb{X}} \mathbf{1}_A(x)\, K(\omega, dx)\, \mu(d\omega).$$

If an investor evaluates a finite number of models, $\Omega = \{\omega_1, \omega_2, ..., \omega_n\}$, then $Q(A) = \sum_{i=1}^{N} \mu(\omega_i)\, K(\omega_i, A)$. In figure 1.2, I illustrate two examples in which a DM evaluates respectively a set of ten and thirty models, that describe the possible distribution of X. The two upper graphs show the mean-measure (in red) and the conditional distributions implied by each model (dotted blue lines) when $X \mid \omega$ follows a Gaussian law $K(\omega, \cdot) \sim N(\omega, 1)$. On the other hand, the histograms and the approximating densities, reported in the lower graphs, show the subjective distributions over the set of models. To give an intuition on how higher-order terms impact the certainty equivalent, I start considering the Subjective expected utility framework in which $u = v$. The following lemma provides an approximation of the certainty equivalent for an ambiguity neutral agent that evaluates a prospect $Y := b + aX$, $a, b \in \mathbb{R}$. I express the ambiguous variable as a linear combination to emphasize the possibility of disentangling its final value between a deterministic component b and an uncertain one aX, with a denoting the degree of exposition of Y towards uncertainty.

Figure 1.2: Mean-measure and distribution over models



The two figures above plot the densities of the models under evaluation (blue dotted lines), that follow a Gaussian law $K(\omega, \cdot) \sim N(\omega, 1)$, and of the associated mean-measure distribution (red lines). The two bottom graphs show the density distributions μ over the two sets of models.

Lemma. *Let $u : D \subseteq \mathbb{X} \to \mathbb{R}$, $u \in C^3(D)$ and X, $u \in L^3(\mathbb{X}, \mathfrak{X}, Q)$. Then*

$$\begin{aligned} C(b + aX) = {} & b + a\mathbb{E}_Q(X) - \frac{a^2}{2} A_u(b) Var_Q(X) \\ & + \frac{a^3}{6} A_u(b) P_u(b) \mathbb{M}_Q^3(X) - \frac{a^3}{2} A_u'(b) \mathbb{E}_Q[X] Var_Q(X) + o(|a^3|), \end{aligned} \tag{1.4}$$

where $\mathbb{M}_Q^3(X) := \mathbb{E}_Q[X - \mathbb{E}_Q[X]]^3$ and $A_u'(b)$ denotes the variation of risk aversion.

The contribution of third-order terms appears in the second line of (1.4) and it is divided into a pure third moment effect, scaled by the product between the indexes of absolute risk prudence and absolute risk aversion, and a mean-variance interaction term scaled by the adjustment in risk aversion. The lemma confirms that, for an ambiguity neutral DM, probabilities over models and outcomes might be compounded and the only relevant statistical moments are the ones of the mean-measure Q. On the other hand, if the DM has non-neutral attitudes towards ambiguity, then v and u have different shapes and the statistical moments of μ contribute with *ad-hoc* corrections.

Theorem 1.2.1. *Let* $u : D \subseteq \mathbb{X} \to \mathbb{R}$, $v : u^{-1}(D) \to \mathbb{R}$ *be three times continuously differentiable and* $X, u, v \in L^3(\mathbb{X}, \mathfrak{X}, Q)$. *Then*

$$\begin{aligned} C(b+aX) &= b + a\mathbb{E}_Q(X) - \frac{a^2}{2}A_u(b)\,Var_Q(X) - \frac{a^2}{2}(A_v(b) - A_u(b))\,Var_\mu(\mathbb{E}_\omega(X)) \\ &+ \frac{a^3}{6}A_u(b)\,P_u(b)\,\mathbb{M}_Q^3(X) + \frac{a^3}{6}(A_v(b)\,P_v(b) - A_u(b)\,P_u(b))\,\mathbb{M}_\mu^3(\mathbb{E}_\omega[X]) \\ &- \frac{a^3}{2}A_u'(b)\,\mathbb{E}_Q[X]\,Var_Q(X) - \frac{a^3}{2}\left(A_v'(b) - A_u'(b)\right)\mathbb{E}_Q[X]\,Var_\mu(\mathbb{E}_\omega[X]) \\ &+ \frac{a^3}{2}A_u(b)(A_v(b) - A_u(b))\,Cov_\mu(\mathbb{E}_\omega[X], Var_\omega(X)) + o\left(\left|a^3\right|\right), \end{aligned} \tag{1.5}$$

where $\mathbb{E}_\omega[X] := \mathbb{E}_{K(\omega,\cdot)}[X]$ *and* $\mathbb{M}_\mu^3(\mathbb{E}_\omega[X]) := \mathbb{E}_\mu[\mathbb{E}_\omega[X] - \mathbb{E}_Q[X]]^3$.

Proof. See proof in the appendix. □

A good feature of the above approximation is the additive separability between the effects related to risk and model uncertainty. Similar to the Subjective expected utility framework, the contribution of risk is evaluated under the mean-measure probability. However, for a smooth investor, model uncertainty provides an additional contribution that depends on two elements: the statistical moments of μ and the strength of ambiguity attitudes embedded in the shape of ϕ.

Since, in many applications, model uncertainty is generated starting from the distributions of models under evaluation, it might be useful to explicit the conditional statistical moments. Exploiting the law of total cumulance,

$$Var_Q(X) = \mathbb{E}_\mu\left[Var_{K(\omega,\cdot)}(X)\right] + Var_\mu(\mathbb{E}_\omega[X]),$$

$$\mathbb{M}_Q^3(X) = E_\mu\left[\mathbb{M}_{K(\omega,\cdot)}^3(X)\right] + \mathbb{M}_\mu^3(\mathbb{E}_\omega[X]) + 3Cov_\mu(\mathbb{E}_\omega[X], Var_\omega(X)),$$

it is possible to rearrange approximation (1.5) as

$$\begin{aligned}C(b+aX) &= b + a\mathbb{E}_Q(X) - \frac{a^2}{2}A_u(b)\,\mathbb{E}_\mu[Var_\omega(X)] - \frac{a^2}{2}A_v(b)\,Var_\mu(\mathbb{E}_\omega(X)) \\ &+ \frac{a^3}{6}A_u(b)\,P_u(b)\,\mathbb{E}_\mu\left[\mathbb{M}^3_{K(\omega,\cdot)}(X)\right] + \frac{a^3}{6}A_v(b)\,P_v(b)\,\mathbb{M}^3_\mu(\mathbb{E}_\omega[X]) \\ &- \frac{a^3}{2}A_u^{'}(b)\,E_Q[X]\,\mathbb{E}_\mu[Var_\omega(X)] - \frac{a^3}{2}A_v^{'}(b)\,\mathbb{E}_Q[X]\,Var_\mu(\mathbb{E}_\omega[X]) \\ &+ \frac{a^3}{2}A_u(b)\,(A_v(b) - A_u(b) + P_u(b))\,Cov_\mu(\mathbb{E}_\omega[X], Var_\omega(X)) + o\left(\left|a^3\right|\right),\end{aligned} \tag{1.6}$$

in which the moments of Q have been replaced by the averages of the conditional moments.

Before discussing the contribution of third-order terms, notice that the first three terms in the first line of (1.5) correspond to the classical Arrow-Pratt approximation. The fourth term, introduce by Maccheroni et al. (2013), embeds in the evaluation of the certainty equivalent the contribution of ambiguity aversion whose relevance is proportional to the variance of the conditional expectation.

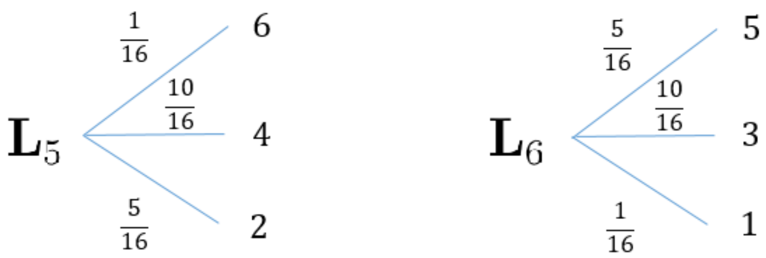
1.2.1 Higher-order effects

While the second central moment of a probability distribution provides information about the size of uncertainty, the third central moment provides information about the size of downside uncertainty. In general, given a probability measure, it is possible to apply a proper combination of mean preserving spreads and mean preserving contractions to shift probability mass from the right to the left side of the distribution, without changing the first two moments.[7] As a consequence of such transformation, the downside uncertainty of the resulting distribution increases but, by definition, the variance is unchanged. From a statical perspective, increments of downside uncertainty are captured by variations in the degree of asymmetry that partially depend on third moments. To see it, notice that if a variable Z follows a distribution P, then the following factorization holds

$$\mathbb{M}^3_P(Z) = Var_P^{\frac{3}{2}}(Z)\,\gamma_{1,P}(Z),$$

[7] This combination belongs to the set of the mean-variance preserving transformations (MVPT).

where $\gamma_{1,P}(Z)$ denotes the Pearson's coefficient of skewness. More properly, Ebert (2013) shows that downside uncertainty relates to all odd moments of the distribution and not only to the third one. The point is showed in the following example



in which, although for both lotteries $\mathbb{M}^3(\mathbf{L}_5) = \mathbb{M}^3(\mathbf{L}_6) = 0$, $\mathbf{L}_6$ exhibits more downside risk since $\mathbb{M}^5(\mathbf{L}_5) > 0 > \mathbb{M}^5(\mathbf{L}_6)$.

Risk prudence

The contribution of risk prudence to the certainty equivalent is due to the first term on the second line of (1.5) where the overall effect depends on the size of $\mathbb{M}_Q^3$ scaled by $A_u(x)\,P_u(x)$. Following the work of Modica and Scarsini (2005), I interpret $A_u(x)\,P_u(x)$ as a local coefficient of downside risk aversion, in which the sign of the third derivative of u governs preferences both towards downside risk and precautionary savings motives.[8] In fact, Kimball (1990) defines an agent to be risk prudent when $u^{'''} > 0$ and a similar interpretation of $u^{'''} > 0$ has been given by Menezes et al. (1980) in terms of aversion towards downside risk.

If DM shows non-satiation and risk prudence, i.e. $A_u(x)\,P_u(x) > 0$, then any increment of $\mathbb{M}_Q^3$ increases the value of certainty equivalent.[9] However, a typical assumption in many applications that account for model uncertainty requires that returns are Gaus-

[8] Although the third derivative of u appears also in the variation of risk adjustment, it as a pure third moment effect, see Eeckhoudt and Schlesinger (2006) page 287 line 10.

[9] A portfolio implication of the convexity of $u^{'}$ is that the risk premium is decreasing in the level of wealth.

sian distributed with a Gaussian-Inverse Gamma prior or an uninformative prior over the unknown parameters. This choice mechanically implies that mean-measure distribution resulting from the Bayesian updating is symmetric, i.e. $\mathbb{M}_Q^3(X) = 0$, and therefore risk prudence becomes irrelevant.

Ambiguity prudence

If a DM has non-neutral attitudes towards ambiguity, then also the third central moment of the distribution of the average quantities has a direct impact on the certainty equivalent. In particular, the magnitude of $\mathbb{M}_\mu(E_\omega[X])$ is scaled by $A_v(x)P_v(x) - A_u(x)P_u(x)$ that, following Proposition 1 in Modica and Scarsini (2005), I interpret as an excess of downside aversion to model uncertainty. If $A_v(x)P_v(x) - A_u(x)P_u(x) > 0$ then a DM will be ready to pay an excess premium to hedge against downside model uncertainty $\mathbb{M}_\mu(E_\omega[X]) < 0$ or, equivalently, she is ambiguity skewness seeking.[10]

Differently from the risk setting, ambiguity prudence and ambiguity skewness seeking are not equivalent properties. However, if a DM displays non-satiation, ambiguity aversion and ambiguity prudence then $sgn(A_v(x)P_v(x) - A_u(x)P_u(x)) > 0$. The idea is that preferences towards positive ambiguity skewness are necessary but not sufficient conditions for ambiguity prudence since DM needs also to consider the co-movements between $\mathbb{E}_\omega[X]$ and $Var_\omega(X)$.

The covariance term in (1.5) implies that a DM evaluates also co-movements of the model-conditional moments. The way these co-movements impact on the certainty equivalent depends on the investor's attitudes both towards risk and model uncertainty. Theorem (1.2.1) shows that the covariance between mean and variance of conditional probabilities is scaled by two factors: the Arrow-Pratt index of risk aversion and the excess of model uncertainty penalization. The more v is concave, the more a DM penalizes variations of the inner certainty equivalent induced by models in the support of μ. Rephrasing, DMs dislike to be uncertain about models that imply low mean and high variance and

[10]The sign of $\mathbb{M}_\mu(\mathbb{E}_\omega[X])$ characterizes the sign of ambiguity skewness.

models that imply high mean and low variance. Therefore, if a DM is risk and ambiguity averse, then a positive covariance increases the certainty equivalent. Instead, if u is a linear function then the conditional variance is irrelevant in computing the inner certainty equivalent $c(\omega, X)$ and the covariance term disappears. In case investors is ambiguity neutral, $u = v$, the two sources of uncertainty compound and the only relevant moments are those under the mean probability Q.

Other terms

The other two terms on the third line of (1.5) display an interaction between mean and variance of X and $\mathbb{E}_\omega[X]$ under the respective probability measures. Since later on I require positive expected excess-return, i.e. $\mathbb{E}_Q[X] > 0$, the sign of these terms depends on the signs of $A_u'(x)$ and $A'_v(x) - A'_u(x)$, that are respectively the variation in the absolute risk aversion and the difference between the variations in model uncertainty and risk aversion. If I explicit such variations as

$$A_f'(x) = A_f(x)\left[A_f(x) - P_f(x)\right],$$

it appears that the sign of $A_f'(x)$ depends also on $f'''(x)$. However, a positive third derivate is not enough to imply a reduction in risk aversion but I need the stronger condition that $f'''(x) \geq f''^2(x)/f'(x)$.[11] If a function f has the CARA form, then $A_f'(x) = 0$; on the other hand, if it has the DARA (or IARA) form, then $A_f'(x) < 0$ $\left(A_f'(x) > 0\right)$.

1.3 Preference parameters

In this section, I exploit the type of evidence collected by Boiney (1993) to propose a calibration of the preferences parameter that governs attitudes towards ambiguity. Consistently with standard empirical works in the literature, I assume utility indexes have

[11] See Eeckhoudt and Schlesinger, 2006, page 287 line 5.

the CRRA form

$$u(x) = \frac{x^{1-\rho}}{1-\rho} \quad v(x) = \frac{x^{1-\eta}}{1-\eta},$$

and I look for the values of η that are consistent with the experimental results. To infer them, I assume DMs preferences are represented by approximation (1.5), from which I neglect the error term. Since for CRRA utilities, the indexes A_u, A_v, P_u, P_v depend on the level of wealth, I consider two alternative values of b that are respectively 50% higher and lower than the average expected outcome. In section 3.1.2, I use the result of this calibration to quantify the relevance of ambiguity attitudes for the optimal portfolio composition.

During his laboratory experiment with 130 MBA students, Boiney tried to elicit individual preferences asking their opinion over three different decision problems. In each, students are asked what they would had rather choose between two given assets: an ambiguous one and a risky one. While the risky asset is invariant among the three decision problems, the characteristics of the ambiguous ones vary. The figure below shows the four assets used in the experiment.

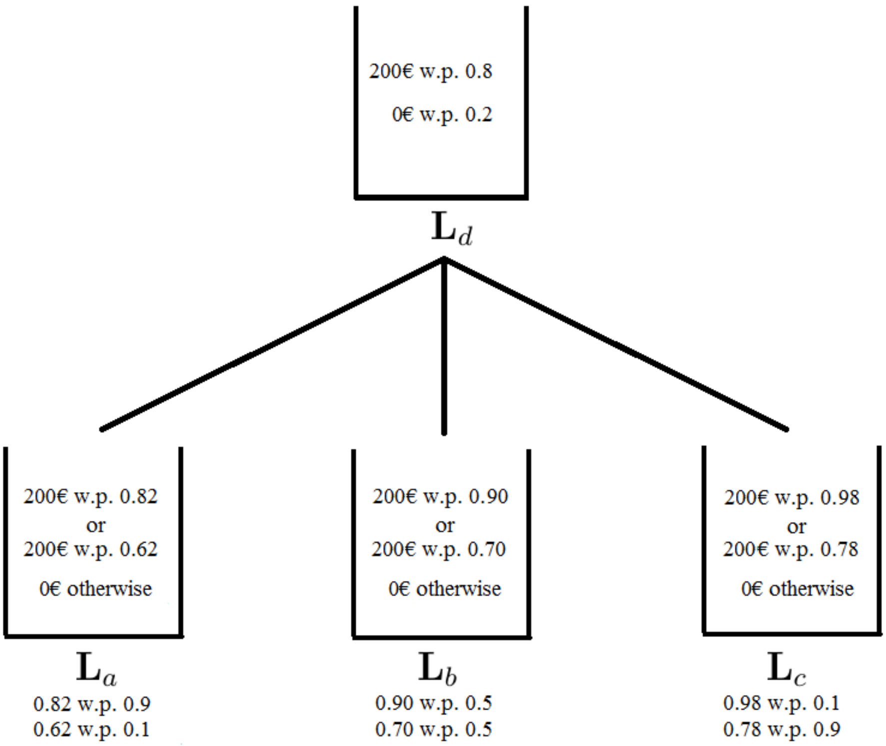


Table 1.1: Statistical moments

Assets	$\mathbb{E}_Q$	Var_Q	$\mathbb{M}^3_Q$	Var_μ	$\mathbb{M}^3_\mu$	Cov_μ
$\mathbf{L}_a$	160	6400	-768000	144	-4608	-12672
$\mathbf{L}_b$	160	6400	-768000	400	0	-48000
$\mathbf{L}_c$	160	6400	-768000	144	4608	-21888
$\mathbf{L}_d$	160	6400	-768000	0	0	0

For each ambiguous asset there are two possible models governing return distribution, that in case of asset $\mathbf{L}_a$ are

$$P(\mathbf{L}_a = 200 \mid \omega_1) = 0.82 \quad P(\mathbf{L}_a = 0 \mid \omega_1) = 0.18,$$
$$P(\mathbf{L}_a = 200 \mid \omega_2) = 0.62 \quad P(\mathbf{L}_a = 0 \mid \omega_2) = 0.38,$$

so that $\mathbb{E}_{\omega_1}[\mathbf{L}_a] = 164$, $\mathbb{E}_{\omega_2}[\mathbf{L}_a] = 124$, $Var_\mu(\mathbb{E}_\omega[\mathbf{L}_a]) = 144$, etc. Table 1.1 summarizes the values of the statistical moments that appear in (1.5).

Experimental results suggest that DMs prefer asset $\mathbf{L}_c$ over the risky one $\mathbf{L}_d$ and both prospects are more favourable than $\mathbf{L}_b$ and $\mathbf{L}_a$, with this latter being the most disliked. Although their pairwise rankings $\mathbf{L}_d \succsim \mathbf{L}_a$ and $\mathbf{L}_d \succsim \mathbf{L}_b$ are consistent with Ellsberg's idea, the fact that $\mathbf{L}_c \succsim \mathbf{L}_d$ suggests that ambiguity aversion might not be the only relevant attitude in decision-making.[12] Moreover, since each asset has the same distribution under the mean-measure Q, these preferences depend only on DM's attitudes towards ambiguity.

To find the values of preference parameters that are consistent with such rankings, I set ρ to take two standard values in portfolio literature, $\rho \in \{2, 5\}$ and I consider the differences between the certainty equivalents of ambiguous and risky assets as a function

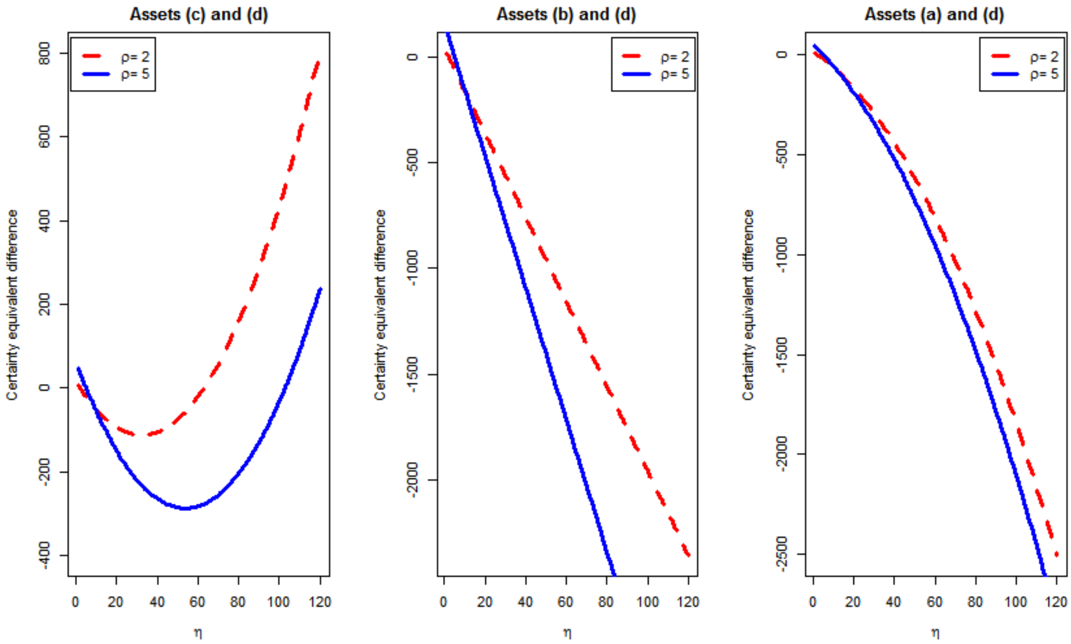
[12] The idea that a DM might prefer a prospect that is relatively more ambiguous as long as it is perceived to be relative more prudent. Among others, Curley et al. (1986) motivate such preferences supporting the "other-evaluation" hypothesis, which claims individuals choose the actions that can be more easily justified by others.

of η. If such difference takes negative values, e.g. $C(\mathbf{L}_d) - C(\mathbf{L}_b) < 0$, I assume DM prefers the risky asset over the ambiguous one.

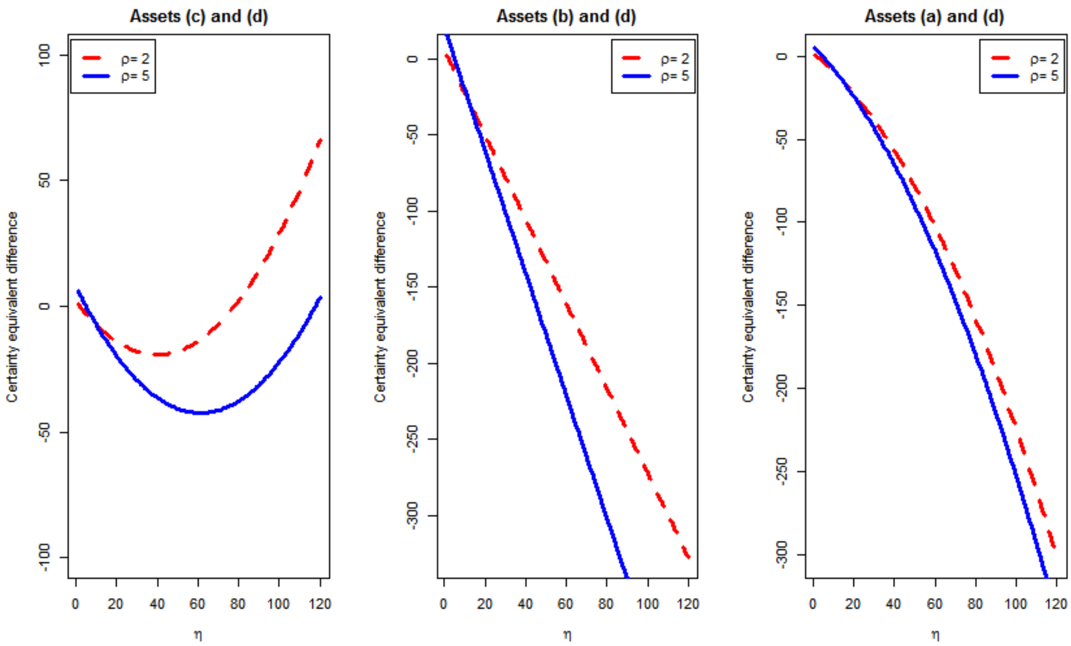
The two left graphs in figure 1.3 show that the difference between $C(\mathbf{L}_c) - C(\mathbf{L}_d)$ is U-shaped and this is due to the higher-order terms. In fact, since $sgn\left(\mathbb{M}^3_\mu(\mathbf{L}_c)\right) \neq sgn\left(Cov_\mu\left(\mathbb{E}_\omega[\mathbf{L}_c], Var_\omega(\mathbf{L}_c)\right)\right)$, the difference attains a positive value when η is large enough and this implies that ambiguity prudent attitudes have to be strong. Differently, assets $\mathbf{L}_a$ and $\mathbf{L}_b$ display negative third-order effects therefore therefore, for increasing value of η, either ambiguity aversion and prudence induce a reduction of the two assets certainty equivalent.

Figure 1.3: Differences between certainty equivalents

(a) $w = 80$



(b) $w = 240$



The graphs show the pairwise difference between the certainty equivalents of the ambiguous assets and of the risky one. Negative values imply that a DM prefers the risky asset.

Chapter 2

Portfolio Allocation

In this chapter, I formally consider the portfolio problem for an investor who has to allocate her wealth W_0, for simplicity normalized to 1, between two assets: an ambiguous stock with gross return X and a risk-free bond R_f. The investor wants to maximize her terminal wealth and, since she ignores the probabilistic law of X, she evaluates a set of possible models indexed by $\omega \in \Omega$, i.e. $\{K(\omega, \cdot), \omega \in \Omega\}$. Portfolio compositions are ranked according to smooth preferences, therefore the investor's problem becomes

$$\max_{\pi \in \mathbb{R}} v^{-1} \left(\int_\Omega v \circ u^{-1} \left(\int_\mathbb{R} u(W) K(\omega, dx) \right) \mu(d\omega) \right) \tag{2.1}$$

$$s.t. \quad W = \pi X + (1 - \pi) R_f. \tag{2.2}$$

In order to quantify the contribution of non-neutral ambiguity attitudes, I compare results obtained under smooth preferences with those implied by subjective expected utility for which the maximization problem simplifies to

$$\max_{\pi \in \mathbb{R}} u^{-1} \left(\int_\mathbb{R} u(W) Q(dx) \right)$$

$$s.t. \; (2.2).$$

A central pillar in portfolio theory, because of its elegance and compelling tractability, is the mean-variance approach. In its classic formulation, the idea is that investors look at the first two moments of the returns distribution and aim at those portfolios with low variability and high expected return. In case the distribution of X is ambiguous and preferences are as in (2.1), Maccheroni et al. (2013) derive a model-free robust mean-variance solution of the optimal stock allocation, in which investors penalize the additional uncertainty over the expected return

$$\pi_{MV} = \frac{\mathbb{E}_Q[X] - R_f}{A_u(w)\,Var_Q(X) + \underbrace{(A_v(w) - A_u(w))\,Var_\mu(\mathbb{E}_\omega[X])}_{=\text{Ambiguity aversion correction}}}, \tag{2.3}$$

with $w = R_f$. If u, v are of the CARA form and the probability measures in the set $\{\mu, K(\omega, \cdot) : \forall \omega \in \Omega\}$ are Gaussian distributed, then the mean-variance approximation coincides with the theoretical optimal allocation and higher-order ambiguity attitudes are irrelevant for the optimal portfolio composition. The tractability of the Gaussian distribution together with the interpretability of the mean-variance approach has induced most of the studies to focus on ambiguity aversion.

In this work, I relax the conjugate Gaussian assumption focusing on the impact of ambiguity prudence on the optimal portfolio allocation. Theorem 1.2.1 defines the third-order approximation of the portfolio certainty equivalent so that, when the utility indexes are given by $v = -e^{-\eta x}$ and $u = -e^{-\rho x}$, the optimal stock allocation π satisfies

$$\pi \approx \pi_{MV} + \frac{\pi^2}{2} \underbrace{\frac{\rho^2 \mathbb{M}^3_Q(X)}{\rho Var_Q(X) + (\eta - \rho)\,Var_\mu(\mathbb{E}_\omega(X))}}_{\text{risk prudence variation}} \tag{2.4}$$

$$+ \frac{\pi^2}{2} \underbrace{\frac{(\eta^2 - \rho^2)\,\mathbb{M}^3_\mu(\mathbb{E}_\omega[X]) + 3\rho(\eta - \rho)\,Cov_\mu(\mathbb{E}_\omega[X], Var_\omega(X))}{\rho Var_Q(X) + (\eta - \rho)\,Var_\mu(\mathbb{E}_\omega(X))}}_{\text{ambiguity prudence variation}}.$$

While the first term on the RHS in (2.4) coincides with robust mean-variance solution reported in (2.3), other terms induce a variation that depends on third-order effects. When investors display aversion and prudence towards ambiguity,

$$\mathbb{M}^3_\mu(\mathbb{E}_\omega[X]) \geq |\, Cov_\mu(\mathbb{E}_\omega[X], Var_\omega(X)) \,|$$

is a sufficient condition to guarantee an increment of the optimal stock allocation. Noteworthy, third-order terms are quadratic in π, therefore a low absolute value of π_{MV} mitigates their relevance and, more importantly, positive and negative higher-order effects have different impacts over the optimal stock investment.[1]

In order to show that the quantitative relevance of the non-linearities induce by ambiguity prudence can be very large, in the following subsections I analyse two different portfolio problems. In section 2.1, I focus on the contribution of ambiguity skewness when an investor believes returns are i.i.d. with unknown expected return. I consider two simple generalizations of the classical conjugate Gaussian setting in which beliefs over the expected return, and later on also over returns, follow a Skew-normal distribution. However, ambiguity prudent behaviours arise also when uncertainty over parameters is neglected but time dependence among the expected returns is unknown. To show it, in section 2.2, I analyse the problem of an investor who is ambiguous about the predictability of the excess-returns.

2.1 I.I.D. returns

2.1.1 Gaussian returns

Consider a portfolio problem in which an investor cares only about her final level of wealth and believes stock return X is Gaussian distributed with known variance σ^2 and unknown expected return m, over which she attaches Skew-normal beliefs, i.e.

$$X = m + \varepsilon, \tag{2.5}$$

$\varepsilon \sim N(0, \sigma^2)$ and $m \sim SN(\xi, \tau_1, \alpha_1)$.[2] Compared to the conjugate Gaussian setting, the skew-normality implies that an investor evaluates the same set of Gaussian models but attaching them a different subjective probability. In particular, if the expected return is negatively skewed, then the left tail of the distribution is thicker and investor believes

[1] The first is a trivial consideration since the error of the robust mean-variance approximation is $o(|\pi^{*2}|)$.

[2] See Appendix for a brief description.

an harmful model is more likely to realize. I provide a twofold motivation for the Skew-normal distribution: first, it is a natural choice to introduce higher-order effects through the perturbation of a Gaussian law. Second, as a side benefit, it is parametrized by a low dimensional vector of parameters and this allows a good trade-off between a more flexible shape of μ and a gain in tractability.

Suppose preferences are represented by (2.1) with utility indexes $v(x) = -e^{-\eta x}$ and $u(x) = -e^{-\rho x}$, then the portfolio problem becomes

$$\max_{\pi_n \in \mathbb{R}} - \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \exp(-\rho W) \, \Phi(dx, m, \sigma) \right)^{\frac{\eta}{\rho}} \Phi^{SN}(dm; \xi, \tau_1, \alpha_1) \tag{2.6}$$

$$s.t. \quad W = \pi_n X + (1 - \pi_n) R_f,$$

and I denote by π_n^* the theoretical optimal stock allocation.

Lemma 2.1.1. *Consider problem* (2.6). *If* $X \mid m \sim N(m, \sigma^2)$ *with* σ^2 *known and* $m \sim SN(\xi, \tau_1, \alpha_1)$, *then the optimal allocation* π_n^* *satisfies*

$$\pi_n^* \left(\rho\sigma^2 + \eta\tau_1^2 \right) - \delta_1 \tau_1 \lambda \left(-\delta_1 \tau_1 \eta \pi_n^* \right) = \xi - R_f. \tag{2.7}$$

with $\delta_1 = \frac{\alpha_1}{\sqrt{1+\alpha_1^2}}$ *and* $\lambda(x) = \frac{\phi(x)}{\Phi(x)}$.[3]

Proof. See proof of the more general equation (2.10). □

The parameter α_1 governs the higher-order moments of the Skew-normal distribution. When $\alpha_1 = 0$, an investor believes that the expected return is Gaussian distributed, the second term on the LHS disappears, and the optimal allocation coincides with the robust mean-variance solution

$$\pi_n = \frac{\xi - R_f}{\rho\sigma^2 + \eta\tau_1^2}.$$

On the other hand, for general values of α_1, the mean and the variance of the expected return differ, respectively, from ξ, τ_1^2. In particular, both the mean and the degree of skewness of the expected return are increasing in α_1, while the variance is decreasing. Therefore, in order to isolate the contribution of higher-order ambiguity attitudes, I rewrite

[3]Here ϕ denotes the pdf of a standard Gaussian distribution.

equation (2.7) as

$$\pi_n^* = \frac{\overbrace{\xi + \delta_1\tau_1\lambda(0)}^{=E_Q[X]} - R_f}{\rho\sigma^2 + \eta\underbrace{\tau_1^2\left(1-\delta_1^2\lambda^2(0)\right)}_{=Var_\mu(E_\omega[X])}} \tag{2.8}$$

$$\underbrace{+\delta_1\tau_1\frac{\lambda\left(-\delta_1\tau_1\eta\pi_n^*\right) - \lambda(0)\left(1+\eta\delta_1\tau_1\lambda(0)\,\pi_n^*\right)}{\rho\sigma^2 + \eta\tau_1^2\left(1-\delta_1^2\lambda^2(0)\right)}}_{\text{higher-order ambiguity contribution}},$$

and I denote by $\pi_{n,MV}$ (respectively, $\pi_{n,B,MV}^*$ when $\eta = \rho$) the first term on the RHS of (2.8), that coincides with the robust mean-variance allocation.

Lemma 2.1.2. *If* η,$\rho > 0$, *then*

$$sgn\left(\pi_n^* - \pi_{n,MV}^*\right) = sgn\left(\pi_{n,B}^* - \pi_{n,B,MV}\right) = sgn\left(\alpha_1\right).$$

Proof. The lemma follows from the properties of a convex functional. Since the inverse Mills ratio $\lambda(x) = \frac{\phi(x)}{\Phi(x)}$ is convex in its argument, the sign of the second term on the RHS of (2.8) coincides with $sgn\left(\alpha_1\right)$. □

The lemma relates the sign of α_1 to the variation of the portfolio composition from the robust mean-variance solution. If $\alpha_1 > 0$, the second term on the LHS of (2.8) is positive and the optimal stock allocation is larger than the one prescribed by the mean-variance solution. On the other hand, if $\alpha < 0$, then the investor reduces the optimal stock investment in order to mitigate the exposure towards the downside model uncertainty. This result is consistent with the intuition given by (1.6). In fact, when the stock returns are Gaussian and α_1 is positive (negative), the expected conditional third central moment $\mathbb{M}_\omega^3(X)$ is null and the expected returns are positively (respectively, negatively) skewed, a characteristic appreciated by an ambiguity prudent investor.

Noteworthy, the same qualitative effect holds also for a Bayesian investor. Indeed, when σ^2 is known, by the law of total cumulance

$$\mathbb{M}_Q^3(X) = \mathbb{E}_\mu\left[\underbrace{\mathbb{M}_\omega^3(X)}_{=0}\right] + \underbrace{\mathbb{M}_\mu^3\left(\mathbb{E}_\omega[X]\right)}_{sgn\left(\mathbb{M}_\mu^3\right)=sgn(\alpha_1)},$$

and the following two statements are true: the distribution of returns under Q exhibits non-null third central moment and, mechanically, the sign of the asymmetry of returns under Q coincides with the one of the expected return, i.e. $sgn\left(\mathbb{M}_Q^3(X)\right) = sgn\left(\mathbb{M}_\mu^3(E_\omega[X])\right)$. However, in section 2.1.3, I show that the quantitative contribution of a skewed expected return for smooth and Bayesian investors is different.

Finally, since the Inverse Mills ratio is convex and strictly decreasing on $\mathbb{R}$, a positive skewed expected return has a larger marginal contribution on the optimal portfolio than a negative skewed expected return. The reason for such non-linearity is due to the fact that a positive α_1 implies a larger share of wealth invested in the stock and this increases the portfolio exposure to model uncertainty, making higher-order ambiguity attitudes more relevant.

2.1.2 Skew-Normal Returns

In order to study how the optimal portfolio depends on the interactions between higher-order attitudes related to different sources of uncertainty, I relax the Gaussian assumption allowing returns to follow a Skew-normal distribution. For simplicity, I assume α_2, τ_2 are known and investor is uncertain only about the location parameter ξ_2 over which she has a skew-normal prior $\xi_2 \sim SN(\xi_1, \tau_1, \alpha_1)$. Problem (2.1) becomes

$$\max_{\pi\in\mathbb{R}} - \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \exp(-\rho W)\, \Phi^{SN}(dx, \xi_2, \tau_2, \alpha_2) \right)^{\frac{\eta}{\rho}} \Phi^{SN}(d\xi_2; \xi_1, \tau_1, \alpha_1) \tag{2.9}$$

$$s.t. \quad W = \pi X + (1-\pi) R_f,$$

and I denote π^*_{sn} the optimal stock allocation. The additional degree of freedom given by α_2 allows $sgn\left(\mathbb{M}_Q^3(X)\right) \neq sgn\left(\mathbb{M}_\mu^3(E_\omega[X])\right)$; therefore, higher-order moments of risk and model uncertainty push π^* towards opposite directions. The following proposition clarifies the point.

Proposition 1. *Consider problem* (2.9). *If* $X \mid \xi_2 \sim SN(\xi_2, \tau_2, \alpha_2)$ *and* $\xi_2 \sim SN(\xi_1, \tau_1, \alpha_1)$, *then the theoretical optimal allocation,* π^*_{sn}, *satisfies*

$$\pi^*_{sn}\left(\rho\tau_2^2+\eta\tau_1^2\right)-\delta_1\tau_1\lambda\left(-\delta_1\tau_1\eta\pi^*_{sn}\right)-\delta_2\tau_2\lambda\left(-\delta_2\tau_2\rho\pi^*_{sn}\right)=\xi_1-R_f, \tag{2.10}$$

with $\delta_i=\frac{\alpha_i}{\sqrt{1+\alpha_i^2}}, i\in\{1,2\}$.

The first two terms on the LHS of (2.10) are those analysed in the previous subsection, while the third one is specific to Skew-normal returns. Similar to Gaussian returns, I isolate the contribution of higher-order attitudes rewriting (2.10) as

$$\pi^*_{sn}=\frac{\overbrace{\xi_1+\left(\delta_1\tau_1+\delta_2\tau_2\right)\lambda\left(0\right)}^{=E_Q[X]}-R_f}{\underbrace{\rho\tau_2^2\left(1-\delta_2^2\lambda^2\left(0\right)\right)}_{=Var_{K(\omega,\cdot)}(X)}+\underbrace{\eta\tau_1^2\left(1-\delta_1^2\lambda^2\left(0\right)\right)}_{=Var_\mu(E_\omega[X])}} \tag{2.11}$$

$$\underbrace{+\delta_1\tau_1\frac{\lambda\left(-\eta\delta_1\tau_1\pi^*_{sn}\right)-\lambda\left(0\right)\left(1+\eta\delta_1\tau_1\lambda\left(0\right)\pi^*_{sn}\right)}{\rho\tau_2^2\left(1-\delta_2^2\lambda^2\left(0\right)\right)+\eta\tau_1^2\left(1-\delta_1^2\lambda^2\left(0\right)\right)}}_{\text{higher-order ambiguity variation}}$$

$$\underbrace{+\delta_2\tau_1\frac{\lambda\left(-\rho\delta_2\tau_2\pi^*_{sn}\right)-\lambda\left(0\right)\left(1+\rho\delta_2\tau_2\lambda\left(0\right)\pi^*_{sn}\right)}{\rho\tau_2^2\left(1-\delta_2^2\lambda^2\left(0\right)\right)+\eta\tau_1^2\left(1-\delta_1^2\lambda^2\left(0\right)\right)},}_{\text{higher-order risk variation}}$$

in which the first term on the LHS of (2.11) coincides with the mean-variance solution, while the other two terms induce a variation that depends on the contribution of higher-order attitudes towards ambiguity and risk.

Equation (2.11) clarifies that, fixed a value of $\alpha_1\in\mathbb{R}$, the sign of the variation of π^*_{sn} from π^*_n coincides with the sign of α_2, i.e. $sgn\left(\pi^*_{sn}-\pi^*_n\right)=sgn\left(\alpha_2\right)$. When $sgn\left(\alpha_1\right)=sgn\left(\alpha_2\right)>0$, both returns and expected returns display positive skewness; therefore, if the investor is prudent towards risk and ambiguity, then $\pi^*_{sn}\geq\pi^*_n$ and both stock allocations are larger than the one implied by the robust mean-variance solution. Analogously, if $\alpha_1,\alpha_2<0$ then returns and expected returns display negative skewness and prudent attitudes imply $\pi^*_{sn}\leq\pi^*_n\leq\pi_{MV}$.

In case $sgn\left(\alpha_1\right)\neq sgn\left(\alpha_2\right)$, the degrees of skewness of returns and of the expected return display opposite signs and the third moment under Q depends on the relative importance of the two sources of uncertainty

$$\mathbb{M}_Q^3(X) = E_\mu\left[\underbrace{\mathbb{M}_{K(\omega,\cdot)}^3(X)}_{sgn\left(\mathbb{M}_{K(\omega,\cdot)}^3(X)\right)=sgn(\alpha_2)}\right] + \underbrace{\mathbb{M}_\mu^3(\mathbb{E}_\omega[X])}_{sgn\left(\mathbb{M}_\mu^3\right)=sgn(\alpha_1)}. \quad (2.12)$$

Typically, for short investment horizons, statistical moments of the risk distribution of returns have a larger order of magnitude than the ones of expected return distribution; therefore, the variation from the mean-variance solution induced by the last term on the RHS of (2.11) is larger. However, as the investment horizon increases, the relevance of model uncertainty grows faster because of the positive covariance among future returns realization: $Cov(X_{t+1}, X_{t+j}) > 0$, $\forall j > 0$.[4] The economic intuition behind such dependence is that, if investors believe that an high value of the expected return is relatively more likely, then the same holds for each period and the importance of such correlations amplifies as the maturity of the investment increases. To see it, consider the following example: if $X_1, ..., X_t \mid m \overset{iid}{\sim} N(m, \sigma^2)$ with $m \sim SN(\xi_1, \tau_1, \alpha_1)$, then

$$E\left[\sum_{i=1}^{t} X_i \mid m\right] = tm, \quad Var\left(\sum_{i=1}^{t} X_i \mid m\right) = t\sigma^2, \quad Var(tm) \propto t^2\tau_1^2. \quad (2.13)$$

The faster growth of model uncertainty, combined with the strength of ambiguity attitudes, rapidly increases the contribution of the second term on the RHS of (2.11). On the other hand, the penalization of model uncertainty, due to ambiguity aversion, reduces the overall stock allocation. When $\pi_{sn} \to 0$, the relevance of higher-order attitudes towards uncertainty is mitigated. Because of these joint effects, I show that ambiguity prudence induces a variation from the robust mean-variance solution that does not have a monotonic pattern.

2.1.3 Simulations

In order to quantify the portfolio relevance of prudence effects at different investment horizons, I analyse the problem for a *buy-and-hold* investor. This perspective is useful

[4] Formally, this is due to the infinite exchangeability of returns.

to isolate the contribution of ambiguity attitudes without confounding their effects with those related to preferences over intertemporal choice, e.g. elasticity of intertemporal substitution.

A *buy-and-hold* investor has to optimally choose the portfolio composition at the beginning of the first period and she is not allowed to change it until periods are over. Consistently with standard choices in the empirical literature, I assume that utility indexes u, v in (2.1) have the CRRA form, $v(x) = \frac{x^{1-\eta}}{1-\eta}$ and $u(x) = \frac{x^{1-\rho}}{1-\rho}$. The portfolio problem for an investment with time to maturity given by T is

$$\max_{\pi \in \mathbb{R}} \int_{\Omega} \left(\int_{\mathbb{R}} \frac{W_{t+T}^{1-\rho}}{1-\eta} K(\omega, dx) \right)^{\frac{1-\eta}{1-\rho}} \mu(d\omega) \tag{2.14}$$

$$s.t.\ W_{t+T} = \pi \exp(r_f T) + (1-\pi) \exp(X_{t,T} + r_f T),$$

where $X_{t,T} = \sum_{i=1}^{T} X_{t+i}$ is the cumulative excess return between periods t and $t+T$. I consider two alternative parsimonious calibrations of preference parameters $(\rho, \eta) \in \{(2, 60), (5, 100)\}$, that are consistent both with the results of section 1.3 and with standard values adopted in the literature.

Optimal allocations are obtained through numerical methods in which integrals are approximated using standard Monte Carlo techniques. Simulations require a two-step sampling: first, I generate a large set of models $\{\omega_i\}_{i=1,..,N}$ sampling from μ and then, for each ω_i, I generate a sequence of cumulative excess return realizations sampling from the conditional distribution $K(\omega, X_{t,T} \in B)$. The mean-measure probability is given by

$$Q(X_{t,T} \in B) \approx \sum_{i=1}^{N} K(\omega_i, X_{t,T} \in B) \mu(\omega_i). \tag{2.15}$$

To estimate statistical parameters, I use data as calculated by the Center for Research in Security Prices (CRSP). Data have yearly frequency with observations that span 92 years from 1927 to 2018. In order to avoid effects related to World War II, I also restrict attention to the period following the Treasury Accord of 1951, for which there are 67 observations from 1952 to 2018. I proxy stock returns using log-returns (cum-dividend) of the value-weighted market portfolio, including stocks traded on the NYSE, AMEX

Table 2.1: Summary Statistics

	Excess Return	
Period	1927 - 2018	1952 - 2018
Number Obs	92	67
Mean	0.058	0.056
Variance	0.039	0.029

and NASDAQ indexes. The risk-free rate is set constant and equal to the mean of the 90 Day-Bill returns of the sample, respectively $r_f = 0.006$ and $r_f = 0.011$. All nominal variables are deflated using the Consumer Price Index. Table 2.1 shows the summary statistics of real excess-returns. While the average excess-return is similar between the two samples, if the observations are restricted to the postwar period then the variance of returns is ≈26.5% lower; this implies a larger allocation in the stock that amplifies the impact of higher-order ambiguity attitudes.

In order to quantify the relevance of ambiguity prudence, I compare the optimal stock allocations for three investors, indexed by $i \in \{1, 2, 3\}$, whose beliefs over the expected return display heterogeneous degrees of skewness. I initially assume that investors believe stock returns are i.i.d. from a Gaussian distribution with known variance, set equal to the sample variance, and unknown expected return m, over which they have Skew-normal beliefs, i.e. $X_{t+1} \mid m \sim N(m, \sigma^2)$ and $m \sim SN(\xi_1^i, \tau_1^i, \alpha_1^i)$.

Since α_1^i is the only parameter governing the asymmetry of the Skew-normal distribution, I assume that for each investor α_1^i takes a different value in $\{-8, 0, +8\}$.[5] The remaining hyper-parameters (ξ^i, τ_1^i) are set in order to match the mean and the variance of the posterior distribution of the expected return, that I obtained updating an

[5]These values of α_1 imply that the degrees of skewness of the expected return distributions take value in $\{-0.9, 0, 0.9\}$.

uninformative prior. The third and fourth columns of table 2.2 show the mean and standard deviation of the posterior distribution of m, while the last three columns report the calibration of the Skew-normal parameters.

Table 2.2: Hyper-parameters calibration

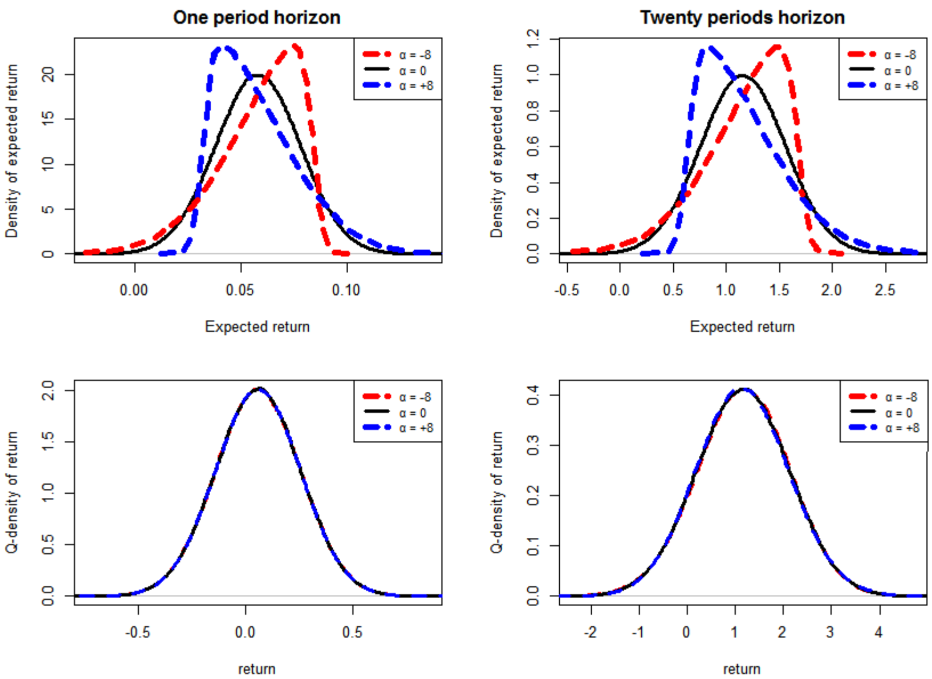
Period		$E\left[m \vert \left(X_k\right)_{k=1}^{K}\right]$	$Sd\left(m \vert \left(X_k\right)_{k=1}^{K}\right)$	α_1^i	ξ^i	τ_1^i
1927 - 2018	$i=1$	0.058	0.02	-8	0.084	0.033
	$i=2$	0.058	0.02	0	0.058	0.02
	$i=3$	0.058	0.02	+8	0.032	0.033
1952 - 2018	$i=1$	0.056	0.02	-8	0.082	0.033
	$i=2$	0.056	0.02	0	0.056	0.02
	$i=3$	0.056	0.02	+8	0.03	0.033

The table shows the posterior mean and variance of the expected return obtained updating an uninformative prior. The last three columns show the calibration of the Skew-normal parameters that I use to match the first two moments of the posterior distribution.

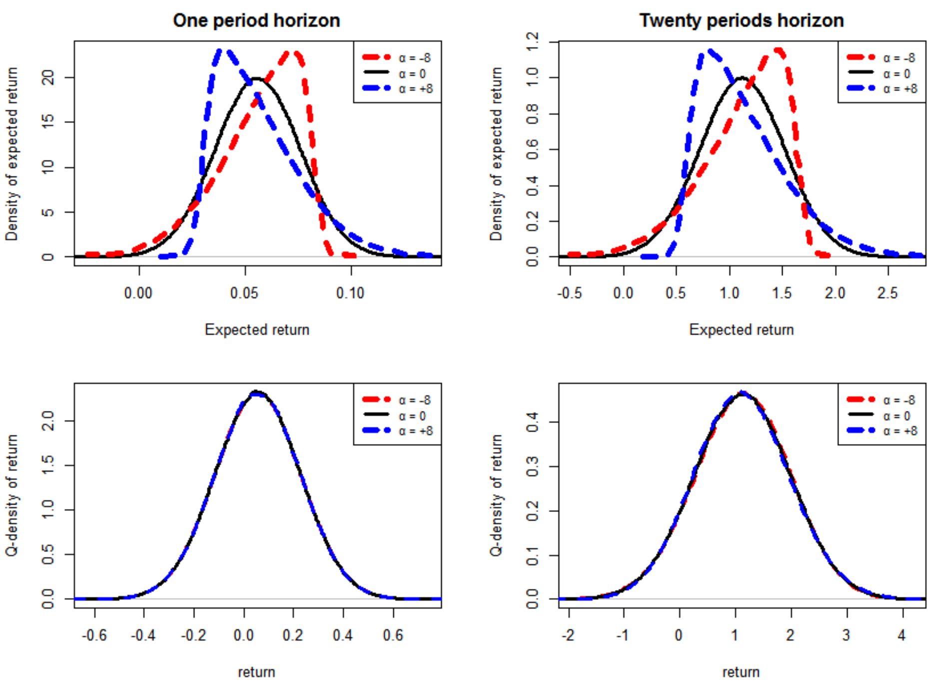
Noteworthy, given any τ_1, the Skew-normal distributions implied by opposite values of α_1 have same variance and excess of kurtosis but opposite degrees of skewness. However, in order to ascribe to ambiguity prudence the variations from the robust mean-variance solution, a crucial point is how the heterogeneity over the higher-moments of m affects the mean-measure distribution of returns. Reassuringly, figure 2.1 shows that, apart for the first moment, the Skew-normal distribution has a negligible impact over the Q-measure, both for short (one-year) and long (twenty-years) investment horizons. The reason is that the contribution of model uncertainty to the Q-measure depends on the relative size of the statistical moments and table 2.2 confirms that the uncertainty over m has a lower order of magnitude than the risk size implied by the Gaussian distribution of returns.

Figure 2.1: Densities of Returns and Expected Return

(a) 1927 – 2018



(b) 1952 – 2018



The figure shows the μ-densities of the expected-return and the Q-densities of returns, when investors believe returns follow a Gaussian distribution.

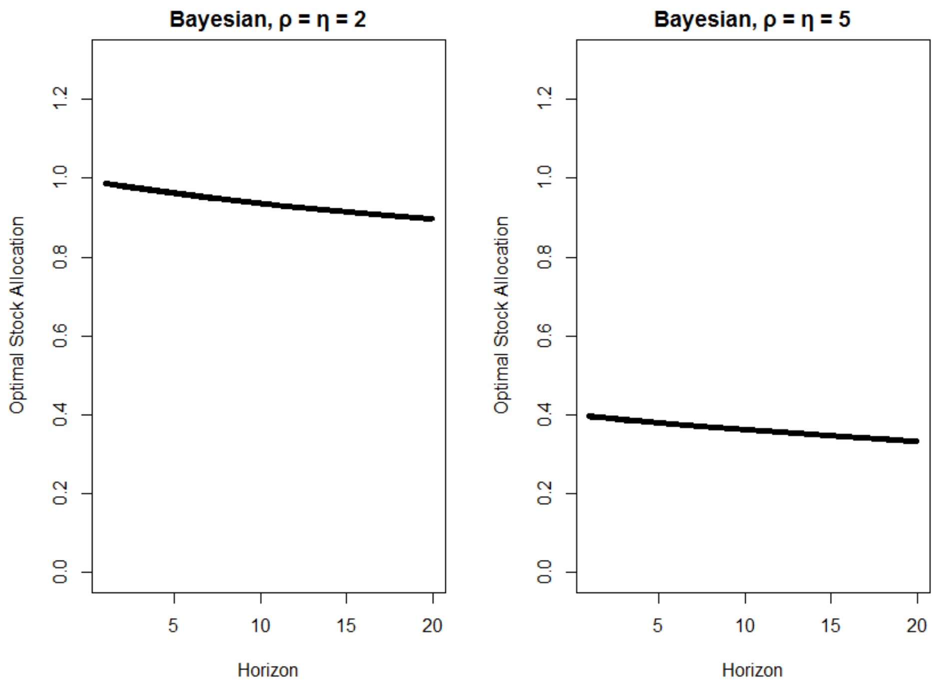
In line with this fact, if an investor is Bayesian then the faster growth of model uncertainty does not have a sizeable impact over the optimal portfolio composition and, indeed, figure 2.2 confirms that the stock investment experiences only a small reduction. Coherently with theoretical results, when I compare the optimal stock allocations for the three investors, figure 2.3 shows that if the expected return is positive (negative) skewed, then the stock reduction is mitigated (strengthen) by risk prudent attitudes. However, because of the limited role of ambiguity, the relevance of heterogeneous beliefs over m is negligible and the portfolio compositions remain similar to those implied by the conjugate Gaussian solution, i.e. $\alpha_1 = 0$.

On the other hand, if an investor has non-neutral attitudes towards ambiguity, then the faster growth of model uncertainty has important portfolio implications. In figure 2.4, I compare the optimal stock allocations for smooth - solid lines - and Bayesian - dashed lines - investors, when they believe that both returns and expected return follow a Gaussian distribution. Graphs show that, as the investment horizon increases, a smooth investor experiences a much sharper reduction of the optimal stock allocation that is due to the *ad-hoc* penalization of model uncertainty induced by ambiguity aversion. For example, if the maturity of the investment is twenty years, then the optimal stock allocation for a smooth investor with $\rho = 2, \eta = 60$ is approximately the 80% lower than the one of a Bayesian investor!

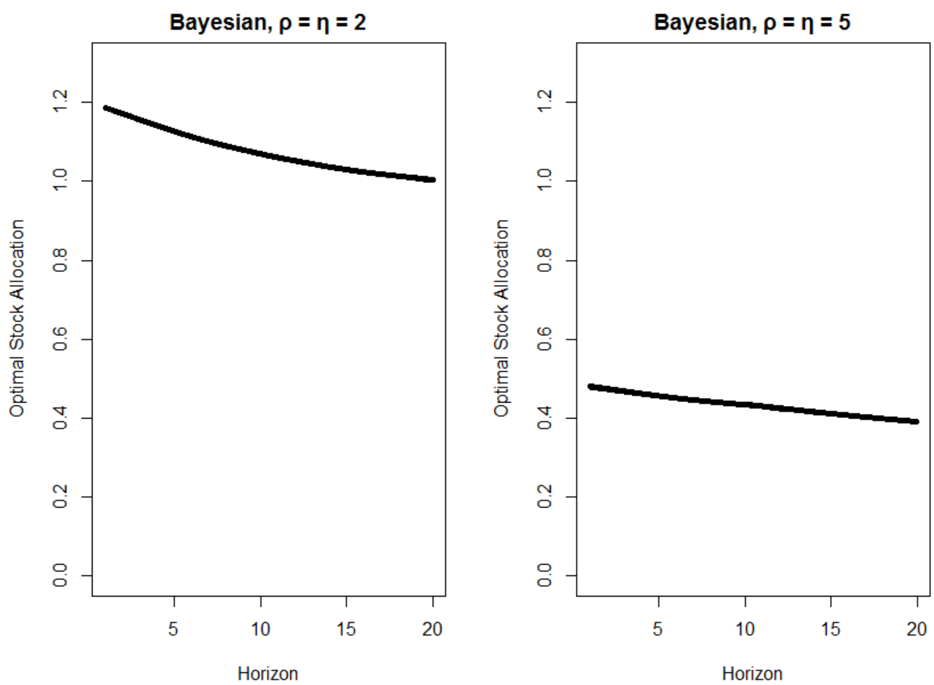
When beliefs over the expected return display heterogeneous degrees of skewness, the reduction of stock investment can be mitigated or strengthened by ambiguity prudent attitudes. Figure 2.5 compares the optimal allocations for the three smooth investor and it clarifies that, differently from the Bayesian setting, heterogeneous beliefs over downside model uncertainty have a sizeable impact over the portfolio composition. Moreover, the marginal contribution of positive and negative values of α_1 is highly non-linear. In fact, the variation from the robust mean-variance solution implied by a positive α_1 is much larger than the variation induced by the opposite value of α_1. For example, for a twenty-

Figure 2.2: Bayesian Investor with Gaussian Returns

(a) 1927 – 2018



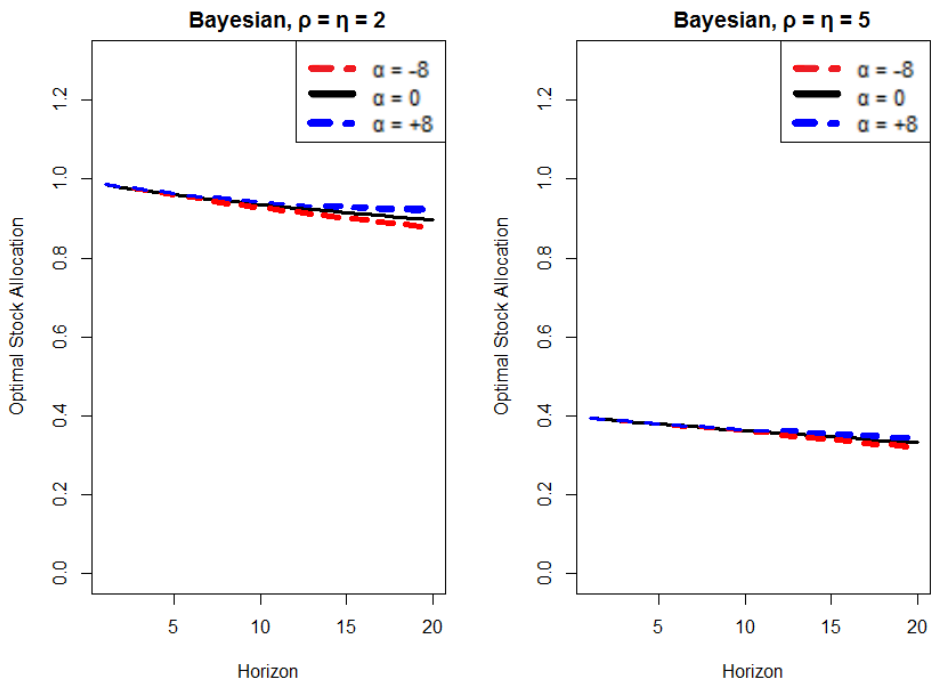
(b) 1952 – 2018



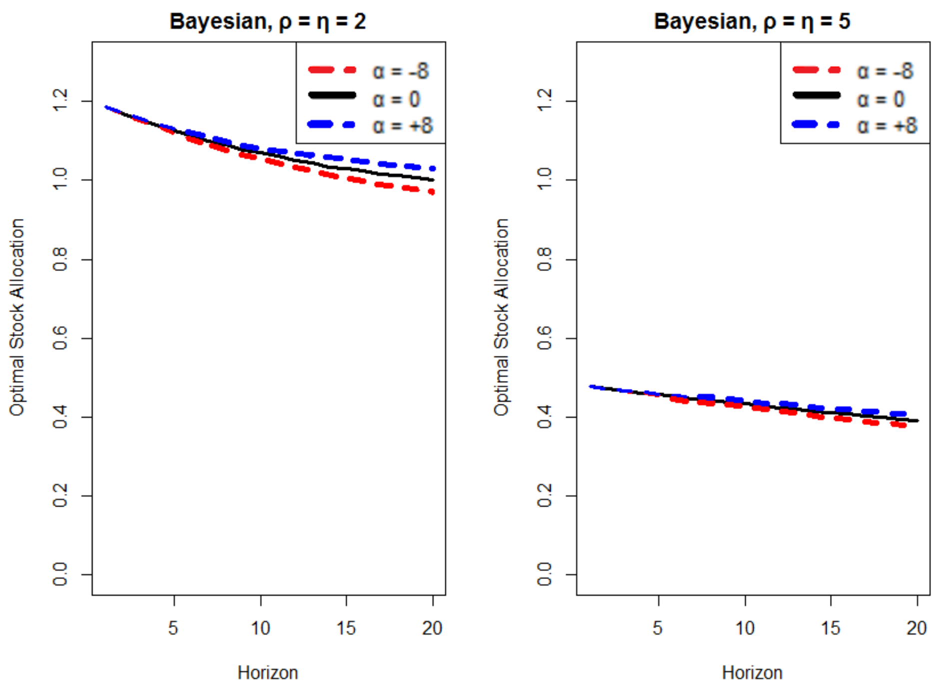
The graphs report the optimal stock allocation for a Bayesian investor, as a function of T, when she believes that both returns and expected return follow a Gaussian law.

Figure 2.3: Bayesian Investors with Gaussian Returns

(a) 1927 – 2018



(b) 1952 – 2018



The graphs report the optimal stock allocation for three Bayesian investors, as a function of T, when they believe that returns follow a Gaussian law and expected return a Skew-normal law.

years horizon, an investor with $\rho = 2$, $\eta = 60$ and positive skewed beliefs over the expected return, i.e. $\alpha_1 = 8$, would invest in the stock around the 55-60% of her total wealth, against the 20% implied by Gaussian beliefs, i.e. $\alpha_1 = 0$, and the 15% implied by negative skewed beliefs over the expected return, i.e. $\alpha_1 = -8$.

In figure 2.6, I quantify the persistence of ambiguity prudence across different investment horizons. Specifically, I compare the range size spanned by the heterogeneous stock allocations, i.e. $\pi_n(\alpha_1 = 8) - \pi_n(\alpha_1 = -8)$, with the stock investment implied by Gaussian beliefs over the expected return. The graphs show that, at very short maturities, the range size is relatively low with respect to the Gaussian solution, therefore ambiguity prudence implies a limited variation from the robust mean-variance solution. However, after few periods, the range size exceeds the Gaussian allocation and its value remains persistently high also when the Gaussian allocation becomes low. This fact suggests that the portfolio contribution induced by ambiguity prudence is persistent across investment horizons.

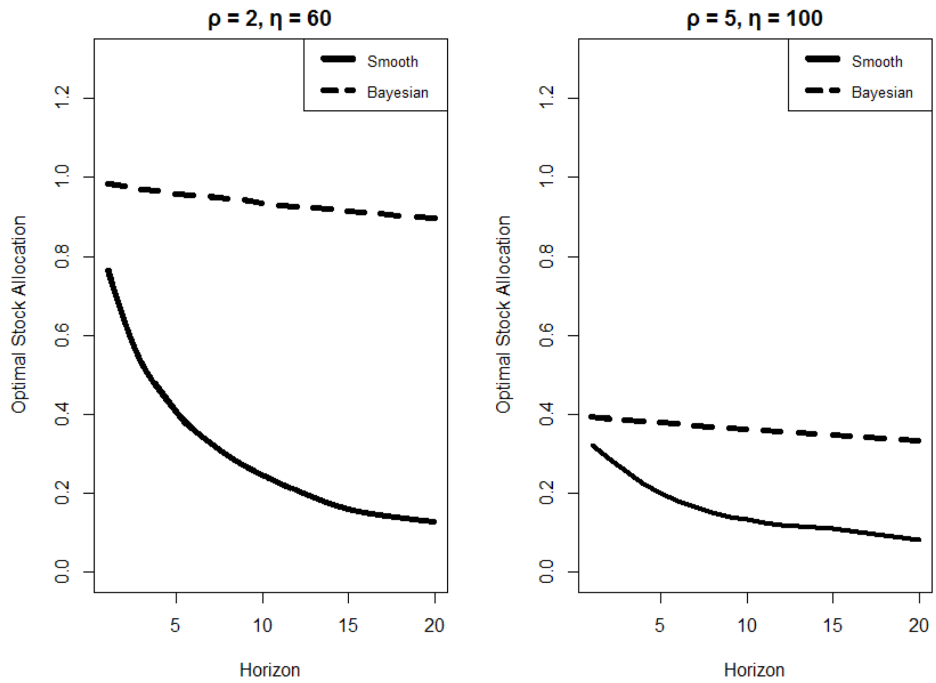
In order to compare the portfolio contribution induced by ambiguity prudence with the one induced by risk prudence, I replace the Gaussian assumption of returns, requiring that they follow a Skew-normal distribution $X \sim SN(\xi_2, \tau_2, \alpha_2)$. In the same spirit as the previous analysis, I compare the optimal portfolio composition of three investors that have heterogeneous beliefs over the degree of skewness of returns, $\alpha_2^i \in \{-8, 0, +8\}$. I assume that investors known the variance of returns, i.e. α_2 and τ_2, but they are uncertain about the expected return, over which they share a common Gaussian belief.[6] In order to make portfolio allocations comparable with those obtained in the previous framework, I set the variance of returns equal to the sample variances, reported in table 2.1, and the values of the hyper-parameters governing beliefs over expected return equal to those reported in table 2.2 when $\alpha_1 = 0$.[7]

[6]Conditioning on the values of α_2 and τ_2, the expected return is a function of the parameter ξ_2. Therefore, a Gaussian belief over the expected return is implied by a Gaussian belief over ξ_2.

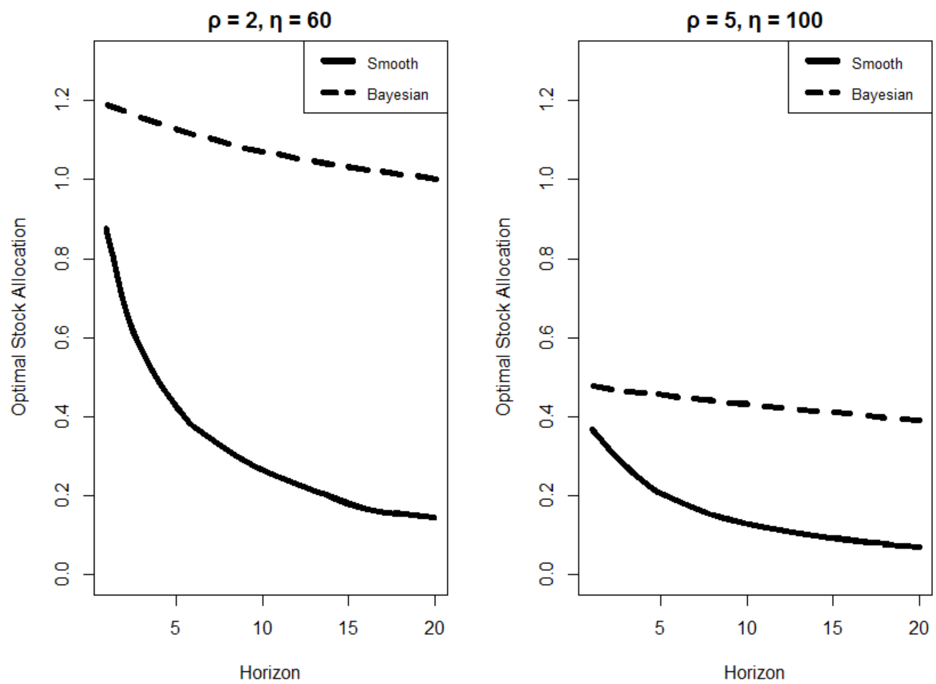
[7]In this case, if $\alpha_2 = 0$ then I retrieve the conjugate Gaussian allocations reported in figure 2.4.

Figure 2.4: Smooth and Bayesian Investors

(a) 1927 – 2018



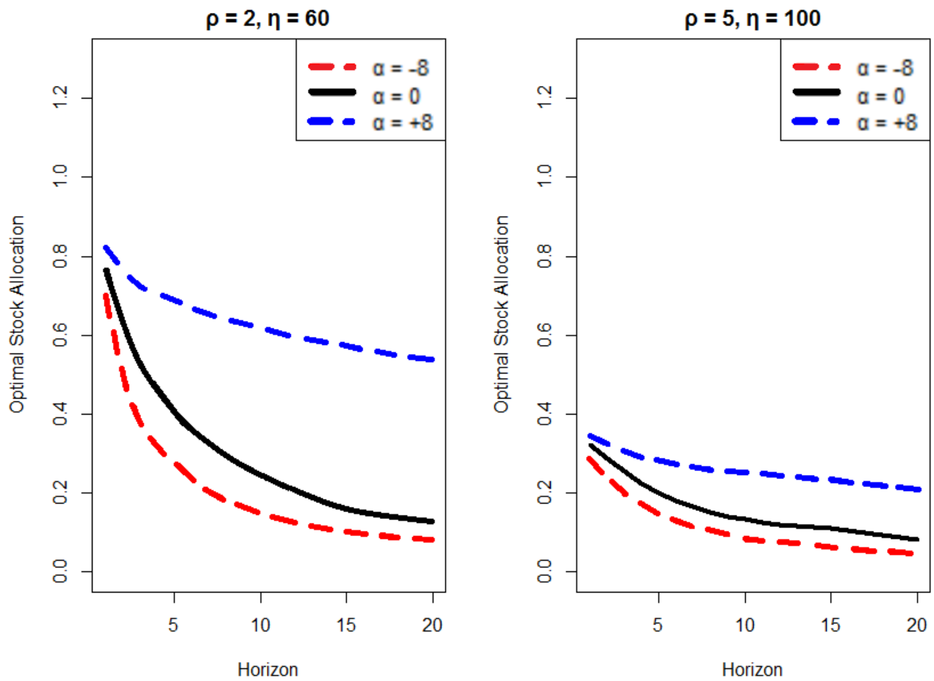
(b) 1952 – 2018



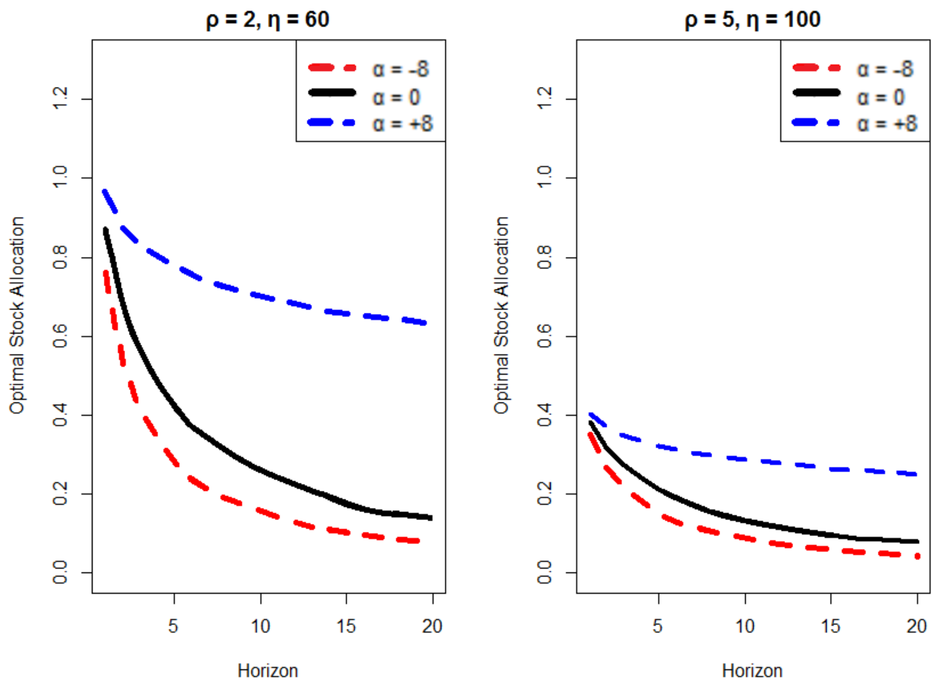
The graphs report the optimal stock allocation for Bayesian (dotted line) and smooth (solid line) investors, as a function of T, when they believe that both returns and expected return follow a Gaussian law.

Figure 2.5: Ambiguity prudence

(a) 1927 – 2018



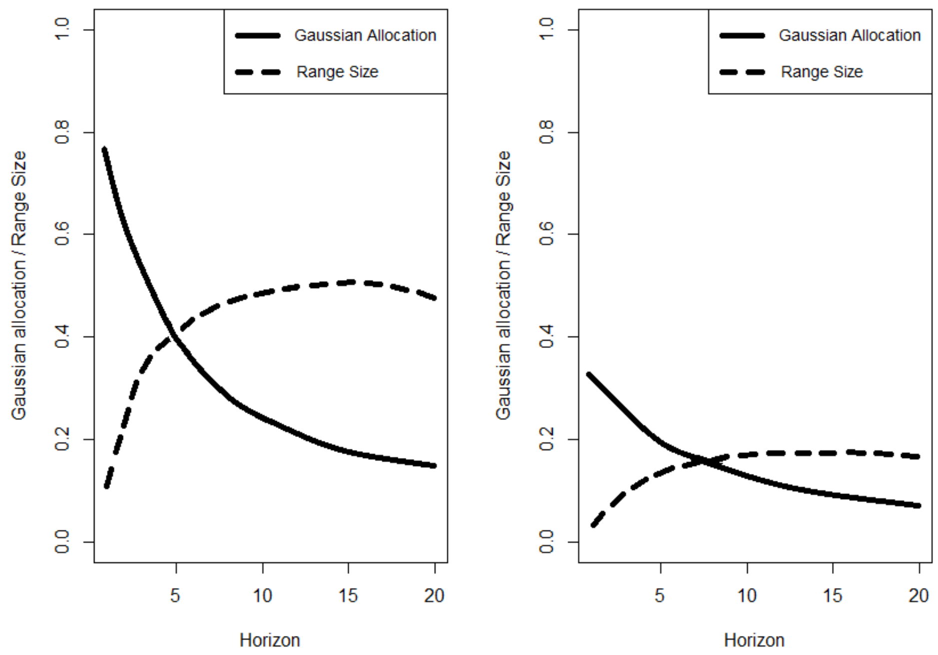
(b) 1952 – 2018



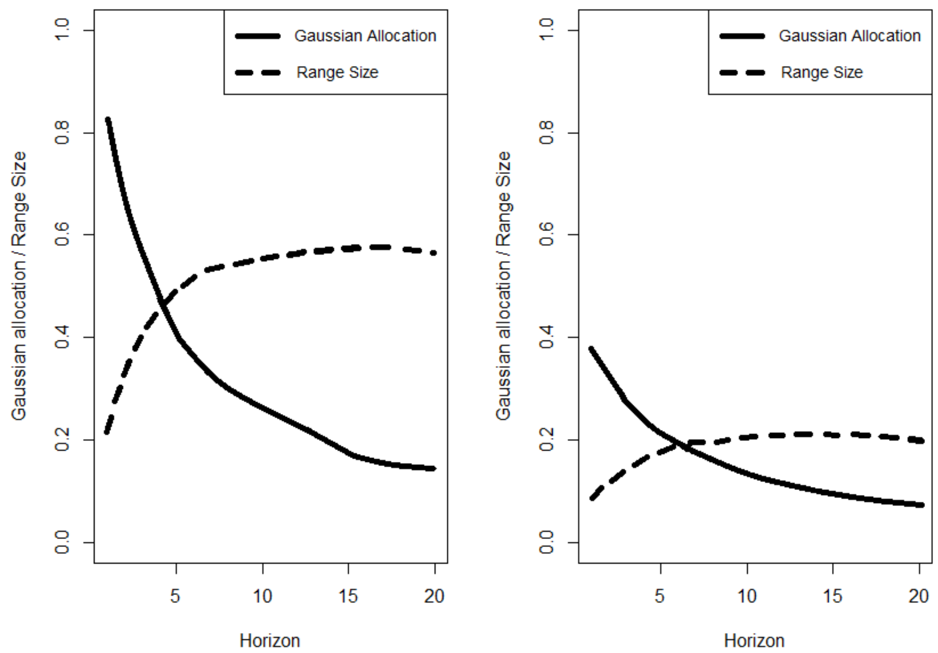
The graphs report the optimal stock allocation for the smooth investors, as a function of T, when they believe returns follow a Gaussian law and expected return a Skew-normal law.

Figure 2.6: Range size and Gaussian allocation

(a) 1927 – 2018



(b) 1952 – 2018



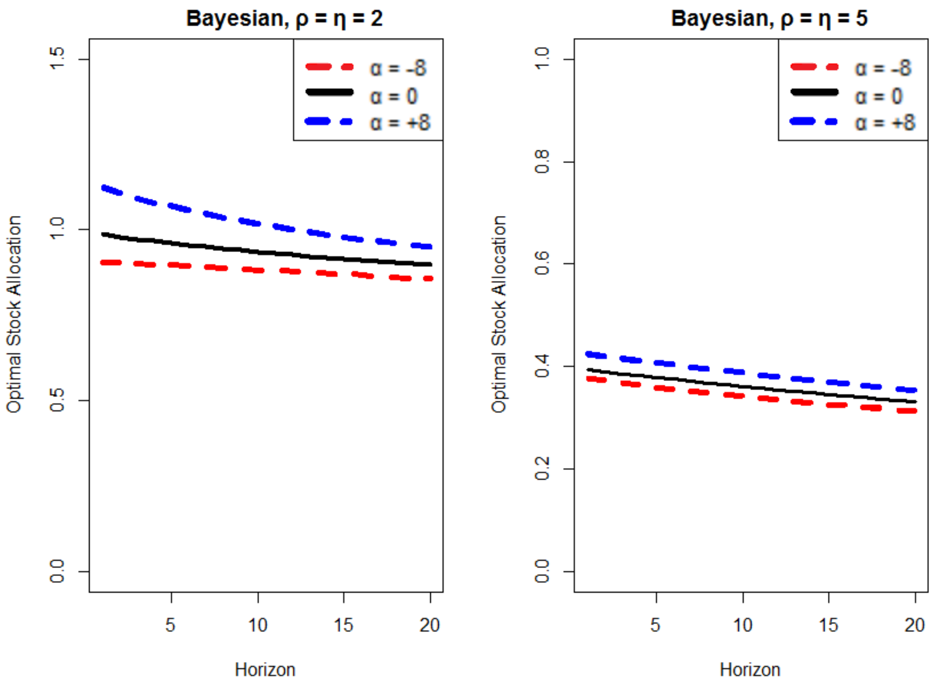
The graphs compare the optimal stock allocation for a smooth investor (solid line) who believes that expected return follows a Gaussian law with the range spanned by the heterogeneous allocations when expected return follows a Skew-normal law (dotted line).

Figure 2.7 shows that if investors are ambiguity neutral, then risk prudent attitudes induce significant variations from the allocations implied by the mean-variance solution.[8] Moreover, since model uncertainty does not receive an *ad-hoc* penalization, these variations are persistent across different investment horizons. On the other hand, if preferences have a smooth representation, then figure 2.8 shows that the prominence of ambiguity aversion mitigates the relevance of heterogeneous beliefs over the degree of skewness of returns. Therefore, after very few horizons, the optimal portfolio compositions coincide with the ones implied by robust mean-variance solution.

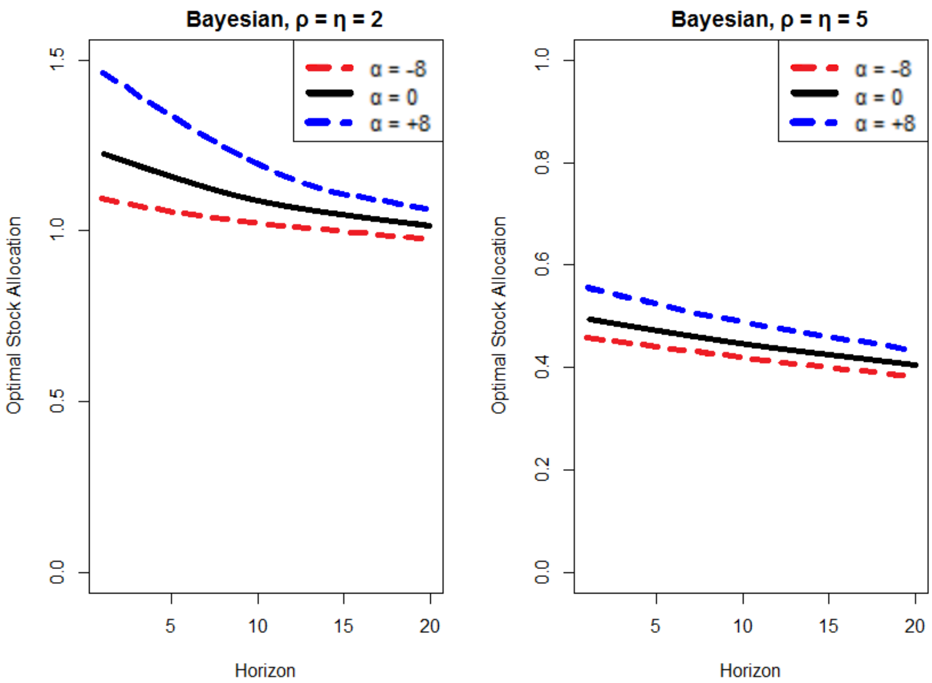
[8]The differences from the optimal allocations report in table 2.3 are due to the relative order of magnitude between risk and model uncertainty.

Figure 2.7: Bayesian investors

(a) 1927 – 2018



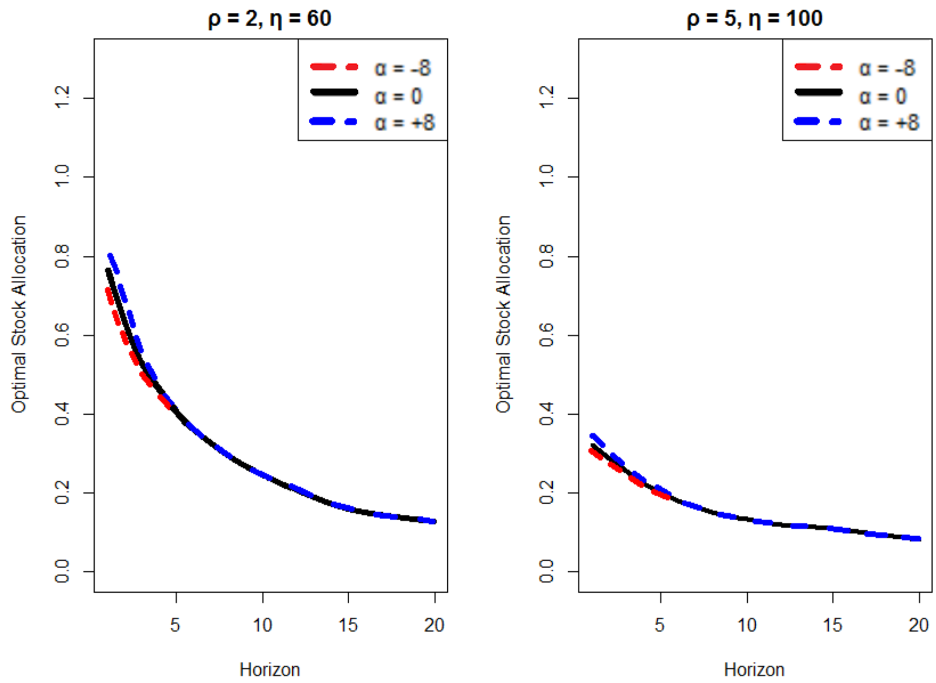
(b) 1952 – 2018



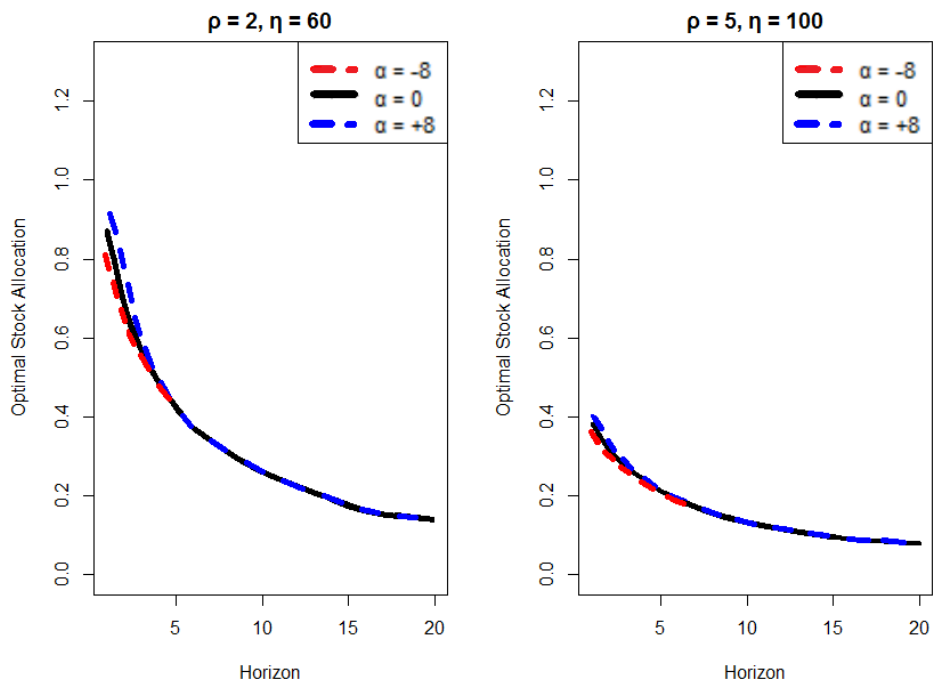
The graphs report the optimal stock allocation for three Bayesian investors, as a function of T, when they believe returns follow a Skew-normal law and expected return a Gaussian law.

Figure 2.8: Smooth investors

(a) 1927 – 2018



(b) 1952 – 2018



The graphs report the optimal stock allocation for three smooth investors, as a function of T*, when they believe returns follow a Skew-normal law and expected return a Gaussian law.*

2.2 Ambiguous predictability

In this section, I investigate how ambiguity prudence contributes to the optimal portfolio composition when an investor is uncertain about the conditional distribution of the excess-returns. I assume that an investor evaluates two alternative models: in the first, excess-returns are i.i.d. while, in the second, they contain a predictable component. The predictability of the excess-returns is the subject of an important academic debate among financial economists. Among others, Welch and Goyal (2008) have pointed out the poor out-of-sample performance of standard predictors proposed in the asset pricing literature, e.g. dividend-yield or earning-price ratio. On the other hand, Campbell and Thompson (2007) have responded that under parameters' restrictions, motivated by economic arguments, the goodness of fit significantly improves and it provides a relevant contribution to the portfolio' Sharpe ratio. More generally, an increasing body of literature supports the idea that excess-returns are predictable.[9] Here, I assume an investor does not exclude a-priori any of the two theories.

The quantitative relevance of this source of model uncertainty has been quantified in a recursive setting by Chen et al. (2014). In this work, I highlight how ambiguity prudence is a key element that underlies such empirical findings and I exploit approximation (2.4) to disentangle different effects among beliefs and ambiguity attitudes. In order to keep a stylized setting, I consider a one-period problem. Results show that, if the certainty equivalents of the two models significantly differ and the investor has asymmetric beliefs over them, then ambiguity prudence induces sizeable non-linearities in the optimal stock allocation. These effects emerge because of the investor's willingness to hedge against downside ambiguity.

Formally, an investor attaches probability $\mu \in (0,1)$ to

- model 1

$$X_{t+1} = m + \varepsilon_{1,t+1}, \tag{2.16}$$

[9] See, for example, Lettau and Ludvigson (2001) or Martin (2017).

in which excess-returns are i.i.d. from a Gaussian distribution $\varepsilon_{1,t+1} \sim N(0, \sigma_1^2)$, and probability $1-\mu$ to

- model 2

$$\begin{cases} X_{t+1} = m + \theta_2 d_t + \varepsilon_{2,t+1} \\ d_{t+1} = \theta_1 d_t + \varepsilon_{d,t+1}, \end{cases} \tag{2.17}$$

$\varepsilon_{2,t+1} \sim N(0, \sigma_2^2)$, $\varepsilon_{d,t+1} \sim N(0, \sigma_d^2)$, $Cov(\varepsilon_{2,t+1}, \varepsilon_{d,t+1}) = \sigma_{x,d}$, in which the excess-returns can be predicted by the dividend-yield and I denote by d_t its demeaned logarithmic transformation.[10]

To estimate the statistical parameters of each model, I use yearly data from CRSP for the period from 1927 to 2018 where, similarly to the previous section, both excess-returns and dividend-yield are in real terms. Table 2.3 reports the results of such estimates and, for simplicity, I neglect about parameters uncertainty.

Table 2.3: Parameters estimation

	$m - R_f$	θ_2	θ_1	$\sigma^2 \times 1000$	$\sigma_d^2 \times 1000$	$\sigma_{x,d} \times 1000$
	0.0582	-	-	39.13	-	-
model 1	(0.02)	-	-	$(-)$	-	-
	0.0582	0.096	0.94	37.45	20.81	-18.73
model 2	(0.02)	(0.047)	(0.036)	$(-)$	$(-)$	$(-)$

Investor's preferences are represented by (1.1) in which the utility indexes u, v have the CARA form. I consider two alternative calibrations of the preference parameters $(\rho, \eta) \in \{(1.5, 46), (3.75, 80)\}$ that are consistent with the allocations implied by CRRA indexes

[10]This choice is consist with standard works of portfolio literature, e.g. Barberis (2000), Xia (2001), Campbell et al. (2002), etc.

of section 1.3. The portfolio problem (2.1) becomes

$$\max_{\pi\in\mathbb{R}} -\mu\left(\int_{\mathbb{R}}\exp\left(-\rho W\right)\Phi_{iid}\left(dx\right)\right)^{\frac{\eta}{\rho}}-\left(1-\mu\right)\left(\int_{\mathbb{R}}\exp\left(-\rho W\right)\Phi_{VAR}\left(dx\right)\right)^{\frac{\eta}{\rho}} \tag{2.18}$$

$$s.t. \quad W=\pi X+\left(1-\pi\right)R_f.$$

Lemma 2.2.1. *Consider an investor facing problem* (2.18)*. Then the theoretical optimal allocation satisfies*

$$\pi^*=\mu\frac{\left(m-R_f\right)}{\rho\left(\mu\sigma_1^2+\left(1-\mu\right)\sigma_2^2c\right)}+\left(1-\mu\right)\frac{\left(m+\theta_2 d_t-R_f\right)c}{\rho\left(\mu\sigma_1^2+\left(1-\mu\right)\sigma_2^2c\right)}. \tag{2.19}$$

with $c=\exp\left(-\eta\pi^*\theta_2 d_t+\frac{\pi^{*2}}{2}\rho\eta\left(\sigma_2^2-\sigma_1^2\right)\right)$.

The factor c in (2.19) embeds in the optimal solution the non-linearities due to higher-order attitudes towards uncertainty. Although from a theoretical perspective $c\neq 1$ also for a classical Bayesian investor, under reasonable values of ρ, the quantitative relevance of c is negligible. To show it, I plot in figure 2.9 the portfolio optimal allocations as a function of the probability over the i.i.d. model for the two degrees of risk-aversion. The two graphs on the left (right) consider the case in which the dividend-yield takes a large negative (positive) value $d_t=-0.75$ ($d_t=0.75$) and therefore model 2 predicts lower (higher) excess return than the i.i.d. model.

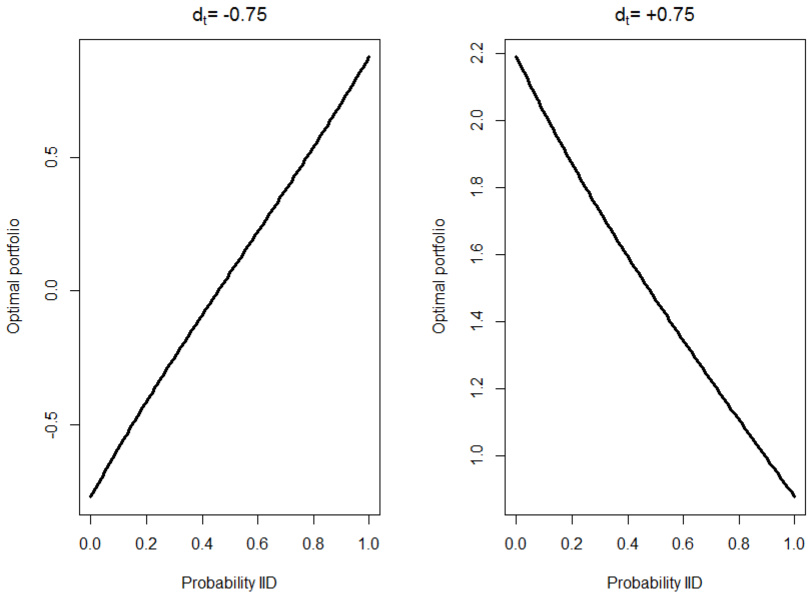
Results show that, when the investor is ambiguity neutral, the curvature due to higher-order risk attitudes is negligible, both for high and low values of the dividend-yield, and the optimal portfolio allocations look like the convex combination of the degenerate optimal solution in case $\mu=0$ or $\mu=1$.

On the other hand, if an investor has non-neutral attitudes towards ambiguity then the previous results do not hold. Indeed, figure 2.10 shows that, for a smooth investor, the non-linearities induced by the factor c have sizeable implications over the optimal allocations and the theoretical solutions look very different from the convex combination of the limiting cases.

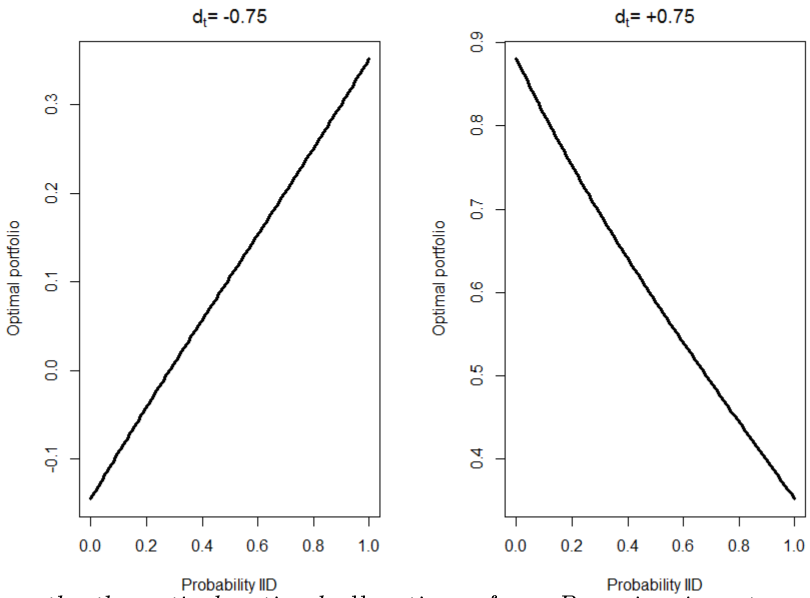
In terms of behavioural attitudes, if $d_t\neq 0$ and $\mu\in(0,1)$, then an ambiguity averse investor would reduce stock allocation because she penalizes the uncertainty over the

Figure 2.9: Uncertainty over predictability: Bayesian

(a) $\rho = 1.5$



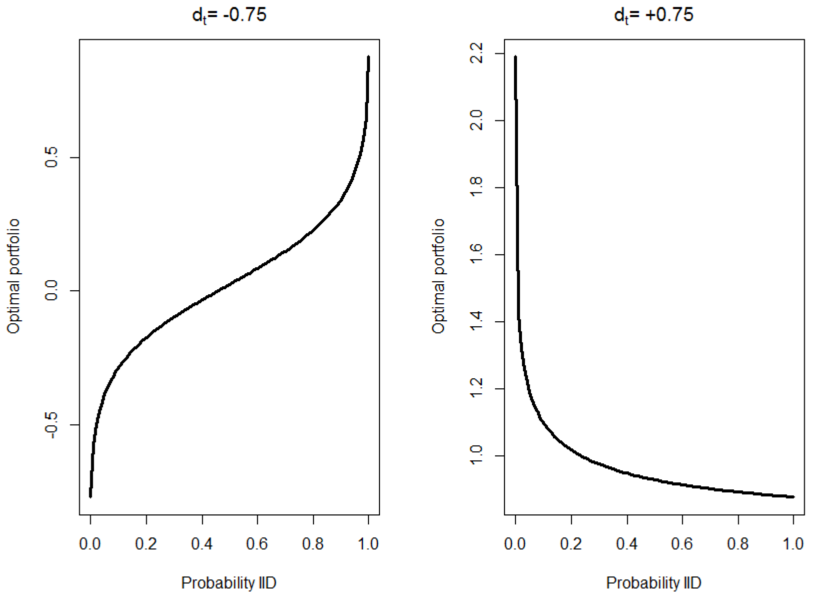
(b) $\rho = 3.75$



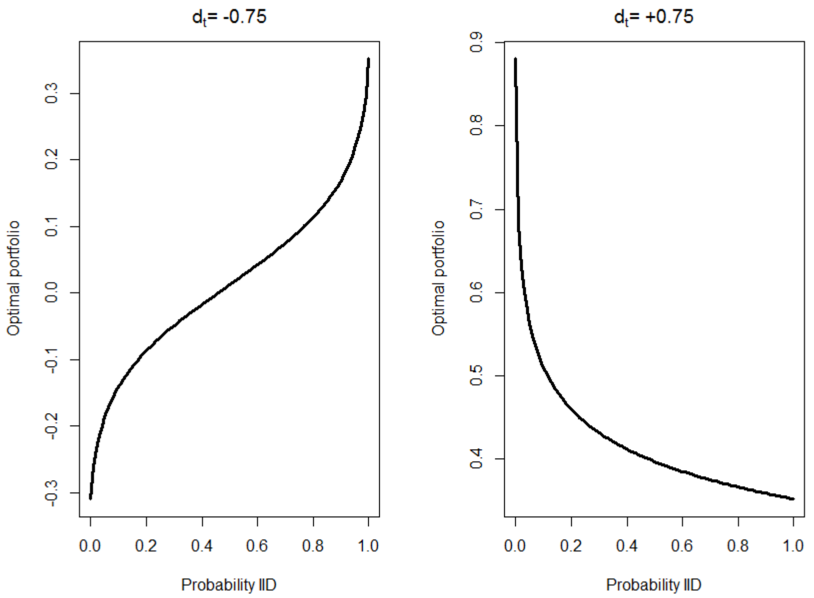
The graphs show the theoretical optimal allocations, for a Bayesian investor, as functions of the probability over the I.I.D. model. In the left ones dividend yield predictor takes a low value, $d_t = -0.75$, while in the right ones it takes an high value, $d_t = 0.75$.

Figure 2.10: Uncertainty over predictability: Smooth

(a) $\rho = 1.5, \eta = 46$



(b) $\rho = 3.75, \eta = 80$



The graphs show the theoretical optimal allocations, for a smooth investor, as a function of the probability over the i.i.d. model. In the left ones dividend yield predictor takes a low value, $d_t = -0.75$*, while in the right ones it takes an high value,* $d_t = 0.75$.

expected excess-return. Figure 2.12 (a) shows that, given the distribution over models, the uncertainty over the expected excess-return increases in the absolute value of the predictor. On the other hand, given the value of d_t, the variance of the expected excess-return attains its maximum when the two models are equally likely.

Although ambiguity aversion is, typically, the prominent behavioural attitude towards model uncertainty, it is not the only characteristic that is relevant to the investor's optimal decision. In fact, if the predictor takes values far from its unconditional mean and the subjective belief over the two models is highly asymmetric, then also ambiguity prudence provides a significant contribution to the optimal portfolio composition. In order to investigate and to disentangle the relevance of the two ambiguity attitudes, I report in figure 2.11 the comparison among the theoretical optimal solutions obtained solving (2.19) numerically - solid black lines, with those implied by the robust mean-variance approach (2.3) - dashed blue lines - and with those implied by the third-order approximation (2.4) - dashed red lines. Results confirm that, for large values of μ and negative values of d_t, or small values of μ and positive values of d_t, an investor that neglects to account for ambiguity prudence effects would over-invest in the stock and her portfolio composition would result to be significantly different from the optimal one.

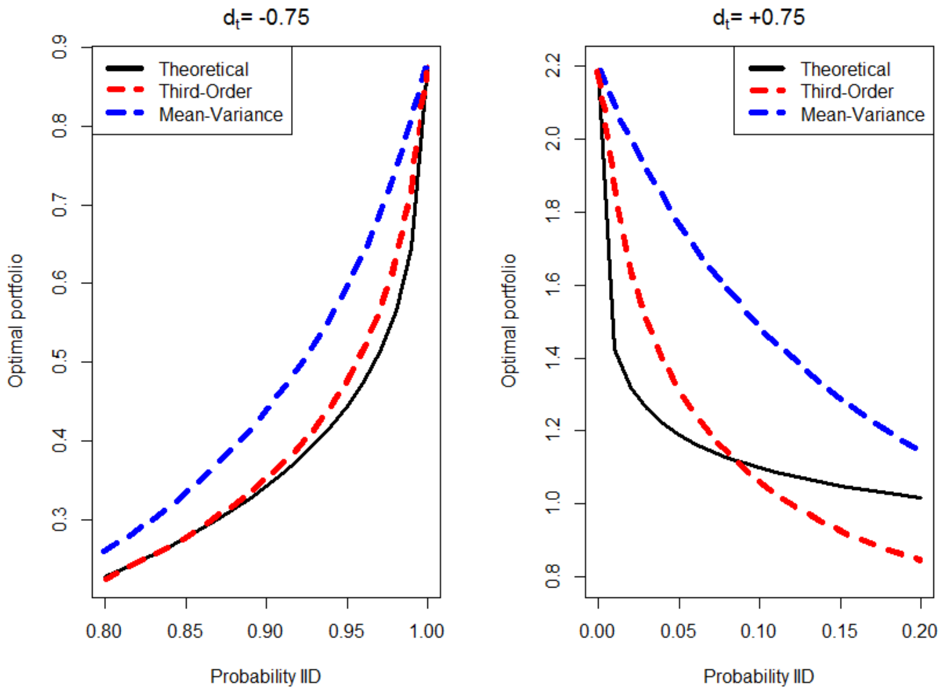
The intuition is that when d_t is significantly lower than its unconditional mean, a high value for excess-return is most likely to realize under the i.i.d. model. Although investor attaches high probability to the i.i.d. model, she might be worried about the possibility to bear too much stock in case excess-returns are predictable, i.e. under the model that is, among the two, the most harmful. Thus, she decides to reduce stock allocation to hedge against downside ambiguity. The same argument applies when the investor attaches a low probability over the i.i.d. model and the dividend-yield is larger than its unconditional mean $d_t > 0$.

Figure 2.12 (b) shows that sign of these effects is given by the third central moment (and, more generally, by odd moments) of the distribution over the expected excess-return. However, since the magnitude of the third moment is relative low with respect to magnitude of the variance, the relevance of ambiguity prudence is driven mainly by the strong preferences' penalization.

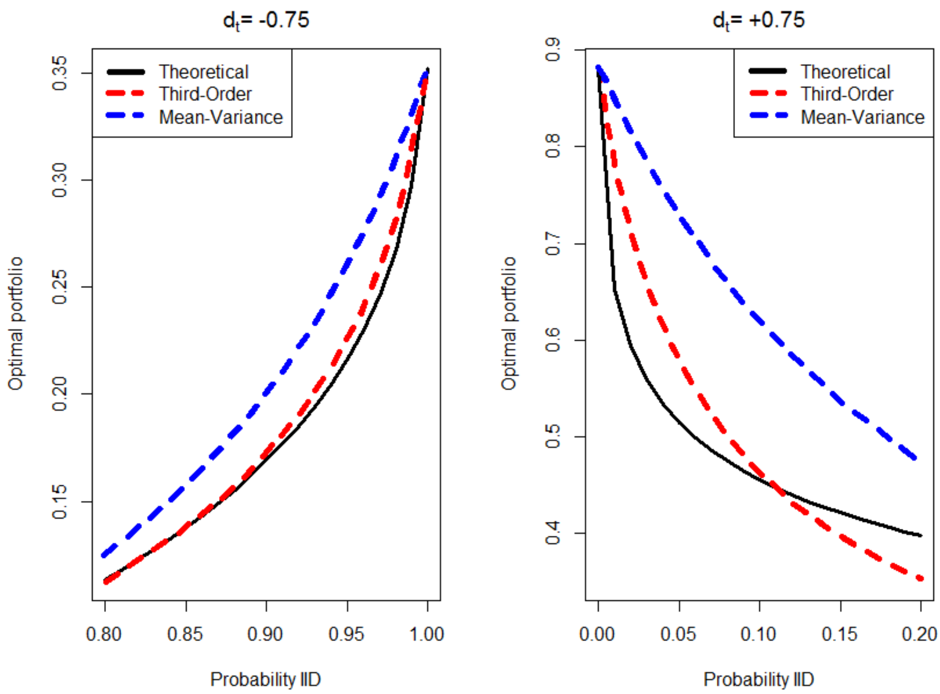
Finally, although the third central moment is symmetric with respect to the point $(d_t, \mu) = (0, 0.5)$, the contribution of ambiguity prudence to the optimal portfolio composition results to be asymmetric. More specifically, if the dividend-yield takes large positive values, then the theoretical optimal solutions, reported on the right graphs of figure 2.10, are kinked only towards the worst-case model. Consistently with theoretical results, I ascribe this effect to the larger stock investment that makes ambiguity attitudes more relevant.

Figure 2.11: Uncertainty over predictability: Smooth

(a) $\rho = 1.5, \eta = 46$



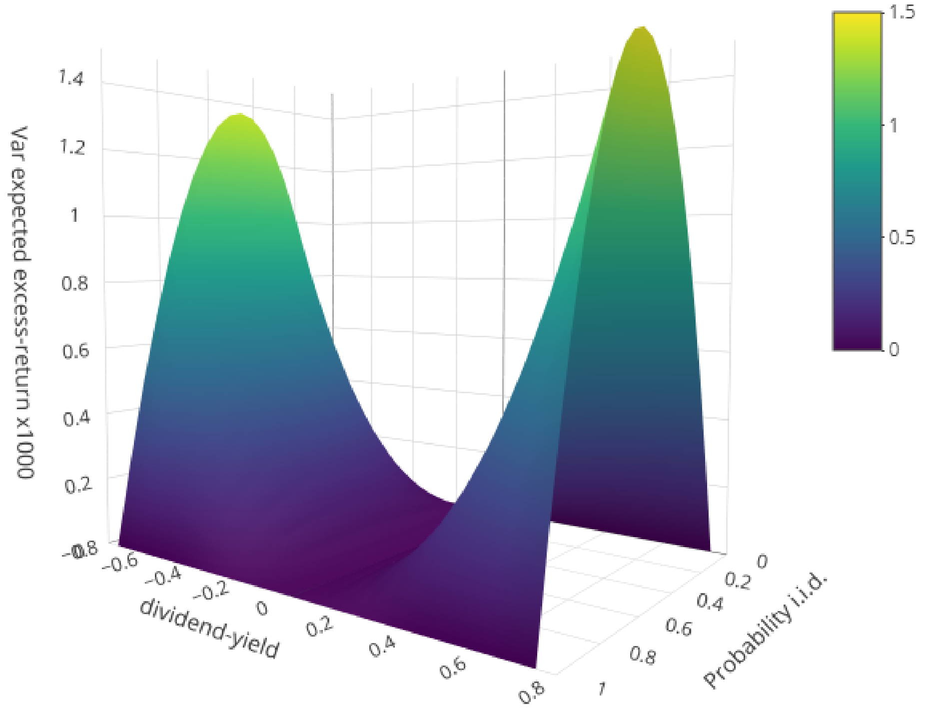
(b) $\rho = 3.75, \eta = 80$



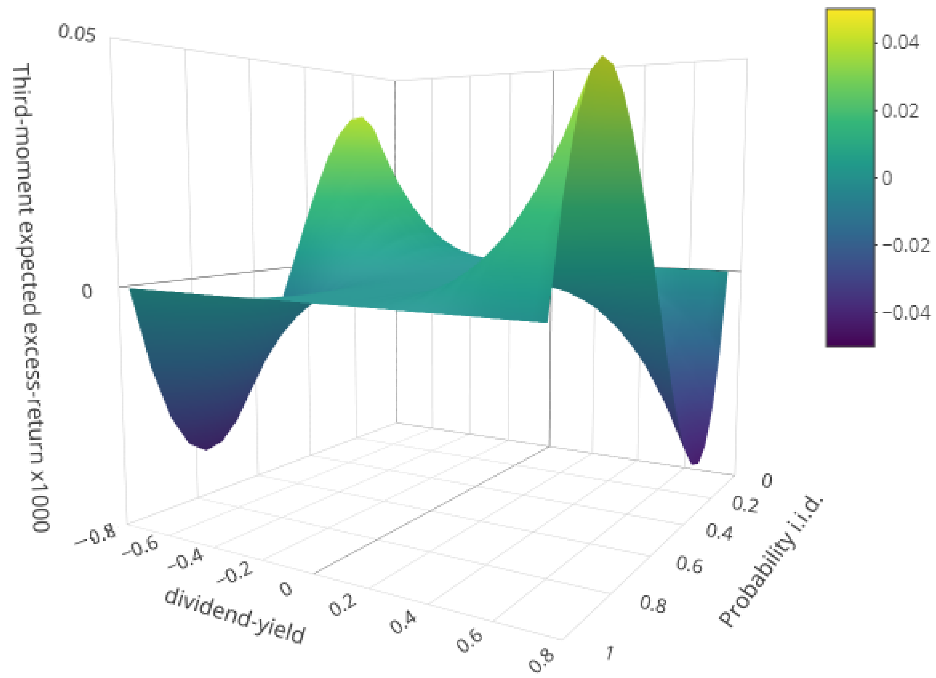
The figures report the theoretical optimal allocation (solid black line), the robust mean-variance solution (dashed blue line) and the solution implied by the third-order approximation (dashed red line).

Figure 2.12: Statistical moments of μ

(a) Variance of the expected excess-return



(b) Third central moment of the expected excess-return



2.3 An asset pricing example

I conclude this work reconciling the role of ambiguity prudence in putting non-linearities in the optimal portfolio composition with empirical results developed within the robust asset pricing literature. For example, Hansen and Sargent (2001) show that if an investor has concerns about the evolution of consumption growth and evaluates future streams according to multiplier preferences, then she acts *as if* her beliefs were endogenously distorted towards the model with the lowest continuation value.

In order to highlight how ambiguity prudence affects the stochastic discount factor, hence equilibrium quantities e.g. prices or risk-free rate, in this section I consider a Lucas type economy. I assume that in each period the economy is endowed with an exogenous aggregate level of consumption C_t. Investor ignores the growth rate of consumption and evaluates the following model

$$\begin{cases} \ln\left(\frac{C_{t+1}}{C_t}\right) = \theta_0 + \theta_2 X_t + \sigma_c \varepsilon_{t+1}, \\ X_t = \theta_1 X_{t-1} + \sigma_x \varepsilon^x_{t+1}, \end{cases} \qquad \varepsilon_{t+1}, \varepsilon^x_{t+1} \sim N(0, I) \tag{2.20}$$

in which the value of θ_2 is ambiguous and the unconditional mean of X_t is zero. Investor attaches probability $\mu(\omega_1) = \Pr(\theta_2 = 0)$ and $\mu(\omega_2) = \Pr\left(\theta_2 = \bar{\theta}\right)$. Preferences over future streams of consumption $(C_s)_{s=t+1}^{+\infty} = (C_{t+1}, C_{t+2}, ...) \in \mathbb{R}_+^\infty$ have a smooth representation

$$U(C_t) = u(C_t) + u \circ v^{-1}\left(\mathbb{E}_{\mu_t}\left[v \circ u^{-1}\left(\mathbb{E}_{K_t(\omega,\cdot)}\left[\beta U(C_{t+1})\right]\right)\right]\right). \tag{2.21}$$

The multiplier preferences of Anderson et al. (1998) and Hansen and Sargent (2001), very popular in the robust asset pricing literature, are a particular case of (1.1) in which $v \circ u^{-1}(x) = -\exp(-\beta x/\eta)$. In this case, the stochastic discount factor is given by

$$SDF_{t,t+1}(\omega, C_{t+1}) = \frac{C_t}{C_{t+1}} \frac{\exp\left(\frac{-\mathbb{E}_{K_t(\omega,\cdot)}[\beta U(C_{t+1})]}{\eta}\right)}{\mathbb{E}_{\mu_t}\left[\exp\left(\frac{-\mathbb{E}_{K_t(\omega,\cdot)}[\beta U(C_{t+1})]}{\eta}\right)\right]}, \tag{2.22}$$

where the first factor on the RHS is the standard pricing kernel for time additive utility while the last factor is the Randon-Nikodym derivative that distorts beliefs to properly account for ambiguity attitudes. Because of this change of measure, the investor acts as if she is Bayesian with beliefs given by

$$\tilde{\mu}_t(\omega_1) = \frac{\mu_t(\omega_1)}{\mu_t(\omega_1) + \mu_t(\omega_2)\exp\left(\frac{\beta}{\eta}\left[\mathbb{E}_{K(\omega_1,\cdot)}\left[U(C_{t+1})\right] - \mathbb{E}_{K(\omega_2,\cdot)}\left[U(C_{t+1})\right]\right]\right)}.$$

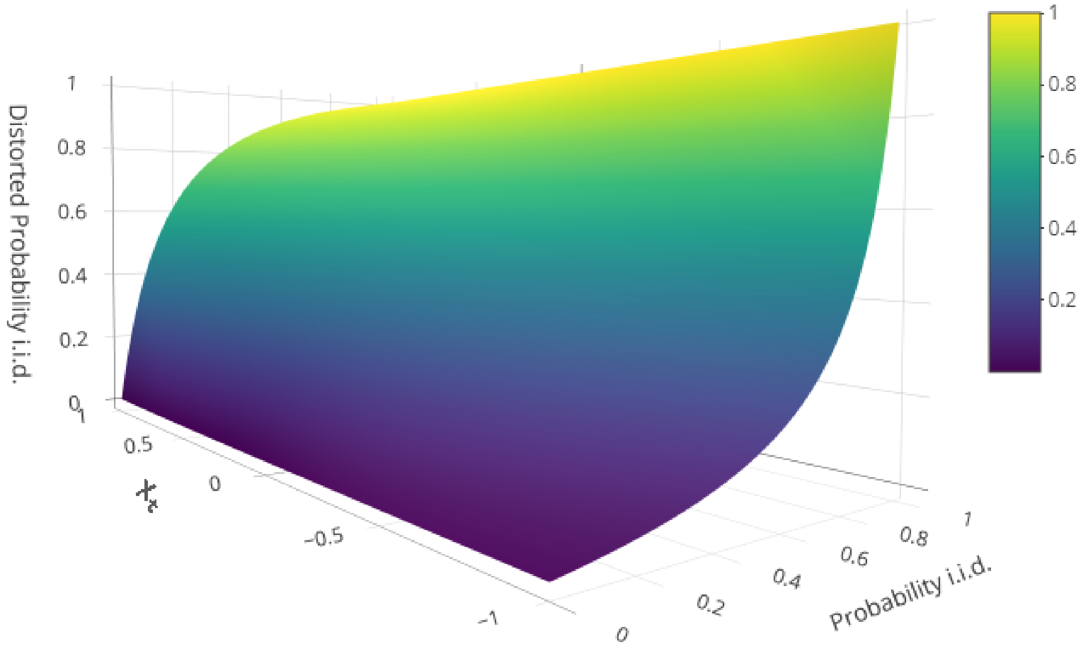
If the shape of the utility index is $u(x) = (1-\beta)\log x$, then the continuation values of the two models are given by

$$\mathbb{E}_{K(\omega_1,\cdot)}\left[\beta U(C_{t+1})\right] = \beta\theta_0 + \ln C_t,$$

$$\mathbb{E}_{K(\omega_2,\cdot)}\left[\beta U(C_{t+1})\right] = \beta\theta_0 + \beta\bar{\theta}(1-\beta\theta_1)^{-1} X_t + \ln C_t.$$

If I calibrate the parameters of model ω_2 as proposed by Bansal and Yaron (2004), i.e. $\bar{\theta} = 1$, $\theta_1 = 0.975$ and $\beta = 0.99$, then figure 2.13 shows the distorted beliefs as a function of the value of the persistent component X_t and of the probability over the I.I.D. model.

Figure 2.13: Distorted beliefs



When the persistent component X_t attains values far from the unconditional mean, the continuation value of the two models significantly differs and figure 2.13 shows that the distortion of beliefs is larger. In particular, if an investor attaches a very low probability over the I.I.D. model, i.e. $\mu_t(\omega_1)$ close to 0, but X_t takes large positive values, then under $\tilde{\mu}$ the I.I.D. model is much more likely to realize. The same holds when $\mu_t(\omega_1)$ is close to one and X_t is largely negative. The twisted probabilities put into the investor evaluation her distrust towards the reference probabilities. Moreover, the non-linear dependence of beliefs distortion is perfectly consistent with the non-linear optimal allocations reported in figure 2.11. For this reason, I ascribe the higher curvature of the SDF to ambiguity prudent attitudes that generate a sharp endogenous distortion of beliefs towards the model that provides the lowest continuation value.

2.4 Conclusion

In this work, I study the contribution of ambiguity prudence to the optimal portfolio composition for an investor whose preferences are represented by the smooth ambiguity criterion of Klibanoff et al. (2005). The motivations of this research question rely on the experimental evidence, e.g. Baillon et al. (2018), supporting the idea that ambiguity prudence is important for individual decision-making, on recent works in the asset-pricing literature suggesting that the equity premium is skewed and, more generally, that higher moments of the distribution over models might help to explain relevant macro-finance puzzles.

In the first chapter, I derive a model-free third-order approximation of the portfolio certainty equivalent that helps to study ambiguity prudence in its full generality. I show that prudence towards ambiguity induces an uncertainty premium that depends on the third central moment of the distribution over models and on the co-movement across models between expected return and risk size. Noteworthy, the contribution of these effects is highly non-linear.

In the second chapter, I focus on two settings in which ambiguity prudence has relevant implications for the portfolio composition. In the first, a *buy-and-hold* investor believes that returns are i.i.d. with unknown expected return, over which she has skewed beliefs. In the second setting, an investor is ambiguous about the predictability of the excess-return. In both settings, I show that if the certainty equivalents implied by models under evaluation are significantly different and the investor has asymmetric beliefs over them, then ambiguity prudence induces sizeable variation from the robust mean-variance solution. These effects emerge because of the investor's willingness to hedge against downside model uncertainty.

I conclude the paper reconciling the portfolio implications of ambiguity prudence with some empirical regularities emerging from the asset-pricing literature. Specifically, if the investor's preferences account for concerns about robustness, then ambiguity prudence puts non-linearities in the marginal evaluation of future streams of consumption. The quantitative relevance of these non-linearities has been rigorously investigated in the literature, see e.g. Hansen and Sargent (2010). Results confirm that non-neutral ambiguity attitudes provide a possible resolution of several puzzling links between macroeconomics and aggregate financial markets.

Appendix

2.A Third-Order Approximation

Definition. *Let $(\mathbb{X}, \mathfrak{X})$ be the measurable Polish space of the possible realizations of X and $(\Omega, \mathcal{F})$ be a comfortable measurable space. A random probability measure is a function $K : \Omega \times \mathfrak{X} \to [0, 1]$ such that:*

- $\omega \longmapsto K(\omega, A)$ *is measurable random variable,*
- $\forall A \in \mathfrak{X}$*;* $A \longmapsto K(\omega, A)$ *is a probability measure over* $(\mathbb{X}, \mathfrak{X})$*,* $\forall \omega \in \Omega$.

The existence of $K(\omega, \cdot)$ in Polish spaces is guaranteed by the following theorem.

Theorem. *If $(X, \mathfrak{X})$ is a standard measurable space, then there exist a regular version. See Çınlar (2011) for details.*

Let $u : \mathbb{X} \to \mathbb{R}$ and $v : u^{-1}(\mathbb{X}) \to \mathbb{R}$ denote the Bernoulli and the second order utility functions that, along with $\phi := v \circ u^{-1}$, $\phi : u(\mathbb{X}) \to v(\mathbb{X})$, $\psi := v^{-1} : v(\mathbb{X}) \to \mathbb{X}$, are N-times differentiable functions, respectively over $int\mathbb{X}$, $u^{-1}(int\mathbb{X})$, $u(int\mathbb{X})$ and $v(int\mathbb{X})$. Let X, $u(X)$, $v(X) \in L^N(\mathbb{X}, \mathfrak{X}, Q)$, where Q is the mean probability measure over $(\mathbb{X}, \mathfrak{X})$, i.e. $Q(A) := \int K(\omega, A)\, \mu(d\omega)$, $\forall A \in \mathfrak{X}$.

Lemma. *If $f \in L^N(\mathbb{X}, \mathfrak{X}, Q)$, then $\mu\left(\left\{\omega : f \in L^N(\mathbb{X}, \mathfrak{X}, K(\omega, \cdot))\right\}\right) = 1$.*

Proof. Let $f \in L^N$, by definition of a L^p space and monotonic property of expectations, it holds that $0 \leq \int_{\mathbb{X}} |f|^N Q(dx) < +\infty$.

Define $A := \left\{\omega \in \Omega : \int_{\mathbb{X}} |f|^N K(\omega, dx) = +\infty\right\}$, I know

$$\int_{\mathbb{X}} |f|^N K(\omega, dx) \geq n 1_A,\ \forall n \in \mathbb{N}, \forall \omega \in \Omega. \tag{2.23}$$

and if $\mu(A) > 0$ then

$$\int_{\Omega} \left(\int_{\mathbb{X}} |f|^N K(\omega, dx)\right) \mu(d\omega) \geq n\mu(A)\ \forall n \in \mathbb{N}, \tag{2.24}$$

implying that

$$\int_{\mathbb{X}} |f|^N Q(dx) = \int_{\Omega} \left(\int_{\mathbb{X}} |f|^N K(\omega, dx)\right) \mu(d\omega) = +\infty, \tag{2.25}$$

which leads to an absurd. □

This result guarantees that, if the *N-th* moment exists under the mean measure, then the same holds almost surely under all the probability measures in the support of μ.

Proof. of Theorem (1.2.1).

The third order approximation of (1.1) around $\pi = 0$ is given

$$C(\pi, X) = C(0, X) + \pi C'(0, X) + \frac{\pi^2}{2!} C''(0, X) + \frac{\pi^3}{3!} C'''(0, X) + o\left(\left|\pi^3\right|\right), \tag{2.26}$$

and the expression of the first three terms of (2.26) is given by Maccheroni et al. (2013). In order to find the value of tha last term, I define

$$C(\pi, X) = \psi(F(0))$$

$$F(\pi, X) := \int_{\Omega} f_\omega(\pi, X) \mu(d\omega),$$

$$f_\omega(\pi, X) := \phi \circ \mathbb{E}_\omega(\pi, X),$$

$$\mathbb{E}_\omega(\pi, X) := \int_{\mathbb{X}} u(\pi, X) K(\omega, dx).$$

so that

$$C'''(0) = \psi'''(F(0)) F'(0)^3 + 3\psi''(F(0)) F'(0) F''(0) + \psi'(F(0)) F'''(0).$$

Under standard integrability conditions

$$F^{'''}(\pi, X) = \int_{\Omega} f_{\omega}^{'''}(\pi, X)\,\mu(d\omega), \tag{2.27}$$

$$\frac{\partial^3 E_{\omega}(\pi, X)}{\partial \pi^3} = \int_{\mathbb{X}} \frac{\partial^3 u(\pi, X)}{\partial \pi^3} X^3\, K(\omega, dx), \tag{2.28}$$

and, in $\pi = 0$, the integrand of (2.27) is given by

$$f_{\omega}^{'''}(0, X) = \phi^{'''}(E_{\omega}(0, X))\, E_{\omega}^{'}(0, X)^3 + 3\phi^{''}(E_{\omega}(0, X))\, E_{\omega}^{'}(0, X)\, E_{\omega}^{''}(0, X)$$

$$+E_{\omega}^{'''}(0, X)\,\phi^{'}(E_{\omega}(0, X)).$$

Therefore,

$$F^{'''}(0) = \int_{\Omega} \phi^{'''}(E(0, \omega)) \left[u^{'}(0) \int_{\mathbb{X}} X K(\omega, dx)\right]^3 \mu(d\omega) \tag{2.29}$$

$$+3 \int_{\Omega} \left[\frac{v^{''}(0)}{u^{'}(0)} - v^{'}(0) \frac{u^{''}(0)}{u^{'}(0)^2}\right] \left[\int_{\mathbb{X}} X\, K(\omega, dx)\right] \left[u^{''}(0) \int_{\mathbb{X}} X^2\, K(\omega, dx)\right] \mu(d\omega)$$

$$+ \int_{\Omega} \left[u^{'''}(0) \int_{\mathbb{X}} X^3\, K(\omega, dx)\right] \frac{v^{'}(0)}{u^{'}(0)} \mu(d\omega).$$

The third order derivatives of ϕ and ψ are given by

$$\phi^{'''}(u(x)) = \frac{v^{'''}(x)}{u'(x)^3} - \frac{u^{'''}(x)\, v'(x)}{u'(x)^4} - \frac{3u^{''}(x)\, v^{''}(x)}{u'(x)^4} + \frac{3u^{''}(x)^2\, v'(x)}{u'(x)^5}.$$

$$\psi^{''}(\phi(u(x))) = -\frac{v^{'''}(x)}{v'(x)^4} + \frac{3v^{''}(x)^2}{v'(x)^5}.$$

Plugging in these terms in (2.29),

$$F^{'''}(0) = \left[v^{'''}(0) - \frac{u^{'''}(0)\, v^{'}(0)}{u^{'}(0)} - \frac{3u^{''}(0)\, v^{''}(0)}{u^{'}(0)} + \frac{3u^{''}(0)^2\, v^{'}(0)}{u^{'}(0)^2}\right] \int_{\Omega} \mathbb{E}_{\omega}^3 [X]\, \mu(d\omega)$$

$$+3 \left[\frac{u^{''}(0)\, v^{''}(0)}{u^{'}(0)} - \frac{u^{''}(0)^2\, v^{'}(0)}{u^{'}(0)^2}\right] \mathbb{E}_{\mu}\left[\mathbb{E}_{\omega}[X]\, \mathbb{E}_{\omega}\left[X^2\right]\right] + \frac{u^{'''}(0)\, v^{'}(0)}{u^{'}(0)} \mathbb{E}_{Q}\left[X^3\right].$$

Finally the new term in the Mclaurin expansion is

$$\frac{C^{'''}(0)}{3!}\pi^3 = \frac{\pi^3}{6} \left[\left(-\frac{v^{'''}(0)}{v^{'}(0)} + \frac{3v^{''}(0)^2}{v^{'}(0)^2}\right) \mathbb{E}_{Q}^3 [X] - 3\left(\mathbb{E}_{Q}[X]\right)\right.$$

$$\left.\left[\left(\frac{v^{''}(0)^2}{v^{'}(0)^2} - \frac{u^{''}(0)\, v^{''}(0)}{u^{'}(0)\, v^{'}(0)}\right) \mathbb{E}_{\mu}\left[\mathbb{E}_{\omega}^2 [X]\right] + \frac{u^{''}(0)\, v^{''}(0)}{u^{'}(0)\, v^{'}(0)} \mathbb{E}_{Q}\left[X^2\right]\right]\right.$$

$$+\left(\frac{v'''(0)}{v'(0)}-\frac{u'''(0)}{u'(0)}-\frac{3u''(0)\,v''(0)}{u'(0)\,v'(0)}+\frac{3u''(0)^2}{u'(0)^2}\right)\mathbb{E}_\mu\left[\mathbb{E}^3_\omega[X]\right]$$

$$+3\left(\frac{u''(0)\,v''(0)}{u'(0)\,v'(0)}-\frac{u''(0)^2}{u'(0)^2}\right)\mathbb{E}_\mu\left[\mathbb{E}_\omega[X]\,\mathbb{E}_\omega\left[X^2\right]\right]+\frac{u'''(0)}{u'(0)}\mathbb{E}_Q\left[X^3\right]\Bigg].$$

Adding and subtracting $3\frac{u''(0)v''(0)}{u'(0)v'(0)}\mathbb{E}_Q[X]\,\mathbb{E}^2_Q[X]$, I obtain

$$\frac{C'''(0)}{3!}\pi^3=\frac{\pi^3}{6}\Bigg[3\frac{u''(0)\,v''(0)}{u'(0)\,v'(0)}\mathbb{E}_Q[X]\left(\mathbb{E}_\mu\left[\mathbb{E}^2_\omega[X]\right]-\mathbb{E}^2_Q[X]\right)$$

$$-3\frac{u''(0)\,v''(0)}{u'(0)\,v'(0)}\mathbb{E}_Q[X]\left(\mathbb{E}_Q\left[X^2\right]-\mathbb{E}^2_Q[X]\right)-3\left(\frac{v''(0)}{v'(0)}\right)^2\mathbb{E}_Q[X]$$

$$\left(\mathbb{E}_\mu\left[\mathbb{E}^2_\omega[X]\right]-\mathbb{E}^2_Q[X]\right)+\frac{v'''(0)}{v'(0)}\left(\mathbb{E}_\mu\left[\mathbb{E}^3_\omega[X]\right]-\mathbb{E}^3_Q[X]\right)$$

$$+\left(-\frac{u'''(0)}{u'(0)}-\frac{3u''(0)\,v''(0)}{u'(0)\,v'(0)}+\frac{3u''(0)^2}{u'(0)^2}\right)\mathbb{E}_\mu\left[\mathbb{E}^3_\omega[X]\right]+3\frac{u''(0)\,v''(0)}{u'(0)\,v'(0)}$$

$$\mathbb{E}_\mu\left[\mathbb{E}_\omega[X]\,\mathbb{E}_\omega\left[X^2\right]\right]-3\left(\frac{u''(0)}{u'(0)}\right)^2\mathbb{E}_\mu\left[\mathbb{E}_\omega[X]\,\mathbb{E}_\omega\left[X^2\right]\right]+\frac{u'''(0)}{u'(0)}\mathbb{E}_Q\left[X^3\right]\Bigg]$$

and, adding and subtracting $-\frac{u'''(0)}{u'(0)}\mathbb{E}^3_Q[X]$,

$$=\frac{\pi^3}{6}\Bigg[3\frac{u''(0)\,v''(0)}{u'(0)\,v'(0)}\mathbb{E}_Q[h]\left(\mathbb{E}_\mu\left[\mathbb{E}^2_\omega[X]\right]-\mathbb{E}^2_Q[X]\right)$$

$$-3\frac{u''(0)\,v''(0)}{u'(0)\,v'(0)}\mathbb{E}_Q[X]\left(\mathbb{E}_Q\left[X^2\right]-\mathbb{E}^2_Q[X]\right)-3\left(\frac{v''(0)}{v'(0)}\right)^2\mathbb{E}_Q[X]$$

$$\left(\mathbb{E}_\mu\left[\mathbb{E}^2_\omega[X]\right]-\mathbb{E}^2_Q[X]\right)+\frac{v'''(0)}{v'(0)}\left(\mathbb{E}_\mu\left[\mathbb{E}^3_\omega[X]\right]-\mathbb{E}^3_Q[X]\right)$$

$$-\frac{u'''(0)}{u'(0)}\left(\mathbb{E}_\mu\left[\mathbb{E}^3_\omega[X]\right]-\mathbb{E}^3_Q[X]\right)+\frac{u'''(0)}{u'(0)}\left(\mathbb{E}_Q\left[X^3\right]-\mathbb{E}^3_Q[X]\right)$$

$$+3\left(\frac{u''(0)\,v''(0)}{u'(0)\,v'(0)}-\frac{u''(0)^2}{u'(0)^2}\right)\left(\mathbb{E}_\mu\left[\mathbb{E}_\omega[X]\,\mathbb{E}_\omega\left[X^2\right]\right]-\mathbb{E}_\mu\left[\mathbb{E}^3_\omega[X]\right]\right)\Bigg]=$$

$$=\frac{\pi^3}{6}\Bigg[-3\left(\left(\frac{v''(0)}{v'(0)}\right)^2-\frac{u''(0)\,v''(0)}{u'(0)\,v'(0)}\right)\mathbb{E}_Q[X]\,Var_\mu\left(\mathbb{E}_\omega[X]\right)-$$

$$-3\frac{u''(0)\,v''(0)}{u'(0)\,v'(0)}\mathbb{E}_Q[X]\,Var_Q(X)+\left(\frac{v'''(0)}{v'(0)}-\frac{u'''(0)}{u'(0)}\right)\left(\mathbb{E}_\mu\left[\mathbb{E}^3_\omega[X]\right]-\mathbb{E}^3_Q[X]\right)+$$

$$+\frac{u'''(0)}{u'(0)}\left(\mathbb{E}_Q\left[X^3\right]-\mathbb{E}^3_Q[X]\right)+3\left(\frac{u''(0)\,v''(0)}{u'(0)\,v'(0)}-\frac{u''(0)^2}{u'(0)^2}\right)$$

$$\left(\mathbb{E}_{\mu}\left[\mathbb{E}_{\omega}[X]\,\mathbb{E}_{\omega}\left[X^{2}\right]\right]-\mathbb{E}_{\mu}\left[\mathbb{E}_{\omega}^{3}[X]\right]\right)\right]=$$

$$=\frac{\pi^{3}}{6}\left[-3\left(\left(\frac{v''(0)}{v'(0)}\right)^{2}-\frac{u''(0)\,v''(0)}{u'(0)\,v'(0)}-\left(\frac{v'''(0)}{v'(0)}-\frac{u'''(0)}{u'(0)}\right)\right)\right.$$

$$\mathbb{E}_{Q}[X]\,Var_{\mu}\left(\mathbb{E}_{\omega}[X]\right)-3\left(\frac{u''(0)\,v''(0)}{u'(0)\,v'(0)}-\frac{u'''(0)}{u'(0)}\right)\mathbb{E}_{Q}[X]\,Var_{Q}(X)$$

$$+\left(\frac{v'''(0)}{v'(0)}-\frac{u'''(0)}{u'(0)}\right)\left(\mathbb{E}_{\mu}\left[\mathbb{E}_{\omega}^{3}[X]\right]-\mathbb{E}_{Q}^{3}[X]-3\mathbb{E}_{Q}[X]\,Var_{\mu}\left(\mathbb{E}_{\omega}[X]\right)\right)$$

$$+\frac{u'''(0)}{u'(0)}\left(\mathbb{E}_{Q}\left[X^{3}\right]-\mathbb{E}_{Q}^{3}[X]-3\mathbb{E}_{Q}[X]\,Var_{Q}(X)\right)$$

$$+3\left(\frac{u''(0)\,v''(0)}{u'(0)\,v'(0)}-\frac{u''(0)^{2}}{u'(0)^{2}}\right)\left(\mathbb{E}_{\mu}\left[\mathbb{E}_{\omega}[X]\,\mathbb{E}_{\omega}\left[X^{2}\right]\right]-\mathbb{E}_{\mu}\left[\mathbb{E}_{\omega}^{3}[X]\right]\right)=$$

$$=\frac{\pi^{3}}{6}\left[-3\left(\left(\frac{v''(0)}{v'(0)}\right)^{2}-\frac{u''(0)\,v''(0)}{u'(0)\,v'(0)}-\left(\frac{v'''(0)}{v'(0)}-\frac{u'''(0)}{u'(0)}\right)\right)\mathbb{E}_{Q}[X]\,Var_{\mu}\left(\mathbb{E}_{\omega}[X]\right)\right.$$

$$-3\left(\frac{u''(0)\,v''(0)}{u'(0)\,v'(0)}-\frac{u'''(0)}{u'(0)}\right)\mathbb{E}_{Q}[X]\,Var_{Q}(X)+$$

$$+\left(\frac{v'''(0)}{v'(0)}-\frac{u'''(0)}{u'(0)}\right)\mathbb{E}_{\mu}\left[\mathbb{E}_{\omega}[X]-\mathbb{E}_{Q}[X]\right]^{3}+\frac{u'''(0)}{u'(0)}\mathbb{E}_{Q}\left[X-\mathbb{E}_{Q}[X]\right]^{3}+$$

$$\left.+3\left(\frac{u''(0)\,v''(0)}{u'(0)\,v'(0)}-\frac{u''(0)^{2}}{u'(0)^{2}}\right)\left(\mathbb{E}_{\mu}\left[\mathbb{E}_{\omega}[X]\,\mathbb{E}_{\omega}\left[X^{2}\right]\right]-\mathbb{E}_{\mu}\left[\mathbb{E}_{\omega}^{3}[X]\right]\right)\right].$$

Finally, if I add and subtract

$$3\left(\frac{u''(0)\,v''(0)}{u'(0)\,v'(0)}-\frac{u''(0)^{2}}{u'(0)^{2}}\right)\mathbb{E}_{Q}[X]\left(\mathbb{E}_{Q}\left[X^{2}\right]+\mathbb{E}_{Q}^{2}[X]+\mathbb{E}_{\mu}\left[\mathbb{E}_{\omega}^{2}[X]\right]\right),$$

then the last term of the above approximation becomes

$$3\left(\frac{u''(0)\,v''(0)}{u'(0)\,v'(0)}-\frac{u''(0)^{2}}{u'(0)^{2}}\right)\left(\mathbb{E}_{\mu}\left[\mathbb{E}_{\omega}[X]\,\mathbb{E}_{\omega}\left[X^{2}\right]\right]\right.$$

$$-\mathbb{E}_{Q}[X]\,\mathbb{E}_{Q}\left[X^{2}\right]+\mathbb{E}_{Q}[X]\,\mathbb{E}_{Q}\left[X^{2}\right]-\mathbb{E}_{Q}[X]\,\mathbb{E}_{Q}^{2}[X]$$

$$\left.+E_{Q}[X]\,\mathbb{E}_{Q}^{2}[X]+\mathbb{E}_{Q}[X]\,\mathbb{E}_{\mu}\left[\mathbb{E}_{\omega}^{2}[X]\right]-\mathbb{E}_{Q}[X]\,\mathbb{E}_{\mu}\left[\mathbb{E}_{\omega}^{2}[X]\right]-\mathbb{E}_{\mu}\left[\mathbb{E}_{\omega}^{3}[X]\right]\right)=$$

$$=3\left(\frac{u''(0)\,v''(0)}{u'(0)\,v'(0)}-\frac{u''(0)^{2}}{u'(0)^{2}}\right)$$

$$\left(Cov\left(\mathbb{E}_{\omega}[X],\mathbb{E}_{\omega}\left[X^{2}\right]\right)+\mathbb{E}_{Q}[X]\,Var_{Q}(X)-\mathbb{E}_{Q}[X]\,Var_{\mu}\left(\mathbb{E}_{\omega}[X]\right)\right.$$

$$\left.-Cov\left(\mathbb{E}_{\omega}[X],\mathbb{E}_{\omega}^{2}[X]\right)\right).$$

I can now substitute it in the general expression in order to get

$$= \frac{\pi^3}{6}\left[-3\left(\left(\frac{v''(0)}{v'(0)}\right)^2 - \frac{u''(0)\,v''(0)}{u'(0)\,v'(0)} - \left(\frac{v'''(0)}{v'(0)} - \frac{u'''(0)}{u'(0)}\right)\right)\right.$$

$$\mathbb{E}_Q[X]\,Var_\mu(\mathbb{E}_\omega[X]) - 3\left(\frac{u''(0)\,v''(0)}{u'(0)\,v'(0)} - \frac{u'''(0)}{u'(0)}\right)\mathbb{E}_Q[X]\,Var_Q(X)$$

$$+\left(\frac{v'''(0)}{v'(0)} - \frac{u'''(0)}{u'(0)}\right)\mathbb{E}_\mu\left[\mathbb{E}_\omega[X] - \mathbb{E}_Q[X]\right]^3 + \frac{u'''(0)}{u'(0)}\mathbb{E}_Q\left[X - \mathbb{E}_Q[X]\right]^3$$

$$+3\left(\frac{u''(0)\,v''(0)}{u'(0)\,v'(0)} - \frac{u''(0)^2}{u'(0)^2}\right)\left(Cov_\mu\left(\mathbb{E}_\omega[X], Var_\omega(X)\right) + \mathbb{E}_Q[X]\,Var_Q(X)\right.$$

$$\left.\left.-\mathbb{E}_Q[X]\,Var_\mu(\mathbb{E}_\omega[X])\right)\right] =$$

$$= \frac{\pi^3}{6}\left[-3\left(\left(\frac{v''(0)}{v'(0)}\right)^2 - \left(\frac{u''(0)}{u'(0)}\right)^2 - \left(\frac{v'''(0)}{v'(0)} - \frac{u'''(0)}{u'(0)}\right)\right)\right.$$

$$\mathbb{E}_Q[X]\,Var_\mu(\mathbb{E}_\omega[X]) - 3\left(\frac{u''(0)^2}{u'(0)^2} - \frac{u'''(0)}{u'(0)}\right)\mathbb{E}_Q[X]\,Var_Q(X) +$$

$$+\left(\frac{v'''(0)}{v'(0)} - \frac{u'''(0)}{u'(0)}\right)\mathbb{E}_\mu\left[\mathbb{E}_\omega[X] - \mathbb{E}_Q[X]\right]^3 + \frac{u'''(0)}{u'(0)}\mathbb{E}_Q\left[X - \mathbb{E}_Q[X]\right]^3 +$$

$$\left.+3\left(\frac{u''(0)\,v''(0)}{u'(0)\,v'(0)} - \frac{u''(0)^2}{u'(0)^2}\right)Cov_\mu\left(\mathbb{E}_\omega[X], Var_\omega(X)\right)\right] =$$

$$= \frac{\pi^3}{6}\left[-3\left(A_v(0)^2 - A_u(0)^2 - \left(P_v(0)\,A_v(0) - P_u(0)\,A_u(0)\right)\right)\right.$$

$$\mathbb{E}_Q[X]\,Var_\mu(\mathbb{E}_\omega[X]) - 3\left(A_u(0)^2 - P_uA_u(0)\right)\mathbb{E}_Q[X]\,Var_Q(X)$$

$$+\left(P_v(0)\,A_v(0) - P_u(0)\,A_u(0)\right)\mathbb{E}_\mu\left[\mathbb{E}_\omega[X] - \mathbb{E}_Q[X]\right]^3 + P_uA_u(0)\,\mathbb{E}_Q\left[X - \mathbb{E}_Q[X]\right]^3 +$$

$$\left.+3A_u(0)\left(A_v(0) - A_u(0)\right)Cov_\mu\left(\mathbb{E}_\omega[X], Var_\omega(X)\right)\right].$$

□

2.B Skew-normal Beliefs

The Skew-normal distribution, introduced by Azzalini (2013), is a special case of the multiplicative contamination of a reference probability of van der Linde et al. (2007). The intuition is that an DM believes that a random variable follows a specific distribution but she evaluates also alternative models, generated through the perturbation of such distribution. The skew-normal is retrieved when the reference density f_0 is Gaussian and the contaminating functions have the form of the cumulative distribution of a standard normal probability, Φ. The set of univariate densities is given by

$$\Gamma^{SN} = \left\{ f \;:\; f\left(x;\xi,\alpha_1,\tau_1\right) = \frac{2}{\tau_1}\phi\left(\frac{x-\xi}{\tau_1}\right)\Phi\left(\alpha_1\frac{x-\xi}{\tau_1}\right), \left(\xi,\alpha_1,\tau_1\right)\in\mathbb{R}^2\times\mathbb{R}_{++}\right\},$$

where ϕ and Φ are respectively the probability density function and cumulative distribution function of a standard normal distribution. Each element of Γ^{SN} is indexed by a three-dimensional vector of parameters with α_1 being the only parameter affecting higher-moments. It is immediate to see that the family of Gaussian distributions is retrieved as a particular case of $\alpha_1 = 0$.

Proof. of Proposition 1.

Since v is increasing and continuous, we can focus on the following problem

$$\max_{\pi\in\mathbb{R}} \int_{\mathbb{R}} v\circ u^{-1}\left(\underbrace{\int_{\mathbb{R}} u\left(\pi X_{t+1} + (1-\pi)R_f\right)\Phi^{SN}\left(dx,m,\tau_2,\alpha_2\right)}_{=\mathbb{E}_\omega[u(W_{t+1})]}\right)\Phi^{SN}\left(dm;\xi,\tau_1,\alpha_1\right)$$

Given the moment generating function of skew normal distribution,

$$\mathbb{E}_\omega\left[u\left(W_{t+1}\right)\right] = -2\exp\left\{-\rho\left(1-\pi\right)R_f - \rho m\pi + \frac{1}{2}\rho^2\tau_2^2\pi^2\right\}\Phi\left(-\frac{\alpha_2}{\sqrt{1+\alpha_2^2}}\tau_2\rho\pi\right),$$

$$u^{-1}\left(\mathbb{E}_\omega\left[u\left(W_{t+1}\right)\right]\right) = -\frac{\ln 2}{\rho} + \left(1-\pi\right)R_f + m\pi - \frac{1}{2}\rho\tau_2^2\pi^2 - \frac{1}{\rho}\ln\Phi\left(-\frac{\alpha_2}{\sqrt{1+\alpha_2^2}}\tau_2\rho\pi\right),$$

$$v\left(u^{-1}\left(\mathbb{E}_\omega\left[u\left(W_{t+1}\right)\right]\right)\right) = -2^{\frac{\eta}{\rho}}\exp\left(-\eta\left(1-\pi\right)R_f - \eta m\pi + \frac{1}{2}\rho\eta\tau_2^2\pi^2\right)$$

$$\Phi\left(-\frac{\alpha_2}{\sqrt{1+\alpha_2^2}}\tau_2\rho\pi\right)^{\frac{\eta}{\rho}}.$$

Using the moment generating function of the Skew-normal, the optimal stock allocation π^* has to maximize

$$\max_{\pi\in\mathbb{R}} -2^{\frac{\eta+\rho}{\rho}} \exp\left(-\eta\left(1-\pi\right)R_f - \eta\xi\pi + \frac{1}{2}\left(\tau_2^2\eta\rho+\tau_1^2\eta^2\right)\pi^2\right)$$

$$\Phi\left(-\frac{\alpha_1}{\sqrt{1+\alpha_1^2}}\tau_1\eta\pi\right)\Phi\left(-\frac{\alpha_2}{\sqrt{1+\alpha_2^2}}\tau_2\rho\pi\right)^{\frac{\eta}{\rho}}.$$

Taking the first order derivative with respect to π, we obtain the following equation

$$\pi^*\left(\rho\tau_2^2+\eta\tau_1^2\right) - \frac{\alpha_1}{\sqrt{1+\alpha_1^2}}\tau_1\frac{\phi\left(-\frac{\alpha_1}{\sqrt{1+\alpha_1^2}}\tau_1\eta\pi^*\right)}{\Phi\left(-\frac{\alpha_1}{\sqrt{1+\alpha_1^2}}\tau_1\eta\pi^*\right)}$$

$$-\frac{\alpha_2}{\sqrt{1+\alpha_2^2}}\tau_2\frac{\phi\left(-\frac{\alpha_2}{\sqrt{1+\alpha_2^2}}\tau_2\rho\pi^*\right)}{\Phi\left(-\frac{\alpha_2}{\sqrt{1+\alpha_2^2}}\tau_2\rho\pi^*\right)} = \xi - R_f.$$

□

Part II

Ambiguity and Consumption Sharing

Chapter 3

Heterogeneous beliefs and ambiguity attitudes

Abstract

This paper studies the optimal consumption allocation when investors have heterogeneous beliefs and ambiguity attitudes towards three sources of uncertainty: the first two concern the conditional distribution of consumption, while the third one concerns the persistence of the shocks hitting the economy. I provide a mechanism to explain why, as economic conditions deteriorate, subjective beliefs polarize. If the economy has a large number of periods, then disagreement over the persistent component of consumption becomes a main driver of the changes in the distribution of wealth.

3.1 Introduction

This chapter investigates the Pareto optimal allocations for an endowment economy populated by two investors that ignore the probabilistic law governing the growth rate of consumption and they react to the ongoing model selection problem evaluating a set of possible data generating processes. I propose a simple model to highlight how different sources of heterogeneity affect the time-varying distribution of wealth. To show it, I consider a social planner problem and I approximate the equation governing time-varying Pareto weights as a function of the realizations of exogenous variables. The social planner perspective is justified by the second welfare theorem that claims how, under proper conditions, a Pareto optimal allocation might be obtained as the result of a decentralized competitive equilibrium.

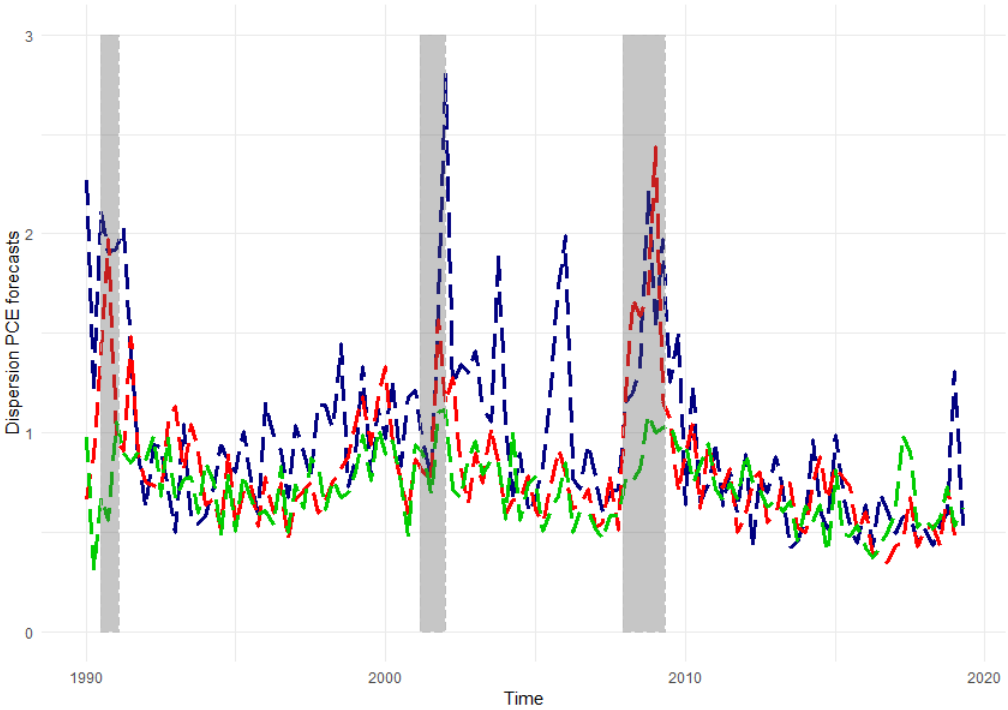
In the spirit of Hansen and Sargent (2010), I assume that investors face three sources of model uncertainty: the first one refers to the distributions of the zero-mean shocks hitting the economy, the second one concerns the time-varying expected growth rate of consumption and the third one concerns the persistence of the shocks. Investors have heterogeneous reference beliefs over these two latter sources of uncertainty and they share a common reference Gaussian belief over the distributions of the shocks. A key feature of this work is the assumption that investors distrust their reference beliefs and they exhibit heterogeneous attitudes towards each source of ambiguity. In order to introduce concerns towards the reference probabilities, I assume that investors' preferences have the multiplier representation of Hansen and Sargent (2001).[1]

Results show that the evolution of the pseudo Pareto weights is driven by the investors' disagreement over the conditional expectations of future states. However, differently from the classical Bayesian setting, disagreement is evaluated under twisted probabilities that reflect the distrust towards reference models. The magnitude of these distortions depends on the strength of attitudes towards ambiguity and on the size of each source of uncertainty. As a consequence, if an investor is highly ambiguity averse and another one is ambiguity neutral, then the share of consumption of the first agent might reduce even if she has (more) accurate beliefs. Noteworthy, since the distortion of beliefs increases in the size of uncertainty, the countercyclical patterns of volatility might explain why in bad times investors' opinions polarize and forecasts have a larger dispersion.

The countercyclical dynamic of investors' disagreement is consistent with the empir-

[1] Multiplier preferences, axiomatized by Strzalecki (2011), lie at the intersection between the Variational preferences of Maccheroni et al. (2006) and the smooth preferences of Klibanoff et al. (2005).

Figure 3.1.1: Forecast dispersion of PCE



The graph shows the dispersion of expected growth rate of personal consumption across different horizons. The blue line refers to the dispersion of the expected growth rate between times t and $t+1$, the red line between $t+2$ and $t+3$, the green line between $t+4$ and $t+5$. Shaded areas denote the NBER recession periods. Data: Professional Forecasters. Frequency: quarterly.

ical evidence collected by many works, e.g. Kandel and Pearson (1995) and Patton and Timmermann (2010). The emerging heterogeneity in investors beliefs has important implications for asset pricing, e.g. Anderson et al. (2005) or Basak (2005). Among others, Cujean and Hasler (2017) develop a dynamic general equilibrium model to investigate the asset pricing implications of spikes in disagreement and they show how, as economic conditions deteriorate, opinions' polarization leads to return predictability to concentrate in bad times.

When the economy admits a large number of periods, heterogeneous ambiguity attitudes towards the unknown persistence of the shocks become a main driver of the variation of wealth distribution. Pohl et al. (2018) highlight the importance for asset prices of heterogeneous beliefs over the persistent component under the assumption that investors have preference for an early resolution of uncertainty.

Related Literature

In this work, I relate to a large literature that studies the dynamic of the optimal consumption allocation. Among others, Blume and Easley (1992), Sandroni (2000) and Blume and Easley (2006) derive important results concerning the survivorship of investors, in complete and incomplete markets, that have heterogeneous beliefs and time additive preferences. On the other hand, Anderson (2005) analyses a stochastic economy in which agents have homogeneous beliefs but heterogeneous risk-sensitive preferences and he defines a set of necessary and sufficient conditions under which Pareto weights are time-invariant.

Recently, Colacito et al. (2019) consider an endowment economy with two goods and risk-sensitive agents and they show that, if investors have heterogeneous preferences over goods, then the sufficient conditions about the survival might not hold. Instead, Borovicka (2018) consider a recursive economy in which agents have heterogeneous but exogenous beliefs and he shows that agents with more incorrect beliefs can survive or dominate the economy in the long-run. Finally, within the literature that studies the economic implications of model uncertainty, Guerdjikova and Sciubba (2015) show that if an economy is populated by subjective expected utility and smooth ambiguity investors then, under a large degree of ambiguity prudence, smooth investors survive and affect the long-run equilibrium price.

Differently from these works, I do not focus on the asymptotic properties of Pareto weights but I am interested in the short-term contribution of different sources of model uncertainty when the investors populating the economy have non-neutral attitudes towards ambiguity.

The methodology of this paper relates to the asset pricing literature that studies the market price of model uncertainty, e.g. Hansen et al. (2006), Hansen (2007) or Hansen and Sargent (2019). In particular, I assume that investors have the same preferences and face the same sources of model uncertainty as in Hansen and Sargent (2010). However, differently from these works, I do not consider a representative agent economy but I analyse how the heterogeneous characteristics of investors affect the optimal consumption sharing.

Finally, since an important source of uncertainty faced by the investors concerns the persistence of the shocks, I relate also to the long-run risk literature, e.g. Bansal and Yaron (2004) or Bansal et al. (2016). In particular, several works, e.g. Bidder and Dew-Becker (2016) or Bansal et al. (2009), have underlined the difficulties in estimating the dynamic properties of an economy; therefore, this evidence can be used to motivate the existence of heterogeneous beliefs over the persistent component.

3.2 Economic Environment

In this section, I study the social planner problem for an endowment economy populated by two investors that ignore the probabilistic law governing the growth rate of aggregate consumption and they evaluate a set of candidate models. I denote by $\mathbb{T} = \{0, 1, 2, ...\}$ the set of time periods and by $(C_t)_{t\in\mathbb{T}}$ the sequence of random variables that, at each period, defines the aggregate consumption level. In a similar way to Hansen and Sargent (2010), I assume there are three sources of ambiguity: the first one concerns the distribution of the zero-mean shocks $\varepsilon_{t+1}, \varepsilon^x_{t+1}$ in

$$c_{t+1} - c_t = \theta_2 X_t + \sigma\varepsilon_{t+1}, \tag{3.1}$$

$$X_{t+1} = \theta_1 X_t + \sigma_x \varepsilon^x_{t+1}, \tag{3.2}$$

where $c_t = \ln C_t$ and I assume σ, σ_x are known.[2] Investors ignore the probabilistic law of the shocks and they share a common reference belief $\left(\varepsilon_{t+1}, \varepsilon^x_{t+1}\right) \sim N(0, I)$. The second source of uncertainty concerns the expected growth rate of consumption and investors have an heterogeneous subjective Gaussian belief $\mu_i : \mathcal{B}(\Theta_2) \to [0, 1]$ over the parameter $\theta_2 \in \Theta_2$ that governs the conditional mean.[3] The third source of uncertainty concerns the value of the parameter $\theta_1 \in \Theta_1$ that governs the persistence of economic shocks and, similarly to θ_2, they have over θ_1 heterogeneous Gaussian subjective beliefs $\nu_i : \mathcal{B}(\Theta_1) \to [0, 1]$.

At each period, investors observe both the realizations of X_t and C_t therefore, in every period $t \in \mathbb{T}$, the information set is given by the sigma-algebra $\mathcal{F}_t := \mathcal{F}(Z^t)$ generated by $Z^t := ((C_1, X_1), ..., (C_t, X_t))$.

In order to captures the idea that investors adopt a cautious evaluation towards the reference probabilities, I assume that preferences over future streams of consumption $(C_{i,k})^{\infty}_{k=t+1} = \{C_{i,t+1}, C_{i,t+2}, ...\}$ have the multiplier representation of Hansen and Sargent

[2] This might be justified when $\Delta t \to 0$ by the convergence in probability of the quadratic variation.

[3] Heterogeneity might arise because investors focus on different predictors X_t or because they have a different prior over θ_2. For simplicity I assume $\Theta_1, \Theta_2 \subset \mathbb{R}$ and that both investors rely on the same X_t.

(2001)

$$V_{i,t} = u_i\left(C_{i,t}\right) + \mathbf{T}^{\nu_{\mathbf{i},\mathbf{t}}}\left[\mathbf{T}^{\mu_{\mathbf{i},\mathbf{t}}}\left[\mathbf{T}^{\mathbf{K_t}(\theta,\cdot)}\left[\beta_i V_{i,t+1}\right]\right]\right], \tag{3.3}$$

where $\beta_i \in (0,1)$ determines time preference and the shapes of the operators $\mathbf{T}^{\mathbf{K_t}(\theta,\cdot)}$, $\mathbf{T}^{\mu_{\mathbf{i},\mathbf{t}}}$ and $\mathbf{T}^{\nu_{\mathbf{i},\mathbf{t}}}$ are given by[4]

$$\mathbf{T}^{\mathbf{K_t}(\theta,\cdot)}\left[\beta_i V_{i,t+1}\right] = -\eta_{i,3}\ln\left(\mathbb{E}_{K_t(\theta,\cdot)}\left[\exp\left(\frac{-\beta_i}{\eta_{i,3}}V_{i,t+1}\right)\right]\right), \tag{3.4}$$

$$\mathbf{T}^{\mu_{\mathbf{i},\mathbf{t}}}\left[\mathbf{T}^{\mathbf{K_t}(\theta,\cdot)}\left[\beta_i V_{i,t+1}\right]\right] = -\eta_{i,2}\ln\left(\mathbb{E}_{\mu_{i,t}}\left[\exp\left(\frac{-1}{\eta_{i,2}}\mathbf{T}^{\mathbf{K_t}(\theta,\cdot)}\left[\beta_i V_{i,t+1}\right]\right)\right]\right), \tag{3.5}$$

$$\mathbf{T}^{\nu_{\mathbf{i},\mathbf{t}}}\left[\mathbf{T}^{\mu_{\mathbf{i},\mathbf{t}}}\left[\mathbf{T}^{\mathbf{K_t}(\theta,\cdot)}\left[\beta_i V_{i,t+1}\right]\right]\right] = -\eta_{i,1}\ln\left(\mathbb{E}_{\nu_{i,t}}\left[\exp\left(\frac{-1}{\eta_{i,1}}\mathbf{T}^{\mu_{\mathbf{i},\mathbf{t}}}\left[\mathbf{T}^{\mathbf{K_t}(\theta,\cdot)}\left[\beta_i V_{i,t+1}\right]\right]\right)\right]\right). \tag{3.6}$$

For any $\theta = (\theta_1, \theta_2)$, I denote by $K_t(\theta, \cdot)$ the time t conditional probability over the random vector (c_{t+1}, X_{t+1}) that determines the continuation value $V_{i,t+1}$. Attitudes towards the three sources of model uncertainty are captured by the values of the preference parameters $\eta_{i,1}, \eta_{i,2}, \eta_{i,3} \in \mathbb{R}$. In this work, I make the following comfortable assumptions over preferences:

Assumption 1. *Investors have the same discount factor, i.e.* $\beta_i = \beta_j = \beta \in (0,1)$.

Assumption 2. *Investors have a logarithmic utility index, i.e.* $\forall k \in \{i,j\}$ $u_k(C) = (1-\beta_k)\ln C$.

Assumption 3. *Investors display aversion towards each source of ambiguity, i.e.* $\forall k \in \{i,j\}$, $\eta_k = (\eta_{k,1}, \eta_{k,2}, \eta_{k,3}) \in \bar{\mathbb{R}}^3_+$.

Positive finite values of η have two important implications: first, the sources of uncertainty can not compound by Bayesian averaging into a single mean-measure distribution; second, the operators twist investors' beliefs with a pessimistic distortion. In the limiting

[4] The idea to apply this operator goes back to Hansen and Sargent (1995), Hansen et al. (1999), Tallarini Jr (2000) and others, who interpret θ as an additional parameter governing the risk preferences for the timing of the resolution of uncertainty.

case in which $\eta_{i,1} = \eta_{i,2} = \eta_{i,3} = +\infty$, investor i has neutral attitudes towards ambiguity and equation (3.3) simplifies to standard time-additive preferences

$$V_{i,t} = u_i\left(C_{i,t}\right) + \mathbb{E}_{q_{i,t}}\left[\beta_i V_{i,t+1}\right]. \tag{3.7}$$

The mean-measure probability $q_{i,t}$ over (c_{t+1}, X_{t+1}) is obtained marginalizing the joint probability $\pi_{i,t} : \mathcal{B}(\Theta_1) \otimes \mathcal{B}(\Theta_2) \otimes \mathcal{B}(\mathbb{R}^2) \to [0,1]$ with respect to the distributions of θ_1, θ_2, i.e.

$$\pi_{i,t}\left(\theta_1 \times \theta_2 \times c_{t+1} \times X_{t+1}\right) = \nu_{i,t}\left(d\theta_1\right)\mu_{i,t}\left(d\theta_2\right)K_t\left(\theta, dc_{t+1} \times dX_{t+1}\right), \tag{3.8}$$

$$q_{i,t}\left(c_{t+1}, X_{t+1}\right) = \int_{\Theta_1}\int_{\Theta_2} \pi_{i,t}\left(d\theta_1 \times d\theta_2 \times c_{t+1} \times X_{t+1}\right). \tag{3.9}$$

3.3 Pareto Optimality

Let $\lambda_t = (\lambda_{i,t}, 1 - \lambda_{i,t})$ be the vector of Pareto weights and C_t the aggregate consumption at time t, the social planner wants to maximize

$$V\left(Z_t\right) = \lambda_{i,t} V_{i,t}\left(Z_t\right) + \left(1 - \lambda_{i,t}\right) V_{j,t}\left(Z_t\right) \tag{3.10}$$

$$s.t. \begin{cases} C_{i,t} + C_{j,t} = C_t \\ C_{k,1}, C_{k,2} \geq 0. \qquad \forall t \in \mathbb{T}, \forall k \in \{i,j\} \\ (3.1), (3.2) \end{cases} \tag{3.11}$$

where $Z_t = (C_t, X_t)$ and the time t consumption allocation for investor i, is a function of her Pareto weight and of the aggregate consumption level $C_{i,t} = C_i\left(C_t, \lambda_{i,t}\right)$. The state contingent dynamic of the next-period Pareto weight for investor i is given by

$$\lambda_{i,t+1}\left(Z^{t+1}\right) = \frac{\lambda_{i,t}\left(Z^t\right)\mathcal{M}_{i,t+1}\left(Z^{t+1}\right)}{\lambda_{i,t}\left(Z^t\right)\mathcal{M}_{i,t+1}\left(Z^{t+1}\right) + \lambda_{j,t}\left(Z^t\right)\mathcal{M}_{j,t+1}\left(Z^{t+1}\right)}, \tag{3.12}$$

where

$$\mathcal{M}_{i,t+1}\left(Z^{t+1}\right) = \frac{\partial V_{i,t}\left(Z^t\right)}{\partial V_{i,t+1}\left(Z^{t+1}\right)} \tag{3.13}$$

is the marginal contribution of the continuation value for a given path Z^{t+1} of state-variables realizations. The dynamic of the pseudo-Pareto weight

$$S_{t+1}\left(Z^{t+1}\right)=\frac{\lambda_{i,t+1}\left(Z^{t+1}\right)}{1-\lambda_{i,t+1}\left(Z^{t+1}\right)} \tag{3.14}$$

is given by

$$S_{t+1}\left(Z^{t+1}\right)=S_t\left(Z^t\right)\frac{\mathcal{M}_{i,t}\left(Z^{t+1}\right)}{\mathcal{M}_{j,t}\left(Z^{t+1}\right)},$$

that, conditional to the time t information set, it depends just on the realization of Z_{t+1}. The first order condition of the planner to optimally allocate present consumption requires that

$$\frac{\lambda_i}{\lambda_j}=\frac{u_j^{'}\left(C_{j,t}\right)}{u_i^{'}\left(C_{i,t}\right)}. \tag{3.15}$$

Under assumptions 1 and 2, each investor consumes a strictly positive share of the aggregate consumption iff she has a positive Pareto weight and the present consumption for investor i is given by $C_{i,t}=\lambda_{i,t}C_t$.

In the following subsections, I illustrate how heterogeneous beliefs and attitudes towards ambiguity affect the dynamic of the optimal consumption sharing. However, because of the difficult analytical tractability, I approximate the evolution of the pseudo Pareto weight.

3.3.1 One-period problem

In order to study how uncertainty over the conditional distribution of consumption growth affects the consumption sharing, I consider a one-period problem $t\in\{0,1\}$ in which the uncertainty over the persistence of the shock hitting the consumption growth does not play any role. Formally, this implies that the continuation utility $V_{i,1}\left(Z_1\right)$ does not vary with respect to X_1 and beliefs over θ_1 are ineffective. Therefore, in a one-period problem, I can think at $K\left(\theta,\cdot\right)$ as a probability measure of c_{t+1} that is independent of θ_1.

The FOC of optimal consumption allocation in $t=1$ requires that

$$\frac{\lambda_i}{\lambda_j}\frac{\beta_i}{\beta_j}\frac{\tilde{q}_i\left(c_1\right)}{\tilde{q}_j\left(c_1\right)}=\frac{u^{'}\left(C_{j,1}\right)}{u^{'}\left(C_{i,1}\right)}, \tag{3.16}$$

where $\tilde{q}_i, \tilde{q}_j$ are the distorted mean-measure probabilities of the two investors. Equation (3.16) shows that investors act as if they were subjective expected utility maximizers with distorted beliefs given by

$$\tilde{q}_i\left(dc_1\right) = \int_{\Theta_2} \tilde{\pi}_i\left(\Theta_1 \times d\theta_2 \times dc_1 \times \mathbb{R}\right), \tag{3.17}$$

$$\tilde{\pi}_i\left(\Theta_1 \times d\theta_2 \times dc_1 \times \mathbb{R}\right) = \tilde{\mu}_i\left(d\theta_2\right) \tilde{K}_i\left(\theta, dc_1\right), \tag{3.18}$$

where

$$\frac{d\tilde{K}_i\left(\theta, c_1\right)}{dK\left(\theta, c_1\right)} = \frac{\exp\left(\frac{-\beta u(C_{i,1})}{\eta_{i,3}}\right)}{\mathbb{E}_{K(\theta,\cdot)}\left[\exp\left(\frac{-\beta u(C_{i,1})}{\eta_{i,3}}\right)\right]}, \tag{3.19}$$

$$\frac{d\tilde{\mu}_i\left(\theta_2\right)}{d\mu_i\left(\theta_2\right)} = \frac{\left(\mathbb{E}_{K(\theta,\cdot)}\left[\exp\left(\frac{-\beta}{\eta_{i,3}} u\left(C_{i,1}\right)\right)\right]\right)^{\frac{\eta_{i,3}}{\eta_{i,2}}}}{\mathbb{E}_{\mu_i}\left[\left(\mathbb{E}_{K(\theta,\cdot)}\left[\exp\left(\frac{-\beta}{\eta_{i,3}} u\left(C_{i,1}\right)\right)\right]\right)^{\frac{\eta_{i,3}}{\eta_{i,2}}}\right]}, \tag{3.20}$$

are the Radon Nikodym derivatives that map the reference beliefs over the distributions of the zero-mean shocks and of the expected growth rate of consumption to beliefs that embed the distrust towards the reference probabilities.[5] Since $K\left(\theta, c_1\right)$ and $\tilde{K}\left(\theta, c_1\right)$ are equivalent measures with full support on $\mathbb{R}$, investor i consumes a positive share of next-period aggregate consumption iff she has a positive Pareto weight.

As $\eta_{i,3} \downarrow 0$, the investor i becomes infinitely ambiguity averse towards the uncertain distribution of the zero-mean shock ε_1 and Laplace's principle implies that her share of next-period consumption converges to zero $\lim_{\eta_{i,3}\downarrow 0} C_{i,1} = 0$. On the other hand, for arbitrary values of $\eta_{i,2}, \eta_{j,3} \in \mathbb{R}_+$, the distortion of beliefs is a function of the optimal allocation through the consumption sharing rules of the investors. In general, the functional $c_1 \mapsto C_{i,1}\left(c_1\right)$ is not linear, therefore it is challenging to derive a close-form solution also for a one-period problem.

To overcome this problem, I follow the approach of Colacito et al. (2019) that consists of log-linearizing the solution around the current-period pseudo Pareto weight $s_0 =$

[5] The distorted beliefs are mathematically equivalent to the probabilities that minimize the variational preferences with penalty function given by the Kullback–Leibler divergence from the reference beliefs.

$\log(S_0)$, where $S_t = \frac{\lambda_{i,t}}{\lambda_{j,t}}$. Furthermore, I assume that the variance of uncertainty over θ_2 faced by each investor is the same.

Lemma 3.3.1. *If assumptions 1,2 hold and* $\forall k \in \{i,j\}$ $Var_{\mu_k}(\theta_2 X_0) = \sigma_\mu^2$, *then*

$$u(C_{i,1}) \approx \kappa_i(s_0) + d_i(s_0)\, c_1 \tag{3.21}$$

$$u(C_{j,1}) \approx \kappa_j(s_0) + d_j(s_0)\, c_1 \tag{3.22}$$

with

$$d_i(s_0) = \frac{1}{1+\exp(s_0)} \frac{d_c(s_0)}{\sigma^2+\sigma_\mu^2} + 1, \quad d_j(s_0) = \frac{-\exp(s_0)}{1+\exp(s_0)} \frac{d_c(s_0)}{\sigma^2+\sigma_\mu^2} + 1,$$

$$d_c(s_0) = \frac{\mathbb{E}_{q_i}[c_1] - \mathbb{E}_{q_j}[c_1] - \beta\left(\frac{1}{\eta_{i,3}} - \frac{1}{\eta_{j,3}}\right)\sigma^2 - \beta\left(\frac{1}{\eta_{i,2}} - \frac{1}{\eta_{j,2}}\right)\sigma_\mu^2}{1 + \frac{\beta}{\sigma^2+\sigma_\mu^2}\left[\frac{1}{1+\exp(s_0)}\left(\frac{\sigma^2}{\eta_{i,3}} + \frac{\sigma_\mu^2}{\eta_{i,2}}\right) + \frac{\exp(s_0)}{1+\exp(s_0)}\left(\frac{\sigma^2}{\eta_{j,3}} + \frac{\sigma_\mu^2}{\eta_{j,2}}\right)\right]}. \tag{3.23}$$

Proof. See Appendix.

Although the first-order approximation of investors' future utility, proposed in lemma 3.3.1, fails to capture crucial features of the dynamics of the Pareto weights, it is useful to provide meaningful analytical insights.[6] In particular, since the next-period utility is Gaussian distributed, the distorted beliefs of investor i are given by

$$\tilde{K}_i(\theta, c_1) \sim N\left(c_0 + \theta_2 X_0 - \frac{\beta d_i(s_0)\sigma^2}{\eta_{i,3}}, \sigma^2\right), \tag{3.24}$$

$$\tilde{\mu}_i(\theta_2 X_0) \sim N\left(E_{\mu_i}[\theta_2 X_0] - \frac{\beta d_i(s_0)\sigma_\mu^2}{\eta_{i,2}}, \sigma_\mu^2\right), \tag{3.25}$$

$$\tilde{q}_i(\Theta_1 \times c_1) \sim N\left(\underbrace{c_0 + \mathbb{E}_{q_i}[\theta_2 X_0]}_{=\mathbb{E}_{q_i}[c_1]} - \frac{\beta d_i(s_0)\sigma_\mu^2}{\eta_{i,2}} - \frac{\beta d_i(s_0)\sigma^2}{\eta_{i,3}}, \sigma^2 + \sigma_\mu^2\right). \tag{3.26}$$

Equation (3.26) shows that the distorted belief $\tilde{q}_i$, that appears in (3.16), is a function both of the reference probabilities and of the strength of ambiguity attitudes. The latter might mitigate or amplify the effects related to heterogeneous beliefs, e.g. optimism or pessimism concerning consumption growth. Remarkably, the second moment of the

[6] For example, Colacito et al. (2019) show how a first-order approximation might fail to capture the mean-reversion in the evolution of the Pareto weights.

reference probability also affects the distorted conditional expectation. Therefore, in periods with high uncertainty, an optimist but ambiguity averse investor might behave as a Bayesian one that is relatively more pessimist about the expected growth rate of consumption.

Given the heterogeneous beliefs and preferences of the investors, the dynamic of the pseudo Pareto weight is

$$s_1 - s_0 = -\frac{\mathbb{E}^2_{\tilde{q}_i}\left[c_1\right] - \mathbb{E}^2_{\tilde{q}_j}\left[c_1\right]}{2\left(\sigma^2 + \sigma^2_\mu\right)} + \frac{1}{\sigma^2 + \sigma^2_\mu}\underbrace{\left(\mathbb{E}_{\tilde{q}_i}\left[c_1\right] - \mathbb{E}_{\tilde{q}_j}\left[c_1\right]\right)}_{=:d_c(s_0)} c_1. \tag{3.27}$$

A good feature of the first order approximation is that the evolution of the pseudo Pareto weight in (3.27) becomes linear in c_1 with a coefficient that is the product of two factors. The first one d_c, whose value is reported in (3.23), corresponds to the disagreement over the expected growth rate of consumption evaluated under the investors' distorted beliefs. The second one is the inverse of the variance of the mean-measure (or predictive) distribution of next-period consumption growth rate. The product of these two effects

$$\frac{\mathbb{E}_{\tilde{q}_i}\left[c_1\right] - \mathbb{E}_{\tilde{q}_j}\left[c_1\right]}{\sigma^2 + \sigma^2_\mu} = \frac{\frac{\mathbb{E}_{q_i}[c_1] - \mathbb{E}_{q_j}[c_1]}{\sigma^2} - \beta\left(\frac{1}{\eta_{i,3}} - \frac{1}{\eta_{j,3}}\right) - \beta\left(\frac{1}{\eta_{i,2}} - \frac{1}{\eta_{j,2}}\right)\frac{\sigma^2_\mu}{\sigma^2}}{1 + \sigma^2_\mu + \beta\left[\frac{1}{1+\exp(s_0)}\left(\frac{1}{\eta_{i,3}} + \frac{\exp(s_0)}{\eta_{j,3}}\right) + \frac{1}{1+\exp(s_0)}\frac{\sigma^2_\mu}{\sigma^2}\left(\frac{1}{\eta_{i,2}} + \frac{\exp(s_0)}{\eta_{j,2}}\right)\right]}, \tag{3.28}$$

shows that, if investors have different attitudes towards ambiguity, then a large size of uncertainty has a twofold effect: on one side, it might strengthen the beliefs' distortion increasing investors disagreement; on the other side, it mitigates the relevance of heterogeneous reference beliefs.[7] In the limiting case in which both investors are Bayesian, i.e. $\eta_{k,3} = \eta_{k,2} = +\infty$ with $k \in \{i, j\}$, disagreement depends only on the difference between investors' reference probabilities; therefore, periods characterized by high levels of uncertainty, e.g. financial crisis or economic recessions, would mitigate the relevance of heterogeneous beliefs. Differently from the classical Bayesian setting, if investors have the same reference models, then the evolution of the pseudo Pareto-weight might not be

[7]When the predictive distribution of the next-period aggregate consumption has a high variance, the investors have spread beliefs over future consumption realizations.

constant but it depends on the relative strength of ambiguity attitudes.

3.3.2 Multi-period problem

I now consider a multi-period economy $t \in \{0, 1, ..., n+1\}$ and I analyse the contribution to the optimal consumption sharing of heterogeneous beliefs and attitudes towards the ambiguous persistence of the shocks.

In this setting, the first order condition for the next-period consumption allocation becomes

$$\frac{\lambda_i}{\lambda_j}\frac{\beta_i}{\beta_j}\frac{\tilde{q}_i(c_1)}{\tilde{q}_j(c_1)} = \frac{u'(C_{j,1})}{u'(C_{i,1})}, \tag{3.29}$$

where the distorted mean-measure probabilities depend also on the evolution of X_{t+1}

$$\tilde{p}_i(c_1) = \int_{\Theta_1 \times \Theta_2 \times \mathbb{R}} \tilde{\pi}_i(d\theta_1 \times d\theta_2 \times c_1 \times dX_1), \tag{3.30}$$

$$\tilde{\pi}_i(d\theta_1 \times d\theta_2 \times c_1 \times dX_1) = \tilde{\nu}_i(d\theta_1)\tilde{\mu}_i(d\theta_2)\tilde{K}_i(\theta, c_1 \times dX_1), \tag{3.31}$$

$$\frac{d\tilde{\nu}_i(\theta_1)}{d\nu_i(\theta_1)} = \frac{\left(\mathbb{E}_{\mu_i}\left[\left(\mathbb{E}_{K(\theta,\cdot)}\left[\exp\left(\frac{-\beta}{\eta_{i,3}}u(V_{i,1})\right)\right]\right)^{\frac{\eta_{i,3}}{\eta_{i,2}}}\right]\right)^{\frac{\eta_{i,2}}{\eta_{i,1}}}}{\mathbb{E}_{\nu_i}\left[\left(\mathbb{E}_{\mu_i}\left[\left(\mathbb{E}_{K(\theta,\cdot)}\left[\exp\left(\frac{-\beta}{\eta_{i,3}}u(V_{i,1})\right)\right]\right)^{\frac{\eta_{i,3}}{\eta_{i,2}}}\right]\right)^{\frac{\eta_{i,2}}{\eta_{i,1}}}\right]}. \tag{3.32}$$

I require the simplifying assumption that model uncertainty vanishes after the first period so that, under assumptions 1,2 and stationarity of the process governing X_t, the continuation value of the investor i in period one is given by

$$V_{i,1} = \left(1-\beta^{n+1}\right)(\ln\lambda_{i,1} + c_1) + \underbrace{\frac{1-\beta}{1-\theta_1}\left[\frac{\beta(1-\beta^n)}{1-\beta} - \frac{\beta\theta_1(1-(\beta\theta_1)^n)}{1-\beta\theta_1}\right]}_{a_n:=}\theta_2 X_1. \tag{3.33}$$

The last term on the RHS of (3.33) embeds in the continuation utility the persistence of shocks whose relevance clearly depends on the number of periods in the economy. I consider the two limiting cases in which $n = 1$ or $n = +\infty$.[8]

[8] Here n denotes the number of missing periods in the economy. Therefore, $n = 1$ implies that the economy has two periods.

Similarly to the one-period problem, I assume investors face the same size of uncertainty towards each source of ambiguity and I log-linearise the continuation values $V_{i,1}, V_{j,1}$ around $s_0 = \ln(S_0)$.

Lemma 3.3.2. *Let assumptions 1,2 hold and* $\forall k \in \{i,j\}$ $Var_{\mu_k}(\theta_2 X_0) = \sigma_\mu^2$, $Var_{\nu_k}(\theta_1 X_0) = \sigma_\nu^2$. *If*

$$V_{i,1} \approx \kappa_i(s_0) + d_{i,1}(s_0)\,c_1 + d_{i,2}(s_0)\,X_1,$$

$$V_{j,1} \approx \kappa_j(s_0) + d_{j,1}(s_0)\,c_1 + d_{j,2}(s_0)\,X_1,$$

then $d_{i,1}, d_{j,1}, d_c$ *are the same as in lemma 3.3.1 and*

$$d_{i,2}(s_0) = \frac{1}{1+\exp(s_0)}\frac{d_x(s_0)}{\sigma_x^2+\sigma_\nu^2} + a_n\theta_2, \quad d_{j,2}(s_0) = \frac{-\exp(s_0)}{1+\exp(s_0)}\frac{d_x(s_0)}{\sigma_x^2+\sigma_\nu^2} + a_n\theta_2, \tag{3.34}$$

$$d_x(s_0) = \frac{E_{v_i}[X_1] - E_{v_j}[X_1] + \left[\frac{-\beta\sigma_x^2}{\eta_{i,3}} + \frac{\beta\sigma_x^2}{\eta_{j,3}} + \frac{-\beta\sigma_\nu^2}{\eta_{i,1}} + \frac{\beta\sigma_\nu^2}{\eta_{j,1}}\right] a_n\theta_2}{\left(1 + \frac{\beta}{\sigma_x^2+\sigma_\nu^2}\left[\frac{1}{1+\exp(s_0)}\left(\frac{\sigma_x^2}{\eta_{i,3}} + \frac{\sigma_\nu^2}{\eta_{i,1}}\right) + \frac{\exp(s_0)}{1+\exp(s_0)}\left(\frac{\sigma_x^2}{\eta_{j,3}} + \frac{\sigma_\nu^2}{\eta_{j,1}}\right)\right]\right)}. \tag{3.35}$$

Proof. See Appendix.

Lemma 3.3.2 shows that, under the first order approximation, the continuation value of each investor has a linear dependence from the states c_1 and X_1. Because of the Gaussian distribution of the sources of uncertainty, the distorted beliefs over the persistence of the shocks are given by

$$\tilde{\nu}_i(\theta_1 X_0) \sim N\left(E_{\nu_i}[\theta_1 X_0] - \frac{\beta d_{i,2}(s_0)\,\sigma_x^2}{\eta_{i,1}}, \sigma_\nu^2\right). \tag{3.36}$$

The dynamic of the pseudo Pareto weight becomes

$$\begin{aligned} s_1 - s_0 = &-\frac{\mathbb{E}^2_{\tilde{\mu}_i}[c_1] - \mathbb{E}^2_{\tilde{\mu}_j}[c_1]}{2\left(\sigma^2+\sigma_\mu^2\right)} + \frac{1}{\sigma^2+\sigma_\mu^2}\left(\underbrace{\mathbb{E}_{\tilde{\mu}_i}[c_1] - \mathbb{E}_{\tilde{\mu}_j}[c_1]}_{d_c:=}\right)c_1 \\ &-\frac{\mathbb{E}^2_{\tilde{\nu}_i}[X_1] - \mathbb{E}^2_{\tilde{\nu}_j}[X_1]}{2\left(\sigma_x^2+\sigma_\nu^2\right)} + \frac{1}{\sigma_x^2+\sigma_\nu^2}\left(\underbrace{\mathbb{E}_{\tilde{\nu}_i}[X_1] - \mathbb{E}_{\tilde{\nu}_j}[X_1]}_{d_x:=}\right)X_1. \end{aligned} \tag{3.37}$$

Equation (3.37) is a natural extension of (3.3.2) and, indeed, the first two terms on the RHS coincide with those in (3.27). Similar to the next-period consumption, the coefficient of X_1 depends on the investors' disagreement over the persistence of the shocks, evaluated under distorted beliefs, $d_x := \mathbb{E}_{\tilde{\nu}_i}[X_1] - \mathbb{E}_{\tilde{\nu}_j}[X_1]$ and on the variance implied by the mean-measure distribution of X_1.

The relevance of uncertainty over the persistence of the shock depends trivially on the number of periods of the economy. In case there are only two periods, i.e. $n = 1$, the value of a_n is given by $a_n = \beta(1-\beta)$. On the other hand, when there are infinite many periods, i.e. $n = +\infty$, the value of a_n becomes $a_n = \frac{\beta}{1-\beta\theta_1}$. If investors believe that X_t is a very persistent variable, then the value of a_n becomes very large and this has important implications for the evolution of the pseudo Pareto weight.

This simple result suggests that investors' disagreement over thr persistence of economic shocks has a prominent role in infinite horizon economies. Therefore, in the spirit of Bansal and Yaron (2004), it is possible to consider the long-run ambiguity as the main driver in the variation of the pseudo Pareto weight.

3.4 Conclusion

In this work, I consider an endowment economy populated by two investors and I study the problem of a social planner that has to optimally allocate them the aggregate consumption. Investors have heterogeneous beliefs over three sources of model uncertainty concerning the expected growth rate of consumption, the distribution and the persistence of the shocks hitting the economy. Moreover, they distrust their reference probabilities exhibiting heterogeneous attitudes towards ambiguity.

I provide a stylized model that explains how different sources of heterogeneity affect the dynamic of the wealth distribution. Results show that, similar to the classical Bayesian setting, the evolution of the Pareto weights depends on investors' disagreement, defined as the difference between the conditional expectations of next-period exogenous state variables. However, since investors have non-neutral attitudes towards ambiguity, disagreement is evaluated under distorted beliefs that embed in the valuation the pessimism of the investors towards the reference models. The relevance of such distortion increases with the strength of ambiguity attitudes and with the size of uncertainty. Therefore, during bad periods, spikes in volatility can induce an opinion polarization. Finally, for an infinite horizon economy, disagreement over the persistence of the economic shock has a prominent role in the evolution of wealth distribution.

Appendix

3.A Proofs

Proof. of Lemma 3.3.1.

Let $s_t = \ln(S_t)$, I want to develop a first order approximation around s_1, that is the ratio of the initial Pareto weights. Similar to Colacito et al. (2019), I log-linearize the period 1 allocation

$$u(C_{i,1}) = \ln\left(\frac{\exp(s_0)}{1+\exp(s_0)}\right) + \frac{1}{1+\exp(s_0)}(s_1 - s_0) + c_1,$$
$$u(C_{j,1}) = \ln\left(\frac{1}{1+\exp(s_0)}\right) - \frac{\exp(s_0)}{1+\exp(s_0)}(s_1 - s_0) + c_1,$$

and I guess

$$u(C_{i,1}) \approx \kappa_i(s_0) + d_i(s_0)c_1,$$

$$u(C_{j,1}) \approx \kappa_j(s_0) + d_j(s_0)c_1.$$

Under assumptions 1 and 2, the next-period pseudo Pareto weight is given by

$$S_1 = S_0 \frac{\mathbb{E}_{\mu_i}\left[K(\theta, dc_1)\left(\mathbb{E}_{K(\theta,\cdot)}\left[\exp\left(\frac{-\beta_i}{\eta_{i,3}}u(C_{i,1})\right)\right]\right)^{\frac{\eta_{i,3}}{\eta_{i,2}}-1}\right]}{\mathbb{E}_{\mu_j}\left[K(\theta, dc_1)\left(\mathbb{E}_{K(\theta,\cdot)}\left[\exp\left(\frac{-\beta_j}{\eta_{j,3}}u(C_{j,1})\right)\right]\right)^{\frac{\eta_{j,3}}{\eta_{j,2}}-1}\right]}$$

$$\frac{\exp\left(\frac{-\beta_i}{\theta_{i,2}}u(C_{i,1})\right)}{\exp\left(\frac{-\beta_j}{\theta_{j,2}}u(C_{j,1})\right)} \frac{\mathbb{E}_{\mu_j}\left[\left(\mathbb{E}_{K(\theta,\cdot)}\left[\exp\left(\frac{-\beta}{\eta_{j,3}}u(C_{j,1})\right)\right]\right)^{\frac{\eta_{j,3}}{\eta_{j,2}}}\right]}{\mathbb{E}_{\mu_i}\left[\left(\mathbb{E}_{K(\theta,\cdot)}\left[\exp\left(\frac{-\beta}{\eta_{i,3}}u(C_{i,1})\right)\right]\right)^{\frac{\eta_{i,3}}{\eta_{i,2}}}\right]}.$$

Given the first-order approximation of the utility function,

$$\frac{K(\theta, dc_1)\exp\left(\frac{-\beta_i}{\eta_{i,3}}u(C_{i,1})\right)}{\mathbb{E}_{K(\theta,\cdot)}\left[\exp\left(\frac{-\beta_i}{\eta_{i,3}}u(C_{i,1})\right)\right]} = \phi\left(\frac{c_1 - \mathbb{E}_{K(\theta,\cdot)}\left[c_1 - \frac{\beta d_i(s_0)\sigma^2}{\eta_{i,3}}\right]}{\sigma}\right),$$

and

$$\frac{\mu_i(\theta_2)\left(\mathbb{E}_{K(\theta,\cdot)}\left[\exp\left(\frac{-\beta_i}{\eta_{i,3}}u(C_{i,1})\right)\right]\right)^{\frac{\eta_{i,3}}{\eta_{i,2}}}}{\mathbb{E}_{\mu_i}\left[\left(\mathbb{E}_{K(\theta,\cdot)}\left[\exp\left(\frac{-\beta}{\eta_{i,3}}u(C_{i,1})\right)\right]\right)^{\frac{\eta_{i,3}}{\eta_{i,2}}}\right]}=\phi\left(\frac{\mathbb{E}_{K(\theta,\cdot)}[c_1]-\mathbb{E}_{\mu_i}\left[c_1-\frac{\beta d_i(s_0)\sigma_\mu^2}{\eta_{i,2}}\right]}{\sigma_\mu}\right).$$

The next-period Pareto weight is given by

$$S_1=S_0\frac{\phi\left(\frac{c_1-\mathbb{E}_{\tilde{q}_i}[c_1]}{\sqrt{\sigma_\mu^2+\sigma^2}}\right)}{\phi\left(\frac{c_1-\mathbb{E}_{\tilde{q}_j}[c_1]}{\sqrt{\sigma_\mu^2+\sigma^2}}\right)},$$

with the distorted mean-measure probability given by

$$\tilde{q}_i(c_1)\sim N\left(\mathbb{E}_{q_i}\left[c_1-\frac{\beta d_i(s_0)\sigma_\mu^2}{\eta_{i,2}}-\frac{\beta d_i(s_0)\sigma^2}{\eta_{i,3}}\right],\sigma_\mu^2+\sigma^2\right).$$

Taking the logarithm, it follows that

$$s_1-s_0=-\frac{\left(\mathbb{E}_{\tilde{q}_i}[c_1]\right)^2-\left(\mathbb{E}_{\tilde{q}_j}[c_1]\right)^2}{2\left(\sigma^2+\sigma_\mu^2\right)}+\frac{1}{\sigma^2+\sigma_\mu^2}\underbrace{\left(\mathbb{E}_{\tilde{q}_i}[c_1]-\mathbb{E}_{\tilde{q}_j}[c_1]\right)}_{=:d_c(s_0)}c_1.$$

The undetermined coefficients are given by

$$d_i(s_0)=\frac{1}{1+\exp(s_0)}\frac{1}{\sigma^2+\sigma_\mu^2}d_c(s_0)+1,$$

$$d_j(s_0)=\frac{-\exp(s_0)}{1+\exp(s_0)}\frac{1}{\sigma^2+\sigma_\mu^2}d_c(s_0)+1,$$

$$d_c(s_0)=\frac{E_{q_i}[c_1]-E_{q_j}[c_1]-\beta\left(\frac{1}{\eta_{i,3}}-\frac{1}{\eta_{j,3}}\right)\sigma^2-\beta\left(\frac{1}{\eta_{i,2}}-\frac{1}{\eta_{j,2}}\right)\sigma_\mu^2}{1+\frac{\beta}{\sigma^2+\sigma_\mu^2}\left[\frac{1}{1+\exp(s_0)}\left(\frac{\sigma^2}{\eta_{i,3}}+\frac{\sigma_\mu^2}{\eta_{i,2}}\right)+\frac{\exp(s_0)}{1+\exp(s_0)}\left(\frac{\sigma^2}{\eta_{j,3}}+\frac{\sigma_\mu^2}{\eta_{j,2}}\right)\right]}.$$

□

Proof. of Lemma 3.3.2. Assume investors live for $n+1$ periods and that uncertainty vanishes after the first period. Then, at time 0, the continuation value for investor i is given by

$$V_{i,1}=(1-\beta)\ln C_{i,1}+\beta(1-\beta)\ln C_{i,2}+...+\beta^{n-1}(1-\beta)\ln C_{i,n}$$

$$=(1-\beta)\ln\lambda_{i,1}C_1+\beta(1-\beta)\ln\lambda_{i,1}C_2+...+\beta^{n-1}(1-\beta)\ln\lambda_{i,1}C_n$$

$$= \left(1-\beta^{n+1}\right)\ln\lambda_{i,1} + (1-\beta)\sum_{k=1}^{n}\beta^{k-1}c_k$$

and, given the processes (3.1)-(3.2), $c_s = c_1 + \theta_2 X_1 \sum_{k=0}^{s-2}\theta_1^k$. Therefore,

$$V_{i,1} = \left(1-\beta^{n+1}\right)(\ln\lambda_{i,1}+c_1) + (1-\beta)\sum_{k=2}^{n}\beta^{k-1}\theta_2 X_1 \sum_{m=0}^{k-2}\theta_1^m$$

$$= \left(1-\beta^{n+1}\right)(\ln\lambda_{i,1}+c_1) + (1-\beta)\,\theta_2 X_1 \sum_{k=2}^{n}\beta^{k-1}\frac{1-\theta_1^{k-1}}{1-\theta_1}$$

$$= \left(1-\beta^{n+1}\right)(\ln\lambda_{i,1}+c_1) + (1-\beta)\,\frac{\theta_2 X_1}{1-\theta_1}\left[\sum_{k=2}^{n}\beta^{k-1} - \sum_{k=2}^{n}(\beta\theta_1)^{k-1}\right]$$

$$= \left(1-\beta^{n+1}\right)(\ln\lambda_{i,1}+c_1) + \frac{1-\beta}{1-\theta_1}\left[\frac{\beta(1-\beta^n)}{1-\beta} - \frac{\beta\theta_1\left(1-(\beta\theta_1)^n\right)}{1-\beta\theta_1}\right]\theta_2 X_1,$$

and I define

$$a_n = \frac{1-\beta}{1-\theta_1}\left[\frac{\beta(1-\beta^n)}{1-\beta} - \frac{\beta\theta_1\left(1-(\beta\theta_1)^n\right)}{1-\beta\theta_1}\right].$$

Now, I log-linearise the continuation value $V_{i,1}$ around $s_0 = \ln(S_0)$

$$V_{i,1} \approx \left(1-\beta^{n+1}\right)\left[\ln\left(\frac{\exp(s_0)}{1+\exp(s_0)}\right) + \frac{1}{1+\exp(s_0)}(s_1-s_0) + c_1\right] + a_n\theta_2 X_1,$$

$$V_{j,1} \approx \left(1-\beta^{n+1}\right)\left[\ln\left(\frac{1}{1+\exp(s_0)}\right) - \frac{\exp(s_0)}{1+\exp(s_0)}(s_1-s_0) + c_1\right] + a_n\theta_2 X_1.$$

and I guess

$$V_{i,1} \approx k_i + d_{i,1}c_1 + d_{i,2}X_1,$$

$$V_{j,1} \approx k_j + d_{j,1}c_1 + d_{j,2}X_1.$$

The next-period pseudo Pareto weight is given by

$$S_1 = S_0\frac{\mathbb{E}_{\nu_i}\left[\dfrac{\left(\mathbb{E}_{\mu_i}\left[\left(\mathbb{E}_{K(\theta,\cdot)}\left[\exp\left(\frac{-\beta}{\eta_{i,3}}V_{i,1}\right)\right]\right)^{\frac{\eta_{i,3}}{\eta_{i,2}}}\right]\right)^{\frac{\eta_{i,2}}{\eta_{i,1}}}}{\mathbb{E}_{\nu_i}\left[\left(\mathbb{E}_{\mu_i}\left[\left(\mathbb{E}_{K(\theta,\cdot)}\left[\exp\left(\frac{-\beta}{\eta_{i,3}}V_{i,1}\right)\right]\right)^{\frac{\eta_{i,3}}{\eta_{i,2}}}\right]\right)^{\frac{\eta_{i,2}}{\eta_{i,1}}}\right]}\right]}{\mathbb{E}_{\nu_j}\left[\dfrac{\left(\mathbb{E}_{\mu_j}\left[\left(\mathbb{E}_{K(\theta,\cdot)}\left[\exp\left(\frac{-\beta}{\eta_{j,3}}V_{j,1}\right)\right]\right)^{\frac{\eta_{j,3}}{\eta_{j,2}}}\right]\right)^{\frac{\eta_{j,2}}{\eta_{j,1}}}}{\mathbb{E}_{\nu_j}\left[\left(\mathbb{E}_{\mu_j}\left[\left(\mathbb{E}_{K(\theta,\cdot)}\left[\exp\left(\frac{-\beta}{\eta_{j,3}}V_{j,1}\right)\right]\right)^{\frac{\eta_{j,3}}{\eta_{j,2}}}\right]\right)^{\frac{\eta_{j,2}}{\eta_{j,1}}}\right]}\right]}$$

$$\frac{\mathbb{E}_{\mu_i}\left[\frac{K(\theta,d(c_1,X_1))\exp\left(\frac{-\beta_i}{\eta_{i,3}}V_{i,1}\right)}{\mathbb{E}_{K(\theta,\cdot)}\left[\exp\left(\frac{-\beta_i}{\eta_{i,3}}V_{i,1}\right)\right]}\frac{\left(\mathbb{E}_{K(\theta,\cdot)}\left[\exp\left(\frac{-\beta_i}{\eta_{i,3}}V_{i,1}\right)\right]\right)^{\frac{\eta_{i,3}}{\eta_{i,2}}}}{\mathbb{E}_{\mu_i}\left[\left(\mathbb{E}_{K(\theta,\cdot)}\left[\exp\left(\frac{-\beta}{\eta_{i,3}}V_{i,1}\right)\right]\right)^{\frac{\eta_{i,3}}{\eta_{i,2}}}\right]}\right]}{\mathbb{E}_{\mu_j}\left[\frac{K(\theta,d(c_1,X_1))\exp\left(\frac{-\beta_j}{\eta_{j,3}}V_{j,1}\right)}{\mathbb{E}_{K(\theta,\cdot)}\left[\exp\left(\frac{-\beta_j}{\eta_{j,3}}V_{j,1}\right)\right]}\frac{\left(\mathbb{E}_{K(\theta,\cdot)}\left[\exp\left(\frac{-\beta_j}{\eta_{j,3}}V_{j,1}\right)\right]\right)^{\frac{\eta_{j,3}}{\eta_{j,2}}}}{\mathbb{E}_{\mu_j}\left[\left(\mathbb{E}_{K(\theta,\cdot)}\left[\exp\left(\frac{-\beta}{\eta_{j,3}}V_{j,1}\right)\right]\right)^{\frac{\eta_{j,3}}{\eta_{j,2}}}\right]}\right]}.$$

Given the above utility approximations,

$$\frac{K\left(\theta,dc_1\times dX_1\right)\exp\left(\frac{-\beta_i}{\eta_{i,3}}V_{i,1}\right)}{\mathbb{E}_{K(\theta,\cdot)}\left[\exp\left(\frac{-\beta_i}{\eta_{i,3}}V_{i,1}\right)\right]}\sim\phi\left(\frac{c_1-\mathbb{E}_{K(\theta,)}\left[c_1-\frac{\beta d_{i,1}\sigma^2}{\eta_{i,3}}\right]}{\sigma}\right)$$

$$\phi\left(\frac{X_1-\mathbb{E}_{K(\theta,)}\left[X_1-\frac{\beta d_{i,2}\sigma_x^2}{\eta_{i,3}}\right]}{\sigma_x}\right),$$

and therefore

$$S_1=S_0\frac{\phi\left(\frac{c_1-\mathbb{E}_{\mu_i}\left[c_1-\frac{\beta d_{i,1}\sigma^2}{\eta_{i,3}}-\frac{\beta d_{i,1}\sigma_\mu^2}{\eta_{i,2}}\right]}{\sqrt{\sigma^2+\sigma_\mu^2}}\right)\phi\left(\frac{X_1-\mathbb{E}_{\nu_i}\left[X_1-\frac{\beta d_{i,2}\sigma_x^2}{\eta_{i,3}}-\frac{\beta d_{i,2}\sigma_\nu^2}{\eta_{i,1}}\right]}{\sqrt{\sigma_x^2+\sigma_\nu^2}}\right)}{\phi\left(\frac{c_1-\mathbb{E}_{\mu_j}\left[c_1-\frac{\beta d_{j,1}\sigma^2}{\eta_{j,3}}-\frac{\beta d_{j,1}\sigma_\mu^2}{\eta_{j,2}}\right]}{\sqrt{\sigma^2+\sigma_\mu^2}}\right)\phi\left(\frac{X_1-\mathbb{E}_{\nu_j}\left[X_1-\frac{\beta d_{j,2}\sigma_x^2}{\eta_{j,3}}-\frac{\beta d_{j,2}\sigma_\nu^2}{\eta_{j,1}}\right]}{\sqrt{\sigma_x^2+\sigma_\nu^2}}\right)}$$

$$=S_0\frac{\phi\left(\frac{c_1-\mathbb{E}_{\tilde{\mu}_i}[c_1]}{\sqrt{\sigma^2+\sigma_\mu^2}}\right)\phi\left(\frac{X_1-\mathbb{E}_{\tilde{\nu}_i}[X_1]}{\sqrt{\sigma_x^2+\sigma_\nu^2}}\right)}{\phi\left(\frac{c_1-\mathbb{E}_{\tilde{\mu}_j}[c_1]}{\sqrt{\sigma^2+\sigma_\mu^2}}\right)\phi\left(\frac{X_1-\mathbb{E}_{\tilde{\nu}_j}[X_1]}{\sqrt{\sigma_x^2+\sigma_\nu^2}}\right)}.$$

Taking the logarithm,

$$s_1=s_0-\frac{\mathbb{E}_{\tilde{\mu}_i}^2\left[c_1\right]-\mathbb{E}_{\tilde{\mu}_j}^2\left[c_1\right]}{2\left(\sigma^2+\sigma_\mu^2\right)}+\frac{\mathbb{E}_{\tilde{\mu}_i}\left[c_1\right]-\mathbb{E}_{\tilde{\mu}_j}\left[c_1\right]}{\sigma^2+\sigma_\mu^2}c_1$$

$$-\frac{\mathbb{E}_{\tilde{\nu}_i}^2\left[X_1\right]-\mathbb{E}_{\tilde{\nu}_j}^2\left[X_1\right]}{2\left(\sigma_x^2+\sigma_\nu^2\right)}+\frac{\mathbb{E}_{\tilde{\nu}_i}\left[X_1\right]-\mathbb{E}_{\tilde{\nu}_j}\left[X_1\right]}{\sigma_x^2+\sigma_\nu^2}X_1.$$

where

$$d_c:=\mathbb{E}_{\tilde{\mu}_i}\left[c_1\right]-\mathbb{E}_{\tilde{\mu}_j}\left[c_1\right]$$

$$d_x:=\mathbb{E}_{\tilde{\nu}_i}\left[X_1\right]-\mathbb{E}_{\tilde{\nu}_j}\left[X_1\right]$$

are the two sources of disagreement. The undetermined coefficients of consumption growth are given by

$$d_{i,1}\left(s_0\right) = \frac{1}{1+\exp\left(s_0\right)}\frac{d_c\left(s_0\right)}{\sigma^2+\sigma_\mu^2}+1, \quad d_{j,1}\left(s_0\right) = \frac{-\exp\left(s_0\right)}{1+\exp\left(s_0\right)}\frac{d_c\left(s_0\right)}{\sigma^2+\sigma_\mu^2}+1,$$

and d_c is given by (3.23). On the other hand,

$$d_{i,2}\left(s_0\right) = \frac{1}{1+\exp\left(s_0\right)}\frac{d_x\left(s_0\right)}{\sigma_x^2+\sigma_\nu^2}+a_n\theta_2, \quad d_{j,2}\left(s_0\right) = \frac{-\exp\left(s_0\right)}{1+\exp\left(s_0\right)}\frac{d_x\left(s_0\right)}{\sigma_x^2+\sigma_\nu^2}+a_n\theta_2,$$

$$d_x\left(s_0\right) = \frac{E_{v_i}\left[X_1\right]-E_{v_j}\left[X_1\right]+\left[\frac{-\beta\sigma_x^2}{\eta_{i,3}}+\frac{\beta\sigma_x^2}{\eta_{j,3}}+\frac{-\beta\sigma_\nu^2}{\eta_{i,1}}+\frac{\beta\sigma_\nu^2}{\eta_{j,1}}\right]a_n\theta_2}{\left(1+\frac{\beta}{\sigma_x^2+\sigma_\nu^2}\left[\frac{1}{1+\exp(s_0)}\left(\frac{\sigma_x^2}{\eta_{i,3}}+\frac{\sigma_\nu^2}{\eta_{i,1}}\right)+\frac{\exp(s_0)}{1+\exp(s_0)}\left(\frac{\sigma_x^2}{\eta_{j,3}}+\frac{\sigma_\nu^2}{\eta_{j,1}}\right)\right]\right)}.$$

□

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