PhD THESIS DECLARATION

The undersigned

SURNAME Girlanda
FIRST NAME Francesco
PhD Registration Number *1612415*

Thesis title: Essays on Financial Economics and Decision Theory

PhD in *Economics and Finance*
Cycle *XXVII*
Student's Tutor *Elena Carletti*
Year of thesis defence *2017*

DECLARES

Under his responsibility:

1) that, according to Italian Republic Presidential Decree no. 445, 28th December 2000, mendacious declarations, falsifying records and the use of false records are punishable under the Italian penal code and related special laws. Should any of the above prove true, all benefits included in this declaration and those of the temporary "embargo" are automatically forfeited from the beginning;

2) that the University has the obligation, according to art. 6, par. 11, Ministerial Decree no. 224, 30th April 1999, to keep a copy of the thesis on deposit at the "Bilioteche Nazionali Centrali" (Italian National Libraries) in Rome and Florence, where consultation will be permitted, unless there is a temporary "embargo" protecting the rights of external bodies and the industrial/commercial exploitation of the thesis;

3) that the Bocconi Library will file the thesis in its "Archivio Istituzionale ad Accesso Aperto" (Institutional Registry) which permits online consultation of the complete text (except in cases of temporary "embargo");

4) that, in order to file the thesis at the Bocconi Library, the University requires that the thesis be submitted online by the student in unalterable format to Società NORMADEC (acting on behalf of the University), and that NORMADEC will indicate in each footnote the following information:

- PhD thesis Essays on Financial Economics and Decision Theory;
- by Girlanda Francesco ;
- defended at Università Commerciale "Luigi Bocconi" – Milano in the year 2017;

- the thesis is protected by the regulations governing copyright (Italian law no. 633, 22nd April 1941 and subsequent modifications). The exception is the right of Università Commerciale "Luigi Bocconi" to reproduce the same, quoting the source, for research and teaching purposes;

5) that the copy of the thesis submitted online to Normadec is identical to the copies handed in/sent to the members of the Thesis Board and to any other paper or digital copy deposited at the University offices, and, as a consequence, the University is absolved from any responsibility regarding errors, inaccuracy or omissions in the contents of the thesis;

6) that the contents and organization of the thesis is an original work carried out by the undersigned and does not in any way compromise the rights of third parties (Italian law, no. 633, 22nd April 1941 and subsequent integrations and modifications), including those regarding security of personal details; therefore the University is in any case absolved from any responsibility whatsoever, civil, administrative or penal, and shall be exempt from any requests or claims from third parties;

7) that the thesis is not subject to "embargo", i.e. that it is not the result of work included in the regulations governing industrial property; it was not written as part of a project financed by public or private bodies with restrictions on the diffusion of the results; is not subject to patent or protection registrations.

Date 24-01-2017

SURNAME Girlanda
FIRST NAME Francesco

Contents

List of Figures

List of Tables

Acknowledgements

Many are the persons who helped me to complete my PhD Thesis and I would like to express my sincere gratitude to all of them. First of all I am deeply indebted to my PhD Thesis advisors Elena Carletti and Massimo Marinacci that provided me continuous guidance and support in all stages of the process. I owe also much to Francesco Corielli who acted as a third advisor and that gave me an incredible support on the econometric part of my thesis. I am particularly thankful to Simone Cerreia Vioglio and Fabio Maccheroni for their help in developing the decision theoretic parts of my work. I would like to express my appreciation to Pierpaolo Battigalli and Nicola Pavoni for their stimulating discussions and insightful comments. I wish to thank also Paolo Colla, Alfredo Di Tillio, Carlo Favero, Mattias Messner, Sujoy Mukerji, Efe Ok and Doriana Ruffino for useful suggestions and overall support.

The staff at Bocconi University and in particular the availability and kindness of Angela Baldassarre and Silvia Acquati has made working here an enjoyable experience on an everyday basis. I am also thankful to all my PhD colleagues with who I shared this long trip.

Finally, I will be forever grateful to my family. Above all, I would like to thank my wife Ekaterina for her love, help and eternal patience she has shown during the writing of my thesis. I can't thank enough my amazing parents Paolo and Emilia, my grandmother Maria, my sister Stefania and my little nephew Michele (who, when he was just one year old, was the first referee of the preliminary draft of my work and he decided to destroy it. I hope to find nicer referees in the future!) for their love, encoroungement and for believing in me.

Abstract

The thesis consists of 4 chapters on financial economics and decision theoretic topics.

Chapter 1 studies if and how mutual fund strategies are affected by the odds of survival. By using the CRSP survivor bias free mutual fund dataset, we develop a two step procedure that consists in first estimating a Cox (1972) semiparametric model and then in interacting the estimated hazard ratios with the factors of several factor models. We find that the market factor and the momentum factor interact in a negative, robust, statistically significant and, most important, economically relevant way with the odds of survival while the interaction of the size factor goes in the opposite direction. We also highlight how the hazard ratio even if always positive and significant, does not affect a lot the estimated coefficients of the classical factor models when it is not interacted with the other factors. All these results read together suggest that there is a structural difference in strategies between funds with different survival probability, hence between survived and dead funds.

Chapter 2 generalizes the model of Nosal and Ordonez (2016) to study the effects that future concerns of the banking entrepreneurs have on the effectiveness of the constructive ambiguity approach. We introduce the possibility that the institutions can steal part of the potentially pledgeable income. We show that if the government can regulate the ex-post behavior of the banks or if the financial institutions can ex-ante commit to no stealing then constructive ambiguity continue to be a successful policy even without future concerns from the side of the banking institutions. On the other side, we show that, in general, uncertainty about government's bailout policy can act as a good commitment device only in the case in which the banks care about the future.

Chapter 3, in an Anscombe and Aumann (1963) setting, generalizes the model of Gilboa, Maccheroni, Marinacci and Schmeidler (2010) by studying the problem of a Decision Maker that considers several potential completion criteria in order to complete an "objective" incomplete preference relation. We show how the attitude of the decision maker toward the potential completion criteria influences the final aggregation process.

Finally Chapter 4 shows that ambiguity aversion modelled through the smooth second-order expected utility is an important phenomenon in a high frequency Brownian information setting. The research question is important because allows to extend the effects of model uncertainty and ambiguity to the most frequently applied information setting in continuous time finance.

Chapter 1

Odds of Survival and Mutual Fund Strategy

This chapter is joint work with Francesco Corielli and Enrico Ferrari. We thank Elena Carletti, Paolo Colla, Carlo Favero, Nicola Pavoni for helpful comments and suggestions. All remaining errors are ours.

1.1 Introduction

Since the classical work by Jensen (1968) , a central problem in finance has been the evaluation of mutual fund performance and of persistence in mutual fund performance. A great part of this literature makes use of static common factor models like the Capital Asset Pricing Model (CAPM) (Sharpe (1964) and Lintner (1965)) and other factor models a là Fama and French (1993) , a là Carhart (1997) or a là Fama and French (2015) . The persistence in mutual fund performance is normally attributed to common investment strategies and several papers document persistence over a short time horizon from one to three years. In particular, during the last three decades an important challenge of this literature has been to deal with all the biases that could skew the evaluation of performance and a core role among all these biases has been played by the survivorship bias.

Following the tradition of Ferson and Schadt (1996) that introduce a conditional and dynamic asset pricing model in the mutual fund literature and by using a so called survivor bias free dataset, the aim of our paper is to study if and how mutual fund strategies are affected by the odds of survival. In order to reach our objective we make use of a two step procedure. We first estimate a Cox (1972) semiparametric model to study the determinants of mutual fund survival and then we interact the estimated hazard ratios with several factors of the different factor models. In this way we are able to capture the dynamics of the mutual fund strategies and to assess how much and toward which direction the odds of survival influence the strategies of mutual fund managers.

The basic version of our Cox model highlights how the main determinants of mutual fund survivorship are fund's dimension and its expense ratio. The bigger is the mutual fund the higher is its probability of surviving while the expense ratio has an opposite effect, i.e. the higher the fund's expense ratio the lower its probability of surviving. These results are robust to several different specifications of the survival model. Other variables that often have a positive and statistically significant impact on mutual fund survivorship are fund's Nav and flow of funds, measured as the growth in fund's assets.

When we add in the factor model an interaction term between the hazard ratio and the exposure to the market factor we get a negative and both economically and statistically significant parameter estimate. The same happens when the interaction is between the hazard ratio and the momentum. On the other hand the interaction between the hazard ratio and the size factor yields a positive parameter.

These results suggest the existence of a relevant strategic behavior of funds concerning the hazard ratio and factor exposures and imply that a survival model is necessary not only for correcting possible biases in, say, performance evaluation but also in order to give a correct characterization of the behavior of funds confronted with different probabilities of survival.

Even though in the present work, for consistence with the literature, we will continue to use the term survivor bias free dataset and of survivorship bias how it has been classically used by the literature, a connected purpose of our research is to make the reader aware of the fact that it would be useful to introduce a new extended notion of survivorship bias free dataset. This new notion of survivorship bias should integrate the classical notion with the concept of attrition bias (see, for instance, Carpenter and Lynch (1999)) to take into account that theoretically, even if obviously impossible, a true completion of the dataset should imply resurrection of dead funds in order to measure their performances after demise. The main role of a "survivorship bias free" dataset is, in our opinion, not that of directly solving the survivorship bias problem. On the contrary, the existence of such a dataset allows estimating a survival model that is the tool for solving problems related to different survival probabilities. In fact the main benefit of having a survivorship bias free dataset that contains the histories of dead funds is not simply to correct estimates that otherwise, under the assumption that mutual funds with poor performance close down, would be upward biased. The main advantage of having such a dataset is the possibility to estimate a full survival model and to evaluate if it is necessary to correct for both the survivorship bias and the attrition effect. Interestingly, we find that the classical estimates are indeed not particularly biased by the attrition effect. In fact, by using an approach that is in the spirit similar to Heckman (1976 , 1979), we find that the hazard ratio when it is not interacted with the other common factors, even if it is always positive and significant, does not affect a lot the estimated coefficients, with the exception of the constant of the model. On the other side, as previously explained, if we consider the interavction of the hazard ratio with the different factors we estimate statistically significant and important economic effects.

We can conclude that even if the attrition bias probably does not represent a big problem for classical performance studies, on the other side survivor funds and dead funds are structurally different in terms of their strategies. For this reason we suggest that performance and persistence studies should be revised by using an approach similar to ours that is able to take into account for this structural difference.

We proceed as follows. In the Related Literature Section we highlight the connections of our work with the existing literature. In the Data and Sample Selection Section we describe our data, sample selection procedure and the descriptive statistics. In the Empirical Methodology Section and the Results Section we first describe our two step procedure and we then presents the results of the empirical analysis. Finally we run several robustness checks in the Robustness Section and we conclude.

1.2 Related Literature

The aim of the present section is to put our paper into proper perspective and it is far from building a complete review of the literature. Recent research on mutual fund has focused on the interplay between persistence of performance and survival. Brown, Goetzmann, Ibbotson, Ross (1992) show both theoretically, under an independence assumption on mutual fund returns, and by a numerical simulation that survivorship bias may induce persistence in mutual fund returns. Since this pioneering study the focus of the literature has been mainly empirical with the important exceptions of Brown, Goetzmann, Ross (1995) that claim that the equity premium may be seriously affected by the survival bias and Li and Xu (2002) that, by using a general survival model, show that this is not the case because an high survival bias could be reconciled only with an extremely small probability of market survival over a long term period.

From an empirical viewpoint, several authors (among others see Grinblatt and Titman (1989) , Hendricks, Patel, and Zeckhauser (1993) , Malkiel (1995) , Elton, Gruber and Blake (1996a , 1996b), Carhart (1997) , Elton, Gruber, Blake (2001) , Bu and Lacey (2007)) built datasets that have been named survivor bias free, i.e. datasets that contained both the histories of dead and survivor funds. Nowadays, the CRSP Survivor Bias Free dataset, originated by the work of Carhart (1997) , has become one of the industry standards for analyzing US mutual fund data and for this reason we make use of this dataset in the present work. Given the purpose of our research question, as explained in the next section, we adopt a population model view and we will be able to exploit all the information contained in our panel sample. For this reason, differently w.r.t. to the studies mentioned above that were mainly focused on persistence and performance, we do not need particularly long timeseries and we can focus on the period for which the CRSP data is more reliable, i.e. from 2003 onwards.

The evidence contained in all these performance studies is often contradictory. For instance, while Grinblatt and Titman (1989) and Wermers (1997) find that mutual fund managers can pick stocks in a successful way w.r.t. their benchmarks, on the other side Carhart (1997) highlights how mutual fund activism is negatively correlated with the benchmark adjusted net returns. Although this ambiguous evidence, investors continue to pour money in actively managed funds and several interesting papers have been written on mutual fund skills with the aim of distinguishing skills from luck. Barras, Scaillet and

Wermers (2010) develop an innovative approach for separating luck from skills in the evaluation of mutual fund performance. In particular they measure the proportion of skilled funds, unskilled funds and zero alpha funds and they find that the proportion of skilled funds decreased dramatically from 1975 to 2006 and that the majority of funds have zero alpha with a minority of funds that produce negative alpha. Fama and French (2010) and Kosowski, Timmermann, Wermers and White (2006) use bootstrap simulations to distinguish skills from luck. The results by Kosowski, Timmermann, Wermers and White (2006) are slightly more positive than the ones by Fama and French (2010) , but the take-home message is always the same: only a minority of managers are able to cover their costs and have alphas that persist through time.

More recently also cross country studies on mutual fund performance have been developed. Ferreira, Keswani, Miguel and Ramos (2013) show that on average mutual funds underperform around the world with important differences in the determinants of mutual fund performance in the US w.r.t. other countries. Large US funds perform worse than smaller ones but in the rest of the world this pattern is reversed and larger funds are the best performers. The diminishing return to scale of US mutual fund industry seems to be related to liquidity constraints faced by US funds that are forced to invest in domestic and small stocks due to their style. The authors find strong positive correlation between the country level of financial development and funds performance and they highlight how investor protection and law enforcement have a positive impact on performance. Cogneau and Hubner (2015) use an international database in order to study the interconnections between several performance measures and the likelihood of a fund to disappear.

In almost all these studies abnormal returns are computed by using classical performance measures based on Jensen's alpha (Jensen (1968)), 3 factors alpha (Fama and French (1993)) and 4 factors alpha (Carhart (1997))[1]. These are standard measures based on static models and they totally disregard the dynamics, i.e. the possibility for a fund of changing its strategy as a function of its survival probability and consequently the interactions between the probability of surviving and the strategy of the mutual fund managers. Our contribution to this literature is to introduce a method that allows to take into account for these dynamics. In particular our 2 step approach is able to capture the structural difference that features survivor and dead funds. It would be worthwhile to investigate the impact of our technique also on the results of papers whose main purpose is not to study the link between performance and survivorship but that use survivor bias free dataset and classical factor models. Among others, Nohel, Wang and Zheng (2010)

[1]Otten and Bams (2004) represents a rare exception to the described pattern. The authors try to investigate the relationship between survivorship bias and performance by using not only static factor models but also conditional dynamic factor models a là Ferson and Schadt (1996).

that study side by side asset management, Evans (2010) that study the impact of mutual fund incubation or Amihud and Goyenko (2013) that proposes R^2 as a performance predictor.

Most of the analyses that aim at investigating the relationship between survivorship and performance have often disregarded the tight link that there exists between survivorship bias and attrition bias. This is probably the result of the fact that in order to correct for both biases it is not enough to add the histories of dead funds to the sample but it would be necessary to know how the dead funds would have performed if they had not been closed. Given that resurrecting dead funds is obviously impossible, a way to attack this research question has been to run a simulated sample (Carpenter and Lynch (1999)) or to construct the true population alpha by using the simulated method of moments (Linnainmaa (2013)). Interestingly, almost no author used standard statistical methods for correcting for sample selection problems[2] and, to be best of our knowledge, we are the first to correct for the attrition bias by first estimating a survival model and then inserting the hazard ratio in the factor model estimation in order to adjust for the bias. We find that the attrition bias alone, in the model where we do not interact the hazard ratios with the factors, does not affect in a relevant way the factor loadings.

A final note on the survival estimation part. The statistical model most commonly used has been the Probit model (among others, Brown and Goetzmann (1995) , Ter Horst, Nijman and Verbeek (2001) , Carhart, Carpenter, Lynch and Musto (2002) and Rohleder, Scholz and Wilkens (2011)). Few authors have used other parametric models like the Logit (Cogneau and Hubner (2015)) or other semiparametric models like the Cox model (Lunde, Timmermann and Blake (2002) and Linnainmaa (2013) [3]). We believe that a semiparametric model like the one developed by Cox, given its flexibility, is better suited to study the determinants of mutual fund survivorship and it is also able to take into account for the dynamics of the covariates. Anyway our procedure is suited to any survival model. In the robustness checks section we show that our results are robust to different specifications of the survival model like the Probit and Logit.

Classically the authors have concentrated their attention on examining the impact of performance on survivorship and the other fund specific variables have often been overlooked. We don't want to impose any relationship ex-ante between performance and survivorship and in our survival model any fund specific variable may in principle have an impact on mutual fund survival.

[2]Carhart, Carpenter, Lynch and Musto (2002) represent a small exception. In fact the authors implement the Heckman selection model in order to correct for look ahead bias.

[3]We notice that in the hedge fund literature the Cox model has been used by Agarwal, Fos and Jiang (2013) in order to study reporting related biases.

1.3 Data and Sample Selection

Our data sample spans the period July 2003-December 2015. We make use of the CRSP survivor bias free dataset. As described in the next section, the aim of our research question allows us to exploit all the information contained in our panel dataset and for this reason we do not need particularly long timeseries. Hence we can concentrate on the period on which the data reach the highest standards. The CRSP dataset becomes particularly reliable from 1998 onwards because the provider starts to collect data electronically from Lipper and Thomson Reuters and the data are even more consistent from July 2003 onwards when the provider starts to assign a unique portfolio identifier to all the funds. We make use of monthly observations and we focus on equity funds with more than 5 mln $ of assets under management.

A classical problem of the CRSP dataset is that it identifies each fund's share class separately and as consequence it is necessary to aggregate the funds that have multiple share classes. We perform this aggregation by looking at the fund's name[4] and by using as weight the total net assets value of each shareclass.

We erase fund's observations for which there are missing values either of monthly total net assets ($Mtna$) or of monthly returns [5] ($Mret$) or of monthly net asset values[6] ($Mnav$). The CRSP returns are before any front-end or back-end loads. In the core part of our analysis described in the next section, in order to maximize the number of observations contained in our sample, together with the variables previously mentioned, we include only a subset of the fund's characteristics available. In particular we select the following fund's characteristics: turnover ratio[7] ($Turn\_ratio$), expense ratio[8] (Exp_ratio), and income yield ($Yield$), this latter variable calculated as the ratio of income distributions

[4]Several authors in order to aggregate for multiple share classes make use of MFLINKS that is available on WRDS. Of course also MFLINKS has its own problematics but this tool is normally preferred to fund's name aggregation when long timeseries are needed. In fact fund's name aggregation is based on the CRSP convention to separate fund's name from the share class by making use of " ;". Prior to 1998 several symbols are used as separator and not just ";" and for this reason writing a code to identify the share classes is problematic. Fortunately we can aggregate in a consistent and correct way given that our sample is from 2003 onwards.

[5]Monthly returns values are net of all management expenses and 12b1-fees and they are calculated as a change in Net Asset Values taking into account reinvested dividends from one period to the next. See the CRSP guide [26] for details.

[6]The *mnav* is measured as the monthly value of the fund's underlying assets (including cash) minus its liabilities (fees, expenses, etc.) divided by the number of shares outstanding. See the CRSP guide [26] for details.

[7]The fund's turnover ratio is calculated as the minimum of aggregated sales or aggregated purchases of securities divided by the average 12-month Total Net Assets of the fund. See the CRSP guide [26] for details.

[8]The expense ratio is calculated as the ratio of total investment that shareholders pay for the fund's operating expenses. See the CRSP guide [26] for details.

and fund's NAV[9]. We erase outliers fund's observations that have either a turnover ratio bigger than 100 or an expense ratio greater than 0.05 or an income yield larger than 0.2. Notice that these choices make the sample analyzed more representative and realistic but our results are robust to the inclusion of these outliers[10].

Sometimes it happens that a completely inactive fund is left opened even if it is characterized by a return that is almost zero. For our purposes the last observations of these funds should be erased. For this reason we drop the last observation of a fund when the sum of the last 2 returns is almost zero[11]. In order to have a consistent sample we also drop funds whose observations are provided less frequently than monthly. Our final sample is made up of 504.709 of the initial 856.448 monthly observations.

We do not further cut our sample in the main analysis but in the robustness checks section we show that our results are robust the inclusion of all the fund's characteristics available, to the exclusion of funds that do not invest primarily on common stocks and to the exclusion of smaller funds, that according to the literature are more affected by the survivor bias concern. The additional fund's characteristics that we add in the robustness checks section are: the management fee ($Mgmt\_fee$), the max (Max_12b1) and the actual ($Actual\_12b1$) 12b-1 fees. In this case we further restrict the sample to 500.835 observations by excluding funds outlier observations featured by a management fee greater than 5% or smaller that -5%[12]. Our sample in this case has 11822 funds and this is a pretty large number when compared with the datasets of other papers that focus their attention at maximum on 2000 funds.[13]

In our analysis, unless otherwise specified, we denote as "survivor" a fund that is alive till December 2015 while we denote as "dead" a fund that disappears before December 2015. In Table 1.1 we report all the descriptive statistics of the fund's characteristics. For completeness we also add the fund's age (Age) expressed in years. The mean age of the funds in our sample is 11.4 years and survivor funds are on average almost 2 years older than dead funds. The returns of survivor funds are characterized by a bigger mean and a smaller variance w.r.t. the ones of dead funds. In general a survivor fund is roughly 6 times bigger than a dead fund and it has an higher NAV and bigger management fees. On the other side dead funds have a bigger expense ratio, a greater turnover ratio and higher distribution fees when compared to survivor funds.

[9]See the CRSP guide [26] for details.

[10]Results available from authors upon request.

[11]In particular we drop the last observation of a fund when the sum of the last 2 returns is smaller than e^{-10}. We iterate the code in order to eliminate all such observations for a given fund.

[12]Reimbursements can lead to negative management fees. See the CRSP guide [26] for details.

[13]Linnainmaa (2013) has a dataset that contains 1853 mutual funds.

Table 1.1: Descriptive statistics

Panel A: All Funds

Variable	Mean	Std. Dev.	Min	Max
Mret	0,006	0,046	-0,547	0,398
Mtna	1378	6482	5	248725
Mnav	17,056	16,317	0,329	906,950
Turn_ratio	0,827	1,489	0,000	91,500
Exp_ratio	0,012	0,005	-0,002	0,050
Yield	0,009	0,011	0,000	0,197
Age	11,403	11,607	0,019	91,523
Mgmt_fee	0,561	0,584	-4,998	4,116
Actual_12b1	0,004	0,002	0,000	0,010
Max_12b1	0,005	0,002	0,000	0,010

Panel B: Survivor Funds

Variable	Mean	Std. Dev.	Min	Max
Mret	0,006	0,045	-0,547	0,398
Mtna	1710	7360	5	248725
Mnav	18,046	18,010	0,750	906,950
Turn_ratio	0,759	1,343	0,000	91,500
Exp_ratio	0,011	0,005	-0,002	0,050
Yield	0,009	0,010	0,000	0,193
Age	11,864	11,990	0,019	91,523
Mgmt_fee	0,568	0,571	-4,998	4,116
Actual_12b1	0,004	0,002	0,000	0,010
Max_12b1	0,004	0,002	0,000	0,010

Panel C: Dead Funds

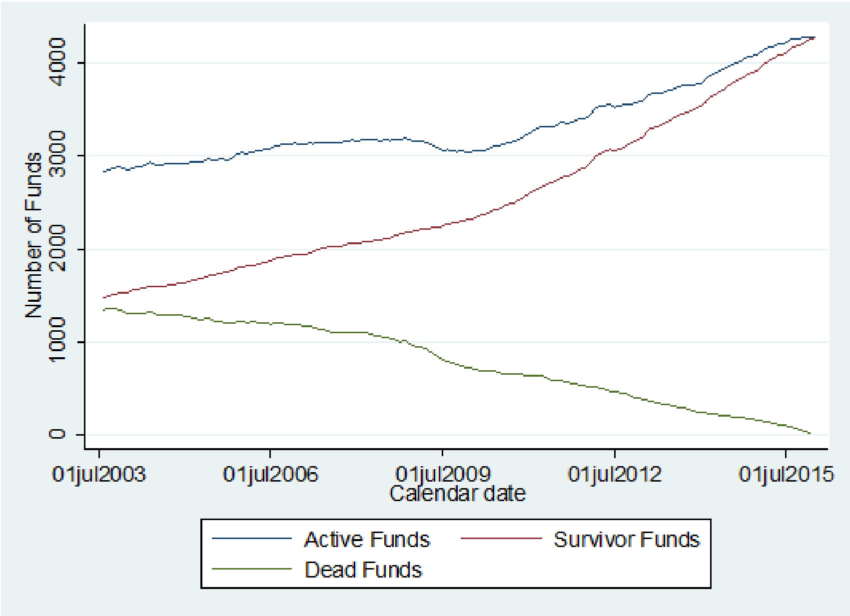
Variable	Mean	Std. Dev.	Min	Max
Mret	0,005	0,047	-0,379	0,389
Mtna	292	811	5	18480
Mnav	13,811	7,878	0,329	146,510
Turn_ratio	1,046	1,865	0,000	49,800
Exp_ratio	0,013	0,005	0,000	0,050
Yield	0,009	0,011	0,000	0,197
Age	9,891	10,103	0,038	80,479
Mgmt_fee	0,538	0,623	-4,944	2,679
Actual_12b1	0,005	0,002	0,000	0,010
Max_12b1	0,005	0,002	0,000	0,010

The table reports the descriptive statistics of the variables of interest. Panel A reports the descriptive statistics for all the sample, while Panel B and Panel C contain the descriptive statistics for survivor funds and dead funds respectively. See the Appendix for a detailed description of all the variables.

Table 1.2 contains the cross correlations of the fund's characteristics. Notice the high level of positive correlation between age and both the size and the NAV of the funds. Older funds have higher management fees and this is particularly true for the sample of dead funds. There is also a positive correlation between the expense ratio and the turnover ratio and again this relationship is particularly strong for the sample of dead funds.

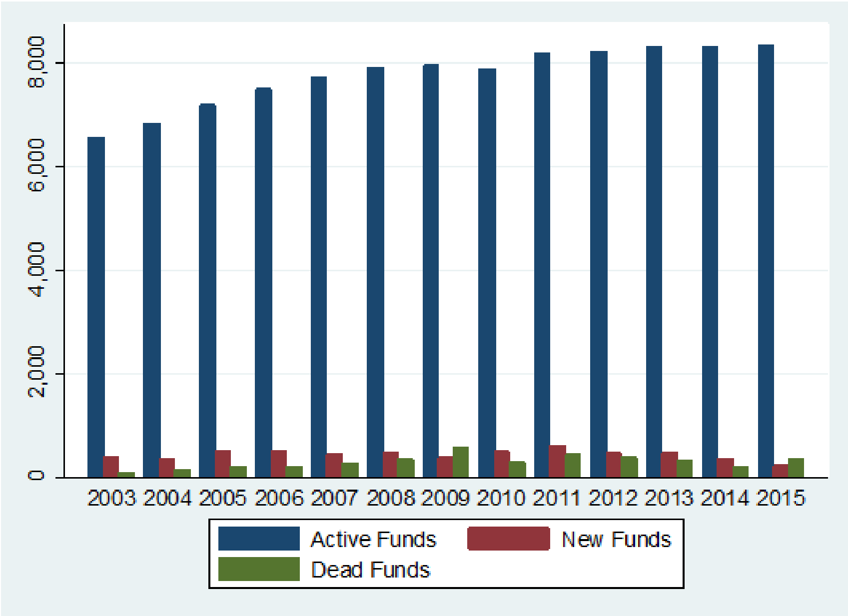
We conclude the section by representing in Figure 1.1 how dead and survivor funds have evolved. We also report in Figure 1.2 the number of funds that each year are either active or new born or dead. Notice that, differently w.r.t. Figure 1.1, here the Dead Funds category contains the funds that are effectively dead in that reference year. From 2003 to 2015 the mutual fund industry has grown and the only year in which the industry decreased has been the 2010 probably due to the consequences of the financial crisis.

Figure 1.1: Evolution of the number of funds



The Figure represents the monthly evolution of the number of funds. The sample period is July 2003 - December 2015. Notice that the Dead Funds category contains the funds that are alive that month and that will be dead before December 2015 and the Survivor Funds category contains the funds that are alive that month and that will be alive till December 2015.

Figure 1.2: Funds that each year are either active or new born or dead



The Figure contains the number of funds that each year are either active or new born or dead. The sample period is July 2003 -December 2015. Differently w.r.t. Figure 1.1 here the Dead Funds category contains the funds that are effectively dead in that reference year.

Table 1.2: Cross Correlations

Panel A: All Funds

Variable	Mret	Mtna	Mnav	Turn_ratio	Exp_ratio	Yield	Age	Mgmt_fee	Actual_12b1	Max_12b1
Mret	1									
Mtna	0,0041	1								
Mnav	0,0797	0,2365	1							
Turn_ratio	-0,0144	-0,043	-0,0627	1						
Exp_ratio	0,0179	-0,109	0,0281	0,1956	1					
Yield	-0,0484	0,0091	-0,0423	0,0409	-0,0321	1				
Age	0,0078	0,3263	0,2224	-0,0369	-0,0166	-0,0939	1			
Mgmt_fee	0,009	-0,01	0,1338	0,1273	0,4396	-0,0514	0,1433	1		
Actual_12b1	-0,0051	-0,048	-0,0499	0,0379	0,2768	0,0108	-0,09	0,0032	1	
Max_12b1	-0,0022	0,0319	-0,0137	0,0286	0,2303	0,0125	-0,015	-0,0012	0,8414	1

Panel B: Survivor Funds

Variable	Mret	Mtna	Mnav	Turn_ratio	Exp_ratio	Yield	Age	Mgmt_fee	Actual_12b1	Max_12b1
Mret	1									
Mtna	0,0023	1								
Mnav	0,0766	0,2405	1							
Turn_ratio	-0,0123	-0,044	-0,0718	1						
Exp_ratio	0,0149	-0,113	0,0361	0,1855	1					
Yield	-0,0439	0,0119	-0,0478	0,0353	-0,0366	1				
Age	0,0103	0,3506	0,2332	-0,0456	-0,02	-0,0951	1			
Mgmt_fee	0,0089	-0,016	0,1317	0,1318	0,4564	-0,0385	0,1278	1		
Actual_12b1	-0,0077	-0,046	-0,0589	0,032	0,2811	0,0056	-0,073	0,0066	1	
Max_12b1	-0,0046	0,0474	-0,0126	0,018	0,2214	0,0117	0,0001	-0,0138	0,8412	1

Panel C: Dead Funds

Variable	Mret	Mtna	Mnav	Turn_ratio	Exp_ratio	Yield	Age	Mgmt_fee	Actual_12b1	Max_12b1
Mret	1									
Mtna	0,0213	1								
Mnav	0,093	0,284	1							
Turn_ratio	-0,0172	-0,047	-0,0012	1						
Exp_ratio	0,0352	-0,052	0,0539	0,2102	1					
Yield	-0,0625	-0,043	-0,0169	0,0579	-0,0203	1				
Age	-0,0062	0,264	0,1365	0,0076	0,0251	-0,0901	1			
Mgmt_fee	0,009	0,12	0,1523	0,1158	0,4006	-0,0927	0,2045	1		
Actual_12b1	0,0082	-0,006	0,0297	0,0376	0,2377	0,0246	-0,133	-0,004	1	
Max_12b1	0,0106	0,0203	0,0276	0,0425	0,2282	0,0134	-0,047	0,0391	0,8376	1

The table reports the cross correlations of the variables of interest. Panel A reports the descriptive statistics for all the sample, while Panel B and Panel C contain the descriptive statistics for survivor funds and dead funds respectively. See the Appendix for a detailed description of all the variables.

1.4 Empirical Methodology

In the present section we describe our two step procedure that allows us to identify the effect of the odds of survival on mutual fund strategy. The first step of our approach consists in estimating the semiparametric Cox proportional hazard model developed by Cox (1972) . In this way we are able to relate the failure event to the fund's characteristics. A failure happens when a fund dies in a specific month.

There are several reasons that can lead to a failure. A fund may be merged with another fund or it may be liquidated. A fund may be liquidated because the manager wants to rationalize the funds within a management group and he closes funds with too similar investment objectives. Other explanations for liquidating a fund may be that the manager wants to close the less performing funds in order to raise the average performance of a group or a fund is closed because it does not reach a certain critical mass of capital. From this perspective the dataset could be improved in order to provide more details on the reasons of fund's closure although we recognize that this type of information is hard to collect given the nature of data.

We have a right censored sample, i.e. some funds are "alive" when the study ends in December 2015, with late entries. Our statistical framework considers and properly model the fact that some funds were already alive when the study starts while some other funds enter in our sample later on because they are born during the period analyzed. An example of a fund of the first type is the Nottingham Investment Trust II offered for the first time on August 1992, while an example of a fund of the second type is Advisers Investment Trust: JOHCM Emerging Markets Small Mid Cap Equity Fund offered for the first time in December 2014.

It is useful to think of the failure time T as a random variable with an associated density function $f(t)$ and cumulative distribution function (CDF) $F(t) = \Pr(T \leq t)$. The complement of the CDF is the survival function $S(t)$. In survival analysis a key role is played by the hazard function $h(t)$ that is defined as follows:

$$h(t) = \lim_{\Delta t \to 0} \frac{\Pr(t \leq T < t + \Delta t | T \geq t)}{\Delta t} = \frac{f(t)}{S(t)}.$$

The hazard rate represents the expected number of events for each fund per unit of time, i.e. it is the instantaneous risk of closing for the population of active funds, while the cumulative hazard function $H(t) = \int_0^t h(z)\, dz$ represents the number of events that have occurred by time t. Proposing a survival model is equivalent to specify a functional form for the hazard rate $h(t)$. The functional form assumed by Cox is the following:

$$h(t) = h_0(t)\, e^{\beta X_i(t)}$$

where we denote with $h_0(t)$ the baseline hazard function, with $X_i(t)$ a vector containing the fund i's characteristics at time t and with β a vector containing the effects of the various fund's specific variables on the hazard function $h(t)$. In our analysis we use as covariates the fund's characteristics lagged of one month. A positive coefficient β_j associated to fund i's covariate X_{ij} implies that the characteristic j has the effect of increasing the instantaneous probability of fund i's death. A common practice in hazard analysis is to report the hazard ratio of the covariate. This variable is obtained by studying the impact on the hazard rate of a unit increment of the covariate X_{ij}, i.e.:

$$hr_{ij}(t) = \frac{h(t|\, X_{ij}+1, X_{i-j})}{h(t|\, X_{ij}, X_{i-j})}$$

where we denote with X_{i-j} all fund i' s characteristics but j. If this ratio is bigger than 1 then the covariate increases the probability of fund's closure and the opposite is true when the hazard ratio is smaller than 1. In our tables we report both the coefficients and the associated hazard ratios.

The great contribution of Cox was not simply to propose the semiparametric form described above but he also proposed a method in order to implement the estimation of the parameters that was otherwise not feasible. Let's assume to have a population of n funds and, just momentarily, that the covariates are not time varying. In this case, by denoting with t_i the time at which fund i either fails or it is right censored and by introducing a dummy variable d_i that it is equal to 1 if a fail is observed, it is possible to write the following likelihood function L as:

$$L(\beta) = \Pi_{i=1}^{n} \left[h_0(t_i)\, e^{\beta X_i}\right]^{d_i} \left[S(t_i|\, X_i, \beta)\right]^{1-d_i}.$$

Given that the baseline function is not specified, it is not feasible to proceed by maximizing the likelihood. The amazing idea of Cox was to maximize what he denoted as partial likelihood L^p, i.e. the following ratio:

$$L^p(\beta) = \Pi_{i=1}^{n} \left[\frac{e^{\beta X_i}}{\sum_{l\in R(t_i)} e^{\beta X_l}}\right]^{d_i}$$

where $R(t_i)$ is the population of funds at risk at time t_i. To sum up estimating the Cox survival model with the different fund's characteristics constitutes the first step of our technique and it produces as output the hazard ratios that are key for our second step.

Our objective is to verify if and how the odds of survival influences funds strategies and the second step of our procedure try to answer this question. Our focus is on the overall population of funds and not on the strategy of a single fund. For this reason, in order to exploit all the information contained in our sample, we estimate fund fixed effects panel regressions. Even if we do not perform the classical performance analysis and we opt for a population view, our approach could be easily adapted to performance studies by leaving the first step how it is and by modifying the second step accordingly.

The factor models that we consider in order to answer to our research question are the followings:

The Jensen (1968)'s one factor model with α representing the average performance of the population of funds, $Mretadj_{it}$ representing the return of fund i at time t less the risk free interest rate (one month treasury bill), $Mktrf_t$ representing the excess return on the CRSP value weight market index and hr_{it} representing the hazard rate of fund i at time t:

$$Mretadj_{it} = \alpha + \gamma hr_{it} + \beta Mktrf_t + \delta^{Mktrf} hr_{it} Mktrf_t.$$

The Fama and French (1993)'s 3 factors model with Smb_t representing the small minus big factor, i.e. the return on a diversified portfolio of small stocks minus the returns of a diversified portfolio of big stocks, and Hml_t representing the high minus low factor, i.e. the return on a diversified portfolio of high B/M stocks minus the returns of a diversified portfolio of low B/M stocks:

$$\begin{aligned} Mretadj_{it} = {} & \alpha + \gamma hr_{it} + \beta Mktrf_t + \eta Smb_t + \theta Hml_t \\ & + \delta^{Mktrf} hr_{it} Mktrf_t + \delta^{Smb} hr_{it} Smb_t + \delta^{Hml} hr_{it} Hml_t. \end{aligned}$$

The Carhart (1997)'s 4 factors model, with Umd_t representing the momentum factor, i.e. the return on a diversified portfolio of the highest performers over the last 12 months minus the return on a diversified portfolio of the lowest performers over the last 12 months:

$$\begin{aligned} Mretadj_{it} = {} & \alpha + \gamma hr_{it} + \beta Mktrf_t + \eta Smb_t + \theta Hml_t + \lambda Umd_t \\ & + \delta^{Mktrf} hr_{it} Mktrf_t + \delta^{Smb} hr_{it} Smb_t + \delta^{Hml} hr_{it} Hml_t + \delta^{Umd} hr_{it} Umd_t. \end{aligned}$$

The Fama and French (2015)'s 5 factors model, with Rmw_t representing the robust minus weak factor, i.e. the return on a diversified portfolio of stocks with robust profitability minus the return on a diversified portfolio of stocks with weak profitability, and Cma_t representing the conservative minus aggressive factor, i.e. the return on a diversified portfolio of stocks of low investment firms minus the return on a diversified portfolio of stocks of high investment firms:

$$\begin{aligned} Mretadj_{it} = &\alpha + \gamma hr_{it} + \beta Mktrf_t + \eta Smb_t + \theta Hml_t + \mu Rmw_t + \nu Cma_t \\ &+ \delta^{Mktrf} hr_{it} Mktrf_t + \delta^{Smb} hr_{it} Smb_t + \delta^{Hml} hr_{it} Hml_t + \delta^{Rmw} hr_{it} Rmw_t \\ &+ \delta^{Cma} hr_{it} Cma_t. \end{aligned}$$

The estimation of these classical factor models augmented with the interaction of the factors with the hazard rate constitutes the second step of our approach. We are particularly interested in the significance and sign of the δ coefficients that allow us to understand how on average the odds of survival influences the strategy of the funds. Notice that in our survival estimation we have used as covariates fund's characteristics lagged of one period. Hence the hazard ratio can be sawn as a function of variables available at time $t-1$. Using this perspective, our factor models remember Ferson and Schadt (1996)'s conditional factor model that conditions on the information available at time $t-1$. Notice also that by setting all the δ coefficients equal to zero it is possible to test whether the sample is affected by an attrition bias. If the sample is affected by a big attrition bias then the γ coefficient has to be significantly different from zero and the factor loadings should change significantly w.r.t. the case in which the hazard rate is completely excluded from the model.

1.5 Results

In the present section we present the results of the empirical methodology previously described. The appendix contains a detailed description of all the variables used in the present analysis. As a preliminary step we first estimate the Jensen (1968)'s one factor model, the Fama and French (1993)'s 3 factors model, the Carhart (1997)'s 4 factors model and the Fama and French (2015)'s 5 factors model for the population of survivor and dead funds. Table 1.3 describes the results of these preliminary regressions.

All the estimated panel regressions yield an R^2 above 70% and all the factor loadings turn out to be statistically significant with the exception of the momentum factor for dead funds. Survivor funds are characterized by a lower exposition to the market factor w.r.t. dead funds. If we focus our attention on the Carhart (1997) 's 4 factors model, survivor funds have a market factor loading of 0.916 while dead funds have a loading of 0.97. Survivor funds have smaller small minus big factor, momentum factor and conservative minus aggressive factor. On the other side this category of funds shows an higher exposition to the high minus low factor and to the robust minus weak factor.

In the survival model estimation, we expect dead funds to be characterized on average by an higher hazard ratio. Hence the results of Table 1.3 will constitute our benchmark in order to understand if in our modified versions of the factor models, that include the hazard ratio and its interaction with the factors, an higher hazard ratio will move us toward the direction of the strategy of the dead funds.

We now proceed with the first step of our methodology and we estimate a Cox survival model with dependent variable funds failure, i.e. a dummy variable that equals 1 when a fund closes and 0 otherwise, and the covariates are the 1 period lag of monthly returns multiplied by 100 ($Mretl1100$), the 1 period lag of the natural logarithm of the monthly total net assets ($Logmtnal1$)[14], the 1 period lag of the monthly net asset values ($Mnavl1$), the 1 period lag of the turnover ratio multiplied by 100 ($Turn\_ratiol1100$), the 1 period lag of the expense ratio multiplied by 100 ($Exp\_ratiol1100$). We further include the 1 period lag of a dummy variable ($Dummyieldl1$) that is equal to 1 when income yield is distributed in the reference period and it is equal to 0 otherwise. The last variable that we include in our core specification is the 1 period lag of the flow variable multiplied by 100 ($Flowl1100$). The classical definition of the flow variable at time t for fund i (see for instance Sirri and Tufano (1998)) is the following:

$$Flow_{it} = \frac{Mtna_{it} - Mtna_{it-1} \times (1 + Mret_{it-1})}{Mtna_{it-1}}$$

and it represents the percentage growth of a fund in excess to the growth attributed to the reinvestment of dividends and to the existing invested funds. In order to maximize the reliability of our results and the number of observations in our sample, at the moment we exclude the other fund's characteristics that seems to be particularly affected by outlier observations. The robustness checks section shows that our results are robust to the inclusion of variables excluded in this section.

[14] In the robustness checks section we show that our final results are not affected by the choice of including the natural logarithm of the assets. We believe that the most correct specification has to include the natural logarithm of the assets in order to not underestimate the important role that size has in mutual fund's survivorship.

Table 1.4 contains the results of the estimated Cox model. The variables that most influence mutual fund survivorship are the assets and the expense ratio. The size has a positive impact on survivorship while the expense ratio has a negative impact on it. Other highly statistically significant variables are the NAV, the flows and the dummyield variable. In particular while the first two variables generate an increase of the probability of surviving, the latter variable is featured by an hazard ratio bigger than 1 and hence it reduces the survivorship probability. The turnover ratio, even if it is weakly statistically significant, has an hazard rate equal to 1 and consequently it does not impact survivorship toward any specific direction. The monthly returns turn out to be not statistically significant.

On average, as expected, the hazard ratio of dead funds is higher than the one of survivor funds. Dead funds hazard ratio has also a bigger variance w.r.t. the one of the survivor funds. Notice that the hazard ratio is on average below 1 also for dead funds. This result is driven by the fact that, in order to not underestimate the role of the size variable we decided to include the natural logarithm of the monthly total net assets, as it has be done classically by the literature. The inclusion of the not transformed size variable, as shown in the robustness checks section, will lead to an average hazard ratio above 1 for the category of dead funds and it will result in a reduction of the impact of assets on survivorship.

We proceed by estimating the Jensen (1968)'s one factor model, the Fama and French (1993)'s 3 factors model, the Carhart (1997)'s 4 factors model and the Fama and French (2015)'s 5 factors model together with their modified versions. .

As it is possible to notice by analyzing Table 1.5, the modified versions of the factor models highlight how mutual funds strategies are indeed dynamic and funds strategies are correlated with their survivorship probability in that factor loadings change when survivorship probability changes. In particular the statistically significant and negative interaction between the hazard ratio and the market factor is a sign of the fact that there is a strong correlation between fund exposition to the market and the probability that the funds are going to close [15]. In order to properly interpret these results it is important to remember that the average hazard ratio is almost 0.1 and, for instance, the average reduction of market factor loading related to the survivorship effect is equal to 0.024 according to the Carhart (1997)'s 4 factors model. This effect is certainly important and it has a size even larger than the momentum factor.

The negative interaction of the hazard ratio with the market factor is surprising and

[15] We can just talk about correllation and not about causality. In fact in order to construct a structural model and to talk about causal relationships we should have data on fund behaviour that are not easily available.

unexpected from an ex-ante viewpoint. In fact from Table 1.3, given that dead funds are featured by an higher market factor loading and they have on average a bigger hazard rate, we would expect ex-ante an opposite sign. From Table 1.6, it is possible to notice how just the introduction of the hazard ratio does not have a sizable effect on factor loadings, it is the interaction term that allows to exploit the proper relationship between the factors and the survival probability.

In the appendix, by using simulated data, we show that, in a simple 1 factor model, if the funds that have an higher probability to die have an higher exposition to the market then our 2 step procedure estimates a positive interaction between the factor and the hazard ratio. But real strategies are certainly much more complicated and this 2 step methodology seems to capture effects that cannot be observed by focusing separately on the population of survivor and dead funds.

Other statistically strong effects captured by our model are the positive interaction of the hazard ratio with the small minus big factor and the negative interaction with the momentum factor. By looking at Table 1.3, we notice that while the positive interaction with the small minus big factor is ex-ante expected the opposite is true for the momentum factor. Table 1.5 shows also some evidence of interaction effects of the hazard ratio with the other factors, but these results are either not robust or they have a lower statistical significance. The robustness checks section will tend to confirm all the results highlighted by our core specification. All panel regressions are characterized by an R^2 above 70%.

Finally, it is really important to understand if our results are driven by an attrition effect or if there are deep structural differences between survivor and dead funds strategies. By using an approach that resembles Heckman (1976, 1979), we estimate the factor models without interaction terms between the hazard ratio and the factors and with just the inclusion of the hazard ratio. By carefully looking at Table 1.6, we can conclude that our results are not due to an attrition bias. In fact, even though the hazard rate is always positive and statistically significant, its inclusion has almost no effect on the factor loadings.

Table 1.3: Factor analysis of survivor and dead funds

VARIABLES	(1) SurvFF1	(2) DeadFF1	(3) SurvFF3	(4) DeadFF3	(5) SurvFF4	(6) DeadFF4	(7) SurvFF5	(8) DeadFF5
Mktrf	0.938***	0.982***	0.923***	0.970***	0.916***	0.970***	0.904***	0.931***
	(0.00424)	(0.00693)	(0.00399)	(0.00675)	(0.00393)	(0.00660)	(0.00383)	(0.00642)
Smb			0.0734***	0.0949***	0.0777***	0.0952***	0.0587***	0.0884***
			(0.00458)	(0.00785)	(0.00456)	(0.00779)	(0.00469)	(0.00786)
Hml			-0.00994**	-0.0723***	-0.0278***	-0.0729***	0.0617***	-0.0149**
			(0.00404)	(0.00714)	(0.00374)	(0.00654)	(0.00450)	(0.00732)
Rmw							-0.0891***	-0.154***
							(0.00359)	(0.00707)
Cma							-0.235***	-0.212***
							(0.00590)	(0.00907)
Umd					-0.0268***	-0.00113		
					(0.00172)	(0.00332)		
Constant	-0.00128***	-0.00139***	-0.00125***	-0.00141***	-0.00114***	-0.00141***	-0.000858***	-0.000834***
	(3.15e-05)	(3.53e-05)	(3.07e-05)	(4.20e-05)	(3.06e-05)	(4.01e-05)	(2.95e-05)	(4.84e-05)
Observations	385,990	118,719	385,990	118,719	385,990	118,719	385,990	118,719
R-squared	0.713	0.731	0.715	0.734	0.715	0.734	0.719	0.737
Number of funds	8,105	3,820	8,105	3,820	8,105	3,820	8,105	3,820

The Table presents panel regressions with fund fixed effects. We denote as "survivor" a fund that is alive till December 2015 while we denote as "dead" a fund that disappears before December 2015. Columns (1) and (2) represents results for the Jensen (1968)'s one factor model for survivor and dead funds respectively. Columns (3) and (4) represents results for the Fama and French (1993)'s 3 factors model for survivor and dead funds respectively. Columns (5) and (6) represents results for the Carhart (1997)'s 4 factors model for survivor and dead funds respectively. Columns (7) and (8) represents results for the Fama and French (2015)'s 5 factors model for survivor and dead funds respectively. The sample period is July 2003 - December 2015. Robust standard errors in parentheses, *** indicates significance at 1%, ** indicates significance at 5% and * indicates significance at 10%. See the Appendix for a detailed description of all the variables.

Table 1.4: Cox analysis and hazard ratio of the main specification

Panel A: Cox Analysis

VARIABLES	(1) Hazard Ratios	(2) Coefficients
Mretl1100	1.003	0.00285
	(0.00387)	(0.00386)
Mnavl1	0.974***	-0.0260***
	(0.00292)	(0.00300)
Logmtnal1	0.617***	-0.483***
	(0.00888)	(0.0144)
Turn_ratiol1100	1.000*	0.000157*
	(8.08e-05)	(8.08e-05)
Exp_ratiol1100	1.257***	0.229***
	(0.0451)	(0.0359)
Dummyieldl1	1.130***	0.122***
	(0.0457)	(0.0404)
Flowl1100	0.970***	-0.0305***
	(0.00148)	(0.00153)
Observations	435,644	435,644

Panel B: Hazard Ratio

Variable	Mean	Std. Dev.
Hr	0,09991	0,12372
Hr_surv	0,08431	0,10851
Hr_dead	0,15038	0,15297

The Table reports the results of the Cox analysis. Panel A reports the estimated hazard ratios and coefficients (Columns (1) and (2)) relative to the variables used in the analysis. The sample period is July 2003 - December 2015. Robust standard errors in parentheses, *** indicates significance at 1%, ** indicates significance at 5% and * indicates significance at 10%. Panel B represents the mean and the standard deviation of the hazard ratio for all the population of funds, for the survivor funds and for the dead funds. We denote as "survivor" a fund that is alive till December 2015 while we denote as "dead" a fund that disappears before December 2015. See the Appendix for a detailed description of all the variables.

Table 1.5: Factor analysis relative to the Cox model of the main specification

VARIABLES	(1) FF1	(2) FF1HR	(3) FF3	(4) FF3HR	(5) FF4	(6) FF4HR	(7) FF5	(8) FF5HR
Mktrf	0.963***	0.977***	0.949***	0.967***	0.943***	0.966***	0.925***	0.943***
	(0.00370)	(0.00459)	(0.00351)	(0.00438)	(0.00344)	(0.00434)	(0.00333)	(0.00422)
Smb			0.0877***	0.0722***	0.0916***	0.0741***	0.0739***	0.0569***
			(0.00431)	(0.00559)	(0.00428)	(0.00555)	(0.00439)	(0.00568)
Hml			-0.0309***	-0.0404***	-0.0436***	-0.0457***	0.0352***	0.0323***
			(0.00382)	(0.00512)	(0.00351)	(0.00515)	(0.00411)	(0.00562)
Umd					-0.0197***	-0.00678***		
					(0.00162)	(0.00236)		
Mktrf*Hr		-0.140***		-0.191***		-0.236***		-0.186***
		(0.0297)		(0.0286)		(0.0281)		(0.0279)
Smb*Hr				0.150***		0.164***		0.167***
				(0.0313)		(0.0312)		(0.0317)
Hml*Hr				0.111***		0.0545		0.0593
				(0.0350)		(0.0375)		(0.0400)
Umd*Hr						-0.105***		
						(0.0168)		
Hr		0.0115***		0.0115***		0.0108***		0.0120***
		(0.00103)		(0.00104)		(0.000990)		(0.00107)
Rmw							-0.101***	-0.109***
							(0.00330)	(0.00542)
Cma							-0.226***	-0.240***
							(0.00533)	(0.00791)
Rmw*Hr								0.0800*
								(0.0448)
Cma*Hr								0.110**
								(0.0559)
Constant	-0.00128***	-0.00243***	-0.00128***	-0.00242***	-0.00120***	-0.00229***	-0.000883***	-0.00207***
	(2.41e-05)	(0.000105)	(2.36e-05)	(0.000105)	(2.32e-05)	(0.000101)	(2.40e-05)	(0.000109)
Observations	435,690	435,690	435,690	435,690	435,690	435,690	435,690	435,690
R-squared	0.731	0.732	0.733	0.734	0.733	0.734	0.737	0.737
Number of funds	10,282	10,282	10,282	10,282	10,282	10,282	10,282	10,282

The Table represents panel regressions with fund fixed effects. The hazard ratios are derived from the Cox model represented in Table 1.4. Columns (1) and (2) represents results for the Jensen (1968)'s one factor model and our modified version with the hazard ratio and the interaction of the factor with the hazard ratio. Columns (3) and (4) represents results for the Fama and French (1993)'s 3 factors model and our modified version with the hazard ratio and the interactions of the factors with the hazard ratio. Columns (5) and (6) represents results for the Carhart (1997)'s 4 factors model and our modified version with the hazard ratio and the interactions of the factors with the hazard ratio. Columns (7) and (8) represents results for the Fama and French (2015)'s 5 factors model and our modified version with the hazard ratio and the interactions of the factors with the hazard ratio. The sample period is July 2003 - December 2015. Robust standard errors in parentheses, *** indicates significance at 1%, ** indicates significance at 5% and * indicates significance at 10%. See the Appendix for a detailed description of all the variables.

Table 1.6: Factor analysis with just the hazard ratio relative to the Cox model of the main specification

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
VARIABLES	FF1	FF1JHR	FF3	FF3JHR	FF4	FF4JHR	FF5	FF5JHR
Mktrf	0.963***	0.962***	0.949***	0.948***	0.943***	0.943***	0.925***	0.925***
	(0.00370)	(0.00369)	(0.00351)	(0.00351)	(0.00344)	(0.00344)	(0.00333)	(0.00333)
Smb			0.0877***	0.0870***	0.0916***	0.0908***	0.0739***	0.0734***
			(0.00431)	(0.00431)	(0.00428)	(0.00428)	(0.00439)	(0.00438)
Hml			-0.0309***	-0.0308***	-0.0436***	-0.0427***	0.0352***	0.0358***
			(0.00382)	(0.00382)	(0.00351)	(0.00351)	(0.00411)	(0.00411)
Hr		0.00999***		0.00956***		0.00901***		0.0101***
		(0.000973)		(0.000941)		(0.000904)		(0.000983)
Umd					-0.0197***	-0.0186***		
					(0.00162)	(0.00161)		
Rmw							-0.101***	-0.101***
							(0.00330)	(0.00330)
Cma							-0.226***	-0.228***
							(0.00533)	(0.00533)
Constant	-0.00128***	-0.00228***	-0.00128***	-0.00223***	-0.00120***	-0.00210***	-0.000883***	-0.00189***
	(2.41e-05)	(0.000100)	(2.36e-05)	(9.71e-05)	(2.32e-05)	(9.33e-05)	(2.40e-05)	(0.000101)
Observations	435,690	435,690	435,690	435,690	435,690	435,690	435,690	435,690
R-squared	0.731	0.732	0.733	0.733	0.733	0.734	0.737	0.737
Number of funds	10,282	10,282	10,282	10,282	10,282	10,282	10,282	10,282

The Table represents panel regressions with fund fixed effects. The hazard ratios are derived from the Cox model represented in Table 1.4. Columns (1) and (2) represents results for the Jensen (1968)'s one factor model and our modified version with just the hazard ratio. Columns (3) and (4) represents results for the Fama and French (1993)'s 3 factors model and our modified version with just the hazard ratio. Columns (5) and (6) represents results for the Carhart (1997)'s 4 factors model and our modified version with just the hazard ratio. Columns (7) and (8) represents results for the Fama and French (2015)'s 5 factors model and our modified version with just the hazard ratio. The sample period is July 2003 - December 2015. Robust standard errors in parentheses, *** indicates significance at 1%, ** indicates significance at 5% and * indicates significance at 10%. See the Appendix for a detailed description of all the variables.

1.6 Robustness Checks

In the present section we run several robustness check in order further test the results of the previous section. We will use several different specifications of the survival model. All the models tend to confirm the negative statistically significant interaction between the hazard ratio and both the market and the momentum factors. Moreover almost all the specifications show a positive and statistically significant interaction between the hazard ratio and the small minus big factor.

1.6.1 Proportional Hazards Assumption

In this subsection we first test the proportional hazard assumption that is implicit in the Cox model and we eventually modify the model in order to take into account of the violations of this hypothesis. Table 1.7 contains the test of the proportional hazards assumption based on the Schoenfeld residuals. Schoenfeld residuals allow to assess how the value of the covariate of a given fund is relative to a weighted mean of the values of the same covariate for the funds that are at risk at the evaluation moment. The test of the proportional hazards assumption is based on the principle that the estimated coefficient for a given regressor of the survival model does not have to be time varying. The real time varying coefficient can be approximated by the sum of the Schoenfeld residuals and the estimated coefficient. Hence the idea is to show that this sum is not time varying.

The test shows that the 1 period lag of the natural logarithm of the monthly total net assets ($Logmtnal1$), the 1 period lag of the monthly net asset values ($Mnavl1$), the 1 period lag of the turnover ratio multiplied by 100 ($Turn\_ratiol1100$) and the 1 period lag of the dummy variable related to the distribution of yields ($Dummyieldl1$) violate the proportional hazards assumption. The global test confirms the rejection of the proportional hazards hypothesis. We modify our model by interacting these variables with time in order to adjust for these violations. The results of the new specification of the survival model are included in Table 1.8. These results are similar to the ones of the static model with the exception that now the variable $Dummyieldl1$ is significant only in its interaction with time. Notice that the hazard ratios related to the interaction of the variables with time are always equal to 1 and consequently these interactions do not add any particular relevant information to the survival analysis. Consequently, our original specification, even if it contained violations of the proportional hazards assumption, was reliable.

For completeness we report the results of the second step when the hazard ratio is derived from the model adjusted for the violations of the proportional hazards hypothesis (Table 1.9). The results are again similar to the ones of the original specification with a

Table 1.7: Test of the proportional hazards assumption for the Cox model of the main specification

Variable	Rho	Chi2	Prob>Chi2
Mretl1100	-0,01887	1,66	19,76%
Mnavl1	0,07758	25,64	0,00%
Logmtnal1	0,07866	22,06	0,00%
Turn_ratiol1100	-0,07321	16,01	0,01%
Exp_ratiol1100	0,00214	0,02	89,60%
Dummyieldl1	0,04931	9,43	0,21%
Flowl1100	-0,02354	1,47	22,60%
Global Test		87,88	0,00%

The table represents the test of the proportional hazards assumption for the Cox model presented in Table 1.4. The test is based on the Schoenfeld residuals. See the Appendix for a detailed description of all the variables.

Table 1.8: Cox analysis and hazard ratio of the main specification adjusted for the proportional hazard assumption

Panel A: Cox Analysis

VARIABLES	(1) Hazard Ratios	(2) Hazard Ratios	(3) Coefficients	(4) Coefficients
Mretl1100	1.003		0.00308	
	(0.00385)		(0.00384)	
Mnavl1	0.965***	1.000***	-0.0360***	1.59e-06***
	(0.00417)	(4.38e-07)	(0.00433)	(4.38e-07)
Logmtnal1	0.584***	1.000***	-0.538***	1.22e-05***
	(0.0117)	(3.14e-06)	(0.0200)	(3.14e-06)
Turn_ratiol1100	1.000***	1.000	0.000255***	-2.37e-08
	(8.67e-05)	(2.05e-08)	(8.67e-05)	(2.05e-08)
Exp_ratiol1100	1.255***		0.227***	
	(0.0450)		(0.0358)	
Dummyieldl1	1.009	1.000***	0.00938	2.86e-05***
	(0.0582)	(1.05e-05)	(0.0576)	(1.05e-05)
Flowl1100	0.970***		-0.0301***	
	(0.00149)		(0.00153)	
Observations	435,644	435,644	435,644	435,644

Panel B: Hazard Ratio

Variable	Mean	Std. Dev.
Hr	0,06865	0,09563
Hr_surv	0,05673	0,08307
Hr_dead	0,10723	0,12024

The Table reports the results of the Cox analysis. Panel A reports the estimated hazard ratios (Columns (1) and (2)) relative to the variables used in the analysis and the interaction of some selected variables with time. Columns (3) and (4) reports the estimated coefficients relative to the variables used in the analysis and the interaction of some selected variables with time. The sample period is July 2003 - December 2015. Robust standard errors in parentheses, *** indicates significance at 1%, ** indicates significance at 5% and * indicates significance at 10%. Panel B represents the mean and the standard deviation of the hazard ratio for all the population of funds, for the survivor funds and for the dead funds. We denote as "survivor" a fund that is alive till December 2015 while we denote as "dead" a fund that disappears before December 2015. See the Appendix for a detailed description of all the variables.

negative and statistically significant interaction with the market factor and the momentum factor and a positive and statistically significant interaction with the small minus big factor.

From Table 1.10 is possible to notice that our results are again not driven by an attrition bias effect given that the factor loadings are almost not affected by the inclusion of just the hazard ratio variable although this latter variable is always statistically significant.

1.6.2 All Fund's Characteristics

We now introduce in the survival model the fund's characteristics that we have previously excluded, i.e. we include in the model the 1 period lag of the the management fee multiplied by 100 ($Mgmt\_feel1100$), the 1 period lag of the max 12b-1 fee multiplied by 100 ($Max\_12b1l1100$) and the 1 period lag of the actual 12b-1 fee multiplied by 100 ($Actual\_12b1l1100$). In the core analysis we excluded these variables because they seemed less reliable and more affected by outlier observations w.r.t. the other fund's characteristics. For the analysis of this subsection we further restrict the sample to 500.835 observations by dropping funds observations featured by a management fee greater than 5% or smaller that -5%.

The size, the NAV, the fund's flows, the expense ratio and the yields continue to have same impact on fund's survivorship (see Table 1.11). Among the new variables, the only one that turns out to be statistically significant is the management fee but its hazard ratio is roughly 1.

Table 1.12 shows that the results, in terms of statistical significance and direction of the interaction terms of the hazard ratios with the factors, are practically the same as before.

1.6.3 Factors and Gdp

We want to add to the survival model's estimation all the factors that are used in the second step of our approach and the gross domestic product (Gdp). We are not interested in the impact of these new variables on survivorship but this addition allow us to address the concern that the impact of our fund's characteristics may be driven by some macroeconomic variables. Table 1.13 testifies that this is not the case. In fact the survival results are unchanged with the greatest impact on survivorship due to the 1 period lag of the natural logarithm of the monthly total net assets and to the 1 period lag of the expense ratio multiplied by 100. The flow variable and the Nav tend to reduce the odds of survival while the dummy variable related to yield distribution has the usual negative effect on fund's survival.

Finally the results relative to the second step of the methodology are similar to previous ones w.r.t. the interaction terms (see Table 1.14) with the exception of the small minus big factor that in this case turns out to be not statistically significant in terms of its interaction with the hazard ratio.

1.6.4 Performance

In this subsection we want to study better the impact that performance has on mutual fund survivorship and specifically on our 2 step methodology. For this reason we substitute the 1 period lag of monthly returns multiplied by 100 ($Mretl1100$) with either the 1 period lag of monthly relative returns multiplied by 100 ($Retrelml1100$) or with the 1 period lag of the 1 year average of the monthly returns multiplied by 100 ($Mretm1y100$). Tables 1.15 and 1.17 shows that these new performance measures seems to have no impact on survivorship when the other fund's characteristics are taken into account. The introduction of these new variables leaves the effect of the other fund's characteristics on survival unchanged. Tables 1.16 and 1.18 present the results related to the factor models and also in this case we get estimations that in terms of their economic interpretation are equivalent to the ones discussed in the previous section.

1.6.5 No Dummy Yield

In this robustness check we want to exclude the dummy variable related to yield. Tables 1.19 and 1.20 shows that both the survival analysis and the factor analysis are robust to this new specification.

1.6.6 No Logarithm of Assets

We now make the same analysis discussed in the Results section without taking the natural logarithm of the size variable. A careful examination of Table 1.21 highlights how this choice underestimate the impact of the size variable and overestimates the impact of the expense ratio. Notice that now the hazard ratio of the dead funds is on average higher than 1 how it is natural to expect from an ex-ante viewpoint. This new specification of the survival model leaves the factor analysis' results equivalent to the ones previously discussed (See table 1.22).

1.6.7 Common Stocks

Several studies (see for instance Fama and French (2010)) tend to focus their attention to active funds and to funds that invests primarily on common stocks. For this reason we want to see if our results are robust to a sample reduction that focuses on funds whose investment on common stocks is higher than 90%. Now the sample is made up of 172609 observations. The survival analysis is not particularly influenced by this sample reduction with the exception of the dummy variable related to the yield distribution that now seems to have an higher statistically significant negative impact on survival (see Table 1.23).

The factor analysis highlights that when we focus on US funds that invests primarily on common stocks the interaction between the market factor and hazard ratio is not particularly robust. In particular this interaction is strongly statistically significant and negative only in the Carhart (1997)'s 4 factors model. On the other side the interactions of the small minus big factor and of the momentum factor with the hazard ratio are strongly statistically significant and with the same signs of the full sample. Table 1.24 shows also that the analysis on the reduced sample made the interaction of the high minus low factor with the hazard ratio always positive and strongly statistically significant.

This case is also an outstanding example of how important is to consider the interactions of the factors with the hazard ratio. In fact, if we focus our attention on the Carhart (1997)'s 4 factors model, we see that in the original model the momentum factor's coefficient is not statistically significant (this result is consistent with the estimations of Fama and French (2010)) while in the modified version of the model the momentum turns out to be strongly statistically significant. In this case only by using a model that takes into account the dynamics of mutual funds strategies it is possible to disentangle the effect of the momentum factor.

1.6.8 Big Funds

Given the important role that size has on mutual fund survivorship, smaller funds seems to be particularly affected by the survivorship and attrition bias problems. For this reason as a further robustness checks we keep in the sample only funds that have assets under management of at least 100 mln \$. The reduced sample contains 290879 observations. The survival analysis is not influenced in terms of the statistical significance and of the signs of the effects of the fund's characteristics (see Table 1.25). The reduction seems to overestimate the impact of the size variable and of the expense ratio w.r.t. the previous analysis.

The sample reduction has a peculiar influence on the factor analysis. While the statistical significance and the signs of the interaction terms relative to the market and momentum factors are not effected, the interactions terms of the other factors are strongly influenced by this sample choice. The small minus big factor interaction is not as robust as before and it turns out to be strongly statistically significant and positive only in the Fama and French (2015)'s 5 factors model. On the other side the high minus low factor interaction term, with the exception of the Fama and French (2015)'s 5 factors model, is now strongly statistically significant and positive. Table 1.26 emphasizes how, in this case, also the interaction terms relative to the robust minus weak factor and to the conservative minus aggressive factor are statistically significant and positive.

1.6.9 Probit

We now want to substitute the semiparametric Cox model with the parametric model most frequently used in the survivorship bias literature, i.e. the Probit model. In this case we can include the age variable, differently w.r.t the Cox case in which the inclusion of this fund's characteristic didn't allow the maximization algorithm to converge. We use as dependent variable a dummy variable relative to fund's failure, i.e. a dummy variable that is equal to 1 when a fund closes, and we include in the analysis all the fund's characteristics. Table 1.27 presents the results. The size, the NAV, the flows, the expense ratio and the dummy variable relative to yield are statistically significant and have the same impact on survivorship as the one estimated by the Cox analysis

Differently w.r.t. the Cox model the management fee variable is not statistically significant while the max 12b-1 fee variable has a negative statistically significant impact on survivorship. The age variable although statistically significant has a positive counter intuitive sign. Notice that this latter result is probably due to the interaction of the age variable with the other fund's characteristics. In fact, if we exclude all other fund's characteristics but age and returns[16], the age variable has the natural negative statistically significant sign.

In the second step of our approach the role that was previously played by the hazard ratio is now taken by the predicted probability estimated by the Probit model. The results of the factor analysis, reported in Table 1.28, are, from an economic perspective, totally equivalent to the ones of the original specification. Also in this case (see Table 1.29) the results seem to be driven by a structural difference between the strategy of survivor and dead funds and not from the attrition bias.

[16]This results are not reported.

1.6.10 Logit

In this section, we use in the first step of our methodology a Logit model instead of the Cox one. The results, described in Table 1.30, are equivalent to the ones described in the Probit subsection with the age variable that continues to have the unexpected positive statistically significant sign.

In the second step of our approach, also in this case, the place of the hazard ratio is taken by the estimated probability from the Logit model and the results of the factor analysis are, as expected, equivalent to the ones with the Cox and Probit specification (see Table 1.31). The results are not driven by the attrition bias as Table 1.32 testifies.

1.6.11 Lehman Brothers Collapse

In this subsection we control for the event of Lehman collapse. We address this issue by restricting the analysis to the period from October 2008 to December 2015. In the estimated Cox model the expense ratio looses its significance while the effect of the other variables does not change (Table 1.33). This sample restriction does not have any impact on the negative interaction of the hazard ratio with both the market factor and the momentum factor. Also the positive interaction between the hazard ratio and the size factor continues to hold toward the same direction in a statistically significant way (see Table 1.34).

1.6.12 Squared Hazard Ratio

In this subsection we try to test a different functional form for the hazard ratio. In particular we introduce in our econometric model a squared hazard ratio. The model highlights how the main effects of the odds of survival are captured by a linear model while the impact of the squared component is really small although statistically significant (see Table 1.35).

Table 1.9: Factor analysis relative to the Cox model of the main specification adjusted for the proportional hazard assumption

VARIABLES	(1) FF1	(2) FF1HR	(3) FF3	(4) FF3HR	(5) FF4	(6) FF4HR	(7) FF5	(8) FF5HR
Mktrf	0.963***	0.975***	0.949***	0.965***	0.943***	0.963***	0.925***	0.941***
	(0.00370)	(0.00446)	(0.00351)	(0.00425)	(0.00344)	(0.00417)	(0.00333)	(0.00409)
Smb			0.0877***	0.0733***	0.0916***	0.0758***	0.0739***	0.0578***
			(0.00431)	(0.00537)	(0.00428)	(0.00535)	(0.00439)	(0.00546)
Hml			-0.0309***	-0.0379***	-0.0436***	-0.0439***	0.0352***	0.0336***
			(0.00382)	(0.00487)	(0.00351)	(0.00470)	(0.00411)	(0.00527)
Umd					-0.0197***	-0.00886***		
					(0.00162)	(0.00239)		
Mktrf*Hr		-0.173***		-0.241***		-0.296***		-0.238***
		(0.0396)		(0.0383)		(0.0364)		(0.0372)
Smb*Hr				0.203***		0.215***		0.229***
				(0.0410)		(0.0409)		(0.0413)
Hml*Hr				0.125***		0.0517		0.0666
				(0.0445)		(0.0453)		(0.0499)
Umd*Hr						-0.122***		
						(0.0236)		
Hr		0.0156***		0.0157***		0.0146***		0.0163***
		(0.00138)		(0.00139)		(0.00132)		(0.00144)
Rmw							-0.101***	-0.107***
							(0.00330)	(0.00490)
Cma							-0.226***	-0.238***
							(0.00533)	(0.00732)
Rmw*Hr								0.0905
								(0.0565)
Cma*Hr								0.122*
								(0.0702)
Constant	-0.00128***	-0.00235***	-0.00128***	-0.00235***	-0.00120***	-0.00221***	-0.000883***	-0.00199***
	(2.41e-05)	(9.74e-05)	(2.36e-05)	(9.84e-05)	(2.32e-05)	(9.41e-05)	(2.40e-05)	(0.000102)
Observations	435,690	435,690	435,690	435,690	435,690	435,690	435,690	435,690
R-squared	0.731	0.732	0.733	0.734	0.733	0.734	0.737	0.737
Number of funds	10,282	10,282	10,282	10,282	10,282	10,282	10,282	10,282

The Table represents panel regressions with fund fixed effects. The hazard ratios are derived from the Cox model represented in Table 1.8. Columns (1) and (2) represents results for the Jensen (1968)'s one factor model and our modified version with the hazard ratio and the interaction of the factor with the hazard ratio. Columns (3) and (4) represents results for the Fama and French (1993)'s 3 factors model and our modified version with the hazard ratio and the interactions of the factors with the hazard ratio. Columns (5) and (6) represents results for the Carhart (1997)'s 4 factors model and our modified version with the hazard ratio and the interactions of the factors with the hazard ratio. Columns (7) and (8) represents results for the Fama and French (2015)'s 5 factors model and our modified version with the hazard ratio and the interactions of the factors with the hazard ratio. The sample period is July 2003 - December 2015. Robust standard errors in parentheses, *** indicates significance at 1%, ** indicates significance at 5% and * indicates significance at 10%. See the Appendix for a detailed description of all the variables.

Table 1.10: Factor analysis with just the hazard ratio relative to the Cox model of the main specification adjusted for the proportional hazard assumption

VARIABLES	(1) FF1	(2) FF1JHR	(3) FF3	(4) FF3JHR	(5) FF4	(6) FF4JHR	(7) FF5	(8) FF5JHR
Mktrf	0.963***	0.962***	0.949***	0.948***	0.943***	0.942***	0.925***	0.925***
	(0.00370)	(0.00369)	(0.00351)	(0.00351)	(0.00344)	(0.00344)	(0.00333)	(0.00333)
Smb			0.0877***	0.0871***	0.0916***	0.0908***	0.0739***	0.0734***
			(0.00431)	(0.00431)	(0.00428)	(0.00428)	(0.00439)	(0.00438)
Hml			-0.0309***	-0.0308***	-0.0436***	-0.0427***	0.0352***	0.0356***
			(0.00382)	(0.00382)	(0.00351)	(0.00351)	(0.00411)	(0.00411)
Hr		0.0135***		0.0130***		0.0122***		0.0136***
		(0.00138)		(0.00134)		(0.00129)		(0.00139)
Umd					-0.0197***	-0.0185***		
					(0.00162)	(0.00161)		
Rmw							-0.101***	-0.101***
							(0.00330)	(0.00330)
Cma							-0.226***	-0.227***
							(0.00533)	(0.00533)
Constant	-0.00128***	-0.00220***	-0.00128***	-0.00216***	-0.00120***	-0.00204***	-0.000883***	-0.00181***
	(2.41e-05)	(9.85e-05)	(2.36e-05)	(9.59e-05)	(2.32e-05)	(9.23e-05)	(2.40e-05)	(9.84e-05)
Observations	435,690	435,690	435,690	435,690	435,690	435,690	435,690	435,690
R-squared	0.731	0.732	0.733	0.733	0.733	0.734	0.737	0.737
Number of funds	10,282	10,282	10,282	10,282	10,282	10,282	10,282	10,282

The Table represents panel regressions with fund fixed effects. The hazard ratios are derived from the Cox model represented in Table 1.8. Columns (1) and (2) represents results for the Jensen (1968)'s one factor model and our modified version with just the hazard ratio. Columns (3) and (4) represents results for the Fama and French (1993)'s 3 factors model and our modified version with just the hazard ratio. Columns (5) and (6) represents results for the Carhart (1997)'s 4 factors model and our modified version with just the hazard ratio. Columns (7) and (8) represents results for the Fama and French (2015)'s 5 factors model and our modified version with just the hazard ratio. The sample period is July 2003 - December 2015. Robust standard errors in parentheses, *** indicates significance at 1%, ** indicates significance at 5% and * indicates significance at 10%. See the Appendix for a detailed description of all the variables.

Table 1.11: Cox analysis and hazard ratio of the main specification with all fund's characteristics

Panel A: Cox Analysis

VARIABLES	(1) Hazard Ratios	(2) Coefficients
Mretl1100	1.003	0.00302
	(0.00422)	(0.00420)
Mnavl1	0.975***	-0.0250***
	(0.00325)	(0.00333)
Logmtnal1	0.637***	-0.450***
	(0.0109)	(0.0170)
Turn_ratiol1100	1.000*	0.000151*
	(8.71e-05)	(8.71e-05)
Exp_ratiol1100	1.286***	0.252***
	(0.0615)	(0.0478)
Dummyieldl1	1.123***	0.116***
	(0.0500)	(0.0446)
Mgmt_feel1100	0.999***	-0.00145***
	(0.000318)	(0.000318)
Max_12b1l1100	1.336	0.290
	(0.248)	(0.185)
Actual_12b1l1100	1.154	0.143
	(0.227)	(0.197)
Flowl1100	0.966***	-0.0346***
	(0.00157)	(0.00163)
Observations	339,657	339,657

Panel B: Hazard Ratio

Variable	Mean	Std. Dev.
Hr	0,14147	0,1911
Hr_surv	0,11865	0,161
Hr_dead	0,20995	0,24916

The Table reports the results of the Cox analysis. Panel A reports the estimated hazard ratios and coefficients (Columns (1) and (2)) relative to the variables used in the analysis. The sample period is July 2003 - December 2015. Robust standard errors in parentheses, *** indicates significance at 1%, ** indicates significance at 5% and * indicates significance at 10%. Panel B represents the mean and the standard deviation of the hazard ratio for all the population of funds, for the survivor funds and for the dead funds. We denote as "survivor" a fund that is alive till December 2015 while we denote as "dead" a fund that disappears before December 2015. See the Appendix for a detailed description of all the variables.

Table 1.12: Factor analysis relative to the Cox model of the main specification with all fund's characteristics

VARIABLES	(1) FF1	(2) FF1HR	(3) FF3	(4) FF3HR	(5) FF4	(6) FF4HR	(7) FF5	(8) FF5HR
Mktrf	0.954***	0.968***	0.941***	0.958***	0.935***	0.958***	0.917***	0.934***
	(0.00424)	(0.00501)	(0.00405)	(0.00484)	(0.00397)	(0.00488)	(0.00385)	(0.00469)
Smb			0.0845***	0.0744***	0.0882***	0.0750***	0.0695***	0.0586***
			(0.00471)	(0.00609)	(0.00469)	(0.00618)	(0.00481)	(0.00612)
Hml			-0.0385***	-0.0472***	-0.0506***	-0.0519***	0.0282***	0.0262***
			(0.00435)	(0.00591)	(0.00401)	(0.00593)	(0.00468)	(0.00653)
Umd					-0.0188***	-0.00655**		
					(0.00180)	(0.00281)		
Mktrf*Hr		-0.0979***		-0.126***		-0.163***		-0.120***
		(0.0216)		(0.0213)		(0.0217)		(0.0210)
Smb*Hr				0.0703***		0.0895***		0.0767***
				(0.0243)		(0.0260)		(0.0236)
Hml*Hr				0.0730**		0.0301		0.0338
				(0.0290)		(0.0307)		(0.0339)
Umd*Hr						-0.0734***		
						(0.0140)		
Hr		0.00642***		0.00626***		0.00590***		0.00649***
		(0.000766)		(0.000788)		(0.000749)		(0.000808)
Rmw							-0.105***	-0.112***
							(0.00375)	(0.00643)
Cma							-0.227***	-0.243***
							(0.00601)	(0.00884)
Rmw*Hr								0.0486
								(0.0380)
Cma*Hr								0.0944**
								(0.0455)
Constant	-0.00145***	-0.00235***	-0.00145***	-0.00233***	-0.00138***	-0.00222***	-0.00104***	-0.00195***
	(2.78e-05)	(0.000110)	(2.73e-05)	(0.000113)	(2.68e-05)	(0.000109)	(2.76e-05)	(0.000117)
Observations	339,657	339,657	339,657	339,657	339,657	339,657	339,657	339,657
R-squared	0.724	0.725	0.726	0.727	0.726	0.727	0.730	0.730
Number of funds	8,587	8,587	8,587	8,587	8,587	8,587	8,587	8,587

The Table represents panel regressions with fund fixed effects. The hazard ratios are derived from the Cox model represented in Table 1.11. Columns (1) and (2) represents results for the Jensen (1968)'s one factor model and our modified version with the hazard ratio and the interaction of the factor with the hazard ratio. Columns (3) and (4) represents results for the Fama and French (1993)'s 3 factors model and our modified version with the hazard ratio and the interactions of the factors with the hazard ratio. Columns (5) and (6) represents results for the Carhart (1997)'s 4 factors model and our modified version with the hazard ratio and the interactions of the factors with the hazard ratio. Columns (7) and (8) represents results for the Fama and French (2015)'s 5 factors model and our modified version with the hazard ratio and the interactions of the factors with the hazard ratio. The sample period is July 2003 - December 2015. Robust standard errors in parentheses, *** indicates significance at 1%, ** indicates significance at 5% and * indicates significance at 10%. See the Appendix for a detailed description of all the variables.

Table 1.13: Cox analysis and hazard ratio of the main specification with all fund's characteristics, factors and GDP

Panel A: Cox Analysis

VARIABLES	(1) Hazard Ratios	(2) Coefficients
Mretl1100	0.998	-0.00164
	(0.00455)	(0.00456)
Mnavl1	0.974***	-0.0259***
	(0.00333)	(0.00342)
Logmtnal1	0.634***	-0.456***
	(0.0108)	(0.0170)
Turn_ratiol1100	1.000*	0.000153*
	(8.59e-05)	(8.59e-05)
Exp_ratiol1100	1.254***	0.226***
	(0.0597)	(0.0476)
Dummyieldl1	1.097**	0.0925**
	(0.0494)	(0.0450)
Mgmt_feel1100	0.999***	-0.00139***
	(0.000318)	(0.000318)
Max_12b1l1100	1.322	0.279
	(0.240)	(0.182)
Actual_12b1l1100	1.146	0.136
	(0.221)	(0.193)
Flowl1100	0.966***	-0.0347***
	(0.00155)	(0.00160)
Mktrf100	0.968***	-0.0325***
	(0.00570)	(0.00589)
Smb100	1.039***	0.0379***
	(0.00962)	(0.00926)
Hml100	1.005	0.00540
	(0.0105)	(0.0105)
Umd100	1.003	0.00296
	(0.00524)	(0.00522)
Rmw100	0.993	-0.00692
	(0.0136)	(0.0137)
Cma100	1.136***	0.128***
	(0.0184)	(0.0162)
Gdp100	1.028***	0.0272***
	(0.00779)	(0.00758)
Observations	339,657	339,657

Panel B: Hazard Ratio

Variable	Mean	Std. Dev.
Hr	0,14417	0,20196
Hr_surv	0,11939	0,16517
Hr_dead	0,21855	0,27192

The Table reports the results of the Cox analysis. Panel A reports the estimated hazard ratios and coefficients (Columns (1) and (2)) relative to the variables used in the analysis. The sample period is July 2003 - December 2015. Robust standard errors in parentheses, *** indicates significance at 1%, ** indicates significance at 5% and * indicates significance at 10%. Panel B represents the mean and the standard deviation of the hazard ratio for all the population of funds, for the survivor funds and for the dead funds. We denote as "survivor" a fund that is alive till December 2015 while we denote as "dead" a fund that disappears before December 2015. See the Appendix for a detailed description of all the variables.

Table 1.14: Factor analysis relative to the Cox model of the main specification with all fund's characteristics, factors and GDP

VARIABLES	(1) FF1	(2) FF1HR	(3) FF3	(4) FF3HR	(5) FF4	(6) FF4HR	(7) FF5	(8) FF5HR
Mktrf	0.954***	0.959***	0.941***	0.947***	0.935***	0.944***	0.917***	0.925***
	(0.00424)	(0.00488)	(0.00405)	(0.00471)	(0.00397)	(0.00466)	(0.00385)	(0.00460)
Smb			0.0845***	0.0818***	0.0882***	0.0826***	0.0695***	0.0681***
			(0.00471)	(0.00563)	(0.00469)	(0.00565)	(0.00481)	(0.00594)
Hml			-0.0385***	-0.0438***	-0.0506***	-0.0522***	0.0282***	0.0259***
			(0.00435)	(0.00529)	(0.00401)	(0.00507)	(0.00468)	(0.00599)
Umd					-0.0188***	-0.0116***		
					(0.00180)	(0.00233)		
Mktrf*Hr		-0.0305		-0.0368**		-0.0548***		-0.0428**
		(0.0189)		(0.0185)		(0.0181)		(0.0189)
Smb*Hr				0.00968		0.0334		-0.0127
				(0.0192)		(0.0203)		(0.0223)
Hml*Hr				0.0284		0.00167		0.0187
				(0.0218)		(0.0214)		(0.0285)
Umd*Hr						-0.0525***		
						(0.0122)		
Hr		0.00320***		0.00238***		0.00278***		0.00576***
		(0.000456)		(0.000433)		(0.000445)		(0.000705)
Rmw							-0.105***	-0.103***
							(0.00375)	(0.00553)
Cma							-0.227***	-0.233***
							(0.00601)	(0.00823)
Rmw*Hr								-0.0210
								(0.0303)
Cma*Hr								-0.0366
								(0.0396)
Constant	-0.00145***	-0.00193***	-0.00145***	-0.00182***	-0.00138***	-0.00180***	-0.00104***	-0.00189***
	(2.78e-05)	(7.37e-05)	(2.73e-05)	(6.76e-05)	(2.68e-05)	(6.92e-05)	(2.76e-05)	(0.000104)
Observations	339,657	339,657	339,657	339,657	339,657	339,657	339,657	339,657
R-squared	0.724	0.724	0.726	0.726	0.726	0.726	0.730	0.730
Number of funds	8,587	8,587	8,587	8,587	8,587	8,587	8,587	8,587

The Table represents panel regressions with fund fixed effects. The hazard ratios are derived from the Cox model represented in Table 1.13. Columns (1) and (2) represents results for the Jensen (1968)'s one factor model and our modified version with the hazard ratio and the interaction of the factor with the hazard ratio. Columns (3) and (4) represents results for the Fama and French (1993)'s 3 factors model and our modified version with the hazard ratio and the interactions of the factors with the hazard ratio. Columns (5) and (6) represents results for the Carhart (1997)'s 4 factors model and our modified version with the hazard ratio and the interactions of the factors with the hazard ratio. Columns (7) and (8) represents results for the Fama and French (2015)'s 5 factors model and our modified version with the hazard ratio and the interactions of the factors with the hazard ratio. The sample period is July 2003 - December 2015. Robust standard errors in parentheses, *** indicates significance at 1%, ** indicates significance at 5% and * indicates significance at 10%. See the Appendix for a detailed description of all the variables.

Table 1.15: Cox analysis and hazard ratio of the main specification with relative returns

Panel A: Cox Analysis

VARIABLES	(1) Hazard Ratios	(2) Coefficients
Retrelml1100	1.001	0.00113
	(0.00720)	(0.00719)
Mnavl1	0.974***	-0.0259***
	(0.00292)	(0.00300)
Logmtnal1	0.617***	-0.482***
	(0.00888)	(0.0144)
Turn_ratiol1100	1.000*	0.000156*
	(8.09e-05)	(8.09e-05)
Exp_ratiol1100	1.258***	0.229***
	(0.0452)	(0.0359)
Dummyieldl1	1.131***	0.123***
	(0.0458)	(0.0405)
Flowl1100	0.970***	-0.0305***
	(0.00148)	(0.00153)
Observations	435,644	435,644

Panel B: Hazard Ratio

Variable	Mean	Std. Dev.
Hr	0,09999	0,1237
Hr_surv	0,08438	0,1086
Hr_dead	0,15051	0,15268

The Table reports the results of the Cox analysis. Panel A reports the estimated hazard ratios and coefficients (Columns (1) and (2)) relative to the variables used in the analysis. The sample period is July 2003 - December 2015. Robust standard errors in parentheses, *** indicates significance at 1%, ** indicates significance at 5% and * indicates significance at 10%. Panel B represents the mean and the standard deviation of the hazard ratio for all the population of funds, for the survivor funds and for the dead funds. We denote as "survivor" a fund that is alive till December 2015 while we denote as "dead" a fund that disappears before December 2015. See the Appendix for a detailed description of all the variables.

Table 1.16: Factor analysis relative to the Cox model of the main specification with relative returns

VARIABLES	(1) FF1	(2) FF1HR	(3) FF3	(4) FF3HR	(5) FF4	(6) FF4HR	(7) FF5	(8) FF5HR
Mktrf	0.963***	0.977***	0.949***	0.967***	0.943***	0.966***	0.925***	0.943***
	(0.00370)	(0.00459)	(0.00351)	(0.00438)	(0.00344)	(0.00434)	(0.00333)	(0.00423)
Smb			0.0877***	0.0718***	0.0916***	0.0738***	0.0739***	0.0566***
			(0.00431)	(0.00557)	(0.00428)	(0.00553)	(0.00439)	(0.00567)
Hml			-0.0309***	-0.0397***	-0.0436***	-0.0453***	0.0352***	0.0334***
			(0.00382)	(0.00508)	(0.00351)	(0.00508)	(0.00411)	(0.00558)
Umd					-0.0197***	-0.00715***		
					(0.00162)	(0.00235)		
Mktrf*Hr		-0.135***		-0.187***		-0.230***		-0.185***
		(0.0294)		(0.0284)		(0.0278)		(0.0278)
Smb*Hr				0.155***		0.168***		0.170***
				(0.0311)		(0.0309)		(0.0317)
Hml*Hr				0.105***		0.0504		0.0516
				(0.0345)		(0.0366)		(0.0393)
Umd*Hr						-0.103***		
						(0.0168)		
Hr		0.0112***		0.0112***		0.0105***		0.0118***
		(0.00100)		(0.00101)		(0.000967)		(0.00105)
Rmw							-0.101***	-0.108***
							(0.00330)	(0.00536)
Cma							-0.226***	-0.241***
							(0.00533)	(0.00789)
Rmw*Hr								0.0766*
								(0.0442)
Cma*Hr								0.111**
								(0.0550)
Constant	-0.00128***	-0.00240***	-0.00128***	-0.00240***	-0.00120***	-0.00227***	-0.000883***	-0.00206***
	(2.41e-05)	(0.000103)	(2.36e-05)	(0.000103)	(2.32e-05)	(9.92e-05)	(2.40e-05)	(0.000107)
Observations	435,690	435,690	435,690	435,690	435,690	435,690	435,690	435,690
R-squared	0.731	0.732	0.733	0.734	0.733	0.734	0.737	0.737
Number of funds	10,282	10,282	10,282	10,282	10,282	10,282	10,282	10,282

The Table represents panel regressions with fund fixed effects. The hazard ratios are derived from the Cox model represented in Table 1.15. Columns (1) and (2) represents results for the Jensen (1968)'s one factor model and our modified version with the hazard ratio and the interaction of the factor with the hazard ratio. Columns (3) and (4) represents results for the Fama and French (1993)'s 3 factors model and our modified version with the hazard ratio and the interactions of the factors with the hazard ratio. Columns (5) and (6) represents results for the Carhart (1997)'s 4 factors model and our modified version with the hazard ratio and the interactions of the factors with the hazard ratio. Columns (7) and (8) represents results for the Fama and French (2015)'s 5 factors model and our modified version with the hazard ratio and the interactions of the factors with the hazard ratio. The sample period is July 2003 - December 2015. Robust standard errors in parentheses, *** indicates significance at 1%, ** indicates significance at 5% and * indicates significance at 10%. See the Appendix for a detailed description of all the variables.

Table 1.17: Cox analysis and hazard ratio of the main specification with 1 year average returns

Panel A: Cox Analysis

VARIABLES	(1) Hazard Ratios	(2) Coefficients
Mretm1y100	0.993	-0.00676
	(0.0119)	(0.0120)
Mnavl1	0.976***	-0.0245***
	(0.00341)	(0.00349)
Logmtnal1	0.600***	-0.511***
	(0.00991)	(0.0165)
Turn_ratiol1100	1.000	0.000146
	(9.20e-05)	(9.20e-05)
Exp_ratiol1100	1.213***	0.193***
	(0.0519)	(0.0428)
Dummyieldl1	1.172***	0.159***
	(0.0543)	(0.0463)
Flowl1100	0.970***	-0.0306***
	(0.00163)	(0.00168)
Observations	356,827	356,827

Panel B: Hazard Ratio

Variable	Mean	Std. Dev.
Hr	.08173	.1095779
Hr_surv	.0691737	.0969462
Hr_dead	.128239	.137562

The Table reports the results of the Cox analysis. Panel A reports the estimated hazard ratios and coefficients (Columns (1) and (2)) relative to the variables used in the analysis. The sample period is July 2003 - December 2015. Robust standard errors in parentheses, *** indicates significance at 1%, ** indicates significance at 5% and * indicates significance at 10%. Panel B represents the mean and the standard deviation of the hazard ratio for all the population of funds, for the survivor funds and for the dead funds. We denote as "survivor" a fund that is alive till December 2015 while we denote as "dead" a fund that disappears before December 2015. See the Appendix for a detailed description of all the variables.

Table 1.18: Factor analysis relative to the Cox model of the main specification with 1 year average returns

VARIABLES	(1) FF1	(2) FF1HR	(3) FF3	(4) FF3HR	(5) FF4	(6) FF4HR	(7) FF5	(8) FF5HR
Mktrf	0.967***	0.977***	0.955***	0.969***	0.947***	0.966***	0.931***	0.945***
	(0.00397)	(0.00481)	(0.00378)	(0.00460)	(0.00371)	(0.00456)	(0.00358)	(0.00443)
Smb			0.0802***	0.0649***	0.0837***	0.0680***	0.0646***	0.0479***
			(0.00466)	(0.00587)	(0.00464)	(0.00581)	(0.00478)	(0.00600)
Hml			-0.0350***	-0.0454***	-0.0529***	-0.0558***	0.0386***	0.0357***
			(0.00414)	(0.00529)	(0.00382)	(0.00538)	(0.00438)	(0.00570)
Umd					-0.0266***	-0.0145***		
					(0.00173)	(0.00252)		
Mktrf*Hr		-0.115***		-0.177***		-0.230***		-0.178***
		(0.0357)		(0.0345)		(0.0340)		(0.0341)
Smb*Hr				0.182***		0.179***		0.200***
				(0.0377)		(0.0373)		(0.0387)
Hml*Hr				0.142***		0.0735		0.0724
				(0.0411)		(0.0452)		(0.0453)
Umd*Hr						-0.117***		
						(0.0207)		
Hr		0.0108***		0.0111***		0.00980***		0.0118***
		(0.00118)		(0.00120)		(0.00110)		(0.00125)
Rmw							-0.0920***	-0.100***
							(0.00354)	(0.00568)
Cma							-0.257***	-0.273***
							(0.00608)	(0.00868)
Rmw*Hr								0.0967*
								(0.0564)
Cma*Hr								0.152**
								(0.0676)
Constant	-0.00143***	-0.00231***	-0.00142***	-0.00232***	-0.00133***	-0.00215***	-0.00106***	-0.00202***
	(2.41e-05)	(9.93e-05)	(2.34e-05)	(0.000100)	(2.32e-05)	(9.38e-05)	(2.35e-05)	(0.000105)
Observations	356,866	356,866	356,866	356,866	356,866	356,866	356,866	356,866
R-squared	0.738	0.738	0.739	0.740	0.740	0.740	0.744	0.744
Number of funds	8,396	8,396	8,396	8,396	8,396	8,396	8,396	8,396

The Table represents panel regressions with fund fixed effects. The hazard ratios are derived from the Cox model represented in Table 1.17. Columns (1) and (2) represents results for the Jensen (1968)'s one factor model and our modified version with the hazard ratio and the interaction of the factor with the hazard ratio. Columns (3) and (4) represents results for the Fama and French (1993)'s 3 factors model and our modified version with the hazard ratio and the interactions of the factors with the hazard ratio. Columns (5) and (6) represents results for the Carhart (1997)'s 4 factors model and our modified version with the hazard ratio and the interactions of the factors with the hazard ratio. Columns (7) and (8) represents results for the Fama and French (2015)'s 5 factors model and our modified version with the hazard ratio and the interactions of the factors with the hazard ratio. The sample period is July 2003 - December 2015. Robust standard errors in parentheses, *** indicates significance at 1%, ** indicates significance at 5% and * indicates significance at 10%. See the Appendix for a detailed description of all the variables.

Table 1.19: Cox analysis and hazard ratio of the main specification with no dummy yield

Panel A: Cox Analysis

VARIABLES	(1) Hazard Ratios	(2) Coefficients
Mretl1100	1.003	0.00307
	(0.00386)	(0.00385)
Mnavl1	0.974***	-0.0264***
	(0.00291)	(0.00299)
Logmtnal1	0.617***	-0.482***
	(0.00887)	(0.0144)
Turn_ratiol1100	1.000*	0.000149*
	(8.09e-05)	(8.09e-05)
Exp_ratiol1100	1.228***	0.205***
	(0.0426)	(0.0347)
Flowl1100	0.970***	-0.0304***
	(0.00148)	(0.00153)
Observations	435,644	435,644

Panel B: Hazard Ratio

Variable	Mean	Std. Dev.
Hr	0,09239	0,11412
Hr_surv	0,07783	0,09998
Hr_dead	0,13951	0,14112

The Table reports the results of the Cox analysis. Panel A reports the estimated hazard ratios and coefficients (Columns (1) and (2)) relative to the variables used in the analysis. The sample period is July 2003 - December 2015. Robust standard errors in parentheses, *** indicates significance at 1%, ** indicates significance at 5% and * indicates significance at 10%. Panel B represents the mean and the standard deviation of the hazard ratio for all the population of funds, for the survivor funds and for the dead funds. We denote as "survivor" a fund that is alive till December 2015 while we denote as "dead" a fund that disappears before December 2015. See the Appendix for a detailed description of all the variables.

Table 1.20: Factor analysis relative to the Cox model of the main specification with no dummy yield

VARIABLES	(1) FF1	(2) FF1HR	(3) FF3	(4) FF3HR	(5) FF4	(6) FF4HR	(7) FF5	(8) FF5HR
Mktrf	0.963***	0.975***	0.949***	0.965***	0.943***	0.964***	0.925***	0.942***
	(0.00370)	(0.00460)	(0.00351)	(0.00440)	(0.00344)	(0.00434)	(0.00333)	(0.00425)
Smb			0.0877***	0.0706***	0.0916***	0.0725***	0.0739***	0.0550***
			(0.00431)	(0.00561)	(0.00428)	(0.00556)	(0.00439)	(0.00568)
Hml			-0.0309***	-0.0389***	-0.0436***	-0.0439***	0.0352***	0.0331***
			(0.00382)	(0.00507)	(0.00351)	(0.00506)	(0.00411)	(0.00551)
Umd					-0.0197***	-0.00688***		
					(0.00162)	(0.00238)		
Mktrf*Hr		-0.127***		-0.183***		-0.231***		-0.183***
		(0.0317)		(0.0308)		(0.0301)		(0.0302)
Smb*Hr				0.181***		0.196***		0.200***
				(0.0347)		(0.0344)		(0.0349)
Hml*Hr				0.0997***		0.0354		0.0492
				(0.0372)		(0.0394)		(0.0417)
Umd*Hr						-0.111***		
						(0.0182)		
Hr		0.0123***		0.0125***		0.0118***		0.0129***
		(0.00111)		(0.00114)		(0.00108)		(0.00118)
Rmw							-0.101***	-0.107***
							(0.00330)	(0.00534)
Cma							-0.226***	-0.239***
							(0.00533)	(0.00786)
Rmw*Hr								0.0639
								(0.0480)
Cma*Hr								0.107*
								(0.0600)
Constant	-0.00128***	-0.00241***	-0.00128***	-0.00242***	-0.00120***	-0.00231***	-0.000883***	-0.00207***
	(2.41e-05)	(0.000105)	(2.36e-05)	(0.000107)	(2.32e-05)	(0.000102)	(2.40e-05)	(0.000110)
Observations	435,690	435,690	435,690	435,690	435,690	435,690	435,690	435,690
R-squared	0.731	0.732	0.733	0.734	0.733	0.734	0.737	0.737
Number of funds	10,282	10,282	10,282	10,282	10,282	10,282	10,282	10,282

The Table represents panel regressions with fund fixed effects. The hazard ratios are derived from the Cox model represented in Table 1.19. Columns (1) and (2) represents results for the Jensen (1968)'s one factor model and our modified version with the hazard ratio and the interaction of the factor with the hazard ratio. Columns (3) and (4) represents results for the Fama and French (1993)'s 3 factors model and our modified version with the hazard ratio and the interactions of the factors with the hazard ratio. Columns (5) and (6) represents results for the Carhart (1997)'s 4 factors model and our modified version with the hazard ratio and the interactions of the factors with the hazard ratio. Columns (7) and (8) represents results for the Fama and French (2015)'s 5 factors model and our modified version with the hazard ratio and the interactions of the factors with the hazard ratio. The sample period is July 2003 - December 2015. Robust standard errors in parentheses, *** indicates significance at 1%, ** indicates significance at 5% and * indicates significance at 10%. See the Appendix for a detailed description of all the variables.

Table 1.21: Cox analysis and hazard ratio of the main specification with no logarithm of assets

Panel A: Cox Analysis

VARIABLES	(1) Hazard Ratios	(2) Coefficients
Mretl1100	1.002	0.00225
	(0.00386)	(0.00385)
Mnavl1	0.968***	-0.0320***
	(0.00301)	(0.00311)
Mtnal1	0.999***	-0.00101***
	(0.000118)	(0.000118)
Turn_ratiol1100	1.000***	0.000232***
	(7.46e-05)	(7.46e-05)
Exp_ratiol1100	1.433***	0.360***
	(0.0488)	(0.0340)
Dummyieldl1	1.136***	0.128***
	(0.0443)	(0.0390)
Flowl1100	0.964***	-0.0364***
	(0.00149)	(0.00155)
Observations	435,644	435,644

Panel B: Hazard Ratio

Variable	Mean	Std. Dev.
Hr	0,73246	0,7066
Hr_surv	0,63757	0,62821
Hr_dead	1,0396	0,84527

The Table reports the results of the Cox analysis. Panel A reports the estimated hazard ratios and coefficients (Columns (1) and (2)) relative to the variables used in the analysis. The sample period is July 2003 - December 2015. Robust standard errors in parentheses, *** indicates significance at 1%, ** indicates significance at 5% and * indicates significance at 10%. Panel B represents the mean and the standard deviation of the hazard ratio for all the population of funds, for the survivor funds and for the dead funds. We denote as "survivor" a fund that is alive till December 2015 while we denote as "dead" a fund that disappears before December 2015. See the Appendix for a detailed description of all the variables.

Table 1.22: Factor analysis relative to the Cox model of the main specification with no logarithm of assets

VARIABLES	(1) FF1	(2) FF1HR	(3) FF3	(4) FF3HR	(5) FF4	(6) FF4HR	(7) FF5	(8) FF5HR
Mktrf	0.963***	0.972***	0.949***	0.964***	0.943***	0.964***	0.925***	0.942***
	(0.00370)	(0.00524)	(0.00351)	(0.00496)	(0.00344)	(0.00488)	(0.00333)	(0.00481)
Smb			0.0877***	0.0650***	0.0916***	0.0657***	0.0739***	0.0497***
			(0.00431)	(0.00602)	(0.00428)	(0.00612)	(0.00439)	(0.00620)
Hml			-0.0309***	-0.0403***	-0.0436***	-0.0441***	0.0352***	0.0357***
			(0.00382)	(0.00595)	(0.00351)	(0.00607)	(0.00411)	(0.00654)
Umd					-0.0197***	-0.00210		
					(0.00162)	(0.00347)		
Mktrf*Hr		-0.0126**		-0.0212***		-0.0296***		-0.0236***
		(0.00568)		(0.00544)		(0.00524)		(0.00527)
Smb*Hr				0.0304***		0.0338***		0.0325***
				(0.00546)		(0.00569)		(0.00568)
Hml*Hr				0.0142**		0.00513		0.00330
				(0.00645)		(0.00681)		(0.00714)
Umd*Hr						-0.0199***		
						(0.00388)		
Hr		0.00142***		0.00146***		0.00141***		0.00158***
		(0.000140)		(0.000149)		(0.000147)		(0.000156)
Rmw							-0.101***	-0.102***
							(0.00330)	(0.00609)
Cma							-0.226***	-0.248***
							(0.00533)	(0.00930)
Rmw*Hr								0.00123
								(0.00727)
Cma*Hr								0.0248**
								(0.00987)
Constant	-0.00128***	-0.00232***	-0.00128***	-0.00235***	-0.00120***	-0.00227***	-0.000883***	-0.00205***
	(2.41e-05)	(0.000104)	(2.36e-05)	(0.000111)	(2.32e-05)	(0.000111)	(2.40e-05)	(0.000116)
Observations	435,690	435,690	435,690	435,690	435,690	435,690	435,690	435,690
R-squared	0.731	0.732	0.733	0.734	0.733	0.734	0.737	0.737
Number of funds	10,282	10,282	10,282	10,282	10,282	10,282	10,282	10,282

The Table represents panel regressions with fund fixed effects. The hazard ratios are derived from the Cox model represented in Table 1.21. Columns (1) and (2) represents results for the Jensen (1968)'s one factor model and our modified version with the hazard ratio and the interaction of the factor with the hazard ratio. Columns (3) and (4) represents results for the Fama and French (1993)'s 3 factors model and our modified version with the hazard ratio and the interactions of the factors with the hazard ratio. Columns (5) and (6) represents results for the Carhart (1997)'s 4 factors model and our modified version with the hazard ratio and the interactions of the factors with the hazard ratio. Columns (7) and (8) represents results for the Fama and French (2015)'s 5 factors model and our modified version with the hazard ratio and the interactions of the factors with the hazard ratio. The sample period is July 2003 - December 2015. Robust standard errors in parentheses, *** indicates significance at 1%, ** indicates significance at 5% and * indicates significance at 10%. See the Appendix for a detailed description of all the variables.

Table 1.23: Cox analysis and hazard ratio of the main specification on a sample of funds that invests primarily on common stocks

Panel A: Cox Analysis

VARIABLES	(1) Hazard Ratios	(2) Coefficients
Mretl1100	1.000	5.12e-05
	(0.00488)	(0.00488)
Mnavl1	0.979***	-0.0208***
	(0.00351)	(0.00359)
Logmtnal1	0.656***	-0.422***
	(0.0130)	(0.0198)
Turn_ratiol1100	1.000**	0.000460**
	(0.000185)	(0.000185)
Exp_ratiol1100	1.475***	0.389***
	(0.0717)	(0.0486)
Dummyieldl1	1.492***	0.400***
	(0.0811)	(0.0544)
Flowl1100	0.974***	-0.0260***
	(0.00253)	(0.00260)
Observations	153,656	153,656

Panel B: Hazard Ratio

Variable	Mean	Std. Dev.
Hr	0,1914	0,20319
Hr_surv	0,14802	0,15327
Hr_dead	0,27233	0,25382

The Table reports the results of the Cox analysis. Panel A reports the estimated hazard ratios and coefficients (Columns (1) and (2)) relative to the variables used in the analysis. The sample period is July 2003 - December 2015. Robust standard errors in parentheses, *** indicates significance at 1%, ** indicates significance at 5% and * indicates significance at 10%. Panel B represents the mean and the standard deviation of the hazard ratio for all the population of funds, for the survivor funds and for the dead funds. We denote as "survivor" a fund that is alive till December 2015 while we denote as "dead" a fund that disappears before December 2015. See the Appendix for a detailed description of all the variables.

Table 1.24: Factor analysis relative to the Cox model of the main specification on a sample of funds that invests primarily on common stocks

VARIABLES	(1) FF1	(2) FF1HR	(3) FF3	(4) FF3HR	(5) FF4	(6) FF4HR	(7) FF5	(8) FF5HR
Mktrf	1.067***	1.059***	1.042***	1.045***	1.043***	1.052***	1.020***	1.023***
	(0.00391)	(0.00528)	(0.00348)	(0.00477)	(0.00326)	(0.00463)	(0.00307)	(0.00428)
Smb			0.141***	0.114***	0.141***	0.106***	0.134***	0.104***
			(0.00724)	(0.00954)	(0.00727)	(0.00968)	(0.00739)	(0.00975)
Hml			-0.0479***	-0.0721***	-0.0469***	-0.0612***	0.00248	-0.0167*
			(0.00699)	(0.00894)	(0.00631)	(0.00861)	(0.00714)	(0.00919)
Umd					0.00159	0.0213***		
					(0.00271)	(0.00397)		
Mktrf*Hr		0.0356*		-0.0228		-0.0580***		-0.0204
		(0.0195)		(0.0165)		(0.0161)		(0.0161)
Smb*Hr				0.137***		0.173***		0.155***
				(0.0292)		(0.0311)		(0.0303)
Hml*Hr				0.120***		0.0818**		0.0984***
				(0.0321)		(0.0322)		(0.0338)
Umd*Hr						-0.0866***		
						(0.0163)		
Hr		0.00635***		0.00562***		0.00543***		0.00628***
		(0.000871)		(0.000803)		(0.000831)		(0.000836)
Rmw							-0.0853***	-0.0974***
							(0.00541)	(0.00782)
Cma							-0.181***	-0.184***
							(0.00744)	(0.0111)
Rmw*Hr								0.0640**
								(0.0305)
Cma*Hr								0.0133
								(0.0393)
Constant	-0.000409***	-0.00162***	-0.000384***	-0.00147***	-0.000391***	-0.00148***	-0.000132***	-0.00135***
	(2.08e-05)	(0.000169)	(2.20e-05)	(0.000157)	(1.87e-05)	(0.000161)	(2.83e-05)	(0.000164)
Observations	153,656	153,656	153,656	153,656	153,656	153,656	153,656	153,656
R-squared	0.804	0.804	0.808	0.808	0.808	0.809	0.810	0.811
Number of funds	5,177	5,177	5,177	5,177	5,177	5,177	5,177	5,177

The Table represents panel regressions with fund fixed effects. The hazard ratios are derived from the Cox model represented in Table 1.23. Columns (1) and (2) represents results for the Jensen (1968)'s one factor model and our modified version with the hazard ratio and the interaction of the factor with the hazard ratio. Columns (3) and (4) represents results for the Fama and French (1993)'s 3 factors model and our modified version with the hazard ratio and the interactions of the factors with the hazard ratio. Columns (5) and (6) represents results for the Carhart (1997)'s 4 factors model and our modified version with the hazard ratio and the interactions of the factors with the hazard ratio. Columns (7) and (8) represents results for the Fama and French (2015)'s 5 factors model and our modified version with the hazard ratio and the interactions of the factors with the hazard ratio. The sample period is July 2003 - December 2015. Robust standard errors in parentheses, *** indicates significance at 1%, ** indicates significance at 5% and * indicates significance at 10%. See the Appendix for a detailed description of all the variables.

Table 1.25: Cox analysis and hazard ratio of the main specification on a sample of big funds

Panel A: Cox Analysis

VARIABLES	(1) Hazard Ratios	(2) Coefficients
Mretl1100	1.009	0.00896
	(0.00694)	(0.00688)
Mnavl1	0.986***	-0.0140***
	(0.00373)	(0.00378)
Logmtnal1	0.520***	-0.654***
	(0.0179)	(0.0344)
Turn_ratiol1100	1.000	7.30e-05
	(0.000199)	(0.000199)
Exp_ratiol1100	1.851***	0.616***
	(0.115)	(0.0624)
Dummyieldl1	1.353***	0.302***
	(0.0854)	(0.0632)
Flowl1100	0.955***	-0.0462***
	(0.00296)	(0.00310)
Observations	261,627	261,627

Panel B: Hazard Ratio

Variable	Mean	Std. Dev.
Hr	0,03502	0,04434
Hr_surv	0,03062	0,04238
Hr_dead	0,05737	0,04723

The Table reports the results of the Cox analysis. Panel A reports the estimated hazard ratios and coefficients (Columns (1) and (2)) relative to the variables used in the analysis. The sample period is July 2003 - December 2015. Robust standard errors in parentheses, *** indicates significance at 1%, ** indicates significance at 5% and * indicates significance at 10%. Panel B represents the mean and the standard deviation of the hazard ratio for all the population of funds, for the survivor funds and for the dead funds. We denote as "survivor" a fund that is alive till December 2015 while we denote as "dead" a fund that disappears before December 2015. See the Appendix for a detailed description of all the variables.

Table 1.26: Factor analysis relative to the Cox model of the main specification on a sample of big funds

VARIABLES	(1) FF1	(2) FF1HR	(3) FF3	(4) FF3HR	(5) FF4	(6) FF4HR	(7) FF5	(8) FF5HR
Mktrf	0.966***	0.977***	0.955***	0.971***	0.950***	0.970***	0.932***	0.946***
	(0.00464)	(0.00577)	(0.00443)	(0.00567)	(0.00435)	(0.00562)	(0.00423)	(0.00548)
Smb			0.0750***	0.0667***	0.0780***	0.0663***	0.0605***	0.0410***
			(0.00556)	(0.0100)	(0.00553)	(0.00951)	(0.00567)	(0.00881)
Hml			-0.0361***	-0.0553***	-0.0458***	-0.0586***	0.0305***	0.0240***
			(0.00475)	(0.00685)	(0.00445)	(0.00653)	(0.00501)	(0.00732)
Umd					-0.0150***	-0.00125		
					(0.00195)	(0.00295)		
Mktrf*Hr		-0.326***		-0.502***		-0.621***		-0.438***
		(0.107)		(0.111)		(0.108)		(0.111)
Smb*Hr				0.237		0.329		0.559***
				(0.233)		(0.215)		(0.187)
Hml*Hr				0.583***		0.448***		0.242
				(0.141)		(0.135)		(0.155)
Umd*Hr						-0.341***		
						(0.0652)		
Hr		0.0291***		0.0289***		0.0301***		0.0334***
		(0.00745)		(0.00708)		(0.00671)		(0.00680)
Rmw							-0.102***	-0.132***
							(0.00410)	(0.00598)
Cma							-0.225***	-0.247***
							(0.00690)	(0.0121)
Rmw*Hr								0.903***
								(0.127)
Cma*Hr								0.606**
								(0.281)
Constant	-0.00101***	-0.00201***	-0.000992***	-0.00200***	-0.000927***	-0.00199***	-0.000623***	-0.00178***
	(3.09e-05)	(0.000261)	(3.02e-05)	(0.000250)	(2.97e-05)	(0.000237)	(3.11e-05)	(0.000240)
Observations	261,627	261,627	261,627	261,627	261,627	261,627	261,627	261,627
R-squared	0.741	0.742	0.743	0.743	0.743	0.743	0.746	0.747
Number of funds	6,003	6,003	6,003	6,003	6,003	6,003	6,003	6,003

The Table represents panel regressions with fund fixed effects. The hazard ratios are derived from the Cox model represented in Table 1.25. Columns (1) and (2) represents results for the Jensen (1968)'s one factor model and our modified version with the hazard ratio and the interaction of the factor with the hazard ratio. Columns (3) and (4) represents results for the Fama and French (1993)'s 3 factors model and our modified version with the hazard ratio and the interactions of the factors with the hazard ratio. Columns (5) and (6) represents results for the Carhart (1997)'s 4 factors model and our modified version with the hazard ratio and the interactions of the factors with the hazard ratio. Columns (7) and (8) represents results for the Fama and French (2015)'s 5 factors model and our modified version with the hazard ratio and the interactions of the factors with the hazard ratio. The sample period is July 2003 - December 2015. Robust standard errors in parentheses, *** indicates significance at 1%, ** indicates significance at 5% and * indicates significance at 10%. See the Appendix for a detailed description of all the variables.

Table 1.27: Probit analysis and predicted probability of the main specification

Panel A: Probit Analysis

VARIABLES	(1) Coefficients
Age	0.00653***
	(0.000679)
Mretl1100	0.00213
	(0.00158)
Mnavl1	-0.00781***
	(0.00114)
Logmtnal1	-0.165***
	(0.00614)
Turn_ratiol1100	2.74e-05
	(3.06e-05)
Exp_ratiol1100	0.0873***
	(0.0174)
Dummyieldl1	0.0459***
	(0.0163)
Mgmt_feel1100	-0.000104
	(0.000113)
Max_12b1l1100	0.147**
	(0.0705)
Actual_12b1l1100	0.0775
	(0.0746)
Flowl1100	-0.0183***
	(0.00113)
Constant	-1.785***
	(0.0402)
Observations	339,657
Number of funds	8,587

Panel B: Predicted Probability

Variable	Mean	Std. Dev.
Pr	0,00842	0,0093
Pr_surv	0,00713	0,008
Pr_dead	0,01227	0,01158

The Table reports the results of the Probit analysis. Panel A reports the coefficients (Column (1)) relative to the variables used in the analysis. The sample period is July 2003 - December 2015. Robust standard errors in parentheses, *** indicates significance at 1%, ** indicates significance at 5% and * indicates significance at 10%. Panel B represents the mean and the standard deviation of the predicted probabilities for all the population of funds, for the survivor funds and for the dead funds. We denote as "survivor" a fund that is alive till December 2015 while we denote as "dead" a fund that disappears before December 2015. See the Appendix for a detailed description of all the variables.

Table 1.28: Factor analysis relative to the Probit model of the main specification

VARIABLES	(1) FF1	(2) FF1HR	(3) FF3	(4) FF3HR	(5) FF4	(6) FF4HR	(7) FF5	(8) FF5HR
Mktrf	0.954***	0.963***	0.941***	0.955***	0.935***	0.956***	0.917***	0.931***
	(0.00424)	(0.00548)	(0.00405)	(0.00531)	(0.00397)	(0.00531)	(0.00385)	(0.00512)
Smb			0.0845***	0.0693***	0.0882***	0.0705***	0.0695***	0.0515***
			(0.00471)	(0.00617)	(0.00469)	(0.00622)	(0.00481)	(0.00629)
Hml			-0.0385***	-0.0495***	-0.0506***	-0.0514***	0.0282***	0.0252***
			(0.00435)	(0.00616)	(0.00401)	(0.00628)	(0.00468)	(0.00678)
Umd					-0.0188***	-0.00226		
					(0.00180)	(0.00283)		
Mktrf*Pr		-1.174**		-1.808***		-2.523***		-1.696***
		(0.459)		(0.450)		(0.451)		(0.435)
Smb* Pr				1.759***		1.964***		2.101***
				(0.435)		(0.451)		(0.441)
Hml* Pr				1.444***		0.485		0.649
				(0.535)		(0.569)		(0.617)
Umd* Pr						-1.600***		
						(0.238)		
Pr		0.140***		0.137***		0.130***		0.141***
		(0.0120)		(0.0120)		(0.0119)		(0.0121)
Rmw							-0.105***	-0.116***
							(0.00375)	(0.00601)
Cma							-0.227***	-0.245***
							(0.00601)	(0.00896)
Rmw*Pr								1.352**
								(0.583)
Cma*Pr								1.778**
								(0.774)
Constant	-0.00145***	-0.00262***	-0.00145***	-0.00260***	-0.00138***	-0.00250***	-0.00104***	-0.00222***
	(2.78e-05)	(0.000103)	(2.73e-05)	(0.000103)	(2.68e-05)	(0.000103)	(2.76e-05)	(0.000104)
Observations	339,657	339,657	339,657	339,657	339,657	339,657	339,657	339,657
R-squared	0.724	0.725	0.726	0.726	0.726	0.727	0.730	0.730
Number of funds	8,587	8,587	8,587	8,587	8,587	8,587	8,587	8,587

The Table represents panel regressions with fund fixed effects. The predicted probabilities are derived from the Probit model represented in Table 1.27. Columns (1) and (2) represents results for the Jensen (1968)'s one factor model and our modified version with the predicted probability and the interaction of the factor with the predicted probability. Columns (3) and (4) represents results for the Fama and French (1993)'s 3 factors model and our modified version with the predicted probability and the interactions of the factors with the predicted probability. Columns (5) and (6) represents results for the Carhart (1997)'s 4 factors model and our modified version with the predicted probability and the interactions of the factors with the predicted probability. Columns (7) and (8) represents results for the Fama and French (2015)'s 5 factors model and our modified version with the predicted probability and the interactions of the factors with the predicted probability. The sample period is July 2003 - December 2015. Robust standard errors in parentheses, *** indicates significance at 1%, ** indicates significance at 5% and * indicates significance at 10%. See the Appendix for a detailed description of all the variables.

Table 1.29: Factor analysis with just the hazard ratio relative to the Probit model of the main specification

VARIABLES	(1) FF1	(2) FF1JHR	(3) FF3	(4) FF3JHR	(5) FF4	(6) FF4JHR	(7) FF5	(8) FF5JHR
Mktrf	0.954***	0.953***	0.941***	0.941***	0.935***	0.935***	0.917***	0.917***
	(0.00424)	(0.00423)	(0.00405)	(0.00404)	(0.00397)	(0.00397)	(0.00385)	(0.00384)
Smb			0.0845***	0.0839***	0.0882***	0.0875***	0.0695***	0.0690***
			(0.00471)	(0.00471)	(0.00469)	(0.00469)	(0.00481)	(0.00480)
Hml			-0.0385***	-0.0382***	-0.0506***	-0.0496***	0.0282***	0.0290***
			(0.00435)	(0.00435)	(0.00401)	(0.00400)	(0.00468)	(0.00468)
Pr		0.129***		0.124***		0.117***		0.131***
		(0.0113)		(0.0110)		(0.0107)		(0.0114)
Umd					-0.0188***	-0.0177***		
					(0.00180)	(0.00179)		
Rmw							-0.105***	-0.105***
							(0.00375)	(0.00374)
Cma							-0.227***	-0.229***
							(0.00601)	(0.00602)
Constant	-0.00145***	-0.00254***	-0.00145***	-0.00249***	-0.00138***	-0.00236***	-0.00104***	-0.00214***
	(2.78e-05)	(9.85e-05)	(2.73e-05)	(9.60e-05)	(2.68e-05)	(9.33e-05)	(2.76e-05)	(9.91e-05)
Observations	339,657	339,657	339,657	339,657	339,657	339,657	339,657	339,657
R-squared	0.724	0.725	0.726	0.726	0.726	0.727	0.730	0.730
Number of funds	8,587	8,587	8,587	8,587	8,587	8,587	8,587	8,587

The Table represents panel regressions with fund fixed effects. The predicted probabilities are derived from the Probit model represented in Table 1.27. Columns (1) and (2) represents results for the Jensen (1968)'s one factor model and our modified version with just the predicted probability. Columns (3) and (4) represents results for the Fama and French (1993)'s 3 factors model and our modified version with just the predicted probability. Columns (5) and (6) represents results for the Carhart (1997)'s 4 factors model and our modified version with just the predicted probability. Columns (7) and (8) represents results for the Fama and French (2015)'s 5 factors model and our modified version with just the predicted probability. The sample period is July 2003 - December 2015. Robust standard errors in parentheses, *** indicates significance at 1%, ** indicates significance at 5% and * indicates significance at 10%. See the Appendix for a detailed description of all the variables.

Table 1.30: Logit analysis and predicted probability of the main specification

Panel A: Logit Analysis

VARIABLES	(1) Coefficients
Age	0.0177***
	(0.00180)
Mretl1100	0.00490
	(0.00404)
Mnavl1	-0.0229***
	(0.00311)
Logmtnal1	-0.440***
	(0.0163)
Turn_ratiol1100	4.99e-05
	(7.31e-05)
Exp_ratiol1100	0.206***
	(0.0444)
Dummyieldl1	0.105**
	(0.0426)
Mgmt_feel1100	-8.22e-05
	(0.000281)
Max_12b1l1100	0.371**
	(0.184)
Actual_12b1l1100	0.204
	(0.193)
Flowl1100	-0.0389***
	(0.00187)
Constant	-3.076***
	(0.105)
Observations	339,657
Number of funds	8,587

Panel B: Predicted Probability

Variable	Mean	Std. Dev.
Pr	0,00867	0,00977
Pr_surv	0,0074	0,00832
Pr_dead	0,01247	0,01245

The Table reports the results of the Logit analysis. Panel A reports the coefficients (Column (1)) relative to the variables used in the analysis. The sample period is July 2003 - December 2015. Robust standard errors in parentheses, *** indicates significance at 1%, ** indicates significance at 5% and * indicates significance at 10%. Panel B represents the mean and the standard deviation of the predicted probabilities for all the population of funds, for the survivor funds and for the dead funds. We denote as "survivor" a fund that is alive till December 2015 while we denote as "dead" a fund that disappears before December 2015. See the Appendix for a detailed description of all the variables.

Table 1.31: Factor analysis relative to the Logit model of the main specification

VARIABLES	(1) FF1	(2) FF1HR	(3) FF3	(4) FF3HR	(5) FF4	(6) FF4HR	(7) FF5	(8) FF5HR
Mktrf	0.954***	0.964***	0.941***	0.956***	0.935***	0.957***	0.917***	0.931***
	(0.00424)	(0.00549)	(0.00405)	(0.00531)	(0.00397)	(0.00534)	(0.00385)	(0.00514)
Smb			0.0845***	0.0694***	0.0882***	0.0704***	0.0695***	0.0516***
			(0.00471)	(0.00623)	(0.00469)	(0.00628)	(0.00481)	(0.00633)
Hml			-0.0385***	-0.0504***	-0.0506***	-0.0530***	0.0282***	0.0241***
			(0.00435)	(0.00629)	(0.00401)	(0.00656)	(0.00468)	(0.00696)
Umd					-0.0188***	-0.00272		
					(0.00180)	(0.00292)		
Mktrf*Pr		-1.157***		-1.804***		-2.520***		-1.685***
		(0.442)		(0.434)		(0.439)		(0.425)
Smb* Pr				1.707***		1.942***		2.028***
				(0.432)		(0.450)		(0.434)
Hml* Pr				1.505***		0.640		0.746
				(0.539)		(0.593)		(0.624)
Umd* Pr						-1.523***		
						(0.241)		
Pr		0.123***		0.121***		0.114***		0.124***
		(0.0124)		(0.0125)		(0.0122)		(0.0126)
Rmw							-0.105***	-0.117***
							(0.00375)	(0.00642)
Cma							-0.227***	-0.245***
							(0.00601)	(0.00920)
Rmw*Pr								1.357**
								(0.620)
Cma*Pr								1.720**
								(0.780)
Constant	-0.00145***	-0.00251***	-0.00145***	-0.00249***	-0.00138***	-0.00239***	-0.00104***	-0.00211***
	(2.78e-05)	(0.000109)	(2.73e-05)	(0.000110)	(2.68e-05)	(0.000109)	(2.76e-05)	(0.000112)
Observations	339,657	339,657	339,657	339,657	339,657	339,657	339,657	339,657
R-squared	0.724	0.725	0.726	0.726	0.726	0.727	0.730	0.730
Number of funds	8,587	8,587	8,587	8,587	8,587	8,587	8,587	8,587

The Table represents panel regressions with fund fixed effects. The predicted probabilities are derived from the Logit model represented in Table 1.30. Columns (1) and (2) represents results for the Jensen (1968)'s one factor model and our modified version with the predicted probability and the interaction of the factor with the predicted probability. Columns (3) and (4) represents results for the Fama and French (1993)'s 3 factors model and our modified version with the predicted probability and the interactions of the factors with the predicted probability. Columns (5) and (6) represents results for the Carhart (1997)'s 4 factors model and our modified version with the predicted probability and the interactions of the factors with the predicted probability. Columns (7) and (8) represents results for the Fama and French (2015)'s 5 factors model and our modified version with the predicted probability and the interactions of the factors with the predicted probability. The sample period is July 2003 - December 2015. Robust standard errors in parentheses, *** indicates significance at 1%, ** indicates significance at 5% and * indicates significance at 10%. See the Appendix for a detailed description of all the variables.

Table 1.32: Factor analysis with just the hazard ratio relative to the Logit model of the main specification

VARIABLES	(1) FF1	(2) FF1JHR	(3) FF3	(4) FF3JHR	(5) FF4	(6) FF4JHR	(7) FF5	(8) FF5JHR
Mktrf	0.954***	0.953***	0.941***	0.941***	0.935***	0.935***	0.917***	0.917***
	(0.00424)	(0.00423)	(0.00405)	(0.00404)	(0.00397)	(0.00397)	(0.00385)	(0.00384)
Smb			0.0845***	0.0840***	0.0882***	0.0876***	0.0695***	0.0691***
			(0.00471)	(0.00471)	(0.00469)	(0.00469)	(0.00481)	(0.00480)
Hml			-0.0385***	-0.0382***	-0.0506***	-0.0497***	0.0282***	0.0289***
			(0.00435)	(0.00435)	(0.00401)	(0.00400)	(0.00468)	(0.00468)
Pr		0.112***		0.107***		0.101***		0.113***
		(0.0114)		(0.0111)		(0.0107)		(0.0116)
Umd					-0.0188***	-0.0178***		
					(0.00180)	(0.00180)		
Rmw							-0.105***	-0.105***
							(0.00375)	(0.00374)
Cma							-0.227***	-0.229***
							(0.00601)	(0.00602)
Constant	-0.00145***	-0.00242***	-0.00145***	-0.00237***	-0.00138***	-0.00225***	-0.00104***	-0.00202***
	(2.78e-05)	(0.000103)	(2.73e-05)	(9.95e-05)	(2.68e-05)	(9.62e-05)	(2.76e-05)	(0.000104)
Observations	339,657	339,657	339,657	339,657	339,657	339,657	339,657	339,657
R-squared	0.724	0.725	0.726	0.726	0.726	0.727	0.730	0.730
Number of funds	8,587	8,587	8,587	8,587	8,587	8,587	8,587	8,587

The Table represents panel regressions with fund fixed effects. The predicted probabilities are derived from the Logit model represented in Table 1.30. Columns (1) and (2) represents results for the Jensen (1968)'s one factor model and our modified version with just the predicted probability. Columns (3) and (4) represents results for the Fama and French (1993)'s 3 factors model and our modified version with just the predicted probability. Columns (5) and (6) represents results for the Carhart (1997)'s 4 factors model and our modified version with just the predicted probability. Columns (7) and (8) represents results for the Fama and French (2015)'s 5 factors model and our modified version with just the predicted probability. The sample period is July 2003 - December 2015. Robust standard errors in parentheses, *** indicates significance at 1%, ** indicates significance at 5% and * indicates significance at 10%. See the Appendix for a detailed description of all the variables.

Table 1.33: Cox analysis and hazard ratio of the main specification after Lehman collapse

Panel A: Cox Analysis

VARIABLES	(1) Hazard Ratios	(2) Coefficients
Mretl1100	0.991*	-0.00931*
	(0.00487)	(0.00491)
Mnavl1	0.959***	-0.0419***
	(0.00478)	(0.00498)
Logmtnal1	0.560***	-0.580***
	(0.0113)	(0.0202)
Turn_ratiol1100	1.000**	0.000233**
	(9.08e-05)	(9.07e-05)
Exp_ratiol1100	0.952	-0.0489
	(0.0495)	(0.0520)
Dummyieldl1	0.888**	-0.119**
	(0.0504)	(0.0567)
Flowl1100	0.967***	-0.0340***
	(0.00174)	(0.00181)
Observations	267,303	267,303

Panel B: Hazard Ratio

Variable	Mean	Std. Dev.
Hr	0,03770	0,06462
Hr_surv	0,03251	0,05919
Hr_dead	0,07061	0,08469

The Table reports the results of the Cox analysis. Panel A reports the estimated hazard ratios and coefficients (Columns (1) and (2)) relative to the variables used in the analysis. The sample period is October 2008 - December 2015. Robust standard errors in parentheses, *** indicates significance at 1%, ** indicates significance at 5% and * indicates significance at 10%. Panel B represents the mean and the standard deviation of the hazard ratio for all the population of funds, for the survivor funds and for the dead funds. We denote as "survivor" a fund that is alive till December 2015 while we denote as "dead" a fund that disappears before December 2015. See the Appendix for a detailed description of all the variables.

Table 1.34: Factor analysis relative to the Cox model of the main specification after Lehman collapse

VARIABLES	(1) FF1	(2) FF1HR	(3) FF3	(4) FF3HR	(5) FF4	(6) FF4HR	(7) FF5	(8) FF5HR
Mktrf	0.948***	0.957***	0.932***	0.946***	0.913***	0.929***	0.914***	0.928***
	(0.00395)	(0.00448)	(0.00364)	(0.00423)	(0.00351)	(0.00406)	(0.00352)	(0.00412)
Smb			0.0818***	0.0746***	0.0863***	0.0815***	0.0753***	0.0655***
			(0.00466)	(0.00551)	(0.00464)	(0.00537)	(0.00470)	(0.00560)
Hml			-0.00383	-0.0142***	-0.0516***	-0.0587***	0.0879***	0.0782***
			(0.00391)	(0.00473)	(0.00374)	(0.00469)	(0.00511)	(0.00617)
Umd					-0.0665***	-0.0626***		
					(0.00191)	(0.00236)		
Mktrf*Hr		-0.223***		-0.358***		-0.414***		-0.373***
		(0.0589)		(0.0577)		(0.0554)		(0.0573)
Smb*Hr				0.196***		0.124**		0.259***
				(0.0686)		(0.0616)		(0.0739)
Hml*Hr				0.305***		0.233***		0.324***
				(0.0757)		(0.0748)		(0.0980)
Umd*Hr						-0.0876***		
						(0.0321)		
Hr		0.0154***		0.0183***		0.0158***		0.0199***
		(0.00208)		(0.00220)		(0.00190)		(0.00233)
Rmw							-0.0158***	-0.0300***
							(0.00348)	(0.00510)
Cma							-0.275***	-0.270***
							(0.00676)	(0.00842)
Rmw*Hr								0.380***
								(0.107)
Cma*Hr								-0.221*
								(0.128)
Constant	-0.00238***	-0.00297***	-0.00232***	-0.00300***	-0.00234***	-0.00294***	-0.00169***	-0.00242***
	(3.70e-05)	(8.54e-05)	(3.76e-05)	(8.96e-05)	(3.76e-05)	(7.92e-05)	(3.68e-05)	(9.28e-05)
Observations	267,303	267,303	267,303	267,303	267,303	267,303	267,303	267,303
R-squared	0.759	0.759	0.760	0.761	0.764	0.764	0.764	0.765
Number of funds	7,258	7,258	7,258	7,258	7,258	7,258	7,258	7,258

The Table represents panel regressions with fund fixed effects. The hazard ratios are derived from the Cox model represented in Table 1.33. Columns (1) and (2) represents results for the Jensen (1968)'s one factor model and our modified version with the hazard ratio and the interaction of the factor with the hazard ratio. Columns (3) and (4) represents results for the Fama and French (1993)'s 3 factors model and our modified version with the hazard ratio and the interactions of the factors with the hazard ratio. Columns (5) and (6) represents results for the Carhart (1997)'s 4 factors model and our modified version with the hazard ratio and the interactions of the factors with the hazard ratio. Columns (7) and (8) represents results for the Fama and French (2015)'s 5 factors model and our modified version with the hazard ratio and the interactions of the factors with the hazard ratio. The sample period is October 2008 - December 2015. Robust standard errors in parentheses, *** indicates significance at 1%, ** indicates significance at 5% and * indicates significance at 10%. See the Appendix for a detailed description of all the variables.

Table 1.35: Factor analysis with the introduction of the squared hazard ratio relative to the Cox model of the main specification

VARIABLES	(1) FF1	(2) FF1HR	(3) FF3	(4) FF3HR	(5) FF4	(6) FF4HR	(7) FF5	(8) FF5HR
Mktrf	0.963***	0.980***	0.949***	0.971***	0.943***	0.971***	0.925***	0.947***
Smb			0.0877***	0.0701***	0.0916***	0.0719***	0.0739***	0.0532***
Hml			-0.0309***	-0.0443***	-0.0436***	-0.0456***	0.0352***	0.0286***
Rmw							-0.101***	-0.113***
Cma							-0.226***	-0.238***
Mktrf*Hr		-0.181***		-0.242***		-0.304***		-0.236***
Smb*Hr				0.167***		0.181***		0.201***
Hml*Hr				0.160***		0.0693*		0.110**
Rmw*Hr								0.137***
Cma*Hr								0.0659
Hr		0.0263***		0.0255***		0.0239***		0.0267***
Mktrf*Hr2		0.0377***		0.0431***		0.0633***		0.0418***
Smb*Hr2				-0.0212		-0.0301**		-0.0280*
Hml*Hr2				-0.0262*		-0.00955		-0.0170
Rmw*Hr2								-0.0281
Cma*Hr2								-0.00226
Hr2		-0.00434***		-0.00414***		-0.00396***		-0.00434***
Umd					-0.0197***	-0.00207		
Umd*Hr						-0.145***		
Umd*Hr2						0.0326***		
Constant	-0.00128***	-0.00379***	-0.00128***	-0.00371***	-0.00120***	-0.00352***	-0.000883***	-0.00343***
Observations	435,690	435,690	435,690	435,690	435,690	435,690	435,690	435,690
R-squared	0.731	0.732	0.733	0.734	0.733	0.735	0.737	0.738
Number of funds	10,282	10,282	10,282	10,282	10,282	10,282	10,282	10,282

The Table represents panel regressions with fund fixed effects. The hazard ratios are derived from the Cox model represented in Table 1.4. Columns (1) and (2) represents results for the Jensen (1968)'s one factor model and our modified version with the hazard ratio and the squared hazard ratio and the interaction of the factor with these two variables. Columns (3) and (4) represents results for the Fama and French (1993)'s 3 factors model and our modified version with the hazard ratio and the squared hazard ratio and the interactions of the factors with these two variables. Columns (5) and (6) represents results for the Carhart (1997)'s 4 factors model and our modified version with the hazard ratio and the squared hazard ratio and the interactions of the factors with these two variables. Columns (7) and (8) represents results for the Fama and French (2015)'s 5 factors model and our modified version with the hazard ratio and the squared hazard ratio and the interactions of the factors with these two variables. The sample period is July 2003 - December 2015. *** indicates significance at 1%, ** indicates significance at 5% and * indicates significance at 10%. See the Appendix for a detailed description of all the variables.

1.7 Conclusions

We developed a novel 2 step procedure in order to capture the dynamics of mutual funds strategies and the influence of the odds of survival on these dynamics. The first step of our methodology consists of the estimation of a Cox survival model and we show that the main determinants of fund's survivorship are its size and its expense ratio. The second step of our technique, based on the estimation of several factor models, highlights the importance of the interaction between the factors and the hazard ratio computed from the survival model. Particularly important is the negative statistically significant interaction of the hazard ratio with the market and the momentum factors. We also estimate a positive and statistically significant interaction of the odds of survival with the size factor. Our results are robust to several different specifications of the survival model and they seem to be driven mainly by structural differences between dead and survivor funds and not by attrition effects.

This implies that both the simple analysis of the survived sample and the analysis of what is commonly referred to as the ”survivorship bias free” sample, cannot, alone, yield a correct picture of the behavior of mutual funds with respect to market factors.

These analyses must be supplemented by a survival model which help along two very important and interconnected lines:

- it gives us the possibility of correcting biases due to no (survived sample) or partial (survivorship bias free sample) observation of dead funds histories;
- it allows a first study of the possible strategic interaction between survival probability and fund strategy which, as can be seen from our results, is by no means negligible. This second line has a clear relevance when the method is applied to economic data where strategic behavior is, often, at the center of interest.

We believe that our novel methodology should be applied when investigating mutual fund performance and its persistence. It would also be important to study the impact of our technique on the results of papers that address different research questions but that use performance measures in their analysis (see among others, Nohel, Wang and Zheng (2010), Evans (2010) and Amihud and Goyenko (2013)).

An important point to notice is that, while we document a not negligible interaction between survival probabilities and factor loadings, such interaction is difficult to explain without recourse to a strategic behavior and we do not model this behavior in any explicit way. The problem in doing this is not that of writing a proper theoretical model but that of lack of information, in the available dataset, about variables which would be essential to estimate it.

The estimation of an actual strategic model would require a much more detailed dataset which, arguably, could be derived not by a ”general census” procedure but by a statistically designed survey sampling .

We consider this a very interesting topic for further research.

1.8 Appendix

1.8.1 Variables

Table 1.36 contains a description of all the variables used in our analysis. For a detailed description of the fund's characteristics it is possible to consult the CRSP guide [26] for details.

1.8.2 Simulation

In this subsection we show, by using simulated data, that in a simple 1 factor model, if the funds that have an higher probability to die have an higher exposition to the market then our 2 step procedure estimates a positive interaction between the factor and the hazard ratio. The objective of this subsection is neither to be realistic nor to be particularly insightful. The only aim of our simulation is to show that with a simple fund's strategy our 2 step methodology captures exactly what we would expect ex-ante by separately looking at survivor and dead funds.

We simulate 15 years of monthly observations for 100000 funds. The model that generates the data is the following:

$$r_{it} = r^f + \beta_i \left(R^{mkt} - r^f\right) + \epsilon_{it}$$

where we have that $\beta_i \sim N\left(\mu_i^{A,\beta}, \sigma^\beta\right)$, $\epsilon_{it} \sim N\left(0, \sigma_i\right)$, $\sigma_i^2 = \left(1-\beta_i\right)^2$, $\left(R^{mkt} - r^f\right) \sim N\left(\mu^{mkt}, \sigma^{mkt}\right)$. Notice $\mu_i^{A,\beta}$ is a function of the asset variable A that is distributed according to the following AR1 model:

$$A_{it} = \mu_i^A + \rho_{it} A_{it-1} + \epsilon_{it}^A$$

with $\mu_i^A \sim N\left(\mu^A, \sigma^A\right)$, $\rho_{it} \sim U\left(0,1\right)$ and $\epsilon_{it}^A \sim N\left(0,1\right)$.

We choose parameters similar to the ones that we observe in the data. In particular we set:

$$
\begin{aligned}
& r^f = 0 \\
& \mu_i^{A,\beta} = \begin{cases} 0.7 & if\ A_{it} > \overline{A}_t \\ 1.05 & if\ A_{it} \leq \overline{A}_t \end{cases} \\
& \overline{A}_t = asset\ average\ at\ time\ t \\
& \sigma^{\beta} = 0.5 \\
& \mu^{mkt} = 0.68 \\
& \sigma^{mkt} = 4 \\
& \mu^{A} = 1365.5 \\
& \sigma^{A} = 10
\end{aligned}
$$

The selection rule is determined by the assets. We select the first two years of observations for all funds. From the third year onward a fund survives if its assets are above a threshold that we set equal to 10500. Table 1.37 represents the Jensen (1968)'s one factor model for survivor and dead funds respectively. We denote as "survivor" a fund that survives during all the 15 year period while we denote as "dead" a fund that disappears before the last simulation period. By looking separately at these two groups, it is possible we see that the average exposition to the market factor is bigger for the dead funds. These funds, given our selection rule, are on average smaller than survivor funds.

The Cox analysis, reported in Table 1.38, highlights how the size reduces the probability of death.

We now proceed with the second step and we estimate both the original Jensen (1968)'s one factor model and our modified version with the interaction between the market factor and the hazard ratio (see Table 1.39). We see that, as ex-ante expected by separately looking at survivor and dead funds, the interaction term is statistically significant and positive. Hence our method properly captures simple funds strategies.

On the contrary, as showed in the paper, our analysis based on real data captures strategies that are ex-ante unexpected. Consequently, in the real world, it is not enough to look separately at the strategies of survivor and dead funds and it is necessary to use a procedure like ours in order to disentangle the complex economic dynamics of mutual funds strategies.

Table 1.36: Variables Description

Variables	Description
Mret	Total month return per share
Mtna	Monthly total net assets (mln $)
Mnav	Monthly net asset value computed as the difference between fund's underlying assets minus its liabilities
Turn_ratio	Fund turnover ratio computed as the minimum of aggregated sales or aggregated purchases of securities divided by the average 1 year total net assets
Exp_ratio	Expense ratio computed as the ratio of total investment that shareholders pay for the fund's operating expenses
Yield	Yield computed as the ratio of income distributions and NAV
Mgmt_fee	Management fee ($) over average net assets ($)
Max_12b1	Maximum contractual 12b-1 fee
Actual_12b1	Actual 12b-1 fee calculated as the ratio of marketing and distribution costs and total assets
Age	Mutual fund's age expressed in years
Mktrf	Excess Return on the Market
Smb	Small minus Big
Hml	High minus Low
Umd	Momentum
Cma	Conservative minus aggressive
Rmw	Robust minus weak
Flow	Flow computed as the percentage increment in fund's assets in excess to one due to returns
Hr	Hazard ratio estimated from the Cox model
Hr_surv	Hazard ratio estimated from the Cox model for the population of survivor funds, i.e. funds alive till December 2015
Hr_dead	Hazard ratio estimated from the Cox model for the population of dead funds, i.e. funds that disappears before December 2015
Pr	Predicted probability from the Probit/Logit model
Pr_surv	Predicted probability from the Probit/Logit model for the population of survivor funds, i.e. funds alive till December 2015
Pr_dead	Predicted probability from the Probit/Logit model for the population of dead funds, i.e. funds that disappears before December 2015
Dummyield	A dummy variable that takes value equal to 1 if yield is positive and 0 otherwise
Logmtna	Natural logarithm of mtna
Gdp	Quarter on quarter percentage change of US gross domestic product (source Baureau of Economic Analysis)
Retrelm	Monthly relative return calculated as the difference of the monthly return and the average return of all funds in that calendar date
Mretm1y	1 year average of the montlhy returns of the specific fund analyzed
Xl1	1 period lag of variable X
X100	Variable X has been multiplied by 100
Xl1100	1 period lag of variable X multiplied by 100

Table 1.37: Factor analysis of survivor and dead funds on simulated data

VARIABLES	(1) SurvFF1	(2) DeadFF1
Mktrf	0.822***	1.050***
	(0.00459)	(0.00170)
Constant	-8.90e-05	0.000835
	(0.00513)	(0.00441)
Observations	2,341,080	2,088,123
R-squared	0.701	0.807
Number of funds	13,006	86,994

The Table presents panel regressions with fund fixed effects. We denote as "survivor" a fund that survives during all the 15 year period while we denote as "dead" a fund that disappears before the last simulation period. Columns (1) and (2) represents results for the Jensen (1968)'s one factor model for survivor and dead funds respectively.

Table 1.38: Cox analysis and hazard ratio on simulated data

Panel A: Cox Analysis

VARIABLES	(1) Hazard Ratios	(2) Coefficients
Asset	.9999***	-9.93e-05***
	(1.21e-06)	(1.21e-06)
Observations	4,429,203	4,429,203

Panel B: Hazard Ratio

Variable	Mean	Std. Dev.
Hr	0,42311	0,32509
Hr_surv	0,13892	0,11886
Hr_dead	0,74173	0,12754

The Table reports the results of the Cox analysis. Panel A reports the estimated hazard ratios and coefficients (Columns (1) and (2)) relative to the variables used in the analysis. Robust standard errors in parentheses, *** indicates significance at 1%, ** indicates significance at 5% and * indicates significance at 10%. Panel B represents the mean and the standard deviation of the hazard ratio for all the population of funds, for the survivor funds and for the dead funds. We denote as "survivor" a fund that survives during all the 15 year period while we denote as "dead" a fund that disappears before the last simulation period.

Table 1.39: Factor analysis relative to the Cox model on simulated data

VARIABLES	(1) FF1	(2) FF1HR
Mktrf	0.910***	0.753***
	(0.00294)	(0.00520)
Mktrfhr		0.423***
		(0.00742)
Hr		11.08
		(14.06)
Constant	0.120***	-4.703
	(0.00533)	(5.948)
Observations	4,429,203	4,429,203
R-squared	0.740	0.756
Number of funds	100,000	100,000

The Table represents panel regressions with fund fixed effects. The hazard ratios are derived from the Cox model represented in Table 1.38. Columns (1) and (2) represents results for the Jensen (1968)'s one factor model and our modified version with the hazard ratio and the interaction of the factor with the hazard ratio.

Chapter 2

Constructive Ambiguity and Banks' Bailouts

We thank Elena Carletti, Paolo Fulghieri and Nicola Pavoni for helpful comments and suggestions. All remaining errors are ours.

2.1 Introduction

The recent financial crisis has been characterized by several types of government interventions and bailouts. One of the main feature of the present crisis is that U.S. government did not immediately intervene and Lehman Brothers filed for bankruptcy before policy makers decided to help distressed institutions such as AIG. These events highlight 2 important facts: first, some governments use what it could be defined as a constructive ambiguity approach. In fact even if the cost of letting banks fail is high, the government, by using the credible excuse of acquiring more information about the nature of the problem, may wait before acting to reduce the moral hazard incentives of the institutions. Second, it is really important for the banks at which time they show distress. In fact if Lehman had shown distress after AIG, probably now it would be still alive.

The relationship between bailouts and moral hazard problems has been deeply explored in the literature. It is well known that there is a tradeoff between trying to restore confidence in order to prevent systemic problems and the fact that government's commitment to bailout induces excessive risk taking by the financial institutions. As emphasized by Nosal and Ordoñez (2016), using a constructive ambiguity approach may be beneficial because uncertainty about government's bailout policy may act as a commitment device and lead banks to self restrain their behavior.

The aim of our research is to bring inside the present debate different sources of moral hazard and to study how investors' trust affects the effectiveness of the constructive ambiguity policy. Hence, in a framework a la Nosal and Ordoñez (2016), we introduce a new source of moral hazard. In particular we consider the possibility that the financial institutions, if they refinance their projects, can ex-post steal part of the potentially pledgeable income of the original project. This fact generates time inconsistency of banks' behavior. Ex-ante the banks would like to commit to no opportunistic attitude toward investors. In this way they will increase the liquidity that they can eventually raise if they are hit by a shock and they need to refinance their projects. But ex-post the financial institutions, if they have no future concerns, want to steal as much as possible of the pledgeable income.

If the government has the power to regulate the ex-post behavior of the financial institutions then it will use this possibility because, how we formally show in the model, by doing so it will reduce the cost of an eventual bailout. We show that when the policy maker has this capability, uncertainty will continue to be a good commitment device and the constructive ambiguity approach will be successful. Interestingly, this latter policy will continue to be a good one also in the case in which the policy maker has no control on the ex-post behavior of the banks but these institutions can ex-ante credibly commit to no opportunistic behavior.

In the case in which the financial institutions do not care about their future reputation, the investors will internalize this information and, in case of necessity, they will lend the minimal amount of liquidity to banks. We show that, in such cases, uncertainty looses its capability to act as a commitment device and an equilibrium may not exist. We formally prove that only when the banks have some future concerns the efficacy of the constructive ambiguity approach is restored.

2.2 Literature Review

In order to develop our model we build on Nosal and Ordoñez (2016), Farhi and Tirole (2012) and Holmstrom and Tirole (1998). Nosal and Ordoñez (2016) provide a theoretical explanation in order to justify constructive ambiguity policies. In their model government bailout uncertainty leads the banks to self restrain their behavior by issuing a Bertrand type competition for not being the first institution to show distress. Farhi and Tirole (2012) construct a framework to jointly analyze maturity mismatches and leverage and they illustrate how these leverage choices are a function of the anticipated reaction of the policy maker to the overall maturity mismatch. The common building block of these two papers is Holmstrom and Tirole (1998) that studies the efficient way to solve the liquidity needs of a firm that is credit constrained due to moral hazard problems. Our contribution is to introduce a second source of moral hazard and future concerns from the side of the banking entrepreneurs and to study how the constructive ambiguity policy is affected by these new elements.

The classical use of the words uncertainty and ambiguity in the economic decision theoretic literature refers to a situation in which the economic agents do not know the probability distribution of the state of the world. Even in our context the word ambiguity is related to a situation in which the financial institutions are uncertain of being bailed out but we don't explicitly model the attitude of these economic agents toward this uncertainty. A paper that studies bailout uncertainty in a way that is much more in line with the economic decision theoretic literature is Cukierman and Izhakian (2014) . In particular, Cukierman and Izhakian (2014), in a micro funded general equilibrium model, make use of the maxmin model of Gilboa and Schmeidler (1989) and they show how bailout uncertainty affects interest rates, default and moral hazard.

Few papers in the literature analyze the constructive ambiguity policy. Freixas (1999) studies this type of policy by focusing on the liability structure of the financial institutions. The optimal policy that arises from the analysis is to impose a critical threshold of uninsured debt above which the government will not rescue anybody. We do not explicitly focus our attention on the balance sheet of the banks and our optimal policy is a function of the informational set available to the policy maker.

In our model government behavior is time inconsistent: ex-ante the government does not want to provide bailout but ex-post, when the problem is systemic, it wants to provide help to the financial system. The literature on this type of commitment problem is really big and often the policy maker has to intervene ex-post for a "too big to fail" reason[1]. In our framework time inconsistency of no bailout policy is not necessarily driven by "too big to fail" issues and, as in Farhi and Tirole (2012), the problem may be related to coordinated actions of small banks. A paper that explicitly models herding behavior of small banks is Acharya and Yorulmazer (2007) . This paper, like Nosal and Ordoñez (2016) and our paper, also introduces the possibility of bank takeovers.

Pasten (2015) studies the interplay between bailout decisions and prudential regulations in a framework inspired by Farhi and Tirole (2012). This paper, together with Chari and Kehoe (2013), is one of the few dynamic papers that studies lack of commitment problems relative to government bailout decisions. Pasten (2015) shows how prudential policies like liquidity requirements may backfire under certain conditions. In our work the continuation value of the banking entrepreneurs plays a leading role for making constructive ambiguity a sustainable policy. We can easily interpret our framework as a reduced form of a repeated game similar in spirit to the ones of Pasten (2015) and Chari and Kehoe (2013).

The high continuation value attributed to future periods may be driven from reputational concerns from the side of the financial institutions and the main forces at play are similar to the ones that make shadow banking sustainable through the use of reputation in Ordoñez (2014).

2.3 The Model

We generalize Nosal and Ordoñez (2016) framework in order to construct a model environment in which the fundamentals of the economy $\theta \in \Theta = \{\theta_1, \ldots, \theta_n\}$ play a key role, where $\forall i \in \{1, \ldots, n\}$, $\theta_i \in [0, 1]$. In particular we denote $\theta_1 = \underline{\theta}$ and $\theta_n = \overline{\theta}$. The parameter space Θ has finite dimension and it is such that $0 \leq \theta_j < \theta_{j+1} \leq 1$ for any $j \in \{1, \ldots, n\}$. The distribution over the state space Θ, $\mu(\cdot) \in \Delta(\Theta)$ is common

[1]Stern and Feldman (2004) provide a fantastic analysis of this topic in their book.

knowledge, where we denote with $\Delta(\Theta)$ the probability simplex over the parameter space Θ. The model is characterized by two types of shocks: an idiosyncratic shock P_1, that is independent of the economic fundamental θ, and an aggregate shock P_2^θ that is decreasing as the fundamental θ grows. P_2^θ is a function of θ. In particular, for a given $\theta \in \Theta$ we partition the state space conditional on the realization of θ, in such a way that $P_0^\theta + 2P_1 + P_2^\theta = 1$, where P_0^θ denotes the probability that no bank is affected by the shock. We assume that $\theta' > \theta$ implies that $P_2^\theta > P_2^{\theta'}$ and consequently $P_0^\theta < P_0^{\theta'}$. The fact that P_1 is independent of the economic fundamentals captures the idea that, in our model, there are project's specific issues, that are independent of the economic environment. In particular these peculiarities are related to how the project is managed and they could lead to a not successful project even when the economy goes well.

The economy is populated by three types of agents: a government, two banks (or banking entrepreneurs) and a continuum of households. Time is continuous $t \in [0, 2]$. The projects of the banks are illiquid and the banks need to borrow money from households at time $t = 0$. If the project is successful it repays full payoff at time $t = 1$ otherwise the banking entrepreneurs need to borrow again short term in order to refinance the project and to get certain full payoff at time $t = 2$. When P_1 attains only one bank needs refinancing while in case P_2^θ realizes both banks need refinancing. We allow an healthy bank to takeover a distressed bank.

2.3.1 Banks

Each bank has its own endowment of assets A and at time $t = 0$ it has to decide the investment scale i to maximize its individual net worth. If $A < i$, the bank needs to borrow money from households/investors. The speed of expense outflows is proportional to the investment size and it is given by idt. This assumption clearly implies that the bank will run out of cash at $t = 1$ and, if it is hit by either an idiosyncratic or a systemic shock, it needs to refinance the project. The speed of expense outflows does not change after refinancing the project. Thus if the banking entrepreneur refinances an amount $j \leq i$ of the investment at time $t = 1$ then the bank will show distress at time $t_d = 1 + \frac{j}{i} < 2$.

We now describe the payoff structure of the investment project assuming that an amount i has been financed. The project repays, independently of the shock, a certain amount πi at time $t = 1$ in addition to an amount $(\rho_0 + \rho_1)\, i$ that is paid only if the bank is not hit by any shock. If the bank is hit by a shock then the project can be refinanced up to size $j \leq i$ and in this case it will pay $(\rho_0 + \rho_1)\, j$ at time $t = 2$. $\rho_0 + \delta\rho_1$ is the pledgeable component[2] of the payoff, where $\delta < 1$, while $(1 - \delta)\, \rho_1$ is a private benefit

[2]We make use of a more general definition of pledgeable income with respect to the one that is

that can be enjoyed only by the banking entrepreneur. If the bank refinances its project then, after that payoffs are realized at time $t = 2$, it chooses $\delta \in [0, \delta^{\max}]$, i.e. it selects which part of ρ_1 to transfer to the investors up to a maximum level of $\delta^{\max}\rho_1$. In choosing δ the banking entrepreneurs also consider their non negative and exogenous[3] continuation value $\gamma V(\delta, \theta)$ per unit of reinvestment that has the following functional form:

$$V(\delta, \theta) = \begin{cases} \theta\sqrt{\delta} & if\ \theta > \theta^V \\ 0 & if\ \theta \leq \theta^V \end{cases}$$

where $0 \leq \gamma < 1$ is a discount factor and $\theta^V \in \Theta$. Notice that the continuation value is weakly increasing in both the fundamental value θ and the transfer parameter δ. A natural interpretation of the functional form chosen for the continuation value is that picking an higher δ increases bank's future refinancing opportunities. The fact that the continuation value is per unit of reinvestment implicitly assumes that a bank will have bigger refinancing opportunities tomorrow if today has already good refinancing opportunities, i.e. it is able to refinance a great part of its investment.

In textbooks[4] pledgeable income is defined as the maximum expected amount that can be promised to investors when the entrepreneurs is paid the minimum rent to exert maximum effort. If we have to conform to this definition we may say that the pledgeable income of the investment is $\rho_0 + \delta^{\max}\rho_1$ and we could justify our model by thinking that at time $t = 2$, before payoffs are realized, investors are exposed to a source of risk that is not related to the specific investment for which the bank originally borrowed money. For instance at time $t = 2$, before repaying investors, the banking entrepreneurs may have the opportunity to make a further short term investment that will allow them to internalize a further part of the payoff of the original investment.

The private benefit that can be internalized only by the financial institution is a function of the parameter $0 \leq \delta \leq \delta^{\max} < \delta^{\sup} < 1$, where $\delta^{\max}$ is given and $\delta^{\sup}$ is specified below. This parameter, from the perspective of the investors, it can be interpreted as a measure of how much they trust the banking entrepreneur or of the type of entrepreneur they expect to meet and it can be justified in several different ways. If we use an exogenous constraints[5] on payouts explanation based on tangible benefits in order to justify the wedge $(1 - \delta)\rho_1$ then we can interpret part of this tangible benefits $\delta\rho_1$ as a potential transfer to

standard in the literature. See below for explanations and interpretations.

[3]The continuation value may be endogenized by considering a full fledged repeated game, where the basic game is repated several times. This model would only add technical complexity without adding any economic insight.

[4]See Holmstrom and Tirole (2011).

[5]See Holmstrom and Tirole (2011) for details on how to justify the wedge between pledgeable and non pledgeable income.

investors that depends on the entrepreneur's will. If we use an endogenous constraints on payouts explanation, in line with the classical theory, we can think that the entrepreneur may in principle reduce part of his private benefits, and hence of his rent, because he cares about future issues that will affect his future refinancing opportunities. It is clear that, in a model that ends at time $t = 2$ without continuation value, i.e. when $\gamma = 0$ or $\theta^V = \overline{\theta}$, the banks will always have the incentive ex-post to retain as much private benefit as possible and to set $\delta = 0$. This fact is internalized by the investors given that they are sure to receive only ρ_0. In a dynamic model with future concerns, a bank may want to set a level of $\delta \neq 0$ because it knows that it will affect its continuation value. For the reasons specified above we will denote δ as a transfer or trust parameter.

The interpretation that we like the most about our modelling structure is that investors may face banking entrepreneurs of different types, where the type is identified by $\delta \in [0, \delta^{\max}]$, and it is characterized by a different minimum rent. Following this interpretation we could imagine that ρ_0 is the pledgeable income that can be promised to investors by the 0-type entrepreneur. In our model, the fundamental value θ has a direct impact on the continuation value and consequently investors' fundamental uncertainty translates into investors' uncertainty about banks' types.

2.3.2 Households/Investors

There are two generations of households that are born at time $t = 0$ and $t = 1$. Each generation contains a continuum of risk neutral households. Households are endowed with a large amount of assets S_t when they are born. The government will eventually collect an amount of taxes T_{t+1} from them in order to finance a bailout policy. Hence the utility function of the household is $U_t = S_t - T_{t+1}$. Each household can either invest in a safe storage technology whose return is R or lend money to the bank. We assume perfect competition of the households as lenders and that borrowing is non contingent at time $t = 0$. Without loss of generality we impose that $R = 1$.

A peculiarity of our model is that investors at time $t = 1$ will lend money to the financial firm only in exchange of a pledgeable component of the payoff that is a function of the endogenous trust parameter δ. If the expected trust parameter is $\widehat{\delta}$, then the pledgeable component of the investment project perceived by the investors will be:

$$\widehat{\rho}_0 = \rho_0 + \widehat{\delta}\rho_1.$$

2.3.3 Government

The objective of the government is to maximize the welfare function W:

$$W = U_0 + U_1 + \beta V + 2\alpha j$$

where U_0 and U_1 denote utilities of the households/investors; $2j$ is the total reinvestment in the economy, i.e. the sum of the reinvestments of the two banking entrepreneurs; V is the sum of the net worth of the two banking entrepreneurs; α,β are non negative real numbers.

The first three components of the welfare function are standard. In particular, as it is common in the literature, we will assume that $\beta < 1$ in order to model the fact that welfare transfers from households to banks are costly.

The term $2\alpha j$ was already introduced by Farhi and Tirole (2012) and it captures the idea that the investment has a positive effect on the economy besides its payoffs. In fact the investment (or reinvestment in case of a crisis) has a positive effect on its stakeholders such as the workers, provided that they are better off working, and industrial firms that are involved in the project. These positive externalities are not internalized by the banking entrepreneurs but are taken into consideration by the government.

2.3.4 Basic Assumptions

In this section we introduce the first set of assumptions that make the model economically interesting.

Assumption 1 $\pi < 1,\ \rho_0 + \delta^{\max}\rho_1 < 1. and\ \forall\theta \in \Theta,\ P_0^\theta > 0$

Assumption 1 implies that both the initial investment and the reinvestment are finite independently from the transfer parameter δ and that there is always strictly positive probability that both banks are not affected by any shock. Notice that $\delta^{\max} < \delta^{\sup} = \frac{1-\rho_0}{\rho_1}$.

Assumption 2 $\rho_0 + \rho_1 > 1.$

Assumption 2 claims that reinvesting is efficient.

2.3.5 Timing

At time $t = 0$, the government decides the bailout policy and, in the δ-regulated framework, it also chooses a minimal level of δ. Then the banks make their investment decisions.

At time $t = 1$, the investors decide how much they trust the banking entrepreneurs, i.e. they define which is their level of perceived $\widehat{\delta}$ and consequently they define banks' refinancing conditions. Then the state of nature $\theta \in \Theta$ realizes according to the probability distribution μ. The fundamental of the economy θ is observed by the government and by the banking entrepreneurs. A shock eventually happens that can be either idiosyncratic with probability $2P_1$ or aggregate with probability P_2^{θ}. The banks affected by the shock decide which is the amount of the reinvestment that they want to make. The banks that are not affected by the shock get $\rho_0 + \rho_1$ in addition to π.

If an idiosyncratic shock happens and one bank shows distress, the government decides whether to bailout or not the distressed bank. In case of no bailout, the healthy banking entrepreneur can, if he wants, takeovers the distressed entrepreneur. It is important to notice that takeovers are always beneficial given that reinvesting creates value by assumption 2.

When an aggregate shock happens, the government takes the decision to bailout or not if it sees a bank in distress. In case of no bailout of the first distressed bank, the government can eventually bail out the second distressed bank.

At time $t = 2$, payoffs realize, the banks decide the level of δ, payoffs are paid and the game ends.

2.4 δ-Regulated Framework

In this section the government can impose a minimal level of transfer $\delta^{\min}$ to the financial firms. This fact will be fully internalized by the investors that will set their perceived level of transfer $\widehat{\delta} = \delta^{\min}$. In order to isolate the main forces at play we will assume that banks have no continuation value and that $\gamma = 0$. All the results that we will get under the δ-regulated framework would continue to be true also in the case in which there are future concerns and $\gamma \neq 0$ because a positive continuation value would increase the incentives of choosing a $\delta \neq 0$. Given that ex-post, in this framework without future concerns, the banks will choose δ as smaller as possible, when the government selects $\delta^{\min}$ it is as if it is selecting δ. For this reason we will simply say that the government imposes to the banks a specific level of δ. This is equivalent to study a modified version of the game in which the government at time $t = 0$ selects the level of δ together with the bailout policy.

2.4.1 Full Information

Our aim is to use the model previously described to study how investors' trust affect the behavior of the banking entrepreneurs and of the government. Notice that if μ was a dirac measure concentrated on a single $\theta \in \Theta$ and $\delta^{\max}$ was equal to 0, our model would be identical to Nosal and Ordoñez (2016)'s framework with a slightly modified social welfare function. Under the full information hypothesis the government knows whether the problem is idiosyncratic or systemic, i.e. it knows exactly how many banks were hit by the shock if any. We further assume that the government can credibly commit to a specific bailout policy.

μ is such that ex-ante banks want to invest in the project:

Assumption 3 $\pi + \rho_0 + \rho_1 > 1 + P_1 + \sum_{\theta \in \Theta} \mu(\theta) P_2^{\theta}$.

Assumption 3 makes investing efficient in expectation for the financial firms. Moreover, we impose that bailouts are socially costly but they are beneficial because they save efficient projects even when $\delta = 0$.

Assumption 4 $\beta\rho_1 > 1 - \rho_0$.

Notice that when the government saves a distressed bank the cost of the bailout net of the pledgeable part of the payoff is fully paid by the consumers while the non pledgeable component $(1-\delta)\rho_1$ is internalized by the bank. These considerations explain the weight β in front of ρ_1 when $\delta = 0$. Although according to assumption 4 the government wants to save a distressed bank instead of letting it fail, yet the best option for the government is that an healthy bank takeovers a distressed bank. In fact, given a level of transfer δ, the expected social gain of a bailout with public funds is $x(\delta) = \beta(1-\delta)\rho_1 + \delta\rho_1 + \rho_0 - 1$, while the expected social gain of a bailout with private funds is $y = \beta(\rho_1 + \rho_0 - 1)$. Notice that this latter term is positive by assumption 2 and that $x(\delta)$ is increasing in the level of δ. It easy to see that $y > x$ independently from the level of δ, in fact the following holds:

$$\begin{aligned} y - x(\delta) &= \beta(\rho_1 + \rho_0 - 1) - (\beta\rho_1(1-\delta) + \delta\rho_1 + \rho_0 - 1) \\ &= \beta\rho_1 + \beta\rho_0 - \beta - \beta\rho_1 + \beta\delta\rho_1 - \delta\rho_1 - \rho_0 + 1 \\ &= (1-\beta)(1 - \delta\rho_1 - \rho_0) > 0 \end{aligned}$$

by assumption 1.

The government bailout policy $G : [1,2) \times [0,1] \to \{0,1\}$, is a function of the time t_d at which it observes the first bank in distress and of its beliefs p^g that the shock is aggregate, where 1 denotes bailout while 0 denotes no bailout. Under full information we have that $p^g \in \{0,1\}$. We denote with t^* the first time at which the government will provide bailout to a distressed bank, i.e. $t^* = \min\{t \in [1,2) \mid G(t,p^g) = 1\}$. Remember that under commitment the government can credibly commit to a specific bailout policy t^*. The government provides targeted bailout to the distressed bank by letting the financial firm borrow at the interest rate $\rho_0 + \widehat{\delta}\rho_1$. Conditional on a perceived level of transfer $\widehat{\delta}$, we will assume that the government has 2 strategies: either provides no bailout or it provides full bailout by setting an interest rate equal to $\rho_0 + \widehat{\delta}\rho_1$. Under the δ-regulated assumption, the bailout interest rate is $\rho_0 + \delta\rho_1$ given that $\widehat{\delta} = \delta$.

The game that we are considering has the same structure of a leader and follower game: the government is the leader and it chooses the bailout policy and the transfer level δ while the banks are the followers that take as given the choice made by the government in order to decide their optimal level of cash savings c. We first study the bank's problem and then we go backward to study the optimal government policy. The type of equilibrium we are looking for is the following:

Definition 1 *A δ-regulated symmetric equilibrium under government's commitment is a bailout policy $G(t,1)$, a level of cash holdings c^* and a transfer level δ^* such that: c^* is the optimal level of cash holdings for the banks given government's policy $G(t,1)$ and the imposed transfer level δ^*; $G(t,1)$ and δ^* are optimal choices for the Government when both financial entrepreneurs select c^*.*

Let's study the bank's maximization problem. At time $t = 0$ the bank makes an investment of size i by using its own assets A and borrowing money from investors. Hence the following relationship has to be satisfied:

$$R(i - A) + ci = \pi i \implies i(c) = \frac{A}{1 - \pi + c}$$

where w.l.o.g. $R = 1$ and c denotes the savings per unit of investment of the entrepreneur for reinvestment purposes. If the bank is hit by a shock at time $t = 1$, it can refinance the project up to size $j \leq i$, where the reinvestment j is such that:

$$j - ci = \left(\rho_0 + \widehat{\delta}\rho_1\right) j \implies j\left(c, \widehat{\delta}\right) = \min\left\{\frac{c}{1 - \rho_0 - \widehat{\delta}\rho_1} i,\ i\right\}$$

where $\widehat{\delta}$ denotes the investors' trust and in our δ-regulated framework, assuming that a level δ of transfer is chosen by the government, we have that $\widehat{\delta} = \delta$. The speed of expense outflows is given by idt and if the bank makes a reinvestment of size $j\left(c, \widehat{\delta}\right)$ it will show distress at time t_d:

$$t_d = \min\left\{1 + \frac{c}{1-\rho_0-\widehat{\delta}\rho_1}, 2\right\}.$$

By inspection of the above equation it is clear that if at time $t = 0$ the bank saves a level of cash $c = 1 - \rho_0 - \widehat{\delta}\rho_1$ then it will never show distress even if it will be hit by a shock.

The bank chooses c in order to maximize its expected utility. If the bank shows distress at time $t_d\left(c, \widehat{\delta}\right)$ before the government bailout time t^* then its expected pay off will be:

$$EV\left(c\right)\big|_{t_d(c)<t^*} = \sum_{\theta\in\Theta}\mu\left(\theta\right)\left[\begin{array}{c} \left(P_0^\theta + P_1\right)\left(c + \rho_0 + \rho_1\right)i\left(c\right) \\ +P_1\left(\left(1-\widehat{\delta}\right)\rho_1 + \left(\widehat{\delta}-\delta\right)\rho_1\right)j\left(c,\widehat{\delta}\right) \\ +P_1\left(i' - j'\right)\left(\rho_0+\rho_1-1\right) \\ +P_2^\theta\left[c + \left(\rho_0+\rho_1-1+\left(\widehat{\delta}-\delta\right)\rho_1\right)\left(t_d-1\right)\right]i\left(c\right) \end{array}\right].$$

In fact fixed the state $\theta \in \Theta$, with probability $P_0^\theta + P_1$ the bank will be healthy and it will have full payoff $\rho_0 + \rho_1$ per unit of investment plus the level of cash holdings that it saved at time $t = 0$. When the bank is healthy, there is probability P_1 that the other bank will be hit by a shock, and the healthy bank can takeover the distressed bank gaining $\left(i' - j'\right)\left(\rho_0 + \rho_1 - 1\right)$, where $i' - j'$ is the amount of the investment that the other distressed financial entrepreneur cannot refinance. With probability P_1 the bank will be hit by an idiosyncratic shock and it will just gain $\left(1-\widehat{\delta}\right)\rho_1 + \left(\widehat{\delta}-\delta\right)\rho_1$, where δ is chosen by the banking entrepreneur at time $t = 2$. The gap $\left(\widehat{\delta}-\delta\right)\rho_1$ is specific of the present framework and it captures the incentives that the entrepreneurs has to shirk w.r.t. to the investors' trust parameter $\widehat{\delta}$. Finally in case the bank is hit by an aggregate shock it gains $ci\left(c\right) + \left(\rho_0+\rho_1-1+\left(\widehat{\delta}-\delta\right)\rho_1\right)\left(t_d-1\right)i\left(c\right)$. Notice that the following relationship holds:

$$\begin{aligned} & ci\left(c\right) + \left(\rho_0+\rho_1-1+\left(\widehat{\delta}-\delta\right)\rho_1\right)\left(t_d-1\right)i\left(c\right) \\ & = j\left(c,\widehat{\delta}\right) - \rho_0 j\left(c,\widehat{\delta}\right) - \widehat{\delta}\rho_1 j\left(c,\widehat{\delta}\right) + \left(\rho_0+\rho_1-1+\left(\widehat{\delta}-\delta\right)\rho_1\right)j\left(c,\widehat{\delta}\right) \\ & = \left(1-\widehat{\delta}\right)\rho_1 j\left(c,\widehat{\delta}\right) + \left(\widehat{\delta}-\delta\right)\rho_1 j\left(c,\widehat{\delta}\right) \\ & = \left(1-\delta\right)\rho_1 j\left(c,\widehat{\delta}\right) \end{aligned}$$

In our δ-regulated framework we have that $\delta = \widehat{\delta}$ and the banking entrepreneur payoff simplifies to:

$$EV(c)|_{t_d(c)<t^*} = \begin{array}{c} \left(P_0^{\mu}+P_1\right)(c+\rho_0+\rho_1)\, i(c) + \left(P_1+P_2^{\mu}\right)\left((1-\delta)\,\rho_1\right) j(c,\delta) \\ +P_1\left(i^{'}-j^{'}\right)(\rho_0+\rho_1-1) \end{array}$$

where $P_0^{\mu} = \sum_{\theta\in\Theta}\mu(\theta)\, P_0^{\theta}$ and $P_2^{\mu} = \sum_{\theta\in\Theta}\mu(\theta)\, P_2^{\theta}$.

If the bank shows distress at time $t_d(c)$ after the government bailout time t^* then its payoff in general will be equal to:

$$EV(c)|_{t_d(c)\geq t^*} = \sum_{\theta\in\Theta}\mu(\theta)\left[\begin{array}{c} \left(P_0^{\theta}+P_1\right)(c+\rho_0+\rho_1)\, i(c) \\ +P_1\left(\left(1-\widehat{\delta}\right)\rho_1+\left(\widehat{\delta}-\delta\right)\rho_1\right) j\left(c,\widehat{\delta}\right) \\ +P_1\left(i^{'}-j^{'}\right)(\rho_0+\rho_1-1) \\ +P_2^{\theta}\left[\begin{array}{c} c+\rho_0+\rho_1-(t^*-1) \\ -\left(\rho_0+\widehat{\delta}\rho_1\right)(2-t^*)+\left(\widehat{\delta}-\delta\right)\rho_1 \end{array}\right] i(c) \end{array}\right].$$

Under the δ-regulated assumption and the usual notation introduced above the expression reduces to the following:

$$EV(c)|_{t_d(c)\geq t^*} = \begin{array}{c} \left(P_0^{\mu}+P_1\right)(c+\rho_0+\rho_1)\, i(c) + P_1\left((1-\delta)\,\rho_1\right) j(c,\delta) \\ +P_1\left(i^{'}-j^{'}\right)(\rho_0+\rho_1-1) \\ +P_2^{\mu}\left[c+\rho_0+\rho_1-(t^*-1)-(\rho_0+\delta\rho_1)(2-t^*)\right] i(c) \end{array}.$$

The first order derivatives of the expected value function $\left.\frac{\partial EV(c)}{\partial c}\right|_{t_d<t^*}$ and $\left.\frac{\partial EV(c)}{\partial c}\right|_{t_d\geq t^*}$ are really important in determining the behavior of the banking entrepreneur and how he will react to the bailout policy announced by the government.

In particular, the prior μ is such that $\forall t^*\in[1,2]$, $\left.\frac{\partial EV(c)}{\partial c}\right|_{t_d<t^*}>0$ and $\left.\frac{\partial EV(c)}{\partial c}\right|_{t_d\geq t^*}<0$ when the pledgeable component perceived by the investors is $\rho_0+\delta^{\max}\rho_1$. This is formally stated in the next two assumptions:

Assumption 5 $\left.\frac{\partial EV(c)}{\partial c}\right|_{t_d<t^*,\widehat{\delta}=\delta^{\max}}>0$

Assumption 6 $\left.\frac{\partial EV(c)}{\partial c}\right|_{t_d\geq t^*,t^*=2,\widehat{\delta}=\delta^{\max}}<0$

These two latter assumptions are explicitly derived as a function of the model's parameters in the appendix in their full generality for a generic $\widehat{\delta}=\delta$. Assumption 5 states that, before being bailed out, the bank cares about the reinvestment size when the investors believe that the pledgeable part of the project is equal to $\rho_0+\delta^{\max}\rho_1$, i.e. the costs of increasing the cash holdings are smaller than the benefits. Assumption 6 models the fact that if the banking entrepreneur shows distress after the bailout date t^* then the financial firm wants to increase leverage.

In the appendix we also show that $\frac{\partial}{\partial \delta}\left(\left.\frac{\partial EV(c)}{\partial c}\right|_{t_d<t^*}\right) > 0$ and $\frac{\partial}{\partial \delta}\left(\left.\frac{\partial EV(c)}{\partial c}\right|_{t_d\geq t^*, t^*=2}\right) > 0$, hence when δ decreases $\left.\frac{\partial EV(c)}{\partial c}\right|_{t_d\geq t^*, t^*=2}$ will continue to be negative. On the contrary $\left.\frac{\partial EV(c)}{\partial c}\right|_{t_d<t^*}$ could become negative as δ decreases and in the appendix we explicitly derive the threshold level of transfer δ^t at which $\left.\frac{\partial EV(c)}{\partial c}\right|_{t_d<t^*, \widehat{\delta}=\delta^t} = 0$.

The next proposition is an extension of proposition 1 of Nosal and Ordoñez (2016) and it shows how the banking entrepreneurs select their optimal level of cash holdings given the bailout policy t^* and the level of transfer δ chosen by the government.

Proposition 2 *If assumptions 1-6 holds and the government chooses a level of $\delta > \delta^t$ and a bailout policy t^*. Then the optimal level of cash holdings of the banking entrepreneurs is $c(t^*) = (1-\rho_0-\delta\rho_1)(t^*-1)$. If $\delta < \delta^t$ then $c(t^*) = 0$. If $\delta = \delta^t$ the optimal level of cash holdings is not determinate.*

Notice that by choosing t^* and δ the government can implicitly select the level of cash holdings that it prefers for the banks. In particular there are cases in which the same level of cash holdings could be obtained with different combinations of t^* and δ.

In order to fully characterize the equilibrium we need to derive the government optimal policy. The government wants to induce the banks to choose an optimal level of cash that maximizes its ex-ante expected welfare. We follow Nosal and Ordoñez (2016) and we consider just the ex-ante expected welfare without the fixed terms that are not influenced by the level of cash holdings:

$$W = 2\left[\beta\left(\pi+\rho_0+\rho_1-1-P_1-P_2^{\mu}\right)i(c)+\alpha i(c)-(1-\beta)P_2^{\mu}(1-\rho_0-\delta\rho_1-c)\,i(c)\right]$$

Notice that this expression would remain the same even with a positive continuation value given that this latter is not a function of the level of cash holdings. The ax-ante expected welfare is obtained by subtracting from the social value that is created by always financing and fully refinancing the project the expected value of taxes that are computed as follows:

$$\begin{aligned} T^{\mu} &= 2P_2^{\mu}\left(i\left(c\right)-j\left(c\right)\right)\left(1-\rho_0-\delta\rho_1\right) \\ &= 2P_2^{\mu}\left(1-\frac{c}{1-\rho_0-\delta\rho_1}\right)\left(1-\rho_0-\delta\rho_1\right)i\left(c\right) \\ &= 2P_2^{\mu}\left(1-\rho_0-\delta\rho_1-c\right)i\left(c\right). \end{aligned}$$

where $i(c)-j(c)$ is the size of the bailout while $1-\rho_0-\delta\rho_1$ is the cost of such bailout per unit of investment. Notice that $\forall c \in [0, 1-\rho_0-\delta\rho_1)$, as δ grows the level of taxes decreases. Moreover if $c = 1-\rho_0-\delta\rho_1$ then $T^{\mu} = 0$ and increasing the level of δ will have the benefit of a bigger level of the investment i. All these considerations lead the government to select $\delta = \delta^{\max}$[6]. It is easy to see that

$$\frac{\partial W}{\partial c} > 0$$

if and only if

$$\beta < \frac{P_2^{\mu}\left(2-\rho_0-\delta\rho_1-\pi\right)-\alpha}{\pi+\rho_0+\rho_1-1-P_1-P_2^{\mu}+P_2^{\mu}\left(2-\rho_0-\delta\rho_1-\pi\right)} = \beta^t.$$

As α increases β^t decreases, in particular if $\alpha \geq P_2^{\mu}\left(2-\rho_0-\delta\rho_1-\pi\right)$ then $\beta^t = 0$. In fact as α becomes bigger the government cares less and less about the cash holdings because it wants to maximize the size of the investment given that it creates big positive externalities.

It is interesting to study the effect that δ has on the threshold β^t:

$$\begin{aligned} \frac{\partial \beta^t}{\partial \delta} &= \frac{\partial}{\partial \delta}\left(\frac{P_2^{\mu}\left(2-\rho_0-\delta\rho_1-\pi\right)-\alpha}{\pi+\rho_0+\rho_1-1-P_1-P_2^{\mu}+P_2^{\mu}\left(2-\rho_0-\delta\rho_1-\pi\right)}\right) \\ &= -P_2^{\mu}\rho_1\frac{\pi+\rho_0+\rho_1-1-P_1-P_2^{\mu}+\alpha}{\left[\pi+\rho_0+\rho_1-1-P_1-P_2^{\mu}+P_2^{\mu}\left(2-\rho_0-\delta\rho_1-\pi\right)\right]^2} < 0 \end{aligned}$$

by assumption 3. Hence as δ grows, the level of trust of the investors in the financial system becomes bigger and the banks acquire importance. In this latter situation it becomes harder for the government to care about the cash holdings instead of the size of the investment because as δ grows the cost of bailout decreases. For a given level of β, the trust parameter δ changes the incentives of the government toward the cash holdings. Imagine that $\forall \delta \in [0, \delta^{\max}]$ we have that $\beta > \beta^t$ and $\frac{\partial W}{\partial c} < 0$. In this case the policy maker

[6] Remember that even with a positive continuation value the government would like to select $\delta = \delta^{\max}$ because the continuation value is a non decreasing function of δ.

wants to reduce the level of cash holdings as much as possible and it provides immediate bailout by setting $t^* = 1$. We want to study situations in which the government care more about cash holdings w.r.t. the size of the investment, for this reason we will assume that the parameters of the model satisfy the following hypothesis:

Assumption 7 *The parameter α and β are such that $0 < \beta < \beta^t$ when $\delta = \delta^{\max}$*

Given the above discussion, it should be clear that β will continue to be smaller than β^t when $\delta = 0$.

Proposition 3 *Under assumptions 1-7 the δ-regulated symmetric equilibrium is given by $\delta^* = \delta^{\max}$, $t^* = 2$ and $c^* = 1 - \rho_0 - \delta^{\max}\rho_1$.*

Remark: notice that when assumptions 4 and 7 hold together, government bailout policy is time inconsistent. Ex-ante the government does not want to provide bailout in order to give incentives to financial institutions to increase their level of cash holdings. But ex-post the bailout is socially beneficial and the government in case of aggregate shock and no ex-ante commitment to no bailout it will intervene.

If we relax the hypothesis that the government can ex-ante commit to a specific bailout policy t^*, by following a line of reasoning essentially equivalent to the one in Nosal and Ordoñez (2016), under assumptions 4 and 6 we could define and find a non commitment equilibrium in which the government selects $\delta^* = \delta^{\max}$, it immediately intervenes when the shock is aggregate and the banks hold no capital, i.e. $c^* = 0$. In fact the government under full information knows the type of shock and if the shock is idiosyncratic it is better do not intervene and let the healthy bank to takeover the distressed bank.

2.4.2 Imperfect Information

In this subsection we will relax the assumption of full information and we will assume that the government does not know the nature of the shock. In particular when it observes a bank in distress in state θ, it updates its belief about having two banks in distress from P_2^θ to $P_2^{\theta,new}$ by using Bayes rule, i.e. the following holds:

$$P_2^{\theta,new} = \frac{P_2^\theta}{P_1 + P_2^\theta} > \frac{P_2^\theta}{P_0^\theta + 2P_1 + P_2^\theta} = P_2^\theta.$$

If

$$\left[P_2^{\theta,new}\left(2x\left(\delta^{\max}\right)\right) + \left(1 - P_2^{\theta,new}\right)x\left(\delta^{\max}\right)\right](2 - t_d)\, i$$
$$< \left[P_2^{\theta,new}\left(x\left(\delta^{\max}\right)\right) + \left(1 - P_2^{\theta,new}\right)y\right](2 - t_d)\, i$$

then the interim welfare from saving the first bank in distress is smaller than the interim welfare of waiting and eventually saving the second bank in distress. The next assumption extends this condition to any θ and δ. When this condition holds, the government wants to wait and it does not intervene when it sees the first bank in distress.

Assumption 8 $P_2^{\theta,new} < 1 - \frac{x(\delta^{\max})}{y}$

Given that we are in a δ-regulated framework, the banks take as given the level of δ imposed by the government and, in order to decide the level of cash c that they want to hold, they have to look at the choice $c^{'}$ made by the opponent institution. In fact, under assumption 8, the expected payoff of a generic financial institution is specified as follows:

if $t_d\left(c,\delta\right) < t_d\left(c^{'},\delta\right)$,

$$EV\left(c\right)|_{t_d(c,\delta)<t_d\left(c^{'},\delta\right)} = \begin{array}{c}\left(P_0^{\mu}+P_1\right)\left(c+\rho_0+\rho_1\right)i\left(c\right)+\left(P_1+P_2^{\mu}\right)\left(\left(1-\delta\right)\rho_1\right)j\left(c,\delta\right)\\ +P_1\left(i^{'}-j^{'}\right)\left(\rho_0+\rho_1-1\right)\end{array} \tag{2.1}$$

if $t_d\left(c,\delta\right) = t_d\left(c^{'},\delta\right)$,

$$EV\left(c\right)|_{t_d(c,\delta)=t_d\left(c^{'},\delta\right)} = \begin{array}{c}\left(P_0^{\mu}+P_1\right)\left(c+\rho_0+\rho_1\right)i\left(c\right)+P_1\left(\left(1-\delta\right)\rho_1\right)j\left(c,\delta\right)\\ +P_1\left(i^{'}-j^{'}\right)\left(\rho_0+\rho_1-1\right)\\ +P_2^{\mu}\left[c+\rho_0+\rho_1-\left(t_d\left(c,\delta\right)-1\right)-\left(\rho_0+\delta\rho_1\right)\left(2-t_d\left(c,\delta\right)\right)\right]\frac{i(c)}{2}\end{array} \tag{2.2}$$

and if $t_d\left(c,\delta\right) > t_d\left(c^{'},\delta\right)$

$$EV\left(c\right)|_{t_d(c,\delta)>t_d\left(c^{'},\delta\right)} = \begin{array}{c}\left(P_0^{\mu}+P_1\right)\left(c+\rho_0+\rho_1\right)i\left(c\right)+P_1\left(\left(1-\delta\right)\rho_1\right)j\left(c,\delta\right)\\ +P_1\left(i^{'}-j^{'}\right)\left(\rho_0+\rho_1-1\right)\\ +P_2^{\mu}\left[c+\rho_0+\rho_1-\left(t_d\left(c,\delta\right)-1\right)-\left(\rho_0+\delta\rho_1\right)\left(2-t_d\left(c,\delta\right)\right)\right]i\left(c\right)\end{array}. \tag{2.3}$$

We assumed that if both banks show distress at the same moment then the government saves only one bank and each bank has a probability 1/2 of being rescued.

The type of equilibrium we are looking for under imperfect information is a modified version of the notion of equilibrium that we previously introduced. Under this new equilibrium notion, the government does not provide bailout to the first bank in distress while it always saves the second bank showing distress because, in this latter situation, it is sure that the shock is aggregate. We denote this government's policy as constructive ambiguity.

Definition 4 *A δ-regulated symmetric equilibrium under imperfect information and delay (assumption 8 holds) is a bailout policy $G(t, p^g)$, a level of cash holdings c^* and a transfer level δ^* such that: c^* is the optimal level of cash holdings for the banks given the imposed transfer level δ^* and government's constructive ambiguity policy $G(t, p^g)$; $G(t, p^g)$ and δ^* are optimal choices for the Government when both financial entrepreneurs select c^*.*

As usual, under assumptions 4 and 7, increasing δ will decrease both the cost and the probability of a bailout. Moreover when the banks choose the maximal level of cash given δ, letting this latter parameter growing will increase the size of the investment without increasing the probability of showing distress of the 2 financial entrepreneurs. As a consequence the government will always select $\delta = \delta^{\max}$ and the banks will take this level of transfer as given in their optimization problem.

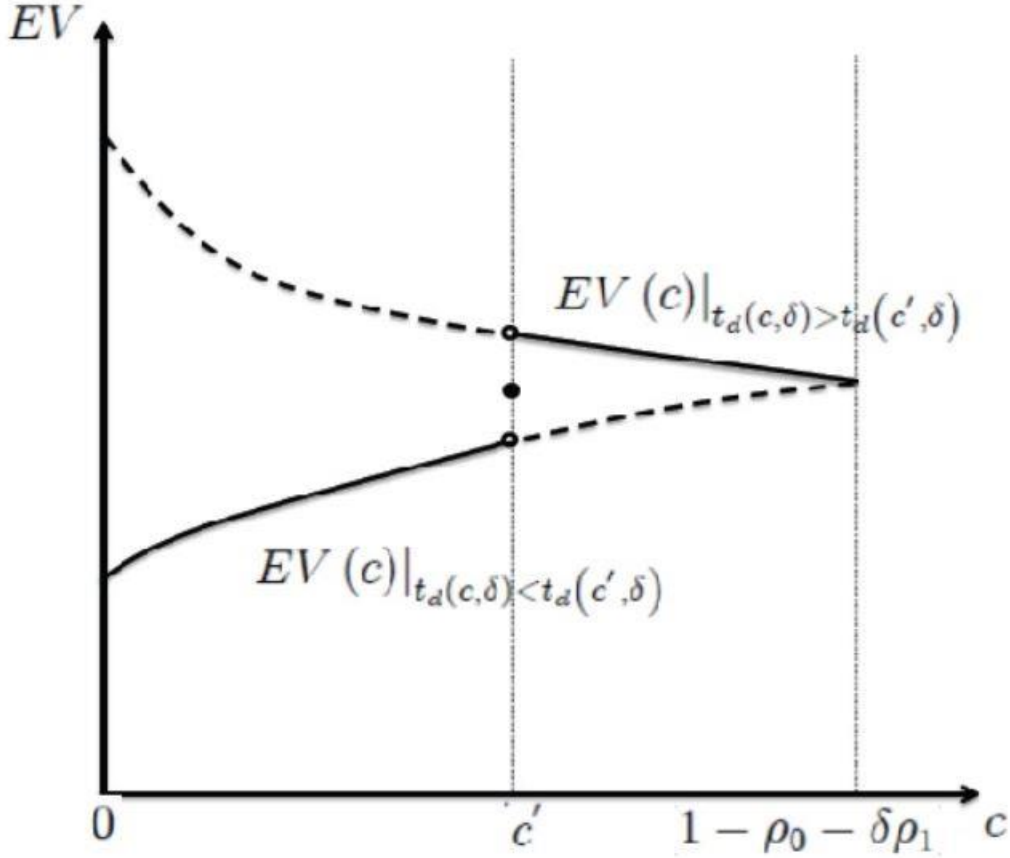
Exactly as in Nosal and Ordoñez (2016), each bank in deciding its level of cash c, by taking as given the level of cash $c^{'} \in [0, 1 - \rho_0 - \delta^{\max}\rho_1)$ of the opponent financial entrepreneur, has an incentive to choose $c^{'} + \varepsilon$ with ε strictly positive and small. In fact by making this choice, the financial entrepreneur will have a discontinuous jump in its payoff at a cost of reducing slightly the amount of the investment. A graphical representation of this situation is contained in figure 2.1, where we plotted the expected payoff of a generic bank as a function of the cash level c.

This sort of race between the financial institutions will continue till both entrepreneurs will choose $c = c^{'} = 1 - \rho_0 - \delta^{\max}\rho_1$. These considerations allow us to extend the main result of Nosal and Ordoñez (2016) to our δ-regulated framework:

Proposition 5 *Under assumptions 1-8 the unique δ-regulated symmetric equilibrium under imperfect information and delay is given by $\delta^* = \delta^{\max}$, government's constructive ambiguity policy $G(t, p^g)$ and $c^* = 1 - \rho_0 - \delta^{\max}\rho_1$.*

According to this result the constructive ambiguity approach under the delay condition seems to be a successful policy. Unfortunately as we will show in the next section, without any considerations about the future from the side of the financial entrepreneurs, the result will not anymore be true once we exit from the δ-regulated framework.

Figure 2.1: Expected Payoff when $\delta > \delta^t$



2.5 General Framework

In the present section we consider situations in which the threshold level δ^t is strictly positive and the government cannot anymore impose to the banks a minimum level of δ.

2.5.1 No Future Concerns

Only for this subsection we will continue to assume that $\gamma = 0$ and that there are no future concerns. A really interesting result is that, under full information and government's commitment, the equilibrium allocation reached in proposition 3 could be obtained also if we relax the assumption that the level of transfer δ is imposed by the government and we assume that the financial entrepreneur can commit ex-ante to a specific level of δ by choosing together both the size of the investment and the transfer after that the government has established its bailout policy. In this modified version of the game, investors' perceived level of trust $\widehat{\delta}$ is still equal to δ because banks' commitment is credible. The next proposition summarize this result. The concept of equilibrium that we use is equivalent to the δ-regulated equilibrium introduced above except for the fact that now the transfer δ is chosen ex-ante by the financial institution, we denote this equilibrium as the full commitment equilibrium. In fact the government commits to a bailout policy while the bank commits to a specific level of δ.

Proposition 6 *Under assumptions 1-7 the full commitment equilibrium is given by,* $t^* = 2$*,* $\delta^* = \delta^{\max}$ *and* $c^* = 1 - \rho_0 - \delta^{\max}\rho_1$*.*

The main message of proposition 6 is that ex-ante the banks have all the incentives to behave properly even without future concerns because in this way they have better refinancing opportunities if they are hit by a shock[7]. The problem is that ex-post, if there are no future concerns, these incentives disappear.

We will now focus our attention to the imperfect information case without ex-ante commitment from the side of the banks.

When the financial entrepreneurs cannot commit ex-ante to a certain level of δ, given that in our model there are no future concerns, at time $t = 2$ they will want to lower δ as much as possible and, independently of the state $\theta \in \Theta$, they will set $\delta = 0$. In equilibrium, the rational investors anticipate banks' behavior and they set their level of perceived transfer $\widehat{\delta} = 0 = \delta$.

[7] By now it should be clear why in our framework it is highly realistic to assume that the continuation value is non decreasing in δ.

The value functions of the banks under imperfect information and assumptions 1-8, are given by equations 2.1-2.3. In order to understand what happens when $\delta = 0$, we graphically show how these functions behave when $\delta = \delta^t$ and $\delta < \delta^t$. In particular we plot how the payoff of a financial institution varies as a function of the chosen level of cash c, conditional the level of cash $c^{'}$ chosen by the opponent financial entrepreneur. If $\delta \geq \delta^t$[8], then the constructive ambiguity approach works fine because the competition between banks will lead them to increase the level of cash holdings till the level of $1 - \rho_0 - \delta\rho_1$ is reached by both the financial institutions.

The problem is that in our general framework without future concerns we end up in the situation depicted in figure 2.3, because $\delta = 0 < \delta^t$. In this case, under assumptions 1-8, an equilibrium does not exist even under the constructive ambiguity policy. Suppose that the opponent bank holds $c^{'} = 0$, then the financial entrepreneur will choose $c = 0 + \varepsilon$ with ε small and strictly positive. The opponent bank reacts to this choice by choosing $c^{'} = c + \varepsilon$ and this type of behavior will continue till one of the two banks reach a point at which $EV(c)|_{t_d(c,\delta) > t_d\left(c^{'},\delta\right)} = EV(0)$. In this situation the other bank does not want to go a bit further in terms of cash holdings and the best option will be to go back to 0 cash holdings and the cycle restarts.

This reasoning showed that an equilibrium does not exist in general in our framework when the banks don't have any concerns about their future continuation value.

Remark: there is a nice parallelism between the present framework without future concerns and an alternative model where the banking entrepreneurs care about the future but investors are uncertainty averse. In fact an equivalent no equilibrium situation can be reached if the investors' preferences exhibits an extreme uncertainty aversion behavior and consequently they always associate to the banks the lowest possible type.

2.5.2 Future Matters

We believe that the situation depicted in the present setting best describes a crisis scenario where markets are illiquid. The objective of this subsection is to show that it is possible to restore the successfulness of the constructive ambiguity policy and to reach a positive equilibrium provided that the banks care enough about the future. Hence an

[8]For the case $\delta > \delta^t$, look at Figure 2.1 of the previous section.

Figure 2.2: Expected Payoff when $\delta = \delta^t$

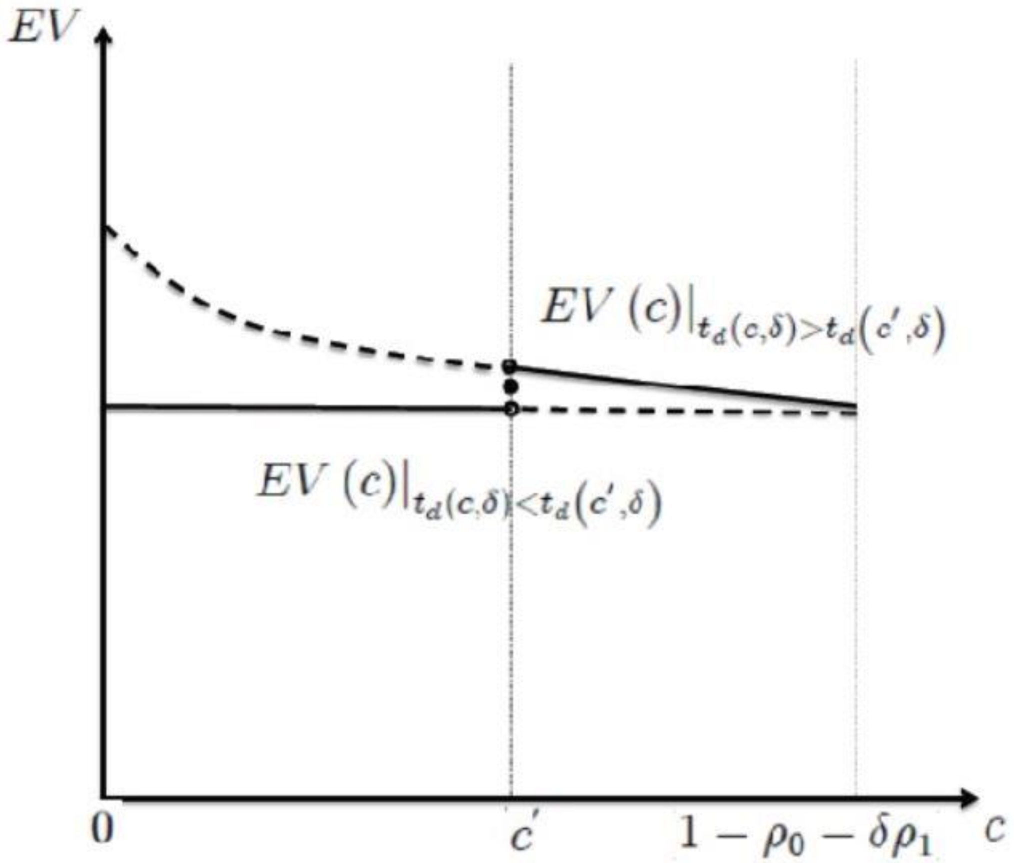
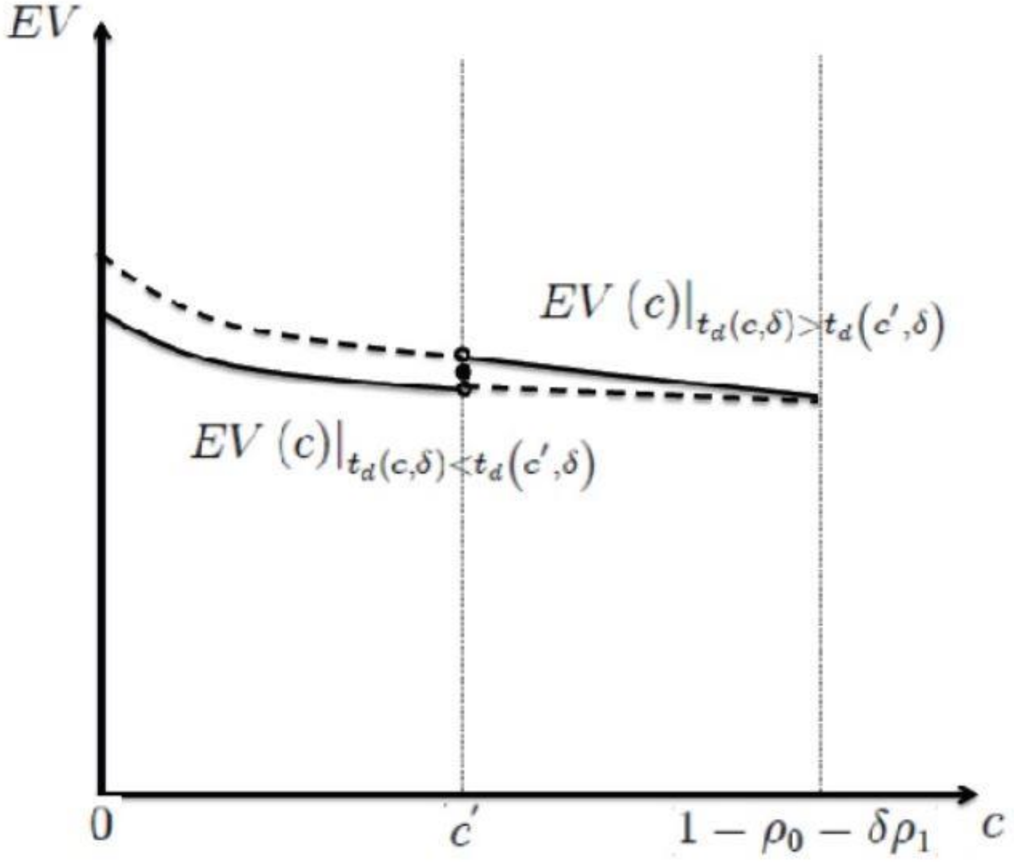


Figure 2.3: Expected Payoff when $\delta < \delta^t$



important policy implication of the present analysis is that if the government wants to use a constructive ambiguity policy in a successful way then it has to complement it with market policies that increase the continuation value of the banks that behave properly.

We start by relaxing the hypothesis of no future concerns and by assuming that $\gamma \neq 0$. As usual we make use of a backward approach. At time $t = 2$ after the realization of the fundamental value $\theta \in \Theta$, the banking entrepreneur solves the following optimization problem:

$$\max_{\delta \in [0,\delta^{\max}]} (1-\delta)\rho_1 + \gamma V(\delta,\theta).$$

Given the above stated functional form for the continuation value, we have that the ex-ante optimal strategy $\delta^* : \Theta \to [0, \delta^{\max}]$ is given by the following expression[9]:

$$\delta^*(\theta) = \begin{cases} \left(\frac{\gamma\theta}{2\rho_1}\right)^2 & if\ \theta > \theta^V \\ 0 & if\ \theta \leq \theta^V \end{cases}$$

Obviously the rational investors internalize this fact and at time $t = 1$ and, before θ realizes, they define the refinancing condition for a distressed bank by setting $\widehat{\delta} = \sum_{\theta \in \Theta} \mu(\theta)\delta^*(\theta)$. We assume that the common prior μ is such that the following holds:

Assumption 9 $\sum_{\theta \in \Theta} \mu(\theta)\delta^*(\theta) \geq \delta^t$.

The above assumption implies that when the banking entrepreneurs care about the future the market becomes enough liquid in such a way that a positive equilibrium can be sustained. The type of equilibrium we are looking for is the following:

Definition 7 *A symmetric equilibrium under imperfect information and delay (assumption 8 holds) is a bailout policy $G(t,p^g)$, a level of cash holdings c^* and a banks' transfer strategy δ^* such that: c^* is the optimal level of cash holdings for the banks given investors' perceived level of transfer $\widehat{\delta} = \sum_{\theta \in \Theta} \mu(\theta)\delta^*(\theta)$, where δ^* denote the banks' optimal transfer strategy, and government's constructive ambiguity policy $G(t,p^g)$; $G(t,p^g)$ is an optimal choice for the Government when both financial entrepreneurs select c^* and investors' perceived level of transfer is $\widehat{\delta}$; δ^* is an optimal transfer strategy for the banking entrepreneurs.*

It is now straightforward to make a type of analysis similar to the one of the previous subsection and to verify the following proposition:

[9] We assume that the parameters of the problem are such that an internal solution exists, i.e. $\left(\frac{\gamma\theta}{2\rho_1}\right)^2 \in [0, \delta^{\max}]$.

Proposition 8 *Under assumptions 1-9 the unique symmetric equilibrium under imperfect information and delay is given by* $\delta^* = \delta^*(\theta) = \begin{cases} \left(\frac{\gamma\theta}{2\rho_1}\right)^2 & if\ \theta > \theta^V \\ 0 & if\ \theta \leq \theta^V \end{cases}$, *government's constructive ambiguity policy* $G(t, p^g)$ *and* $c^* = 1 - \rho_0 - \widehat{\delta}\rho_1$.

2.6 Conclusions

By introducing the possibility that the banks can ex-post steal part of the potentially pledgeable income, we have shown that, when the banking entrepreneurs do not have future concerns, the constructive ambiguity approach is successful only when the government can regulate banks' ex-post behavior in a δ-regulated framework or when the financial entrepreneurs have credible commitment devices.

In general when the banks do not have the possibility to commit ex-ante to no stealing then an equilibrium may not exist and the constructive ambiguity policy may fail. Only the introduction of future concerns considerations from the perspective of the financial institutions allow us to restore the successfulness of the constructive ambiguity policy.

2.7 Appendix: δ-regulated framework

2.7.1 Derivation of $\left.\frac{\partial EV(c)}{\partial c}\right|_{t_d<t^*}$

By using the fact that $ci(c) = (\pi - 1)\, i(c) + A$ and $j(c,\delta)(1-\rho_0-\delta\rho_1) = ci(c)$ we have that the following holds:

$$\left.\frac{\partial EV(c)}{\partial c}\right|_{t_d<t^*} = \frac{\partial}{\partial c}\left(\begin{array}{c}(P_0^\mu + P_1)(c+\rho_0+\rho_1)\, i(c) + (P_1+P_2^\mu)((1-\delta)\rho_1)\, j(c,\delta) \\ +P_1\left(i' - j'\right)(\rho_0+\rho_1-1)\end{array}\right)$$

$$= \frac{\partial}{\partial c}\left(\begin{array}{c}(P_0^\mu + P_1)(\rho_0+\rho_1)\, i(c) + P_1\left(i'-j'\right)(\rho_0+\rho_1-1) \\ +\left[P_0^\mu + P_1 + \frac{\left(P_1+P_2^\mu\right)((1-\delta)\rho_1)}{(1-\rho_0-\delta\rho_1)}\right] ci(c)\end{array}\right)$$

$$= \frac{\partial}{\partial c}\left(\begin{array}{c}(P_0^\mu + P_1)(\rho_0+\rho_1)\, i(c) + P_1\left(i'-j'\right)(\rho_0+\rho_1-1) \\ +\left[P_0^\mu + P_1 + \frac{\left(P_1+P_2^\mu\right)((1-\delta)\rho_1)}{(1-\rho_0-\delta\rho_1)}\right]((\pi-1)\, i(c) + A)\end{array}\right)$$

$$= \left(\begin{array}{c}(P_0^\mu + P_1)(\rho_0+\rho_1) \\ +\left[P_0^\mu + P_1 + \frac{\left(P_1+P_2^\mu\right)((1-\delta)\rho_1)}{(1-\rho_0-\delta\rho_1)}\right](\pi-1)\end{array}\right)\frac{\partial i(c)}{\partial c}$$

Notice that

$$\frac{\partial i(c)}{\partial c} = \frac{\partial}{\partial c}\left(\frac{A}{1-\pi+c}\right) = -\frac{A}{(1-\pi+c)^2} < 0$$

Hence $\left.\frac{\partial EV(c)}{\partial c}\right|_{t_d<t^*} > 0$ if and only if

$$(P_1+P_2^\mu)((1-\delta)\rho_1)\frac{(1-\pi)}{(1-\rho_0-\delta\rho_1)} - (P_0^\mu + P_1)(\rho_0+\rho_1+\pi-1) > 0$$

2.7.2 Derivation of $\frac{\partial}{\partial\delta}\left(\left.\frac{\partial EV(c)}{\partial c}\right|_{t_d<t^*}\right) > 0$

Let's compute $\frac{\partial}{\partial\delta}\left(\left.\frac{\partial EV(c)}{\partial c}\right|_{t_d<t^*}\right) > 0$.

$$\frac{\partial}{\partial \delta}\left(\left.\frac{\partial EV(c)}{\partial c}\right|_{t_d<t^*}\right) = \frac{\partial}{\partial \delta}\left[\frac{(P_1+P_2^{\mu})\,((1-\delta)\,\rho_1)}{(1-\rho_0-\delta\rho_1)}(\pi-1)\frac{\partial i(c)}{\partial c}\right]$$
$$= \frac{-(P_1+P_2^{\mu})\,\rho_1\,(1-\rho_0-\delta\rho_1)}{(1-\rho_0-\delta\rho_1)^2}(\pi-1)\frac{\partial i(c)}{\partial c}$$
$$+\frac{\rho_1\,(P_1+P_2^{\mu})\,(1-\delta)\,\rho_1}{(1-\rho_0-\delta\rho_1)^2}(\pi-1)\frac{\partial i(c)}{\partial c}$$
$$= \frac{(P_1+P_2^{\mu})\,\rho_1\,(\rho_1+\rho_0-1)}{(1-\rho_0-\delta\rho_1)^2}(\pi-1)\frac{\partial i(c)}{\partial c}$$

By assumptions 1 and 2 and the fact $\frac{\partial i(c)}{\partial c} < 0$, this derivative is always positive.

2.7.3 Derivation of $\left.\frac{\partial EV(c)}{\partial c}\right|_{t_d \geq t^*, t^*=2}$

$$\left.\frac{\partial EV(c)}{\partial c}\right|_{t_d\geq t^*} = \frac{\partial}{\partial c}\begin{pmatrix} (P_0^{\mu}+P_1)\,(c+\rho_0+\rho_1)\,i(c)+P_1\,((1-\delta)\,\rho_1)\,j(c,\delta) \\ +P_1\left(i'-j'\right)(\rho_0+\rho_1-1) \\ +P_2^{\mu}\left[c+\rho_0+\rho_1-(t^*-1)-(\rho_0+\delta\rho_1)\,(2-t^*)\right]i(c) \end{pmatrix}$$
$$= \frac{\partial}{\partial c}\begin{pmatrix} (P_0^{\mu}+P_1+P_2^{\mu})\,(\rho_0+\rho_1)\,i(c)+P_1\,((1-\delta)\,\rho_1)\,j(c,\delta) \\ (P_0^{\mu}+P_1+P_2^{\mu})\,ci(c)+P_1\left(i'-j'\right)(\rho_0+\rho_1-1) \\ +P_2^{\mu}\left[-(t^*-1)-(\rho_0+\delta\rho_1)\,(2-t^*)\right]i(c) \end{pmatrix}$$

By using again the fact that $ci(c) = (\pi-1)\,i(c)+A$ and $j(c,\delta)\,(1-\rho_0-\delta\rho_1) = ci(c)$ we get:

$$\left.\frac{\partial EV(c)}{\partial c}\right|_{t_d\geq t^*} = \frac{\partial}{\partial c}\begin{pmatrix} (P_0^{\mu}+P_1+P_2^{\mu})\,(\rho_0+\rho_1)\,i(c)+\frac{P_1((1-\delta)\rho_1)}{1-\rho_0-\delta\rho_1}\,((\pi-1)\,i(c)+A) \\ +(P_0^{\mu}+P_1+P_2^{\mu})\,((\pi-1)\,i(c)+A)+P_1\left(i'-j'\right)(\rho_0+\rho_1-1) \\ +P_2^{\mu}\left[-(t^*-1)-(\rho_0+\delta\rho_1)\,(2-t^*)\right]i(c) \end{pmatrix}$$
$$= \begin{pmatrix} (P_0^{\mu}+P_1+P_2^{\mu})\,(\rho_0+\rho_1)+\frac{P_1((1-\delta)\rho_1)}{1-\rho_0-\delta\rho_1}(\pi-1) \\ +(P_0^{\mu}+P_1+P_2^{\mu})\,(\pi-1) \\ +P_2^{\mu}\left[-(t^*-1)-(\rho_0+\delta\rho_1)\,(2-t^*)\right] \end{pmatrix}\frac{\partial i(c)}{\partial c}$$
$$= \begin{pmatrix} (1-P_1)\,(\rho_0+\rho_1)+\frac{P_1((1-\delta)\rho_1)}{1-\rho_0-\delta\rho_1}(\pi-1) \\ +(1-P_1)\,(\pi-1) \\ +P_2^{\mu}\left[-(t^*-1)-(\rho_0+\delta\rho_1)\,(2-t^*)\right] \end{pmatrix}\frac{\partial i(c)}{\partial c}$$

We want to restrict the parameters of the model in such a way that after bailout is provided for the bank is optimal to increase leverage, i.e. $\left.\frac{\partial EV(c)}{\partial c}\right|_{t_d\geq t^*} < 0$. Given that

$\frac{\partial i(c)}{\partial c} < 0$ we have that if this condition is satisfied for $t^* = 2$ then it is satisfied for any t^*. By imposing this restriction we get that:

$$\left.\frac{\partial EV(c)}{\partial c}\right|_{t_d \geq t^*} = \left(\begin{array}{c} (1-P_1)(\rho_0+\rho_1) + \frac{P_1((1-\delta)\rho_1)}{1-\rho_0-\delta\rho_1}(\pi-1) \\ (1-P_1)(\pi-1) - P_2^{\mu} \end{array} \right) \frac{\partial i(c)}{\partial c}$$

and we have that $\left.\frac{\partial EV(c)}{\partial c}\right|_{t_d \geq t^*} < 0$ if and only if

$$(1-P_1)(\pi+\rho_0+\rho_1-1) - \frac{P_1((1-\delta)\rho_1)}{1-\rho_0-\delta\rho_1}(1-\pi) - P_2^{\mu} > 0$$

2.7.4 Derivation of $\frac{\partial}{\partial \delta}\left(\left.\frac{\partial EV(c)}{\partial c}\right|_{t_d \geq t^*, t^*=2}\right) > 0$

We now derive the explicit expression for $\frac{\partial}{\partial \delta}\left(\left.\frac{\partial EV(c)}{\partial c}\right|_{t_d \geq t^*}\right) > 0$:

$$\begin{aligned}
\frac{\partial}{\partial \delta}\left(\left.\frac{\partial EV(c)}{\partial c}\right|_{t_d \geq t^*, t^*=2}\right) &= \frac{\partial}{\partial \delta}\left[\left(\begin{array}{c} (1-P_1)(\rho_0+\rho_1) + \frac{P_1((1-\delta)\rho_1)}{1-\rho_0-\delta\rho_1}(\pi-1) \\ +(1-P_1)(\pi-1) - P_2^{\mu} \end{array} \right) \frac{\partial i(c)}{\partial c}\right] \\
&= \frac{\partial}{\partial \delta}\left[\frac{P_1((1-\delta)\rho_1)}{1-\rho_0-\delta\rho_1}(\pi-1)\frac{\partial i(c)}{\partial c}\right] \\
&= \left(\frac{-P_1\rho_1(1-\rho_0-\delta\rho_1) + \rho_1(P_1\rho_1 - P_1\delta\rho_1)}{(1-\rho_0-\delta\rho_1)^2}\right)(\pi-1)\frac{\partial i(c)}{\partial c} \\
&= \left(\frac{P_1\rho_1(\rho_1+\rho_0-1)}{(1-\rho_0-\delta\rho_1)^2}\right)(\pi-1)\frac{\partial i(c)}{\partial c}
\end{aligned}$$

By assumptions 1 and 2 and the fact $\frac{\partial i(c)}{\partial c} < 0$ and $t^* \in [1,2]$, this derivative is always positive.

2.7.5 Proof of Proposition 2

If $\delta > \delta^t$, we have that $\left.\frac{\partial EV(c)}{\partial c}\right|_{t_d \geq t^*} < 0$ and $\left.\frac{\partial EV(c)}{\partial c}\right|_{t_d < t^*} > 0$.

If $c > (1-\rho_0-\delta\rho)(t^*-1)$, then $t_d = 1 + \frac{c}{1-\rho_0-\delta\rho} > t^*$ and by assumption 6 $\left.\frac{\partial EV(c)}{\partial c}\right|_{t_d \geq t^*} < 0$. Consequently the bank will reduce its level of cash.

If $c < (1-\rho_0-\delta\rho_0)(t^*-1)$ then $t_d < t^*$ and by assumption 5 $\left.\frac{\partial EV(c)}{\partial c}\right|_{t_d < t^*} > 0$ Consequently the bank will increase its level of cash. These considerations hold when $t_d < 2$. If $t_d \geq 2$, then

$$
\begin{aligned}
EV(c) &= \sum_{\theta\in\Theta}\mu(\theta)\left[\begin{array}{c}\left(P_0^{\theta}+P_1\right)(c+\rho_0+\rho_1)\,i(c)+\left(P_1+P_2^{\theta}\right)(c+\rho_0+\rho_1-1)\,i(c)\\ +P_1\left(i'-j'\right)(\rho_0+\rho_1-1)\end{array}\right]\\
&= \sum_{\theta\in\Theta}\mu(\theta)\left[\begin{array}{c}ci(c)+\left(\rho_0+\rho_1-\left(P_1+P_2^{\theta}\right)\right)i(c)\\ +P_1\left(i'-j'\right)(\rho_0+\rho_1-1)\end{array}\right]\\
&= \sum_{\theta\in\Theta}\mu(\theta)\left[\begin{array}{c}A+\left(\pi-1+\rho_0+\rho_1-\left(P_1+P_2^{\theta}\right)\right)i(c)\\ +P_1\left(i'-j'\right)(\rho_0+\rho_1-1)\end{array}\right]\\
&= A+\left(\pi-1+\rho_0+\rho_1-\left(P_1+P_2^{\mu}\right)\right)i(c)+P_1\left(i'-j'\right)(\rho_0+\rho_1-1)
\end{aligned}
$$

By taking the derivative w.r.t. c we obtain:

$$
\frac{\partial EV(c)}{\partial c}=\left(\pi-1+\rho_0+\rho_1-\left(P_1+P_2^{\mu}\right)\right)\frac{\partial i(c)}{\partial c}
$$

and $\frac{\partial EV(c)}{\partial c}<0$ by the fact that $\frac{\partial i(c)}{\partial c}<0$ and by assumption 3. Hence the bank also in this case wants to reduce the level of cash holdings. ■

2.7.6 Derivation of the threshold δ^t

By solving $\left.\frac{\partial EV(c)}{\partial c}\right|_{t_d<t^*}=0$ as a function of δ we obtain that

$$
\left(P_0^{\mu}+P_1\right)(\rho_0+\rho_1)+\left[P_0^{\mu}+P_1+\frac{\left(P_1+P_2^{\mu}\right)\left((1-\delta)\rho_1\right)}{(1-\rho_0-\delta\rho_1)}\right](\pi-1)=0 \tag{2.4}
$$

$$
\left(P_0^{\mu}+P_1\right)(\rho_0+\rho_1+\pi-1)-\frac{\left(P_1+P_2^{\mu}\right)\left((1-\delta)\rho_1\right)}{(1-\rho_0-\delta\rho_1)}(1-\pi)=0
$$

$$
\left(P_0^{\mu}+P_1\right)(\rho_0+\rho_1+\pi-1)-\frac{\left(P_1+P_2^{\mu}\right)(\rho_1-\delta\rho_1)}{(1-\rho_0-\delta\rho_1)}(1-\pi)=0
$$

$$
\left(P_0^{\mu}+P_1\right)(\rho_0+\rho_1+\pi-1)(1-\rho_0-\delta\rho_1)-\left(P_1+P_2^{\mu}\right)(\rho_1-\delta\rho_1)(1-\pi)=0
$$

$$
\left[\left(P_0^{\mu}+P_1\right)(\rho_0+\rho_1)+\pi-1\right](1-\rho_0-\delta\rho_1)+\left[\left(P_1+P_2^{\mu}\right)(1-\rho_0-\rho_1)\right](1-\pi)=0
$$

$$
\delta^t=\frac{1}{\rho_1}\left[1-\frac{\left(P_1+P_2^{\mu}\right)(\rho_0+\rho_1-1)(1-\pi)}{\left(P_0^{\mu}+P_1\right)(\rho_0+\rho_1)-(1-\pi)}-\rho_0\right]
$$

Notice that $\delta^t < \delta^{\max}$ because the left hand side of equation 2.4 increases as δ decreases and it is negative at $\delta = \delta^{\max}$ by assumption 5. Notice that by assumption 2:

$$\rho_0 + \rho_1 - 1 > 0.$$

■

2.7.7 Proof of Proposition 3

By assumption 7 we know that $\forall \delta \in [0, \delta^{\max}]$,

$$\frac{\partial W}{\partial c} > 0.$$

Consequently the government wants to induce the financial firms to choose $c^* = 1-\rho_0-\delta\rho_1$ and by proposition 2 if $\delta > \delta^t$ this can be obtained by setting $t^* = 2$, i.e. by never providing bailout. Assume that $\delta > \delta^t$. We will show that when $c^*(\delta) = 1-\rho_0-\delta\rho_1$ the government has always the incentive to increase δ. The ex-ante expected social welfare is:

$$\begin{aligned} W &= 2\left[\beta\left(\pi+\rho_0+\rho_1-1-P_1-P_2^{\mu}\right)i(c)+\alpha i(c)-(1-\beta)P_2^{\mu}\left(1-\rho_0-\delta\rho_1-c\right)i(c)\right] \\ &= \left[\beta\left(\pi+\rho_0+\rho_1-1-P_1-P_2^{\mu}\right)+\alpha\right]\frac{2A}{1-\pi+1-\rho_0-\delta\rho_1} \end{aligned}$$

where we used the fact that $i(1-\rho_0-\delta\rho_1) = \frac{A}{1-\pi+1-\rho_0-\delta\rho_1}$. By derivating this latter expression w.r.t. δ we get that:

$$w\frac{\partial W}{\partial \delta} = \left[\beta\left(\pi+\rho_0+\rho_1-1-P_1-P_2^{\mu}\right)+\alpha\right]\frac{2\rho_1 A}{\left(1-\pi+1-\rho_0-\delta\rho_1\right)^2} > 0$$

by assumption 3.■

2.7.8 Proof of Proposition 6

Let's assume that the government chooses $t^* = 2$.

If $\delta < \delta^t$ then $c^* = 0$ and

$$EV(0) = \left(P_0^{\mu}+P_1\right)\left(\rho_0+\rho_1\right)\frac{A}{1-\pi} + P_1\left(i^{'}-j^{'}\right)\left(\rho_0+\rho_1-1\right).$$

Given that $EV(0)$ is constant and independent from δ, we know that when $\delta > \delta^t$ the bank by setting $c^* = 1-\rho_0-\delta\rho_1$ obtains a payoff bigger than $EV(0)$. For this reason for the bank ex-ante is optimal to set $\delta > \delta^t$. When $\delta > \delta^t$ the optimal choice of the bank is $c^* = 1 - \rho_0 - \delta\rho_1$. At this value for the financial entrepreneur is optimal to further increase δ if $\frac{\partial}{\partial\delta}\left(EV\left(1-\rho_0-\delta\rho_1\right)|_{t_d<t^*}\right) > 0$. Notice that

$$EV\left(1-\rho_0-\delta\rho_1\right) = A\frac{\left(P_0^\mu + P_1\right)\left(1+\left(1-\delta\right)\rho_1\right) + \left(P_1 + P_2^\mu\right)\left(\left(1-\delta\right)\rho_1\right)}{1-\pi+1-\rho_0-\delta\rho_1} + P_1\left(i^{'}-j^{'}\right)\left(\rho_0+\rho_1-1\right).$$

By derivating this expression w.r.t. δ we get that:

$$\frac{\partial}{\partial\delta}\left[\frac{\left(P_0^\mu + P_1\right) + \left(\left(1-\delta\right)\rho_1\right)}{1-\pi+1-\rho_0-\delta\rho_1}\right] = \rho_1\frac{P_0^\mu + P_1 + \rho_1 + \rho_0 + \pi - 1 - 1}{\left(1-\pi+1-\rho_0-\delta\rho_1\right)^2} > 0$$

by assumption 3.

Chapter 3

From Preferences to Choice: a Completion Approach

We especially thank Simone Cerreia Vioglio, Fabio Maccheroni and Massimo Marinacci. We would also like to thank Pierpaolo Battigalli, Jos Heleno Faro, Matthias Messner, Efe Ok, Nicola Pavoni, Gil Riella, Kota Saito, Marciano Siniscalchi and Games and Decisions 2013 Conference Audience (Scuola Normale Superiore, Pisa) for their helpful suggestions. The financial support of the European Research Council (advanced grant, BRSCDP-TEA) is gratefully acknowledged. All remaining errors are ours.

3.1 Introduction

Since the seminal paper by Aumann (1962), modelling incomplete preferences has become a standard problem in the field of decision theory. Several authors, Aumann (1962), Bewley (2002), Shapley and Baucells (1998) and Dubra, Maccheroni and Ok (2004) among others, found a way of representing an incomplete preference relation either in an ambiguity setup or in a risk based framework.

An apparently independent problem in decision theory under uncertainty has been the resolution of Ellsberg type paradoxes like the one of Ellsberg (1961). Some of the most satisfactory and elegant resolutions of this latter problem were proposed by Schmeidler (1989) and by Gilboa and Schmeidler (1989) (GS hereafter). The paper by Gilboa, Maccheroni, Marinacci and Schmeidler (2010) (GMMS hereafter) provides a bridge between these two type of issues. In fact the authors have found a way of completing an initially incomplete preference relation in such a way that the resulting complete preference relation has a maxmin functional representation a là GS. The key axiom behind the GMMS's representation is an axiom called "caution". A Decision Maker (DM hereafter) satisfies the caution axiom if she prefers constant acts whenever two acts are not comparable with respect to the incomplete preference relation.

Our paper is aimed at studying, in a generalized version of the GMMS's framework, how an agent can complete in a consistent way an incomplete piece of information. Our agent is embedded with a finite number of potential completion criteria and, since a choice has to be performed, the agent aggregates these potential completion criteria in a unique choice correspondence that represents the choices that will be eventually made[1]. We can think about the potential completion criteria as competing theories of which the DM is aware of and that are consistent with the initial piece of information.

The present framework has also an interesting interpretation from a statistical decision theoretic viewpoint. There is in fact a tight link between robust statistics and decision making under ambiguity[2]. Following the suggestion provided by Cerreia-Vioglio, Maccheroni, Marinacci and Montrucchio (2013a), we can interpret the "objective" incomplete preference relation $\succsim^*$ in terms of a datum of the problem that has to be inserted inside the DM's subjective framework. The "objective" information is commonly accepted [3]

[1]The fact that the DM considers only a finite number of criteria can be interpreted by thinking that our agent is computationally bounded and it is not able to contemplate an infinite number of potential completion criteria.

[2]For a complete discussion and proper formalization of the topic see Cerreia-Vioglio, Maccheroni, Marinacci and Montrucchio (2013b).

[3]According to the classical interpretation provided by GMMS, it is as if the DM has a proof of the correctness of her "objective" decisions.

and, as a consequence, all the potential completion criteria have to be consistent with this incomplete preference relation. It is as if the DM knows that the correct model belongs to the class of models consistent with the "objective" incomplete preference relation. Hence she has to choose a subset of models from this class.

We now provide a simple clarifying example.

Example 9 A DM considers 3 potential completion criteria, 2 of which have a representation a là GS and the remaining one has an invariant biseparable preference a là Ghirardato, Maccheroni and Marinacci (2004) (GMM hereafter). All 3 potential completion criteria have to be consistent with $\succsim^*$ and they are aggregated by the agent in the"subjective" choice correspondence C^o. The situation just described is represented in Figure 3.1.

▲

The attitude that the DM has toward the potential completion criteria plays a leading role in our paper. A first basic Harsanyi type result is introduced in Proposition 24 in which we show how consistency of the final aggregator w.r.t. the completion criteria only implies that the final functional representation is a (not unique) combination of the potential completion criteria.

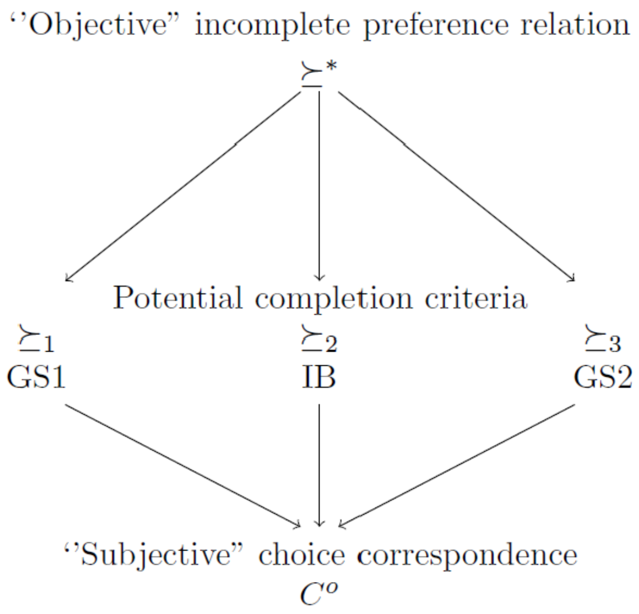
The main representation results of the paper are characterized by a negative attitude of the agent toward the potential completion criteria. This pessimistic behavior is axiomatically formalized by Criteria Uncertainty Aversion Axiom 25. In Proposition 26 and Proposition 30 we show that only a really "cautious" DM will end up in a representation a là GMMS and that in general a distrustful agent is characterized by the following criteria:

$$C^o(A) = \underset{f \in A}{argmax} \left\{ \underset{\gamma \in \Gamma}{min} \sum_{j=1}^{N} \gamma_j I_j(f) \right\}$$

where $\{I_j : B_0(\Sigma) \to \mathbb{R}\}_{j=1}^N$ are monotonic, constant additive and positively homogenous linear functional and $\Gamma \subseteq \triangle(\{1, 2, \ldots, N\})$ is a closed and convex set.

In Theorem 33 we give a variational type representation w.r.t. the potential completion criteria by considering an agent that satisfies Criteria Uncertainty Aversion Axiom and two novel axioms: Variability and Criteria Betweenness. Criteria Betweenness axiomatically captures the idea that the final valuation $I_o(u(f))$ of a generic non constant act f has to be always between the valuations of 2 potential completion criteria, i.e. there always exist 2 criteria i and j such that $I_i(u(f)) \leq I_o(u(f)) \leq I_j(u(f))$. On the other side the second axiom formalizes the idea that the potential completion criteria considered needs to have enough variability in their valuations in such a way that for any

Figure 3.1: A simple example



The Figure represents the relationship between the incomplete preference relationship, the potential completion criteria and the choice correspondence.

2 non constant acts f, g there always exists a criteria i such that $\mid I_i(u(f)) - I_i(u(g)) \mid \geq \mid I_o(u(f)) - I_o(u(g)) \mid$. Although the different axiomatic structure, this latter result is clearly inspired by the work of Maccheroni, Marinacci and Rustichini (2006) (MMR hereafter) and we obtain the following representation:

$$C^o(A) = \underset{f \in A}{argmax} \left\{ \underset{\gamma \in \triangle(\{1,2,\ldots,N\})}{min} \left(\sum_{j=1}^{N} \gamma_j I_j(u(f)) + c(\gamma) \right) \right\}$$

where $c: \triangle(\{1, 2, \ldots, N\}) \to [0, \infty]$ is a grounded, convex and lower semicontinuous function.

The paper consists overall of 5 sections. In the "Framework" section we describe the setup and we introduce the preliminaries necessary to properly understand the model. In the sections "A simple aggregation process", "A pessimistic attitude", "A variational approach" we axiomatically characterize a DM that has different attitudes toward the potential completion criteria. Finally we try to insert our paper inside the current literature in the "Related Literature" section and we conclude. All the proofs are contained inside the "Appendix" section.

3.2 Framework

We make use of a version of the Anscombe and Aumann (1963) model as restated by Fishburn (1970). A von-Neumann-Morgenstern lottery is a finite support probability distribution over the set of outcomes X. The set of lotteries L over X, is endowed with a mixing operation: $\forall P, Q \in L$ and $\forall \alpha \in [0, 1]$, we define $\alpha P + (1 - \alpha) Q \in L$ pointwise over X. The finite set of states of the world is S and it is endowed with an algebra Σ of events. The set of finitely additive probabilities on Σ is denoted as $\Delta(\Sigma)$ and it is endowed with the eventwise convergence topology. The set of simple acts $\mathcal{F}$ consists of all simple measurable functions $f: S \to L$ and it is convexified by performing a pointwise mixture operation on S. We denote with $\mathcal{F}_c$ the set of constant acts[4] and with $\mathcal{H}$ the set of all non empty finite subsets of $\mathcal{F}$. Analogously we define with $\mathcal{H}_c$ the set of all non empty finite subsets of $\mathcal{F}_c$. We denote by $B_0(\Sigma)$ the vector space, endowed with the supnorm, generated by the indicators functions of the elements of Σ.

DM is characterized by a preference relation $\succsim^*$ that represents "objective rationality"[5] and she evaluates several "subjective" preference relations $\{\succsim_i\}_{i=1}^N$ representing potential completion criteria that she considers possible in order to complete the incomplete relation $\succsim^*$. The choice correspondence $C^o: \mathcal{H} \to \mathcal{H}$ represents the final choices made by DM

[4]An act f is constant iff $\exists P \in L$ such that $\forall s \in S$ we have that $f(s) = P$.

[5]When $f \succsim^* g$, it is as if the DM has a correct proof of the fact that f is better than g.

and it is the result of the aggregation of the criteria $\{\succsim_i\}_{i=1}^N$.

3.3 A simple aggregation process

In this section we explore the implications of the simplest form of consistency that we can impose in order to link the potential completion criteria $\{\succsim_i\}_{i=1}^N$ with the choice correspondence C^o. We say that the incomplete preference relation $\succsim^*$is Bewley if it satisfies the following set of axioms:

Axiom 10 *(Reflexivity)* $\forall f \in \mathcal{F}$ *we have that* $f \succsim^* f$*;*

Axiom 11 *(Transitivity)* $\forall f,\, g,\, h \in \mathcal{F}$*, if* $f \succsim^* g$ *and* $g \succsim^* h$ *then* $f \succsim^* h$*;*

Axiom 12 *(Nontriviality)* $\exists f, g \in \mathcal{F}$ *such that* $f \succ^* g$*;*

Axiom 13 *(Monotonicity)* $\forall f, g \in \mathcal{F}$*, if* $\forall s \in S$ *we have that* $f(s) \succsim^* g(s)$ *then* $f \succsim^* g$*;*

Axiom 14 *(Continuity)* $\forall f,\, g,\, h, \in \mathcal{F}$*, the sets* $\{\lambda \in [0,\, 1] :\, \lambda f + (1-\lambda)\, g \succsim^* h\}$ *and*

$\{\lambda \in [0,\, 1] :\, h \succsim^* \lambda f + (1-\lambda)\, g\}$ are closed in $[0,\, 1]$;

Axiom 15 *(Independence)* $\forall f, g, h \in \mathcal{F}$ *and* $\forall \alpha \in (0,\, 1)$*, we have that* $f \succsim^* g$ *if and only if* $\alpha f + (1-\alpha)\, h \succsim^* \alpha g + (1-\alpha)\, h$*;*

Axiom 16 *(C-completeness)* $\forall f,\, g, \in \mathcal{F}_c$ *either* $f \succsim^* g$ *or* $g \succsim^* f$*.*

Moreover we claim that the set of potential completion criteria $\{\succsim_i\}_{i=1}^N$ and the choice correspondence C^o are Invariant Biseparable[6] if they satisfy Reflexivity Axiom 10, Transitivity Axiom 11, Nontriviality Axiom 12, Monotonicity Axiom 13, Continuity Axiom 14 together with the following axioms:

Axiom 17 *(Completeness)* $\forall f,\, g \in \mathcal{F}$*, either* $f \succsim_i g$ *or* $g \succsim_i f$*;*

Axiom 18 *(C-Independence)* $\forall f, g \in \mathcal{F}$*,* $\forall h \in \mathcal{F}_c$ *and* $\forall \alpha \in (0,\, 1)$ *we have that* $f \succsim_i g$ *if and only if* $\alpha f + (1-\alpha)\, h \succsim_i \alpha g + (1-\alpha)\, h$*;*

We assume that each potential completion criteria $\succsim_i$is related to the incomplete preference relation $\succsim^*$ through the following consistency assumption:

[6] In Appendix A we provide the axioms that characterize an Invariant Biseparable choice correspondence. These type of preferences were deeply studied by Ghirardato, Maccheroni and Marinacci (2004).

Axiom 19 *(Consistency)* $\forall f,\, g \in \mathcal{F}$, *if* $f \succsim^* g$ *then* $f \succsim_i g$;

A minimal consistency requirement that can be imposed in order to link the "subjective" choice correspondence $C^o : \mathcal{H} \to \mathcal{H}$ with the potential completion criteria $\{\succsim_i\}_{i=1}^N$ is described by the following axiom:

Axiom 20 *(Consistency Toward Criteria)* $\forall f,\, g \in \mathcal{F}$, *if* $\forall i \in \{1, 2, \dots, N\}$ *we have that* $f \succsim_i g$ *then* $f \in C^o(\{f,\, g\})$.

The issue of this section is to understand which type of functional representation can be obtained by using this latter axiom. Before stating the first result of the paper we need to introduce the concept of joint convexity and of unambiguous preference relation.

Definition 21 *We say that a finite set of preferences* $\{\succsim_i\}_{i=1}^K$ *over a set* $\mathcal{F}$ *satisfies* ***joint convexity*** *if they admit real valued functional representations*[7] $\{I_i\}_{i=1}^K$ *and the range of the functional* $I = (I_1, \dots, I_K)$[8] *is a convex subset of* $\mathbb{R}^K$.

Definition 22 *We say that a finite set containing both preferences* $\{\succsim_i\}_{i=1}^K$ *over a set* $\mathcal{F}$ *and choice correspondences* $\{C^i\}_{i=1}^M$ *over the set of all non empty finite subsets of* $\mathcal{F}$ *satisfies* ***joint convexity*** *if the set obtained by putting together the preferences* $\{\succsim_i\}_{i=1}^K$ *and the revealed preferences associated to the choice correspondences* $\{C^i\}_{i=1}^M$ *satisfies joint convexity.*

If all the potential completion criteria $\{\succsim_i\}_{i=1}^N$ have a functional representation in terms of subjective expected utility then the assumption of joint convexity is satisfied thanks to a direct application of the Lyapunov Theorem. In Appendix B we show that there are other interesting non trivial cases in which joint convexity holds and in particular we focus our attention on the case of unanimity games. In fact given the tight link that there is between invariant biseparable preferences and the Choquet capacities [9] studying the cases in which joint convexity holds requires understanding up to which point Lyapunov Theorem can be extended to capacities.

Definition 23 *(GMM (2004)) Let* $f, g \in \mathcal{F}$. *The unambiguous preference* $\succsim_i^*$ *w.r.t. to the potential completion criteria* $\succsim_i$ *is such that* $\forall h \in \mathcal{F}$ *and* $\forall \lambda \in (0, 1]$ *if* $\lambda f + (1 - \lambda)\, h \succsim_i^* \lambda g + (1 - \lambda)$ *then* $f \succsim_i^* g$.

[7] This is the case if $\forall i \in \{1, 2, \dots, K\}$ we have that $\exists I_i : B_o(\Sigma) \to \mathbb{R}$ and $\exists u_i : X \to \mathbb{R}$ such that $\forall f, g \in \mathcal{F}$ we have that $f \succsim_i g$ if and only if $I_i(u_i(f)) \geq I_i(u_i(g))$.

[8] I is obtained by associating all the functional representations $\{I_i\}_{i=1}^K$.

[9] A preference relation is invariant biseparable if it can be represented by a functional I on acts such that $I(x_A y) = \nu(A)\, u(x) + (1 - \nu(A))\, u(y)$ for some utility function $u(\cdot)$ and some capacity $\nu(\cdot)$ where we assume that $\nu(\Omega) = 1$.

We now have all the elements that allow us to properly formalize Proposition 24.

Proposition 24 *The following statements are equivalent:*

1. $\succsim^*$ *is a Bewley preference;* $\{\succsim_i\}_{i=1}^N$ *and* C^o *are Invariant Biseparable and satisfy joint convexity;* $\{\succsim_i\}_{i=1}^N$ *satisfy Consistency Axiom 19 w.r.t.* $\succsim^*$*;* C^o *satisfies Consistency Toward Criteria Axiom 20 w.r.t.* $\{\succsim_i\}_{i=1}^N$*;*

2. *There exists a nonempty closed and convex set* $\mathcal{C}^*$ *of probabilities on* Σ*, a set of closed and convex sets* $\{\mathcal{C}^i\}_{i=1}^N \subseteq \mathcal{C}^*$ *of probabilities on* Σ*, a non constant affine function* $u : X \to \mathbb{R}$*, several monotonic, constant additive and positively homogenous linear functionals* $\{I_i : B_0(\Sigma) \to \mathbb{R}\}_{i=1}^N$ *and* $I_o : B_0(\Sigma) \to \mathbb{R}$ *and a set of non negative weights* $\{\gamma_i\}_{i=1}^N$ *such that* $\forall f, g \in \mathcal{F}$ *and* $\forall A \in \mathcal{H}$ *the following holds:*

$$f \succsim^* g \;\; \Leftrightarrow \int_S E_{f(s)} u dp(s) \geq \int_S E_{g(s)} u dp(s) \;\; \forall p \in \mathcal{C}^* \tag{3.1}$$

$$f \succsim_i g \;\; \Leftrightarrow I_i(u(f)) \geq I_i(u(g)) \;\; i = 1, \ldots, N \tag{3.2}$$

$$C^o(A) = \underset{f \in A}{argmax} \{I_o(u(f))\} = \underset{f \in A}{argmax} \left\{ \sum_{j=1}^{N} \gamma_j I_j(u(f)) \right\} \tag{3.3}$$

$$f \succsim_i^* g \iff \int_S E_{f(s)} u dp(s) \geq \int_S E_{g(s)} u dp(s) \;\; \forall p \in \mathcal{C}^i$$

Moreover in this case, $\mathcal{C}^*$ *and* $\{\mathcal{C}^i\}_{i=1}^N$ *are unique,* u *is unique up to positive affine transformations. If the set of outcomes* X *is finite we have that* $\sum_{j=1}^{N} \gamma_j = 1$.

Proposition 24 shows that the Consistency Toward Criteria Axiom 20 simply implies that the final implemented choice correspondence is a linear combination with non negative weights of the representation functionals associated to the potential completion criteria. Hence we reached the intuitive result that if DM wants to be consistent she needs to evaluate act f by combining the different evaluations of the criteria that she considers possible in the first stage of the completion process. This result is in line with the one obtained by Harsanyi (1955) for social welfare functions by imposing a Pareto condition that plays the role of our consistency assumption. The key technical ideas behind this first basic Proposition are contained in De Meyer and Mongin (1995). From an historical perspective this result can be sawn as a generalization and an axiomatic foundation of the Hodges and Lehmann (1952)'s paper that, for the first time in a statistical decision theoretic framework, provided a decision criteria that was a combination of the Wald's minimax principle and of the Bayesian approach.

3.4 A pessimistic attitude

In this section we will assume that we face a pessimistic DM that has a negative attitude toward the potential completion criteria. In particular, we formalize this pessimistic behavior by using an axiom that is the straightforward adaptation to our framework of the Expert Uncertainty Aversion axiom of Cres, Gilboa and Vieille (2011) (CGV hereafter). In order to properly state the axiom we need to introduce some notation. For each act $f \in \mathcal{F}$ we denote by $c_i^f \in \mathcal{F}_c$ the certainty equivalent of the act f with respect to the preference relation $\succsim_i$. The certainty equivalent of act $f \in \mathcal{F}$ with respect to the choice correspondence $C^o : \mathcal{H} \to \mathcal{H}$ is defined as the constant act $c_o^f \in \mathcal{F}_c$ such that we have both $c_o^f \in C^o\left(\left\{f, c_o^f\right\}\right)$ and $f \in C^o\left(\left\{f, c_o^f\right\}\right)$.

Axiom 25 *(Criteria Uncertainty Aversion)* $\forall f,\ f_j \in \mathcal{F},\ \forall \alpha_j \geq 0$ *such that* $\sum_{j=1}^{J} \alpha_j = 1$, *if* $\forall i \in \{1, \ldots, N\}$ *we have that* $f \succsim_i \sum_{j=1}^{J} \alpha_j c_i^{f_j}$ *then* $f \in C^o\left(\left\{f,\ \sum_{j=1}^{J} \alpha_j c_o^{f_j}\right\}\right)$.

Notice that by simply setting $J = 1$, $f_J = g$ and $\alpha_J = 1$ we have that Criteria Uncertainty Aversion Axiom 25 implies Consistency Toward Criteria Axiom 20. The next Proposition 26 formalizes from an analytical viewpoint the attitude of our pessimistic DM.

Proposition 26 *The following statements are equivalent:*

1. $\succsim^*$ *is a Bewley preference;* $\{\succsim_i\}_{i=1}^N$ *and* C^o *are Invariant Biseparable;* $\{\succsim_i\}_{i=1}^N$ *satisfy Consistency Axiom 19 w.r.t.* $\succsim^*$*;* C^o *satisfies Criteria Uncertainty Aversion Axiom 25 w.r.t.* $\{\succsim_i\}_{i=1}^N$*;*

2. *There exists a nonempty closed and convex set* $\mathcal{C}^*$ *of probabilities on* Σ*, a set of closed and convex sets* $\{\mathcal{C}^i\}_{i=1}^N \subseteq \mathcal{C}^*$ *of probabilities on* Σ*, a non constant affine function* $u : X \to \mathbb{R}$*, several monotonic, constant additive and positively homogenous linear functionals* $\{I_i : B_0(\Sigma) \to \mathbb{R}\}_{i=1}^N$ *and* $I_o : B_0(\Sigma) \to \mathbb{R}$ *and a closed and convex set* $\Gamma \subseteq \triangle(\{1, 2, \ldots, N\})$ *such that* $\forall f, g \in \mathcal{F}$ *and* $\forall A \in \mathcal{H}$ *the following holds:*

$$f \succsim^* g \;\Leftrightarrow\; \int_S E_{f(s)} u dp(s) \geq \int_S E_{g(s)} u dp(s) \;\; \forall p \in \mathcal{C}^*$$

$$f \succsim_i g \;\Leftrightarrow I_i(u(f)) \geq I_i(u(g)) \quad i = 1, \ldots, N$$

$$C^o(A) = \underset{f \in A}{argmax}\ \{I_o(u(f))\} = \underset{f \in A}{argmax} \left\{\underset{\gamma \in \Gamma}{min} \sum_{j=1}^{N} \gamma_j I_j(u(f))\right\}$$

$$f \succsim_i^* g \;\Longleftrightarrow\; \int_S E_{f(s)} u dp(s) \geq \int_S E_{g(s)} u dp(s) \;\; \forall p \in \mathcal{C}^i$$

Moreover in this case, $\mathcal{C}^*$ *and* $\{\mathcal{C}^i\}_{i=1}^N$ *are unique,* u *is unique up to positive affine transformations.*

Proposition 26 generalizes and extends to our framework the main representation theorem of CGV. Notice that CGV assume GS's preferences while we make the weaker assumption of Invariant Biseparable preference relations. The other results contained in Proposition 26 are standard and in line with what showed by GMMS and GMM. We will now introduce a corollary of Proposition 26 that clarifies under which conditions we end up in a representation theorem a là GMMS. Before stating the corollary we need to recall the definition of Caution Axiom introduced by GMMS. In particular two preference relations $\succsim^*$ and $\succsim_i$ satisfy Caution Axiom if the following holds:

Axiom 27 *(Caution)* $\forall f \in \mathcal{F}$ *and* $\forall g \in \mathcal{F}_c$*, if* $f \not\succsim^* g$ *then* $g \succsim_i f$*.*

Corollary 28 *Under the assumptions of Proposition 26, if* $\exists \bar{i} \in \{1, \ldots, N\}$ *such that* $\succsim_{\bar{i}}$ *satisfies Caution Axiom 27 w.r.t.* $\succsim^*$ *and the standard vector*[10] $e_{\bar{i}} \in \Gamma \subseteq \triangle(\{1, 2, \ldots, N\})$*, then there exists a nonempty closed and convex set* $\mathcal{C}^*$ *of probabilities on* Σ*, a non constant affine function* $u : X \to \mathbb{R}$ *such that:*

$$f \succsim^* g \quad \Leftrightarrow \int_S E_{f(s)} u dp(s) \geq \int_S E_{g(s)} u dp(s) \;\; \forall p \in \mathcal{C}^*$$

$$C^o(A) = \underset{f \in A}{argmax} \left\{ \min_{p \in \mathcal{C}^*} \int_S E_{f(s)} u dp(s) \right\}$$

Moreover, in this case, $\mathcal{C}^*$ *is unique and* u *is unique up to positive affine transformations.*

Corollary 28 delivers a representation identical to the main representation Theorem of GMMS. Notice that ex ante neither the potential completion criteria nor the choice correspondence satisfies Uncertainty Aversion axiom[11] a là GS. The only assumption that we made on C^o is the pessimistic attitude toward the potential completion criteria.

Hence Criteria Uncertainty Aversion Axiom 25 alone is not sufficient for having a representation result a là GMMS even if $N-1$ of the potential criteria satisfy Caution axiom 27. In order to collapse in GMMS's main theorem, it is necessary that our DM considers possible to use only the cautious completion criteria and consequently only a really cautious agent will satisfy this type of representation theorem.

We will now introduce a novel behavioral axiom that we will use in order to characterize the behavior of an agent a là GMMS. One of the most pessimistic attitude that a DM can have toward the potential completion criteria is described by the following axiom:

[10] $e_{\bar{i}}$ is the standard vector of $\mathbb{R}^N$ that assigns weight 1 to the element in position $\bar{i}$ and 0 otherwise.
[11] Uncertainty Aversion is one of the key axioms behind GS's maxmin representation.

Axiom 29 *(Caution Toward Criteria)* $\forall f \in \mathcal{F}$ *and* $\forall g \in \mathcal{F}_c$ *if* $\exists i \in \{1, 2, \ldots, N\}$ *such that* $f \not\succsim_i g$ *then* $g \in C^o(\{f, g\})$.

Proposition 30 does not require that C^o satisfies Criteria Uncertainty Aversion Axiom 25 and by assuming that the DM satisfies Consistency Toward Criteria Axiom 20 and Caution Toward Criteria Axiom 29 it can be sawn as an axiomatic foundation of the behavior of a really cautious agent.

Proposition 30 *The following statements are equivalent:*

1. *If* $\succsim^*$ *is a Bewley preference;* $\{\succsim_i\}_{i=1}^N$ *and* C^o *are Invariant Biseparable;* $\{\succsim_i\}_{i=1}^N$ *satisfy Consistency Axiom 19 w.r.t.* $\succsim^*$*;* C^o *satisfies Consistency Toward Criteria Axiom 20 and Caution Toward Criteria Axiom 29 w.r.t.* $\{\succsim_i\}_{i=1}^N$*;* $\exists \bar{i} \in \{1, \ldots, N\}$ *such that* $\succsim_{\bar{i}}$ *satisfies Caution Axiom 27 w.r.t.* $\succsim^*$*;*

2. *There exists a nonempty closed and convex set* $\mathcal{C}^*$ *of probabilities on* Σ*, a set of closed and convex sets* $\{\mathcal{C}^i\}_{i=1}^N \subseteq \mathcal{C}^*$ *of probabilities on* Σ*, a non constant affine function* $u : X \to \mathbb{R}$*, several monotonic, constant additive and positively homogenous linear functionals* $\{I_i : B_0(\Sigma) \to \mathbb{R}\}_{i=1}^N$ *and* $I_o : B_0(\Sigma) \to \mathbb{R}$ *and a set of non negative weights* $\{\gamma_i\}_{i=1}^N$ *such that:*

$$f \succsim^* g \;\; \Leftrightarrow \int_S E_{f(s)} u dp(s) \geq \int_S E_{g(s)} u dp(s) \;\; \forall p \in \mathcal{C}^*$$

$$f \succsim_i g \;\; \Leftrightarrow I_i(u(f)) \geq I_i(u(g)) \;\; i = 1, \ldots, N$$

$$C^o(A) = \underset{f \in A}{argmax} \{I_o(u(f))\} = \underset{f \in A}{argmax} \left\{ \sum_{j=1}^N \gamma_j I_j(u(f)) \right\}$$
$$= \underset{f \in A}{argmax} \left\{ \underset{p \in \mathcal{C}^*}{min} \int_S E_{f(s)} u dp(s) \right\}$$

$$f \succsim_i^* g \iff \int_S E_{f(s)} u dp(s) \geq \int_S E_{g(s)} u dp(s) \;\; \forall p \in \mathcal{C}^i$$

Moreover in this case, $\mathcal{C}^*$ *and* $\{\mathcal{C}^i\}_{i=1}^N$ *are unique,* u *is unique up to positive affine transformations.*

3.5 A variational approach

In this section we are going to axiomatically characterize the behavior of a DM that is naturally biased toward some weights given to the potential completion criteria. In fact, even in the case in which the agent knows several potential completion criteria, he could find easier to implement only a specific subset of these criteria. In order to mathematically formalize this bias of the DM we need to give to the choice correspondence C^o a variational representation a là MMR.

We introduce two axioms that are key for the representation contained in Theorem 33. The first axiom, called Criteria Betweenness, wants to capture the idea that the final valuation $I_o(u(f))$ of a generic non constant act f has to be always between the valuations of 2 potential completion criteria:

Axiom 31 *(Criteria Betweenness)* $\forall f \in \mathcal{F} \setminus \mathcal{F}_c$, $\exists i, j \in \{1, 2, \ldots, N\}$ *such that the following holds:*

$$f \in C^o\left(\left\{f, c_i^f\right\}\right)$$

and

$$c_j^f \in C^o\left(\left\{f, c_j^f\right\}\right)$$

While Variability Axiom 32 formalizes the idea that the potential completion criteria considered need to have enough variability in their valuations in such a way that for any 2 non constant acts f, g there always exists a criteria i such that $|I_i(u(f)) - I_i(u(g))| \geq |I_o(u(f)) - I_o(u(g))|$:

Axiom 32 *(Variability)* $\forall f, g \in \mathcal{F} \setminus \mathcal{F}_c$, $\exists i \in \{1, 2, \ldots, N\}$ *such that the following holds:*

$$\frac{1}{2}c_o^f + \frac{1}{2}c_i^g \in C^o\left(\left\{\frac{1}{2}c_o^f + \frac{1}{2}c_i^g, \frac{1}{2}c_o^g + \frac{1}{2}c_i^f\right\}\right)$$

We say that the choice correspondence C^o is Weak Variational [12] if it satisfies the Weak Axiom of Revealed Preferences 36, Non Triviality Axiom 37, Monotonicity Axiom 38, Continuity Axiom 39, Weak Certainty Independence Axiom 41.

Theorem 33 *Assume that* $\{\succsim_i\}_{i=1}^N$ *satisfy joint convexity. The following statements are equivalent:*

[12] Axioms' statements that characterize a weak variational choice correspondence are contained in Appendix A. The appendix contains also the precise formulation of Weak Certainty Independence Axiom 41.

1. *$\succsim^*$ is a Bewley preference; $\{\succsim_i\}_{i=1}^N$ are Invariant Biseparable; $\{\succsim_i\}_{i=1}^N$ satisfy Consistency Axiom 19 w.r.t. $\succsim^*$; C^o is Weak Variational and it satisfies Criteria Uncertainty Aversion Axiom 25, Criteria Betweenness Axiom 31 and Variability Axiom 32 w.r.t. $\{\succsim_i\}_{i=1}^N$.*

2. *There exists a nonempty closed and convex set $\mathcal{C}^*$ of probabilities on Σ, , a set of closed and convex sets $\{\mathcal{C}^i\}_{i=1}^N \subseteq \mathcal{C}^*$ of probabilities on Σ, a non constant function $u : X \to \mathbb{R}$, several monotonic, constant additive and positively homogenous linear functionals $\{I_i : B_0(\Sigma) \to \mathbb{R}\}_{i=1}^N$; a normalized, monotonic, translation invariant functional $I_o : B_0(\Sigma) \to \mathbb{R}$ and a lower semicontinuous, grounded and convex function $c : \triangle(\{1, 2, \dots, N\}) \to [0, \infty]$ such that $\forall f, g \in \mathcal{F}$ and $\forall A \in \mathcal{H}$ the following holds:*

$$f \succsim^* g \quad \Leftrightarrow \int_S E_{f(s)} u dp(s) \geq \int_S E_{g(s)} u dp(s) \;\; \forall p \in \mathcal{C}^*$$

$$f \succsim_i g \quad \Leftrightarrow I_i(u(f)) \geq I_i(u(g)) \quad i = 1, \dots, N$$

$$C^o(A) = \underset{f\in A}{argmax}\, \{I_o(u(f))\} = \underset{f\in A}{argmax} \left\{ \min_{\gamma\in\triangle(\{1,2,\dots,N\})} \left(\sum_{j=1}^N \gamma_j I_j(u(f)) + c(\gamma) \right) \right\}$$

$$f \succsim_i^* g \quad \Longleftrightarrow \quad \int_S E_{f(s)} u dp(s) \geq \int_S E_{g(s)} u dp(s) \;\; \forall p \in \mathcal{C}^i$$

Moreover, in this case, $\mathcal{C}^*$ is unique and u is unique up to positive affine transformations.

For coherence with what we developed in the previous sections we decided to retain Criteria Uncertainty Aversion Axiom 25 in the statement of Theorem 33, even though it could be enough to have a weakened version of Criteria Uncertainty Aversion Axiom 25 of the following type:

Axiom 34 *(Weak Criteria Uncertainty Aversion) $\forall f, g, h \in \mathcal{F}$ and $\forall \alpha \in (0, 1)$ if $\forall i \in \{1, 2, \dots, N\}$ we have that $f \succsim_i \alpha c_i^g + (1 - \alpha) c_i^h$ then $f \in C^o\left(\left\{f, \alpha c_o^g + (1 - \alpha) c_o^h\right\}\right)$.*

As usual Proposition 26 can be viewed as almost [13] a special case of Theorem 33 when the cost function $c(\cdot)$ has the following functional form:

$$c(\gamma) = \begin{cases} 0 & \gamma \in \Gamma \\ +\infty & \gamma \notin \Gamma \end{cases}$$

.

[13] Notice that in Proposition 26 we don't have the assumption of joint convexity.

3.6 Related Literature

The papers that inspired the present work are mainly related either to the decision theoretic literature or to the social choice literature (in particular the problem of social aggregation). Our paper, like those of CGV and of Nascimento (2012), lies at the intersection of these two streams of literature. In this section we will briefly sum up the papers that are tightly related with our work and we will try to highlight the main connections.

As mentioned in the Introduction section, our paper can be viewed as a generalization of GMMS. In particular GMMS, by using a framework that involves one incomplete "objective" preference relation and one complete "subjective" preference relation, provide a new axiomatic foundation of the maxmin functional representation à la GS without using the uncertainty aversion axiom. In order to link the "objective" and the "subjective" preference relations they make use of a standard Consistency Axiom and for the first time in the decision theoretic literature they introduce the Caution Axiom. This latter axiom states that the Decision Maker, when comparing a constant act with a generic non constant act, prefers the constant act whenever the choice is undecidable from an "objective" viewpoint.

Cerreia-Vioglio (2012) generalizes the results of GMMS by weakening the Caution axiom and by using the Risk Independence axiom instead of the Constant Independence axiom. The cost that the author has to pay in order to have very general results is to restrict attention to the class of preferences that satisfy the Unboundedness axiom [14].

CGV consider the problem of a DM that has to aggregate several experts opinions in order to make a decision in a framework characterized by uncertainty. Given that the experts have to advice a single individual, all of them use the same utility function when evaluating the different acts and by assumption all the experts have GS's maxmin preferences. Notice that in our work the fact that all preference relations have the same utility function is not imposed but it is a result of the construction of the model. Moreover the potential completion criteria that we study, i.e. invariant biseparable preferences, are generalizations of the GS's maxmin preferences. CGV introduce a really interesting axiom, whose name is Expert Uncertainty Aversion, that axiomatically formalize a pessimistic behavior toward experts' opinions. This axiom delivers the concavity of the functional

---

[14]

Axiom 35 *(Unboundedness) A preference relation $\succeq$ satisfies unboundness if for each $x,\ y \in \mathcal{F}_c$ with $x \succ y$ then $\exists z,\ w \in \mathcal{F}_c$ such that the following holds:*

$$\frac{1}{2}z + \frac{1}{2}y \succeq x \succ y \succeq \frac{1}{2}w + \frac{1}{2}x$$

that it is necessary in order to make a separation argument à la GS. Our Axiom 25 is an adaptation to potential completion criteria of CGV's Expert Uncertainty Aversion.

Nascimento (2012) studies the problem of aggregating preference ordering under "subjective" uncertainty. The author considers an "ex ante" preference defined over set of lotteries of acts $\triangle(\mathcal{F})$, that represents the choice set of the DM, and a set of admissible "ex-post" preferences, that represents all experts' opinions. Like CGV, he needs to assume, by using a Weak Agreement Axiom, that the admissible class of preferences agrees over the set of constant acts. Some of the Propositions proposed by Nascimento are highly in line with the Propositions and Theorem presented in our paper. In particular our Proposition 24, Proposition 26 and Theorem 33 can be viewed as a version of Theorem 1, Proposition 8 and Theorem 2 respectively of Nascimento (2012). Nascimento (2012) has also a version of the main representation theorem of GMMS that goes under the name of Proposition 7. Although there are some similarities between our work and the one by Nascimento (2012), our paper is characterized by a totally different axiomatic structure. Moreover even if Nascimento (2012) allows for an infinite dimensional class of opinions while our work is completely based on a finite dimensional setup, his model needs to use an outer layer of randomization in a Seo (2009) style. Nascimento (2012) himself claims to be "... mainly interested in the restriction of the ex ante preference to the set of acts..." and not in the lotteries over acts.

We want to conclude this section by briefly describing two papers that, like our paper, try to insert an "objective" datum inside the "subjective" framework of the DM.

Cerreia-Vioglio, Maccheroni, Marinacci and Montrucchio (2013a) build up a model that merges a "subjective" choice framework a là Savage with a Waldean "objective" piece of information. In fact, even though the work of Savage was prevalently inspired by the Waldean decision theoretic approach, Savage had a purely subjective setup and he didn't consider the classical "objective" datum of the problem. The authors, by using a consistency axiom, are able from one side to represent the fact that the DM is indeed aware of the datum of the problem (see the structurally rich representation of their Proposition 1 for instance) and from the other side they are able to discriminate among models (see their Proposition 2). As a consequence the classical Savage criterion can be simply considered as the information that an outside observer, that doesn't know the "objective" datum, could collect from the agent's behavior. The structural representations proposed by Cerreia-Vioglio, Maccheroni, Marinacci and Montrucchio (2013a) in Proposition 1, Proposition 3 and Proposition 4 are particularly insightful because they make clear the distinction between state uncertainty and model uncertainty. This latter type of uncertainty, even if it is not directly payoff relevant, plays an instrumental role relative to state uncertainty.

Cerreia-Vioglio, Maccheroni, Marinacci and Montrucchio (2013b) propose a framework in which they try to join the decision theoretic literature on ambiguity and the robust statistics literature on prior uncertainty. In particular they study the conditions under which a problem of ambiguity can be rephrased and reinterpreted in terms of prior uncertainty. The main tools that the authors use in order to join ambiguity and prior uncertainty are a Consistency Axiom and the concept of Dynkin space. Their Consistency Axiom, that is a version of the GMMS's Consistency Axiom, allow to make choices coherent with the probabilistic information by linking the family of "objective" rational beliefs with the DM's "subjective" preference relation. The Dynkin space is the mathematical tool that Cerreia-Vioglio, Maccheroni, Marinacci and Montrucchio (2013b) use in order to formally model the lack of information that affects the economic agent and the relevant probabilistic information is formalized as a sub $\sigma-$algebra of the $\sigma-$algebra of events. Cerreia-Vioglio, Maccheroni, Marinacci and Montrucchio (2013b) have several representation results and in Theorem 4 they also have a variational representation a là MMR.

3.7 Conclusions

We have proposed a framework where it is possible to study the behavior of a DM that wants to complete his incomplete piece of information by considering several potential completion criteria. The attitude that the DM has toward these potential completion criteria determines the functional form associated to the final aggregator represented by the choice correspondence.

We have shown how the main representation result of GMMS is a particular case of our general setup when the DM is really cautious. In our final representation theorem we have introduced two novel axioms, Variability and Criteria Betweenness, that together with Criteria Uncertainty Aversion Axiom allow to obtain a variational type of representation for the choice correspondence aggregator.

3.8 Appendix A

In this section of the Appendix we describe the axioms that we use in order to characterize our choice correspondence $C^o : \mathcal{H} \to \mathcal{H}$.

Axiom 36 *(Weak Axiom of Revealed Preferences)* $\forall A,\, B \in \mathcal{H}$ *if* $B \subseteq A$ *and* $C^o(A) \cap B \neq \emptyset$ *then* $C^o(B) = C^o(A) \cap B$*;*

Axiom 37 *(Nontriviality)* $\exists f, g \in \mathcal{F}$ *such that* $f = C^o(\{f,\, g\})$*;*

Axiom 38 *(Monotonicity)* $\forall f, g \in \mathcal{F}$*, if* $\forall s \in S$ *we have that* $f(s) \in C^o(\{f(s),\, g(s)\})$ *then* $f \in C^o(\{f,\, g\})$*;*

Axiom 39 *(Continuity)* $\forall f,\, g,\, h \in \mathcal{F}$ *the sets*

$$\{\lambda \in [0,\, 1] : \lambda f + (1-\lambda) g \in C^o(\{\lambda f + (1-\lambda) g, h\})\}$$

and

$$\{\lambda \in [0,\, 1] : h \in C^o(\{\lambda f + (1-\lambda) g, h\})\}$$

are closed in $[0,\, 1]$*;*

Axiom 40 *(Constant Independence)* $\forall A \in \mathcal{H}$*,* $\forall h \in \mathcal{F}_c$ *and* $\forall \alpha \in (0,\, 1)$ *we have that* $C^o(\alpha A + (1-\alpha) h) = \alpha C^o(A) + (1-\alpha) h$*;*

Axiom 41 *(Weak Certainty Independence)* $\forall A \in \mathcal{H}$*,* $\forall h,\, z \in \mathcal{F}_c$ *and* $\forall \alpha \in (0,\, 1]$*, if* $C^o(\alpha A + (1-\alpha) h) = \alpha C^o(A) + (1-\alpha) h$ *then* $C^o(\alpha A + (1-\alpha) z) = \alpha C^o(A) + (1-\alpha) z$*;*

Axiom 42 *(Risk Independence)* $\forall A \in \mathcal{H}_c$[15]*,* $\forall h \in \mathcal{F}_c$ *and* $\forall \alpha \in (0,\, 1)$ *we have that* $C^o(\alpha A + (1-\alpha) h) = \alpha C^o(A) + (1-\alpha) h$.

We conclude this section by defining 2 axioms for a generic preference relation $\succsim$ that will be used in the proofs contained in Appendix C.

Axiom 43 *(Risk Independence)*$\forall f, g, h \in \mathcal{F}_c$ *and* $\forall \alpha \in (0,\, 1]$ *we have that* $f \succ g$ *if and only if* $\alpha f + (1-\alpha) h \succ \alpha g + (1-\alpha) h$*;*

Axiom 44 *(Weak Certainty Independence)* $\forall f, g \in \mathcal{F}$*,* $\forall h,\, z \in \mathcal{F}_c$ *and* $\forall \alpha \in (0,\, 1)$ *if* $\alpha f + (1-\alpha) h \succsim \alpha g + (1-\alpha) h$ *then* $\alpha f + (1-\alpha) z \succsim \alpha g + (1-\alpha) z$.

[15] $\mathcal{H}_c$ is the set of all non empty finite subsets of $\mathcal{F}_c$.

3.9 Appendix B

In this part of the appendix we will show that there are interesting non trivial cases in which the assumption of Joint Convexity holds by focusing our attention on the case of unanimity games. In particular we will try to understand under which conditions we can extend the Lyapunov Theorem when we deal with unanimity games, i.e. the simplest types of capacities we can deal with.

3.9.1 Simple results for unanimity games

Fix $A \in \Sigma$ and define the unanimity game $\nu_A : \Sigma \to \mathbb{R}$ as:

$$\nu_A(B) = \begin{cases} 1 & if\ A \subseteq B \\ 0 & else \end{cases}$$

The Choquet integral w.r.t. the unanimity capacity is given by $\int f d\nu_A = \inf_{\omega \in A} f(\omega)$.

Definition 45 *We say that a set $\{\nu_{A_i}\}_{i=1}^n$ of unanimity games have sufficient disjoint support if $\forall i \in \{1, 2, \ldots, n\}$ we have that A_i is not a subset of $\underset{j \neq i}{\cup} A_j$.*

Proposition 46 *Let $\{\nu_{A_i}\}_{i=1}^n$ be a set of unanimity games with sufficient disjoint support. Then the set I defined as follows:*

$$I = \left\{ \left(\int u(f)\, d\nu_{A_1}, \ldots, \int u(f)\, d\nu_{A_n} \right) \mid u(f)\ \ is\ \Sigma - measurable\ and\ 0 \leq u(f) \leq 1 \right\}$$

is a convex subset of $\mathbb{R}^n$.

Proof. We will show by construction that the set obtained under the assumptions of the proposition is the hypercube of length 1 in $\mathbb{R}^n$. Consider the following function:

$$u(f) = \begin{cases} \alpha_1 & if\ A_1 \setminus \underset{j \neq 1}{\cup} A_j \\ \alpha_2 & if\ A_2 \setminus \underset{j \neq 2}{\cup} A_j \\ \quad \vdots & \\ \alpha_n & if\ A_n \setminus \underset{j \neq n}{\cup} A_j \\ 1 & otherwise \end{cases}$$

and notice that $\forall i \in \{1, 2, \ldots, n\}$ we have that $\int u(f)\, d\nu_{A_i} = \min_{\omega \in A} u(f(\omega)) = \alpha_i$ and by letting vary each α_i in the interval $[0,\ 1]$ we obtains each point of the hypercube. ■

Example 47 *As a simple example we show that, differently w.r.t. the general case of Proposition 46, for the special case of* 2 *unanimity games we don't need the assumption of sufficient disjoint support*[16].

Proposition 48 *Let* ν_{A_1} *and* ν_{A2} *be two unanimity games. Then the set* I *defined as follows:*

$$I = \left\{ \left(\int u(f)\, d\nu_{A_1}, \int u(f)\, d\nu_{A_2} \right) \mid u(f) \;\; is\ \Sigma - measurable\ and\ 0 \leq u(f) \leq 1 \right\}$$

is a convex subset of $\mathbb{R}^2$.

Proof. Notice that in general we can have 2 cases: either the sufficient support assumption holds and we know that the result is true by Proposition 46 or the sufficient support assumption doesn't hold. Hence the strategy will be to assume that the sufficient support assumption doesn't hold and we will show that I is still a convex subset of $\mathbb{R}^2$. Suppose w.l.o.g. that $A_1 \subseteq A_2$. Assume that we want to plot I on a plane where we have on the axes $I_1(u(f)) = \int u(f)\, d\nu_{A_1}$ and $I_2(u(f)) = \int u(f)\, d\nu_{A_2}$. If $A_1 = A_2$ we get that I is the segment joining the points $(0,0)$ and $(1,1)$ on the line $I_1(u(f)) = I_2(u(f))$ and of course it is convex. If $A_1 \subset A_2$, then in this case we have that $I_2(u(f)) \leq I_1(u(f))$ and I is equal to a square triangle that is of course a convex set. ■

Example 49 *After seeing the result contained in Example 47 we could think that the assumption of disjoint supports is not needed. Indeed the next Proposition provide a negative answer to this latter intuition.*

Proposition 50 *Let* $\{\nu_{A_i}\}_{i=1}^n$ *be a set of unanimity games. Then the set* I *defined as follows:*

$$I = \left\{ \left(\int u(f)\, d\nu_{A_1}, \ldots, \int u(f)\, d\nu_{A_n} \right) \mid u(f) \;\; is\ \Sigma - measurable\ and\ 0 \leq u(f) \leq 1 \right\}$$

is not in general a convex subset of $\mathbb{R}^n$.

Proof. The strategy of the proof is to construct a counterexample in $\mathbb{R}^3$ in which I is not a convex set. Fix $n = 3$ and consider the case in which $A_1 \subset (A_2 \cup A_3)$, $A_2 \cap A_3 =$

[16]We thank Simone Cerreia Vioglio who focused our attention on this particular case.

$\emptyset,\ A_1 \cap A_3 \neq \emptyset,\ A_1 \cap A_2 \neq \emptyset$. Notice that the points $(0,\ 1,\ 0)\,,\ (0,\ 0,\ 1) \in I$. To see that the latter statement is true it is enough to consider the following functions:

$$u\left(f^{(0,\ 1,\ 0)}\right) = \begin{cases} 1 & if\ A_1 \cap A_2 \\ 1 & if\ A_2 \backslash A_1 \\ 0 & if\ A_1 \cap A_3 \\ 0 & if\ A_3 \backslash A_1 \\ 1 & otherwise \end{cases}$$

$$u\left(f^{(0,\ 0,\ 1)}\right) = \begin{cases} 0 & if\ A_1 \cap A_2 \\ 0 & if\ A_2 \backslash A_1 \\ 1 & if\ A_1 \cap A_3 \\ 1 & if\ A_3 \backslash A_1 \\ 1 & otherwise \end{cases}$$

On the other side if we combine with weight $\frac{1}{2}$ the points $(0,\ 1,\ 0)$ and $(0,\ 0,\ 1)$ we obtain the point $\left(0, \frac{1}{2},\ \frac{1}{2}\right) \notin I$ because from how we constructed the capacities the first component of the vector has to be always bigger than the minimum of the other two components of the vector. ∎

3.10 Appendix C

Lemma 51 *. Define the relation $\succsim^o$ as follows:*

$$f \succsim^o g \Leftrightarrow f \in C^o(\{f, g\})$$

The roles of $\succsim^o$ and of C^o are perfectly interchangeable

Proof. By Weak Axiom of Revealed Preferences 36, $\succsim^o$ is a preference relation that satisfies Completeness Axiom 17, Transitivity Axiom 11 and Reflexivity Axiom 10. If the choice correspondence satisfies Constant Independence Axiom 40, Weak Certainty Independence Axiom 41, Risk Independence Axiom 42, Continuity Axiom 39 and Monotonicity Axiom 38 we have that $\succsim^o$ satisfies Constant Independence Axiom 18, Weak Certainty Independence Axiom 44, Risk Independence Axiom 43, Continuity Axiom 14 and Monotonicity Axiom 13. Moreover $\succsim^o$ has to satisfy Uncertainty Toward Criteria Axiom 25, Uniformity Toward Criteria Axiom 20, Cautiousness Toward Criteria Axiom 29 if the choice correspondence satisfies the corresponding axioms. ■

Given that by Lemma 51 $\succsim^o$ perfectly represents C^o all the proofs will be given in terms of $\succsim^o$.

Proof. of Proposition 24.

We prove only the sufficiency part given that the necessity part is standard.

The core part of the proof is an adaptation to our framework of the proof of Proposition 1 of De Meyer and Mongin (1995). For the standard representations of the Knightian preference relation $\succsim^*$ and of the invariant biseparable preferences check the proof of Proposition 26. Let's define $I^{full} = (I_o(\cdot), I_1(\cdot), \ldots, I_N(\cdot))$ and notice that the range of I^{full} is convex because $\{\succsim_i\}_{i=1}^N$ and C^o satisfy joint convexity. Let's define $M = I^{full} - I^{full} = \left\{x - y \mid x, y \in I^{full}\right\}$ and $Z = \{z_0 < 0, z_1 \geq 0, \ldots, z_N \geq 0\}$. Notice that M is nonempty, convex, symmetric w.r.t. the origin and it contains the origin. Moreover the affine hull of M, denoted as $aff(M)$, by Theorem 1.1 of Rockafellar (1970) coincides with the vector subspace spanned by M. The fact that C^o satisfies consistency toward criteria implies that $aff(M) \cap Z = \emptyset$. To see this assume by way of contradiction that $\exists z \in aff(M) \cap Z$ and given that $z \in aff(M)$ we have that $z = \sum_{j=1}^{N} \lambda_j z_j$ with $z_j \in M \;\; for \; j = 1, \ldots, N$. The fact that M is symmetric w.r.t. the origin allows us to assume that $\lambda_j \geq 0 \, for \, j = 1, \ldots, N$, in fact $-z_j \in M$ if $z_j \in M$. Now notice that by the convexity of the set M we have that $\frac{z}{\sum_{j=1}^{N} \lambda_j} = \sum_{j=1}^{N} \frac{\lambda_j}{\sum_{j=1}^{N} \lambda_j} z_j \in M \cap Z$ and we reached a contradiction with the fact that C^o satisfies consistency toward criteria. Let's now notice that $\left(\overline{Z}-e_o\right)\subset Z$, where $\overline{Z}$ denotes the closure of the set Z and e_o is the standard vector of R^{N+1} that assigns value 1 to the first component and 0 otherwise. Hence we have that $aff(M)\cap\left(\overline{Z}-e_o\right)=\emptyset$ and both these sets are polyhedral non empty convex sets. As a consequence we can strictly separate $aff(M)$ and $\left(\overline{Z}-e_o\right)$ and we get that $\exists\mu\in R^{N+1}$ such that $\mu(x-e_o)>\mu y$ for any $x\in\overline{Z}$ and $y\in aff(M)$. Given that $0\in aff(M)$ we have that $\mu_o<0$ (notice that $0\in\overline{Z}$). Moreover the fact that $aff(M)$ is a vector subspace implies that $\mu y=0$ for any $y\in aff(M)$. Hence it is possible to fix $x\in aff(M)$ and we have that for any $y\in aff(M)$ the following holds:

$$y_o=\sum_{j=1}^{N}\left(\frac{\mu_j}{-\mu_o}\right)y_j+\sum_{j=1}^{N}\left(\frac{\mu_j}{-\mu_o}\right)(-x_j)+x_o=\sum_{j=1}^{N}\gamma_j y_j+kost$$

where $\gamma_j=\frac{\mu_j}{-\mu_o}$ for any $j=1,\ldots,N$, $kost=\sum_{j=1}^{N}\left(\frac{\mu_j}{-\mu_o}\right)(-x_j)+x_o$. By Lemma 1 of GMM $I_o(\cdot)$ is unique up to positive affine transformations allow us to assume w.l.o.g. that $kost=0$.

If we assume that X is finite, by c-completeness of $\succsim^*$ and by the fact that $\{\succsim_i\}_{i=1}^N$ are consistent with respect to $\succsim^*$ we can do a standard normalization thus obtaining $\sum_{j=1}^{N}\gamma_j=1$. ■

Proof. of Proposition 26.

We want to show that if we have that

$$C^o(A)=\underset{f\in A}{argmax}\ \{I_o(u(f))\}=\underset{f\in A}{argmax}\left\{\min_{\gamma\in\Gamma}\sum_{i=1}^{N}\gamma_i I_i(u(f))\right\}$$

then C^o satisfies Criteria Uncertainty Aversion Axiom 27. Fix $f\in\mathcal{F}$, $f_j\in\mathcal{F}$ and $\alpha_j\geq 0$ for $j=1,\ldots J$ such that $\forall i\in\{1,2,\ldots,N\}$ we have that $\sum_{j=1}^{J}\alpha_j=1$ and $f\succsim_i\sum_{j=1}^{J}\alpha_j c_i^{f_j}$. Set $A=\left\{f;\ \sum_{j=1}^{J}\alpha_j c_i^{f_j}\right\}$ and by Lemma 51 it is enough to show the result for $\succsim^o$. Notice that if $f\succsim_i\sum_{j=1}^{J}\alpha_j c_i^{f_j}$ then $I_i(u(f))\geq\sum_{j=1}^{J}\alpha_j I_i(u(f_j))$ hence we have that for any set of weights $\gamma_i\geq 0$ for $i=1,\ldots N$ such that $\sum_{i=1}^{N}\gamma_i=1$ the following holds:

$$\sum_{i=1}^{N}\gamma_i I_i(u(f))\geq\sum_{i=1}^{N}\gamma_i\left(\sum_{j=1}^{J}\alpha_j I_i(u(f_j))\right)=\sum_{j=1}^{J}\alpha_j\left(\sum_{i=1}^{N}\gamma_i I_i(u(f_j))\right)$$

. By taking the *min* of both sides over the set of weights Γ and by using the properties of the min function we obtain the following chain of inequalities:

$$\min_{\gamma\in\Gamma}\sum_{i=1}^{N}\gamma_i I_i\left(u\left(f\right)\right)=I_o\left(u\left(f\right)\right)\geq\min_{\gamma\in\Gamma}\sum_{j=1}^{J}\alpha_j\left(\sum_{i=1}^{N}\gamma_i I_i\left(u\left(f_j\right)\right)\right)$$
$$\geq\sum_{j=1}^{J}\alpha_j\min_{\gamma\in\Gamma}\left(\sum_{i=1}^{N}\gamma_i I_i\left(u\left(f_j\right)\right)\right)=\sum_{j=1}^{J}\alpha_j I_o\left(u\left(f_j\right)\right)$$

.

The rest of the necessity part is standard and it is left to the reader.

We now prove the sufficiency part.

By Theorem 1 of GMMS there exists u^* and $\mathcal{C}^*$ such that:

$$f\succsim^* g\ \ \Leftrightarrow\int_S E_{f(s)}u^*dp\left(s\right)\geq\int_S E_{g(s)}u^*dp\left(s\right)\ \ \forall p\in\mathcal{C}^*$$

By Lemma 1, Proposition 5 and Proposition 7 of GMM we know that for each $i=o,1,\ldots,N$ there exists a non empty closed and convex set $\mathcal{C}^i$ of probabilities on Σ, a non constant function $u^i:X\to\mathbb{R}$ and a monotonic, constant additive and positively homogenous linear functional $I_i:B_0\left(\Sigma\right)\to\mathbb{R}$ (where with $B_0\left(\Sigma\right)$ we denote the vector space generated by the indicator functions of the elements of Σ endowed with the supnorm metric) such that $\forall f,g\in\mathcal{F}$ we have that:

$$f\succsim_i^* g\ \ \Leftrightarrow\int_S E_{f(s)}u^idp\left(s\right)\geq\int_S E_{g(s)}u^idp\left(s\right)\ \ \forall p\in\mathcal{C}^{\mathrm{i}}$$

$$f\succsim_i g\ \ \Leftrightarrow I_i\left(u^i\left(f\right)\right)\geq I_i\left(u^i\left(g\right)\right)$$

$$\min_{p\in\mathcal{C}^i}\int_S E_{f(s)}u^idp\left(s\right)\leq I_i\left(u^i\left(f\right)\right)\leq\max_{p\in\mathcal{C}^i}\int_S E_{f(s)}u^idp\left(s\right)$$

.

Remember that Criteria Uncertainty Aversion Axiom 25 implies Consistency Toward Criteria Axiom 20 and that all the relations $\{\succsim_i\}_{i=1}^N$ are consistent with $\succsim^*$ and as a consequence also $\succsim^o$ is consistent with $\succsim^*$. Given that $\succsim^*$ is a non trivial subrelation of $\{\succsim_i\}_{i=1}^N$ and $\succsim^o$on constant acts $\mathcal{F}_c$, by Proposition 4 of GMM we have that $u^*=u^o=u^1=\cdots=u^N:=u$. Without loss of generality we will assume that 0 is in the interior of the range of the utility function u.

By applying Proposition 4 of GMM we have that each one of $\succsim_i^*$ is the maximal subrelation satisfying independence of the corresponding potential completion criteria $\succsim_i$. By consistency we have that $\succsim^*\subseteq\succsim_i$ and Proposition A.1 of GMM delivers that $\mathcal{C}^i\subseteq\mathcal{C}^*$.

Hence $\forall f \in \mathcal{F}$ the following holds:

$$\underset{p\in\mathcal{C}^*}{min}\int_S E_{f(s)}udp\left(s\right) \leq \underset{p\in\mathcal{C}^i}{min}\int_S E_{f(s)}udp\left(s\right) \leq I_i\left(u\left(f\right)\right)$$

We denote by $R = R\left(I\right)$ the range of the vector $I = \left(I_1\left(\cdot\right), \ldots, I_N\left(\cdot\right)\right)$ and we can find a function $\phi : R \to \mathbb{R}$ such that $\forall f \in \mathcal{F}$ we have that $I_o\left(f\right) = \phi\left(I\left(u\left(f\right)\right)\right)$. By easily adapting the arguments contained in the proof of Theorem 1 of CGV, we can proceed by several extension of the functional ϕ up to $\mathbb{R}^N$ by preserving monotonicity, positive homogeneity, constant additivity and concavity. For any $x \in \mathbb{R}^N$, let's denote with $\bar{\Psi}\left(x\right)$ the final extension ϕ up to $\mathbb{R}^N$.

Finally by a standard argument using the supporting hyperplane theorem, i.e. as in Lemma 3.5 of GS, for any fixed $x^* \in \mathbb{R}^N$ we can find $h^{x^*} \in \mathbb{R}^N$ and $\gamma^{x^*} \in \mathbb{R}$ such that $\forall x \in \mathbb{R}^N$ the following holds:

$$h^{x^*}x + \gamma^{x^*} \geq \bar{\Psi}\left(x\right)$$

and

$$h^{x^*}x^* + \gamma^{x^*} = \bar{\Psi}\left(x^*\right)$$

. Notice that by positive homogeneity of the function $\bar{\Psi}\left(\cdot\right)$ we have that for any $\alpha > 0$ it is true that $\alpha h^{x^*}x^* + \gamma^{x^*} = \alpha\bar{\Psi}\left(x^*\right) \Rightarrow \gamma^{x^*} = 0$. Monotonicity of $\bar{\Psi}\left(\cdot\right)$ implies that $h^{x^*} \geq 0$, because if by contradiction we assume that $h_i^{x^*} < 0$ then we would have a violation of monotonicity, i.e. $\bar{\Psi}\left(x^*\right) = h^{x^*}x^* + \gamma^{x^*} > h^{x^*}\left(x^* + e_i\right) + \gamma^{x^*} \geq \bar{\Psi}\left(x^* + e_i\right)$. Finally if $\bar{\Psi}\left(x^*\right) = c$ we have that for any $\rho \in \mathbb{R}$ the following relationship holds $\bar{\Psi}\left(\rho x^* + \left(1-\rho\right)\vec{c}\right) = \bar{\Psi}\left(\rho x^*\right) + \left(1-\rho\right)c = \rho h^{x^*}x^* + \left(1-\rho\right)c = \rho\bar{\Psi}\left(x^*\right) + \left(1-\rho\right)c = c$ because $\bar{\Psi}\left(\cdot\right)$ satisfies constant additivity. This latter consideration in turn it implies that $h^{x^*}\vec{c} = c$ and it allow us to interpret h^{x^*} as a vector of positive weights that sum up to 1. Finally by setting $\Gamma := closure\left(co\left\{h^{x^*} \mid x^* \in \mathbb{R}^N\right\}\right)$, we have shown that there exists a closed and convex set $\Gamma \subseteq \triangle\left(\{1, 2, \ldots, N\}\right)$ such that $\forall x \in \mathbb{R}^N$ we have that:

$$\bar{\Psi}\left(x\right) = \underset{\gamma\in\Gamma}{min}\sum_{i=1}^N \gamma_i x_i$$

and in particular $\forall x \in R$ we have that $\phi\left(x\right) = \underset{\gamma\in\Gamma}{min}\sum_{i=1}^N \gamma_i x_i$. ■

Proof. of Corollary 28. If there exists $\bar{\imath}$ such that $\succsim_{\bar{\imath}}$ satisfies Caution Axiom 27 then by Theorem 3 of GMMS we have that:

$$\underset{p\in\mathcal{C}^*}{min}\int_S E_{f(s)}udp\left(s\right) = \underset{p\in\mathcal{C}^{\bar{\imath}}}{min}\int_S E_{f(s)}udp\left(s\right) = I_{\bar{\imath}}\left(u\left(f\right)\right)$$

Notice that $\underset{p\in\mathcal{C}^{\bar{\imath}}}{min}\int_S E_{f(s)}udp\left(s\right) = I_{\bar{\imath}}\left(u\left(f\right)\right) = \underset{p\in\mathcal{C}^{*}}{min}\int_S E_{f(s)}udp\left(s\right) \leq \underset{p\in\mathcal{C}^{i}}{min}\int_S E_{f(s)}udp\left(s\right) \leq I_i\left(u\left(f\right)\right)$ for $i = 1,\ldots,N$. Moreover, given that $e_{\bar{i}} \in \Gamma \subseteq \triangle\left(\{1,2,\ldots,N\}\right)$, $\forall\ f \in \mathcal{F}$ the following holds:

$$I_o\left(f\right) = \underset{\gamma\in\Gamma}{min}\sum_{j=1}^{N}\gamma_j I_j\left(u\left(f\right)\right) = \underset{p\in\mathcal{C}^{*}}{min}\int_S E_{f(s)}udp\left(s\right)$$

.

■

Proof. of Proposition 30.

The necessity part is standard and it is proved with the same arguments used in the previous propositions.

For the sufficiency argument we prove only the part that contains new ideas and that was not shown previously. $\succsim^o$ is consistent with respect to $\succsim^*$because $\succsim^o$ satisfies Uniformity Toward Criteria Axiom 20 and $\{\succsim_i\}_{i=1}^N$ satisfy Consistency Axiom 19 with respect to $\succsim^*$. Following the same type of arguments showed in Lemma 26 it is possible to show that there exists $\mathcal{C}^*$ and a unique u such that $\forall f \in \mathcal{F}$ the following holds:

$$I_o\left(u\left(f\right)\right) \geq \underset{p\in\mathcal{C}^{*}}{min}\int_S E_{f(s)}udp\left(s\right)$$

If there exists $\bar{\imath}$ such that $\succsim_{\bar{\imath}}$ satisfies caution then by Theorem 3 of GMMS we have that:

$$I_o\left(u\left(f\right)\right) \geq \underset{p\in\mathcal{C}^{*}}{min}\int_S E_{f(s)}udp\left(s\right) = \underset{p\in\mathcal{C}^{\bar{\imath}}}{min}\int_S E_{f(s)}udp\left(s\right) = I_{\bar{\imath}}\left(u\left(f\right)\right)$$

Suppose by contra that $I_o\left(u\left(f\right)\right) > \underset{p\in\mathcal{C}^{*}}{min}\int_S E_{f(s)}udp\left(s\right)$, then it is possible to find a constant act $g \in \mathcal{F}_c$ such that the following holds:

$$I_o\left(u\left(f\right)\right) > u\left(g\right) > \underset{p\in\mathcal{C}^{*}}{min}\int_S E_{f(s)}udp\left(s\right) = I_{\bar{\imath}}\left(u\left(f\right)\right)$$

. But this latter inequality contradicts Caution Toward Criteria Axiom 29 because for $\bar{\imath}$ we have that $f \not\succsim_{\bar{\imath}} g$ but $f \succsim^o g$. ■

Proof. of Theorem 33.

Let' start with the necessity part. Given the functional representation of I_o we will show that the axiom betweenness is satisfied. Otherwise by way of contradiction assume that:

$$I_o(u(f)) = \underset{\gamma\in\triangle(\{1,2,\dots,N\})}{min}\left(\sum_{j=1}^{N}\gamma_j I_j(u(f)) + c(\gamma)\right) > \max\ I_i(u(f))$$

and assume that the minimum is attained at $\overline{\gamma}$. Hence for any $\gamma \in \triangle(\{1,2,\dots,N\})$ we have that:

$$\sum_{j=1}^{N}\gamma_j I_j(u(f)) + c(\gamma) \geq \sum_{j=1}^{N}\overline{\gamma}_j I_j(u(f)) + c(\overline{\gamma}) > \max_{i\in\{1,2,\dots,N\}} I_i(u(f))$$

. Clearly $c(\overline{\gamma}) > 0$ and, given that c is grounded, $\exists\gamma^g \in \triangle(\{1,2,\dots,N\})$ such that $c(\gamma^g) = 0$. These considerations lead to the following chain of inequalities:

$$\sum_{j=1}^{N}\gamma_j^g I_j(u(f)) \geq \sum_{j=1}^{N}\overline{\gamma}_j I_j(u(f)) + c(\overline{\gamma}) > \max_{i\in\{1,2,\dots,N\}} I_i(u(f))$$

and we have that:

$$\sum_{j=1}^{N}\gamma_j^g I_j(u(f)) > \max_{i\in\{1,2,\dots,N\}} I_i(u(f))$$

that is absurd. Let's verify the Variability Axiom 32. By way of contradiction suppose that there exist $f, g \in \mathcal{F}\backslash\mathcal{F}_c$ such that either

$$\underset{\gamma\in\triangle(\{1,2,\dots,N\})}{min}\left(\sum_{j=1}^{N}\gamma_j I_j(u(f)) + c(\gamma)\right) - \underset{\gamma\in\triangle(\{1,2,\dots,N\})}{min}\left(\sum_{j=1}^{N}\gamma_j I_j(u(g)) + c(\gamma)\right)$$
$$< \min_{i\in\{1,2,\dots,N\}}\left(I_i(u(f)) - I_i(u(g))\right)$$

or

$$\max_{i\in\{1,2,\dots,N\}}\left(I_i(u(f)) - I_i(u(g))\right)$$
$$< \underset{\gamma\in\triangle(\{1,2,\dots,N\})}{min}\left(\sum_{j=1}^{N}\gamma_j I_j(u(f)) + c(\gamma)\right) - \underset{\gamma\in\triangle(\{1,2,\dots,N\})}{min}\left(\sum_{j=1}^{N}\gamma_j I_j(u(g)) + c(\gamma)\right)$$

. W.l.o.g. suppose that:

$$\underset{\gamma\in\triangle(\{1,2,\dots,N\})}{min}\left(\sum_{j=1}^{N}\gamma_j I_j(u(f)) + c(\gamma)\right) = \sum_{j=1}^{N}\overline{\gamma}_j I_j(u(f)) + c(\overline{\gamma})$$

and

$$\underset{\gamma\in\triangle(\{1,2,\dots,N\})}{min}\left(\sum_{j=1}^{N}\gamma_j I_j(u(g)) + c(\gamma)\right) = \sum_{j=1}^{N}\widetilde{\gamma}_j I_j(u(g)) + c(\widetilde{\gamma})$$

If

$$\underset{\gamma\in\triangle(\{1,2,\ldots,N\})}{min}\left(\sum_{j=1}^{N}\gamma_j I_j\left(u\left(f\right)\right)+c\left(\gamma\right)\right)-\underset{\gamma\in\triangle(\{1,2,\ldots,N\})}{min}\left(\sum_{j=1}^{N}\gamma_j I_j\left(u\left(g\right)\right)+c\left(\gamma\right)\right)$$
$$<\min_{i\in\{1,2,\ldots,N\}}\left(I_i\left(u\left(f\right)\right)-I_i\left(u\left(g\right)\right)\right)$$

then the contradiction is reached by the following chain of inequalities:

$$\min_{i\in\{1,2,\ldots,N\}}\left(I_i\left(u\left(f\right)\right)-I_i\left(u\left(g\right)\right)\right)$$
$$\leq\sum_{j=1}^{N}\overline{\gamma}_j I_j\left(u\left(f\right)\right)+c\left(\overline{\gamma}\right)-\sum_{j=1}^{N}\overline{\gamma}_j I_j\left(u\left(g\right)\right)-c\left(\overline{\gamma}\right)$$
$$\leq\sum_{j=1}^{N}\overline{\gamma}_j I_j\left(u\left(f\right)\right)+c\left(\overline{\gamma}\right)-\underset{\gamma\in\triangle(\{1,2,\ldots,N\})}{min}\left(\sum_{j=1}^{N}\gamma_j I_j\left(u\left(g\right)\right)+c\left(\gamma\right)\right)$$
$$<\min_{i\in\{1,2,\ldots,N\}}\left(I_i\left(u\left(f\right)\right)-I_i\left(u\left(g\right)\right)\right)$$

. On the other side if

$$\max_{i\in\{1,2,\ldots,N\}}\left(I_i\left(u\left(f\right)\right)-I_i\left(u\left(g\right)\right)\right)$$
$$<\underset{\gamma\in\triangle(\{1,2,\ldots,N\})}{min}\left(\sum_{j=1}^{N}\gamma_j I_j\left(u\left(f\right)\right)+c\left(\gamma\right)\right)-\underset{\gamma\in\triangle(\{1,2,\ldots,N\})}{min}\left(\sum_{j=1}^{N}\gamma_j I_j\left(u\left(g\right)\right)+c\left(\gamma\right)\right)$$

then we can reach a contradiction as follows

$$\max_{i\in\{1,2,\ldots,N\}}\left(I_i\left(u\left(f\right)\right)-I_i\left(u\left(g\right)\right)\right)$$
$$<\underset{\gamma\in\triangle(\{1,2,\ldots,N\})}{min}\left(\sum_{j=1}^{N}\gamma_j I_j\left(u\left(f\right)\right)+c\left(\gamma\right)\right)-\sum_{j=1}^{N}\widetilde{\gamma}_j I_j\left(u\left(g\right)\right)-c\left(\widetilde{\gamma}\right)$$
$$\leq\sum_{j=1}^{N}\widetilde{\gamma}_j I_j\left(u\left(f\right)\right)+c\left(\widetilde{\gamma}\right)-\sum_{j=1}^{N}\widetilde{\gamma}_j I_j\left(u\left(g\right)\right)-c\left(\widetilde{\gamma}\right)\leq\max_{i\in\{1,2,\ldots,N\}}\left(I_i\left(u\left(f\right)\right)-I_i\left(u\left(g\right)\right)\right)$$

.

Let's finally verify Criteria Uncertainty Aversion Axiom 25. Fix $f\in\mathcal{F}$, $f_j\in\mathcal{F}$ and $\alpha_j\geq 0$ for $j=1,\ldots J$ such that $\sum_{j=1}^{J}\alpha_j=1$ and $f\succsim_i\sum_{j=1}^{J}\alpha_j c_i^{f_j}$ for $i=1,\ldots,N$. Set $A=\left\{f;\ \sum_{j=1}^{J}\alpha_j c_i^{f_j}\right\}$ and by Lemma 51 it is enough to show the result for $\succsim^o$. Notice that if $f\succsim_i\sum_{j=1}^{J}\alpha_j c_i^{f_j}$ then $I_i\left(u\left(f\right)\right)\geq\sum_{j=1}^{J}\alpha_j I_i\left(u\left(f_j\right)\right)$ hence we have that for any set of weights $\gamma_i\geq 0$ for $i=1,\ldots N$ such that $\sum_{i=1}^{N}\gamma_i=1$ and any lower semicontinuous, grounded and convex function $c:\,\triangle\left(\{1,2,\ldots,N\}\right)\rightarrow[0,\,\infty]$ the following holds:

$$\sum_{i=1}^{N}\gamma_i I_i\left(u\left(f\right)\right)+c\left(\gamma\right)\geq\sum_{i=1}^{N}\gamma_i\left(\sum_{j=1}^{J}\alpha_j I_i\left(u\left(f_j\right)\right)\right)+\sum_{j=1}^{J}\alpha_j c\left(\gamma\right)=\sum_{j=1}^{J}\alpha_j\left(\sum_{i=1}^{N}\gamma_i I_i\left(u\left(f_j\right)\right)+c\left(\gamma\right)\right)$$

. By taking the *min* of both sides over the set of weights Γ and by using the properties of the min function we obtain the following chain of inequalities:

$$\begin{aligned}\underset{\gamma\in\Gamma}{min}\left(\sum_{i=1}^{N}\gamma_i I_i\left(u\left(f\right)\right)+c\left(\gamma\right)\right)=I_o\left(u\left(f\right)\right)&\geq\underset{\gamma\in\Gamma}{min}\sum_{j=1}^{J}\alpha_j\left(\sum_{i=1}^{N}\gamma_i I_i\left(u\left(f_j\right)\right)+c\left(\gamma\right)\right)\\&\geq\sum_{j=1}^{J}\alpha_j\underset{\gamma\in\Gamma}{min}\left(\sum_{i=1}^{N}\gamma_i I_i\left(u\left(f_j\right)\right)+c\left(\gamma\right)\right)=\sum_{j=1}^{J}\alpha_j I_o\left(u\left(f_j\right)\right)\end{aligned}$$

.

The rest of the necessity arguments are standard and are left to the reader.

Let's now show the sufficiency part. As usual Criteria Uncertainty Aversion Axiom 25 implies Consistency Toward Criteria Axiom 20. For the standard representations of the Knightian preference relation $\succsim^*$ and of the Invariant Biseparable preferences check the proof of Proposition 26. We will focus our attention on the representation of the choice correspondence. By the Lemma 51 the revealed preference relation $\succsim^o$ associated to the choice correspondence C^o satisfies Completeness Axiom 17, Transitivity Axiom 11, Nontriviality Axiom 12, Monotonicity Axiom 13, Continuity Axiom 14, Weak Certainty Independence Axiom 44. Notice that Weak Certainty Independence Axiom 44 implies Risk Independence Axiom 43. The existence of a normalized, monotonic, continuous functional $I_o:B_0\left(\Sigma\right)\rightarrow\mathbb{R}$, given that our continuity axiom implies the archimedean continuity axiom, is the result of Proposition 1 of Cerreia Vioglio, Ghirardato, Maccheroni, Marinacci, Siniscalchi (2011). By the proof of Proposition 26 we know that there exists a unique utility function $u:\,X\rightarrow\mathbb{R}$ for all the functional representations of the preference relations $\succsim^*$, $\{\succsim_i\}_{i=1}^N$ and $\succsim^o$. We denote with $R=R\left(I\right)$ the range of the vector $I=\left(I_1\left(\cdot\right),\ldots,I_N\left(\cdot\right)\right)$ and given that $\succsim^o$ satisfies Consistency Toward Criteria Axiom 20 we can find a well defined function $\phi:\,R\rightarrow\mathbb{R}$ such that for each $f\in\mathcal{F}$ we have that $I_o\left(u\left(f\right)\right)=\phi\left(I\left(u\left(f\right)\right)\right)$. Moreover $\phi\left(\cdot\right)$ is monotone: fix 2 acts $f,\,g\in\mathcal{F}$ such that $I\left(u\left(f\right)\right)\geq I\left(u\left(g\right)\right)$ and by Consistency Toward Criteria Axiom 20 we have that $\phi\left(I\left(u\left(f\right)\right)\right)=I_o\left(u\left(f\right)\right)\geq I_o\left(u\left(g\right)\right)=\phi\left(I\left(u\left(g\right)\right)\right)$. $\phi\left(\cdot\right)$ is also normalized, given that for any $k\in\mathbb{R}$ such that $k\mathbf{1}_N\in R$ we have that there exists a constant act $c^k\in\mathcal{F}_c$ such that for all $i\in\{o,1,\ldots,N\}$ $I_i\left(u\left(c^k\right)\right)=u\left(c^k\right)=k$. It is important to notice that by assumption the domain of ϕ is convex subset of $B_o\left(\Gamma\right)$, where $\Gamma=2^N$, given that $\{\succsim_i\}_{i=1}^N$ satisfy joint convexity. Now we want to show that ϕ is a niveloid, i.e. for any $x,\,y\in R$

we have that $\phi(x) - \phi(y) \leq \underset{i \in \{1,2,\ldots,N\}}{max} (x_i - y_i)$. If $f^x, f^y \in \mathcal{F}_c$ then the relation trivially holds by Consistency Toward Criteria Axiom 20:

$$\phi(x) - \phi(y) = I_o(u(f^x)) - I_o(u(f^y)) = \underset{i \in \{1,2,\ldots,N\}}{max} (x_i - y_i) =$$

$$\underset{i \in \{1,2,\ldots,N\}}{max} (I_i(u(f^x)) - I_i(u(f^y))) = u(f^x) - u(f^y)$$

If $f^x,\ f^y \in \mathcal{F} \setminus \mathcal{F}_c$ then by Variability Axiom 32 we have that $\exists i \in \{1,2,\ldots,N\}$ such that the following holds:

$$\frac{1}{2}c_o^{f^y} + \frac{1}{2}c_i^{f^x} \succsim^o \frac{1}{2}c_o^{f^x} + \frac{1}{2}c_i^{f^y}$$

and we have that $u\left(\frac{1}{2}c_o^{f^y} + \frac{1}{2}c_i^{f^x}\right) \geq u\left(\frac{1}{2}c_o^{f^x} + \frac{1}{2}c_i^{f^y}\right)$.

Notice that $I_o\left(u\left(c_i^{f^x}\right)\right) = \sum_{x \in X} c_i^{f^x}(x)\, u(x) = I_i\left(u\left(c_i^{f^x}\right)\right)$ and by using the affinity of the utility function we have that:

$$\frac{1}{2}u\left(c_o^{f^y}\right) + \frac{1}{2}u\left(c_i^{f^x}\right) \geq \frac{1}{2}u\left(c_o^{f^x}\right) + \frac{1}{2}u\left(c_i^{f^y}\right)$$

that it is equivalent to the following:

$$\frac{1}{2}I_o\left(u\left(c_o^{f^y}\right)\right) + \frac{1}{2}I_i\left(u\left(c_i^{f^x}\right)\right) \geq \frac{1}{2}I_o\left(u\left(c_o^{f^x}\right)\right) + \frac{1}{2}I_i\left(u\left(c_i^{f^y}\right)\right)$$

and as a consequence we have that:

$$\frac{1}{2}I_o(u(f^y)) + \frac{1}{2}I_i(u(f^x)) \geq \frac{1}{2}I_o(u(f^x)) + \frac{1}{2}I_i(u(f^y))$$

By simplifying and rearranging the last expression we obtain:

$$\phi(x) - \phi(y) = I_o(u(f^x)) - I_o(u(f^y)) \leq I_i(u(f^x)) - I_i(u(f^y))$$

$$\leq \underset{i \in \{1,2,\ldots,N\}}{max} (I_i(u(f^x)) - I_i(u(f^y))) = \underset{i \in \{1,2,\ldots,N\}}{max} (x_i - y_i)$$

Finally if $f^x \in \mathcal{F} \setminus \mathcal{F}_c$ and $f^y \in \mathcal{F}_c$ by Criteria Betweenness Axiom 31 we have that $\exists i \in \{1,2,\ldots,N\}$ such that $c_i^{f^x} \succsim^o f^x$, i.e. $I_o(u(f^x)) \leq I_o\left(u\left(c_i^{f^x}\right)\right) = u\left(c_i^{f^x}\right) = I_i(u(f^x))$. Notice that by assumption $I_o(u(f^y)) = u(f^y) = I_i(u(f^y))$ and we have that $I_o(u(f^x)) + u(f^y) - u(f^y) \leq I_i(u(f^x))$. The last expression implies that the following is true:

$$\phi(x) - \phi(y) = I_o(u(f^x)) - I_o(u(f^y)) \leq I_i(u(f^x)) - I_i(u(f^y))$$

$$\leq \max_{i\in\{1,2,\ldots,N\}} \left(I_i\left(u\left(f^x\right)\right) - I_i\left(u\left(f^y\right)\right)\right) = \max_{i\in\{1,2,\ldots,N\}} \left(x_i - y_i\right)$$

Let's now verify that $\phi(\cdot)$ is concave. For any $x,\, y \in R$ and $\alpha \in (0,\, 1)$ we want to show that $\phi(\alpha x + (1-\alpha)y) \geq \alpha\phi(x) + (1-\alpha)\phi(y)$. By the convexity of R we know that there exists $f^{xy} \in \mathcal{F}$ such that $I(u(f^{xy})) = \alpha x + (1-\alpha)y$. For any $i \in \{1, 2, \ldots, N\}$ we have that $I_i(u(f^{xy})) = \alpha I_i(u(f^x)) + (1-\alpha) I_i(u(f^y)) = \alpha I_i\left(u\left(c_i^{f^x}\right)\right) + (1-\alpha) I_i\left(u\left(c_i^{f^y}\right)\right)$, i.e. $f^{xy} \sim^i \alpha c_i^{f^x} + (1-\alpha) c_i^{f^y}$. By Criteria Uncertainty Aversion Axiom 25 we have that $f^{xy} \succsim^o \alpha c_o^{f^x} + (1-\alpha) c_o^{f^y}$, i.e. (by using the affinity of u) $I_o(u(f^{xy})) \geq \alpha I_o(u(f^x)) + (1-\alpha) I_o(u(f^y))$ and as a consequence the following is true:

$$\begin{aligned}\phi(\alpha x + (1-\alpha)y) &= \phi(I(u(f^{xy}))) \geq \alpha\phi(I(u(f^x))) + (1-\alpha)\phi(I(u(f^y))) \\ &= \alpha\phi(x) + (1-\alpha)\phi(y)\end{aligned}$$

Hence $\phi(\cdot)$ is a concave niveloid defined over a convex subset of $B(\Gamma)$ and by following Dolecki and Greco (1995) we know that the least niveloid on $B_o(\Gamma)$ that extends $\phi(\cdot)$ for any $x \in B_o(\Gamma)$ is:

$$\hat{\phi}(x) = \sup_{y\in R} \left[\phi(y) + \inf_{i\in\{1,\ldots,\,N\}} (x_i - y_i)\right]$$

By following the same approach of MMR, we can apply the Fenchel-Moreau theorem to $\hat{\phi}(\cdot)$ and we have that the following functional representation holds:

$$\hat{\phi}(x) = \min_{\mu\in ba(\Gamma)} \left[\langle x,\, \mu\rangle - \hat{\phi}^*(\mu)\right]$$

where $ba(\Gamma)$ denote the set of bounded and finitely additive set functions $\mu : \Gamma \to \mathbb{R}$ endowed with the total variation norm and $\hat{\phi}^*(\mu) = \inf_{x\in B(\Gamma)} \left(\langle x,\, \mu\rangle - \hat{\phi}(x)\right)$ is the Fenchel conjugate of $\hat{\phi}(\cdot)$. If μ is not positive then, given that w.l.o.g. we can assume that $0 \in int(u(X))$ and $[-1,\, 1] \subseteq u(X)$, it is possible to find $x \geq 0$ such that $\langle x,\, \mu\rangle < 0$. As a consequence for all $\alpha \geq 0$ we have that the monotonicity of $\hat{\phi}(\cdot)$ implies that $\langle \alpha x,\, \mu\rangle - \hat{\phi}(\alpha x) \leq \alpha\langle x,\, \mu\rangle - \hat{\phi}(0)$ and $\hat{\phi}^*(\mu) = -\infty$. If $\mu(\{1, 2, \ldots, N\}) \neq 1$ then for any $x \in B(\Gamma)$ and $a \in \mathbb{R}$ by using the constant additivity of the niveloid we have that the following holds:

$$\langle x + a,\, \mu\rangle - \hat{\phi}(x + a) = \langle x,\, \mu\rangle - \hat{\phi}(x) + a(\mu(\{1, 2, \ldots, N\}) - 1)$$

and again $\hat{\phi}^*(\mu) = -\infty$. As a consequence of the previous argument we have the following representation:

$$\hat{\phi}(x) = \min_{\gamma \in \triangle(\Gamma)} \left[\langle x, \gamma \rangle - \hat{\phi}^*(\gamma) \right] = \min_{\gamma \in \triangle(\{1,2,\ldots,N\})} \left(\sum_{j=1}^{N} \gamma_j I_j \left(u \left(f^x \right) \right) + c(\gamma) \right)$$

where $c(\gamma) = -\hat{\phi}^*(\gamma)$. The fact that $c(\cdot)$ is a lower semicontinuous, grounded and convex function is a direct application of Lemma 26 of MMR. ■

Chapter 4

Uncertainty Does Not Vanish in the Small

This chapter is joint work with Fabio Maccheroni, Massimo Marinacci, Sujoy Mukerji and Doriana Ruffino. The financial support of the European Research Council (advanced grant Indimacro) is gratefully acknowledged. The views in this paper are solely the responsability of the authors and should not be interpreted as reflecting the views of the Board of Governors of the Federal Reserve System or of any other person associated with the Federal Reserve System. All remaining errors are ours.

4.1 Introduction

Decision making under ambiguity has recently gained much importance thanks to both the development of theoretical models, and the applications of such models to many fields in economics and finance. The smooth ambiguity model of Klibanoff, Marinacci, and Mukerji (2005) (KMM hereafter) plays an especially important role because of its analytical tractability. Given this tractability, the smooth model lends itself to an extension to continuous time. Skiadas (2013) considers this extension and writes: "A central conclusion of this paper is the irrelevance of ambiguity aversion modeled through smooth second-order expected utility in a high frequency Brownian information setting."

Skiadas' result for the Brownian information setting (Theorem 1 in his paper and the basis for his conclusion), applies not only to "ambiguity aversion modeled through smooth second-order expected utility" but equally across the gamut of ambiguity sensitive preferences. For example, his result applies to the class of variational preferences of Maccheroni, Marinacci, and Rustichini (2006), which includes the multiple priors/maxmin expected utility (MEU) preferences of Gilboa and Schmeidler (1989). Since the Brownian information setting is ubiquitous in continuous-time finance models, Skiadas' conclusion implies that the effects of ambiguity cannot be incorporated in the most frequently applied information setting in continuous-time finance.

This paper shows that Skiadas' conclusion is not robust. The standard way to obtain the Brownian setting (Skiadas' Theorem 1 and our analysis here) is through a random walk with a stochastic binomial "up or down" increment at each node (observation). Then, one would allow the time interval between consecutive observations to become arbitrarily small and pass to the limit. The defining feature of the random walk is the transition probability, p (that is, the probability of transiting to, say, an up increment from a given node). Ambiguity, or model uncertainty, enters this setting when one allows the decision maker to be uncertain about this probability and, in particular, to believe that $p = 1/2$ is not the only possibility but that there may be a bias. Skiadas assumes a parametric form of the transition probability which severely restricts how the bias may be introduced: his bias is strictly increasing in the interval between consecutive observations and, thus, it does not survive when passing to the limit (specifically, in the limit the possible ps lie in a singleton set, $\left\{\frac{1}{2}\right\}$). This assumption rules out the very possibility of ambiguity and, consequently, the possibility that the decision maker's attitude toward ambiguity may affect her evaluation of the limiting uncertainty.

In contrast, we assume a more elaborate parametric form of the transition probability which allows for a bias *independent* of the frequency of observation. Such a bias survives in the limit so that ambiguity aversion remains relevant (shown in our Theorem 1). Further, and very importantly, we demonstrate that the diffusion limit of the random walk with our assumed transition probability takes the same form as the diffusion limit under Skiadas' assumed transition probability: by the main theorem of Gruber and Schweizer (2006), such diffusion limit is a Gaussian process with constant drift and constant volatility (Appendix 4.5.1).

Our analysis relies on the approximation of the smooth ambiguity model studied by Maccheroni, Marinacci and Ruffino (2013) and fully developed in Maccheroni, Marinacci and Ruffino (2011, henceforth MMR). The proof of our main result, Theorem 54, can be found in Appendix 4.5.2 and a simple heuristic argument is presented in Appendix 4.5.3.

4.2 A binomial model of small uncertainties

Diffusion limits à la Donsker (see Billingsley, 1968) are based on discrete processes defined on finite grids. As the grid mesh gets smaller, continuous-time interpolations of such discrete processes converge to diffusion processes that solve suitable stochastic differential equations. In Appendix 4.5.1, we specify the diffusion limit corresponding to the discrete process that we assume to model the uncertainty. Here, we take the vantage point of a single arbitrary node in a given grid and we study what happens in this cell of the grid when the mesh, ε, gets smaller.

One may think of the grid as a timeline where nodes represent observations of the evolving value of a stock holding. Then, a smaller ε translates to a higher frequency of observation. At each node, the observed random increment in the stock-holding value from the previous node is a binary random variable: *given* an ε, such increment is an up or down movement of fixed size.

For each mesh $\varepsilon \in (0,1]$, we consider a $\{1,-1\}$-valued binary random variable Z_ε such that

$$\mathbb{P}\left(Z_\varepsilon = 1 \,|Z_0 = z_0\right) = p\left(\varepsilon|z_0\right),$$

where Z_0 is the binary random variable at the current node, $z_0 = \pm 1$, and $p : (0,1] \times \{1,-1\} \to [0,1]$ is a transition function. In particular, $p\left(\varepsilon|z_0\right)$ is the (binomial) probability of observing a positive increment at the next node, conditional on the *sign* of the increment observed at the current node and in function of the mesh size. Specifying this transition function is the core, and the defining feature, of our analysis. We adopt the following specification

$$p\left(\varepsilon|z_0\right) = \frac{1}{2}\left(1 + z_0 a + \sqrt{\varepsilon} b\right), \tag{4.1}$$

where the parameters $a \in (-1, 1)$ and $b \in \mathbb{R}$ are such that $p\left(\varepsilon|z_0\right) \in [0, 1]$. The specified transition function—a special case of one studied by Gruber and Schweizer (2006)—defines a process through the increments

$$W_\varepsilon - W_0 = \alpha\sqrt{\varepsilon} Z_\varepsilon + \beta\varepsilon \tag{4.2}$$

which form a *correlated random walk* with drift $\beta \in \mathbb{R}$ and volatility $\alpha > 0$. It is so called because, for $a \neq 0$, consecutive random increments are correlated (with the value of a determining the correlation). When $a = 0$, we have the case of an *uncorrelated random walk.*

To ease matters, we assume that the drift of the random walk, β, is known and equal to 0. As to the volatility of the random walk, α, we consider two cases. In the first case (section 4.3.1), α is known and equal to 1. In the second case (section 4.3.2), there exists a function $h : A \longrightarrow (0, +\infty)$ such that $\alpha = h(a)$. In either case, and given the specification of the transition function p, we represent ambiguity—uncertainty about p—as the uncertainty about the values of a and b—the parameters of the random walk probability model. In particular, we allow a and b to range in two finite sets of possible values, A and B respectively. That is, we consider the parametric family of transition functions $\left\{p_{(a,b)} : (a, b) \in A \times B\right\}$, within the parametric family defined by (4.1). If $a = 0 = b$, then $p = 1/2$. Combinations of (non-zero) values of a and b, however, allow p to be different from 1/2 and introduce biases to the random walk. *Ambiguity amounts to uncertainty about the biases to the random walk.*

We formulate *small uncertainties* by considering the limit case as $\varepsilon \downarrow 0$. In this case, observations are arbitrarily frequent and random increments—obtained by scaling the unit increment Z_ε by $\alpha\sqrt{\varepsilon}$ (see (4.2))—are arbitrarily small. The limit transitions

$$p_a\left(z_0\right) \equiv \lim_{\varepsilon \downarrow 0} p_{a,b}\left(\varepsilon|z_0\right) = \frac{1}{2}\left(1 + z_0 a\right) \quad \forall a \in A \text{ (and } b \in B) \tag{4.3}$$

describe the limit behavior of transition probabilities as the mesh, ε, goes to zero.[1] If $a = 0$, then $p_a\left(z_0\right) = 1/2$ irrespective of the value of b. That is, uncertainty about the bias to the random walk survives in the limit transition *if and only if* it originates (at least in part) from uncertainty about the value of the parameter a. Indeed, since $\sqrt{\varepsilon} b \downarrow 0$,

[1]Note that $\lim_{\varepsilon \to 0} p_{a,b}\left(\varepsilon|1\right) \neq 1$ and $\lim_{\varepsilon \to 0} p_{a,b}\left(\varepsilon|-1\right) \neq 0$. In words, we do not require the states 1 and -1 to be standard (Kingman, 1972).

uncertainty about the bias that *purely* originates from the value of b—because $a = 0$ or, more generally, because its value is known—does not survive in the small. These observations are crucial to our findings.

Next, we consider a net $\{\pi_{\varepsilon|z_0}\}_{\varepsilon\in(0,1]}$ of subjective prior beliefs $\pi_{\varepsilon|z_0} : A \times B \to [0,1]$ on the transition parameters for every initial value z_0 and mesh ε. We denote the marginal distribution of $\pi_{\varepsilon|z_0}$ on A by

$$\pi_{\varepsilon|z_0}(a) = \sum_{b\in B} \pi_{\varepsilon|z_0}(a,b) \quad \forall a \in A.$$

We assume that priors converge like the transitions above. Formally,

$$\pi_{z_0}(a) \equiv \lim_{\varepsilon\downarrow 0} \pi_{\varepsilon|z_0}(a) \quad \forall a \in A \tag{4.4}$$

exist and they are finite. In addition, we denote by

$$\overline{\pi}_{z_0} = \sum_{a\in A} a\pi_{z_0}(a), \quad \overline{\overline{\pi}}_{z_0} = \sum_{a\in A} a^2\pi_{z_0}(a), \quad \text{and} \quad \sigma^2_{\pi_{z_0}} = \overline{\overline{\pi}}_{z_0} - \overline{\pi}^2_{z_0}$$

the first moment, second moment, and variance of π_{z_0}, respectively. Note that $\sigma^2_{\pi_{z_0}} = 0$ if and only if the support of π_{z_0} is a singleton.

Remark 52 For every initial value $z_0 = \pm 1$ and mesh ε, consider a net $\{\pi_{\varepsilon|z_0}\}_{\varepsilon\in(0,1]}$ on the transition parameters a and b such that $\pi_{\varepsilon|z_0}(a) = \hat{\pi}_{z_0}(a)$ for every $a \in A$. By (4.4), $\pi_{z_0}(a) = \hat{\pi}_{z_0}(a)$ for every $a \in A$. Hence, if a convergent sequence of subjective priors is such that its marginal on A (corresponding to each prior) is independent of ε, then the limit prior shares the same marginal on A with elements of the sequence. If the priors in the sequence have the property that the common marginal on A has non-singleton support, then so does the marginal of the limit prior.

Let B_t be a standard Brownian motion on $[0,1]$. By Gruber and Schweizer's (2006) main result (Appendix 4.5.1), the correlated random walk specified by transition (4.1) and increments (4.2) converges in distribution to a Brownian motion W_t on $[0,1]$. The drift, μ, and volatility, σ, of W_t are given by

$$\mu = \beta + \frac{\alpha b}{1-a} = \frac{\alpha b}{1-a} \quad \text{and} \quad \sigma = \alpha\sqrt{\frac{1+a}{1-a}}. \tag{4.5}$$

When $b \neq 0$, uncertainty about a translates into uncertainty about both the drift and the volatility of the Brownian motion.[2] If both volatilities α and σ are known, the relation $\sigma = \alpha\sqrt{(1+a)/(1-a)}$ identifies the value of a: in this case, no model uncertainty survives in the limit. We consider two other cases:

(1) α is known and equal to 1, σ is unknown. Setting $\mu(a,b) = b/(1-a)$ and $\sigma(a) = \sqrt{(1+a)/(1-a)}$, here we have that

$$W_t = W_0 + \mu(a,b)\,t + \sigma(a)\,B_t; \tag{4.6}$$

(2) α is unknown and given by $\alpha = h(a)$ with $h : A \to (0,+\infty)$ a known function, σ is known. Setting $h(a) = \sigma\sqrt{(1-a)/(1+a)}$ and $\mu(a,b) = h(a)\,b/(1-a)$, we have that

$$W_t = W_0 + \mu(a,b)\,t + \sigma B_t.$$

We study these two cases separately and show that, in both, the effects of ambiguity aversion are not negligible. That said, our assumption that the volatility σ is unknown (Case 1) is empirically questionable: volatilities can be identified, almost surely, through the high frequency data underlying the convergence to the continuous-time process W_t (see Anderson, Hansen and Sargent (2003, p. 104)).[3] In Case 2, we take σ to be known—a sounder assumption empirically.

Remark 53 The binomial model of small uncertainties set out above is the "high frequency Brownian information setting"studied by Skiadas (2013, sections i-iv and included in Theorem 1), with α known and equal to 1. Skiadas, however, considers the special case of $a = 0$ in transition probability (4.1): in his case, the only source of ambiguity is uncertainty about b (ρ in his paper). Thus, the Brownian motion W_t has drift $\mu = b$ and volatility $\sigma = 1$ (given $a = 0$); that is, $W_t = W_0 + bt + B_t$. Here, the sole source of uncertainty about the drift (through the transition function of the underlying random walk) need be uncertainty about b—this is the sole source of model uncertainty that Skiadas considers for the Brownian case. As shown in (4.3), model uncertainty that is driven purely by uncertainty about b does not survive in the limit transition. This remark is most significant since it is the limit transition which is relevant to a decision maker's evaluation of small uncertainties (section 4.3).[4]

[2] When $a = 0$, β and α, the drift and volatility of the random walk, coincide with μ and σ, the drift and volatility of the Brownian motion obtained as the diffusion limit of the random walk.

[3] A similar issue arises in all continuous-time analyses that assume mutually singular models.

[4] Appendix 4.5.1 gives a fuller account of the binomial model of uncertainty presented in this section and its diffusion limit. Table 1 at the end of this appendix shows the exact correspondence between the binomial models studied here and in Skiadas (2013).

4.3 Effects of small uncertainties

4.3.1 Case 1: known α and unknown σ

We assume that α is equal to 1. In this case small uncertainties are described by the net $\{W_\varepsilon\}_{\varepsilon\in(0,1]}$, where $W_\varepsilon = \sqrt{\varepsilon} Z_\varepsilon, \forall\varepsilon \in (0,1]$ scales the binary "tilt" Z_ε à la Donsker. To study the effects of risk and ambiguity for small uncertainties, we consider an ambiguity sensitive decision maker who evaluates (smaller and smaller) perturbations W_ε to his wealth w. Let $I \subseteq \mathbb{R}$ be a non-singleton interval of outcomes of the prospect $w + W_\varepsilon$. According to the smooth ambiguity model of KMM, with ambiguity attitude denoted by ϕ and risk attitude by u, and where $u : I \to \mathbb{R}$ and $\phi : u(I) \to \mathbb{R}$ are strictly increasing and continuous functions, the prospect $(w + W_\varepsilon \,|z_0)$ is evaluated as

$$\sum_{a,b} \pi_{\varepsilon|z_0}(a,b)\, \phi\left(p_{a,b}(\varepsilon|z_0)\, u\left(w + \sqrt{\varepsilon}\right) + (1 - p_{a,b}(\varepsilon|z_0))\, u\left(w - \sqrt{\varepsilon}\right)\right).$$

Writing $v = \phi \circ u$, the certainty equivalent $\mathrm{C}_\varepsilon(w + W_\varepsilon \,|z_0)$ of $w + W_\varepsilon$ conditional on z_0 is

$$v^{-1}\left(\sum_{a,b} \pi_{\varepsilon|z_0}(a,b)\, v\left(u^{-1}\left(p_{a,b}(\varepsilon|z_0)\, u\left(w + \sqrt{\varepsilon}\right) + (1 - p_{a,b}(\varepsilon|z_0))\, u\left(w - \sqrt{\varepsilon}\right)\right)\right)\right).$$

The expectation $\mathrm{E}_\varepsilon(w + W_\varepsilon \,|z_0)$ of $w + W_\varepsilon$ conditional on z_0 is

$$\sum_{a,b} \pi_{\varepsilon|z_0}(a,b)\left(p_{a,b}(\varepsilon|z_0)\left(w + \sqrt{\varepsilon}\right) + (1 - p_{a,b}(\varepsilon|z_0))\left(w - \sqrt{\varepsilon}\right)\right),$$

so that the *uncertainty premium* is given by

$$\mathrm{UP}_\varepsilon(w + W_\varepsilon \,|z_0) = \mathrm{E}_\varepsilon(w + W_\varepsilon \,|z_0) - \mathrm{C}_\varepsilon(w + W_\varepsilon \,|z_0).$$

Next, we establish the quadratic approximation of $\mathrm{UP}_\varepsilon(w + W_\varepsilon \,|z_0)$.[5] The Arrow-Pratt coefficients of u and v are $\lambda_u(w) = -u''(w)/u'(w)$ and $\lambda_v(w) = -v''(w)/v'(w)$, respectively.

Theorem 54 *Take the functions u and v thrice continuously differentiable in a neighborhood of w, with $u' > 0$, $v' > 0$, $\lambda_u(w) \neq 0$, $\lambda_v(w) \neq 0$, $\lambda_u(w) \neq \lambda_v(w)$. If $\pi_{\varepsilon|z_0}(a)$ converges as $\varepsilon \downarrow 0$ for all $a \in A$, then*

$$\mathrm{UP}_\varepsilon(w + W_\varepsilon \,|z_0) = \frac{\varepsilon}{2}\lambda_u(w)\left(1 - \overline{\overline{\pi}}_{z_0}\right) + \frac{\varepsilon}{2}\lambda_v(w)\,\sigma^2_{\pi_{z_0}} + o(\varepsilon), \quad z_0 = \pm 1. \tag{4.7}$$

[5] See Appendix 4.5.2 for a formal proof or Appendix 4.5.3 for a heuristic derivation.

The following observations follow immediately from the theorem and are key to the conclusion of our paper.

(i) $1-\overline{\overline{\pi}}_{z_0} = \sum_{a\in A} (1-a^2)\,\pi_{z_0}(a) > 0$ because $A \subseteq (-1,1)$: *the effects of risk aversion are never negligible.*

(ii) $\sigma^2_{\pi_{z_0}} = 0$ if and only if $\pi_{z_0} = \delta_{a^*}$, for some $a^* \in A$.

(iii) The Brownian motion considered by Skiadas (2013) belongs to (ii) because $A = \{0\}$ is a singleton and, thus, $\pi_{z_0} = \delta_0$. In particular, since $\overline{\overline{\pi}}_{z_0} = 0$, Skiadas' approximation becomes

$$\mathrm{UP}_\varepsilon\left(w + W_\varepsilon \,|z_0\right) = \frac{\varepsilon}{2}\lambda_u(w) + o(\varepsilon).$$

In this case, the smooth ambiguity certainty equivalent for a small uncertainty is indistinguishable from the expected utility certainty equivalent.

(iv) In the KMM evaluation functional, ambiguity neutrality corresponds to the case where $v = u$ (i.e., ϕ is affine). In this case, the uncertainty premium reflects purely the effect of risk aversion and reduces to:

$$\mathrm{RP}_\varepsilon\left(w + W_\varepsilon \,|z_0\right) \equiv \frac{\varepsilon}{2}\lambda_u(w)\left(1-\overline{\overline{\pi}}_{z_0}\right) + \frac{\varepsilon}{2}\lambda_u(w)\,\sigma^2_{\pi_{z_0}} + o(\varepsilon).$$

Noting that $\lambda_\phi(w) = \lambda_v(w) - \lambda_u(w)$, we derive the part of the uncertainty premium that is affected purely by ambiguity attitude

$$\mathrm{UP}_\varepsilon\left(w + W_\varepsilon \,|z_0\right) - \mathrm{RP}_\varepsilon\left(w + W_\varepsilon \,|z_0\right) = \frac{\varepsilon}{2}\lambda_\phi(w)\,\sigma^2_{\pi_{z_0}} + o(\varepsilon).$$

For the net of priors identified in Remark 52 (with a common marginal on A invariant to ε and non-singleton support on A), $\lambda_\phi(w)\,\sigma^2_{\pi_{z_0}} > 0$ and *the effects of ambiguity aversion are not negligible* for small uncertainties. Furthermore, to ensure that the effects of ambiguity aversion survive, ambiguity aversion needs *not* increase arbitrarily as the mesh $\varepsilon \downarrow 0$—$\lambda_\phi(w)$ may be constant.

4.3.2 Case 2: unknown α and known σ

We assume that $\alpha = h(a)$, with $h : A \to (0,+\infty)$ a known function. In this case $W_\varepsilon = \sqrt{\varepsilon}h(a)\,Z_\varepsilon$ and the certainty equivalent, $\mathrm{C}_\varepsilon\left(w + W_\varepsilon \,|z_0\right)$, becomes

$$v^{-1}\left(\sum_{a,b}\pi_{\varepsilon|z_0}(a,b)\,v\left(u^{-1}\left(p_{a,b}(\varepsilon|z_0)\,u\left(w+h(a)\sqrt{\varepsilon}\right) + \left(1-p_{a,b}(\varepsilon|z_0)\right)u\left(w-h(a)\sqrt{\varepsilon}\right)\right)\right)\right).$$

In Appendix 4.5.3, we provide a heuristic derivation of the quadratic approximation of the uncertainty premium $\mathrm{UP}_{\varepsilon}\left(w+W_{\varepsilon}\left|z_0\right.\right)$ corresponding to $\mathrm{C}_{\varepsilon}\left(w+W_{\varepsilon}\left|z_0\right.\right)$. We obtain

$$\mathrm{UP}_{\varepsilon}\left(w+W_{\varepsilon}\left|z_0\right.\right)=\frac{\lambda_u}{2}\varepsilon E_{\pi_{z_0}}\left(\left(h\left(a\right)\right)^2\left(1-a^2\right)\right)+\frac{\lambda_v}{2}\varepsilon\sigma^2_{\pi_{z_0}}\left(h\left(a\right)a\right).^6 \tag{4.8}$$

Quadratic approximation (4.8) shows that even in Case 2 the effects of ambiguity aversion are not negligible—the observations we made following Case 1 continue to hold.

4.4 Conclusion

We overturn Skiadas' (2013) finding on the irrelevance of ambiguity aversion "modeled through smooth second-order expected utility in a high frequency Brownian information setting". We do so by defining the random walk model that describes the information setting through a more general transition function than the one assumed by Skiadas.

The required increased generality is the allowance for consecutive random increments to be correlated: for example, an "up" movement is more likely if preceded by an "up" movement. Crucially, such an allowance generalizes the ways of introducing bias to the transition probability. In particular, it is possible to have a class of biases that survive in the limit as random increments are scaled down because of increased frequency of observation. If the decision maker's ambiguity about the transition probability were driven (at least partly) by biases from this class, then that ambiguity survives in the limit and the effects of ambiguity aversion remains relevant in the small.

In addition, the parameter that drives the ambiguity of the transition probability in the random walk also makes the drift and volatility parameters of the diffusion limit uncertain. In the event that data is rich enough to identify the volatility parameter of the diffusion limit, our key conclusion that the effects of ambiguity aversion are not negligible in the small still holds.

Skiadas' finding created a seemingly impassable block in the path of pursuing a recursive continuous-time model of ambiguity aversion, where ambiguity aversion is modeled through smooth second-order expected utility as in KMM. Now, that block seems passable.

[6] When $h\left(a\right)$ is constant and equal to 1, (4.8) reduces to (4.7).

4.5 Appendix

4.5.1 Diffusions

Diffusion limit

By using a result of Gruber and Schweizer (2006), we show that there exists a continuos-time diffusion process that is the limit of the binomial model considered in our framework.[7] Consider a unit time interval $[0,1]$. Each $n \geq 1$ determines a grid $\{k/n\}_{k=0}^{n}$ of the unit time interval, i.e.,

$$0 < \frac{1}{n} < \cdots < \frac{n-1}{n} < 1.$$

Fix n and a probability space $(\Omega_n, \Sigma_n, \mathbb{P}_n)$. Consider the *tilt process* $\{Z_k^n\}_{k=0}^{n}$ where each $Z_k^n : \Omega_n \to \{-1, 1\}$ is a binary random variable for each $k = 0, ..., n$. Given two coefficients $\alpha_n, \beta_n \in \mathbb{R}$, define the process $\{W_k^n\}_{k=0}^{n}$ by $W_k^n - W_{k-1}^n = \alpha_n Z_k^n + \beta_n$ so that $W_k^n = W_0^n + \sum_{j=1}^{k} \left(\alpha_n Z_j^n + \beta_n\right)$, for each $k = 0, ..., n$. The process $\{W_k^n\}_{k=0}^{n}$ is a possibly correlated random walk. The coefficients α_n and β_n are its volatility and drift, respectively.

Let $\{\Sigma_k^n\}_{k=0}^{n}$, with $\Sigma_k^n = \sigma\left(W_0^n, ..., W_{k-1}^n\right) \subseteq \Sigma_n$, be the filtration determined by the random walk $\{W_k^n\}_{k=0}^{n}$. For $k = 1, ..., n$, it holds

$$\mathbb{P}_n\left(Z_k^n = 1 \left| \Sigma_{k-1}^n\right.\right) = p_n\left(k, W_{k-1}^n, Z_{k-1}^n\right)$$

where $p_n : \{1, ..., n\} \times \mathbb{R} \times \{-1, 1\} \to [0, 1]$ is the transition function of the random walk. In particular, $p_n(k, x, z)$ is the probability that the tilt process takes value $+1$ at k/n given that the random walk and the tilt process took values x and z at $(k-1)/n$, respectively.

Through linear interpolation, the random walk $\{W_k^n\}_{k=0}^{n}$ defines a random function $\widetilde{W}^n : \Omega_n \times [0,1] \to \mathbb{R}$ given by $\widetilde{W}_\varepsilon^n(\omega_n) = W_{[n\varepsilon]}^n(\omega_n)$. We have that

$$\widetilde{W}_\varepsilon^n(\omega_n) = \begin{cases} W_0^n & \text{if } \varepsilon = 0 \\ W_k^n(\omega_n) & \text{if } \varepsilon \in \left[\frac{k}{n}, \frac{k+1}{n}\right) \text{ and } 1 < k < n \\ W_n^n & \text{if } \varepsilon = 1 \end{cases} . \tag{4.9}$$

The paths of $\widetilde{W}^n$ are right-continuous functions on $[0,1]$ with left-hand limits, i.e., $\widetilde{W}^n(\omega_n) \in$

[7] Section 4.5.1 summarizes the relationships between the notation used in this appendix, the notation in the main text, and the notation in Skiadas' (2013) paper.

D for all $\omega_n \in \Omega_n$.[8] Let $\left\{\widetilde{\Sigma}_\varepsilon^n\right\}_{\varepsilon\in[0,T]}$ be the filtration generated by $\widetilde{W}^n$, so that

$$\widetilde{\Sigma}_\varepsilon^n = \Sigma_k^n \qquad \forall \varepsilon \in \left[\frac{k-1}{n}, \frac{k}{n}\right), \forall k = 1, ..., n.$$

Denote by P_n the distribution on the Skorohod measurable space $(D, \mathcal{D})$ of the random function $\widetilde{W}^n$ under $\mathbb{P}_n$,[9] that is

$$P_n(A) = \mathbb{P}_n\left(\omega_n \in \Omega_n : \widetilde{W}^n(\omega_n) \in A\right) \qquad \forall A \in \mathcal{D}.$$

Suppose there are:

1) constant $\alpha > 0$ and $\beta \in \mathbb{R}$ such that $\alpha_n = \alpha/\sqrt{n}$ and $\beta_n = \beta/n$;[10]

2) $a \in (-1, 1)$ and $b \in \mathbb{R}$ such that

$$p_n(z) = \frac{1}{2}\left(1 + za + \frac{1}{\sqrt{n}}b\right). \tag{4.10}$$

Under more general hypotheses than 1) and 2) above (including, for example, time-varying coefficients a and b), Gruber and Schweizer (2006) prove the following Donsker-type invariance principle.

Theorem 55 *Under hypotheses 1), 2), and a martingale assumption, if $\{W_0^n\}$ converges in distribution to some W_0, then $\left\{\widetilde{W}^n\right\}$ converges in distribution to a Brownian motion $W_t = W_0 + \mu t + \sigma B_t$ on $[0, 1]$ with initial value W_0,*

$$\mu = \beta + \frac{\alpha b}{1-a} \quad , \text{ and } \quad \sigma = \alpha\sqrt{\frac{1+a}{1-a}}.$$

Cells and grids

In the main text we study the initial cell of the grid, with a smaller and smaller scalar mesh. Thus, the mesh ε corresponds to $1/n$, the binary random variable Z_ε to Z_1^n, the transition $p(\varepsilon|z_0)$ to $p_n(1|z) = \mathbb{P}_n(Z_1^n = 1|Z_0^n = z_0)$, and the random variable W_ε to $(1/\sqrt{n}) Z_1^n$. These relationships, and their equivalent in Skiadas' (2013) paper, are summarized in Table 4.1. This Table is the Rosetta stone of our exercise.

---

[8] D is the space of right-continuous functions on $[0, 1]$ with left-hand limits (Billingsley, 1968).

[9] See Billingsley (1968) p. 121.

[10] $1/\sqrt{n}$ and $1/n$ are the scaling factors of volatility and drift, respectively.

Table 4.1: Rosetta stone

	Appendix	**Main text**	**Skiadas (2013)**
Mesh	$\frac{1}{n}$	$\varepsilon \in (0,1]$	h
Tilt	Z_1^n	Z_ε	± 1
Scaled tilt	$\frac{1}{\sqrt{n}} Z_1^n$	$W_\varepsilon = \sqrt{\varepsilon} Z_\varepsilon$	$\pm\sqrt{h}$
Transition	$p_n(1 \mid z) = \frac{1}{2}\left(1 + za + \frac{1}{\sqrt{n}} b\right)$	$p(\varepsilon \mid z_0) = \frac{1}{2}(1 + z_0 a + \sqrt{\varepsilon} b)$	$Q(1) = \frac{1}{2}\left(1 + \rho\sqrt{h}\right)$

The table is the Rosetta stone of our exercise.

4.5.2 Proof of Theorem 54 and related material

Each of the beliefs, $\pi_{\varepsilon|z_0}$, is a probability distribution on $A \times B$ and it induces a corresponding discrete probability distribution on $[0,1]$

$$\mu_{\varepsilon|z_0}(q) = \sum_{a,b} \pi_{\varepsilon|z_0}(a,b)\,\delta_{p_{a,b}(\varepsilon|z_0)}(q) \quad \forall q \in [0,1].$$

In what follows, we write $\mu_\varepsilon \Rightarrow \mu$ to indicate that the net of Borel probability measures on $[0,1]$, μ_ε, weakly converges to μ as $\varepsilon \downarrow 0$. "Weakly" refers to the weak convergence of probability measures (see Billingsley, 1968).

Lemma 56 *Let $z_0 = \pm 1$. If the limit $\lim_{\varepsilon\downarrow 0} \pi_{\varepsilon|z_0}(a) = \pi_{z_0}(a)$ exists and is finite for every $a \in A$, then π_{z_0} is a probability measure on A and*

$$\mu_{\varepsilon|z_0} \Rightarrow \mu_{z_0} = \sum_{a\in A} \pi_{z_0}(a)\,\delta_{\frac{1}{2}(1+z_0 a)}.$$

Proof of Lemma 56 Let $z_0 = \pm 1$, $\varepsilon \in (0,1]$, and $n \in \mathbb{N}$. Then

$$\begin{aligned}
\int_0^1 q^n d\mu_{\varepsilon|z_0}(q) &= \int_0^1 q^n d\left(\sum_{(a,b)\in A\times B} \pi_{\varepsilon|z_0}(a,b)\,\delta_{p_{a,b}(\varepsilon|z_0)}(q)\right) \\
&= \sum_{(a,b)\in A\times B} \pi_{\varepsilon|z_0}(a,b) \int_0^1 q^n d\delta_{p_{a,b}(\varepsilon|z_0)}(q) \\
&= \sum_{(a,b)\in A\times B} \pi_{\varepsilon|z_0}(a,b)\,(p_{a,b}(\varepsilon|z_0))^n \\
&= \sum_{(a,b)\in A\times B} \pi_{\varepsilon|z_0}(a,b) \left(\frac{1}{2}\left(1 + z_0 a + \sqrt{\varepsilon} b\right)\right)^n \\
&= \sum_{a\in A} \left[\sum_{b\in B} \pi_{\varepsilon|z_0}(a,b) \left(\frac{1}{2}\left(1 + z_0 a + \sqrt{\varepsilon} b\right)\right)^n\right].
\end{aligned}$$

Denote by b' and b'' the minimum and maximum elements of B, respectively. For every $a \in A$ and all $(\varepsilon, b) \in (0,1] \times B$,

$$0 \leq \frac{1}{2}\left(1 + z_0 a + \sqrt{\varepsilon} b'\right) \leq \frac{1}{2}\left(1 + z_0 a + \sqrt{\varepsilon} b\right) \leq \frac{1}{2}\left(1 + z_0 a + \sqrt{\varepsilon} b''\right).$$

Therefore, for every $n \in \mathbb{N}$

$$\begin{aligned}\pi_{\varepsilon|z_0}(a)\left(\frac{1}{2}\left(1+z_0a+\sqrt{\varepsilon}b'_a\right)\right)^n &= \sum_{b\in B}\pi_{\varepsilon|z_0}(a,b)\left(\frac{1}{2}\left(1+z_0a+\sqrt{\varepsilon}b'_a\right)\right)^n \\ &\le \sum_{b\in B}\pi_{\varepsilon|z_0}(a,b)\left(\frac{1}{2}\left(1+z_0a+\sqrt{\varepsilon}b\right)\right)^n \\ &\le \sum_{b\in B}\pi_{\varepsilon|z_0}(a,b)\left(\frac{1}{2}\left(1+z_0a+\sqrt{\varepsilon}b''_a\right)\right)^n \\ &= \pi_{\varepsilon|z_0}(a)\left(\frac{1}{2}\left(1+z_0a+\sqrt{\varepsilon}b''_a\right)\right)^n.\end{aligned}$$

Keeping a and n fixed, as $\varepsilon \downarrow 0$

$$\begin{gathered}\pi_{\varepsilon|z_0}(a) \to \pi_{z_0}(a), \\ \left(\frac{1}{2}\left(1+z_0a+\sqrt{\varepsilon}b'\right)\right)^n \to \left(\frac{1}{2}\left(1+z_0a\right)\right)^n, \\ \left(\frac{1}{2}\left(1+z_0a+\sqrt{\varepsilon}b''\right)\right)^n \to \left(\frac{1}{2}\left(1+z_0a\right)\right)^n,\end{gathered}$$

and then

$$\sum_{b\in B}\pi_{\varepsilon|z_0}(a,b)\left(\frac{1}{2}\left(1+z_0a+\sqrt{\varepsilon}b\right)\right)^n \to \pi_{z_0}(a)\left(\frac{1}{2}\left(1+z_0a\right)\right)^n.$$

Since this is true for all $a \in A$ and $n \in \mathbb{N}$,

$$\begin{aligned}\lim_{\varepsilon\downarrow 0}\int_0^1 q^n d\mu_{\varepsilon|z_0}(q) &= \lim_{\varepsilon\downarrow 0}\sum_{a\in A}\left[\sum_{b\in B}\pi_{\varepsilon|z_0}(a,b)\left(\frac{1}{2}\left(1+z_0a+\sqrt{\varepsilon}b\right)\right)^n\right] \\ &= \sum_{a\in A}\pi_{z_0}(a)\left(\frac{1}{2}\left(1+z_0a\right)\right)^n = \int_0^1 q^n d\left(\sum_{a\in A}\pi_{z_0}(a)\,\delta_{p_a(z_0)}(q)\right)\end{aligned}$$

and $\mu_{\varepsilon|z_0} \Rightarrow \mu_{z_0} = \sum_{a\in A}\pi_{z_0}(a)\,\delta_{p_a(z_0)}$. ■

Finally, given a Borel probability measure μ on $[0,1]$, we denote by $\overline{\mu} = \int_0^1 q d\mu(q)$ and $\overline{\overline{\mu}} = \int_0^1 q^2 d\mu(q)$ its first and the second moments, respectively.

Proof of Theorem 54 Let $z_0 = \pm 1$. Since the proof builds on Proposition 48 of MMR, we begin by setting up some translation:

- Write t instead of ε in $(0,1]$.

- Set $\Omega = \{1, -1\}$ and $h_t(\omega) = \omega\sqrt{t}$.
- Denote by q both an element of $[0, 1]$ and the corresponding probability measure on Ω, assigning weight q to state 1 (and, hence, weight $1-q$ to state -1). For example,

$$\|h_t\|_q = \sqrt{\int_\Omega h_t^2 dq} = \sqrt{q\left(\sqrt{t}\right)^2 + (1-q)\left(-\sqrt{t}\right)^2} = \sqrt{t} \tag{4.11}$$

  for all q (MMR consider the $L^2(q)$ norm).
- Denote by μ both a Borel probability measure on $[0, 1]$ and the corresponding probability measure on the one-dimensional simplex of probability measures on Ω. Recall that MMR write $\overline{q}$ instead of $\overline{\mu}$ (that is, $\overline{q} = \int_0^1 qd\mu(q) = \bar{\mu}$) and ϑ instead of $\overline{\overline{\mu}}$ (that is, $\vartheta = \int_0^1 q^2 d\mu(q) = \overline{\overline{\mu}}$).
- Write interchangeably μ_t or $\mu_{t|z_0}$ and μ or μ_{z_0} (z_0 is fixed).

Consider the nets $\{h_t, \mu_t\}_{t\in(0,1]}$ and $\{h_t, \overline{\mu}_t\}_{t\in(0,1]}$. Then, by (4.11) $\|h_t\|^2_{\overline{\mu}_t} = t > 0$ and h_t is never $\overline{\mu}_t$-a.e. null. By Lemma 56, as $t \downarrow 0$

$$\int_0^1 qd\mu_t(q) = \overline{\mu}_t(1) = \overline{\mu}_t \to \overline{\mu} = \overline{\mu}(1) = \int_0^1 qd\mu(q)$$
$$\int_0^1 (1-q)\, d\mu_t(q) = \overline{\mu}_t(-1) = 1 - \overline{\mu}_t \to 1 - \overline{\mu} = \overline{\mu}(-1) = \int_0^1 (1-q)\, d\mu(q)$$
$$\int_0^1 q^2 d\mu_t(q) = \overline{\overline{\mu}}_t \to \overline{\overline{\mu}} = \int_0^1 q^2 d\mu(q)$$
$$\int_0^1 (1-q)^2\, d\mu_t(q) = 1 - 2\overline{\mu}_t + \overline{\overline{\mu}}_t \to 1 - 2\overline{\mu} + \overline{\overline{\mu}} = \int_0^1 (1-q)^2\, d\mu(q)\,.$$

Thus $\{h_t, \overline{\mu}_t\}$ is a *risky monetary net* (MMR p. 16) and $\{h_t, \mu_t\}$ is an *uncertain monetary net* (MMR p. 21). Since $\lim_{t\to 0}\|h_t\|^2_{\overline{\mu}_t} = \lim_{t\to 0} t = 0$, then $\{h_t, \overline{\mu}_t\}$ is a *small risk* (MMR p. 16) and $\{h_t, \mu_t\}$ is a *small uncertainty* (MMR p. 21). Moreover, since

$$\max_{\omega\in \operatorname{supp}\overline{\mu}_t} |h_t(\omega)| = \sqrt{t} = \|h_t\|_{\overline{\mu}_t} = O\left(\|h_t\|_{\overline{\mu}_t}\right)$$

as $t \downarrow 0$, $\{h_t, \overline{\mu}_t\}$ is a *small controllable risk* (MMR p. 18) and $\{h_t, \mu_t\}$ is a *small controllable uncertainty* (MMR p. 22). Also, since

$$\overline{\mu}_t \to \overline{\mu} = \sum_{a\in A} \pi_{z_0}(a)\,\frac{1}{2}(1 + z_0 a) = \frac{1}{2} + \frac{1}{2} z_0 \sum_{a\in A} a\pi_{z_0}(a) \in (0, 1)$$

as $t \downarrow 0$, the condition $\overline{\mu}_t \in (0, 1)$ of MMR p. 25 is satisfied for t small enough.

Next, note that

$$
\begin{aligned}
&\mathrm{C}_t\left(w+W_t\,|z_0\right)\\
&=v^{-1}\left(\sum_{a,b}\pi_{t|z_0}\left(a,b\right)v\left(u^{-1}\left(p_{a,b}\left(t|z_0\right)u\left(w+\sqrt{t}\right)+\left(1-p_{a,b}\left(t|z_0\right)\right)u\left(w-\sqrt{t}\right)\right)\right)\right)\\
&=v^{-1}\left(\sum_{a,b}\pi_{t|z_0}\left(a,b\right)\int_0^1 v\left(u^{-1}\left(qu\left(w+\sqrt{t}\right)+\left(1-q\right)u\left(w-\sqrt{t}\right)\right)\right)d\delta_{p_{a,b}(t|z_0)}\left(q\right)\right)\\
&=v^{-1}\left(\int_0^1 v\left(u^{-1}\left(qu\left(w+\sqrt{t}\right)+\left(1-q\right)u\left(w-\sqrt{t}\right)\right)\right)d\left(\sum_{a,b}\pi_{t|z_0}\left(a,b\right)\delta_{p_{a,b}(t|z_0)}\left(q\right)\right)\right)\\
&=v^{-1}\left(\int_0^1 v\left(u^{-1}\left(qu\left(w+\sqrt{t}\right)+\left(1-q\right)u\left(w-\sqrt{t}\right)\right)\right)d\mu_{t|z_0}\left(q\right)\right)\\
&=v^{-1}\left(\int_0^1 v\left(u^{-1}\left(qu\left(w+h_t\left(1\right)\right)+\left(1-q\right)u\left(w+h_t\left(-1\right)\right)\right)\right)d\mu_t\left(q\right)\right).
\end{aligned}
$$

The certainty equivalent $\mathrm{C}_t\left(w+W_t\,|z_0\right)$ coincides with the certainty equivalent considered in MMR p. 6 and, since

$$
\begin{aligned}
\mathrm{E}_t\left(w+W_t\,|z_0\right)&=\sum_{a,b}\pi_{t|z_0}\left(a,b\right)\left(p_{a,b}\left(t|z_0\right)\left(w+\sqrt{t}\right)+\left(1-p_{a,b}\left(t|z_0\right)\right)\left(w-\sqrt{t}\right)\right)\\
&=\int_0^1 q\left(w+h_t\left(1\right)\right)+\left(1-q\right)\left(w+h_t\left(-1\right)\right)d\mu_t\left(q\right),
\end{aligned}
$$

the uncertainty premium $\mathrm{UP}_t\left(w+W_t\,|z_0\right)$ also coincides with the one considered in MMR p. 6.

By (4.11) and by Proposition 48 of MMR, as $t\downarrow 0$

$$
\begin{aligned}
\mathrm{UP}_t\left(w+W_t\,|z_0\right)&=\frac{1}{2}\lambda_u\left(w\right)\left(\overline{\mu}_{z_0}-\overline{\overline{\mu}}_{z_0}\right)\left(2\sqrt{t}\right)^2+\frac{1}{2}\lambda_v\left(w\right)\left(\overline{\overline{\mu}}_{z_0}-\overline{\mu}_{z_0}^2\right)\left(2\sqrt{t}\right)^2+o\left(\left\|h_t\right\|_{\overline{\mu}_t}^2\right)\\
&=2\lambda_u\left(w\right)\left(\overline{\mu}_{z_0}-\overline{\overline{\mu}}_{z_0}\right)t+2\lambda_v\left(w\right)\left(\overline{\overline{\mu}}_{z_0}-\overline{\mu}_{z_0}^2\right)t+o\left(t\right).
\end{aligned}
$$

From the proof of Lemma 56, $\sum_{a\in A}\pi_{z_0}\left(a\right)\left(\frac{1}{2}\left(1+z_0a\right)\right)^n=\int_0^1 q^n d\mu_{z_0}$. Hence

$$
\overline{\mu}_{z_0}=\sum_{a\in A}\pi_{z_0}\left(a\right)\left(\frac{1}{2}\left(1+z_0a\right)\right)\quad\text{and}\quad\overline{\overline{\mu}}_{z_0}=\sum_{a\in A}\pi_{z_0}\left(a\right)\left(\frac{1}{2}\left(1+z_0a\right)\right)^2,
$$

so that

$$\begin{aligned}\overline{\mu}_{z_0} - \overline{\overline{\mu}}_{z_0} &= \sum_{a\in A} \pi_{z_0}(a)\left(\frac{1}{2}(1+z_0 a)\right) - \sum_{a\in A} \pi_{z_0}(a)\left(\frac{1}{2}(1+z_0 a)\right)^2 \\ &= \sum_{a\in A} \pi_{z_0}(a)\left(\frac{1}{2}+\frac{1}{2}z_0 a - \left(\frac{1}{4}+\frac{1}{4}a^2+\frac{1}{2}z_0 a\right)\right) \\ &= \sum_{a\in A} \pi_{z_0}(a)\left(\frac{1}{4}-\frac{1}{4}a^2\right) = \frac{1}{4} - \frac{1}{4}\sum_{a\in A} a^2 \pi_{z_0}(a) = \frac{1}{4}\left(1-\overline{\overline{\pi}}_{z_0}\right)\end{aligned}$$

and

$$\begin{aligned}\overline{\overline{\mu}}_{z_0} - \overline{\mu}_{z_0}^2 &= \sum_{a\in A} \pi_{z_0}(a)\left(\frac{1}{2}(1+z_0 a)\right)^2 - \left(\sum_{a\in A} \pi_{z_0}(a)\left(\frac{1}{2}(1+z_0 a)\right)\right)^2 \\ &= \sum_{a\in A} \pi_{z_0}(a)\left(\frac{1}{4}+\frac{1}{4}a^2+\frac{1}{2}z_0 a\right) - \left(\frac{1}{2}+\frac{1}{2}z_0\overline{\pi}_{z_0}\right)^2 \\ &= \frac{1}{4}+\frac{1}{4}\overline{\overline{\pi}}_{z_0}+\frac{1}{2}z_0\overline{\pi}_{z_0} - \left(\frac{1}{4}+\frac{1}{4}\overline{\pi}_{z_0}^2+\frac{1}{2}z_0\overline{\pi}_{z_0}\right) = \frac{1}{4}\overline{\overline{\pi}}_{z_0} - \frac{1}{4}\overline{\pi}_{z_0}^2 = \frac{1}{4}\sigma_{\pi_{z_0}}^2.\end{aligned}$$

Finally

$$\begin{aligned}\mathrm{UP}_t\left(w+W_t\,|z_0\right) &= 2\lambda_u(w)\left(\frac{1}{4}\left(1-\overline{\overline{\pi}}_{z_0}\right)\right)t + 2\lambda_v(w)\left(\frac{1}{4}\sigma_{\pi_{z_0}}^2\right)t + o(t) \\ &= \frac{t}{2}\lambda_u(w)\left(1-\overline{\overline{\pi}}_{z_0}\right) + \frac{t}{2}\lambda_v(w)\,\sigma_{\pi_{z_0}}^2 + o(t),\end{aligned}$$

which together with $t=\varepsilon$ concludes the proof. ■

4.5.3 Heuristic proof of approximations (4.7) and (4.8)

We consider the general case $W_\varepsilon = \sqrt{\varepsilon}h(a)\,Z_\varepsilon$, where $h: A \to (0,+\infty)$. Consequently

$$\begin{aligned}&\mathrm{C}_\varepsilon\left(w+W_\varepsilon\,|z_0\right) \\ &= v^{-1}\left(\sum_{a,b} \pi_{\varepsilon|z_0}(a,b)\, v\left(u^{-1}\left(p_{a,b}(\varepsilon|z_0)\, u\left(w+h(a)\sqrt{\varepsilon}\right) + \left(1-p_{a,b}(\varepsilon|z_0)\right) u\left(w-h(a)\sqrt{\varepsilon}\right)\right)\right)\right).\end{aligned}$$

Recalling the convergence as $\varepsilon \downarrow 0$ of $\pi_{\varepsilon|z_0}(a)$ to $\pi_{z_0}(a)$ and of $p_{a,b}(\varepsilon|z_0)$ to $p_a(z_0)$ for all $a \in A$, we have

$$\begin{aligned}
&\mathrm{C}_\varepsilon\left(w + W_\varepsilon\,|z_0\,\right)\\
&\approx v^{-1}\left(\sum_a \pi_{z_0}(a)\, v\left(u^{-1}\left(p_a(z_0)\, u\left(w + h(a)\sqrt{\varepsilon}\right) + (1 - p_a(z_0))\, u\left(w - h(a)\sqrt{\varepsilon}\right)\right)\right)\right).
\end{aligned}$$

By the standard Arrow-Pratt approximation

$$\begin{aligned}
&u^{-1}\left(p_a(z_0)\, u\left(w + h(a)\sqrt{\varepsilon}\right) + (1 - p_a(z_0))\, u\left(w - h(a)\sqrt{\varepsilon}\right)\right)\\
&= u^{-1}\left(E_a u\left(w + \sqrt{\varepsilon} h(a)\, Z\right)\right) \approx w + \sqrt{\varepsilon} h(a)\, E_a[Z] - \frac{\lambda_u}{2}\varepsilon h^2(a)\, \sigma_a^2[Z],
\end{aligned}$$

where $Z(\omega) = \omega$ is the identity on $\Omega = \{1, -1\}$. Then

$$\mathrm{C}_\varepsilon\left(w + W_\varepsilon\,|z_0\,\right) \approx v^{-1}\left(\sum_a \pi_{z_0}(a)\, v\left(w + \sqrt{\varepsilon} h(a)\, E_a[Z] - \frac{\lambda_u}{2}\varepsilon h^2(a)\, \sigma_a^2[Z]\right)\right)$$

Another application of the Arrow-Pratt approximation and the elimination of infinitesimals of higher order, delivers

$$\mathrm{C}_\varepsilon\left(w + W_\varepsilon\,|z_0\,\right) \approx w + \sqrt{\varepsilon} E_{\pi_{z_0}}(hE[Z]) - \frac{\lambda_u}{2}\varepsilon E_{\pi_{z_0}}\left(h^2\sigma^2[Z]\right) - \frac{\lambda_v}{2}\varepsilon \sigma^2_{\pi_{z_0}}(hE[Z]).$$

Above, $E[Z] : A \to \mathbb{R}$ and $\sigma^2[Z] : A \to \mathbb{R}$ are defined by $E[Z](a) = E_a[Z]$ and $\sigma^2[Z](a) = \sigma_a^2[Z]$ for all $a \in A$. In particular, $E_a[Z] = \frac{1}{2}(1 + z_0 a) - \left(1 - \frac{1}{2}(1 + z_0 a)\right) = z_0 a$ and we can explicitly compute the coefficients

$$\begin{aligned}
E_{\pi_{z_0}}(hE[Z]) &= \sum_{a\in A} \pi_{z_0}(a)\, h(a)\, z_0 a = z_0 \sum_{a\in A} \pi_{z_0}(a)\, h(a)\, a = z_0 E_{\pi_{z_0}}(h(a)\, a)\\
E_{\pi_{z_0}}\left(h^2\sigma^2[Z]\right) &= \sum_{a\in A} \pi_{z_0}(a)\, h^2(a)\left(1 - (z_0 a)^2\right) = \sum_{a\in A} \pi_{z_0}(a)\, h^2(a)\left(1 - a^2\right)\\
&= E_{\pi_{z_0}}\left(h(a)^2\left(1 - a^2\right)\right)\\
\sigma^2_{\pi_{z_0}}(hE[Z]) &= E_{\pi_{z_0}}\left(h^2 E^2[Z]\right) - E^2_{\pi_{z_0}}(hE[Z])\\
&= \sum_{a\in A} \pi_{z_0}(a)\, h^2(a)\,(z_0 a)^2 - \left(z_0 \sum_{a\in A} \pi_{z_0}(a)\, h(a)\, a\right)^2\\
&= \sum_{a\in A} \pi_{z_0}(a)\, h^2(a)\, a^2 - \left(\sum_{a\in A} \pi_{z_0}(a)\, h(a)\, a\right)^2 = \sigma^2_{\pi_{z_0}}(h(a)\, a).
\end{aligned}$$

Hence

$$\mathrm{C}_{\varepsilon}\left(w+W_{\varepsilon}\left|z_0\right.\right)\approx w+\sqrt{\varepsilon}z_0E_{\pi_{z_0}}\left(h\left(a\right)a\right)-\frac{\lambda_u}{2}\varepsilon E_{\pi_{z_0}}\left(h\left(a\right)^2\left(1-a^2\right)\right)-\frac{\lambda_v}{2}\varepsilon\sigma^2_{\pi_{z_0}}\left(h\left(a\right)a\right),$$

which establishes approximation (4.8).

If $h\left(a\right)$ is constant and equal to 1, we have

$$\begin{aligned}
E_{\pi_{z_0}}\left(hE\left[Z\right]\right) &= E_{\pi_{z_0}}\left(E\left[Z\right]\right)=\sum_{a\in A}\pi_{z_0}\left(a\right)z_0a=z_0\overline{\pi}_{z_0}\\
E_{\pi_{z_0}}\left(h^2\sigma^2\left[Z\right]\right) &= E_{\pi_{z_0}}\left(\sigma^2\left[Z\right]\right)=\sum_{a\in A}\pi_{z_0}\left(a\right)\left(1-a^2\right)=1-\sum_{a\in A}\pi_{z_0}\left(a\right)a^2=1-\overline{\overline{\pi}}_{z_0}\\
\sigma^2_{\pi_{z_0}}\left(hE\left[Z\right]\right) &= \sigma^2_{\pi_{z_0}}\left(E\left[Z\right]\right)=E_{\pi_{z_0}}\left(E^2\left[Z\right]\right)-E^2_{\pi_{z_0}}\left(E\left[Z\right]\right)\\
&= \sum_{a\in A}\pi_{z_0}\left(a\right)a^2-\left(z_0\sum_{a\in A}\pi_{z_0}\left(a\right)a\right)^2=\overline{\overline{\pi}}_{z_0}-\left(\sum_{a\in A}\pi_{z_0}\left(a\right)a\right)^2\\
&= \overline{\overline{\pi}}_{z_0}-\overline{\pi}^2_{z_0}=\sigma^2_{\pi_{z_0}}.
\end{aligned}$$

Approximation (4.7) follows from

$$\mathrm{C}_{\varepsilon}\left(w+W_{\varepsilon}\left|z_0\right.\right)\approx w+\sqrt{\varepsilon}z_0\overline{\pi}_{z_0}-\frac{\lambda_u}{2}\varepsilon\left(1-\overline{\overline{\pi}}_{z_0}\right)-\frac{\lambda_v}{2}\varepsilon\sigma^2_{\pi_{z_0}}.$$

Bibliography

[1] V. Acharya and T. Yorulmazer, Too many to fail-An analysis of time-inconsistency in bank closure policies, *Journal of financial intermediation*, 16, 1, 1-31, 2007.

[2] V. Agarwal, V. Fos, and W. Jiang, Inferring reporting-related biases in hedge fund databases from hedge fund equity holdings, *Management Science,* 59,.6, 1271-1289, 2013.

[3] Y. Amihud and R. Goyenko, Mutual fund's R2 as predictor of performance. *Review of Financial Studies*, 26, 3, 667-694, 2013.

[4] E. W. Anderson, L. P. Hansen, and T. J. Sargent, A quartet of semigroups for model specification, robustness, prices of risk, and model detection, *Journal of the European Economic Association,* 1, 68-123, 2003.

[5] F. Anscombe and R. Aumann, A definition of subjective probability, *The Annals of Mathematical Statistics*, 34, 1, 199-205, 1963.

[6] R. Aumann, Utility theory without the completeness axiom, *Econometrica*, 30,445–462, 1962.

[7] L. Barras, O. Scaillet and R. Wermers, False discoveries in mutual fund performance: Measuring luck in estimated alphas, *The Journal of Finance*, 65, 1, 179-216, 2010.

[8] T. Bewley, Knightian decision theory. Part I, *Decisions in Economics and Finance*, 25, 2, 79-110, 2002.

[9] P. Billingsley, *Convergence of probability measures,* Wiley, New York, 1968.

[10] S. J. Brown and W. N. Goetzmann, Performance persistence, *The Journal of Finance*, 50, 2, 679-698, 1995.

[11] S. J. Brown, W. N. Goetzmann, R. G. Ibbotson and S. A. Ross, Survivorship bias in performance studies, *Review of Financial Studies*, 5, 4, 553-580, 1992.

[12] S. J. Brown, W. N. Goetzmann and S. A. Ross, Survival, *The Journal of Finance*, 50, 3, 853-873, 1995.

[13] Q. Bu and L. Nelson, Exposing survivorship bias in mutual fund data, *The Journal of Business and Economic Studies*, 13, 1, 22, 2007.

[14] M. M. Carhart, On persistence in mutual fund performance, *The Journal of finance*, 52, 1, 57-82, 1997.

[15] M. M. Carhart, J. N. Carpenter, A. W. Lynch and D. K. Musto, Mutual fund survivorship, *Review of Financial Studies*, 15, 5, 1439-1463, 2002.

[16] J. N. Carpenter and A. W. Lynch, Survivorship bias and attrition effects in measures of performance persistence, *Journal of Financial Economics*, 54, 3, 337-374, 1999.

[17] S. Cerreia-Vioglio, Objective Rationality and Uncertainty Averse Preferences, *mimeo*, 2012.

[18] S. Cerreia-Vioglio, P. Ghirardato, F. Maccheroni, M. Marinacci and M. Siniscalchi, Rational preferences under ambiguity, *Economic Theory*, 48, 2-3, 341-375, 2011.

[19] S. Cerreia-Vioglio, F. Maccheroni, M. Marinacci and L. Montrucchio, Ambiguity and robust statistics, *Journal of Economic Theory*, 148, 3, 974-1049, 2013b.

[20] S. Cerreia-Vioglio, F. Maccheroni, M. Marinacci and L. Montrucchio, Classical subjective expected utility, *Proceedings of the National Academy of Sciences*, 110,17, 6754-6759, 2013a.

[21] S. Cerreia-Vioglio, F. Maccheroni, M. Marinacci and A. Rustichini, Niveloids and Their Extensions: Risk Measures on Small Domains, *Igier Working Paper, 458,* 2012.

[22] V. Chari and P. Kehoe, Bailouts, time inconsistency, and optimal regulation, *Working Paper*, National Bureau of Economic Research, 2013.

[23] P. Cogneau and G. Hübner, The prediction of fund failure through performance diagnostics, *Journal of Banking & Finance*, 50, 224-241, 2015.

[24] D.R. Cox, Regression models and life-tables with discussion, *Journal of the Royal Statistical Society B*, 34, 187–220, 1972.

[25] H. Cres, I. Gilboa and N. Vieille, Aggregation of multiple prior opinions, *Journal of Economic Theory*, 146, 6, 2563-2582, 2011.

[26] CRSP, Survivor Bias Free U.S. Mutual Fund Guide, Booth School of Business, The University of Chicago, 2016.

[27] A. Cukierman and Y. Izhakian, Bailout uncertainty in a microfounded general equilibrium model of the financial system, *Journal of Banking & Finance*, 52, 160-179, 2015.

[28] B. De Meyer and P. Mongin, A note on affine aggregation, *Economics Letters*, 47, 2, 177–183, 1995.

[29] S. Dolecki and G. H. Greco, Niveloids, *Topological Methods in Nonlinear Analysis*, 5, 1–22, 1995.

[30] J. Dubra, F. Maccheroni and E. Ok, Expected utility theory without the completeness axiom, Journal of Economic Theory, 115, 1, 118-133, 2004.

[31] D. Ellsberg, Risk, ambiguity, and the Savage axioms, *The Quarterly Journal of Economics*, 643-669, 1961.

[32] E. J. Elton, M. J. Gruber and C. R. Blake, Survivor bias and mutual fund performance, *Review of Financial Studies*, 9, 4, 1097-1120, 1996.

[33] E. J. Elton, M. J. Gruber and C. R. Blake, The persistence of risk-adjusted mutual fund performance, *The Journal of Business*,.69, 2, 133-157, 1996.

[34] E. J. Elton, M. J. Gruber and C. R. Blake, A first look at the accuracy of the CRSP mutual fund database and a comparison of the CRSP and Morningstar mutual fund databases, *The Journal of Finance*, 56, 6, 2415-2430, 2001.

[35] R. B. Evans, Mutual fund incubation, *The Journal of Finance*, 65, 4, 1581-1611, 2010.

[36] E. F. Fama and K. R. French, Common risk factors in the returns on stocks and bonds, *Journal of Financial Economics*, 33, 1, 3-56, 1993.

[37] E. F. Fama and K. R. French, Luck versus skill in the cross-section of mutual fund returns, *The Journal of Finance*, 65, 5, 1915-1947, 2010.

[38] E. F. Fama and K. R. French, A five-factor asset pricing model, *Journal of Financial Economics*, 116, 1, 1-22, 2015.

[39] E. Farhi and J. Tirole, Collective moral hazard, maturity mismatch and systemic bailouts, *American Economic Review*. 102, 1, 60-93, 2012.

[40] M. A. Ferreira, A. Keswani, A. F. Miguel and S. B. Ramos, The determinants of mutual fund performance: A cross-country study, *Review of Finance*, 17, 483-525, 2013.

[41] W. E. Ferson and R. W. Schadt, Measuring fund strategy and performance in changing economic conditions, *The Journal of Finance*, 51, 2, 425-461, 1996.

[42] P. Fishburn, Utility theory for decision making, *New York: Wiley*, 1970.

[43] X. Freixas, Optimal Bail-Out, Conditionality and Constructive Ambiguity, *CEPR Discussion Paper No. 2238*, 1999.

[44] P. Ghirardato, F. Maccheroni and M. Marinacci, Differentiating ambiguity and ambiguity attitude, *Journal of Economic Theory*, 118, 2, 133-173, 2004.

[45] I. Gilboa, F. Maccheroni, M. Marinacci and D. Schmeidler, Objective and subjective rationality in a multiple prior model, *Econometrica*, 78, 2, 755-770, 2010.

[46] I. Gilboa and D. Schmeidler, Maxmin expected utility with non-unique prior, *Journal of Mathematical Economics*, 18, 2, 141-153, 1989.

[47] M. Grinblatt and S. Titman, Mutual fund performance: An analysis of quarterly portfolio holdings, *Journal of Business*, 62, 3, 393-416, 1989.

[48] U. Gruber and M. Schweizer, A diffusion limit for generalized correlated random walks, *Journal of Applied Probability*, 43, 60-73, 2006.

[49] J. C. Harsanyi, Cardinal welfare, individual ethics, and interpersonal comparisons of utility, *The Journal of Political Economy*, 309-321, 1955.

[50] J. J. Heckman, The common structure of statistical models of truncation, sample selection and limited dependent variables and a simple estimator for such models, *Annals of Economic and Social Measurement*, 5, 4, 475-492, 1976.

[51] J. J. Heckman, Sample selection as a specification error, *Econometrica*, 47, 1, 153-161, 1979.

[52] D. Hendricks, J. Patel and R. Zeckhauser, Hot hands in mutual funds: Short-run persistence of relative performance, 1974–1988, *The Journal of Finance*, 48, 1, 93-130, 1993.

[53] J. Hodges and E. Lehmann, The use of previous experience in reaching statistical decisions, *The Annals of Mathematical Statistics*, 23, 3, 396-407,1952.

[54] B. Holmström and J. Tirole, Private and Public Supply of Liquidity, *Journal of Political Economy*,106, 1-40,1998.

[55] B. Holmström and J. Tirole, Inside and outside liquidity, *MIT press*, 2011.

[56] M. C. Jensen, The performance of mutual funds in the period 1945–1964, *The Journal of Finance*, 23, 2, 389-416, 1968

[57] T. Keister, Bailouts and financial fragility, *The Review of Economic Studies*, 83, 2, 704-736, 2016.

[58] J. F. C. Kingman, *Regenerative phenomena*, Wiley, New York, 1972.

[59] P. Klibanoff, M. Marinacci, and S. Mukerji, A smooth model of decision making under ambiguity, *Econometrica*, 73, 1849-1892, 2005.

[60] R. Kosowski, A. Timmermann, R. Wermers and H. White, Can mutual fund "stars" really pick stocks? New evidence from a bootstrap analysis, *The Journal of Finance*, 61, 6, 2551-2595, 2006.

[61] H. Li and Y. Xu, Survival bias and the equity premium puzzle, *The Journal of Finance*, 57, 5, 1981-1995, 2002.

[62] J. T. Linnainmaa, Reverse survivorship bias, *The Journal of Finance*, 68, 3, 789-813, 2013.

[63] J. Lintner, The valuation of risk assets and the selection of risky investments in stock portfolios and capital budgets, *The Review of Economics and Statistics*, 47, 13-37, 1965.

[64] A. Lunde, A. Timmermann and D. Blake, The hazards of mutual fund underperformance: A Cox regression analysis, *Journal of Empirical Finance*, 6, 2, 121-152, 1999.

[65] F. Maccheroni, M. Marinacci and D. Ruffino, Does uncertainty vanish in the small? The smooth ambiguity case, IGIER WP 391, 2011.

[66] F. Maccheroni, M. Marinacci, and D. Ruffino, Alpha as ambiguity: robust mean-variance portfolio analysis, *Econometrica,* 81, 1075-1113, 2013.

[67] F. Maccheroni, M. Marinacci and A. Rustichini, Ambiguity aversion, robustness, and the variational representation of preferences, *Econometrica*, 74, 6, 1447-1498, 2006.

[68] B. G. Malkiel, Returns from investing in equity mutual funds 1971 to 1991, *The Journal of Finance*, 50, 2, 549-572, 1995

[69] L. Nascimento, The ex ante aggregation of opinions under uncertainty, *Theoretical Economics*, 7, 3, 535–570, 2012.

[70] T. Nohel, Z. J. Wang and L. Zheng, Side-by-side management of hedge funds and mutual funds, *Review of Financial Studies*, 23, 6, 2342-2373, 2010.

[71] J. Nosal and G. Ordoñez, Uncertainty as commitment, *Journal of Monetary Economics*, 80, 124-140, 2016.

[72] G. Ordoñez, Sustainable Shadow Banking, *Working Paper*, 2015.

[73] R. Otten and D. Bams, How to measure mutual fund performance: economic versus statistical relevance, *Accounting & Finance*, 44, 2, 203-222, 2004.

[74] E. Pastén, Bailouts and Prudential Policies: A Delicate Interaction, *Working Paper*, Central Bank of Chile, 2015.

[75] R. Rockafellar, Convex Analysis, *Princeton University Press*, 1970.

[76] M. Rohleder, H. Scholz and M. Wilkens, Survivorship bias and mutual fund performance: Relevance, significance, and methodical differences, *Review of Finance*, 15, 441-474, 2011.

[77] D. Schmeidler, Subjective probability and expected utility without additivity, *Econometrica*, 57, 3, 571–587, 1989.

[78] K. Seo, Ambiguity and Second-Order Belief, *Econometrica*, 77, 5, 1575–1605, 2009.

[79] W. F. Sharpe, Capital asset prices: A theory of market equilibrium under conditions of risk, *The Journal of Finance*, 19, 3, 425-442, 1964.

[80] L.S. Shapley and M. Baucells, Multiperson Utility, *Working Paper*, N. 779, Department of Economics, UCLA, 1998.

[81] E. R. Sirri and P. Tufano, Costly search and mutual fund flows, *The Journal of Finance*, 53, 5, 1589-1622, 1998.

[82] C. Skiadas, Smooth ambiguity aversion toward small risks and continuous-time recursive utility, *Journal of Political Economy*, 121, 775-792, 2013.

[83] G. Stern and R. Feldman, Too Big to Fail: The Hazards of Bank Bailouts, *The Brooking Institution*, 2004.

[84] J. R. Ter Horst, T. E. Nijman and M. Verbeek, Eliminating look-ahead bias in evaluating persistence in mutual fund performance, *Journal of Empirical Finance*, 8, 4, 345-373, 2001.

[85] R. Wermers, Momentum investment strategies of mutual funds, performance persistence, and survivorship bias, *University of Colorado Working Paper*, 1997.