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Contents

List of Figures

List of Tables

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Abstract

The recent availability of high-frequency financial data has made the measurement of realized financial risks possible with a good degree of statistical accuracy. While the literature has largely focused on measures of second-order realized risks captured by proxies of quadratic variation, this thesis studies with a unifying approach the statistical properties of proxies of second and higher-order realized risks generated by linear combinations of (power) realized Bregman divergences. In two separate papers, we systematically study the asymptotic properties of financially relevant definitions of scale-invariant and scale-dependent realized divergences, which are tradeable in liquid option markets using dynamic and static option portfolios, respectively.

We obtain laws of large numbers and functional central limit theorems for scale-invariant and scale-dependent realized divergence under general semimartingale conditions. The laws of large numbers provide a proper definition of the hidden realized risk estimated by second and higher-order realized divergences, together with an identification of the risk contributions of continuous and discontinuous semimartingale components. The central limit theorems provide a description of the asymptotic distribution of estimated second- and higher order realized divergence, and its dependence on continuous and discontinuous semimartingale components. We produce feasible asymptotic confidence intervals for second- and higher-order realized divergence based on different approaches. In the scale-independent case, we obtain feasible Gaussian confidence intervals in absence of co-jumps in returns and volatility, and feasible bootstrap confidence intervals in presence of co-jumps. In the scale-dependent case, we obtain feasible bootstrap confidence intervals for the underlying non-Gaussian asymptotic distributions.

Reliable confidence intervals are useful to quantify the estimation risk hidden in second- and higher-order realized divergence, which is relevant for financial applications in which the forecasting and/or the trading of second- and higher-order risks is an issue, e.g., in risk management or portfolio choice applications. They are also important for quantifying the state-dependent noisiness of realized risks, e.g., in contexts where cross-sectional differences in asset risk premia are explained by different cross-sectional exposures to second- and higher-order realized risks. We quantify the accuracy of our feasible approximations of the asymptotic distributions of realized divergence by Monte Carlo simulation, based on a general two-factor stochastic volatility specification allowing for common jumps in returns and volatility. Overall, we find that estimation uncertainty on second- and especially higher-order realized divergence may be successfully incorporated in settings where sampling at frequencies of less than one minute is feasible.

Chapter 1

Introductory remarks

The recent literature on high-frequency measures of price and return variation has had a great impact on the understanding and the measurement of time-varying financial risks. It allowed a more precise study of how risks borne by investors change over time and also gave rise to a new literature, which studies how time-varying realized risks can be traded and which is the risk premium for an exposure to such risks.

The literature initially focused on measures of realized second-order risk, such as realized variance. One early application of high-frequency variance measurement was in Andersen and Bollerslev [1998], who evaluated the quality of GARCH-based variance forecasts. Consistently with the theory of stochastic processes with continuous trajectories (e.g., Karatzas and Shreve [1998]), they observed that recording asset prices at a sufficiently high frequency allows to recover the price or return quadratic variation very accurately. Recognizing the limitations of the continuous-trajectory semimartingale assumption for many relevant applications in finance, the literature later studied asset price processes driven by a general semimartingale, with the aim to estimate the semimartingale characteristics, or functions thereof, from discretely-observed paths. Two key quantities of interest in this context were the process quadratic variation (estimated, e.g., by realized variance) and the integrated variation (estimated, e.g., by multipower variation), where the former can be interpreted as a proxy of total second-order riskiness and the latter as a proxy of the scale of hedgeable second-order risk. A comprehensive treatment of the statistical inference on realized variance can be found, e.g., in Jacod [2007] for Levy processes and in Jacod [2008] for general semimartingales. Inference on multipower variation is treated in Woerner [2006], among others, while Veraart [2010] studies in more detail the quadratic variation component due to jumps and corresponding inference methods. This literature also addressed methods for overcoming the difficulties generated by the microstructure noise inherent in discretely-observed asset prices and the estimation of the spot volatility process, which is necessary for obtaining a feasible statistical inference; see Jacod et al. [2009], Podolskij and Vetter [2009] and Mancini et al. [2015], among others.

The above developments in the econometric literature on realized variance have found wide application in different areas of finance, e.g., in the context of studies on the time-varying risk return tradeoff and in cross-sectional analyses investigating the role of time-varying second-order risk as a priced risk factor for the cross-section of individual stock excess returns. A key issue in this context is the possibility to trade a given realized risk proxy with corresponding financial instruments that create the desired risk exposure. Neuberger [1994] showed that under the assumption of continuous price trajectories the return quadratic variation can be exactly replicated by the payoff of a delta-hedged log contract on the underlying asset price at a given future maturity. The static option replication theory in Carr and Madan [2001] allows a direct replication of such payoffs by means of delta-hedged static option portfolios, which effectively renders the realized return quadratic variation tradeable in liquid option markets under the assumption of continuous price trajectories. These findings were essential for the development of different types of over-the-counter variance swap markets, in which investors can trade different measures of realized second-order risk by means of particular swap contracts on realized variance. Related research on the model-free replication of realized variance measures also contributed to the redefinition of the CBOE [2000] VIX volatility index and to the introduction of VIX futures; see, e.g., Jiang and Tian [2005], Britten-Jones and Neuberger [2000] and Speth et al. [2004], among others.

While the above research largely focuses on realized measures of second-order risk, Schneider and Trojani [2015a] introduce a general model-free approach based on realized Bregman [1967] divergences, which allows to measure and trade in a unified setting second- and higher-order realized risks, based on corresponding divergence swap contracts and option replicating portfolios. The floating legs of suitable portfolios of divergence swaps correspond exactly to a broad family of realized variation measures, which capture to the leading order the (scaled or unscaled) second-order, third-order and fourth-order power variation of log returns. These measures of price and return variation are therefore natural tools to measure at essentially arbitrary frequency the realized risks identified by quadratic variation, realized jump skewness and realized jump kurtosis. Schneider and Trojani [2015b] study the empirical properties of realized price divergence, which is tradeable by means of static option portfolios. Orlowski et al. [2016] address by means of appropriate dynamic option portfolios the replication of realized return divergence, showing that higher-order jump return divergence can be isolated and traded accordingly.

To get a first intuition on the construction of realized divergence measures, let F_t be the forward price of an asset with time to maturity $T - t$ and Φ be a convex function. The Bregman divergence induced by function Φ between the forward prices F_t and F_s at times $s < t$ is defined by

$$D_\Phi (F_t, F_s) := \Phi(F_t) - \Phi(F_s) - \Phi'(F_s) (F_t - F_s) \ ,$$

where Φ' is the derivative of Φ. For a given time grid $0 = t_0 < t_1 \cdots < t_n = T$ and weight function W, the realized divergence of process F over this time grid is defined as:

$$D^n_\Phi(F) := \sum_{i=1}^{n} W(F_{i-1}) D_\Phi(F_i, F_{i-1}) \ .$$

By construction, realized divergence measures the realized dispersion of process F in a way that is compatible with second-order stochastic dominance. Realized divergence is tradeable with suitable option portfolios, which gives rise to different forward contracts on realized divergence, i.e., divergence swaps. The floating leg of a divergence swap is the realized divergence $D^n_\Phi(F)$ at maturity T. In parallel, the buyer of the swap contract pays at maturity T the forward price of $D^n_\Phi(F)$, denoted by $S(D^n_\Phi(F))$. Therefore, the divergence swap payoff is simply the difference of the swap floating and fixed legs: $D^n_\Phi(F) - S(D^n_\Phi(F))$.

To get an intuition for the way how divergence swaps are traded, we introduce the family $\{O_k(K) : K \geq 0\}$ of payoffs of out-of-the money options at time $0 \leq t_k \leq T$ for maturity T, where we assume all these options to be tradeable at any time $0 \leq t_k \leq T$:

$$O_k(K) := \begin{cases} (K - F_T)^+ & \text{if } K \leq F_{t_k} \\ (F_T - K)^+ & \text{otherwise} \ . \end{cases}$$

Let $Q_k(K)$ denote the forward price of payoff $O_k(K)$, at time t_k and for maturity T. Divergence swap payoffs are then replicable by means of the following dynamic option strategy (Schneider and Trojani [2015a], Result 2.3):

(a) At each time $i = 0, \ldots, n-1$ buy forwards on payoff $W(F_i) \int_0^\infty \Phi''(K) O(K) dK$, with forward price
$$W(F_i) \int_0^\infty \Phi''(K) Q_i(K) dK \ .$$

(b) At each time $i = 1, \ldots, n-1$ buy forwards on payoff $W(F_{i-1}) \int_0^\infty \Phi''(K) O(K) dK$, with forward price
$$W(F_{i-1}) \int_0^\infty \Phi''(K) Q_i(K) dK \ .$$

(c) At each time $i = 1, \ldots, n-1$ sell $W(F_{i-1}) \left(\Phi'(F_{i-1}) - \Phi'(F_i) \right)$ forwards on F_T.

Schneider and Trojani [2015a] introduce a flexible and parsimonious family of power divergence swaps, using power generating functions $\Phi_p(x) = \frac{x^p - 1 - p(x-1)}{p(p-1)}$, where $p \in \mathbb{R}$. For $p \in \mathbb{R} \backslash \{0, 1\}$, the p-power divergence is defined by:

$$D_p(F_t, F_s) := D_{\Phi_p}(F_t, F_s) = \frac{F_t^p - F_s^p - p F_s^{p-1}(F_t - F_s)}{p(p-1)} \ ,$$

while for $p \in \{0,1\}$ the definition follows by continuity:

$$D_0(F_t, F_s) := \lim_{p\to 0} D_p(F_t, F_s) = -\log\frac{F_t}{F_s} + \left(\frac{F_t}{F_s} - 1\right) ,$$
$$D_1(F_t, F_s) := \lim_{p\to 1} D_p(F_t, F_s) = F_t \log\frac{F_t}{F_s} - F_s\left(\frac{F_t}{F_s} - 1\right) .$$

Accordingly, realized power divergence is defined by:

$$D_p^n(F) := \sum_{i=1}^{n} W(F_{i-1}) D_p(F_i, F_{i-1}) .$$

In this thesis, we focus on the weight functions $W(x) := 1/F_0^p$ and $W(x) = 1/x^p$, which induce the so-called power realized price and return divergences, respectively. In contrast to the realized return divergence, realized price divergnece can be traded using static option portfolios. This is a direct consequence of the identity:

$$\sum_{i=1}^{p} D_{\Phi_p}(F_i, F_{i-1}) = D_{\Phi_p}(F_T, F_0) - \sum_{i=1}^{p}(\Phi_p'(F_{i-1}) - \Phi_p'(F_0))(F_i - F_{i-1}) . \tag{1.1}$$

Interesting cases of realized power divergence are related to existing variance swap payoffs in the literature. For instance, for $p = 0$ one obtains a realized Itakura and Saito [1968] divergence,

$$D_0^n(F) = -\ln(F_T/F_0) + \sum_{i=1}^{n}(F_i/F_{i-1} - 1) ,$$

which is studied in Bondarenko [2014] and Neuberger [1994], among others, and gives rise to a swap rate proportional to the CBOE volatility index (VIX)

$$S(D_0^n(F)) = T \cdot VIX^2/2 .$$

Moreover, derivatives of power divergences with respect to the power parameter systematically induce a wide family of tradeable higher-order realized divergences that isolate the leading contribution of tradeable realized higher moments.

In this thesis, we systematically develop the econometrics of realized power divergence and provide the foundations for a unified statistical measurement of second- and higher-order realized divergences. Due to their methodological distinction and the distinct tools needed for their treatment, we study realized return divergence in Chapter 2 and realized price divergence in Chapter 3, respectively. For both cases, we obtain appropriate laws of large numbers and functional central limit theorems under general semimartingale conditions. These laws of large numbers provide a proper definition of the hidden realized risk estimated by second and higher-order realized divergences, together with an identification of the risk contributions generated by continuous and discontinuous semimartingale components. The central limit theorems provide a theoretical description of the asymptotic distribution of estimated second- and higher order realized divergence, and its dependence on continuous and discontinuous semimartingale components. In Chapter 2 and Chapter 3 we also produce feasible asymptotic confidence intervals for several proxies of second- and higher-order realized divergence, using either analytical asymptotic approximations or finite sample bootstrap approximations. We also demonstrate by Monte Carlo simulation that these approximations provide reliable information on the noisiness of point estimates of realized divergence.

The remainder of the thesis proceeds as follows. Chapter 2 and Chapter 3 comprehensively address the econometrics of scale-invariant return divergence and scale-dependent price divergence, respectively, while Chapter 4 concludes and discusses directions for future research.

Chapter 2

The econometrics of realized return divergence

2.1 Introduction

This paper addresses the statistical properties of realized return divergence, by focusing on the decomposition or realized divergence into the distinct contributions of realized risks of arbitrary integer order $m \geq 2$. Realized divergence was introduced in Schneider and Trojani [2015a], starting from a convenient definition of realized Bregman [1967] divergence. They show that a parsimonious power parametrization of realized Bregman [1967] divergence gives rise to a unifying approach for measuring and trading realized risks of essentially arbitrary order. Their approach measures realized risks over time increments of a convex transformation of an underlying time-varying risk, where increments can be weighted by a function of past risk information. The property of the weighting has implications both for the definition and interpretation of the corresponding realized risk and for the type of option strategies that can be used to trade it.[1]

This paper focuses on the statistical properties of power realized divergence of returns, which is obtained by scaling the power increments of a price process by the power of its initial value, in order to induce a conditionally scale-independent realized divergence. Orlowski et al. [2016] study in detail the option replicating portfolios of realized return divergence of order two, three and four, showing that the higher-order realized divergence of return jumps can be isolated and traded accordingly using suitable high-frequency replicating portfolios. Their approach is based on proxies of realized divergence computed on a discrete time grid, which theoretically are more conveniently interpreted as point estimates of an underlying realized risk, evolving on a continuos time index under weak semimartingale assumptions. Such a theoretical semimartingale framework is convenient

[1]In particular, while a constant weighting implied a static option replicating portfolio, a stochastic weighting induces a dynamic option replication strategy.

from a financial perspective, because a semimartingale assumption on the price processes of financial assets can be invoked under different no-arbitrage conditions in the literature. This framework also opens the study of the estimation uncertainty embedded in conceptually distinct measures of realized return divergence, which is intrinsically state-dependent and thus potentially systematically related to the level of the realized risk itself.[2] Such a state dependence typically implies a highly time-varying reliability of point estimates of realized risks, which is itself partly persistent and usually lower in states where realized risks are large. It is therefore important to quantify the state-dependent reliability of measured realized risks before using them as inputs for financial decisions related to, e.g., risk forecasting, risk management or risk allocation.

We characterize the asymptotic properties of return divergence of arbitrary order $m \geq 2$, by establishing corresponding Laws of Large Numbers and Central Limit Theorems as the sampling frequency increases. In this way, we provide the foundations for a unified statistical measurement of second- and higher-order realized risks induced by linear combinations of realized return divergences. We then provide corresponding feasible asymptotic confidence intervals that are computable in applications and study their reliability in a number of Monte Carlo simulation experiments. Overall, we find that the estimation uncertainty on realized return divergence of order two, three and four may be successfully measured and incorporated in applications where higher-order realized return risk measuring, forecasting or trading is relevant.

The remainder of the paper proceeds as follows. We first introduce in section 2 the realized return power divergence of integer order $m \geq 2$. This quantity defines our point estimates for an underlying realized risk under general semimartingale conditions. Section 3 states the relevant Laws of Large Numbers for realized return divergence, while section 4 derives the corresponding Central Limit Theorems. Starting from these findings, we develop in section 5 empirically feasible Central Limit Theorems and confidence intervals. The Monte Carlo study on the accuracy of our asymptotic approximations for the finite sample distribution of realized return divergence of order two, three and four is reported in Section 6. Section 7 concludes. Proofs, tables and figures are collected in the appendix.

[2] This aspect is essential for financial applications in which a precise measuring, forecasting or trading of realized risks is an issue, such as studies on the time-varying risk return tradeoff or cross-sectional analyses on the role of time-varying realized risks as risk factors explaining the cross-section of individual returns.

2.2 Bregman divergence and realized return divergence

We follow Schneider and Trojani [2015a] and introduce second-order realized divergence by means of Bregman [1967] power divergences. We then obtain higher-order realized return divergences using linear combinations (derivatives) of realized power divergences. For the whole section, let $(Y_t)_{t\in I}$ be a strictly positive price process on time index set $I \subset [0,\infty)$, such that $Y_0 = 1$ without loss of generality. We detail in a second step the precise semimartingale assumptions needed on process $(Y_t)_{t\in I}$ for our characterizations of realized return divergence to hold.

2.2.1 Power divergence

Given a generating convex function Φ and times $s < t$ in I, the Bregman [1967] divergence between Y_t and Y_s induced by function Φ is:[3]

$$D_\Phi\left(Y_t, Y_s\right) := \Phi(Y_t) - \Phi(Y_s) - \Phi'(Y_s)\left(Y_t - Y_s\right) , \tag{2.1}$$

where Φ' is the derivative of Φ. D_Φ is nonnegative, convex in its first argument and invariant to affine transformations of generating function Φ. A parsimonious very useful family of divergences is obtained using power generating functions.

Definition 2.2.1 (Power divergence). *Given generating function $\Phi_p(x) = \frac{x^p-1}{p(p-1)}$ for $p \in \mathbb{R}\backslash\{0,1\}$, the p-power divergence between Y_s and Y_s is defined by:*

$$D_p(Y_t, Y_s) := D_{\Phi_p}(Y_t, Y_s) = \frac{Y_t^p - Y_s^p - pY_s^{p-1}(Y_t - Y_s)}{p(p-1)} . \tag{2.2}$$

The cases $p \in \{0,1\}$ are defined by continuity:

$$D_0(Y_t, Y_s) := \lim_{p\to 0} D_p(Y_t, Y_s) = -\log\frac{Y_t}{Y_s} + \left(\frac{Y_t}{Y_s} - 1\right) , \tag{2.3}$$

$$D_1(Y_t, Y_s) := \lim_{p\to 1} D_p(Y_t, Y_s) = Y_t \log\frac{Y_t}{Y_s} - Y_s\left(\frac{Y_t}{Y_s} - 1\right) . \tag{2.4}$$

An obvious reparameterization of power divergence that will be useful for the formulation of our Central Limit Theorems is based on a log transform of the underlying price process. Given $X_t := \log Y_t$ for any $t \in I$, it then follows for any $p \in \mathbb{R}$:

$$D_p\left(Y_t, Y_s\right) = F_{D_p}\left(X_s, X_t - X_s\right) , \tag{2.5}$$

where

$$F_{D_p}\left(y, x\right) := e^{py} f_{\overline{D}_p}\left(x\right) \ ; \ f_{\overline{D}_p}\left(x\right) := \frac{(e^{px} - 1) - p(e^x - 1)}{p(p-1)} . \tag{2.6}$$

[3] Convexity of Φ ensures positivity of the Bregman [1967] divergence, which is essential to obtain a good measure of realized risk.

2.2.2 Return divergence

In general, i.e., for $p \neq 0$, power divergence is not independent of the scale of Y. Scale-freeness is obtained by means of power return divergence.

Definition 2.2.2 (p–power return divergence)**.** *For any $p \in \mathbb{R}$, the return power divergence between F_t and F_s is defined by by:*

$$\overline{D}_p\left(Y_t, Y_s\right) := D_p(Y_t/Y_s, 1) = f_{\overline{D}_p}\left(X_t - X_s\right) . \tag{2.7}$$

It is useful to note that to the leading order power return divergence measures the quadratic variation of log returns, as for any $p \in \mathbb{R}$ it follows:

$$f_{\overline{D}_p}\left(x\right) = \frac{x^2}{2} + \frac{1+p}{3!}x^3 + \frac{1+p(p+1)}{4!}x^4 + O(x^5)\,, \tag{2.8}$$

i.e., the leading contribution to $f_{\overline{D}_p}\left(X_t - X_s\right)$ comes from the square of the log-return $(X_t - X_s)^2$, independently of power parameter p. However, higher-order contributions depend on parameter p. This key property can be exploited to systematically generate higher-order return divergences using linear combinations, or derivatives, of power return divergences. Indeed, equation (2.8) suggests that derivatives of divergence function $f_{\overline{D}_p}\left(x\right)$ with respect to parameter p give rise to generating functions with higher order leading terms.[4] This intuition motivates the following definition.

Definition 2.2.3. *Given a power parameter $p_0 \in \mathbb{R}$ and an integer $m \geq 2$, the p_0–th m-th order return divergence is defined by:*

$$\overline{D}_{p_0}(m)\left(Y_t, Y_s\right) := \left.\frac{\partial^{(m-2)}\overline{D}_p\left(Y_t, Y_s\right)}{\partial p^{(m-2)}}\right|_{p=p_0} := f_{\overline{D}_{p_0}(m)}\left(X_t - X_s\right)\,, \tag{2.9}$$

where

$$f_{\overline{D}_{p_0}(m)}\left(x\right) := \left.\frac{\partial^{(m-2)} f_{\overline{D}_p}\left(x\right)}{\partial p^{(m-2)}}\right|_{p=p_0} . \tag{2.10}$$

By definition, $\overline{D}_p(2) = \overline{D}_p$, i.e., the p–th second-order return divergence is simply the p–power return divergence. More broadly, identity (2.8) shows that $f_{\overline{D}_p(3)}$ and $f_{\overline{D}_p(4)}$ induce higher-order return divergences with a leading contribution generated by the third-order and the fourth-order variation of log returns, respectively. Therefore, these divergences have a natural interpretation as realized third and fourth moments of log returns, respectively. In general, the return divergence induced by generating function $f_{\overline{D}_p(m)}$ has the natural interpretation of a realized m–th order variation of log returns.

By construction, all m–th order return divergences of power $p \in \mathbb{R}$ imply an identical leading contribution of the m–th order variation of log returns. However, they differ in

[4] These derivatives can also be interpreted as limit long-short portfolios of power divergences of different order.

the contribution of variations of log returns of order strictly larger than m and in the symmetry properties of such contributions with respect to positive and negative returns.

Schneider and Trojani [2015a] show that the power parameter choice $p = 1/2$ is the only one inducing realized second- and higher-order divergences that are compatible with a convenient notion of (put-call) symmetry in arbitrage-free markets, which gives rise to the family of so-called second- and higher-order Hellinger divergences.[5] Second- and higher-order Hellinger divergences induce the following decomposition of any p–th power divergence:

$$f_{\overline{D}_p}(x) = \sum_{k=0}^{\infty} f_{\overline{D}_{1/2}(k+2)}(x) \frac{(p-1/2)^k}{k!} \,. \tag{2.11}$$

Using this decomposition, every power divergence can be interpreted with good approximation as a particular linear combination of Hellinger divergences of order two, three and four, in which the absolute contribution of higher-order divergences increases with the absolute value of parameter $p - 1/2$. Among all power divergences, the divergence for parameter choice $p = -1$ is the only one that is symmetric in log returns, for which the contribution of any odd variation in identity (2.8) equals zero. Figure 2.1 illustrates more systematically the shape of different return power divergences of order $m = 2, 3, 4$.

2.2.3 Realized return divergence

Given a time step $\Delta_n > 0$ and a discrete set of observed prices $Y_{t_0}, Y_{t_1}, \ldots, Y_{t_n}$ at times $t_0 = 0, t_1 = \Delta_n, \ldots, t_n = n\Delta_n$, we now introduce a convenient definition of realized return divergence of oder m. Let for brevity for any $i = 1, \ldots, n$ the i-th log return be denoted by

$$\Delta_i^n X := X_{i\Delta_n} - X_{(i-1)\Delta_n} = \log(Y_{i\Delta_n}/Y_{(i-1)\Delta_n}) \,.$$

Give a fixed function f of log increment $\Delta_i^n X$, it is convenient to measure the corresponding process variation as the sum of the values of this function over subsequent log increments, which induces the following family of realized variation measures:

$$V^n(f, X)_t = \sum_{i=1}^{[t/\Delta_n]} f(\Delta_i^n X) \,, \tag{2.12}$$

where for any $x \geq 0$ we define $[x] := \inf\{y \in \mathbb{N} : y \leq x\}$. This approach is fully compatible with our specification of return divergence in Definition 2.2.3, which induces the following definition of m-th order realized return divergence.

[5]See, Carr and Lee [2009] for a comprehensive treatment of put-call symmetry. Kitamura et al. [2013] exploit the symmetry properties of Hellingr divergence to obtain robust inference procedures in overidentified moment conditions models.

Definition 2.2.4. *For given integer $m \geq 2$, power parameter $p \in \mathbb{R}$ and corresponding generating function $f_{\overline{D}_p(m)}$ in equation (2.10), the realized m–th order return divergence for parameter p is defined by:*

$$V^n(f_{\overline{D}_p(m)}, X)_t := \sum_{i=1}^{[t/\Delta_n]} \overline{D}_p(m)\left(Y_{i\Delta_n}, Y_{(i-1)\Delta_n}\right) = \sum_{i=1}^{[t/\Delta_n]} f_{\overline{D}_p(m)}(\Delta_i^n X)\,. \tag{2.13}$$

2.2.4 Semimartingale assumptions

In order to formulate our asymptotic theory for realized return divergence we rely on a general semimartingale setting for log price process X; see, e.g., Jacod [2008].

Assumption 2.2.5. *The underlying log price process X is a one dimensional semimartingale on filtered space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\right)$ and it admits the decomposition:*

$$X_t = X_0 + B_t + X_t^c + M_t + \sum_{s\leq t} \triangle X_s 1_{\{\|\triangle X_s\|>1\}}\,, \tag{2.14}$$

where $B_0 = M_0 = 0$, B is a predictable process of locally finite variation, M is a purely discontinuous local martingale and X^c is a continuous local martingale. (B, C, ν) is the predictable characteristics of X, where ν is the compensator of the jump measure μ of X, $C = \langle X^c, X^c\rangle$ and B is defined as above.

Absolute continuity of the X characteristics with respect to Lebesgue measure will be required in the sequel, which induces the class of Itô semimartingales. Precisely, let

$$B_t := \int_0^t b_s ds\ ,\ C_t = \int_0^t c_s ds\ ,\ \nu\left(\omega; dt, dx\right) = F_t\left(\omega, dx\right) dt. \tag{2.15}$$

Using the notations $\sigma_s := \sqrt{c_s}$ and $f\left(x\right) \star \mu_t := \int_0^t \int_E f\left(x\right) \mu\left(ds, dx\right)$, where $(E, \mathcal{E})$ is an auxiliary Polish space, the Itô semimartingale X satisfies the Grigelionis representation:

$$X_t = X_0 + \int_0^t b_s ds + \int_0^t \sigma_s dW_s + \left(\delta 1_{\{\|\delta\|\leq 1\}}\right) \star (\mathfrak{p} - q)_t + \left(\delta 1_{\{\|\delta\|>1\}}\right) \star \mathfrak{p}_t\,. \tag{2.16}$$

In this representation, W is a Brownian motion process and $\mathfrak{p}$ a Poisson random measure on $\mathbb{R}_+ \times E$, having compensator $q\left(dt, dx\right) = dt \otimes \lambda\left(dx\right)$. δ is a predictable function on $\Omega \times \mathbb{R}_+ \times E$ and $F_t\left(\omega, .\right)$ is the image of measure λ induced by the mapping $x \mapsto \delta\left(\omega, t, x\right)$. The following assumptions on the structure of Itô semimartingale X are adopted in the sequel.

Assumption 2.2.6 (H–r). *Given $r \in [0, 2]$, process X is an Itô semimartingale of the form (2.16) such that:*

- *Process b is locally bounded and process σ is cadlag adapted;*

- *There is a localizing sequence (τ_n) of stopping times and a sequence of deterministic nonnegative functions (Γ_n) on E, satisfying $\int \Gamma_n^r(z)\lambda(dz) < \infty$ and such that $\|\delta(\omega,t,z)\| \wedge 1 \leq \Gamma_n(z)$ for all (ω,t,z) with $t \leq \tau_n(\omega)$.*

Assumption (H–r) is stronger than the Itô semimartingale assumption and (H–2) is the weakest such assumption. For any $0 < r < 2$, Assumption (H–r) is equivalent to (H–2) and the requirement that the sum of the $r-$th absolute powers of the process jumps is almost surely finite on any time interval: $\sum_{s\leq t} |\Delta X_s|^r < \infty$ for any $t > 0$.

In order to derive our central limit theorems, we require additional assumptions on the stochastic volatility process σ. To specify these assumptions we introduce a common filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\right)$ that supports a second Brownian motion W', independent of W.

Assumption 2.2.7 (**L**). *Assumption (**H–2**) holds and process σ in equation (2.16) satisfies the following Grigelionis representation:*

$$\sigma_t = \sigma_0 + \int_0^t \tilde{b}_s ds + \int_0^t \tilde{\sigma}_s dW_s + \int_0^t \tilde{\sigma}'_s dW'_s + \left(\tilde{\delta} 1_{\{\|\delta\|\leq 1\}}\right) \star (\mathfrak{p} - q)_t + \left(\tilde{\delta} 1_{\{\|\delta\|> 1\}}\right) \star \mathfrak{p}_t\,, \quad (2.17)$$

where

- *Process $\left(\tilde{b}_t\right)$ is optional and locally bounded;*

- *Processes (b_t), $(\tilde{\sigma}_t)$ and $(\tilde{\sigma}'_t)$ are adapted, left-continuous with right limits and locally bounded;*

- *Functions $\delta(\omega,t,x)$ and $\tilde{\delta}(\omega,t,x)$ are predictable and left-continuous with right limits in t. Moreover, there exists a sequence of localizing stopping times (τ_n) and deterministic functions (Γ_n) and (Υ_n) on E with $\int \Gamma_n^2(z)\lambda(dz) < \infty$, and $\int \Upsilon_n^2(z)\lambda(dz) < \infty$ such that $\|\delta(\omega,t,z)\| \wedge 1 \leq \Gamma_n(z)$ and $\left\|\tilde{\delta}(\omega,t,z)\right\| \wedge 1 \leq \Upsilon_n(z)$ for all (ω,t,z) with $t \leq \tau_n(\omega)$.*

Assumption 2.2.7 implies that both X and σ are Itô semimartingales driven by the same Poisson measure. This assumption is fairly general, as it is satisfied by most specifications of stochastic volatility models in the literature.

2.3 Laws of Large Numbers

The theoretical limit of m–th order realized return divergences under the semimartingale assumption is detailed in the next result.[6]

Theorem 2.3.1 (Laws of Large Numbers). *Let Assumption 2.2.5 be satisfied. We then have the following Skorohod convergence in probability as $\Delta_n \to 0$:*

(i) For p–th ($p \in \mathbb{R}$) second-order realized return divergence ($m = 2$):

$$V^n\left(f_{\overline{D}_p(2)}, X\right)_t \overset{\mathbb{P}}{\Longrightarrow} V\left(f_{\overline{D}_p(2)}, X\right)_t := \frac{1}{2}C_t + \sum_{s\leq t} f_{\overline{D}_p(2)}\left(\Delta X_s\right) . \tag{2.18}$$

(ii) For p–th ($p \in \mathbb{R}$) m–order ($m \geq 3$) realized return divergence:

$$V^n\left(f_{\overline{D}_p(m)}, X\right)_t \overset{\mathbb{P}}{\Longrightarrow} V\left(f_{\overline{D}_p(m)}, X\right)_t := \sum_{s\leq t} f_{\overline{D}_p(m)}\left(\Delta X_s\right) . \tag{2.19}$$

Theorem 2.3.1 provides the foundation for the theoretical quantities estimated by realized return divergence in semimartingale settings. Second-order realized power divergence converges to the sum of a p–independent and a p–dependent process in equation (2.18). The first process equals half the continuous quadratic variation of log returns and is identical for all second-order realized return divergences. The second component is the realized power divergence of return jumps. It is dominated by second-order realized jump variations, but it also reflects a $p-$th dependent contribution of higher-order jump return variations. Higher-order realized return divergence in equation (2.19) converges to a limit that is independent of continuous return variations. By construction, this limit is also independent of jump return variations of order less than m and it equals the m–th order realized divergence of return jumps.

In summary, realized second-oder return divergence converges to the sum of one half the continuous quadratic variation of log returns and the second-order realized jump return divergence. In contrast, realized return divergence of order $m \geq 3$ directly converges to the m–th order realized jump return divergence. In particular, in absence of variations due to jumps all realized jump return divergences are zero and the second-order realized return divergence converges to one half the quadratic variation of log returns, which is the natural continuous-time realized power divergence for semimartingales with continuous trajectories, as under such conditions log returns are conditionally Gaussian. Consistently with Definitions 2.2.2 and 2.2.3, this discussion motivates the following compact formulation of Theorem 2.3.1 based on a gross return divergence notation.

[6] The Laws of Large Numbers we present in this section follow from Jacod and Protter [2012], Theorem 2.3.1. We refer to Appendix A for the proofs.

Corollary 2.3.2. *Let Assumptions 2.2.5 be satisfied. We then have the following Skorokhod convergence in probability as $\Delta_n \to 0$:*

(i) For p–th ($p \in \mathbb{R}$) second-order realized return divergence ($m = 2$):

$$D_p^n(X)_t := \sum_{i=1}^{[t/\Delta_n]} D_p\left(Y_{i\Delta_n}/Y_{(i-1)\Delta_n}, 1\right) \stackrel{\mathbb{P}}{\Longrightarrow} D_p(X)_t\,, \tag{2.20}$$

where

$$D_p(X)_t := \frac{1}{2}C_t + \sum_{s\leq t} D_p\left(Y_s/Y_{s-}, 1\right)\,. \tag{2.21}$$

(ii) For p–th ($p \in \mathbb{R}$) m–order ($m \geq 3$) realized return divergence:

$$D_p^n(m, X)_t := \sum_{i=1}^{[t/\Delta_n]} D_p(m)\left(Y_{i\Delta_n}/Y_{(i-1)\Delta_n}, 1\right) \stackrel{\mathbb{P}}{\Longrightarrow} D_p(m, X)_t\,, \tag{2.22}$$

where

$$D_p(m, X)_t := \sum_{s\leq t} D_p(m)\left(Y_s/Y_{s-}, 1\right)\,. \tag{2.23}$$

Finally, recall that Skorokhod convergence in probability implies convergence in probability for any time $t > 0$ that is not a fix time of discontinuity, i.e., such that $\mathbb{P}(\Delta X_t \neq 0) = 0$. Therefore, the following corollary also follows.

Corollary 2.3.3. *For any integer $m \geq 2$, $p \in \mathbb{R}$ and $t > 0$ that is not a fix time of discontinuity of process X, it follows:*

$$D_p^n(m, X)_t \stackrel{\mathbb{P}}{\to} D_p(m, X)_t\,. \tag{2.24}$$

As virtually all models relevant for applications in finance have no fix discontinuities, Corollary 2.3.3 establishes convergence in probability for virtually all practically relevant purposes.

2.4 Functional Central Limit Theorems

In this section, we use the concept of stable convergence in law introduced by Renyi (1963) to characterize the asymptotic distribution of realized return divergence by means of corresponding Central Limit Theorems. The Central Limit Theorem for $V^n(f, X)_t$ for the class of function f that are quadratic in a neighborhood of zero and for Levy (semimartingale) processes X can be found in Jacod [2007] (Jacod [2008]). As divergence function $f_{\overline{D}_p}$ is not quadratic in a neighborhood of zero (see, e.g., expansion (2.8)), we prove a corresponding Central Limit Theorem for this case. The CLT for third-order return divergence is also not available in the literature, but it follows once we have obtained the CLT for second-order divergence. Finally, the CLT for higher order ($m \geq 4$) divergences are obtained by applying the results in Jacod [2007] and Jacod [2008]. In order to specify the limiting processes in our CLT's for realized divergence, we first introduce the following standard notation.

Notation 2.4.1. *We denote by $(\Omega', \mathcal{F}', \mathbb{P}')$ an auxiliary probability space on which the following mutually independent processes are defined: a Brownian motion $\overline{W}$ and three mutually independent i.i.d. sequences $(\psi_{n-})_{n\in\mathbb{N}}$, $(\psi_{n+})_{n\in\mathbb{N}}$ and $(\kappa_n)_{n\in\mathbb{N}}$ such that $\psi_{n-} \sim N(0,1)$, $\psi_{n+} \sim N(0,1)$ and $\kappa_n \sim U(0,1)$. We obtain a very good filtered extension $\left(\tilde{\Omega}, \tilde{\mathcal{F}}, (\tilde{\mathcal{F}}_t)_{t\geq 0}, \tilde{\mathbb{P}}\right)$ of the probability space in Assumption 2.2.6 by setting*

$$\tilde{\Omega} := \Omega \times \Omega' \ ; \ \tilde{\mathcal{F}} := \mathcal{F} \otimes \mathcal{F}' \ ; \ \tilde{\mathbb{P}} := \mathbb{P} \otimes \mathbb{P}' \, .$$

$(\tilde{\mathcal{F}}_t)_{t\geq 0}$ is defined as the smallest right-continuous filtration such that $\mathcal{F}_t \subset \tilde{\mathcal{F}}_t$ for any $t \geq 0$, $\overline{W}$ is $(\tilde{\mathcal{F}}_t)_{t\geq 0}-$adapted and variable $(\psi_{n-}, \psi_{n+}, \kappa_n)$ is $\tilde{\mathcal{F}}_{T_n}-$measurable for each $n \in \mathbb{N}$, where $(T_n)_{n\in\mathbb{N}}$ is any sequence of stopping times exhausting the jumps of semimartingale X in Assumption 2.2.6. Finally, we define:

$$R_{n-} := \sqrt{\kappa_n}\sigma_{T_n-}\psi_{n-} \ ; \ R_{n+} := \sqrt{1-\kappa_n}\sigma_{T_n}\psi_{n+} \ ; \ R_n = R_{n-} + R_{n+} \, , \tag{2.25}$$

where σ is the volatility process in Assumption 2.2.6.

The next theorem characterizes the asymptotic distribution of second-order realized return divergence.

Theorem 2.4.2 (Central Limit Theorem for Second Order Return Divergence)**.** *Let Assumption (L) be satisfied. We then have the following stable convergence in law for processes as $\Delta_n \to 0$:*

$$\frac{1}{\sqrt{\Delta_n}}\left(V^n\left(f_{\overline{D}_p(2)}, X\right)_t - \frac{1}{2}\int_0^t \sigma_s^2 ds - \sum_{s\leq \Delta_n[t/\Delta_n]} f_{\overline{D}_p(2)}\left(\Delta X_s\right)\right) \stackrel{\mathcal{L}-s}{\Longrightarrow} Z_t^{\overline{D}_p(2)} \, , \tag{2.26}$$

where

$$Z_t^{\overline{D}_p(2)} := \frac{1}{\sqrt{2}}\int_0^t \sigma_s^2 d\overline{W}_s + \sum_{n\geq 1} R_n \partial f_{\overline{D}_p(2)}\left(\triangle X_{T_n}\right) 1_{\{T_n\leq t\}} \, . \tag{2.27}$$

For fixed time $t > 0$, the following stable convergence in law for random variables holds:

$$\frac{1}{\sqrt{\Delta_n}}\left(V^n\left(f_{\overline{D}_p(2)}, X\right)_t - V\left(f_{\overline{D}_p(2)}, X\right)_t\right) \stackrel{\mathcal{L}-s}{\longrightarrow} Z_t^{\overline{D}_p(2)} \, . \tag{2.28}$$

In Theorem 2.4.2, the $\mathcal{F}$–conditional law of stable limit process $Z_t^{\overline{D}_p(2)}$ is independent of the choice of the weakly exhausting sequence of stopping times in Notation 2.4.1. Conditional on $\mathcal{F}$, $Z_t^{\overline{D}_p(2)}$ is the sum of a continuous process and a pure jump process, both with zero-mean independent increments, implying a conditional second moment given by:

$$\Lambda_t^{\overline{D}_p(2)} := \tilde{E}\left[\left(Z_t^{\overline{D}_p(2)}\right)^2 \Big| \mathcal{F}\right] = \frac{1}{2}\int_0^t \sigma_s^4 ds + \frac{1}{2}\sum_{s\le t}(\sigma_{s-}^2 + \sigma_s^2)\partial f^2_{\overline{D}_p(2)}(\Delta X_s) \ . \qquad (2.29)$$

Moreover, the conditional distribution of $Z_t^{\overline{D}_p(2)}$ is Gaussian for realizations of processes X and σ having no common jump in the time interval $[0,t]$:

$$Z_t^{\overline{D}_p(2)}|\mathcal{F} \sim_{\tilde{\mathbb{P}}} N\left(0, \frac{1}{2}\int_0^t \sigma_s^4 ds + \sum_{s\le t}\sigma_s^2\partial f^2_{\overline{D}_p(2)}(\Delta X_s)\right) . \qquad (2.30)$$

While these asymptotic characterizations are useful to define asymptotic confidence intervals for $V\left(f_{\overline{D}_p(2)}, X\right)_t$, they depend on nuisance parameters, such as the integrated quarticity $\int_0^t \sigma_s^4 ds$ and the sequence of spot volatilities at jump times smaller than t. For the implementation of feasible confidence intervals for second-order realized divergence, we will therefore introduce different estimators of these nuisance parameters in Section 2.5. The next theorem characterizes the asymptotic distribution of higher-order realized divergence.

Theorem 2.4.3 (Central Limit Theorem for Higher Order Return Divergence ($m \ge 3$))**.** *Let Assumption (H–2) be satisfied. We then have for any integer $m \ge 3$ the following stable convergence in law for processes as $\Delta_n \to 0$:*[7]

$$\frac{1}{\sqrt{\Delta_n}}\left(V^n\left(f_{\overline{D}_p(m)}, X\right)_t - \sum_{s\le \Delta_n[t/\Delta_n]} f_{\overline{D}_p(m)}\left(\Delta X_s\right)\right) \stackrel{\mathcal{L}-s}{\Longrightarrow} Z_t^{\overline{D}_p(m)} \ , \qquad (2.31)$$

where

$$Z_t^{\overline{D}_p(m)} := \sum_{n\ge 1} R_n \partial f_{\overline{D}_p(m)}\left(\triangle X_{T_n}\right) 1_{\{T_n \le t\}} \ . \qquad (2.32)$$

For fixed time $t > 0$, the following stable convergence in law for random variables holds:

$$\frac{1}{\sqrt{\Delta_n}}\left(V^n\left(f_{\overline{D}_p(m)}, X\right)_t - V\left(f_{\overline{D}_p(m)}, X\right)_t\right) \stackrel{\mathcal{L}-s}{\longrightarrow} Z_t^{\overline{D}_p(m)} . \qquad (2.33)$$

In Theorem 2.4.3, the pure jump stable limit process $Z^{\overline{D}_p(m)}$ has zero-mean independent increments conditional on $\mathcal{F}$, with a conditional second moment given by:

$$\Lambda_t^{\overline{D}_p(m)} := \tilde{E}\left[\left(Z_t^{\overline{D}_p(m)}\right)^2 \Big| \mathcal{F}\right] = \frac{1}{2}\sum_{s\le t}(\sigma_{s-}^2 + \sigma_s^2)\partial f^2_{\overline{D}_p(m)}(\Delta X_s) \ . \qquad (2.34)$$

[7]Similar to Theorem 2.4.2, the $\mathcal{F}$–conditional law of stable limit process $Z_t^{\overline{D}_p(m)}$ is independent of the choice of the weakly exhausting sequence of stopping times in Notation 2.4.1.

In contrast to the results for second-order realized divergence, the conditional asymptotic distribution of higher-order realized divergence is degenerate for any realization with no jumps of process X in time interval $[0,t]$. A nondegenerate conditional stable limit arises for any realization featuring some jump of process X in interval $[0,t]$. In this case, whenever the jumps are not common with process σ, the conditional distribution of $Z_t^{\overline{D}_p(m)}$ is Gaussian:

$$Z_t^{\overline{D}_p(m)}|\mathcal{F} \sim_{\tilde{\mathbb{P}}} N\left(0, \sum_{s\leq t} \sigma_s^2 \partial f^2_{\overline{D}_p(m)}(\Delta X_s)\right) . \tag{2.35}$$

Similar to the findings for second-order realized divergence, the stable asymptotic distributions in Theorem 2.4.3 depend on the sequence of spot volatilities at jump times smaller than t, which need to be estimated to obtain feasible confidence intervals for higher-order realized divergence. In principle, such confidence intervals could be naturally computed conditional on the rejection of the null hypothesis of no process jumps in time interval $[0,t]$. Consistently with the gross return divergence notation in Corollary 2.3.2, the following compact formulation of Theorems 2.4.2 and 2.4.3 also holds.

Corollary 2.4.4. *If Assumption (L) and Assumption (H–2) are satisfied for integers $m = 2$ and $m \geq 3$, respectively, then for any fixed time $t > 0$ the following stable convergence in law for random variables hold:*

$$\frac{1}{\sqrt{\Delta_n}}\left(D_p^n(m,X)_t - D_p(m,X)_t\right) \xrightarrow{\mathcal{L}-s} Z_t^{\overline{D}_p(m)} . \tag{2.36}$$

2.5 Asymptotic Inference

After having developed the foundations for a unified inference on realized return divergences of arbitrary order, we address in this section approaches for a feasible inference.

As the asymptotic distribution of realized return divergence depends on nuisance parameters, a feasible inference requires consistent estimators of these parameters. Existing estimators in the literature for different integrated powers of the spot volatility and the spot volatility are mainly based on two approaches: truncated power variation (e.g., Mancini [2001], Mancini [2006] and Mancini [2009], among others) and realized multipower variation (e.g., Barndorff-Nielsen et al. [2003], Barndorff-Nielsen and Shephard [2004a] and Barndorff-Nielsen et al. [2006]).[8] In this context, spot volatility estimation is usually achieved by means of locally averaged realized multipower variation or threshold realized variance.[9]

[8]Other approaches include, e.g., the range-based realized variation in Andersen et al. [2009] and the truncated realized multipower variation in Corsi et al. [2010].

[9]Spot volatility estimation is also addressed in Kristensen [2010] using a kernel weighted average

In this section, we first briefly review a number of consistent estimators for the nuisance parameters appearing in the asymptotic distributions of realized return divergence. Based on these estimators, we then present feasible Central Limit Theorems for settings with no co-jumps in returns and volatility. This feasible inference exploits the properties of stable convergence in law, which allows to substitute the asymptotic variance by a consistent estimator without affecting the relevant limit, and the fact that the asymptotic distribution is normal in absence of co-jumps. The last part of the section considers the empirically relevant case of co-jumps,[10] which is solved with a different approach based on parametric bootstrap approximations; see Jacod and Todorov [2009] and Jacod and Todorov [2010], among others.

2.5.1 Nuisance parameter estimation: Integrated powers of spot volatility

A first estimator of the integrated spot variance is based on a truncation approach. Intuitively, this is motivated by the fact that large returns associated with sufficiently large jumps are identifiable as $\Delta_n \to 0$ after benchmarking them to some asymptotically vanishing threshold $u_n > 0$. The truncated realized variance is defined by:

$$TRV(\Delta_n, u_n)_t := \sum_{i=1}^{[t/\Delta_n]} (\Delta_i^n X)^2 1_{\{|\Delta_i^n X| \leq u_n\}} \ . \tag{2.37}$$

Under Assumption (H-2) and whenever $u_n \asymp \Delta_n^{\varpi}$ for some $\varpi \in (0, \frac{1}{2})$, this is a consistent estimator of the integrated spot variance in the following sense:[11]

$$TRV(\Delta_n, u_n) \stackrel{u.c.p.}{\Longrightarrow} C = \int_0^{\cdot} \sigma_s^2 ds \ , \tag{2.38}$$

where $u.c.p.$ denotes locally uniform convergence in probability (see, Jacod [2008]).[12] More generally, the truncated realized power variation, which is defined by

$$TPV(p, \Delta_n, u_n)_t := \sum_{i=1}^{[t/\Delta_n]} |\Delta_i^n X|^p 1_{\{|\Delta_i^n X| \leq u_n\}} \ , \tag{2.39}$$

estimator. An estimator of spot volatility based on delta sequences has been introduced by Mancini et al. [2015] for finite activity jump processes. Aït-Sahalia and Jacod [2014] argue that the indicator kernel is preferred to general kernels for spot volatility estimation, because of its simplicity, asymptotic efficiency and robustness to infinite activity jump components.

[10] See, e.g., Bandi and Reno [2016] and Todorov and Tauchen [2011] for some empirical evidence on this topic.

[11] By definition, $u_n \asymp v_n$ if both ratios u_n/v_n and v_n/u_n are bounded.

[12] A sequence of cadlag processes (Y_t^n) converges locally uniformly in probability if $\sup_{s \leq t} |Y_s^n - Y_s^n| \stackrel{\mathbb{P}}{\longrightarrow} 0$ for all $t > 0$.

can be used to consistently estimate integrated powers of the spot volatility (see, Jacod and Protter [2012]). In this case, whenever Assumption (H-r) holds for some $r \in [0, 2)$ and $u_n \asymp \Delta_n^{\varpi}$ for some $\frac{p-2}{2(p-r)} \leq \varpi < \frac{1}{2}$, the following limit holds for any $p > 2$:

$$\Delta_n^{1-p/2} TPV(p, \Delta_n, u_n) \overset{u.c.p.}{\Longrightarrow} \mu_p \int_0^{\cdot} \sigma_s^p ds \ , \tag{2.40}$$

where μ_p is the p-th absolute moment of a standard normal random variable, given by:

$$\mathbb{E}(|U|^p) = \pi^{-1/2} 2^{p/2} \Gamma\left(\frac{p+1}{2}\right) \ .$$

For $p = 4$, this gives rise to the following consistent estimator of integrated quarticity:

$$\frac{1}{6\Delta_n} TPV(4, \Delta_n, u_n)_t = \frac{1}{6\Delta_n} \sum_{i=1}^{[t/\Delta_n]} (\Delta_i^n X)^4 1_{\{|\Delta_i^n X| \leq u_n\}} \overset{\mathbb{P}}{\to} \frac{1}{2} \int_0^t \sigma_s^4 ds \ . \tag{2.41}$$

Another approach to obtain consistent estimators of integrated powers of the spot volatility is based on Barndorff-Nielsen and Shephard [2004b]'s multipower variation, which for any $p > 0$ and integer $k \geq 2$ is defined by:

$$RMV([p,k], \Delta_n)_t = \sum_{i=1}^{[t/\Delta_n]-k+1} |\Delta_i^n X|^{p/k} |\Delta_{i+1}^n X|^{p/k} \times \ldots \times |\Delta_{i+k-1}^n X|^{p/k} \ . \tag{2.42}$$

The intuition for this estimator is that two or more large jumps are unlikely to occur within k-consecutive intervals. Therefore, the effect of a large increment due to a large jump is compensated by other increments in equation (2.42). Under Assumption (H-2) and for $p < 2k$ realized multipower variation satisfies the following limit (Aït-Sahalia and Jacod [2014]):

$$\Delta_n^{1-p/2} RMV([p,k], \Delta_n) \overset{u.c.p.}{\Longrightarrow} (\mu_{p/k})^k \int_0^{\cdot} \sigma_s^p ds \ . \tag{2.43}$$

Therefore, for $p = 4$ and $k = 3$, the following consistent estimator of realized quarticity follows:

$$\frac{\mu_{4/3}^{-3}}{2\Delta_n} RMV([4,3], \Delta_n)_t = \frac{\mu_{4/3}^{-3}}{2\Delta_n} \sum_{i=1}^{[t/\Delta_n]-2} |\Delta_i^n X|^{4/3} |\Delta_{i+1}^n X|^{4/3} |\Delta_{i+2}^n X|^{4/3} \overset{\mathbb{P}}{\to} \frac{1}{2} \int_0^t \sigma_s^4 ds \ . \tag{2.44}$$

A tradeoff between consistent integrated volatility estimators based on truncated realized variance or multipower variation arises, because of the choice of the threshold parameter u_n in the first approach, which is usually a more difficult parameter to tune than the number of lags k in the second approach. However, $1/\sqrt{\Delta_n}$−consistent estimators induced by multipower variation methods are based on stronger consistency assumptions and are asymptotically less efficient than $1/\sqrt{\Delta_n}$−consistent estimators based on truncated realized variance; see, e.g., Aït-Sahalia and Jacod [2014]. This fact motivates our choice of truncation-based estimators, e.g., for estimation of the spot volatility.

2.5.2 Nuisance parameter estimation: Spot volatility

Given a consistent estimator of the integrated spot variance $\int_0^t \sigma_s^2 ds$, the cadlag property of process σ suggests to estimate σ_t^2 and σ_{t-}^2 with consistent estimators of the derivatives:

$$\sigma_t^2 = \lim_{n\to\infty} \frac{\int_t^{t+h_n} \sigma_s^2 ds}{h_n} \ ; \ \sigma_{t-}^2 = \lim_{n\to\infty} \frac{\int_{t-h_n}^{t} \sigma_s^2 ds}{h_n} \ , \tag{2.45}$$

where positive sequence h_n converges to zero at a suitable rate as $n \to \infty$. This intuition motivates a local average estimation approach for the spot variance.

Let $h_n = k_n \Delta_n$, where k_n is a strictly increasing sequence of integers such that $k_n \Delta_n \to 0$. Using truncated realized variance as a consistent and efficient estimator of the integrated spot variance, we obtain the following local average estimator of the spot variance at time $t > 0$:

$$\begin{aligned} \hat{\sigma}^2(u_n)_t &:= \frac{1}{k_n\Delta_n}\left[TRV(\Delta_n, u_n)_{\Delta_n([t/\Delta_n]+k_n)} - TRV(\Delta_n, u_n)_{\Delta_n([t/\Delta_n]+1)}\right] \\ &= \frac{1}{k_n\Delta_n} \sum_{j=[t/\Delta_n]+1}^{[t/\Delta_n]+k_n} \Delta_j^n X^2 1_{\{|\Delta_j^n X| \le u_n\}} \ . \end{aligned} \tag{2.46}$$

Analogously,

$$\begin{aligned} \hat{\sigma}^2(u_n)_t^- &:= \frac{1}{k_n\Delta_n}\left[TRV(\Delta_n, u_n)_{\Delta_n([t/\Delta_n]-1)} - TRV(\Delta_n, u_n)_{\Delta_n([t/\Delta_n]-k_n)}\right] \\ &= \frac{1}{k_n\Delta_n} \sum_{j=[t/\Delta_n]-k_n}^{[t/\Delta_n]-1} \Delta_j^n X^2 1_{\{|\Delta_j^n X| \le u_n\}} \ . \end{aligned} \tag{2.47}$$

Under Assumption (H-2) and for the same threshold choice as in equation (2.38), these are consistent estimators of the spot variance (Jacod and Protter [2012], Theorem 9.3.2):[13]

$$\hat{\sigma}^2(u_n)_t \xrightarrow{\mathbb{P}} \sigma_t^2 \ ; \ \hat{\sigma}^2(u_n)_t^- \xrightarrow{\mathbb{P}} \sigma_{t-}^2 \ . \tag{2.48}$$

In Theorem 2.4.3 and Theorem 2.4.2, the jump contribution to the asymptotic variance of the realized return divergence of order $m \geq 2$ is given by:

$$\frac{1}{2}\sum_{s\le t} (\sigma_{s-}^2 + \sigma_s^2) \partial f_{D_p(m)}^2 (\Delta X_s) \ . \tag{2.49}$$

Therefore, for the threshold choice $\tilde{u}_n = \tilde{\beta}\Delta_n^{\tilde{\varpi}}$ and $\tilde{\beta} > 0$ with $0 < \tilde{\varpi} < \frac{1}{2}$ and a parameter k_n as in equation (2.46) and (2.47), we consider following estimator of the asymptotic

[13]With the same approach, a local average of multipower variation with $p = 2$ and $k \geq 2$ provides a consistent estimator for spot volatility. However, we focus on spot volatility estimators based on truncations for the reasons discussed above.

quantity (2.49):

$$\widehat{JV}(f_{\overline{D}_p}(m),\tilde{u}_n)_t := \frac{1}{2}\sum_{i=1+k_n}^{[t/\Delta_n]-k_n} \partial f^2_{\overline{D}_p(m)}(\Delta_i^n X)1_{\{|\Delta_i^n X|>\tilde{u}_n\}}\left(\hat{\sigma}^2(\tilde{u}_n)_{i\Delta_n}+\hat{\sigma}^2(\tilde{u}_n)^-_{i\Delta_n}\right) . \tag{2.50}$$

Under Assumption (H–2) it then follows (Jacod [2012], Theorem 6.5):

$$\widehat{JV}(f_{\overline{D}_p}(m),\tilde{u}_n)_t \xrightarrow{\mathbb{P}} \frac{1}{2}\sum_{s\le t}(\sigma^2_{s-}+\sigma^2_s)\partial f^2_{\overline{D}_p(m)}(\Delta X_s) . \tag{2.51}$$

2.5.3 Feasible inference in absence of co-jumps

When volatility and returns do not jump simultaneously, the relevant central limit theorems for $m-$th order realized divergence are (2.28) and (2.33), with a limit random variable $Z_t^{\overline{D}_p(m)}$ such that:

$$\left.\frac{Z_t^{\overline{D}_p(m)}}{\sqrt{\Lambda_t^{\overline{D}_p(m)}}}\right| \mathcal{F} \sim_{\tilde{\mathbb{P}}} N(0,1) , \tag{2.52}$$

where the asymptotic variance $\Lambda_t^{\overline{D}_p(m)}$ is explicitly given for $m=2$ by:

$$\Lambda_t^{\overline{D}_p(2)} = \frac{1}{2}\int_0^t \sigma_s^4 ds + \sum_{s\le t}\sigma_s^2\partial f^2_{\overline{D}_p(2)}(\Delta X_s) . \tag{2.53}$$

For $m\ge 3$, it is given by:

$$\Lambda_t^{\overline{D}_p(m)} = \sum_{s\le t}\sigma_s^2\partial f^2_{\overline{D}_p(m)}(\Delta X_s) . \tag{2.54}$$

Therefore, an obvious requirement for the nondegeneracy of the Gaussian distribution of $Z_t^{\overline{D}_p(m)}$ is that $\Lambda_t^{\overline{D}_p(m)} > 0$ almost surely, conditional on $\mathcal{F}$. For $m=2$, a sufficient nondegeneracy condition is that process σ is nonzero on a set of positive Lebesgue measure in the time interval $[0,t]$. For $m\ge 3$, an equivalent condition for nondegeneracy is the presence of a jump in process X at some time s before time t where the volatility σ_s is not zero.

To obtain feasible confidence intervals for realized divergence, we borrow from the results of the last two sections. In this way, we obtain following consistent estimator of $\Lambda_t^{\overline{D}_p(2)}$:

$$\hat{\Lambda}_t^{\overline{D}_p(2)} := \frac{1}{6\Delta_n}TPV(4,\Delta_n,u_n)_t + \widehat{JV}(f_{\overline{D}_p}(2),\tilde{u}_n)_t \xrightarrow{\mathbb{P}} \Lambda_t^{\overline{D}_p(2)} , \tag{2.55}$$

under the conditions given in Section 2.5.1. Similarly,

$$\hat{\Lambda}_t^{\overline{D}_p(m)} := \widehat{JV}(f_{\overline{D}_p}(m),\tilde{u}_n)_t \xrightarrow{\mathbb{P}} \Lambda_t^{\overline{D}_p(m)} , \tag{2.56}$$

for $m \geq 3$ under the condition of Section 2.5.2.

Combining the conditions for the unfeasible stable CLTs of Section 2.4 and the conditions for a consistent estimation of the asymptotic variance of realized return divergence, we obtain the following feasible stable CLT.

Theorem 2.5.1 (Feasible Central Limit Theorem for Realized Return Divergence in Absence of Co-Jumps)**.** *Given the threshold choices* $u_n = \beta\Delta_n^{\varpi}$ *and* $\tilde{u}_n = \tilde{\beta}\Delta_n^{\tilde{\varpi}}$, *where* $\beta, \tilde{\beta} > 0$, $\frac{1}{(4-r)} \leq \varpi < \frac{1}{2}$ *and* $0 < \tilde{\varpi} < \frac{1}{2}$, *assume that either condition (i) or condition (ii) below is satisfied:*

(i) For $m = 2$*: Assumption (L) holds and Assumption (H-r) holds for* $r \in [0, 2)$.

(ii) For integers $m \geq 3$*: Assumption (H-2) holds.*

It then follows for any $A \in \mathcal{F}$ *such that* $\mathbb{P}(A) > 0$ *and* $A \subset \{\omega : \Lambda_t^{\overline{D}_p(m)}(\omega) > 0\}$*:*

$$\mathcal{L}\left(\frac{1}{\sqrt{\Delta_n}}\left(\frac{V^n(f_{\overline{D}_p(m)}, X)_t - V(f_{\overline{D}_p(m)}, X)_t}{\sqrt{\hat{\Lambda}_t^{\overline{D}_p(m)}}}\right)\Bigg| A\right) \longrightarrow N(0,1) \ .$$

Central Limit Theorem 2.5.1 provides an obvious construction of feasible confidence intervals for realized return divergence, which imply a target asymptotic confidence level $1-\alpha$. Let $q_{\alpha/2}$ be the $1-\alpha/2$ quantile of a standard normal distribution and introduce the following feasible confidence interval:

$$\mathcal{I}_n := \left[V^n(f_{\overline{D}_p(m)}, X)_t - q_{\alpha/2}\sqrt{\Delta_n\hat{\Lambda}_t^{\overline{D}_p(m)}}, V^n(f_{\overline{D}_p(m)}, X)_t + q_{\alpha/2}\sqrt{\Delta_n\hat{\Lambda}_t^{\overline{D}_p(m)}}\right] \ . \quad (2.57)$$

By construction, this confidence interval has the correct asymptotic level, in the sense that for any $A \in \mathcal{F}$ such that $\mathbb{P}(A) > 0$ and $A \subset \{\omega : \Lambda_t^{\overline{D}_p(m)}(\omega) > 0\}$ it follows:

$$\lim_{n\to\infty}\tilde{\mathbb{P}}(V(f_{\overline{D}_p(m)}, X)_t \notin \mathcal{I}_n | A) = \alpha \ . \quad (2.58)$$

This ensures control of the conditional asymptotic size for process trajectories that are consistent with a nondegenerate stable limit for realized return divergence, but not in general for arbitrary process trajectories, as discussed above.

2.5.4 Feasible inference in presence of co-jumps

When returns and volatility co-jump, the limit random variable $Z_t^{\overline{D}_p(m)}$ appearing in the unfeasible stable Central Limit Theorems of Section 2.4 follows an unknown distribution. In this case, a Central Limit Theorem can be formulated under identical conditions as in Theorem 2.5.1, i.e., for any set $A \in \mathcal{F}$ such that $\mathbb{P}(A) > 0$ and $A \subset \{\omega : \Lambda_t^{\overline{D}_p(m)}(\omega) > 0\}$ we obtain:

$$\mathcal{L}\left(\frac{1}{\sqrt{\Delta_n}}\left(\frac{V^n(f_{\overline{D}_p(m)}, X)_t - V(f_{\overline{D}_p(m)}, X)_t}{\sqrt{\hat{\Lambda}_t^{\overline{D}_p(m)}}}\right)\Bigg| A\right) \longrightarrow \mathcal{L}_A := \mathcal{L}\left(\frac{Z_t^{D_p(m)}}{\sqrt{\Lambda_t^{D_p(m)}}}\Bigg| A\right), \quad (2.59)$$

for some unknown limit distribution $\mathcal{L}_A$ depending on set A. Thus, $\mathcal{L}_A$ is effectively an infinite dimensional nuisance parameter for the standardized asymptotic distribution of realized return divergence in presence of co-jumps.

Conservative confidence intervals

Recalling the conditional second-order properties of $\Lambda_t^{\overline{D}_p(m)}$ stated in equations (2.29) and (2.34), $\mathcal{L}_A$ is a distribution with variance normalized to 1. This observation directly motivates a conservative asymptotic confidence interval based on Chebyshev inequality. Precisely, for any given $\alpha > 0$ consider the confidence interval:

$$\mathcal{I}_n = \left[V^n(f_{\overline{D}_p(m)}, X)_t - \sqrt{\Delta_n \hat{\Lambda}_t^{\overline{D}_p(m)}/\alpha}, V^n(f_{\overline{D}_p(m)}, X)_t + \sqrt{\Delta_n \hat{\Lambda}_t^{\overline{D}_p(m)}/\alpha} \right] . \quad (2.60)$$

Under the conditions of Theorem 2.5.1, it then follows:

$$\limsup_{n\to\infty} \tilde{\mathbb{P}}(V(f_{\overline{D}_p(m)}, X)_t \notin \mathcal{I}_n | A) \leq \alpha \ . \quad (2.61)$$

While this straightforward confidence interval construction might be useful in some applications, the inequality in equation (2.61) is usually strict and might imply excessively conservative confidence intervals. This motivates the construction of sharper confidence intervals in the next section.

Monte Carlo confidence intervals

Sharper confidence intervals for realized return divergence in presence of co-jumps require a more precise estimation of the quantiles of random variable $Z_t^{\overline{D}_p(m)}$. Following Jacod and Todorov [2009] and Jacod and Todorov [2010], among others, such an estimation can be based on a parametric bootstrap approach.

We borrow from the intuition provided by Theorems 2.4.2 and 2.4.3 and recognize that for any sequence (T_n) of stopping times exhausting the jumps of X random variable $Z_t^{\overline{D}_p(m)}$ is identical in law to random variable:

$$\tilde{Z}_t^{\overline{D}_p(m)} := \mathbf{1}_{\{m=2\}}(m) \frac{\sqrt{\int_0^t \sigma_s^4 ds}}{\sqrt{2}} \psi + \sum_{n\geq 1} R_n \partial f_{\overline{D}_p(m)}(\Delta X_{T_n}) 1_{\{T_n \leq t\}} \ ,$$

where ψ is a standard normal random variable, independent of the sequence of variables $R_n := \sqrt{\kappa_n}\sigma_{T_n-}\psi_{n-} + \sqrt{1-\kappa_n}\sigma_{T_n}\psi_{n+}$ in Theorems 2.4.2 and 2.4.3.[14] The law of $\tilde{Z}_t^{\overline{D}_p(m)}$ depends on all jump times prior to t, the integrated quarticity, as well as the spot volatilities at and before all jump times prior to t. These quantities are not observable from discrete-time returns, but can be estimated consistently with the methods introduced in

[14] See also Notation 2.4.1 for details.

Section 2.5.1 and Section 2.5.2. We exploit this insight to motivate the following Monte Carlo approach for the estimation of asymptotic confidence intervals for realized divergence.

1. First, we fix a threshold $\tilde{u}_n > 0$ and let $i_1, i_2, \ldots, i_{r(n)}$ be the sequence of jump times of process X, for jumps prior to t having absolute size larger than $\tilde{u}_n$. These jumps are denoted by $\Delta^n_{i_1}X, \Delta^n_{i_2}X, \ldots, \Delta^n_{i_{r(n)}}X$. In this way, we introduce for $n \in \mathbb{N}$ the random variables:

$$\begin{aligned} \hat{Z}_t^{\overline{D}_p(m)}(n) \;\; &:= \;\; \mathbf{1}_{\{m=2\}}(m)\sqrt{\frac{1}{6\Delta_n}TPV(4,\Delta_n,u_n)}\,\psi \\ &+\sum_{q=1}^{r(n)}\left(\sqrt{\kappa_q}\hat{\sigma}(\tilde{u}_n)^-_{i_q\Delta_n}\psi_{q-}+\sqrt{1-\kappa_q}\hat{\sigma}(\tilde{u}_n)_{i_q\Delta_n}\psi_{q+}\right)\partial f_{\overline{D}_p(m)}(\Delta^n_{i_q}X)\ , \end{aligned}$$

where threshold choices u_n and $\tilde{u}_n$ are those in the definition of consistent estimators (2.39) and (2.50), respectively. Intuitively, under appropriate conditions the $\mathcal{F}$-conditional law of $\hat{Z}_t^{\overline{D}_p(m)}(n)$ should converge in probability to the conditional law of $\tilde{Z}_t^{\overline{D}_p(m)}$ as $n \to \infty$, which gives us a way to consistently estimate the conditional law of $\tilde{Z}_t^{\overline{D}_p(m)}$ from the finite sample law of $\hat{Z}_t^{\overline{D}_p(m)}(n)$.

2. Second, we compute the quantiles of the law of $\hat{Z}_t^{\overline{D}_p(m)}(n)$ using a parametric bootstrap procedure. To this end, we simulate N_n independent realizations of random vectors $(\psi, k_1, \ldots, k_{r(n)}, \psi_{1-}, \ldots, \psi_{r(n)-}, \psi_{1+}, \ldots, \psi_{r(n)+})$ and obtain a sequence of N_n independent realizations $\{\hat{Z}_t^{\overline{D}_p(m),1}(n), \ldots, \hat{Z}_t^{\overline{D}_p(m),N_n}(n)\}$ of $\hat{Z}_t^{\overline{D}_p(m)}(n)$. We then compute the $\alpha/2-$ and the $(1-\alpha/2)-$quantiles of the empirical distribution of $\{\hat{Z}_t^{\overline{D}_p(m),1}(n), \ldots, \hat{Z}_t^{\overline{D}_p(m),N_n}(n)\}$, denoted by $\Upsilon_n^{\alpha/2}$ and $\Upsilon_n^{1-\alpha/2}$, respectively.

3. Finally, we compute the following $(1-\alpha)-$confidence interval for $m-$th order realized divergence, defined by:

$$\mathcal{I}_n = \left[V^n(f_{\overline{D}_p(m)}, X)_t - \sqrt{\Delta_n}\Upsilon_n^{\alpha/2},\ V^n(f_{\overline{D}_p(m)}, X)_t + \sqrt{\Delta_n}\Upsilon_n^{1-\alpha/2}\right]\ . \qquad (2.62)$$

We expected confidence interval (2.62) to imply a correct asymptotic confidence level, when the involved Monte Carlo error can be ensured to be arbitrarily small and the conditions for a feasible Central Limit Theorem of form (2.59) apply. We borrow from Aït-Sahalia and Jacod [2014], Theorem B.11, and make use of the following assumptions to obtain in the next theorem Monte Carlo confidence intervals with the desired asymptotic confidence level.

Theorem 2.5.2. *Given the threshold choices $u_n = \beta\Delta_n^{\varpi}$ and $\tilde{u}_n = \tilde{\beta}\Delta_n^{\tilde{\varpi}}$, where $\beta, \tilde{\beta} > 0$, $\frac{1}{(4-r)} \leq \varpi < \frac{1}{2}$ and $0 < \tilde{\varpi} < \frac{1}{2}$, assume that either condition (i) or condition (ii) below is satisfied:*

(i) For $m = 2$: Assumption (L) holds and Assumption (H-r) holds for $r \in [0, 1)$.

(ii) For integers $m \geq 3$: Assumption (H-r) holds for $r \in [0, 2)$.

It then follows, whenever $N_n \to \infty$ as $n \to \infty$:

$$\lim_{n\to\infty} \tilde{\mathbb{P}}(V(f_{\overline{D}_p(m)}, X)_T \notin \mathcal{I}_n | A) = \alpha \ , \tag{2.63}$$

for any set $A \in \mathcal{F}$ such that $\mathbb{P}(A) > 0$ and $A \subset \{\omega : \Lambda_t^{\overline{D}_p(m)}(\omega) > 0\}$.

2.6 Monte Carlo evidence and extension of main CLT Results

In this section, we study the finite sample properties of realized return divergence and our asymptotic inference approach, in the context of a general stochastic volatility model with jumps in return and volatilities. We first focus on the properties of our feasible and unfeasible CLT approximations with respect to relevant parameters including, e.g., the sampling frequency and the frequency of large versus small jumps in returns or volatility. In a second step, we introduce a simple adaptation of our feasible CLTs, based on an asymptotically negligible modification of our main results. We show by Monte Carlo simulation that this adaptation induces a more accurate finite-sample inference for realized divergences of order $m \geq 3$, especially for settings where small jumps are relatively frequent. From a different perspective, this asymptotically negligible adaptation of our main CLT findings motivates a more appropriate definition of the relevant limit of higher-order realized return divergence, especially for applications where the sampling frequency cannot be naturally taken as arbitrarily small without additional corrections, e.g., because of the presence of microstructure noise.

For brevity, we focus on realized divergence of order $m = 2, 3, 4$ and a power parameter $p = 1$ (Kullback Leibler divergence). All feasible confidence intervals are estimated with spot volatility estimators based on an indicator kernel.

2.6.1 Monte Carlo setting

We borrow from the Monte Carlo settings in Jacod and Todorov [2009] and Jacod and Todorov [2010] and consider following two-factor stochastic volatility model for log returns:

$$\begin{aligned} dX_t &= \sqrt{V_t^1 + V_t^2} dW_t + \textstyle\int_{\mathbb{R}} x\mu(dt, dx, dy) \ , \\ dV_t^1 &= \kappa_1(\theta - V_t^1)dt + \sigma\sqrt{V_t^1} dW_t' \ , \\ dV_t^2 &= -\kappa_2 V_t^2 dt + \alpha_1 \textstyle\int_{\mathbb{R}} y\mu(dt, dx, dy) + \alpha_2 \textstyle\int_{\mathbb{R}} y\mu'(dt, dy) \ . \end{aligned} \tag{2.64}$$

In this model, W and W' are two independent Brownian motions, while μ and μ' are two independent (finite activity) Poisson measures with time-homogenous compensators:

$$\nu(dt, dx, dy) = \frac{\lambda}{2(h-l)(u-d)} 1_{[-h,-l]\cup[l;h]}(x) 1_{[d,u]}(y) dtdxdy \ ; \ (0 < l < h) \ , \tag{2.65}$$

and

$$\nu'(dt, dy) = \frac{\lambda}{(u-d)} 1_{[d,u]}(y) dtdy \ ; \ (0 < d < u) \ , \tag{2.66}$$

respectively. As shown in Todorov [2010], this stochastic volatility model provides a good fit to the main characteristics of the high-frequency dynamics of different time series of financial returns.

Volatility factors V_1 and V_2 follow mean-reverting pure-diffusion and pure-jump processes, respectively, which are useful to account for the empirical persistence properties of the volatility. Diffusive return and volatility shocks are independent and for $\alpha_1 \neq 0$ returns and volatility can co-jump.[15] In addition, when $\alpha_2 \neq 0$, the volatility also exhibits an idiosyncratic jump component.

Return jumps are symmetrically distributed, with a uniform mass on the positive and negative intervals $[l, h]$ and $[-h, -l]$, while volatility jumps are uniformly distributed on the interval $[d, u]$. The choice of the supports of these jump distributions parsimoniously parameterizes processes with frequent small or large jumps in returns or volatility. Finally, the (constant) jump intensity λ parameterizes the jump arrival rate.

2.6.2 Class of models and parameter choices

Table 2.1 summarizes the parameter choices considered in the simulation, together with corresponding model labels. We consider in the upper part of the table models, in which the diffusive volatility components V_1 has a weaker mean reversion than the pure-jump volatility component V_2 ($k_1 = 0.02 < 0.5 = k_2$). In these models, we fix the long term mean θ and the volatility of volatility σ of diffusive volatility factor V_1, as well as the variance of return jumps. The model parameters are set to imply a fraction of total return variation due to jumps between 0.2 and 0.34, which is comparable to the findings in, e.g., Huang and Tauchen [2005]. Therefore, models with larger jump intensities λ in Table 2.1 imply return and volatility jumps having smaller maximal absolute sizes h and u. Parameters α_1 and α_2 control the jump structure of the volatility. No volatility jumps arise for Models I-c to III-c ($\alpha_1 = \alpha_2 = 0$). Purely idiosyncratic volatility jumps arise in Models I-d to III-d ($\alpha_1 = 0$, $\alpha_2 = 1$). Pure co-jumps emerge in Models I-j to III-j ($\alpha_1 = 1$, $\alpha_2 = 0$), while models I-m to III-m incorporate both idiosyncratic volatility jumps and

[15] See, e.g., Bandi and Reno [2016] for a related empirical evidence on co-jumps.

	Parameters										
Case	κ_1	θ	σ	κ_2	α_1	α_2	λ	l	h	d	u
I-c	0.02	0.4	0.04	0.5	0	0	0.5	0.1	1.0420		
II-c	0.02	0.4	0.04	0.5	0	0	1.0	0.1	0.7197		
III-c	0.02	0.4	0.04	0.5	0	0	4.0	0.1	0.3275		
I-d	0.02	0.4	0.04	0.5	0	1	0.5	0.1	1.0420	0.04	0.76
II-d	0.02	0.4	0.04	0.5	0	1	1.0	0.1	0.7197	0.04	0.36
III-d	0.02	0.4	0.04	0.5	0	1	4.0	0.1	0.3275	0.04	0.06
I-j	0.02	0.4	0.04	0.5	1	0	0.5	0.1	1.0420	0.04	0.76
II-j	0.02	0.4	0.04	0.5	1	0	1.0	0.1	0.7197	0.04	0.36
III-j	0.02	0.4	0.04	0.5	1	0	4.0	0.1	0.3275	0.04	0.06
I-m	0.00	0.0	0.00	0.5	1	1	0.5	0.1	1.0420	0.04	0.76
II-m	0.00	0.0	0.00	0.5	1	1	1.0	0.1	0.7197	0.04	0.36
III-m	0.00	0.0	0.00	0.5	1	1	4.0	0.1	0.3275	0.04	0.06

Table 2.1: Model labels and parameter choices for the Monte Carlo simulation.

co-jumps ($\alpha_1 = \alpha_2 = 1$). Since V_1 is constant in this last setting, the volatility follows a pure-jump process in these specifications.

For each model, we jointly simulate the time series of returns and volatilities over 2000 trading of days. We then condition on days on which a jump was realized to compute confidence intervals for second-, third- and fourth-order return divergence. We sample returns using sampling frequencies of 1 minute and 15 seconds second in each trading day, where a trading day consists of 6.5 hours. This Monte Carlo setting gives rise to samples of $n = 390$ and $n = 1560$ return observations, respectively, on each trading day. For spot volatility estimation, we apply a window $k_n = [\Delta_n^{-0.49}]$ and for truncation parameters $u_n = a\Delta_n^{\varpi}$ and $\tilde{u}_n = a\Delta_n^{\tilde{\varpi}}$ we fix $a = 5 \times \sqrt{RMV([2,2],\Delta_n)_1}$ and $\tilde{\varpi} = \varpi = 0.49$, where $RMV([2,2],\Delta_n)_1$ is the bi-power variation on each trading day. Therefore, this choice of the truncation parameter incorporates the effects of the time-variation of the spot volatility on each day.

2.6.3 Comparison of feasible and unfeasible CLTs

Figure 2.2–2.7 illustrate for the different models in Table 2.1 the time series properties of 95%−confidence intervals for daily realized return divergence of order two, three and four, based on sampling frequency of 1 minute. Red points in these plots highlight observations outside of the daily 95%−confidence interval.

For all models in Table 2.1, Tables 2.2–2.13 report the Monte Carlo empirical coverage of 98%, 95% and 90% confidence intervals for daily realized return divergence of order $m = 2, 3, 4$, using sampling frequencies Δ_n of 15 seconds and 1 minute. We compute the confidence intervals using both the unfeasible exact limit in the Central Limit Theorems 2.4.2 and 2.4.3 and the corresponding feasible CLTs in Theorem 2.5.1 and 2.5.2, which are based on a consistent estimator of the asymptoptic variance that incorporates consistent estimators of quarticity and spot volatilities at jump times.

We find that for all models in Table 2.1 Central Limit Theorems 2.4.2 provides a quite good approximation of the tail quantiles in the asymptotic distribution of realized divergence, even for sampling frequencies of 1 minute. For Central Limit Theorem 2.4.3 and higher-order divergences of order $m = 3, 4$ the evidence is more mixed. Indeed, while in most cases the asymptotic approximations for third-order realized divergence sampled at 15 seconds frequencies are still reasonably accurate, the approximations for fourth-order realized divergence are often significantly distorted.

As intuitively expected, the finite-sample accuracy of the asymptotic confidence intervals tends to deteriorate with the divergence order and to increase in the sampling frequency, which is an obvious consequence of the difficulty in measuring higher-order realized moments from unfrequent sampling when jumps are not large enough. As a consequence, the distortions relative to the asymptotic confidence level can be particularly important for divergences of order $m = 4$ that are estimated in models with large jump intensities and smaller average jump sizes. We find that this feature typically induces a too liberal finite-sample inference. For instance, in Models III-d, III-j and III-m, respectively, the unfeasible 90% (95%) confidence intervals for realized divergence of order $m = 4$ in Theorem 2.4.3 imply a finite sample coverage of only 72%, 75% and 61% (79%, 80% and 69%) already for the highest sampling frequency.

2.6.4 A useful reinterpretation of the CLT for higher-order divergence

The challenges highlighted by our Monte Carlo simulations in providing accurate finite sample inference for higher order realized return divergences motivate the following reinterpretation of Central Limit Theorem 2.4.3, which is inspired by expansion (2.8) of the divergence generating function $f_{\overline{D}_p(m)}$ and by limit (2.41) for the truncated fourth-order variation $TPV(4, \Delta_n, u_n)$.

Theorem 2.6.1 (Adjusted Central Limit Theorem for Higher Order Return Divergence $(m = 3, 4)$)**.** *Let Assumption (H–2) be satisfied. We then have for integers* $m = 3, 4$ *the* *following stable convergence in law for processes as $\Delta_n \to 0$:*

$$\frac{1}{\sqrt{\Delta_n}}\left(V^n\left(f_{\overline{D}_p(m)},X\right)_t - \Delta_n K_p(m)\int_0^t \sigma_s^4 ds - \sum_{s\leq \Delta_n[t/\Delta_n]} f_{\overline{D}_p(m)}\left(\Delta X_s\right)\right) \overset{\mathcal{L}-s}{\Longrightarrow} Z_t^{\overline{D}_p(m)} \ , \tag{2.67}$$

where $K_p(3) = (2p+1)/8$ and $K_p(4) = 1/4$. For fixed time $t > 0$, the following stable convergence in law for random variables holds:

$$\frac{1}{\sqrt{\Delta_n}}\left(V^n\left(f_{\overline{D}_p(m)},X\right)_t - \Delta_n K_p(m)\int_0^t \sigma_s^4 ds - V\left(f_{\overline{D}_p(m)},X\right)_t\right) \overset{\mathcal{L}-s}{\longrightarrow} Z_t^{\overline{D}_p(m)} \ . \tag{2.68}$$

In Theorem 2.6.1, the correction term $\Delta_n K_p(m)\int_0^t \sigma_s^4$ ds is asymptotically negligible. However, it can be useful to better capture the finite-sample contribution of small $m-$th order return variations to the $m-$th order realized divergence, as these contributions might arise from small process variations that can be difficult to isolate from the process jump variation using a finite sample frequency. From expansion (2.8), the return divergences of order $m = 3, 4$ for increment $\Delta_i^n X$ are such that:

$$f_{\overline{D}_p(3)}(\Delta_i^n X) \ = \ \frac{1}{6}(\Delta_i^n X)^3 + \frac{2p+1}{24}(\Delta_i^n X)^4 + O(|\Delta_i^n X|^5) \ , \tag{2.69}$$

$$f_{\overline{D}_p(3)}(\Delta_i^n X) \ = \ \frac{1}{12}(\Delta_i^n X)^4 + O(|\Delta_i^n X|^5) \ , \tag{2.70}$$

which implies with limit (2.41):

$$\frac{1}{\Delta_n}\sum_{i=1}^{[t/\Delta_n]} f_{\overline{D}_p(m)}(\Delta_i^n X) 1_{\{|\Delta_i^n X|\leq u_n\}} \overset{\mathbb{P}}{\to} K_p(m)\int_0^t \sigma_s^4 ds \ . \tag{2.71}$$

Therefore, we obtain following adaptation of the limit CLT for higher-order realized return divergence, which accounts for the difficulty of identifying small jumps with finite sampling frequencies:

$$\frac{1}{\sqrt{\Delta_n}}\left(V^n\left(f_{\overline{D}_p(m)},X\right)_t - V\left(f_{\overline{D}_p(m)},X,\Delta_n\right)_t\right) \overset{\mathcal{L}-s}{\longrightarrow} Z_t^{\overline{D}_p(m)} \ . \tag{2.72}$$

where

$$V\left(f_{\overline{D}_p(m)},X,\Delta_n\right)_t := \Delta_n K_p(m)\int_0^t \sigma_s^4 ds + V\left(f_{\overline{D}_p(m)},X\right)_t \ . \tag{2.73}$$

With this modification, we can easily reinterpret the confidence intervals for third- and fourth-order realized divergence in the previous sections as intervals for the new quantity $V\left(f_{\overline{D}_p(m)},X,\Delta_n\right)_t$.

We verify the finite-sample coverage of the adjusted confidence intervals implied by CLT (2.72) for higher-order divergence in a set of additional Monte Carlo simulations, reported in Tables 2.14–2.17, where we focus for brevity on asymptotic intervals of confidence 95%. We find that the coverage of the adjusted confidence intervals is overall quite accurate and clearly better than the one of the unadjusted intervals. Indeed, even in the most challenging settings for the unadjusted CLTs, i.e., Models III-c, III-d, III-j and III-d for a divergence of order $m = 4$, the finite sample coverage of the adjusted unfeasible (feasible) confidence intervals for a sampling frequency Δ_n =15 seconds is 93.5% (93.9%), 93.5% (94.1%), 93% (94.6%) and 92% (93%), respectively. For comparison, the corresponding coverage of the unadjusted confidence intervals is 90.7% (93.2%), 78,7% (85.5%), 79.9% (87.6%) and 69.0% (77.2%), respectively. We conclude that for applied goals definition (2.73) for the limit of higher-order realized return divergence is more appropriate, as it incorporates a good approximation of the divergence contribution of small return variations, which can be difficult to isolate in finite samples from the underlying pure-jump component.

2.7 Conclusion

In this chapter, we developed laws of large numbers and stable functional central limit theorems for realized (scale-invariant) return divergence under general semimartingale conditions. The laws of large numbers give rise to a natural definition of the realized risk estimated by second and higher-order realized divergences, together with the identification of risk contributions generated by continuous and discontinuous semimartingale components. The central limit theorems characterize the asymptotic distribution of second- and higher-order realized divergence, together with its dependence on the properties of the underlying continuous and discontinuous semimartingale components.

As our Central Limit Theorems depend on nuisance parameters that need to be estimated, we obtain feasible asymptotic confidence intervals for second- and higher-order divergence, using either analytical approximations or suitable Monte Carlo techniques. We demonstrate by simulation that our asymptotic approximations provide reliable information on the noisiness of realized divergence of order two, three and four. In the last two cases, we introduce a natural asymptotically negligible adjustment of the relevant limit processes, which is effective in improving the finite sample coverage of unadjusted confidence intervals for realized return divergence.

Overall, our findings indicate that the estimation uncertainty hidden in realized return divergences of order two, three and four may be successfully incorporated in financial contexts where the measuring, forecasting or trading of such realized risks is relevant.

2.8 Appendix A: Proofs

Proofs for the LLNs

Proof of Theorem 2.3.1(i) We recall the Taylor expansion of $f_{\overline{D}_p(2)}$ around $x = 0$:

$$f_{\overline{D}_p}(x) = \frac{x^2}{2} + \frac{1+p}{6}x^3 + \frac{1+p+p^2}{24}x^4 + O(x^5),$$

Therefore $f_{\overline{D}_p(2)}(x) = \frac{x^2}{2} + o(|x|^2)$ as $x \to 0$ and the convergence in probability follows from Jacod and Protter [2012] (see theorem 3.3.1-A).

Proof of Theorem 2.3.1(ii) It follows from the theorem of 3.3.1-B in Jacod and Protter [2012].

Proofs for the CLTs

Proof of Theorem 2.4.2 Jacod [2008] proved the central limit theorem under assumption (L) for a class of functions which are $f(x) = x^2$ on some neighborhood of zero and with at most polynomial growth. When one of these conditions is not satisfied, the convergence might not hold. For instance, consider the function $f(x) = x^2 + |x|^r$ with $2 < r < 3$ that satisfies growth condition but it is not quadratic on the neighborhood of zero. In this case central limit theorem is not valid. Therefore, we consider $f_{\overline{D}_p}(x) = \frac{x^2}{2} + \frac{1+p}{6}x^3 + o(|x|^3)$ instead of $f_{\overline{D}_p(2)}(x) = \frac{x^2}{2} + o(|x|^2)$ as $x \to 0$ and we develop central limit theorems for such functions. In general, the CLTs hold for the family of $\mathcal{C}^2$ functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(0) = f'(0) = 0$, $f''(0) = K$ and $f^{(3)}$ (third derivative) exists and $K \neq 0$. For simplification of the notations, we assume $f''(0) = 1$. Before proceeding further, we give a brief review on criteria for convergence stably in law.

Suppose we have a sequence $(\mathcal{Y}^n)$ of d-dimensional processes and want to prove that it converges to $\mathcal{Y}$. For this we define the following decomposition for any $\varepsilon > 0$

$$\mathcal{Y}^n = \mathcal{Y}(\varepsilon)^n + \mathcal{Y}'(\varepsilon)^n \tag{2.74}$$

such that we can easily show the convergence of $\mathcal{Y}(\varepsilon)^n$ (as $n \to \infty$) to a limit $\mathcal{Y}(\varepsilon)$. One can prove the convergence of $(\mathcal{Y}^n)$ to $\mathcal{Y}$ by proving $\mathcal{Y}'(\varepsilon)^n$ goes to 0 as $\varepsilon \to 0$, "uniformly in n", and that $\mathcal{Y}(\varepsilon)$ goes to $\mathcal{Y}$, .

Proposition 1 (Jacod and Protter [2012], Proposition 2.2.4)**.** *Let $\mathcal{Y}^n$ and $\mathcal{Y}$ be defined on the same probability space. For $\mathcal{Y}^n \overset{\mathcal{L}-s}{\Longrightarrow} \mathcal{Y}$ it is enough that there are the decompositions (2.74) and also $\mathcal{Y} = \mathcal{Y}(\varepsilon) + \mathcal{Y}'(\varepsilon)$, with the following properties:*

$$\forall\varepsilon \geq 0, \quad \mathcal{Y}(\varepsilon)^n \overset{\mathcal{L}-s}{\Longrightarrow} \mathcal{Y}(\varepsilon), \quad as\ n \to \infty \tag{2.75}$$

$$\mathcal{Y}(\varepsilon) \stackrel{\mathcal{L}-s}{\Longrightarrow} \mathcal{Y}, \quad as\ \varepsilon \to 0 \tag{2.76}$$

$$\forall \eta, t > 0, \quad \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \mathbb{P}\left(\sup_{s \le t} \|\mathcal{Y}'(\varepsilon)^n_s\| > \eta \right) = 0 \tag{2.77}$$

In order to establish the central limit theorem, we introduce notations and the decomposition (2.74) in step 1. The aim of step 2 is to show (2.76) of proposition 1, whereas in steps 3-4 we prove (2.77). Finally, we prove the stable convergence in law for fixed time (convergence of random variable) in step 5.

Before proceeding further, we introduce the localization technique that plays an important role in proving limit theorem for discretized processes. The basic idea is to replace the locally bounded assumptions in (H–2) and (L) by global boundedness. More precisely, we define the following assumptions.

Assumption 2.8.1 (**SH**). *Assumption* ***(H–2)*** *holds, and there exists a constant Π and a nonnegative function Γ on E, such that*

$$\begin{array}{ccc} |b_t| \le \Pi, & |\sigma_t| \le \Pi, & |X_t| \le \Pi \\ |\delta(\omega, t, z)| \le \Gamma(z), & \Gamma(z) \le \Pi, & \int \Gamma^2(z)\lambda(dz) \le \Pi \end{array} \tag{2.78}$$

Assumption 2.8.2 (**SL**). *One can define assumption* ***(SL)*** *where locally boundedness conditions in Assumption 2.2.7 are substituted by bounded conditions.*

Note that K denotes a constant and it might change across different lines.
Upon localization lemma (Jacod and Protter [2012], lemma 4.4.9), it suffices to prove the convergence when process X satisfies stronger assumption (SL).

Step 1:
The set $D(\omega) = \{t : \mu(\omega; \{t\} \times E) = 1\}$ is countable, and we construct an exhausting sequence (S_p) of stopping times for the set D as following. Let $A_0 = \varnothing$, $A_m = \{z : \Gamma(z) \ge 1/m\}$ for all $m \ge 1$ and $(S(m,j))$ be successive jump times of Poisson process $1_{A_m \setminus A_{m-1}} \star \mathfrak{p}$. Then we build the sequence of stopping times $(S_p)_{p \ge 1}$ by reordering of the double sequence $(S(m,j))$.
We assume $\mathcal{P}_m$ is the set of all indices p such that $S_p = S(m', j)$ for some $j \ge 1$ and some $m' \le m$. Moreover we define

$$\left.\begin{aligned} &S_{-}(n,p) = (i-1)\,\triangle_n,\ S_{+}(n,p) = i\,\triangle_n \\ &R_{-}(n,p) = \tfrac{1}{\sqrt{\triangle_n}}\left(X_{S_p-} - X_{(i-1)\triangle_n}\right) \\ &R_{+}(n,p) = \tfrac{1}{\sqrt{\triangle_n}}\left(X_{i\triangle_n} - X_{S_p}\right) \end{aligned}\right\} \text{if } (i-1)\,\triangle_n < S_p \leq i\,\triangle_n \tag{2.79}$$

$$\left.\begin{aligned} &b(m)_t = b_t - \textstyle\int_{A_m\cap\{z:\|\delta(t,z)\|\leq 1\}} \delta(t,z)\,\lambda(dz) \\ &X(m)_t = X_0 + \textstyle\int_0^t b(m)_s\,ds + \int_0^t \sigma_s dW_s + \left(\delta 1_{A_m^c}\right)\star(\mathfrak{p}-q)_t \\ &X'(m) = (\delta 1_{A_m})\star\mathfrak{p} \end{aligned}\right\} \tag{2.80}$$

$\Omega_n(T,m) =$ the set of all ω such that each interval $[0,T]\cap[(i-1)\,\triangle_n,\ i\,\triangle_n]$ contains at most one jumps of the $X'(m)_{(\omega)}$, and that $\left\|X(m)_{t+s} - X(m)_t\right\| \leq 2/m$ for all $t\in[0,T]$, $s\in[0,\triangle_n]$.
We also set

$$R(n,p) = R_{-}(n,p) + R_{+}(n,p) = \frac{1}{\sqrt{\triangle_n}}\left(X_{S_{+}(n,p)} - X_{S_{-}(n,p)} - \triangle X_{S_p}\right) \tag{2.81}$$

$$\zeta_p^n = \frac{1}{\sqrt{\triangle_n}}\left(f\left(\triangle X_{S_p} + \sqrt{\triangle_n}R(n,p)\right) - f\left(\triangle X_{S_p}\right) - f\left(\sqrt{\triangle_n}R(n,p)\right)\right) \tag{2.82}$$

$$Y^n(m)_t = \sum_{p\in\mathcal{P}_m : S_p\leq\triangle_n[t/\triangle_n]} \zeta_p^n. \tag{2.83}$$

We define function h_d from $\mathbb{R}$ to $\mathbb{R}$ for any $d\in\mathbb{N}$

$$h_d(x) = x^d. \tag{2.84}$$

When p is fixed, we have (Jacod and Protter [2012], proposition 4.4.10) that the sequence $R(n,p)$ is bounded (in probability). Since f is $\mathcal{C}^2$ and $f(0) = f'(0) = 0$, from Taylor expansion of f around $\triangle X_{S_p}$ (which is bounded under SH) we obtain

$$|\zeta_p^n - \partial f\left(\triangle X_{S_p}\right)R(n,p)| \leq \sqrt{\triangle_n}KR(n,p)^2,$$

which implies that $\zeta_p^n - \partial f\left(\triangle X_{S_p}\right)R(n,p) \xrightarrow{\mathbb{P}} 0$ as $n\to\infty$. If we apply again Proposition 4.4.10 we get

$$\zeta_p^n \xrightarrow{\mathcal{L}-s} \partial f\left(\triangle X_{S_p}\right)R_p\,,$$

where the variables R_p are similar to R_n in (2.25)[16]. The set $\{S_p : p \in \mathcal{P}_m\} \cap [0\, t]$ is finite for all t, therefore we have $Y^n(m) \stackrel{\mathcal{L}-s}{\Longrightarrow} \bar{V}\left(f, X'\left(m\right)\right)$, where

$$\bar{V}\left(f, X'\left(m\right)\right)_t := \sum_{p \in \mathcal{P}_m} \partial f(\Delta X'(m)_{S_p}) R_p 1_{\{S_p \leq t\}}.$$

Based on the result in Jacod [2008] (page-555), we have the following joint convergence

$$\left(\frac{1}{\sqrt{\Delta_n}}\left(V^n\left(X^c, h_2\right) - C\right), Y^n\left(m\right)\right) \stackrel{\mathcal{L}-s}{\Longrightarrow} \left(\sqrt{2}\int_0^t c_u d\bar{W}_u, \bar{V}\left(f, X'\left(m\right)\right)_t\right), \qquad (2.85)$$

where $c_u = \sigma_u^2$ and $\bar{W}$ is a Brownian motion and it is independent of underlying Brownian motion W.
Because the first stochastic integral on the right side of (2.85) is continuous, we deduce

$$\frac{1}{\sqrt{\Delta_n}}\left(V^n\left(X^c, h_2\right)_t - C_t\right) + Y^n\left(m\right)_t \stackrel{\mathcal{L}-s}{\Longrightarrow} \sqrt{2}\int_0^t c_u d\bar{W}_u + \bar{V}\left(f, X'\left(m\right)\right)_t . \qquad (2.86)$$

In the following remark we discuss the continuity condition in more detail.

Remark 2.8.3. *The Skorokhod topology is not compatible with natural linear structure of the space in a sense that $x_n \stackrel{Sk}{\to} x$ and $y_n \stackrel{Sk}{\to} y$ do not necessarily imply $x_n + y_n \stackrel{Sk}{\to} x + y$. However we have (Jacod and Shiryaev [2003], Proposition 1.23): If $x_n \to x$ and $y_n \to y$ for the Skorokhod topology and if y is continuous, then $x_n + y_n \to x + y$ for the Skorokhod topology.*

Step 2:

Now we vary m. Since on the right side of equation (2.86) only the second part depends on m, it is sufficient to prove

$$\bar{V}\left(f, X'\left(m\right)\right) \stackrel{u.c.p.}{\Longrightarrow} \bar{V}\left(f, X\right) \quad as\, m \to \infty\,. \qquad (2.87)$$

That is equivalent to show $\bar{V}\left(f, X\left(m\right)\right) \stackrel{u.c.p.}{\Longrightarrow} 0$ as $m \to \infty$, because we have that $\bar{V}\left(f, X\right) - \bar{V}\left(f, X'\left(m\right)\right) = \bar{V}\left(f, X\left(m\right)\right)$. The process $\bar{V}\left(f, X\left(m\right)\right)$ under assumption (SH) is a square-integrable martingale, conditionally on $\mathcal{F}$. Therefore we deduce from Doob's inequality that

$$\begin{aligned}
&\tilde{\mathbb{E}}\left(\sup_{s \leq t}\left\|\bar{V}\left(f, X\left(m\right)\right)_s\right\|^2\right) \\
&= \mathbb{E}\left(\tilde{\mathbb{E}}\left(\sup_{s \leq t}\left\|\bar{V}\left(f, X\left(m\right)\right)_s\right\|^2 \mid \mathcal{F}\right)\right) \\
&\leq K\mathbb{E}\left(\sum_{s \leq t}\left\|\partial f\left(\triangle X\left(m\right)_s\right)\right\|^2\left(\|c_{s-}\| + \|c_{s+}\|\right)\right) .
\end{aligned} \qquad (2.88)$$

[16] Note that the variables R_p are defined on extended probability space given in Notation 2.4.1

Boundedness of c_t and $\partial f(x)$ on $\{x : \|x\| \leq 1\}$ yields that

$$\mathbb{E}\left(\sum_{s\leq t} \|\partial f\left(\triangle X\left(m\right)_s\right)\|^2 \left(\|c_{s-}\| + \|c_{s+}\|\right)\right) \leq K \sum_{s\leq t} \|\triangle X_s\|^2 1_{\{\|\triangle X_s\|\leq 1/m\}}$$

Furthermore

$$\mathbb{E}\left(\sum_{s\leq t} \|\triangle X_s\|^2 1_{\{\|\triangle X_s\|\leq 1/m\}}\right) = \mathbb{E}\left(\left(\|x\|^2 1_{\{\|x\|\leq 1/m\}}\right) \star \mu\right) = \mathbb{E}\left(\left(\|x\|^2 1_{\{\|x\|\leq 1/m\}}\right) \star \nu\right)$$
$$\leq t \int_{\{z:\Gamma(z)\leq 1/m\}} \Gamma\left(z\right)^2 \lambda\left(dz\right) \tag{2.89}$$

where the last expression goes to zero as $m \to \infty$ (by dominated convergence theorem).

Step 3:
Since $\lim_{n\to\infty} \mathbb{P}\left(\Omega_n\left(t, m\right)\right) = 1$ for all $m \geq 1$, it remains to prove that, for all $\eta > 0$, we have

$$\lim_{m\to\infty} \limsup_{n\to\infty} \mathbb{P}\left(\sup_{s\leq t} \left\|1_{\Omega_n(t,m)} Q^n\left(m\right)_s\right\| > \eta\right) = 0 \tag{2.90}$$

where

$$Q^n\left(m\right)_t = \frac{1}{\sqrt{\Delta_n}}\left(V^n\left(f, X\right)_t - C_t^{(n)} - f \star \mu_{\triangle_n[t/\triangle_n]} - V^n\left(h_2, X^c\right)_t + C_t\right) - Y^n\left(m\right)_t\ , \tag{2.91}$$

and

$$f \star \mu_{\triangle_n[t/\triangle_n]} = \sum_{s\leq \Delta_n[t/\Delta_n]} \left(f\left(\triangle X\left(m\right)_s\right) + f\left(\triangle X'\left(m\right)_s\right)\right).$$

The third order Taylor expansion of function f can be written as

$$f\left(x\right) = h_2\left(x\right) + k h_3\left(x\right) + h_e\left(x\right) \tag{2.92}$$

where k is a constant, h_2 and h_3 as defined in (2.84) and function $h_e : \mathbb{R} \to \mathbb{R}$ is $h_e\left(x\right) = o\left(\|x\|^3\right)$ as $x \to 0$.

On the set $\Omega_n\left(t, m\right)$ we have:

$$\frac{1}{\sqrt{\Delta_n}}\left(V^n\left(f, X\right)_t - f \star \mu_{\triangle_n[t/\triangle_n]} - C_t^{(n)}\right) = Y^n\left(m\right)_t + \tag{2.93}$$
$$\frac{1}{\sqrt{\Delta_n}}\left(V^n\left(f, X\left(m\right)\right)_t - C_t^{(n)} - \sum_{s\leq t} f\left(\triangle X\left(m\right)_s\right)\right).$$

Therefore, we can define the following decomposition of $Q^n(m)_t$ on Ω_n :

$$Q^n(m)_t = Q^n_{h_2}(m)_t + Q^n_{h_3}(m)_t + Q^n_{h_e}(m)_t \, . \tag{2.94}$$

where

$$Q^n_{h_2}(m)_t = \frac{1}{\sqrt{\Delta_n}}\left(V^n(h_2, X(m))_t - C_t^{(n)} - \sum_{s\le\Delta_n[t/\Delta_n]} h_2(\triangle X(m)_s) - V^n(h_2, X^c)_t + C_t\right) \tag{2.95}$$

$$Q^n_{h_3}(m)_t = \frac{1}{\sqrt{\Delta_n}}\left(V^n(kh_3, X(m))_t - k\sum_{s\le\Delta_n[t/\Delta_n]} h_3(\triangle X(m)_s)\right) \tag{2.96}$$

$$Q^n_{h_e}(m)_t = \frac{1}{\sqrt{\Delta_n}}\left(V^n(h_e, X(m))_t - \sum_{s\le\Delta_n[t/\Delta_n]} h_e(\triangle X(m)_s)\right) \tag{2.97}$$

Thus it is enough to prove (2.90) for three cases above, where $Q^n(m)_t$ is replaced by $Q^n_{h_2}(m)_t$, $Q^n_{h_3}(m)_t$ and $Q^n_{h_e}(m)_t$. This will be shown in the next step but before we give some important estimates for Itô semimartingales and a version of Itô's formula for Itô semimartingales based on their characteristics.

Lemma 2.8.4. *(Jacod and Protter [2012], lemma 2.1.8) Consider the Itô semimartingales X of the form (2.16) . Then if T is a finite stopping time, $s > 0$ and $p \ge 2$, in the bounded case, we have*

$$\|b_t(\omega)\| \le \beta, \quad \|\sigma_t(\omega)\| \le \alpha, \quad \|\delta(\omega,t,z)\| \le \Gamma(z) \Longrightarrow$$

$$\begin{aligned}\mathbb{E}\left(\sup_{u\le s}\|X_{T+u} - X_T\|^p \mid \mathcal{F}_T\right) \quad &\le K\left(s^p\beta^p + s^{p/2}\alpha^p + s\int\Gamma(z)^p\lambda(dz)\right.\\ &+ s^{p/2}\left(\int\Gamma(z)^2 1_{\{\Gamma(z)\le1\}}\lambda(dz)\right)^{p/2}\\ &\left.+ s^p\left(\int\Gamma(z)1_{\{\Gamma(z)>1\}}\lambda(dz)\right)^p\right).\end{aligned} \tag{2.98}$$

Lemma 2.8.5. *(Jacod [2008], lemma 5.13) Under Assumption (SH) for each $\mathcal{C}^2$ function g which fulfills $h(0) = 0$ and $|h'| \le B$ and $|h''| \le B$, there exits a constant α such that,* *we have for all i, n and all $m>1$*

$$t \leq_{\Delta_n} \Rightarrow \begin{cases} |\mathbb{E}^n_{i-1}(h(X(m)_{(i-1)\Delta_n+t} - X(m)_{(i-1)\Delta_n}))| \leq \alpha B \Delta_n, \\ \mathbb{E}^n_{i-1}(h(X(m)_{(i-1)\Delta_n+t} - X(m)_{(i-1)\Delta_n})^2) \leq \alpha(B+B^2)\Delta_n. \end{cases} \tag{2.99}$$

furthermore let Assumption (SL) be satisfied then we have

$$t \leq_{\Delta_n} \Rightarrow \begin{cases} |\mathbb{E}^n_{i-1}(c_{(i-1)\Delta_n+t} - c(i-1)\ \Delta_n)| \leq \alpha B \Delta_n, \\ \mathbb{E}^n_{i-1}((c_{(i-1)\Delta_n+t} - c_{(i-1)\Delta_n})^2) \leq \alpha B \Delta_n. \end{cases} \tag{2.100}$$

Lemma 2.8.6 (Jacod [2008], lemma 5.12)**.** *Under Assumption (SH) there is an increasing functions I_n on $(0, \infty)$ satisfying*

$$\lim_{\delta \to 0} \limsup_n I_n(\delta) = 0. \tag{2.101}$$

Given $X(m)' = X(m) - X_0 - X^c$, for all $i, n, m \in \mathbb{N}, \delta > 0$ we have

$$t \leq \Delta_n \Rightarrow \mathbb{E}^n_{i-1}\left(|X(m)'_{(i-1)\Delta_n+t} - X(m)'_{(i-1)\Delta_n}|^2 \bigwedge \delta^2\right) \leq \Delta_n I_n(\delta). \tag{2.102}$$

We define the following $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$ functions associated with function h

$$\begin{aligned} k(x,y) &= h(x+y) - h(x) - h(y) \\ g(x,y) &= h(x+y) - h(x) - h(y) - \partial h(x)\, y\,. \end{aligned} \tag{2.103}$$

In the remark below, we discuss the general approach which is used to evaluate the function of increments of process. It plays a crucial role in deriving the remaining results.

Remark 2.8.7. *Recall (2.80) and suppose that h is $\mathcal{C}^2$. Applying Itô's formula to the process $X(m)_t - X(m)_{(i-1)\Delta_n}$ and the function h, we get for $i\Delta_n \geq t > (i-1)\Delta_n$:*

$$\frac{1}{\sqrt{\Delta_n}}\left(h(X(m)_t - X(m)_{(i-1)\Delta_n}) - \sum_{(i-1)\Delta_n < s \leq t} h(\Delta X(m)_s)\right) = A(n,m,i)_t + M(n,m,i)_t \tag{2.104}$$

where $M(n,m,i)$ is square-integrable martingale, and with

$$\begin{aligned} A(n,m,i)_t &= \textstyle\int_{(i-1)\Delta_n}^t a(n,m,i)_u\, du, \\ A'(n,m,i)_t := \langle M(n,m,i), M(n,m,i)\rangle_t &= \textstyle\int_{(i-1)\Delta_n}^t a'(n,m,i)_u\, du, \end{aligned} \tag{2.105}$$

$$\begin{cases} a(n,m,i)_t = \frac{1}{\sqrt{\Delta_n}} & case\left(\partial h\left(X(m)_t - X(m)_t^{(n)}\right) b(m)_t + \frac{1}{2}\partial^2 h\left(X(m)_t - X(m)_t^{(n)}\right) c_t \right. \\ & \left. + \int_{A_m^c} g(X(m)_t - X(m)_t^{(n)}, \delta(t,z))\lambda(dz)\right), \\ \\ a'(n,m,i)_t = \frac{1}{\Delta_n} & \left(\int_{A_m^c} k\left(X(m)_t - X(m)_t^{(n)}, \delta(t,z)\right)^2 \lambda(dz)\right. \\ & \left. + h'\left(X(m)_t - X(m)_t^{(n)}\right)^2 c_t\right). \end{cases}$$

Step 4:
From decomposition (2.94), we obtain

$$\|1_{\Omega_n(t,m)}Q^n(m)_t\| \leq \|1_{\Omega_n(t,m)}Q^n_{h_2}(m)_t\| + \|1_{\Omega_n(t,m)}Q^n_{h_3}(m)_t\| + \|1_{\Omega_n(t,m)}Q^n_{h_e}(m)_t\| \tag{2.106}$$

Therefore, it is enough to show

$$\lim_{m\to\infty}\limsup_{n\to\infty}\mathbb{P}\left(\Omega_n(T,m)\cap\left\{\sup_{s\leq t}\|Q^n_h(m)_s\|>\eta\right\}\right)=0\,. \tag{2.107}$$

when function h is h_2, h_3 and h_e. We demonstrate (2.107) for each of these functions through following three lemmas.

Lemma 2.8.8. *(Jacod [2008], Theorem 2.11) For the function $h_2(x)=x^2$, then (2.107) holds.*

Proof. [17] Applying Itô's formula to the process $X(m)_t - X(m)^n_t$ in the virtue of Remark 2.8.7., we get $Q^n_{h_2}(m) = A_{h_2}(n,m)^{(n)} + M_{h_2}(n,m)^{(n)}$, where $M_{h_2}(n,m)_t$ is a square-integrable with predictable bracket $A'_{h_2}(n,m)$. In this case (2.105) is equivalent to

$$\begin{cases} a_{h_2}(n,m)_t = & \frac{1}{\sqrt{\Delta_n}}\left(2\left(X(m)_t - X(m)^{(n)}_t\right)b(m)_t + c_t - c^{(n)}_t\right) \\ \\ a'_{h_2}(n,m)_t = & \frac{4}{\Delta_n}\left(\left(X(m)_t - X(m)^{(n)}_t - \left(X^c_t - (X^c)^{(n)}_t\right)\right)^2 c_t \right. \\ & \left. + \int_{A^c_m}\left(X(m)_t - X(m)^{(n)}_t\right)^2 \delta(t,z)^2\lambda(dz)\right). \end{cases}$$

If one proves for all $\eta>0$ and $t>0$

$$\lim_{m\to\infty}\limsup_{n\to\infty}\mathbb{P}\left(\sup_{s\leq t}\left(\left|A_{h_2}(n,m)^{(n)}_s\right| + A'_{h_2}(n,m)^{(n)}_s\right)>\eta\right)=0 \tag{2.108}$$

then (2.107) yields. We introduce a decomposition of $A_{h_2}(n,m)^{(n)}$and $A'_{h_2}(n,m)^{(n)}$

$$A_{h_2}(n,m)^{(n)} = \sum_{j=1}^{3} D^n(m,j)\,, \qquad A'_{h_2}(n,m)^{(n)} = \sum_{j=4}^{5} D^n(m,j)\,, \tag{2.109}$$

where $D^n(m,j)_t = \sum_{i=1}^{[t/\Delta_n]}\zeta^n_i(m,j)$. In order to get (2.107) it is sufficient to show that for each j, $\lim\limits_{m\to\infty}\limsup\limits_{n\to\infty}\mathbb{P}\left(\sup\limits_{s\leq t}|D^n(m,j)_s|>\eta\right)=0$ holds. This is obtained if either we have

$$\lim_{m\to\infty}\limsup_{n\to\infty}\mathbb{E}\left(\sum_{i=1}^{[t/\Delta_n]}|\zeta^n_i(m,j)|\right)=0 \tag{2.110}$$

[17] We present the proof here since it is simpler for special case of h_2 and also we want to adjust it to our notation.

or the following two properties are fulfilled, as $n \to \infty$:

$$\mathbb{E}\left(\sum_{i=1}^{[t/\Delta_n]} \left|\mathbb{E}_{i-1}^n(\zeta_i^n(m,j))\right|\right) \to 0, \quad \mathbb{E}\left(\sum_{i=1}^{[t/\Delta_n]} \left|\zeta_i^n(m,j)\right|\right) \to 0. \tag{2.111}$$

Recalling that the process $b(m)$ is caglad and predictable (left continuous with right limit) and the right limit $b(m)_{t+}$ of $b(m)$ exists we set

$$\zeta_i^n(m,1) = \frac{1}{\sqrt{\Delta_n}} \int_{(i-1)\Delta_n}^{i\Delta_n} \left(c_t - c_{(i-1)\Delta_n}\right) dt$$

$$\zeta_i^n(m,2) = \frac{2}{\sqrt{\Delta_n}} b(m)_{(i-1)\Delta_n+} \int_{(i-1)\Delta_n}^{i\Delta_n} \left(X(m)_t - X(m)_t^{(n)}\right) dt$$

$$\zeta_i^n(m,3) = \frac{2}{\sqrt{\Delta_n}} \int_{(i-1)\Delta_n}^{i\Delta_n} \left(X(m)_t - X(m)_t^{(n)}\right) \left(b(m)_t - b(m)_{(i-1)\Delta_n+}\right) dt$$

$$\zeta_i^n(m,4) = \frac{4}{\Delta_n} \int_{(i-1)\Delta_n}^{i\Delta_n} \left(X(m)_t - X(m)_t^{(n)} - \left(X_t^c - X_{(i-1)\Delta_n}^c\right)\right)^2 c_t dt$$

$$\zeta_i^n(m,5) = \frac{4}{\Delta_n} \int_{(i-1)\Delta_n}^{i\Delta_n} \int_{A_m^c} \left(X(m)_t - X(m)_t^{(n)}\right)^2 \delta(t,z)^2 \lambda(dz)\, dt$$

We prove (2.108) holds by showing that for each $j = 1, ..., 5$ we have either (2.110) or (2.111).

Case $j = 1$**:** Knowing $\zeta_i^n(m,1) = \zeta_i^n$ is independent of m, we obtain (2.111) for $j = 1$ from (2.100).

Case $j = 2$**:** Note that $\|b(m)\| \le Km$. Then by applying (2.99) we get

$$|\mathbb{E}_{i-1}^n(\zeta_i^n(m,2))| \le mK|\Delta_n^{3/2}, \quad \mathbb{E}_{i-1}^n(\zeta_i^n(m,2))^2 \le m^2 K \Delta_n^2,$$

which implies (2.111) for $j = 2$.

Case $j = 3$**:** The discretized version of (b_{t+}) is denoted by $b_{t_+}^{(n)}$. Using (2.99) and Cauchy-Schwarz inequality, one can get

$$\mathbb{E}\left(\sum_{i=1}^{[t/\Delta_n]}|\zeta_i^n(m,3)|\right) \leq \frac{2}{\sqrt{\Delta_n}}\mathbb{E}_{i-1}^n\left(\int_0^t |X(m)_s - X(m)_s^{(n)}||b(m)_s - b(m)_{s+}^{(n)}|ds\right)$$

$$\leq \frac{2}{\sqrt{\Delta_n}}\left(\mathbb{E}_{i-1}^n\left(\int_0^t |X(m)_s - X(m)_s^{(n)}|^2 ds\right)\mathbb{E}_{i-1}^n\left(\int_0^t |b(m)_s - b(m)_{s+}^{(n)}|^2 ds\right)\right)^{1/2}$$

$$\leq \left(\mathbb{E}_{i-1}^n\left(\int_0^t |b(m)_s - b(m)_{s+}^{(n)}|^2 ds\right)\right)^{1/2}$$

The last inequality follows from (2.99). Since $b(m)_s - b(m)_{s+}^{(n)}$ converges (pointwise) to 0 and it is bounded, the last expression converges to 0. Thus (2.110) follows for $j = 3$.

Case $j = 4$**:** Based on the definition of $\Omega_n(t,m)$ for $0 < t \leq \Delta_n$ we have $|X(m)'_{(i-1)\Delta_n+t} - X(m)'_{(i-1)\Delta_n}|^2 1_{\Omega_n(t,m)} \leq |X(m)'_{(i-1)\Delta_n+t} - X(m)'_{(i-1)\Delta_n}|^2 \bigwedge \frac{4}{m^2}$. Therefore from (2.102) we obtain

$$\mathbb{E}_{i-1}^n(|\zeta_i^n(m,4)|) \leq K\Delta_n I_n(\frac{1}{m}).$$

Then (2.110) follows from (2.101) for $j = 4$.

Case $j = 5$**:** Let introduce $\gamma_2(m) = \int_{A_m^c}\Gamma(z)^2\lambda(dz)$ with $\Gamma(z)$ as in (2.78). Note that $\lim_{m\to\infty}\gamma_2(m) = 0$. Using (2.98) we get

$$\mathbb{E}_{i-1}^n(|\zeta(m,5)|) \leq K(\Delta_n + m^2\Delta_n^2)\gamma_2(m)$$

Since $\gamma_2(m) \to 0$ as $m \to \infty$, we deduce (2.110) for $j = 5$. □

Lemma 2.8.9. *For the function $h_3(x) = x^3$, then (2.107) holds.*

Proof. We follow the approach of remark 2.8.7 and we apply Itô's formula to the function h_3 for the process $X(m)_t - X(m)_t^{(n)}$. Then we obtain $Q_{h_3}^n(m) = A_{h_3}(n,m)^{(n)} + M_{h_3}(n,m)^{(n)}$. The process $M_{h_3}(n,m,i)$ is square-integrable martingale and we have

$$A_{h_3}(n,m,i)_t = \int_{(i-1)\Delta_n}^t a(n,m,i)_u\, du,$$
$$A'_{h_3}(n,m,i)_t := \langle M_{h_3}(n,m,i), M_{h_3}(n,m,i)\rangle_t = \int_{(i-1)\Delta_n}^t a'(n,m,i)_u\, du,$$

where

$$\begin{cases} a_{h_3}(n,m)_t = & \frac{1}{\sqrt{\Delta_n}}\left(3\left(X(m)_t - X(m)_t^{(n)}\right)^2 b(m)_t + 3\left(X(m)_t - X(m)_t^{(n)}\right)c_t\right) \\ & +\frac{1}{\sqrt{\Delta_n}}\int_{A_m^c} g\left(X(m)_t - X(m)_t^{(n)}, \delta(t,z)\right)\lambda(dz) \\ & \\ a'_{h_3}(n,m)_t = \frac{1}{\Delta_n} & \left(\int_{A_m^c} k\left(X(m)_t - X(m)_t^{(n)}, \delta(t,z)\right)^2\lambda(dz)\right. \\ & \left. +9\left(X(m)_t - X(m)_t^{(n)}\right)^4 c_t\right). \end{cases}$$

In order to yield (2.107), one needs to show that for all $\eta > 0$, $t > 0$ we have:

$$\lim_{m\to\infty} \limsup_{n\to\infty} \mathbb{P}\left(\sup_{s\le t} \left(\left| A_{h_3}(n,m)_s^{(n)} \right| + A'_{h_3}(n,m)_s^{(n)} \right) > \eta \right) = 0 \tag{2.112}$$

$A_{h_3}(n,m)^{(n)}$ and $A'_{h_3}(n,m)^{(n)}$ can be decomposed as following:

$$A_{h_3}(n,m)^{(n)} = \sum_{j=6}^{10} D^n(m,j), \qquad A'_{h_3}(n,m)^{(n)} = \sum_{j=11}^{12} D^n(m,j), \tag{2.113}$$

where $D^n(m,j)_t = \sum_{i=1}^{[t/\triangle_n]} \zeta_i^n(m,j)$. We can prove (2.112) if we show that for each $j = 6, ..., 12$

$$\lim_{m\to\infty} \limsup_{n\to\infty} \mathbb{P}\left(\sup_{s\le t} \left| D^n(m,j)_s \right| > \eta \right) = 0 \tag{2.114}$$

Equivalently one can prove that

$$\lim_{m\to\infty} \limsup_{n\to\infty} \mathbb{E}\left(\sum_{i=1}^{[t/\triangle_n]} \left| \zeta_i^n(m,j) \right| \right) = 0 \tag{2.115}$$

Before proceeding further we introduce the following decomposition of $X(m)$ in (2.80):

$$\begin{cases} \tilde{X}(m) = \int_0^t \sigma_s dW_s + \left(\delta 1_{A_m^c}\right) \star (\mathfrak{p} - q)_t \\ X(m)_t = X_0 + \int_0^t b_s(m)\, ds + \tilde{X}(m)_t \end{cases} \tag{2.116}$$

We define

$$\zeta_i^n(m,6) = \frac{3}{\sqrt{\Delta_n}} c_{(i-1)\Delta_n} \int_{(i-1)\Delta_n}^{i\Delta_n} \left(\tilde{X}(m)_t - \tilde{X}(m)_t^{(n)}\right) dt$$

$$\zeta_i^n(m,7) = \frac{3}{\sqrt{\Delta_n}} \int_{(i-1)\Delta_n}^{i\Delta_n} \left(\tilde{X}(m)_t - \tilde{X}(m)_t^{(n)}\right)\left(c_t - c_t^{(n)}\right) dt$$

$$\zeta_i^n(m,8) = \frac{3}{\sqrt{\Delta_n}} \int_{(i-1)\Delta_n}^{i\Delta_n} \left(\int_{(i-1)\Delta_n}^{t} b(m)_s \, ds\right) c_t dt$$

$$\zeta_i^n(m,9) = \frac{3}{\sqrt{\Delta_n}} \int_{(i-1)\Delta_n}^{i\Delta_n} \left(X(m)_t - X(m)_t^{(n)}\right)^2 b(m)_t \, dt$$

$$\zeta_i^n(m,10) = \frac{1}{\sqrt{\Delta_n}} \int_{(i-1)\Delta_n}^{i\Delta_n} \int_{A_m^c} g\left(X(m)_t - X(m)_t^{(n)}, \delta(t,z)\right) \lambda(dz)\, dt$$

$$\zeta_i^n(m,11) = \frac{9}{\Delta_n} \int_{(i-1)\Delta_n}^{i\Delta_n} \left(X(m)_t - X(m)_t^{(n)}\right)^4 c_t dt$$

$$\zeta_i^n(m,12) = \frac{1}{\Delta_n} \int_{(i-1)\Delta_n}^{i\Delta_n} \int_{A_m^c} k\left(X(m)_t - X(m)_t^{(n)}, \delta(t,z)\right)^2 \lambda(dz)\, dt$$

We say a triangular array (ζ_i^n) is asymptotically negligible that is $\sum_{i=1}^{N_n(t)} \zeta_i^n \overset{u.c.p.}{\Longrightarrow} 0 \; \forall t > 0$, where N_n is stopping rule. Here we give lemma 2.2.12 of Jacod and Protter [2012] for triangular arrays that will be used in the sequel .
The array (ζ_i^n) is asymptotically negligible if one of the two properties below holds:

$$\mathbb{E}\left(\sum_{i=1}^{N_n(t)} |\zeta_i^n|\right) \longrightarrow 0 \quad \forall t > 0 \tag{2.117}$$

$$\mathbb{E}\left(\sum_{i=1}^{N_n(t)} |\zeta_i^n|^2\right) \longrightarrow 0 \quad \forall t > 0 \quad \text{and} \quad \mathbb{E}\left(\zeta_i^n \mid \mathcal{F}_{(i-1)\Delta_n}\right) = 0 \quad \forall i, n \tag{2.118}$$

Case $j = 6$: In order to prove that $D^n(m,6)$ satisfies (2.114) , we show that the triangular array $(\zeta_i^n(m,6))$ is asymptotically negligible for all $m \geq 1$. We obtain this by showing (2.118). Note that the process $\tilde{X}(m)$, defined in (2.116) is square-integrable martingale, Then we have .

$$\mathbb{E}_{i-1}^{n}\left(\zeta_{i}^{n}\left(m,6\right)\right)=\mathbb{E}_{i-1}^{n}\left(\frac{3}{\sqrt{\Delta_{n}}}c_{(i-1)\Delta_{n}}\int\limits_{(i-1)\Delta_{n}}^{i\Delta_{n}}\left(\tilde{X}\left(m\right)_{t}-\tilde{X}\left(m\right)_{t}^{(n)}\right)dt\right)=0 \qquad (2.119)$$

Boundedness of process c together with estimation (2.98) lead to

$$\mathbb{E}_{i-1}^{n}\left(\zeta_{i}^{n}\left(m,6\right)^{2}\right)\leq K\ \Delta_{n}^{2} \qquad (2.120)$$

Therefore the array $(\zeta_{i}^{n}\left(m,6\right))$ is asymptotically negligible.

Case $j=7$: Using (2.98) and Cauchy-Schwartz inequality one can get

$$\begin{aligned}\mathbb{E}\sum_{i=1}^{[t/\Delta_{n}]}\left|\zeta_{i}^{n}\left(m,7\right)\right| &\leq\mathbb{E}\left(\frac{1}{\sqrt{\Delta_{n}}}\int\limits_{0}^{t}\left|\tilde{X}\left(m\right)_{t}-\tilde{X}\left(m\right)_{t}^{(n)}\right|\left|c_{s}-c_{s}^{(n)}\right|ds\right)\\ &\leq\frac{K}{\sqrt{\Delta_{n}}}\left(\mathbb{E}\left[\int\limits_{0}^{t}\left(\tilde{X}\left(m\right)_{t}-\tilde{X}\left(m\right)_{t}^{(n)}\right)^{2}ds\right]\mathbb{E}\left[\int\limits_{0}^{t}\left(c_{s}-c_{s}^{(n)}\right)^{2}ds\right]\right)^{1/2}\\ &\leq K\left(\mathbb{E}\left[\int\limits_{0}^{t}\left(c_{s}-c_{s}^{(n)}\right)^{2}ds\right]\right)^{1/2}\end{aligned} \qquad (2.121)$$

The last inequality follows from (2.98). The last term converges to 0 because of dominated convergence theorem for ordinary integrals, pointwise convergence of $c_{s}-c_{s}^{(n)}$ to 0 and the fact that the process c_{t} is cadlag and bounded. Thus we obtain (2.115) for $j=7$.

Case $j=8$**:** We show in the following that $\zeta_{i}^{n}\left(m,8\right)$ satisfies (2.115) by using the assumption $\left|b\left(m\right)_{s}\right|\leq Km$ and boundedness of process c.

$$\begin{aligned}&\mathbb{E}\sum_{i=1}^{[t/\Delta_{n}]}\left|\zeta_{i}^{n}\left(m,8\right)\right|\\ &=\mathbb{E}\left(\sum_{i=1}^{[t/\Delta_{n}]}\frac{3}{\sqrt{\Delta_{n}}}\left|\int_{(i-1)\Delta_{n}}^{i\Delta_{n}}\left(\int_{(i-1)\Delta_{n}}^{t}b\left(m\right)_{s}ds\right)c_{t}dt\right|\right)\\ &\leq\sum_{i=1}^{[t/\Delta_{n}]}\frac{1}{\sqrt{\Delta_{n}}}\mathbb{E}_{i-1}^{n}\left(\int_{(i-1)\Delta_{n}}^{i\Delta_{n}}\left\|c_{u}\right\|du\int_{(i-1)\Delta_{n}}^{i\Delta}\left\|b\left(m\right)_{s}\right\|ds\right)\\ &\leq\left(\frac{1}{\sqrt{\Delta_{n}}}\right)\left(\frac{t}{\Delta_{n}}\right)\left(K\ \Delta_{n}\right)\left(m\ \Delta_{n}\right)\rightarrow0\quad\text{as } n\rightarrow\infty\end{aligned} \qquad (2.122)$$

Case $j=9$**:** From the fact that $\|b_t(m)\| \leq Km$ and the estimate (2.98) under assumption (SH) we obtain

$$\mathbb{E}_{i-1}^{n}\left(\left|\zeta_i^n(m,9)\right|\right) \leq \frac{K}{\sqrt{\triangle_n}} m \triangle_n \mathbb{E}_{i-1}^{n}\left(\left(X(m)_{i\triangle_n} - X(m)_{(i-1)\triangle_n}\right)^2\right) \leq Km\, \triangle_n^{3/2} \quad (2.123)$$

Therefore (2.115) follows for $j=9$.
Case $j=10$**:** Under the assumption (SH) we have

$$\mathbb{E}_{i-1}^{n}\left(\left|\zeta_i^n(m,10)\right|\right) = \mathbb{E}_{i-1}^{n}\left(\frac{3}{\sqrt{\triangle_n}}\left|\int_{(i-1)\triangle_n}^{i\triangle_n}\left(X(m)_t - X(m)_t^{(n)}\right)\int_{A_m^c}\delta(t,z)^2\lambda(dz)\,dt\right|\right)$$

$$\leq \frac{K}{\sqrt{\triangle_n}}\gamma_2(m)\,\mathbb{E}_{i-1}^{n}\left(\int_{(i-1)\triangle_n}^{i\triangle}\left|X(m)_t - X(m)_t^{(n)}\right| dt\right)$$

$$\leq \frac{K}{\sqrt{\triangle_n}}\triangle_n\left(\sqrt{\triangle_n} + m\,\triangle_n\right)\gamma_2(m)$$

The first inequality above is obtained from (2.78) since $\int_{A_m^c}\delta(t,z)^2\lambda(dz) \leq \int_{A_m^c}\Gamma(z)^2\lambda(dz) < \infty$. The second inequality follows from (2.98). Therefore we get

$$\lim_{m\to\infty}\limsup_{n\to\infty}\mathbb{E}\left(\sum_{i=1}^{[t/\triangle_n]}\left|\zeta_i^n(m,10)\right|\right) = 0 \quad (2.124)$$

Case $j=11$**:** The boundedness of c_t together with (2.98) yield

$$\begin{aligned}\mathbb{E}_{i-1}^{n}\left(\left|\zeta_i^n(m,11)\right|\right) &\leq K\mathbb{E}_{i-1}^{n}\left(\sup_{(i-1)\triangle_n\leq t\leq i\triangle_n}\left(X(m)_t - X(m)_{(i-1)\triangle_n}\right)^4\right)\\ &\leq K\,\triangle_n\int_{A_m^c}\Gamma(z)^4\lambda(dz) \leq K\,\triangle_n\,\gamma_2(m)\end{aligned} \quad (2.125)$$

Then $\lim_{m\to\infty}\gamma_2(m)=0$ implies (2.115) holds for $j=11$.

Case $j=12$**:** We set $x = X(m)_t - X(m)_t^{(n)}$ and we follow the same procedure as previous case to get:

$$
\begin{aligned}
\mathbb{E}_{i-1}^{n}\left(\left|\zeta_{i}^{n}\left(m,12\right)\right|\right) &= \mathbb{E}_{i-1}^{n}\left(\frac{9}{\Delta_n}\int_{(i-1)\Delta_n}^{i\Delta_n}\int_{A_m^c}\left(x^2\delta\left(t,z\right)+x\delta\left(t,z\right)^2\right)^2\lambda\left(dz\right)dt\right)\\
&\leq \frac{K}{\Delta_n}\int_{(i-1)\Delta_n}^{i\Delta_n}\int_{A_m^c}\left(X\left(m\right)_t - X\left(m\right)_{(i-1)\Delta_n}\right)^2\delta\left(t,z\right)^2\lambda\left(dz\right)dt\\
&\leq \frac{K}{\Delta_n}\gamma_2\left(m\right)\int_{(i-1)\Delta_n}^{i\Delta_n}\left(X\left(m\right)_t - X\left(m\right)_{(i-1)\Delta_n}\right)^2 dt\\
&\leq K\ \Delta_n\ \gamma_2\left(m\right)
\end{aligned} \tag{2.126}
$$

Then $\lim_{m\to\infty}\gamma_2\left(m\right) = 0$ implies (2.115) fulfills for $j = 12$. This completes the proof of lemma 2.8.9.

From Jacod [2008] we have the following result.

Lemma 2.8.10. *For the class of $\mathcal{C}^2$ function h_e, with $h_e(0) = \partial h_e(0) = 0$ and $h_e(x) = o(\|x\|^3)$ as $x \to 0$, (2.107) holds.*

This ends the proof of functional stable convergence in law of (2.26) in Theorem 2.4.2.

Step 5:
We need to prove (2.28), that is the convergence for fixed time t. Because any time t is not a fixed time of discontinuity of the limiting process (due to the property of Itô semimartingale process), from (2.26) we obtain for any t that

$$
\frac{1}{\sqrt{\Delta_n}}\left(V^n\left(f,X\right)_t - f\star\mu_{\Delta_n[t/\Delta_n]} - C_t^{(n)}\right)\xrightarrow{\mathcal{L}-s}\bar{V}\left(f,X\right)_t + \sqrt{2}\int_0^t c_u d\bar{W}_u
$$

Therefore, it remains to prove that

$$
U_n := \frac{1}{\sqrt{\Delta_n}}\left|f\star\mu_{\Delta_n[t/\Delta_n]} - f\star\mu_t + C_t^{(n)} - C_t\right|\xrightarrow{\mathbb{P}}0 \tag{2.127}
$$

We introduce the set Ψ_n of ω such that the jumps of $X(\omega)$ between the times $t-\ \Delta_n$ and t are less than 1 . Knowing $\Psi_n \uparrow \Omega$, as $n\to\infty$, we need to show that $\mathbb{E}\left(U_n 1_{\Psi_n}\right)\to 0$. We have $U_n \leq \frac{K}{\sqrt{\Delta_n}}\sum_{t-\Delta_n<s\leq t}\left\|\triangle X_s\right\|^2 + \frac{1}{\sqrt{\Delta_n}}\int_{t-\Delta_n}^{t}\left(c_s^n - c_s\right)ds$ on the set Ψ_n since $\left|f\left(x\right)\right| \leq K\left\|x\right\|^2$ when $\|x\| \leq 1$. Thus

$$\begin{aligned}\mathbb{E}\left(U_n 1_{\Psi_n}\right) &\leq \tfrac{K}{\sqrt{\triangle_n}}\mathbb{E}\left(\sum_{t-\triangle_n<s\leq}\left\|\triangle X_s\right\|^2\right)+K\sqrt{\triangle_n} \\ &= \tfrac{K}{\sqrt{\triangle_n}}\mathbb{E}\left(\int_{t-\triangle_n}^{t} ds \int \left\|\delta\left(t,z\right)\right\|^2 \lambda\left(dz\right)\right)+K\sqrt{\triangle_n}\end{aligned} \tag{2.128}$$

Therefore $\mathbb{E}\left(U_n 1_{\Psi_n}\right) \to 0$ because the last term is smaller than $KA\sqrt{\triangle_n}$. This completes the proof of Theorem 2.4.2. □

Proof of Theorem 2.4.3

Proof. The central limit theorem for third order realized return divergence is not available in literature, but it follows directly from the result of second order, as we proved above, however lemma 2.8.8 is not needed. The central limit theorem for higher order ($m \geq 4$) follows from Theorem 2.11 in Jacod [2008] . □

2.9 Appendix B: Figures and Tables

	Unfeasible CLT		Feasible CLT	
$1-\alpha=0.98$				
	$15-sec$	$1-min$	$15-sec$	$1-min$
$V^n\left(f_{\bar{D}_1(2)},X\right)$	0.9804	0.9830	0.9778	0.9765
$V^n\left(f_{\bar{D}_1(3)},X\right)$	0.9713	0.9504	0.9621	0.9374
$V^n\left(f_{\bar{D}_1(4)},X\right)$	0.9400	0.9022	0.9426	0.9009
$1-\alpha=0.95$				
	Unfeasible CLT		Feasible CLT	
	$15-sec$	$1-min$	$15-sec$	$1-min$
$V^n\left(f_{\bar{D}_1(2)},X\right)$	0.9517	0.9465	0.9504	0.9387
$V^n\left(f_{\bar{D}_1(3)},X\right)$	0.9322	0.9087	0.9335	0.8891
$V^n\left(f_{\bar{D}_1(4)},X\right)$	0.9139	0.8565	0.9178	0.8487
$1-\alpha=0.90$				
	Unfeasible CLT		Feasible CLT	
	$15-sec$	$1-min$	$15-sec$	$1-min$
$V^n\left(f_{\bar{D}_1(2)},X\right)$	0.9126	0.8787	0.9074	0.8696
$V^n\left(f_{\bar{D}_1(3)},X\right)$	0.8852	0.8422	0.8904	0.8435
$V^n\left(f_{\bar{D}_1(4)},X\right)$	0.8565	0.7900	0.8722	0.7979

Table 2.2: Simulation study: coverage of confidence intervals for realized return divergence measures. D_1: Kullback Leibler divergence; $\overline{D}_1(2)$, $\overline{D}_1(3)$, $\overline{D}_1(4)$: second, third and fourth order return divergence. Frequencies: 15 seconds and 1 minute. 2000 days of simulated data. Model: Table 2.1 mod I-c.

$1-\alpha=0.98$				
	Unfeasible CLT		Feasible CLT	
	$15-sec$	$1-min$	$15-sec$	$1-min$
$V^n\left(f_{\bar{D}_1(2)},X\right)$	0.9856	0.9832	0.9864	0.9761
$V^n\left(f_{\bar{D}_1(3)},X\right)$	0.9745	0.957	0.9713	0.9570
$V^n\left(f_{\bar{D}_1(4)},X\right)$	0.9498	0.8964	0.9562	0.9275
$1-\alpha=0.95$				
	Unfeasible CLT		Feasible CLT	
	$15-sec$	$1-min$	$15-sec$	$1-min$
$V^n\left(f_{\bar{D}_1(2)},X\right)$	0.9562	0.9498	0.9554	0.9466
$V^n\left(f_{\bar{D}_1(3)},X\right)$	0.9442	0.9267	0.9371	0.9108
$V^n\left(f_{\bar{D}_1(4)},X\right)$	0.9092	0.8702	0.9243	0.8941
$1-\alpha=0.90$				
	Unfeasible CLT		Feasible CLT	
	$15-sec$	$1-min$	$15-sec$	$1-min$
$V^n\left(f_{\bar{D}_1(2)},X\right)$	0.9108	0.8957	0.9068	0.8845
$V^n\left(f_{\bar{D}_1(3)},X\right)$	0.8901	0.8678	0.8845	0.8622
$V^n\left(f_{\bar{D}_1(4)},X\right)$	0.8566	0.8264	0.8662	0.8335

Table 2.3: Simulation study: coverage of confidence intervals for realized return divergence measures. D_1: Kullback Leibler divergence; $\overline{D}_1(2)$, $\overline{D}_1(3)$, $\overline{D}_1(4)$: second, third and fourth order return divergence. Frequencies: 15 seconds and 1 minute. 2000 days of simulated data. Model: Table 2.1 mod II-c.

$1-\alpha=0.98$				
	Unfeasible CLT		Feasible CLT	
	15−sec	1−min	15−sec	1−min
$V^n\left(f_{\bar{D}_1(2)}, X\right)$	0.9806	0.9847	0.9816	0.9832
$V^n\left(f_{\bar{D}_1(3)}, X\right)$	0.9745	0.9547	0.9710	0.9659
$V^n\left(f_{\bar{D}_1(4)}, X\right)$	0.9409	0.8901	0.9689	0.9537
$1-\alpha=0.95$				
	Unfeasible CLT		Feasible CLT	
	15−sec	1−min	15−sec	1−min
$V^n\left(f_{\bar{D}_1(2)}, X\right)$	0.9562	0.9521	0.9547	0.9516
$V^n\left(f_{\bar{D}_1(3)}, X\right)$	0.9318	0.9226	0.9394	0.9336
$V^n\left(f_{\bar{D}_1(4)}, X\right)$	0.9074	0.8382	0.9323	0.9165
$1-\alpha=0.90$				
	Unfeasible CLT		Feasible CLT	
	15−sec	1−min	15−sec	1−min
$V^n\left(f_{\bar{D}_1(2)}, X\right)$	0.8997	0.8967	0.9003	0.8962
$V^n\left(f_{\bar{D}_1(3)}, X\right)$	0.8794	0.8662	0.8835	0.8814
$V^n\left(f_{\bar{D}_1(4)}, X\right)$	0.8575	0.7772	0.8687	0.8519

Table 2.4: Simulation study: coverage of confidence intervals for realized return divergence measures. D_1: Kullback Leibler divergence; $\overline{D}_1(2)$, $\overline{D}_1(3)$, $\overline{D}_1(4)$: second, third and fourth order return divergence. Frequencies: 15 seconds and 1 minute. 2000 days of simulated data. Model: Table 2.1 mod III-c.

$1-\alpha=0.98$				
	Unfeasible CLT		Feasible CLT	
	$15-sec$	$1-min$	$15-sec$	$1-min$
$V^n\left(f_{\bar{D}_1(2)},X\right)$	0.9851	0.9764	0.9826	0.9752
$V^n\left(f_{\bar{D}_1(3)},X\right)$	0.9553	0.9405	0.9578	0.9281
$V^n\left(f_{\bar{D}_1(4)},X\right)$	0.8996	0.8723	0.9144	0.8698
$1-\alpha=0.95$				
	Unfeasible CLT		Feasible CLT	
	$15-sec$	$1-min$	$15-sec$	$1-min$
$V^n\left(f_{\bar{D}_1(2)},X\right)$	0.9516	0.9479	0.9566	0.9318
$V^n\left(f_{\bar{D}_1(3)},X\right)$	0.9306	0.9157	0.9293	0.8736
$V^n\left(f_{\bar{D}_1(4)},X\right)$	0.8748	0.8364	0.8785	0.8228
$1-\alpha=0.90$				
	Unfeasible CLT		Feasible CLT	
	$15-sec$	$1-min$	$15-sec$	$1-min$
$V^n\left(f_{\bar{D}_1(2)},X\right)$	0.9033	0.8909	0.8959	0.8773
$V^n\left(f_{\bar{D}_1(3)},X\right)$	0.8983	0.8686	0.8872	0.8302
$V^n\left(f_{\bar{D}_1(4)},X\right)$	0.8265	0.7819	0.8389	0.7719

Table 2.5: Simulation study: coverage of confidence intervals for realized return divergence measures. D_1: Kullback Leibler divergence; $\overline{D}_1(2)$, $\overline{D}_1(3)$, $\overline{D}_1(4)$: second, third and fourth order return divergence. Frequencies: 15 seconds and 1 minute. 2000 days of simulated data. Model: Table 2.1 mod I-d.

$1-\alpha=0.98$				
	Unfeasible CLT		Feasible CLT	
	$15-sec$	$1-min$	$15-sec$	$1-min$
$V^n\left(f_{\bar{D}_1(2)},X\right)$	0.9781	0.9727	0.9774	0.9727
$V^n\left(f_{\bar{D}_1(3)},X\right)$	0.9579	0.919	0.9556	0.9166
$V^n\left(f_{\bar{D}_1(4)},X\right)$	0.8746	0.8076	0.8956	0.8348
$1-\alpha=0.95$				
	Unfeasible CLT		Feasible CLT	
	$15-sec$	$1-min$	$15-sec$	$1-min$
$V^n\left(f_{\bar{D}_1(2)},X\right)$	0.9478	0.9501	0.9431	0.9470
$V^n\left(f_{\bar{D}_1(3)},X\right)$	0.919	0.887	0.9213	0.8753
$V^n\left(f_{\bar{D}_1(4)},X\right)$	0.8411	0.7757	0.8473	0.7998
$1-\alpha=0.90$				
	Unfeasible CLT		Feasible CLT	
	$15-sec$	$1-min$	$15-sec$	$1-min$
$V^n\left(f_{\bar{D}_1(2)},X\right)$	0.8917	0.9034	0.8933	0.8894
$V^n\left(f_{\bar{D}_1(3)},X\right)$	0.8637	0.8348	0.8629	0.8091
$V^n\left(f_{\bar{D}_1(4)},X\right)$	0.7975	0.7320	0.7850	0.75

Table 2.6: Simulation study: coverage of confidence intervals for realized return divergence measures. D_1: Kullback Leibler divergence; $\overline{D}_1(2)$, $\overline{D}_1(3)$, $\overline{D}_1(4)$: second, third and fourth order return divergence. Frequencies: 15 seconds and 1 minute. 2000 days of simulated data. Model: Table 2.1 mod II-d.

$1-\alpha=0.98$				
	Unfeasible CLT		Feasible CLT	
	$15-sec$	$1-min$	$15-sec$	$1-min$
$V^n\left(f_{\bar{D}_1(2)}, X\right)$	0.9765	0.9796	0.9781	0.9755
$V^n\left(f_{\bar{D}_1(3)}, X\right)$	0.9597	0.9007	0.9608	0.9292
$V^n\left(f_{\bar{D}_1(4)}, X\right)$	0.8524	0.6564	0.9048	0.8188
$1-\alpha=0.95$				
	Unfeasible CLT		Feasible CLT	
	$15-sec$	$1-min$	$15-sec$	$1-min$
$V^n\left(f_{\bar{D}_1(2)}, X\right)$	0.9491	0.9460	0.9486	0.9394
$V^n\left(f_{\bar{D}_1(3)}, X\right)$	0.9211	0.8524	0.9221	0.8961
$V^n\left(f_{\bar{D}_1(4)}, X\right)$	0.7877	0.5806	0.8559	0.7389
$1-\alpha=0.90$				
	Unfeasible CLT		Feasible CLT	
	$15-sec$	$1-min$	$15-sec$	$1-min$
$V^n\left(f_{\bar{D}_1(2)}, X\right)$	0.8982	0.8916	0.8982	0.8951
$V^n\left(f_{\bar{D}_1(3)}, X\right)$	0.8580	0.7852	0.8722	0.8407
$V^n\left(f_{\bar{D}_1(4)}, X\right)$	0.7231	0.4951	0.7781	0.6269

Table 2.7: Simulation study: coverage of confidence intervals for realized return divergence measures. D_1: Kullback Leibler divergence; $\overline{D}_1(2)$, $\overline{D}_1(3)$, $\overline{D}_1(4)$: second, third and fourth order return divergence. Frequencies: 15 seconds and 1 minute. 2000 days of simulated data. Model: Table 2.1 mod III-d.

$1-\alpha=0.98$				
	Unfeasible CLT		Feasible CLT	
	$15{-}sec$	$1{-}min$	$15{-}sec$	$1{-}min$
$V^n\left(f_{\bar{D}_1(2)}, X\right)$	0.9868	0.9644	0.9749	0.9631
$V^n\left(f_{\bar{D}_1(3)}, X\right)$	0.9578	0.9156	0.9472	0.9130
$V^n\left(f_{\bar{D}_1(4)}, X\right)$	0.9011	0.8405	0.9104	0.8695
$1-\alpha=0.95$				
	Unfeasible CLT		Feasible CLT	
	$15{-}sec$	$1{-}min$	$15{-}sec$	$1{-}min$
$V^n\left(f_{\bar{D}_1(2)}, X\right)$	0.9499	0.9328	0.9499	0.9301
$V^n\left(f_{\bar{D}_1(3)}, X\right)$	0.9156	0.8801	0.9196	0.8722
$V^n\left(f_{\bar{D}_1(4)}, X\right)$	0.8748	0.7997	0.8814	0.8287
$1-\alpha=0.90$				
	Unfeasible CLT		Feasible CLT	
	$15{-}sec$	$1{-}min$	$15{-}sec$	$1{-}min$
$V^n\left(f_{\bar{D}_1(2)}, X\right)$	0.9011	0.8906	0.8901	0.8919
$V^n\left(f_{\bar{D}_1(3)}, X\right)$	0.8577	0.8129	0.8577	0.8168
$V^n\left(f_{\bar{D}_1(4)}, X\right)$	0.8287	0.7562	0.8234	0.7575

Table 2.8: Simulation study: coverage of confidence intervals for realized return divergence measures. D_1: Kullback Leibler divergence; $\overline{D}_1(2)$, $\overline{D}_1(3)$, $\overline{D}_1(4)$: second, third and fourth order return divergence. Frequencies: 15 seconds and 1 minute. 2000 days of simulated data. Model: Table 2.1 mod I-j.

$1-\alpha=0.98$				
	Unfeasible CLT		Feasible CLT	
	$15-sec$	$1-min$	$15-sec$	$1-min$
$V^n\left(f_{\bar{D}_1(2)},X\right)$	0.9808	0.9846	0.9754	0.9762
$V^n\left(f_{\bar{D}_1(3)},X\right)$	0.9547	0.9209	0.9508	0.9048
$V^n\left(f_{\bar{D}_1(4)},X\right)$	0.8856	0.8142	0.8994	0.8426
$1-\alpha=0.95$				
	Unfeasible CLT		Feasible CLT	
	$15-sec$	$1-min$	$15-sec$	$1-min$
$V^n\left(f_{\bar{D}_1(2)},X\right)$	0.9547	0.9531	0.9478	0.9508
$V^n\left(f_{\bar{D}_1(3)},X\right)$	0.9240	0.8871	0.9178	0.8718
$V^n\left(f_{\bar{D}_1(4)},X\right)$	0.8503	0.7766	0.8595	0.8035
$1-\alpha=0.90$				
	Unfeasible CLT		Feasible CLT	
	$15-sec$	$1-min$	$15-sec$	$1-min$
$V^n\left(f_{\bar{D}_1(2)},X\right)$	0.9071	0.9109	0.8956	0.9056
$V^n\left(f_{\bar{D}_1(3)},X\right)$	0.8764	0.8326	0.8603	0.8227
$V^n\left(f_{\bar{D}_1(4)},X\right)$	0.8004	0.7283	0.7981	0.7513

Table 2.9: Simulation study: coverage of confidence intervals for realized return divergence measures. D_1: Kullback Leibler divergence; $\overline{D}_1(2)$, $\overline{D}_1(3)$, $\overline{D}_1(4)$: second, third and fourth order return divergence. Frequencies: 15 seconds and 1 minute. 2000 days of simulated data. Model: Table 2.1 mod II-j.

$1-\alpha=0.98$				
	Unfeasible CLT		Feasible CLT	
	$15-sec$	$1-min$	$15-sec$	$1-min$
$V^n\left(f_{\bar{D}_1(2)}, X\right)$	0.9816	0.9801	0.9775	0.9836
$V^n\left(f_{\bar{D}_1(3)}, X\right)$	0.9678	0.9061	0.9699	0.9377
$V^n\left(f_{\bar{D}_1(4)}, X\right)$	0.8679	0.7083	0.9219	0.8643
$1-\alpha=0.95$				
	Unfeasible CLT		Feasible CLT	
	$15-sec$	$1-min$	$15-sec$	$1-min$
$V^n\left(f_{\bar{D}_1(2)}, X\right)$	0.9484	0.9571	0.9510	0.9495
$V^n\left(f_{\bar{D}_1(3)}, X\right)$	0.9265	0.8618	0.9331	0.8969
$V^n\left(f_{\bar{D}_1(4)}, X\right)$	0.7995	0.6404	0.8760	0.7939
$1-\alpha=0.90$				
	Unfeasible CLT		Feasible CLT	
	$15-sec$	$1-min$	$15-sec$	$1-min$
$V^n\left(f_{\bar{D}_1(2)}, X\right)$	0.9107	0.9046	0.9041	0.9061
$V^n\left(f_{\bar{D}_1(3)}, X\right)$	0.8765	0.8001	0.8720	0.8439
$V^n\left(f_{\bar{D}_1(4)}, X\right)$	0.7496	0.5619	0.7929	0.6889

Table 2.10: Simulation study: coverage of confidence intervals for realized return divergence measures. D_1: Kullback Leibler divergence; $\overline{D}_1(2)$, $\overline{D}_1(3)$, $\overline{D}_1(4)$: second, third and fourth order return divergence. Frequencies: 15 seconds and 1 minute. 2000 days of simulated data. Model: Table 2.1 mod III-j.

$1-\alpha=0.98$				
	Unfeasible CLT		Feasible CLT	
	15−*sec*	1−*min*	15−*sec*	1−*min*
$V^n\left(f_{\bar{D}_1(2)}, X\right)$	0.9817	0.9765	0.9895	0.9843
$V^n\left(f_{\bar{D}_1(3)}, X\right)$	0.9438	0.8851	0.9516	0.8864
$V^n\left(f_{\bar{D}_1(4)}, X\right)$	0.8603	0.7924	0.8929	0.8407
$1-\alpha=0.95$				
	Unfeasible CLT		Feasible CLT	
	15−*sec*	1−*min*	15−*sec*	1−*min*
$V^n\left(f_{\bar{D}_1(2)}, X\right)$	0.9569	0.9503	0.9516	0.9569
$V^n\left(f_{\bar{D}_1(3)}, X\right)$	0.9099	0.8537	0.9125	0.8472
$V^n\left(f_{\bar{D}_1(4)}, X\right)$	0.8302	0.7637	0.8550	0.8041
$1-\alpha=0.90$				
	Unfeasible CLT		Feasible CLT	
	15−*sec*	1−*min*	15−*sec*	1−*min*
$V^n\left(f_{\bar{D}_1(2)}, X\right)$	0.9073	0.9046	0.9164	0.9046
$V^n\left(f_{\bar{D}_1(3)}, X\right)$	0.8563	0.7937	0.8616	0.7793
$V^n\left(f_{\bar{D}_1(4)}, X\right)$	0.7806	0.718	0.8015	0.7467

Table 2.11: Simulation study: coverage of confidence intervals for realized return divergence measures. D_1: Kullback Leibler divergence;$\overline{D}_1(2)$, $\overline{D}_1(3)$, $\overline{D}_1(4)$: second, third and fourth order return divergence. Frequencies: 15 seconds and 1 minute. 2000 days of simulated data. Model: Table 2.1 mod I-m.

$1-\alpha=0.98$				
	Unfeasible CLT		Feasible CLT	
	15−*sec*	1−*min*	15−*sec*	1−*min*
$V^n\left(f_{\bar{D}_1(2)},X\right)$	0.9792	0.9816	0.9777	0.9785
$V^n\left(f_{\bar{D}_1(3)},X\right)$	0.9434	0.8773	0.9355	0.8837
$V^n\left(f_{\bar{D}_1(4)},X\right)$	0.8041	0.7181	0.8375	0.7675
$1-\alpha=0.95$				
	Unfeasible CLT		Feasible CLT	
	15−*sec*	1−*min*	15−*sec*	1−*min*
$V^n\left(f_{\bar{D}_1(2)},X\right)$	0.9410	0.9474	0.9450	0.9418
$V^n\left(f_{\bar{D}_1(3)},X\right)$	0.8980	0.8296	0.8996	0.8343
$V^n\left(f_{\bar{D}_1(4)},X\right)$	0.7667	0.6799	0.7961	0.7261
$1-\alpha=0.90$				
	Unfeasible CLT		Feasible CLT	
	15−*sec*	1−*min*	15−*sec*	1−*min*
$V^n\left(f_{\bar{D}_1(2)},X\right)$	0.8941	0.8941	0.8893	0.8949
$V^n\left(f_{\bar{D}_1(3)},X\right)$	0.8415	0.7714	0.8399	0.7778
$V^n\left(f_{\bar{D}_1(4)},X\right)$	0.7213	0.6289	0.7364	0.6648

Table 2.12: Simulation study: coverage of confidence intervals for realized return divergence measures. D_1: Kullback Leibler divergence; $\overline{D}_1(2)$, $\overline{D}_1(3)$, $\overline{D}_1(4)$: second, third and fourth order return divergence. Frequencies: 15 seconds and 1 minute. 2000 days of simulated data. Model: Table 2.1 mod II-m.

$1-\alpha=0.98$				
	Unfeasible CLT		Feasible CLT	
	15−*sec*	1−*min*	15−*sec*	1−*min*
$V^n\left(f_{\bar{D}_1(2)}, X\right)$	0.9761	0.9787	0.9761	0.9787
$V^n\left(f_{\bar{D}_1(3)}, X\right)$	0.9401	0.8575	0.9432	0.8945
$V^n\left(f_{\bar{D}_1(4)}, X\right)$	0.7641	0.4994	0.8529	0.7003
$1-\alpha=0.95$				
	Unfeasible CLT		Feasible CLT	
	15−*sec*	1−*min*	15−*sec*	1−*min*
$V^n\left(f_{\bar{D}_1(2)}, X\right)$	0.9518	0.9492	0.9513	0.9492
$V^n\left(f_{\bar{D}_1(3)}, X\right)$	0.8995	0.7875	0.9046	0.8498
$V^n\left(f_{\bar{D}_1(4)}, X\right)$	0.6901	0.4122	0.7728	0.5826
$1-\alpha=0.90$				
	Unfeasible CLT		Feasible CLT	
	15−*sec*	1−*min*	15−*sec*	1−*min*
$V^n\left(f_{\bar{D}_1(2)}, X\right)$	0.9026	0.8869	0.8985	0.8930
$V^n\left(f_{\bar{D}_1(3)}, X\right)$	0.8367	0.7155	0.8529	0.7834
$V^n\left(f_{\bar{D}_1(4)}, X\right)$	0.6105	0.3336	0.6830	0.4229

Table 2.13: Simulation study: coverage of confidence intervals for realized return divergence measures. D_1: Kullback Leibler divergence; $\overline{D}_1(2)$, $\overline{D}_1(3)$, $\overline{D}_1(4)$: second, third and fourth order return divergence. Frequencies: 15 seconds and 1 minute. 2000 days of simulated data. Model: Table 2.1 mod III-m.

Model I-c				
	Unfeasible CLT		Feasible CLT	
	$15-sec$	$1-min$	$15-sec$	$1-min$
$V^n\left(f_{\bar{D}_1(3)}, X\right)$	0.9387	0.9165	0.9348	0.9009
$V^n\left(f_{\bar{D}_1(4)}, X\right)$	0.9478	0.9152	0.9374	0.9087
Model II-c				
	Unfeasible CLT		Feasible CLT	
	$15-sec$	$1-min$	$15-sec$	$1-min$
$V^n\left(f_{\bar{D}_1(3)}, X\right)$	0.9450	0.9267	0.9355	0.9132
$V^n\left(f_{\bar{D}_1(4)}, X\right)$	0.9442	0.9355	0.9426	0.9347
Model III-c				
	Unfeasible CLT		Feasible CLT	
	$15-sec$	$1-min$	$15-sec$	$1-min$
$V^n\left(f_{\bar{D}_1(3)}, X\right)$	0.9328	0.9221	0.9374	0.9359
$V^n\left(f_{\bar{D}_1(4)}, X\right)$	0.9354	0.917	0.9394	0.945

Table 2.14: Simulation study: coverage of 95% confidence intervals of adjusted CLT for realized return divergence measures. D_1: Kullback Leibler divergence; $\overline{D}_1(3)$, $\overline{D}_1(4)$: third and fourth order return divergence. Frequencies: 15 seconds and 1 minute. 2000 days of simulated data.

Model I-d				
	Unfeasible CLT		Feasible CLT	
	$15-sec$	$1-min$	$15-sec$	$1-min$
$V^n\left(f_{\bar{D}_1(3)}, X\right)$	0.9380	0.9169	0.9343	0.8909
$V^n\left(f_{\bar{D}_1(4)}, X\right)$	0.9417	0.9268	0.9417	0.8983
Model II-d				
	Unfeasible CLT		Feasible CLT	
	$15-sec$	$1-min$	$15-sec$	$1-min$
$V^n\left(f_{\bar{D}_1(3)}, X\right)$	0.9236	0.8940	0.919	0.8831
$V^n\left(f_{\bar{D}_1(4)}, X\right)$	0.9454	0.9026	0.9353	0.8987
Model III-d				
	Unfeasible CLT		Feasible CLT	
	$15-sec$	$1-min$	$15-sec$	$1-min$
$V^n\left(f_{\bar{D}_1(3)}, X\right)$	0.9297	0.8717	0.9328	0.9012
$V^n\left(f_{\bar{D}_1(4)}, X\right)$	0.9353	0.8636	0.9409	0.9323

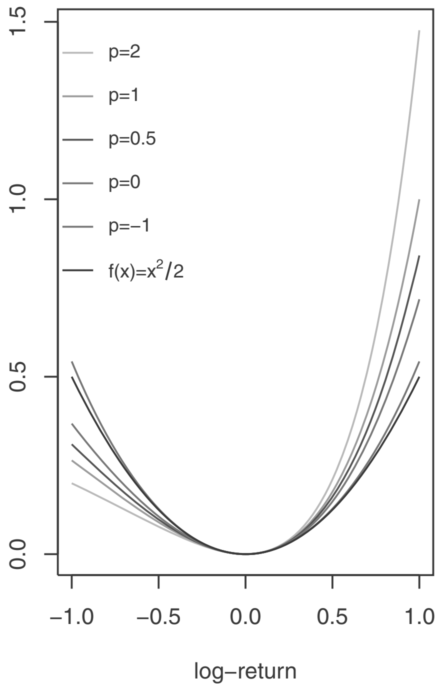
Table 2.15: Simulation study: coverage of 95% confidence intervals of adjusted CLT for realized return divergence measures. D_1: Kullback Leibler divergence; $\overline{D}_1(3)$, $\overline{D}_1(4)$: third and fourth order return divergence. Frequencies: 15 seconds and 1 minute. 2000 days of simulated data.

Model I-j				
	Unfeasible CLT		Feasible CLT	
	$15-sec$	$1-min$	$15-sec$	$1-min$
$V^n\left(f_{\bar{D}_1(3)}, X\right)$	0.9288	0.8959	0.9222	0.8761
$V^n\left(f_{\bar{D}_1(4)}, X\right)$	0.9459	0.9156	0.9367	0.9077
Model II-j				
	Unfeasible CLT		Feasible CLT	
	$15-sec$	$1-min$	$15-sec$	$1-min$
$V^n\left(f_{\bar{D}_1(3)}, X\right)$	0.9286	0.8902	0.9224	0.8864
$V^n\left(f_{\bar{D}_1(4)}, X\right)$	0.9493	0.9079	0.9370	0.8887
Model III-j				
	Unfeasible CLT		Feasible CLT	
	$15-sec$	$1-min$	$15-sec$	$1-min$
$V^n\left(f_{\bar{D}_1(3)}, X\right)$	0.9291	0.8765	0.9352	0.9026
$V^n\left(f_{\bar{D}_1(4)}, X\right)$	0.9301	0.8720	0.9464	0.9306

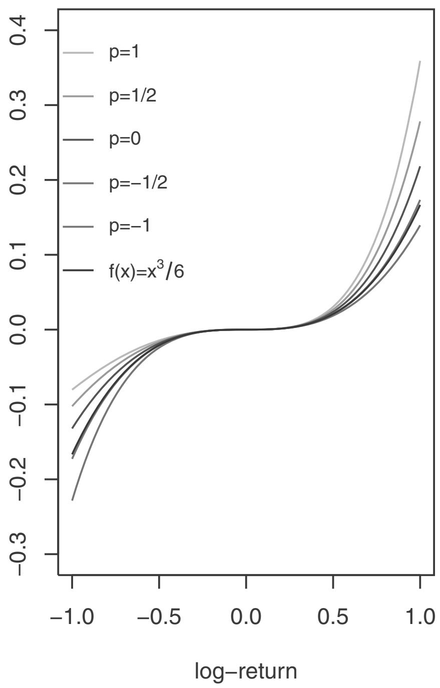
Table 2.16: Simulation study: coverage of 95% confidence intervals of adjusted CLT for realized return divergence measures. D_1: Kullback Leibler divergence; $\overline{D}_1(3)$, $\overline{D}_1(4)$: third and fourth order return divergence. Frequencies: 15 seconds and 1 minute. 2000 days of simulated data.

Model I-m				
	Unfeasible CLT		Feasible CLT	
	$15-sec$	$1-min$	$15-sec$	$1-min$
$V^n\left(f_{\bar{D}_1(3)}, X\right)$	0.9112	0.8563	0.9177	0.8629
$V^n\left(f_{\bar{D}_1(4)}, X\right)$	0.9347	0.8851	0.9347	0.9033
Model II-m				
	Unfeasible CLT		Feasible CLT	
	$15-sec$	$1-min$	$15-sec$	$1-min$
$V^n\left(f_{\bar{D}_1(3)}, X\right)$	0.9124	0.8566	0.9116	0.8503
$V^n\left(f_{\bar{D}_1(4)}, X\right)$	0.9299	0.8781	0.9307	0.8726
Model III-m				
	Unfeasible CLT		Feasible CLT	
	$15-sec$	$1-min$	$15-sec$	$1-min$
$V^n\left(f_{\bar{D}_1(3)}, X\right)$	0.9148	0.8321	0.9092	0.8620
$V^n\left(f_{\bar{D}_1(4)}, X\right)$	0.9198	0.8483	0.9330	0.9127

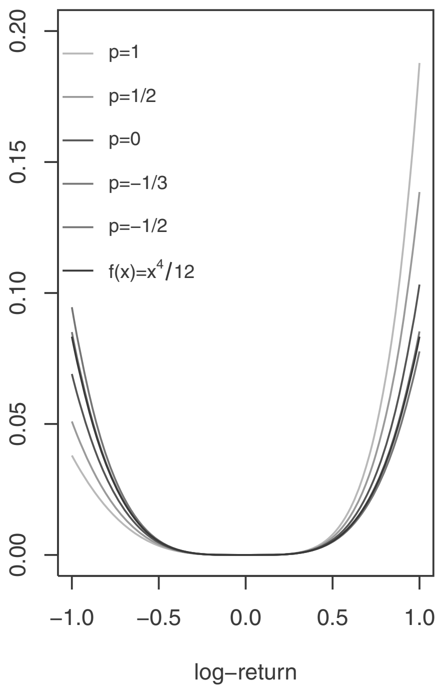
Table 2.17: Simulation study: coverage of 95% confidence intervals of adjusted CLT for realized return divergence measures. D_1: Kullback Leibler divergence; $\overline{D}_1(3)$, $\overline{D}_1(4)$: third and fourth order return divergence. Frequencies: 15 seconds and 1 minute. 2000 days of simulated data.



(a) Second Order Divergence, i.e., $f_{\bar{D}_p(2)}$



(b) Third Order Divergence, i.e., $f_{\bar{D}_p(3)}$



(c) Fourth Order Divergence, i.e., $f_{\bar{D}_p(4)}$

Figure 2.1: m–order $(m = 2, 3, 4)$ return power divergences

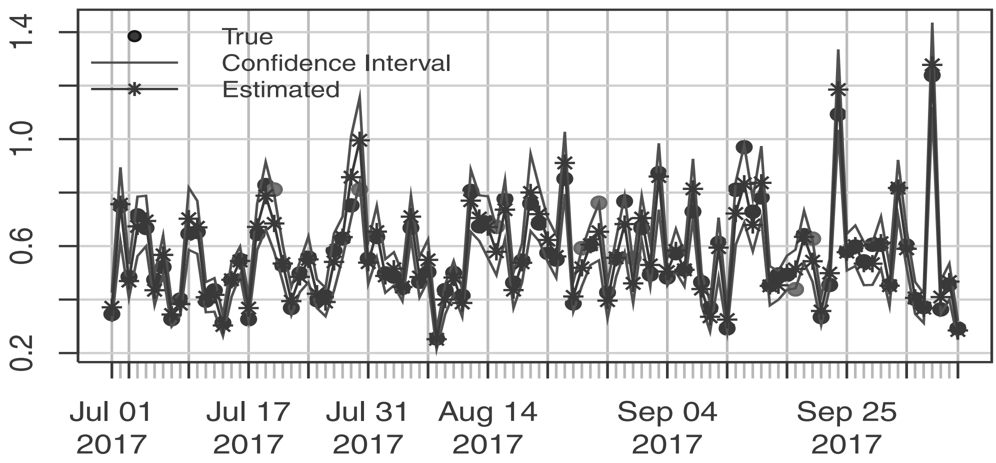


Figure 2.2: 95% confidence interval for daily realized second order return Kullback Leibler divergence. Model: Table 2.1 mod II-d (the case without co-jump). The sampling frequency=1 minute.

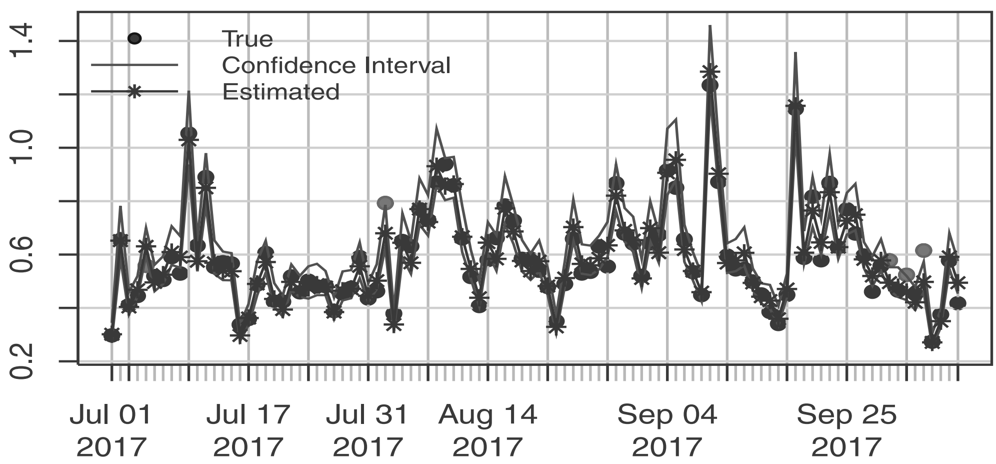


Figure 2.3: 95% confidence interval for daily realized second order return Kullback Leibler divergence. Model: Table 2.1 mod II-j (the case with co-jump). The sampling frequency=1 minute.

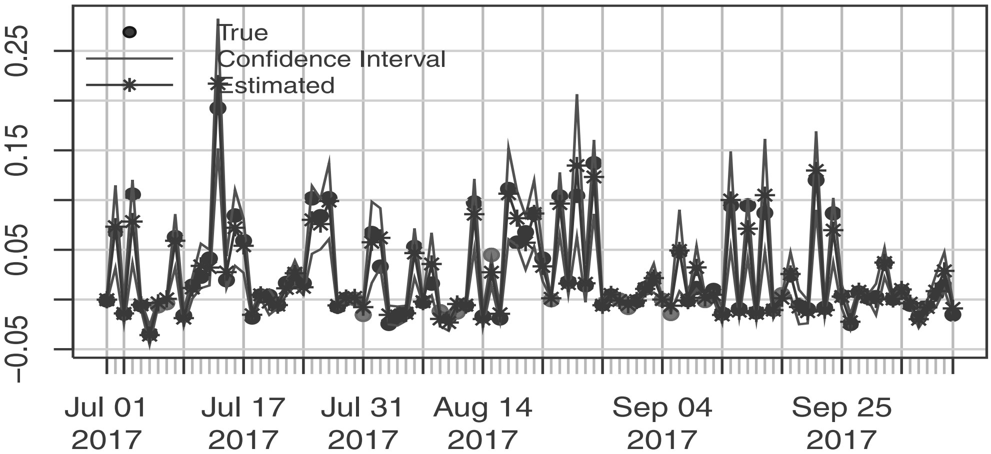


Figure 2.4: 95% confidence interval for daily realized third order return Kullback Leibler divergence. Model: Table 2.1 mod II-d (the case without co-jump). The sampling frequency= 1 minute.

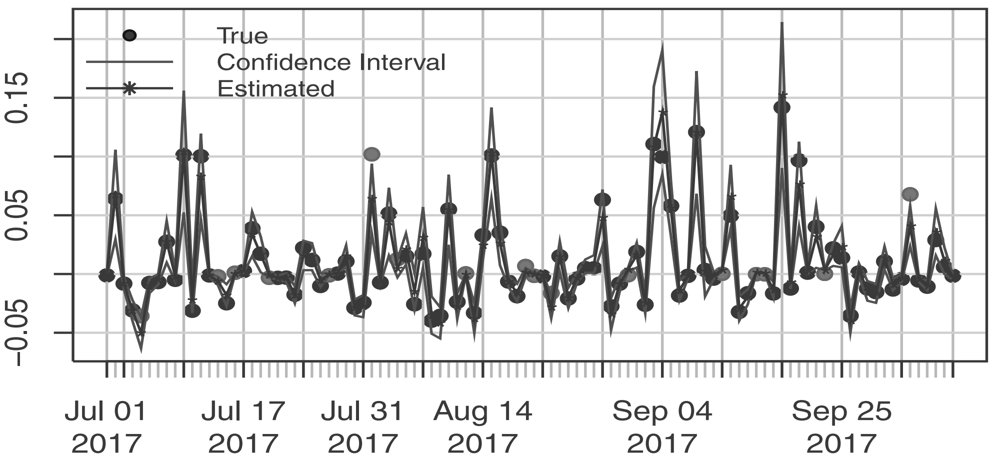


Figure 2.5: 95% confidence interval for daily realized third order return Kullback Leibler divergence. Model: Table 2.1 mod II-j (the case with co-jump). The sampling frequency= 1 minute.

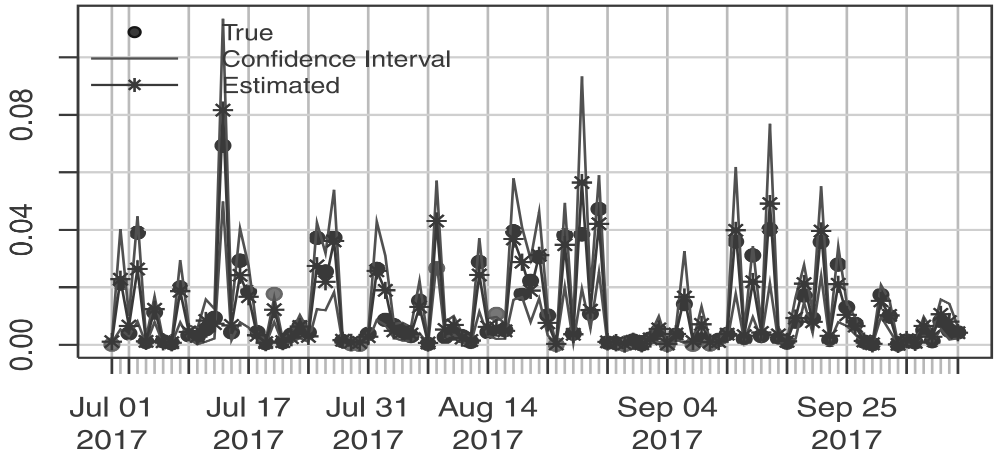


Figure 2.6: 95% confidence interval for daily realized fourth order return Kullback Leibler divergence. Model: Table 2.1 mod II-d (the case without co-jump). The sampling frequency= 1 minute.

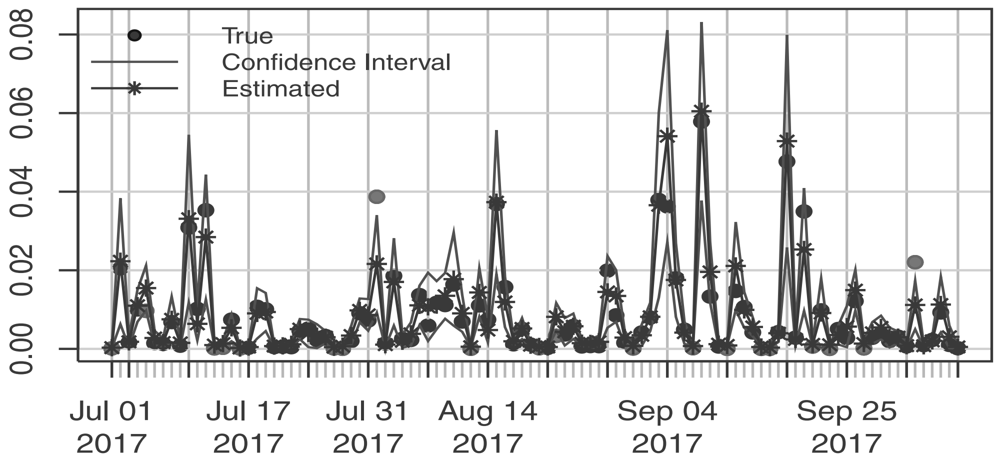


Figure 2.7: 95% confidence interval for daily realized fourth order return Kullback Leibler divergence. Model: Table 2.1 mod II-j (the case with co-jump). The sampling frequency= 1 minute.

Chapter 3

The econometrics of realized price divergence

3.1 Introduction

This paper studies the econometric properties of realized price divergence and its decomposition into the leading contributions of realized risks of arbitrary integer order $m \geq 2$. Using Bregman [1967] divergences, Schneider and Trojani [2015a] introduce realized price power divergence based on a parsimonious power parametrization that provides a unified way of trading realized risks of essentially arbitrary order using static delta-hedged option portfolios. Schneider and Trojani [2015b] extend that approach to incomplete option markets with a finite set of traded options and characterize the empirical properties of realized price divergence.

From a statistical perspective, realized price divergence simply arises from particular (unweighted) sums of the increments of a convex function of asset prices over time. The value of each increment depends of the scale of the underlying semimartingale at each point in time, which makes realized price divergence dependent on both the realized path of returns and the levels of asset prices. Therefore, the econometric properties of price divergence need to be studied with tools and methods that can accommodate this path-dependence. We address this task in this paper.

Under weak semimartingale assumptions, we interpret realized price divergence as a finite sample estimate of an underlying continuous-time price divergence, based on observations that are generated on a discrete time grid. In this way, we can quantify the estimation uncertainty rooted in conceptually distinct measures of realized price divergence. Scale dependent price divergence is potentially even more state-dependent than realized return divergence. Therefore, it is natural to expect that the estimation uncertainty associated with realized price divergence is at least similarly highly time-varying and potentially quite different from the one associated with return divergence.

We characterize the asymptotic properties of price divergence of arbitrary order $m \geq 2$, by deriving Laws of Large Numbers and Central Limit Theorems that explicitly reflect the underlying scale dependence. These results give rise to a unified approach for the statistical measurement of second- and higher-order realized risks induced by linear combinations of realized price divergences that are tradeable with static option portfolios. We consider in a second step feasible asymptotic confidence intervals that are computable in applications and study their accuracy in a number of Monte Carlo simulations for a general two-factor stochastic volatility model with co-jumps in returns and volatility. Overall, we find that the estimation uncertainty on realized price divergence of order two and three can be successfully incorporated in applications where higher-order realized risk measuring, forecasting and trading is relevant.

The rest of the paper is organized as follows. Section 2 introduces the realized price divergence, while Section 3 develops the relevant laws of large numbers. Stable functional central limit theorem are introduced in in Section 4, while the feasible asymptotic inference is addressed in Section 5. The finite sample reliability of our asymptotic approximations is investigated in the Monte Carlo study of Section 6. Section 7 concludes. Proofs, tables and figures are collected in the Appendix.

3.2 Realized price divergence

Following the same power divergence approach of Sections 2.2.1–2.2.3, we recall the following Bregman power divergence applied to price process Y, normalized to $Y_0 = 1$ without loss of generality:

$$D_p\left(Y_t, Y_s\right) = F_{D_p}\left(X_s, X_t - X_s\right) \,, \tag{3.1}$$

where

$$F_{D_p}\left(y, x\right) := e^{py} f_{\overline{D}_p}\left(x\right) \ ; \ f_{\overline{D}_p}\left(x\right) := \frac{(e^{px} - 1) - p(e^x - 1)}{p(p-1)} \,. \tag{3.2}$$

Note that

$$F_{D_p}\left(y, x\right) = e^{py}\left(\frac{x^2}{2} + \frac{1+p}{3!}x^3 + \frac{1+p(p+1)}{4!}x^4 + O(|x|^5)\right) \,, \tag{3.3}$$

for any $p \in \mathbb{R}$. Therefore, as expected, all realized moments of future log returns contribute to the power divergence, but the leading contribution comes from the second realized moments of returns.

In contrast to the return divergence $f_{\overline{D}_p}(X_t - X_s)$, the price divergence $F_{D_p}(X_s, X_t - X_s)$ depends on the scale of Y_s. Similar to the case of return divergence, derivatives of the price power divergence with respect to the power parameter isolate the divergence contribution of the higher realized moments of returns. This motivates the next definition.

Definition 3.2.1. *For any power parameter $p_0 \in \mathbb{R}$ and integer $m \geq 2$, the p_0–th m-th order price divergence is defined by:*

$$D_{p_0}(m)\left(Y_t, Y_s\right) := \left.\frac{\partial^{(m-2)} D_p\left(Y_t, Y_s\right)}{\partial p^{(m-2)}}\right|_{p=p_0} := F_{D_{p_0}(m)}\left(X_s, X_t - X_s\right), \tag{3.4}$$

where

$$F_{D_{p_0}(m)}\left(y, x\right) := \left.\frac{\partial^{(m-2)} F_{D_p}\left(y, x\right)}{\partial p^{(m-2)}}\right|_{p=p_0}. \tag{3.5}$$

The realized m–th order price divergence for parameter p_0 is defined by:

$$V^n(F_{D_{p_0}(m)}, X)_t := \sum_{i=1}^{[t/\Delta_n]} D_{p_0}(m)(Y_{i\Delta_n}, Y_{(i-1)\Delta_n}) = \sum_{i=1}^{[t/\Delta_n]} F_{D_{p_0}(m)}(X_{(i-1)\Delta_n}, \Delta_i^n X). \tag{3.6}$$

3.3 Laws of Large Numbers

We are now ready to formulate our Laws of Large Numbers for realized price divergence, using the general semimartingale setting for process X introduced in Section 2.2.4; see again, e.g., Jacod [2008]. These Laws of Large Numbers are obtained by applying the results in Jacod and Protter [2012]; The proof is given in Appendix A.

Theorem 3.3.1 (Laws of Large Numbers)**.** *Let Assumption 2.2.5 be satisfied. We then have for any $p \in \mathbb{R}$ and any integer divergence order $m \geq 2$ the following Skorokhod convergence in probability, as $\Delta_n \to 0$:*

$$V^n(F_{D_p(m)}, X)_t \overset{\mathbb{P}}{\Longrightarrow} V(F_{D_p(m)}, X)_t := \frac{1}{2}\int_0^t \partial_x^2 F_{D_p(m)}(X_s, 0)\sigma_s^2 ds + \sum_{s\leq t} F_{D_p(m)}\left(X_{s-}, \Delta X_s\right). \tag{3.7}$$

Equivalently, in term of price process Y, it follows:

$$\sum_{i=1}^{[t/\Delta_n]} D_p(m)\left(Y_{i\Delta_n}, Y_{(i-1)\Delta_n}\right) \overset{\mathbb{P}}{\Longrightarrow} \frac{1}{2}\int_0^t (\log Y_s)^{(m-2)} Y_s^p \sigma_s^2 ds + \sum_{s\leq t} D_p(m)\left(Y_s, Y_{s-}\right). \tag{3.8}$$

Finally, for any $t > 0$ that is not a fixed point of discontinuity of process X, convergence in probability applies:

$$V^n(F_{D_p(m)}, X)_t \overset{\mathbb{P}}{\to} V(F_{D_p(m)}, X)_t. \tag{3.9}$$

Theorem 3.3.1 provides the foundations for the theoretical objects estimated by realized price divergence in semimartingale settings. The limit realized power price divergence of order m is the sum of two $p-$dependent processes in equation (3.8). The first process is given by an integrated weighted quadratic variation of continuous price changes. Note that for any $p \in \mathbb{R}$ and for any even order m the weight is positive. For odd orders, the weight is asymmetric, as it is positive (negative) whenever $X_s > 0$ ($X_s < 0$). For positive powers p the weight is also always monotonically increasing in the region of positive total log returns ($X_s > 0$). The second process is the realized $p-$th power divergence of order m of price jumps. In general, realized price divergences are scale dependent. The single exception is the $0-$th second-order price divergence, which coincides with the $0-$th second order return divergence, due to the scale-independence of the divergence induced by a logarithmic generating function: $D_0(2)(Y_t, Y_s) = -\ln(Y_t/Y_s) + (Y_t/Y_s - 1)$.

3.4 Functional Central Limit Theorem

In this section, we adopt the probabilistic setting of Notation 2.4.1 and formulate our Central Limit Theorems for realized price divergence of arbitrary order. This requires an extension of our previous CLTs for (scale-independent) return divergence. Diop [2012] proves a CLT associated with variation functionals $V^n(F, X)_t$ under an Itô semimartingale assumption, when function $F(x, y)$ is approximately $o(|x|^3)$ in a neighborhood of zero, but this setting does not include the price divergence function $F_{D_{p(m)}}$ for arbitrary integer order $m \geq 2$. Therefore, we extend that result to obtain a CLT for realized price divergences of arbitrary order; see also Dohnal [1987] and Genon-Catalot and Jacod [1993] for relevant related references.

Theorem 3.4.1 (Central Limit Theorem for realized m–th order price divergence, $m \geq 2$)**.** *Let Assumption (L) be satisfied. We then have the following stable convergence in law for the processes as $\Delta_n \to 0$:*

$$\frac{1}{\sqrt{\Delta_n}}\left(V^n(F_{D_p(m)}, X)_t - \frac{1}{2}\int_0^t \partial_x^2 F_{D_p(m)}(X_s, 0)\sigma_s^2 ds - \sum_{s\leq\Delta_n[t/\Delta_n]} F_{D_p(m)}\left(X_{s-}, \Delta X_s\right)\right) \stackrel{\mathcal{L}-s}{\Longrightarrow} Z_t^{D_p(m)}, \tag{3.10}$$

where

$$\begin{aligned} Z_t^{D_p(m)} :=& \frac{1}{\sqrt{2}}\int_0^t \partial_x^2 F_{D_p(m)}(X_s, 0)\sigma_s^2 d\overline{W}_s \\ &+ \sum_{n\geq 1}\left[\left(\sqrt{\kappa_n}\sigma_{T_n-}\psi_{n-} + \sqrt{1-\kappa_n}\sigma_{T_n}\psi_{n+}\right)\partial_x F_{D_p(m)}\left(X_{T_n-}, \triangle X_{T_n}\right)\right. \\ &\left. -\sqrt{\kappa_n}\sigma_{T_n-}\psi_{n-}\partial_y F_{D_p(m)}\left(X_{T_n-}, \triangle X_{T_n}\right)\right] 1_{\{T_n\leq t\}}. \end{aligned} \tag{3.11}$$

For any fixed time $t > 0$, the following stable convergence in law for random variables holds:

$$\frac{1}{\sqrt{\Delta_n}}\left(V^n\left(F_{D_p(m)}, X\right)_t - V\left(F_{D_p(m)}, X\right)_t\right) \stackrel{\mathcal{L}-s}{\longrightarrow} Z_t^{D_p(m)} \tag{3.12}$$

As in our previous findings, the $\mathcal{F}$–conditional law of stable limit process $Z_t^{D_p(m)}$ in Theorem 3.4.1 is independent of the choice of the weakly enumerating sequence of stopping times T_n in Notation 2.4.1. Conditional on $\mathcal{F}$, $Z_t^{D_p(m)}$ is the sum of a continuous process and a pure jump process, both with zero-mean independent increments, implying a conditional second moment given by:

$$\begin{aligned}\Lambda_t^{D_p(m)} &:= \tilde{\mathbb{E}}\left((Z_t^{D_p(m)})^2|\mathcal{F}\right)\\ &= \frac{1}{2}\int_0^t \partial_x^2 F_{D_p(m)}(X_s,0)^2\sigma_s^4 ds + \frac{1}{2}\sum_{s\le t}\sigma_s^2\partial_x F_{D_p(m)}\left(X_{s-},\triangle X_s\right)^2\\ &\quad +\frac{1}{2}\sum_{s\le t}\sigma_{s-}^2\left(\partial_x F_{D_p(m)}\left(X_{s-},\triangle X_s\right)-\partial_y F_{D_p(m)}\left(X_{s-},\triangle X_s\right)\right)^2 .\end{aligned} \tag{3.13}$$

Similar to the case of realized return divergence, the distribution of $Z_t^{D_p(m)}$ depends on nuisance parameters that need to be estimated, such as jump times, spot volatilities at jump times and a weighted integrated quarticity. Therefore, Central Limit Theorem 3.4.1 is unfeasible for practical purposes. Moreover, in contrast to the CLTs for realized return divergence, the conditional distribution of limit random variable $Z_t^{D_p(m)}$ in equation (3.11) is not available in closed form, even when the underlying semimartingale X does not include co-jumps in returns and volatility. This feature is due to the last term appearing on the right-hand side of equation (3.11). Therefore, to develop a feasible inference for realized price divergence in general, we need to rely in the next section on a suitable Monte Carlo simulation approach.

3.5 Asymptotic inference

To produce feasible confidence intervals for realized price divergence, we first need a consistent estimator of the asymptotic variance $\Lambda_t^{D_p(m)}$ in equation (3.13), which is a sum of three distinct components. In a second step, we can then develop a suitable Monte Carlo simulation-based inference approach.

3.5.1 Asymptotic variance estimation

The first component of $\Lambda_t^{D_p(m)}$ is proportional to a weighted quarticity of log returns, which is estimated using following truncation-based estimator:

$$\hat{\Lambda}_1^{D_p(m)}(\Delta_n,\nu_n)_t := \frac{1}{6\Delta_n}\sum_{i=1}^{[t/\Delta_n]}\partial_x^2 F_{D_p(m)}(X_{(i-1)\Delta_n},0)^2(\Delta_i^n X)^4 1_{\{|\Delta_i^n X|\le\nu_n\}} . \tag{3.14}$$

Under Assumption (H–r) ($r\in[0,2)$) and for a truncation threshold $\nu_n=\tilde{\beta}\Delta_n^{\tilde{\varpi}}$, where $\frac{1}{(4-r)}\le\tilde{\varpi}<\frac{1}{2}$ and $\tilde{\beta}>0$, consistency follows (see Jacod and Protter [2012], Theorem 9.2.1):

$$\hat{\Lambda}_1^{D_p(m)}(\Delta_n,\nu_n)_t \stackrel{u.c.p}{\Longrightarrow} \frac{1}{2}\int_0^t \partial_x^2 F_{D_p(m)}(X_s,0)^2\sigma_s^4 ds , \tag{3.15}$$

as $\Delta_n \to 0$. The second and third components of $\Lambda_t^{D_p(m)}$ in equation (3.13) are generated by jump price divergence and depend on spot volatilities at jump times, which can be consistently estimated using the spot volatility estimators in equation (2.48) of Section 2.5.2. Therefore, we estimate these two components of $\Lambda_t^{D_p(m)}$ with following estimators:

$$\hat{\Lambda}_2^{D_p(m)}(\Delta_n, u_n)_t \;:=\; \frac{1}{2}\sum_{i=1+k_n}^{[t/\Delta_n]-k_n} \partial_x F_{D_p(m)}(X_{(i-1)\Delta_n}, \Delta_i^n X)^2 1_{\{|\Delta_i^n X|>u_n\}} \hat{\sigma}^2(u_n)_{(i\Delta_n)}\;,$$
$$\hat{\Lambda}_3^{D_p(m)}(\Delta_n, u_n)_t \;:=\; \frac{1}{2}\sum_{i=1+k_n}^{[t/\Delta_n]-k_n} (\partial_x F_{D_p(m)}(X_{(i-1)\Delta_n}, \Delta_i^n X) - \partial_y F_{D_p(m)}(X_{(i-1)\Delta_n}, \Delta_i^n X))^2 \times 1_{\{|\Delta_i^n X|>u_n\}} \hat{\sigma}^2(u_n)^-_{(i\Delta_n)}\;.$$

Under Assumption (H–r) ($r \in [0,2)$) and for the same parameters u_n, k_n as in limit (2.48), consistency applies (see, Jacod and Protter [2012], chapter 9):

$$\hat{\Lambda}_2^{D_p(m)}(\Delta_n, u_n)_t \xrightarrow{\mathbb{P}} \frac{1}{2}\sum_{s\le t} \sigma_s^2 \partial_x F_{D_p(m)}\left(X_{s-}, \triangle X_s\right)^2\;,$$
$$\hat{\Lambda}_3^{D_p(m)}(\Delta_n, u_n)_t \xrightarrow{\mathbb{P}} \frac{1}{2}\sum_{s\le t} \sigma_{s-}^2 \left(\partial_x F_{D_p(m)}\left(X_{s-}, \triangle X_s\right) - \partial_y F_{D_p(m)}\left(X_{s-}, \triangle X_s\right)\right)^2\;,$$

as $\Delta_n \to 0$. As a consequence, estimator

$$\hat{\Lambda}^{D_p(m)}(\Delta_n, \nu_n, u_n)_t := \hat{\Lambda}_1^{D_p(m)}(\Delta_n, \nu_n)_t + \hat{\Lambda}_2^{D_p(m)}(\Delta_n, u_n)_t + \hat{\Lambda}_3^{D_p(m)}(\Delta_n, u_n)_t\;, \tag{3.16}$$

is a consistent estimator of the asymptotic variance $\Lambda_t^{D_p(m)}$ in equation (3.13), under the above conditions for the consistency of $\hat{\Lambda}_i^{D_p(m)}(\Delta_n, \nu_n)_t$ ($i = 1, 2, 3$).

3.5.2 Feasible inference

Combining the conditions for the unfeasible stable Central Limit Theorem 3.4.1 and the conditions for a consistent estimation of the asymptotic variance of realized price divergence, we obtain the following feasible stable CLT.

Theorem 3.5.1. *Let Assumption (L) and Assumption (H–r) be satisfied for some $r \in [0,2)$. Moreover, fix a threshold choice $\nu_n = \tilde{\beta}\Delta_n^{\tilde{\varpi}}$, where $\frac{1}{(4-r)} \le \tilde{\varpi} < \frac{1}{2}$ and $\tilde{\beta} > 0$, and parameters u_n and k_n as in limit (2.48). For any $A \in \mathcal{F}$ such that $\mathbb{P}(A) > 0$ and $A \subset \{\omega : \Lambda_t^{D_p(m)}(\omega) > 0\}$ it then follows as $\Delta_n \to 0$:*

$$\mathcal{L}\left(\frac{1}{\sqrt{\Delta_n}}\left(\frac{V^n(F_{D_p(m)}, X)_t - V(F_{D_p(m)}, X)_t}{\sqrt{\hat{\Lambda}_t^{D_p(m)}}}\right)\middle| A\right) \longrightarrow \mathcal{L}_A := \mathcal{L}\left(\frac{Z_t^{D_p(m)}}{\sqrt{\Lambda_t^{D_p(m)}}}\middle| A\right)\,.$$

In Theorem 3.5.1, the form of the conditional law $\mathcal{L}_A$ is unknown, i.e., $\mathcal{L}_A$ is an infinite dimensional nuisance parameter for the standardazed asymptotic distribution of realized price divergence. Therefore, in order to estimate the quantiles of the conditional law of random variable $Z_t^{D_p(m)}$, we extend the Monte Carlo simulation approach of Section 2.5.4 to the case of realized price divergence.

We borrow from Theorem 3.4.1 and recognize that for any sequence of stopping times (T_n) exhausting the jumps of X random variable $Z_t^{D_p(m)}$ is identical in law to random variable:

$$\begin{aligned}\tilde{Z}_t^{D_p(m)} =& \sqrt{\frac{1}{2}\int_0^t \partial_x^2 F_{D_p(m)}(X_s,0)^2\sigma_s^4 ds}\ \psi \\ &+ \sum_{n\geq 1}\left(\left(\sqrt{\kappa_n}\sigma_{T_n-}\psi_{n-} + \sqrt{1-\kappa_n}\sigma_{T_n}\psi_{n+}\right)\partial_x F_{D_p(m)}\left(X_{T_n-}, \triangle X_{T_n}\right)\right. \\ &\left. - \sqrt{\kappa_n}\sigma_{T_n-}\psi_{n-}\partial_y F_{D_p(m)}\left(X_{T_n-}, \triangle X_{T_n}\right)\right) 1_{\{T_n\leq t\}}\ , \end{aligned} \tag{3.17}$$

where ψ is a standard normal variable independent of variables $(k_n, \psi_{n-}, \psi_{n+})$ in Theorem 3.4.1.[1] Note that the distribution of $\tilde{Z}_t^{D_p(m)}$ depends on nuisance parameters that can be consistently estimated with the estimators introduced in Section 3.5.1. We exploit this insight to motivate following Monte Carlo technique for the estimation of asymptotic confidence intervals for realized price divergence.

1. First, we fix a threshold $u_n > 0$ and let $i_1, i_2, \ldots, i_{r(n)}$ be the sequence of jump times of process X, for jumps prior to t having absolute size larger than u_n. These jumps are denoted by $\Delta_{i_1}^n X, \Delta_{i_2}^n X, \ldots, \Delta_{i_r(n)}^n X$. In this way, we introduce for $n \in \mathbb{N}$ the random variables:

$$\begin{aligned}\hat{Z}_t^{D_p(m)}(n) =& \sqrt{\hat{\Lambda}_1^{D_p(m)}(\Delta_n, \nu_n)_t}\ \psi \\ &+ \sum_{q=1}^{r(n)}\left(\sqrt{\kappa_n\hat{\sigma}^2(u_n)^-_{i_q\Delta_n}}\,\psi_{q-} + \sqrt{(1-\kappa_n)\hat{\sigma}^2(u_n)_{i_q\Delta_n}}\,\psi_{q+}\right) \\ &\qquad \times \partial_x F_{D_p(m)}\left(X_{(i_q-1)\Delta_n}, \Delta_{i_q}^n X\right) \\ &- \sum_{q=1}^{r(n)} \partial_y F_{D_p(m)}\left(X_{(i_q-1)\Delta_n}, \Delta_{i_q}^n X\right)\sqrt{\kappa_n\hat{\sigma}^2(u_n)^-_{i_q\Delta_n}}\,\psi_{q-}\end{aligned} \tag{3.18}$$

where threshold choices u_n and ν_n are those ensuring the consistency of estimators $\hat{\Lambda}_i^{D_p(m)}(\Delta_n, \nu_n)_t$ $(i = 1, 2, 3)$ in Section 3.5.1. Intuitively, under appropriate conditions the $\mathcal{F}$-conditional law of $\tilde{Z}_t^{D_p(m)}$ can be consistently estimated by the conditional law of $\hat{Z}_t^{D_p(m)}(n)$, that is the $\mathcal{F}$-conditional law of $\hat{Z}_t^{D_p(m)}(n)$ should converge in probability to the conditional law of $\tilde{Z}_t^{D_p(m)}$ as $n \to \infty$.

2. Second, we compute the quantiles of the law of $\hat{Z}_t^{D_p(m)}(n)$ using a parametric bootstrap procedure. To this end, we simulate N_n independent realizations of random

[1]See again Notation 2.4.1 for the definition of sequence $(k_n, \psi_{n-}, \psi_{n+})$.

vectors $(\psi, k_1, \ldots, k_{r(n)}, \psi_{1-}, \ldots, \psi_{r(n)-}, \psi_{1+}, \ldots, \psi_{r(n)+})$ and obtain a sequence of N_n independent realizations $\{\hat{Z}_t^{D_p(m),1}(n), \ldots, \hat{Z}_t^{D_p(m),N_n}(n)\}$ of $\hat{Z}_t^{D_p(m)}(n)$. We then compute the $\alpha/2-$ and the $(1-\alpha/2)-$quantiles of the empirical distribution of $\{\hat{Z}_t^{D_p(m),1}(n), \ldots, \hat{Z}_t^{D_p(m),N_n}(n)\}$, denoted by $\Upsilon_n^{\alpha/2}$ and $\Upsilon_n^{1-\alpha/2}$, respectively.

3. Finally, we compute the following $(1-\alpha)-$confidence interval for $m-$th order realized price divergence, defined by:

$$\mathcal{I}_n = \left[V^n(F_{D_p(m)}, X)_t - \sqrt{\Delta_n}\Upsilon_n^{\alpha/2},\ V^n(F_{D_p(m)}, X)_t + \sqrt{\Delta_n}\Upsilon_n^{1-\alpha/2}\right] . \tag{3.19}$$

We finally borrow from Aït-Sahalia and Jacod [2014], Theorem B.11, and make use of the following assumptions to obtain in the next theorem Monte Carlo confidence intervals with correct asymptotic level.

Theorem 3.5.2. *Let Assumptions (L) and (H–r) be satisfied for some* $r \in [0, 1)$. *Moreover, fix the threshold choices* $u_n = \beta\Delta_n^{\varpi}$ *and* $\nu_n = \tilde{\beta}\Delta_n^{\tilde{\varpi}}$*, where* $\varpi \in (0, \frac{1}{2})$ *and* $\frac{1}{(4-r)} \leq \tilde{\varpi} < \frac{1}{2}$. *It then follows, whenever* $N_n \to \infty$ *as* $n \to \infty$*:*

$$\lim_{n\to\infty} \tilde{\mathbb{P}}(V(F_{D_p(m)}, X)_t \notin \mathcal{I}_n | A) = \alpha \ , \tag{3.20}$$

for any set $A \in \mathcal{F}$ *such that* $\mathbb{P}(A) > 0$ *and* $A \subset \{\Lambda_t^{D_p(m)} > 0\}$.

3.6 Simulation study

We study the finite sample properties of realized price divergence and our asymptotic inference approach, focusing for brevity on a power parameter $p = 1$ (Kullback Leibler divergence) and on divergences of order $m = 2, 3$, which are associated with realized second- and third-moment price risk. The Monte Carlo simulation setting is identical to the one of Section 2.6, which is based on stochastic volatility model (2.64) and the parameter choices in Table 2.1, for models of class -c, -d, -j and -m, respectively. Figure 3.1– 3.4 illustrate for the different models in Table 2.1 the time series properties of 95%−confidence intervals for daily realized price divergence of order two and three, based on sampling frequency of 1 minute. Red points in these plots highlight observations outside of the daily 95%−confidence interval.

Tables 3.1-3.12 report the Monte Carlo empirical coverage of 90%, 95% and 98% confidence intervals for daily realized price divergence of order $m = 2, 3$, using sampling frequencies Δ_n of 15 seconds and 1 minute. We find that for all models in Table 2.1 and all sampling frequencies both Theorem 3.4.1 and Theorem 3.5.2 imply a quite accurate inference. In contrast to the evidence obtained using the unadjusted CLTs for realized return divergence, the evidence obtained using the confidence intervals for third-order realized price diverge also appears less dependent on the sampling frequency and the jump structure of the simulated model. This feature might be a consequence of the fact that higher-order price divergence still depends on continuous price variations, which can be estimated more accurately using all sample observations.

3.7 Conclusion

In this chapter, we presented laws of large numbers and stable functional central limit theorems for realized (scale-dependent) price divergence under general semimartingale conditions. These laws of large numbers provide a proper definition of the hidden realized risk estimated by second and higher-order realized price divergences, together with an identification of the risk contributions generated by continuous and discontinuous semimartingale components. The central limit theorems provide a theoretical description of the asymptotic distribution of estimated second- and higher order realized price divergence, and its dependence on continuous and discontinuous semimartingale components. As our central limit theorems depend on nuisance parameters that need to be estimated and all imply a non Gaussian asymptotic distribution in presence of jumps, we obtain feasible asymptotic confidence intervals for second- and higher-order divergence using suitable Monte Carlo techniques. We demonstrate by Monte Carlo simulation that these feasible asymptotic approximations provide reliable information on the noisiness of point estimates of realized price divergence of order two and three. Overall, the findings of this chapter suggest that estimation uncertainty on second-order and higher-order realized price divergence may be successfully incorporated in financial contexts where the measuring, the forecasting or the static trading of second- or higher-order realized risks is relevant.

3.8 Appendix A: Proofs

Proof of LLNs

In order to prove Theorem 3.3.1, we first state Assumption 7.3.4 and Theorem 7.3.6 from Jacod and Protter [2012] (see also Diop [2012]).

Assumption 3.8.1. *Let the function F be $\mathcal{F} \otimes \mathcal{R} \otimes \mathcal{R}$ measurable and for some $\eta > 0$ there exists a sequence of nonnegative functions, g_n defined on $\mathbb{R}$ and a localizing sequence of stopping times (τ_n), such that $||F(\omega, x)_t - \varrho(\omega)_t x^2|| \leq g_n(x)$ for $t \leq \tau_n$ and $||x|| \leq \eta$, in which the ϱ is measurable caglad process.*

Theorem 3.8.2. *Assume X be a semimartingale and Δ be any random discretization scheme and F be a function on $\Omega \times \mathbb{R}_+ \times \mathbb{R}$ and F satisfy Assumption 3.8.1 with $g_n(x) = o(||x||^2)$ as $x \to 0$ for any n. Then*

$$V^n(F, X) \xrightarrow{\mathbb{P}} \int_0^t \varrho_s dC_s + F \star \mu_t \tag{3.21}$$

Now we proceed to the proof. We consider the following representation in term of log price

$$D_p(2)(Y_{i\Delta}, Y_{(i-1)\Delta}) = F_{D_P(2)}\left(X_{(i-1)\Delta_n}, \Delta_i^n X\right) = e^{X_{(i-1)\Delta_n}} f_{\overline{D}_p(2)}(\Delta_i^n X)$$

where

$$F_{D_P}(y, x) = e^{py} f_{\overline{D}_p}(x)$$

$$f_{\overline{D}_p}(x) = \frac{x^2}{2} + \frac{1+p}{6}x^3 + \frac{1+p+p^2}{24}x^4 + O(x^5),$$

and recalling that

$$F_{D_p(m)}(y, x) = \frac{\partial^{(m-2)} F_{D_p}(y, x)}{\partial p^{(m-2)}}$$

By using the fact that $f_{\overline{D}_p(2)}(x) = \frac{x^2}{2} + o(||x||^2)$ and $f_{\overline{D}_p(m)}(x) = O(||x||^m)$ as $x \to 0$ we get

$$F_{D_p(m)}(y, x) = y^{m-2} e^{py} \left(\frac{1}{2}x^2 + K_1(p, y)x^3 + K_2(p, y)o(||x||^3) \right). \tag{3.22}$$

If we define $F(\omega, x)_t := F_{D_p(m)}(X_{(i-1)\Delta_n}(\omega), x)$ then it satisfies the Assumption 3.8.1 (note that the process X_{t-} is caglad). Therefore, Theorem 3.8.2 provides the proof of Theorem 3.3.1.

Proof of Theorem 3.4.1

The proof relies on the results we showed in Theorem 2.4.2, Theorem 3.6 of Diop [2012] and Theorem 10.2.3 of Jacod and Protter [2012].

Step 1 We take F as a shorthand of $F_{D(m)_p}$, since the form of limits is independent of choice of p and m.
We consider Assumption (SL) again which implies that X is bounded. Therefore, from local boundedness of function F we have that the values of F are relevant on some compact set. Thus, F and its derivatives can be assumed bounded and we obtain:

$$\begin{aligned} &\| F(y,x) \| \leq K \left(\| x \|^2 \right) \\ &\| \frac{\partial F}{\partial y}(y,x) \| \leq K \left(\| x \|^2 \right) \end{aligned} \tag{3.23}$$

We use the same specific choice S_p for the sequence of stopping times that weakly enumerates the jumps of X as in the proof of Theorem 2.4.2, and also we take the notation as introduced in step 1 there. However ζ_p^n is defined differently as

$$\begin{aligned} \zeta_p^n = \frac{1}{\sqrt{\Delta_n}} \Big(& F(X_{S_{-(n,p)}}, \Delta X_{S_p} + \sqrt{\Delta_n} R(n,p)) \\ & - F(X_{S_{p-}}, \Delta X_{S_p}) - F(X_{S_{-(n,p)}}, \sqrt{\Delta_n} R(n,p)) \Big) \end{aligned} \tag{3.24}$$

We can introduce the decomposition $\zeta_p^n = \zeta_p^{n,1} + \zeta_p^{n,2}$, where

$$\begin{aligned} \zeta_p^{n,1} = & \frac{1}{\sqrt{\Delta_n}} \left(F\left(X_{S_p} - \sqrt{\Delta_n} R(n,p), \Delta X_{S_p} + \sqrt{\Delta_n} R(n,p) \right) - F(X_{S_{p-}}, \Delta X_{S_p}) \right) \\ \zeta_p^{n,2} = & \frac{-1}{\sqrt{\Delta_n}} \left(F(X_{S_{-(n,p)}}, \sqrt{\Delta_n} R(n,p)) \right) . \end{aligned} \tag{3.25}$$

Based on proposition 4.4.10 in Jacod and Protter [2012] the sequences of $R_-(n,p)$ and $R_+(n,p)$ are bounded in probability and $\| F(y,x) \| \leq K \left(\| x \|^2 \right)$ so we obtain $\zeta_p^{n,2} \xrightarrow{P} 0$. The Taylor's expansion in $\zeta_p^{n,1}$ yields

$$\zeta_p^n - \left(\frac{\partial F}{\partial x}(X_{S_p-}, \Delta X_{S_p}) R(n,p) - \frac{\partial F}{\partial y}(X_{S_p-}, \Delta X_{S_p}) R_-(n,p) \right) \xrightarrow{P} 0 \tag{3.26}$$

We introduced the following definitions in Notation 2.4.1:

$$R_{p-} := \sqrt{\kappa_p} \sigma_{T_p-} \psi_{p-} \; ; \; R_{p+} := \sqrt{1-\kappa_p} \sigma_{T_p} \psi_{p+} \; ; \; R_p = R_{p-} + R_{p+} \, , \tag{3.27}$$

Then by using proposition 4.4.10 in Jacod and Protter [2012] again, one can obtain from (3.26) that

$$\zeta_p^n \xrightarrow{\mathcal{L}-s} \left(\frac{\partial F}{\partial x}(X_{S_p-}, \Delta X_{S_p}) R_p - \frac{\partial F}{\partial y}(X_{S_p-}, \Delta X_{S_p}) R_{p-} \right) \quad \text{as } \Delta_n \to 0 \tag{3.28}$$

Recalling that $Y^n(m)_t = \sum_{p \in \mathcal{P}_m : S_p \leq \Delta_n [t/\Delta_n]} \zeta_p^n$, the same argument as in the proof of Theorem 2.4.2 yields that

$$Y^n(m) \overset{\mathcal{L}-s}{\Longrightarrow} \overline{V}(F(X), X'(m)) \tag{3.29}$$

where

$$\overline{V}\left(F(X),X'(m)\right)_t := \sum_{p\in\mathcal{P}_m}\left(\frac{\partial F}{\partial x}(X_{S_p-},\Delta X_{Sp})R_p - \frac{\partial F}{\partial y}(X_{S_p-},\Delta X_{Sp})R_{p-}\right)1_{\{S_p\le t\}}$$

Step 2

By Taylor's expansion of F we have

$$F(y,x) = \frac{1}{2}\partial_x^2 F(y,0)x^2 + \frac{1}{3!}\partial_x^3 F(y,0)x^3 + H_e(y,x)$$

where for any fixed value of $y\in\mathbb{R}$, then $H_e(y,x) = o\left(\| x \|^3\right)$ as $x\to 0$. We set

$$H_2(y,x) = \frac{1}{2}\partial_x^2 F(y,0)x^2 \qquad\qquad H_3(y,x) = \frac{1}{3!}\partial_x^3 F(y,0)x^3 \tag{3.30}$$

Similar to the joint convergence of (2.85) in Theorem 2.4.2, we have

$$\begin{aligned}&\left(\frac{1}{\sqrt{\Delta_n}}\left(V^n(H_2(X),X^c)_t - \frac{1}{2}\int_0^t \partial_x^2 H_2(X_s,0)c_s ds\right), Y^n(m)_t\right)\\ &\qquad\stackrel{\mathcal{L}-s}{\Longrightarrow}\left(\frac{\sqrt{2}}{2}\int_0^t \partial_x^2 H_2(X_s,0)c_s d\overline{W}_s, \overline{V}\left(F(X),X'(m)\right)_t\right)\end{aligned} \tag{3.31}$$

where the convergence of first component is deduced from Theorem 3.6 in Diop [2012]. Because the first stochastic integral on the right side of (3.31) is continuous and the fact that $\partial_x^2 H_2(X_s,0) = \partial_x^2 F(X_s,0)$, we get

$$\begin{aligned}&\tfrac{1}{\sqrt{\Delta_n}}\left(V^n(H_2(X),X^c)_t - \tfrac{1}{2}\int_0^t \partial_x^2 F(X_s,0)c_s ds\right) + Y^n(m)_t\\ &\stackrel{\mathcal{L}-s}{\Longrightarrow}\ \tfrac{\sqrt{2}}{2}\int_0^t \partial_x^2 F(X_s,0)c_s d\overline{W}_s + \overline{V}\left(F(X),X'(m)\right)_t\end{aligned} \tag{3.32}$$

From the first property of F in (3.23) and step 2 in the proof of Theorem 2.4.2 we deduce that $\overline{V}\left(F(X),X'(m)\right) \stackrel{u.c.p}{\Longrightarrow} \overline{V}\left(F(X),X\right)$ as $m\to\infty$, where

$$\overline{V}\left(F(X),X\right)_t := \sum_{p\ge 1}\left(\frac{\partial F}{\partial x}(X_{S_p-},\Delta X_{Sp})R_p - \frac{\partial F}{\partial y}(X_{S_p-},\Delta X_{Sp})R_{p-}\right)1_{\{S_p\le t\}}$$

Therefore it is enough to prove

$$\lim_{m\to\infty}\limsup_{n\to\infty}\mathbb{P}\left(\Omega_n\left(T,m\right)\cap\left\{\sup_{s\le t}\|Q^n\left(m\right)_s\| > \eta\right\}\right) = 0 \tag{3.33}$$

where

$$\begin{aligned}Q^n\left(m\right)_t &= \tfrac{1}{\sqrt{\Delta_n}}\left(V^n\left(F,X\right)_t - \tilde{C}_t^{(n)} - F\star\mu_{\Delta_n[t/\Delta_n]} - V^n\left(H_2,X^c\right)_t + \tilde{C}_t\right) - Y^n\left(m\right)_t\ ,\\ \tilde{C}_t &:= \tfrac{\sqrt{2}}{2}\int_0^t \partial_x^2 H_2(X_s,0)c_s d\overline{W}_s\end{aligned} \tag{3.34}$$

Similar to the decomposition (2.94) on $\Omega_n(t,m)$ we define $Q^n(m) = Q^n_{H_2}(m) + Q^n_{H_3}(m) + Q^n_{H_e}(m)$, where

$$Q^n_{H_2}(m)_t = \tfrac{1}{\sqrt{\Delta n}} \quad \begin{array}{l}\Big(V^n(H_2, X(m))_t - \tilde{C}^{(n)}_t - V^n(H_2, X^c)_t + \tilde{C}_t \\ -\sum_{s\leq \Delta_n[t/\Delta_n]} H_2(X_{s-}, \triangle X(m)_s)\Big)\end{array} \tag{3.35}$$

$$Q^n_{H_3}(m)_t = \frac{K}{\sqrt{\Delta_n}}\left(V^n(H_3, X(m))_t - \sum_{s\leq \Delta_n[t/\Delta_n]} H_3(X_{s-}, \triangle X(m)_s)\right) \tag{3.36}$$

$$Q^n_{H_e}(m)_t = \frac{1}{\sqrt{\Delta_n}}\left(V^n(H_e, X(m))_t - \sum_{s\leq \Delta_n[t/\Delta_n]} H_e(X_{s-}, \triangle X(m)_s)\right) \tag{3.37}$$

Step 3

In order to show (3.33) we proceed again as in Theorem 2.4.2 with some modification. We rewrite (2.103) for $Q^n_{H_3}(m)$ by

$$\begin{array}{l} k^n_i(w;x,y) = F\left(X_{(i-1)\Delta_n}(w), x+y\right) - F\left(X_{(i-1)\Delta_n}(w), x\right) - F\left(X_{(i-1)\Delta_n}(w), y\right) \\ g^n_i(w;x,y) = k^n_i(w;x,y) - \frac{\partial}{\partial x}F\left(X_{(i-1)\Delta_n}(w), x\right) y \end{array} \tag{3.38}$$

Since $H_3(X_{(i-1)\Delta_n}(w), x)$ are $\mathcal{F}_{(i-1)\Delta_n}$ measurable we can apply It$\hat{o}$'s formula to the process $X(m)_t - X(m)_{(i-1)\Delta_n}$ for $t > (i-1)\Delta_n$ to get

$$\xi(m)^n_i := H_3\left(X_{(i-1)\Delta_n}(w), \Delta^n_i X(m)\right) - \sum_{s\in I(n,i)} H_3\left(X_{(i-1)\Delta_n}(w), \Delta X(m)_s\right) \tag{3.39}$$

$$= A(n,m,i)_{i\Delta_n} + M(n,m,i)_{i\Delta_n} \tag{3.40}$$

where $A(n,m,i)_{i\Delta_n}$ and $M(n,m,i)_{i\Delta_n}$ are as introduced in proof of Theorem 2.4.2 however one needs to substitute k and g with k^n_i and g^n_i in the definition of $a(n,m,i)_t$ and $a'(n,m,i)_t$. The estimates (2.119) to (2.126) uniformly in $\omega, n and i$ are fulfilled. It implies that, if we define $U^n_{H_3}(m)_t = \frac{1}{\sqrt{\Delta_n}}\sum_{i=1}^{[t/\Delta_n]}\xi(m)^n_i$ we obtain as in Theorem 2.4.2 that

$$\lim_{m\to\infty}\limsup_{n\to\infty}\mathbb{P}\left(\Omega_n(T,m)\cap\left\{\sup_{s\leq t}\left\|U^n_{H_3}(m)_s\right\| > \eta\right\}\right) = 0 \tag{3.41}$$

Similar to $U^n_{H_3}(m)_t$, based on $Q^n_{H_2}(m)$ and $Q^n_{H_e}(m)$ one can define $U^n_{H_2}(m)_t$ and $U^n_{H_e}(m)_t$, respectively and show that (3.41) holds.

We denote $U^n(m)_t = U^n_{H_2}(m)_t + U^n_{H_3}(m)_t + U^n_{H_e}(m)_t$. Hence it remains to prove that

$$\lim_{m\to\infty}\limsup_{n\to\infty}\mathbb{P}\left(\Omega_n(T,m)\cap\left\{\sup_{s\leq t}\|Q^n(m)_s - U^n(m)_s\| > \eta\right\}\right) = 0 \tag{3.42}$$

where

$$
\begin{aligned}
Q^n(m)_t - U^n(m)_t &= \frac{1}{\sqrt{\Delta_n}} \sum_{i=1}^{[t/\Delta_n]} \xi'(m)_i^n \\
\xi'(m)_i^n &= \sum_{s \in I(n,i)} F(X_{(i-1)\Delta_n}, \Delta X(m)_s) - F(X_{s-}, \Delta X(m)_s)
\end{aligned}
$$

On the compact set we have $\| \frac{\partial F}{\partial y}(y,x) \| \leq K\left(\| x \|^2\right)$. This implies $\| F(y',x) - F(y,x) \| \leq K \| x \|^2 \| y' - y \|$. We know that $\Delta X(m)_s = \int_{A_m^c} \delta(s,z)\, \mathfrak{p}(\{s\}, dz)$ and $\| \delta(t,z) \| \leq \Gamma(z)$. We denote the compensator of $\mathfrak{p}$ by $\mathfrak{q}$. Then with $\gamma_2(m) = \int_{A_m^c} \Gamma(z)^2 \lambda(dz)$, we get

$$
\begin{aligned}
\mathbb{E}\left(|\xi'(m)_i^n|\right) &\leq K \mathbb{E}\left(\sum_{s \in I(n,i)} \|X_{(i-1)\Delta_n} - X_{s-}\| \|\Delta X(m)_s\|^2 \right) \\
&= K \mathbb{E}\left(\int_{I(n,i)} \int_{A_m^c} \|X_{(i-1)\Delta_n} - X_{s-}\| \|\delta(s,z)\|^2 \mathfrak{p}(ds,dz) \right) \\
&= K \mathbb{E}\left(\int_{I(n,i)} \int_{A_m^c} \|X_{(i-1)\Delta_n} - X_{s-}\| \|\delta(s,z)\|^2 \mathfrak{q}(ds,dz) \right) \\
&\leq K \gamma_2(m)\, \mathbb{E}\left(\int_{I(n,i)} \|X_{(i-1)\Delta_n} - X_{s-}\| ds \right)
\end{aligned}
$$

Under Assumption (SH), from Lemma 2.8.4 we obtain that $\mathbb{E}(\|X_{(i-1)\Delta_n} - X_{s-}\|) \leq K\sqrt{\Delta_n}$ with $(i-1)\Delta_n \leq s < i\Delta_n$. This yields that $\mathbb{E}(\xi'(m)_i^n) \leq K\gamma_2(m)\, \Delta_n^{3/2}$, therefore we get

$$
\mathbb{E}\left(\sum_{i=1}^{[t/\Delta_n]} |\xi'(m)_i^n| \right) \leq Kt\gamma_2(m) \sqrt{\Delta_n}
$$

and $\lim_{m \to \infty} \gamma_2(m) = 0$ implies that (3.42) satisfies. This ends the proof.

3.9 Appendix B: Figures and Tables

$1-\alpha=0.98$				
	Unfeasible CLT		Feasible CLT	
	$15-sec$	$1-min$	$15-sec$	$1-min$
$V^n\left(F_{D_1(2)},X\right)$	0.9895	0.9804	0.9895	0.97
$V^n\left(F_{D_1(3)},X\right)$	0.9752	0.9765	0.9765	0.9674
$1-\alpha=0.95$				
	Unfeasible CLT		Feasible CLT	
	$15-sec$	$1-min$	$15-sec$	$1-min$
$V^n\left(F_{D_1(2)},X\right)$	0.9647	0.9426	0.9634	0.9308
$V^n\left(F_{D_1(3)},X\right)$	0.9517	0.9361	0.9478	0.9335
$1-\alpha=0.90$				
	Unfeasible CLT		Feasible CLT	
	$15-sec$	$1-min$	$15-sec$	$1-min$
$V^n\left(F_{D_1(2)},X\right)$	0.9100	0.8748	0.9061	0.8787
$V^n\left(F_{D_1(3)},X\right)$	0.9035	0.8852	0.8865	0.8644

Table 3.1: Simulation study: coverage of confidence intervals for realized price divergence measures. D_1: Kullback Leibler divergence; $D_1(2)$ and $D_1(3)$: second and third order divergence. Frequencies: 15 seconds and 1 minute. 2000 days of simulated data. Model: Table 2.1 mod I-c.

$1-\alpha=0.98$				
	Unfeasible CLT		Feasible CLT	
	$15\!-\!sec$	$1\!-\!min$	$15\!-\!sec$	$1\!-\!min$
$V^n\left(F_{D_1(2)},X\right)$	0.9880	0.9816	0.9864	0.9785
$V^n\left(F_{D_1(3)},X\right)$	0.9697	0.9785	0.9657	0.9729
$1-\alpha=0.95$				
	Unfeasible CLT		Feasible CLT	
	$15\!-\!sec$	$1\!-\!min$	$15\!-\!sec$	$1\!-\!min$
$V^n\left(F_{D_1(2)},X\right)$	0.9681	0.9562	0.9649	0.9474
$V^n\left(F_{D_1(3)},X\right)$	0.9402	0.9378	0.9378	0.9347
$1-\alpha=0.90$				
	Unfeasible CLT		Feasible CLT	
	$15\!-\!sec$	$1\!-\!min$	$15\!-\!sec$	$1\!-\!min$
$V^n\left(F_{D_1(2)},X\right)$	0.9219	0.9004	0.9179	0.8957
$V^n\left(F_{D_1(3)},X\right)$	0.8964	0.8789	0.8941	0.8710

Table 3.2: Simulation study: coverage of confidence intervals for realized price divergence measures. D_1: Kullback Leibler divergence; $D_1(2)$ and $D_1(3)$: second and third order divergence. Frequencies: 15 seconds and 1 minute. 2000 days of simulated data. Model: Table 2.1 mod II-c.

$1-\alpha=0.98$				
	Unfeasible CLT		Feasible CLT	
	$15-sec$	$1-min$	$15-sec$	$1-min$
$V^n\left(F_{D_1(2)}, X\right)$	0.9801	0.9821	0.9806	0.9837
$V^n\left(F_{D_1(3)}, X\right)$	0.9689	0.9832	0.9628	0.9842
$1-\alpha=0.95$				
	Unfeasible CLT		Feasible CLT	
	$15-sec$	$1-min$	$15-sec$	$1-min$
$V^n\left(F_{D_1(2)}, X\right)$	0.9608	0.9496	0.9608	0.9577
$V^n\left(F_{D_1(3)}, X\right)$	0.9384	0.9608	0.9348	0.9562
$1-\alpha=0.90$				
	Unfeasible CLT		Feasible CLT	
	$15-sec$	$1-min$	$15-sec$	$1-min$
$V^n\left(F_{D_1(2)}, X\right)$	0.9221	0.8957	0.9201	0.9120
$V^n\left(F_{D_1(3)}, X\right)$	0.8896	0.9038	0.8870	0.9150

Table 3.3: Simulation study: coverage of confidence intervals for realized price divergence measures. D_1: Kullback Leibler divergence;$D_1(2)$ and $D_1(3)$: second and third order divergence. Frequencies: 15 seconds and 1 minute. 2000 days of simulated data. Model: Table 2.1 mod III-c.

$1-\alpha=0.98$				
	Unfeasible CLT		Feasible CLT	
	$15{-}sec$	$1{-}min$	$15{-}sec$	$1{-}min$
$V^n\left(F_{D_1(2)},X\right)$	0.9814	0.9752	0.9789	0.9665
$V^n\left(F_{D_1(3)},X\right)$	0.9628	0.9714	0.9491	0.9541
$1-\alpha=0.95$				
	Unfeasible CLT		Feasible CLT	
	$15{-}sec$	$1{-}min$	$15{-}sec$	$1{-}min$
$V^n\left(F_{D_1(2)},X\right)$	0.9578	0.9392	0.9578	0.9355
$V^n\left(F_{D_1(3)},X\right)$	0.9231	0.9169	0.9144	0.9219
$1-\alpha=0.90$				
	Unfeasible CLT		Feasible CLT	
	$15{-}sec$	$1{-}min$	$15{-}sec$	$1{-}min$
$V^n\left(F_{D_1(2)},X\right)$	0.9021	0.8872	0.9033	0.8859
$V^n\left(F_{D_1(3)},X\right)$	0.8698	0.8649	0.8661	0.8438

Table 3.4: Simulation study: coverage of confidence intervals for realized price divergence measures. D_1: Kullback Leibler divergence; $D_1(2)$ and $D_1(3)$: second and third order divergence. Frequencies: 15 seconds and 1 minute. 2000 days of simulated data. Model: Table 2.1 mod I-d.

$1-\alpha=0.98$				
	Unfeasible CLT		Feasible CLT	
	$15-sec$	$1-min$	$15-sec$	$1-min$
$V^n\left(F_{D_1(2)}, X\right)$	0.9789	0.9774	0.9774	0.9711
$V^n\left(F_{D_1(3)}, X\right)$	0.9727	0.9735	0.9665	0.9641
$1-\alpha=0.95$				
	Unfeasible CLT		Feasible CLT	
	$15-sec$	$1-min$	$15-sec$	$1-min$
$V^n\left(F_{D_1(2)}, X\right)$	0.9517	0.9509	0.9524	0.9415
$V^n\left(F_{D_1(3)}, X\right)$	0.9400	0.9501	0.9376	0.9384
$1-\alpha=0.90$				
	Unfeasible CLT		Feasible CLT	
	$15-sec$	$1-min$	$15-sec$	$1-min$
$V^n\left(F_{D_1(2)}, X\right)$	0.8901	0.9026	0.8933	0.8995
$V^n\left(F_{D_1(3)}, X\right)$	0.8808	0.8894	0.8855	0.8823

Table 3.5: Simulation study: coverage of confidence intervals for realized price divergence measures. D_1: Kullback Leibler divergence; $D_1(2)$ and $D_1(3)$: second and third order divergence. Frequencies: 15 seconds and 1 minute. 2000 days of simulated data. Model: Table 2.1 mod II-d.

$1-\alpha=0.98$				
	Unfeasible CLT		Feasible CLT	
	$15{-}sec$	$1{-}min$	$15{-}sec$	$1{-}min$
$V^n\left(F_{D_1(2)},X\right)$	0.9801	0.9781	0.9786	0.9837
$V^n\left(F_{D_1(3)},X\right)$	0.9740	0.9715	0.9725	0.9821
$1-\alpha=0.95$				
	Unfeasible CLT		Feasible CLT	
	$15{-}sec$	$1{-}min$	$15{-}sec$	$1{-}min$
$V^n\left(F_{D_1(2)},X\right)$	0.9577	0.9470	0.9557	0.9613
$V^n\left(F_{D_1(3)},X\right)$	0.9430	0.9394	0.9384	0.9541
$1-\alpha=0.90$				
	Unfeasible CLT		Feasible CLT	
	$15{-}sec$	$1{-}min$	$15{-}sec$	$1{-}min$
$V^n\left(F_{D_1(2)},X\right)$	0.9139	0.8997	0.9155	0.9195
$V^n\left(F_{D_1(3)},X\right)$	0.8895	0.8890	0.8875	0.9160

Table 3.6: Simulation study: coverage of confidence intervals for realized price divergence measures. D_1: Kullback Leibler divergence; $D_1(2)$ and $D_1(3)$: second and third order divergence. Frequencies: 15 seconds and 1 minute. 2000 days of simulated data. Model: Table 2.1 mod III-d.

$1-\alpha=0.98$				
	Unfeasible CLT		Feasible CLT	
	$15-sec$	$1-min$	$15-sec$	$1-min$
$V^n\left(F_{D_1(2)},X\right)$	0.9815	0.9776	0.9776	0.9710
$V^n\left(F_{D_1(3)},X\right)$	0.9710	0.9776	0.9710	0.9644
$1-\alpha=0.95$				
	Unfeasible CLT		Feasible CLT	
	$15-sec$	$1-min$	$15-sec$	$1-min$
$V^n\left(F_{D_1(2)},X\right)$	0.9525	0.9433	0.9449	0.9367
$V^n\left(F_{D_1(3)},X\right)$	0.9420	0.9433	0.9393	0.9314
$1-\alpha=0.90$				
	Unfeasible CLT		Feasible CLT	
	$15-sec$	$1-min$	$15-sec$	$1-min$
$V^n\left(F_{D_1(2)},X\right)$	0.9090	0.9011	0.8972	0.9038
$V^n\left(F_{D_1(3)},X\right)$	0.8827	0.8866	0.8774	0.8642

Table 3.7: Simulation study: coverage of confidence intervals for realized price divergence measures. D_1: Kullback Leibler divergence; $D_1(2)$ and $D_1(3)$: second and third order divergence. Frequencies: 15 seconds and 1 minute. 2000 days of simulated data. Model: Table 2.1 mod I-j.

$1-\alpha=0.98$				
	Unfeasible CLT		Feasible CLT	
	$15-sec$	$1-min$	$15-sec$	$1-min$
$V^n\left(F_{D_1(2)},X\right)$	0.9823	0.9800	0.9754	0.9777
$V^n\left(F_{D_1(3)},X\right)$	0.9723	0.9792	0.9654	0.9746
$1-\alpha=0.95$				
	Unfeasible CLT		Feasible CLT	
	$15-sec$	$1-min$	$15-sec$	$1-min$
$V^n\left(F_{D_1(2)},X\right)$	0.9539	0.9531	0.9501	0.9493
$V^n\left(F_{D_1(3)},X\right)$	0.9339	0.9547	0.9324	0.9508
$1-\alpha=0.90$				
	Unfeasible CLT		Feasible CLT	
	$15-sec$	$1-min$	$15-sec$	$1-min$
$V^n\left(F_{D_1(2)},X\right)$	0.9063	0.9117	0.9102	0.9109
$V^n\left(F_{D_1(3)},X\right)$	0.8825	0.8994	0.8810	0.8971

Table 3.8: Simulation study: coverage of confidence intervals for realized price divergence measures. D_1: Kullback Leibler divergence;$D_1(2)$ and $D_1(3)$: second and third order divergence. Frequencies: 15 seconds and 1 minute. 2000 days of simulated data. Model: Table 2.1 mod II-j.

$1-\alpha=0.98$				
	Unfeasible CLT		Feasible CLT	
	$15-sec$	$1-min$	$15-sec$	$1-min$
$V^n\left(F_{D_1(2)},X\right)$	0.9821	0.9841	0.9790	0.9862
$V^n\left(F_{D_1(3)},X\right)$	0.9714	0.9765	0.9688	0.9785
$1-\alpha=0.95$				
	Unfeasible CLT		Feasible CLT	
	$15-sec$	$1-min$	$15-sec$	$1-min$
$V^n\left(F_{D_1(2)},X\right)$	0.9627	0.9576	0.9597	0.9653
$V^n\left(F_{D_1(3)},X\right)$	0.9439	0.9500	0.9444	0.9592
$1-\alpha=0.90$				
	Unfeasible CLT		Feasible CLT	
	$15-sec$	$1-min$	$15-sec$	$1-min$
$V^n\left(F_{D_1(2)},X\right)$	0.9138	0.9046	0.9082	0.9255
$V^n\left(F_{D_1(3)},X\right)$	0.8903	0.9051	0.8969	0.90

Table 3.9: Simulation study: coverage of confidence intervals for realized price divergence measures. D_1: Kullback Leibler divergence; $D_1(2)$ and $D_1(3)$: second and third order divergence. Frequencies: 15 seconds and 1 minute. 2000 days of simulated data. Model: Table 2.1 mod III-j.

$1-\alpha=0.98$				
	Unfeasible CLT		Feasible CLT	
	$15-sec$	$1-min$	$15-sec$	$1-min$
$V^n\left(F_{D_1(2)},X\right)$	0.9751	0.9778	0.9843	0.9817
$V^n\left(F_{D_1(3)},X\right)$	0.9765	0.9882	0.9804	0.9830
$1-\alpha=0.95$				
	Unfeasible CLT		Feasible CLT	
	$15-sec$	$1-min$	$15-sec$	$1-min$
$V^n\left(F_{D_1(2)},X\right)$	0.9530	0.9569	0.9530	0.9503
$V^n\left(F_{D_1(3)},X\right)$	0.9477	0.9634	0.9464	0.9569
$1-\alpha=0.90$				
	Unfeasible CLT		Feasible CLT	
	$15-sec$	$1-min$	$15-sec$	$1-min$
$V^n\left(F_{D_1(2)},X\right)$	0.9086	0.9099	0.9086	0.8968
$V^n\left(F_{D_1(3)},X\right)$	0.9112	0.9046	0.8942	0.9073

Table 3.10: Simulation study: coverage of confidence intervals for realized price divergence measures. D_1: Kullback Leibler divergence; $D_1(2)$ and $D_1(3)$: second and third order divergence. Frequencies: 15 seconds and 1 minute. 2000 days of simulated data. Model: Table 2.1 mod I-m.

1 − α = 0.98				
	Unfeasible CLT		Feasible CLT	
	$15{-}sec$	$1{-}min$	$15{-}sec$	$1{-}min$
$V^n\left(F_{D_1(2)}, X\right)$	0.9753	0.9832	0.9705	0.9697
$V^n\left(F_{D_1(3)}, X\right)$	0.9697	0.9832	0.9601	0.9713
$1-\alpha=0.95$				
	Unfeasible CLT		Feasible CLT	
	$15{-}sec$	$1{-}min$	$15{-}sec$	$1{-}min$
$V^n\left(F_{D_1(2)}, X\right)$	0.9482	0.9498	0.9482	0.9498
$V^n\left(F_{D_1(3)}, X\right)$	0.9386	0.9633	0.9315	0.9474
$1-\alpha=0.90$				
	Unfeasible CLT		Feasible CLT	
	$15{-}sec$	$1{-}min$	$15{-}sec$	$1{-}min$
$V^n\left(F_{D_1(2)}, X\right)$	0.8901	0.9092	0.8957	0.9044
$V^n\left(F_{D_1(3)}, X\right)$	0.8853	0.9124	0.8845	0.8925

Table 3.11: Simulation study: coverage of confidence intervals for realized price divergence measures. D_1: Kullback Leibler divergence; $D_1(2)$ and $D_1(3)$: second and third order divergence. Frequencies: 15 seconds and 1 minute. 2000 days of simulated data. Model: Table 2.1 mod II-m.

$1-\alpha=0.98$				
	Unfeasible CLT		Feasible CLT	
	$15-sec$	$1-min$	$15-sec$	$1-min$
$V^n\left(F_{D_1(2)},X\right)$	0.9751	0.9766	0.9771	0.9822
$V^n\left(F_{D_1(3)},X\right)$	0.9650	0.9771	0.9736	0.9817
$1-\alpha=0.95$				
	Unfeasible CLT		Feasible CLT	
	$15-sec$	$1-min$	$15-sec$	$1-min$
$V^n\left(F_{D_1(2)},X\right)$	0.9492	0.9452	0.9523	0.9655
$V^n\left(F_{D_1(3)},X\right)$	0.9376	0.9472	0.9340	0.9634
$1-\alpha=0.90$				
	Unfeasible CLT		Feasible CLT	
	$15-sec$	$1-min$	$15-sec$	$1-min$
$V^n\left(F_{D_1(2)},X\right)$	0.9061	0.8940	0.9072	0.9229
$V^n\left(F_{D_1(3)},X\right)$	0.8788	0.8960	0.8798	0.9239

Table 3.12: Simulation study: coverage of confidence intervals for realized price divergence measures. D_1: Kullback Leibler divergence; $D_1(2)$ and $D_1(3)$: second and third order divergence. Frequencies: 15 seconds and 1 minute. 2000 days of simulated data. Model: Table 2.1 mod III-m.

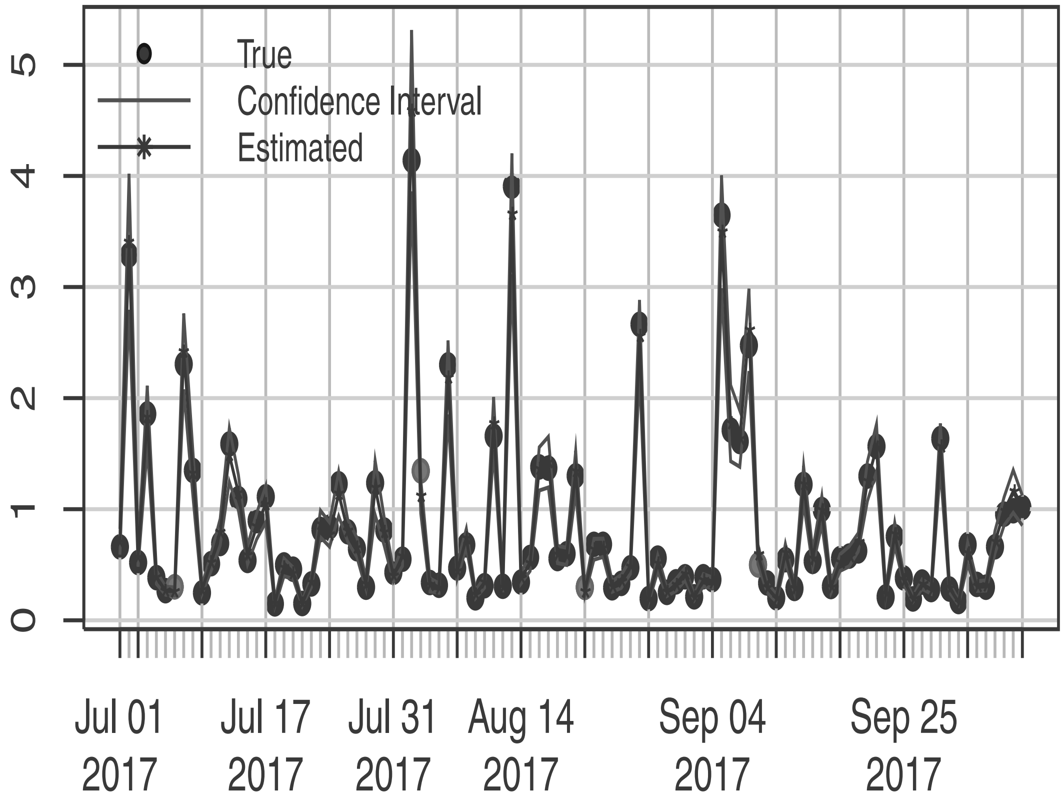


Figure 3.1: 95% confidence interval for daily realized second order Kullback Leibler divergence (without co-jump) with sampling frequency= 1 minute. Model: Table 2.1 mod II-d.

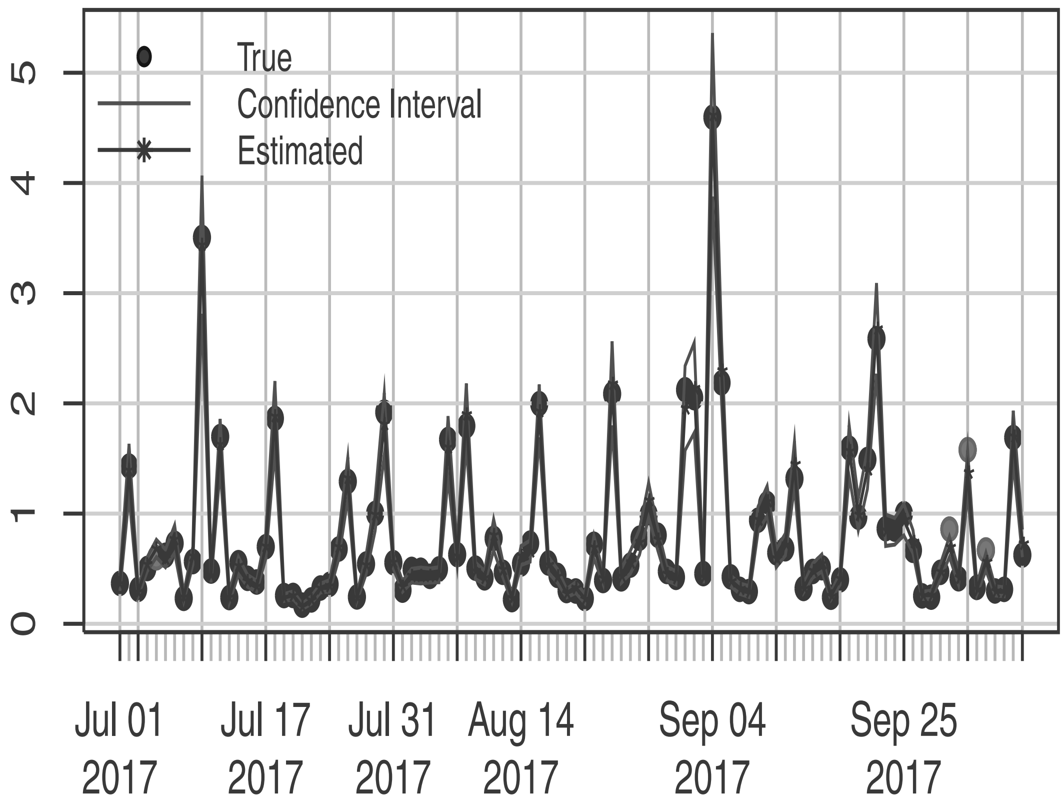


Figure 3.2: 95% confidence interval for daily realized second order Kullback Leibler divergence (with co-jump) with sampling frequency= 1 minute. Model: Table 2.1 mod II-j.

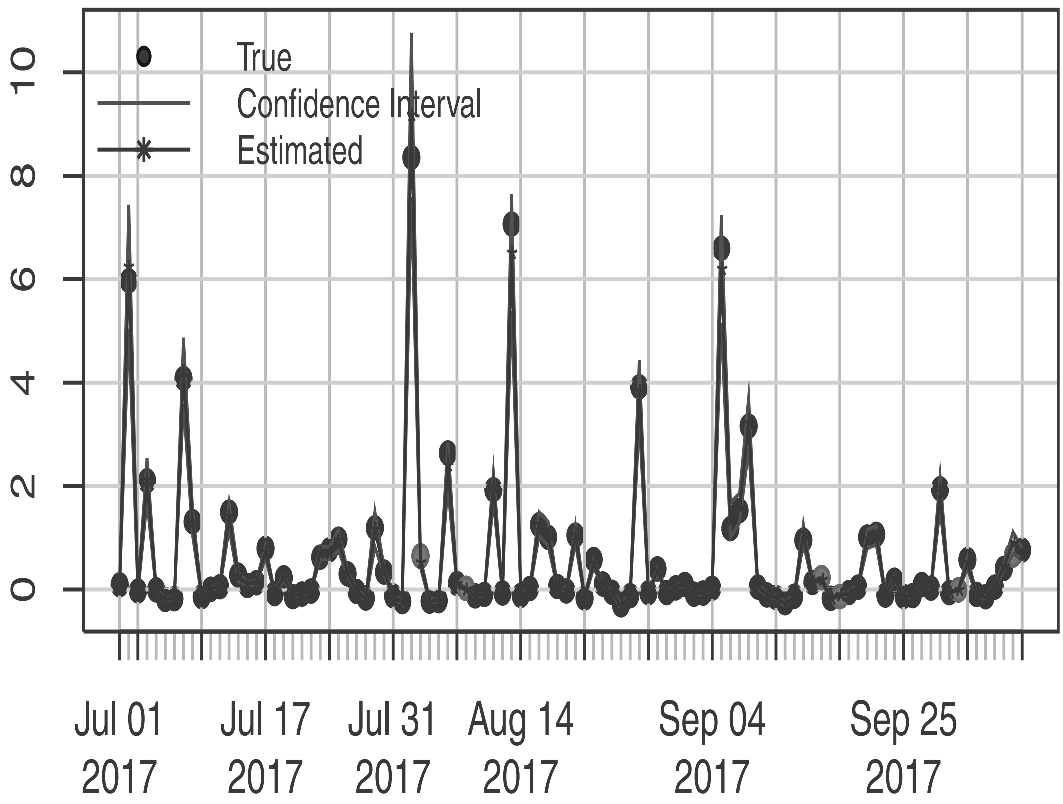


Figure 3.3: 95% confidence interval for daily realized **third order** Kullback Leibler divergence (without co-jump) with sampling frequency= 1 minute. Model: Table 2.1 mod II-d.

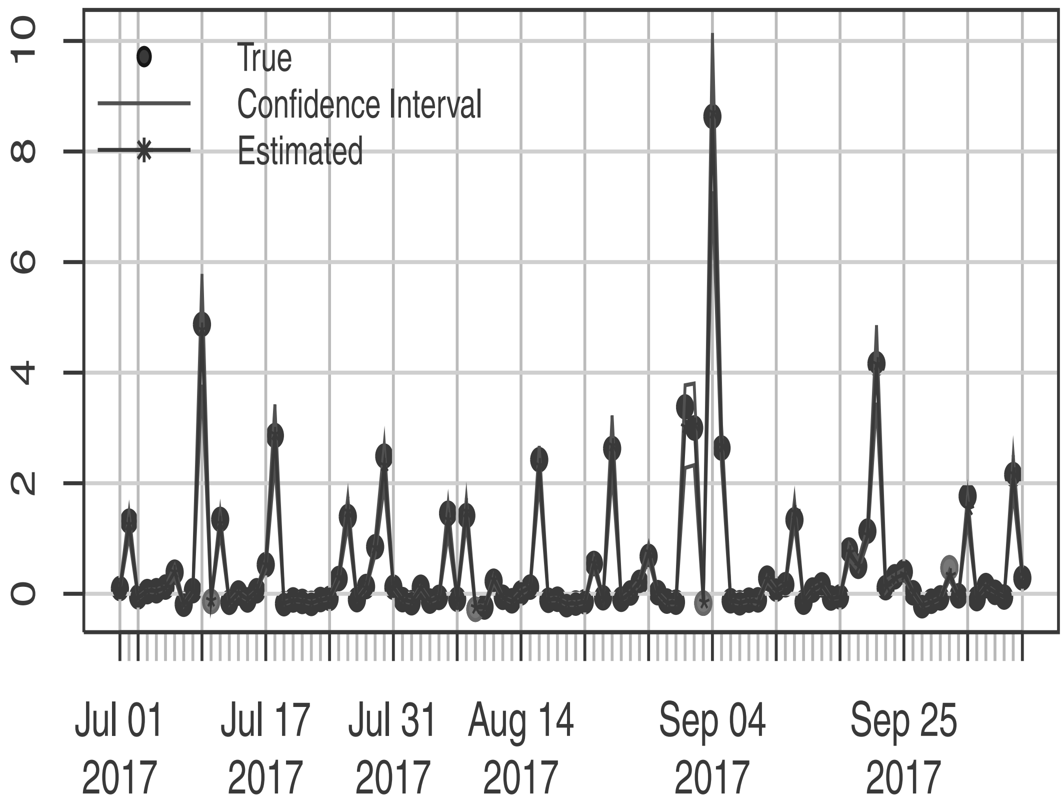


Figure 3.4: 95% confidence interval for daily realized **third order** Kullback Leibler divergence (with co-jump) with sampling frequency= 1 minute. Model: Table 2.1 mod II-j.

Chapter 4

Conclusion and outlook

In this thesis, we developed the econometrics of realized divergence and provided the foundations for a unified statistical measurement of second- and higher-order realized risks induced by tradeable portfolios of realized divergences. We obtained in Chapter 2 and Chapter 3, respectively, appropriate laws of large numbers and functional central limit theorems for realized (scale-invariant) return and (scale-dependent) price divergence under general semimartingale conditions. These laws of large numbers give rise to a proper definition of the hidden realized risk approximated by second and higher-order realized divergences, together with an identification of the risk contributions of continuous and discontinuous semimartingale components. Our functional central limit theorems give rise to a theoretical description of the asymptotic distribution of estimated second- and higher order realized divergence, together with its dependence on continuous and discontinuous semimartingale components, which is a necessary starting point to quantify the uncertainty in realized divergence measures. In each chapter, we produced feasible asymptotic confidence intervals for several proxies of second- and higher-order realized divergence, using either analytic asymptotic approximations or parametric bootstrap techniques. We have finally shown with extensive Monte Carlo simulations of a flexible two-factor double-jump stochastic volatility model that our asymptotic approximations provide reliable information on the noisiness of point estimates of realized divergence of order up to four. Overall, our findings indicate that estimation uncertainty on second-order and higher-order realized divergence may be successfully incorporated in many financial contexts, where the measuring, forecasting or trading of second- and higher-order realized risks is relevant. In this sense, our work provides a comprehensive foundation for an accurate understanding of the nature of realized risks of distinct orders and their tradeable risk premia.

Bibliography

Yacine Aït-Sahalia and Jean Jacod. *High-frequency financial econometrics*. Princeton University Press, 2014.

Torben Andersen and Tim Bollerslev. Answering the skeptics: Yes, standard volatility models do provide accurate forecasts. *International Economic Review*, 39(4):885–905, 1998.

Torben G Andersen, Dobrislav Dobrev, and Ernst Schaumburg. Jump robust volatility estimation using nearest neighbor truncation, nber working paper no. 2009.

Federico M Bandi and Roberto Reno. Price and volatility co-jumps. *Journal of Financial Economics*, 119(1):107–146, 2016.

Ole E Barndorff-Nielsen and Neil Shephard. Power and bipower variation with stochastic volatility and jumps. *Journal of financial econometrics*, 2(1):1–37, 2004a.

Ole E Barndorff-Nielsen and Neil Shephard. Power and bipower variation with stochastic volatility and jumps. *Journal of financial econometrics*, 2(1):1–37, 2004b.

Ole E Barndorff-Nielsen, Neil Shephard, et al. Realized power variation and stochastic volatility models. *Bernoulli*, 9(2):243–265, 2003.

Ole E Barndorff-Nielsen, Svend Erik Graversen, Jean Jacod, Mark Podolskij, and Neil Shephard. A central limit theorem for realised power and bipower variations of continuous semimartingales. In *From stochastic calculus to mathematical finance*, pages 33–68. Springer, 2006.

Oleg Bondarenko. Variance trading and market price of variance risk. *Journal of Econometrics*, 180(1):81–97, 2014.

Lev M Bregman. The relaxation method of finding the common point of convex sets and its application to the solution of problems in convex programming. *USSR computational mathematics and mathematical physics*, 7(3):200–217, 1967.

Mark Britten-Jones and Anthony Neuberger. Option prices, implied price processes, and stochastic volatility. *Journal of Finance*, 55(2):839–866, 2000.

Peter Carr and Roger Lee. Put-call symmetry: Extensions and applications. *Mathematical Finance*, 19(4):523–560, 2009.

Peter Carr and Dilip Madan. Optimal positioning in derivative securities. 2001.

2000 CBOE. The CBOE volatility index–VIX. Working paper, CBOE, 2000.

Fulvio Corsi, Davide Pirino, and Roberto Reno. Threshold bipower variation and the impact of jumps on volatility forecasting. *Journal of Econometrics*, 159(2):276–288, 2010.

Assane Diop. Convergence of some random functionals of discretized semimartingales. *Bernoulli*, 18(4):1188–1222, 2012.

Gejza Dohnal. On estimating the diffusion coefficient. *Journal of Applied Probability*, pages 105–114, 1987.

Valentine Genon-Catalot and Jean Jacod. On the estimation of the diffusion coefficient for multi-dimensional diffusion processes. In *Annales de l'IHP Probabilités et statistiques*, volume 29, pages 119–151, 1993.

Xin Huang and George Tauchen. The relative contribution of jumps to total price variance. *Journal of financial econometrics*, 3(4):456–499, 2005.

Fumitada Itakura and Shuzo Saito. Analysis synthesis telephony based on the maximum likelihood method. In *Proceedings of the 6th International Congress on Acoustics*, volume 17, pages C17–C20. pp. C17–C20, 1968.

Jean Jacod. Asymptotic properties of power variations of levy processes. *ESAIM: PS*, 11:173–196, 2007.

Jean Jacod. Asymptotic properties of realized power variations and related functionals of semimartingales. *Stochastic Processes and their Applications*, 118(4):517 – 559, 2008.

Jean Jacod. Statistics and high frequency data. *Statistical methods for stochastic differential equations*, 124:191–310, 2012.

Jean Jacod and Philip Protter. Discretization of processes. stochastic modelling and applied probability, 67, 2012.

Jean Jacod and Albert Shiryaev. *Limit theorems for stochastic processes*, volume 288. Springer Science & Business Media, 2003.

Jean Jacod and Viktor Todorov. Testing for common arrivals of jumps for discretely observed multidimensional processes. *The Annals of Statistics*, pages 1792–1838, 2009.

Jean Jacod and Viktor Todorov. Do price and volatility jump together? *The Annals of Applied Probability*, 20(4):1425–1469, 2010.

Jean Jacod, Yingying Li, Per A Mykland, Mark Podolskij, and Mathias Vetter. Microstructure noise in the continuous case: the pre-averaging approach. *Stochastic processes and their applications*, 119(7):2249–2276, 2009.

George J Jiang and Y.S. Tian. The model-free implied volatility and its information content. *Review of Financial Studies*, 18(4):1305–1342, 2005.

Ioannis Karatzas and Steven E Shreve. *Methods of mathematical finance*, volume 39. Springer Science & Business Media, 1998.

Yuichi Kitamura, Taisuke Otsu, and Kirill Evdokimov. Robustness, infinitesimal neighborhoods, and moment restrictions. *Econometrica*, 81(3):1185–1201, 2013.

Dennis Kristensen. Nonparametric filtering of the realized spot volatility: A kernel-based approach. *Econometric Theory*, 26(01):60–93, 2010.

Cecilia Mancini. Disentangling the jumps of the diffusion in a geometric jumping brownian motion. LXIV 1947, Giornale dellInstituto Italiano degli Attuari, 2001.

Cecilia Mancini. Estimating the integrated volatility in stochastic volatility models with lévy type jumps. *Preprint*, 2006.

Cecilia Mancini. Non-parametric threshold estimation for models with stochastic diffusion coefficient and jumps. *Scandinavian Journal of Statistics*, 36(2):270–296, 2009.

Cecilia Mancini, Vanessa Mattiussi, and Roberto Reno. Spot volatility estimation using delta sequences. *Finance and Stochastics*, 19(2):261–293, 2015.

Anthony Neuberger. The log contract. *The Journal of Portfolio Management*, 20(2): 74–80, 1994.

Piotr Orlowski, Paul Schneider, and Fabio Trojani. Big risks. Working paper, USI Lugano and SFI, 2016.

Mark Podolskij and Mathias Vetter. Estimation of volatility functionals in the simultaneous presence of microstructure noise and jumps. *Bernoulli*, 15(3):634–658, 08 2009. doi: 10.3150/08-BEJ167.

Paul Schneider and Fabio Trojani. Divergence and the price of uncertainty. Working paper, USI Lugano and SFI, 2015a.

Paul Schneider and Fabio Trojani. Fear trading. Research Paper 15-03, USI Lugano and SFI, 2015b.

William Speth, Joseph Levin, Sandy Rattray, and Devesh Shah. Volatility index and derivative contracts based thereon, October 6 2004. US Patent App. 10/959,528.

Viktor Todorov. Variance risk-premium dynamics: The role of jumps. *Review of Financial Studies*, 23(1):345–383, 2010. doi: 10.1093/rfs/hhp035.

Viktor Todorov and George Tauchen. Volatility jumps. *Journal of Business & Economic Statistics*, 29(3):356–371, 2011.

Almut. E. D. Veraart. Inference for the jump part of quadratic variation of Ito semimartingales. *Econometric Theory*, 26:331–368, 4 2010. ISSN 1469-4360.

Jeannette H. C. Woerner. *Power and Multipower Variation: inference for high frequency data*, pages 343–364. Springer US, Boston, MA, 2006.