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Thesis title:

Essays in Market Microstructure

PhD in	Economics and Finance
Cycle	30
Student's Advisor	Barbara Rindi
Calendar year of thesis defence	2020

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Abstract

The first chapter of the thesis describes price discovery and liquidity provision in a dynamic limit order market with asymmetric information and non-Markovian learning. Investors condition on information in both the current limit order book and also, unlike in previous research, on the prior order history when deciding whether to provide or take liquidity. Numerical examples show that the information content of the prior order history can be substantial. In addition, the information content of arriving orders can differ from order direction and aggressiveness.

The second chapter of the thesis provides a theoretical explanation for the widespread use of rebate-based access pricing — maker-taker and taker-maker — in present-day securities markets. Given a standard model of trading frictions, we show that exchanges optimally use rebate-based access pricing when dispersion of investor asset valuation is low (and thus potential gains from trade are low), but strictly positive fees for both liquidity makers and liquidity takers with high investor valuation dispersion. In addition, when the trading frequency increases, the incentive to use rebate-based pricing decreases. However, rebate-based pricing is more likely in markets with high frequency trading. When rebate-based access pricing is optimal for an exchange, total welfare increases (decreases) when investor valuation dispersion is low (high) without HFTs. However, with HFTs, optimal rebate-based access pricing strictly improves total welfare, although Pareto transfers from exchanges to investors may be needed to improve investor welfare. In addition, we identify an asymmetry in how make fees and take fees affect the trading process. Thus, the effect of maker-taker and taker-maker pricing need not always be symmetric.

In the third chapter of the thesis we analyze the strategic trading behavior of a manipulator and how the market reacted to his trades. We find that the market on average was not able to identify the alleged manipulator's trades and that his trading costs were lower than those of the other market participants. Consistent with Allen and Gale 1992 we find that the manipulator exhibits the same behavior as informed investors in Collin-Dufresne and Fos 2015, Kacperczyk and Pagnotta 2018, Garriott and Riordan 2019. We argue that Regulation SHO mandatory settlement deadline easily binds for small-cap stocks, making manipulation in these stocks more likely.

Contents

Chapter 1

Information, Liquidity, and Dynamic Limit Order Markets

1.1 Introduction

The aggregation of private information and the dynamics of liquidity supply and demand are closely intertwined in financial markets. In dealer markets, informed and uninformed investors trade via market orders and, thus, take liquidity, while dealers provide liquidity and try to extract information from the arriving order flow (e.g., as in Kyle 1985 and Glosten and P. R. Milgrom 1985). However, in limit order markets — the dominant form of securities market organization today — the relation between who has information and who is trying to learn it and who supplies and demands liquidity is not well understood theoretically.[1] Recent empirical research highlights the role of informed traders not only as liquidity takers but also as liquidity suppliers. O'Hara 2015 argues that fast informed traders use market and limit orders interchangeably and often prefer limit orders to marketable orders. Fleming, Mizrach, and Nguyen 2017 and Brogaard, Hendershott, and Riordan 2016 find that limit orders play a significant empirical role in price discovery.[2]

Our paper presents the first rational expectations model of a dynamic limit order market with asymmetric information and history-dependent Bayesian learning. In particular, learning is not constrained to be Markovian in the limit order book. The model represents a trading day with market opening and closing effects. Our model lets us investigate the information content of different types of market and limit orders, the dynamics of who provides and demands liquidity, and the non-Markovian information content of the order history. In addition, we study how changes in the amount of adverse selection — in terms of both asset-value volatility and the arrival probability of informed investors — affect equilibrium trading strategies, liquidity, price discovery, and welfare. We have four main results:

- Increased adverse selection does not always worsen market liquidity as in Kyle 1985. Liquidity can improve if informed traders with better information trade more aggressively by submitting more limit-orders at the inside quotes rather than by using market orders.

[1] See Jain 2005 about the prevalence of limit order markets. See Parlour and Seppi 2008 for a survey of theoretical models of limit order markets. See Rindi 2008 and Boulatov and George 2013 for models of informed traders as liquidity providers.

[2] Gencay et al. 2016 investigate brief episodes of high-intensity/extreme behavior of quotation process in the U.S. equity market (bursts in liquidity provision that happen several hundreds of time a day for actively traded stocks) and find that limit orders during these bursts significantly impact prices.

- The information content of arriving orders can be opposite both order direction and aggressiveness. These patterns happen in markets in which value-shock volatility is small relative to the price grid, and when informed investors have private value shocks as well as information.
- The learning dynamics are non-Markovian in that the order history has information in addition to the current state of the limit order book.[3] In particular, the incremental information content of arriving limit and market orders is history-dependent.
- The conditional price impact of market and limit order flow (as estimated in Hasbrouck VARs) can depend on time, the current standing limit order book, and the prior order history.

Dynamic limit order markets with uninformed investors are studied in a large literature. This includes Foucault 1999, Parlour 1998, Foucault, Kadan, and Kandel 2005, Goettler, Parlour, and Rajan 2005 and Roşu 2009. There is some previous theoretical research that allows informed traders to supply liquidity. Kumar and Seppi 1994 is a static model in which optimizing informed and uninformed investors use profiles of multiple limit and market orders to trade. Kaniel and H. Liu 2006 extend the Glosten and P. R. Milgrom 1985 dealership market to allow informed traders to post limit orders. Aıt-Sahalia and Saglam 2013 also allow informed traders to post limit orders, but they do not allow them to choose between limit and market orders. Moreover, the limit orders posted by their informed traders are always at the best bid and ask prices. Goettler, Parlour, and Rajan 2009 allow informed and uninformed traders to post limit or market orders, but their model is stationary and assumes Markovian learning. Roşu 2016b studies a steady-state limit order market equilibrium in continuous-time also assuming Markovian learning with some additional information-processing restrictions. These last two papers are closest to ours. Our model differs from them in two ways: First, they assume Markovian learning in order to study dynamic trading strategies with order cancellation, whereas we simplify the strategy space (by not allowing dynamic order cancellations and submissions) in order to investigate non-Markovian learning (i.e., our model has a larger state space with full order histories). Second, we model a non-stationary trading day with opening and closing effects. Market opens and closes are important daily events in the dynamics of liquidity in financial markets. Bloomfield, O'Hara, and Saar 2005 show in an experimental market analysis that informed traders sometimes provide more liquidity than uninformed traders. Our model provides equilibrium examples of liquidity provision by informed investors.

A growing literature investigates the relation between information and trading speed (e.g., Biais, Foucault, and Moinas 2015; Foucault, Hombert, and Roşu 2016; and Roşu 2016a). However, these models assume Kyle or Glosten-Milgrom market structures and, thus, cannot consider the roles of informed and uninformed traders as endogenous liquidity providers and demanders. We argue that understanding price discovery dynamics in limit order markets is

[3] To be clear about terminology, we say a stochastic process followed by a set of variables x is *non-Markovian* if the conditional probability distributions $f[x_s|\, x_t, x_{t-1}, \ldots]$ and $f[x_s|\, x_t]$ are different for some times t and $s > t$. If a summary function $g(x_{t-1}, \ldots)$ exists such that $f[(x_s, g(x_{s-1}, \ldots))|\, (x_t, g(x_{t-1}, \ldots)), (x_{t-1}, g(x_{t-2}, \ldots)), \ldots] = f[(x_s, g(x_{s-1}, \ldots))|(x_t, g(x_{t-1}, \ldots)]$, then we say the augmented process (x, g) is Markovian but not that the unaugmented process x is Markovian.

an essential precursor to understanding speedbumps and cross-market competition given the real-world prevalence of limit order markets.

1.2 Model

We consider a limit order market in which a risky asset is traded at N discrete times $t_j \in \{t_1, \ldots, t_N\}$ over a trading day. The fundamental value of the asset at the end of the day after time t_N is

$$\tilde{v} = v_0 + \Delta = \begin{cases} \bar{v} = v_0 + \delta & \text{with } Pr(\bar{v}) = \frac{1}{3} \\ v_0 & \text{with } Pr(v_0) = \frac{1}{3} \\ \underset{\sim}{v} = v_0 - \delta & \text{with } Pr(\underset{\sim}{v}) = \frac{1}{3} \end{cases} \tag{1.1}$$

where v_0 is the ex ante expected asset value, and Δ is a symmetrically distributed value shock. The limit order market allows for trading through two types of orders: Limit orders are price-contingent orders that are collected in a limit order book. Market orders are executed immediately at the best available price in the limit order book. The limit order book has a price grid with four prices, $P_i \in \{A_2, A_1, B_1, B_2\}$, two each on the ask and bid sides of the market. The tick size is equal to $\kappa > 0$, and the ask prices are $A_1 = v_0 + \frac{\kappa}{2}$, $A_2 = v_0 + 1.5\kappa$, ; and by symmetry the bid prices are $B_1 = v_0 - \frac{\kappa}{2}$, $B_2 = v_0 - 1.5\kappa$. For simplicity, we normalize the tick size to $\kappa = 1$.

Order execution follows time and price priority. Thus, at each time t_j, seven possible actions x_{t_j} are available to investors: One possibility is to submit a market order $MBA_{i_{t_j}}$ or $MSB_{i_{t_j}}$ to buy or sell immediately at the best available ask $A_{i_{t_j}}$ or bid $B_{i_{t_j}}$ (indexed by i_{t_j}) in the limit order book at time t_j. A subscript $i_{t_j} = 1$ indicates that the best standing quote at time t_j is at an inside price A_1 or B_1, and $i_{t_j} = 2$ means the best quote is at an outside price A_2 or B_2. Alternatively, the investor can submit one of four possible limit orders LBB_i and LSA_i to buy or sell at the different prices on the ask or bid side of the book. A subscript $i = 1$ denotes an aggressive limit order posted at the inside quote, and $i = 2$ is a less aggressive limit order at the outside quotes.[4] Yet another alternative is to do nothing (NT).

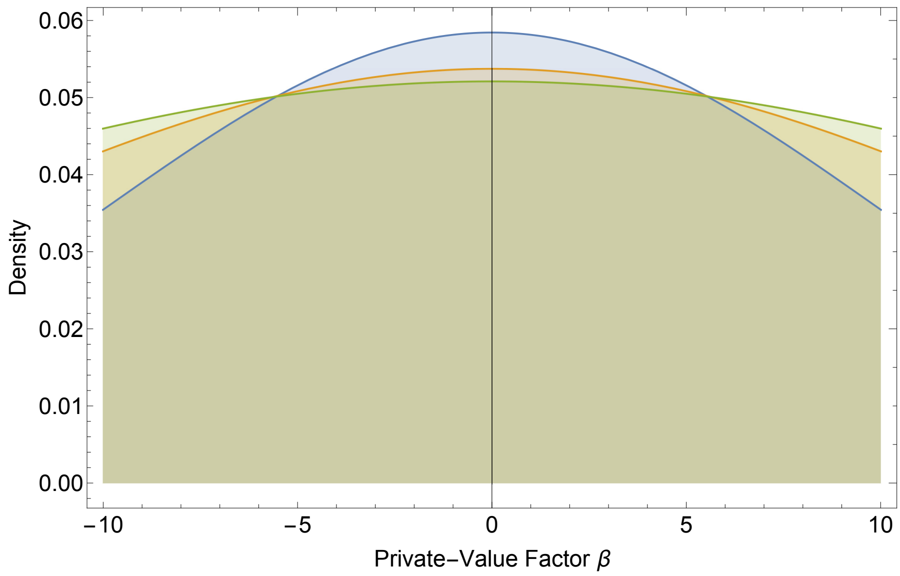
Two types of investors trade in the market. The first are a sequence of arriving active traders with potential gains-from-trade due to private information and/or random private values. One active investor arrives at each time t_j. They are risk-neutral and asymmetrically informed. The active investor arriving at time t_j is informed with probability α and uninformed with probability $1 - \alpha$. Informed investors know the realized value shock Δ perfectly. A generic informed investor is denoted as I. When we want to make explicit the specific information known by the informed investor, then we denote the informed investor as $I_{\bar{v}}$ if the value shock is positive ($\Delta = \delta$), as $I_{\underset{\sim}{v}}$ if the shock is negative ($\Delta = -\delta$), and as I_{v_0} if the shock is zero ($\Delta = 0$). Informed investors arriving at different times during the day all have the identical asset-value information (i.e., there is only one realized Δ). Uninformed investors do not know Δ, so they use Bayes' Rule and their knowledge of the equilibrium to learn about Δ from the observable order history over time. Uninformed investors are denoted as U.

[4]For tractability, it is assumed investors cannot post buy limit orders at A_1 and sell limit orders at B_1. This is one way in which the investor action space is simplified in our model.

An investor arriving at time t_j may also have an additive random personal private-value trading motive β_{t_j}. Non-informational private-value motives include preference shocks, hedging needs, and taxation. The absence of a non-informational trading motive would lead to the P. Milgrom and Stokey 1982 no-trade result. In our analysis, the factor β_{t_j} at time t_j is drawn from a truncated-Normal distribution, $Tr[\mathcal{N}(\mu, \sigma^2)]$, with support over the interval $[-10, 10]$, which corresponds to private valuations of up to plus or minus 10 ticks. The mean, $\mu = 0$, is a neutral private factor. The parameter σ determines the dispersion of an investor's private-value factor β_{t_j}, as shown in Figure 1.1, and, thus, the probability of large private gains-from-trade due to extreme private valuations.

The sequence of arriving active investors is independently and identically distributed in terms of whether investors are informed or uninformed and in terms of their individual private-value factors β_{t_j}. In one specification of our model, only uninformed investors have private valuations, while in a second richer specification both informed and uninformed investors have private valuations.

FIGURE 1.1: **Distribution of Investor Private-Value Factors - $\beta \sim Tr[\mathcal{N}(\mu, \sigma^2)]$.** This figure shows the truncated-Normal probability density function (PDF) of trader private-value factors β_{t_j} with a mean $\mu = 0$ and three different possible values of dispersion σ.



The second type of investors in the market are a group of passive liquidity providers with no active motive to trade. These investors, who we call the *trading crowd*, submit limit orders to provide liquidity. By assumption, the crowd just posts single limit orders at the outside prices A_2 and B_2. The market opens with an initial book submitted by the crowd at time t_0. After the order-submission by the arriving active investor at each time t_j, the crowd replenishes the book at the outside prices, as needed, when either side of the book is empty. Otherwise, if there are limit orders on both sides of the book, the crowd does nothing. The trading crowd effectively establishes a lower bound on the liquidity available in the market.[5]

For tractability, we make four additional simplifying assumptions. First, limit orders cannot be modified or canceled after submission. Thus, each arriving investor has one and only one opportunity to submit an order. Second, there is no quantity decision. Orders are to buy or sell one share. Third, arriving active investors can only submit one single order. Fourth, limit orders by the active investors have priority over limit orders from the crowd. The focus of our model is on market dynamics involving information and liquidity given the behavior of optimizing informed and uninformed investors. We justify this departure from time priority relative to the crowd in that we want arriving active investors to have a non-trivial choice between aggressive and less aggressive limit orders (as well as between market and limit orders) and because the crowd is simply a modeling device to insure it is always possible for arriving active investors to trade with market orders if they so choose.[6] Taken together, these assumptions let us express the action set for arriving active investors at time t_j as $X_{t_j} = \{MSB_{i_{t_j}}, LSA_1, LSA_2, NT, LBB_2, LBB_1, MBA_{i_{t_j}}\}$, where each of the orders denotes an order for one share.[7]

Our model is intentionally non-stationary over the trading day in order to capture market opening and closing effects and intraday dynamics. When the market opens at t_1, the only standing limit orders in the book are those at prices A_2 and B_2 from the trading crowd.[8] At the end of the day all unexecuted limit orders are cancelled. The state of the limit order book at a generic time t_j during the day is

$$L_{t_j} = [q_{t_j}^{A_2}, q_{t_j}^{A_1}, q_{t_j}^{B_1}, q_{t_j}^{B_2}] \tag{1.2}$$

where $q_{t_j}^{A_i}$ and $q_{t_j}^{B_i}$ indicate the total depths at prices A_i and B_i at time t_j. The limit order book changes over time due to the arrival of new limit orders (which augment the depth of the book) and market orders (which remove depth from the book) from arriving informed and uninformed investors and due to the submission of limit orders from the crowd. The resulting dynamics are:

$$L_{t_j} = L_{t_{j-1}} + Q_{t_j} + C_{t_j} \quad j = 1, \ldots, N \tag{1.3}$$

[5]The trading crowd can endogenized as HFT investors in a Budish, Cramton, and Shim 2015 style model with picking-off risk due to immediate public intraday shocks to v_0 that is in addition to the terminal shock Δ that is private information during the day.

[6]In a richer model, we could assume the crowd submits limit orders at prices three ticks from the unconditional common value v_0 and that their limit orders also have time priority.

[7]The action space X_{t_j} of orders that can be submitted at time t_j includes market orders at the standing best bid or offer at time t_j. Our notation $MSB_{i_{t_j}}$ and $MBA_{i_{t_j}}$ reflects the fact that the bid or offer at time t_j is not a fixed number but rather depends on the incoming state of the limit order book. There is no time script in the limit order notation LSA_1, ... because these are just limit orders at particular fixed prices $A_1, \ldots$ in the price grid.

[8]In practice, daily opening limit order books include uncancelled orders from the previous day and new limit orders from opening auctions. For simplicity, we abstract from these interesting features of markets.

where Q_{t_j} is the change in the book due to an arriving investor's action $x_{t_j} \in X_{t_j}$ at t_j:[9]

$$Q_{t_j} = [Q_{t_j}^{A_2}, Q_{t_j}^{A_1}, Q_{t_j}^{B_1}, Q_{t_j}^{B_2}] = \begin{cases} [-1,0,0,0] & \text{if } x_{t_j} = MBA_2 \\ [0,-1,0,0] & \text{if } x_{t_j} = MBA_1 \\ [+1,0,0,0] & \text{if } x_{t_j} = LSA_2 \\ [0,+1,0,0] & \text{if } x_{t_j} = LSA_1 \\ [0,0,0,0] & \text{if } x_{t_j} = NT \\ [0,0,+1,0] & \text{if } x_{t_j} = LBB_1 \\ [0,0,0,+1] & \text{if } x_{t_j} = LBB_2 \\ [0,0,-1,0] & \text{if } x_{t_j} = MSB_1 \\ [0,0,0,-1] & \text{if } x_{t_j} = MSB_2 \end{cases} \tag{1.4}$$

where "+1" with a limit order denotes the arrival of an additional order at a particular limit price and "−1" with a market order denotes execution of an earlier BBO limit order and where C_{t_j} is the change in the limit order book due to any limit orders submitted by the crowd

$$C_{t_j} = \begin{cases} [1,0,0,0] & \text{if } q_{t_{j-1}}^{A_2} + Q_{t_j}^{A_2} = 0 \\ [0,0,0,1] & \text{if } q_{t_{j-1}}^{B_2} + Q_{t_j}^{B_2} = 0. \\ [0,0,0,0] & \text{otherwise.} \end{cases} \tag{1.5}$$

A potentially important source of information at time t_j is the observed history of orders at prior times $t_1, .., t_{j-1}$. In particular, when traders arrive in the market, they observe the history of market activity up through the current standing limit order book at the time they arrive. However, since orders from the crowd have no incremental information beyond that in the arriving investor orders, we exclude them from the notation for the portion of the order-flow history used for informational updating of investor beliefs, which we denote by $\mathcal{L}_{t_{j-1}} = \{Q_{t_1}, \ldots, Q_{t_{j-1}}\}$.

Investors trade using optimal order-submission strategies given their information and any private-value motive. If an uninformed investor arrives at time t_j, then his order x_{t_j} is chosen to maximize his expected terminal payoff

$$\begin{aligned} \max_{x \in X_{t_j}} w^U(x \,|\beta_{t_j}, \mathcal{L}_{t_{j-1}}) &= E[(v_0 + \Delta + \beta_{t_j} - p(x))\, f(x) | \beta_{t_j}, \mathcal{L}_{t_{j-1}}] \\ &= \begin{cases} [v_0 + E[\Delta \,|\mathcal{L}_{t_{j-1}}, \theta_{t_j}^x] + \beta_{t_j} - p(x)]\, Pr(\theta_{t_j}^x | \mathcal{L}_{t_{j-1}}) & \text{if } x \text{ is a buy order} \\ [p(x) - (v_0 + E[\Delta \,|\mathcal{L}_{t_{j-1}}, \theta_{t_j}^x] + \beta_{t_j})]\, Pr(\theta_{t_j}^x | \mathcal{L}_{t_{j-1}}) & \text{if } x \text{ is a sell order} \end{cases} \end{aligned} \tag{1.6}$$

where $p(x)$ is the price at which order x trades, and $f(x)$ denotes the amount of the submitted order that is actually "filled." If x is a market order, then $p(x)$ is the best standing quote on the other side of the market at time t_j, and $f(x) = 1$ for a market buy and $f(x) = -1$ for a market sell (i.e., all of the order is executed). If x is a non-marketable limit order, then the execution price $p(x)$ is its limit price, but the fill amount $f(x)$ is random variable equal to zero if the limit order is never executed and equal to 1 if a limit buy is filled and −1 if a limit sell is

[9] There are nine alternatives in (1.4) because we allow separately for cases in which the best bid and ask for market sells and buys at time t_j are at the inside and outside quotes.

filled. If the investor does not trade — either because no order is submitted (NT) or because a limit order is not filled — then $f(x)$ is zero. In the second line of (1.6), the expression $\theta^x_{t_j}$ denotes the set of future trading states in which an order x submitted at time t_j is executed.[10] This conditioning matters for limit orders because the sequence of subsequent orders in the market, which may or may not result in the execution of a limit order submitted at time t_j, is correlated with the asset value shock Δ. For example, future market buy orders are more likely if the Δ shock is positive (since the average $I_{\overline{v}}$ investors will want to buy but not the average $I_{\underline{v}}$ investor). Uninformed investors rationally take the relation between future orders and Δ into account when forming their expectation $E[\Delta\,|\mathcal{L}_{t_{j-1}}, \theta^x_{t_j}]$ of what the asset will be worth in states in which their limit orders are executed. The second line of (1.6) also makes clear that uninformed investors use the prior order history $\mathcal{L}_{t_{j-1}}$ in two ways: It affects their beliefs about limit order execution probabilities $Pr(\theta^x_{t_j}|\mathcal{L}_{t_{j-1}})$ and their execution-state-contingent asset-value expectations $E[\Delta\,|\mathcal{L}_{t_{j-1}}, \theta^x_{t_j}]$.

An informed investor who arrives at t_j chooses an order x_{t_j} to maximize her expected payoff

$$
\begin{aligned}
\max_{x \in X_{t_j}} w^I(x\,|v, \beta_{t_j}, \mathcal{L}_{t_{j-1}}) &= E[(v_0 + \Delta + \beta_{t_j} - p(x))\, f(x)|\beta_{t_j}, \mathcal{L}_{t_{j-1}}] \\
&= \begin{cases} [v_0 + \Delta + \beta_{t_j} - p(x)]\, Pr(\theta^x_{t_j}|v, \mathcal{L}_{t_{j-1}}) & \text{if } x \text{ is a buy order} \\ [p(x) - (v_0 + \Delta + \beta_{t_j})]\, Pr(\theta^x_{t_j}|v, \mathcal{L}_{t_{j-1}}) & \text{if } x \text{ is a sell order} \end{cases}
\end{aligned}
\tag{1.7}
$$

The only uncertainty for informed investors is about whether any limit orders they submit will be executed. Their belief about order-execution probabilities $Pr(\theta^x_{t_j}|\,v, \mathcal{L}_{t_{j-1}})$ are conditioned on both the trading history up through the current book and on their knowledge about the ending asset value. Thus, informed traders condition on $\mathcal{L}_{t_{j-1}}$, not to learn about the value shock Δ (which they already know) or about future investor private-value factors β_{t_j} (which are i.i.d. over time), but rather because they understand that the trading history is an input in the trading behavior of future uninformed investors (with whom they might trade in the future) and, thus, also in the trading behavior of future informed investors (who will also take history-contingent uninformed-investor learning behavior into account when deciding whether to undercut earlier limit orders). Our analysis considers two model specifications for the informed investors. In the first, informed investors have no private-value motive, so that their β factors are equal to 0. In the second specification, their β factors are random and are independently drawn from the same truncated-Normal distribution $Tr[\mathcal{N}(\mu, \sigma^2)]$ as the uninformed investors.

The optimization problem in (1.6) defines sets of actions $x_{t_j} \in X_{t_j}$ that are optimal for the uninformed investor at different times t_j given different private-value factors β_{t_j} and order histories $\mathcal{L}_{t_{j-1}}$. These optimal orders can be unique, or there may be multiple orders which make the uninformed investor equally well-off. The *optimal order-submission strategy* for the uninformed investor is a probability function $\varphi^U_{t_j}(x|\beta_{t_j}, \mathcal{L}_{t_{j-1}})$ that is zero if the order x is suboptimal and equals a mixing probability over optimal orders. If an optimal order x is unique, then $\varphi_{t_j}(x|\beta_{t_j}, \mathcal{L}_{t_{j-1}}) = 1$. Mixed strategies are also allowed. Similarly, the optimization problem in (1.7) leads to an optimal order-submission strategy $\varphi^I_{t_j}(x|\beta_{t_j}, v, \mathcal{L}_{t_{j-1}})$ for informed investors at

[10] A market orders x_{t_j} is executed immediately at time t_j and so is executed for sure.

time t_j given their factor β_{t_j}, their knowledge about the asset value v, and the order history $\mathcal{L}_{t_{j-1}}$.

Based on the foregoing, our model has four sources of potential order-flow randomness. First, orders are random due to the random arrival of informed vs uninformed investors. Second, they are random due to the asset-value shock Δ. Third, orders are random due to randomness in investors' personal private values β_{t_j}. This is illustrated in Figure 1.2 for a numerical example of our model that is considered in detail in Section 1.3.2 and Appendix A. The plot shows where the order-submission probabilities come from for an informed investor $I_{\overline{v}}$ at time t_1 by superimposing the upper envelope of the expected payoffs for the different optimal orders at time t_1 for the case of good news about a positive value shock δ on the truncated Normal β distribution. It shows how different β subranges correspond to a discrete set of optimal orders delimited by the β thresholds. Similar constructions at other dates for informed investors and also for uninformed investors who must update their asset-value beliefs using Bayes Rule. Fourth and lastly, orders are sometimes random due to possible mixed strategies $\varphi^U_{t_j}$ and $\varphi^I_{t_j}$. However, this only happens when an investor is indifferent between a set of orders.

1.2.1 Equilibrium

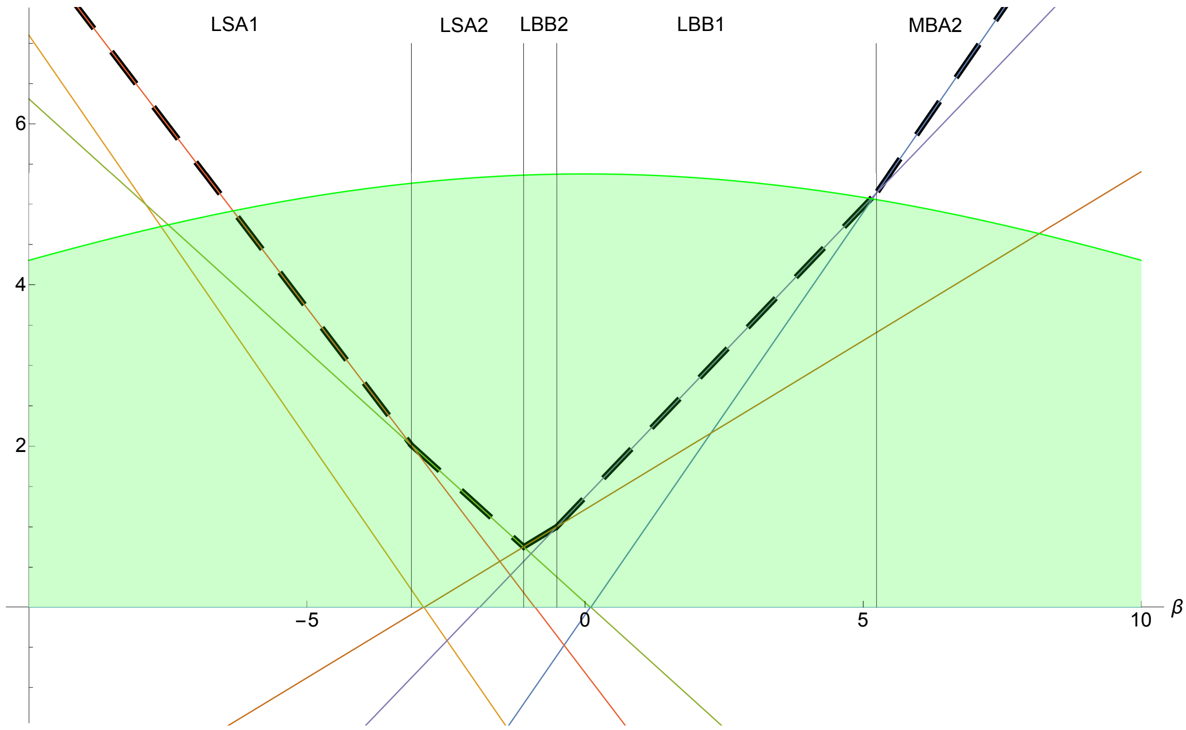
An equilibrium is a set of mutually consistent optimal strategy functions and beliefs for uninformed and informed investors for each time t_j, given each order history $\mathcal{L}_{t_{j-1}}$, private-value factor β_{t_j}, and (for informed traders) private information v. This section explains what "mutually consistent" means and then gives a formal definition of an equilibrium.

A central feature of our model is asymmetric information. The presence of informed investors means that, by observing orders over time, uninformed traders can infer information about the asset value v and use it in their order-submission strategies. More precisely, uninformed traders rationally learn from the trading history about the probability that v will go up, stay constant, or go down. However, investors cannot learn about the private values (β) or information status (I or U) of future traders since, by assumption, these are both i.i.d over time. Informed investors do not need to learn about v since they know it directly. However, they do condition their orders on v (both because v is the final stock value and also because v tells them what type of informed investors I_v will arrive in the future along with the uninformed U traders). Informed investors also condition on the order-flow history $\mathcal{L}_{t-1}$, since $\mathcal{L}_{t-1}$ affects the trading behavior of future investors.[11]

The underlying *economic state* in our model is the realization of the asset value v and a realized sequence of investors who arrive in the market. The investor who arrives at time t_j is described by two characteristics: their status as being informed or uninformed, I or U, and their private-value factor β_{t_j}. The underlying economic state is exogenously chosen over time by Nature. More formally, it follows an exogenous stochastic process described by the model parameters δ, α, μ, and σ. A sequence of arriving investors together with a pair of strategy functions — which we denote here as $\Phi = \{\varphi^U_{t_j}(x|\beta_{t_j}, \mathcal{L}_{t_{j-1}}), \varphi^I_{t_j}(x|\beta_{t_j}, v, \mathcal{L}_{t_{j-1}})\}$ — induce a sequence of trading actions x_{t_j} which — together with the predictable actions of the trading crowd — results

[11] The order history $\mathcal{L}_{t-1}$ is an input in the uninformed-investor learning problem and, thus, is an input in their order-submission strategy. In addition, since future informed investors know that $\mathcal{L}_{t-1}$ can affect uninformed investor trading behavior, it also enters the order-submission strategies of future informed investors.

FIGURE 1.2: **β Distribution and Upper Envelope for Informed Investor $I_{\bar{v}}$ at time t_1.** This figure shows the private-value factor $\beta \sim Tr[\mathcal{N}(\mu, \sigma^2)]$ distribution superimposed on the plot of the expected payoffs the informed investor $I_{\bar{v}}$ with good news at time t_1 for each equilibrium order type MBA_2, MSB_2, LSA_2, LSA_1, LBB_1, LBB_2, NT, (solid colored lines) when the total book (including crowd limit orders) opens $L_{t_0} = [1\ 0\ 0\ 1]$. The dashed line shows the investor's upper envelope for the optimal orders. The vertical black lines show the β-thresholds at which two adjacent optimal strategies yield the same expected payoffs. For example LSA_1 is the optimal strategy for values of β between 0 and the first vertical black line; LSA_2 is instead the optimal strategy for the values of beta between the first and the second vertical lines; and so forth. The parameters are $\alpha = 0.8$, $\delta = 1.6$, $\mu = 0$, $\sigma = 15$, and $\kappa = 1$.



in a sequence of observable changes in the state L_{t_j} of the limit order book. Thus, the stochastic process generating paths of order histories is induced by the economic state process and the strategy functions. Given the order-path process, several probabilistic quantities can be computed directly: First, we can compute the unconditional probabilities of different paths $Pr(\mathcal{L}_{t_j})$ and the conditional probabilities $Pr(Q_{t_j}|\mathcal{L}_{t_{j-1}})$ of particular order book changes Q_{t_j} due to arriving investors given a prior history $\mathcal{L}_{t_{j-1}}$. Certain paths of orders are *possible* (i.e., have positive probability $Pr(\mathcal{L}_{t_j})$) given the strategy functions $\{\varphi^U_{t_j}(x|\beta, \mathcal{L}_{t_{j-1}}), \varphi^I_{t_j}(x|\beta, v, \mathcal{L}_{t_{j-1}})\}$, and certain paths of orders are not possible (i.e., for which $Pr(\mathcal{L}_{t_j}) = 0$). Second, the endogenous order-path process also determines the order-execution probabilities $Pr(\theta^x_{t_j}|\, v, \mathcal{L}_{t_{j-1}})$ and $Pr(\theta^x_{t_j}|\, \mathcal{L}_{t_{j-1}})$ for informed and uninformed investors for various orders x submitted at time t_j. Computing each of these probabilities is simply a matter of listing all of the possible underlying economic states, mechanically applying the order-submission rules, identifying the relevant outcomes path-by-path, and then taking expectations across paths.

Let ℓ denote the set of all feasible histories $\{\mathcal{L}_{t_j} : j = 1, \dots, 4\}$ of physically available orders of lengths up to four trading periods. A four-period long history is the longest history a order-submission strategy can depend on in our model. In this context, *feasible* paths are simply sequences of actions from the action choice sets X_{t_j} over time without regard to whether they are *possible* in the sense that they occur with positive probability given the strategy functions Φ. Let $\ell^{in,\Phi}$ denote the subset of all possible trading paths in ℓ that have positive probability, $Pr(\mathcal{L}_{t_j}) > 0$, given a pair of order strategies Φ. Let $\ell^{off,\Phi}$ denote the complementary set of trading paths that are *feasible* but *not possible* given Φ. This notation will be useful when discussing "equilibrium" beliefs on order paths that have positive probability and "off equilibrium" beliefs on paths that have zero probability given investor strategies. In our analysis, strategy functions Φ are defined for all feasible paths in ℓ. In particular, this includes all of the possible paths in $\ell^{in,\Phi}$ given Φ and also the paths in $\ell^{off,\Phi}$. As a result, the probabilities $Pr(Q_{t_j}|\mathcal{L}_{t_{j-1}})$, $Pr(\theta^x_{t_j}|\, v, \mathcal{L}_{t_{j-1}})$ and $Pr(\theta^x_{t_j}|\, \mathcal{L}_{t_{j-1}})$ are always well-defined, because the continuation trading process going forward — even after an unexpected order-arrival event (i.e., a path $\mathcal{L}_{t_{j-1}} \in \ell^{off,\Phi}$) — is still well-defined.

The stochastic process for order paths and its relation to the underlying economic state also determine the uninformed-investor expectations $E[v\,|\mathcal{L}_{t_{j-1}}, \theta^x_{t_j}]$ of the terminal asset value given the previous order history ($\mathcal{L}_{t_{j-1}}$) and conditional on future execution of a limit order x submitted at time t_j (denoted here by the set of future states $\theta^x_{t_j}$ in which this happens). In particular, belief and expectation formation for the uninformed investor involve backward conditioning on the prior order history $\mathcal{L}_{t_{j-1}}$ and forward conditioning on the endogenous set of future states $\theta^x_{t_j}$ in which limit orders are executed. These beliefs and expectations are determined as follows:

- Step 1: The conditional probabilities $\pi^v_{t_j} = Pr(v|\mathcal{L}_{t_j})$ of a particular final asset value $v = \bar{v}, v_0$ or $\underline{v}$ given a possible trading history $\mathcal{L}_{t_j} \in \ell^{in,\Phi}$ up through time t_j is given by Bayes' Rule. At time t_1, this probability is

$$\begin{aligned}
\pi^v_{t_1} &= \frac{Pr(v, \mathcal{L}_{t_1})}{Pr(\mathcal{L}_{t_1})} = \frac{Pr(\mathcal{L}_{t_1}|v)Pr(v)}{Pr(\mathcal{L}_{t_1})} = \frac{Pr(Q_{t_1}|v)Pr(v)}{Pr(Q_{t_1})} \\
&= \frac{Pr(Q_{t_1}|v, I)Pr(I) + Pr(Q_{t_1}|U)Pr(U)}{Pr(Q_{t_1})} Pr(v) \\
&= \frac{E^\beta[\varphi^I_{t_1}(x_{t_1}|\beta^I_{t_1}, v)|v]\alpha + E^\beta[\varphi^U_{t_1}(x_{t_1}|\beta^U_{t_1})](1-\alpha)}{Pr(Q_{t_1})}\, \pi^v_{t_0}
\end{aligned} \tag{1.8}$$

  where the prior is the unconditional probability $\pi^v_{t_0} = Pr(v)$, x_{t_1} is the order at time t_1 that leads to the order book change Q_{t_1}, and $\beta^I_{t_1}$ and $\beta^U_{t_1}$ are independently distributed private-value β realizations for informed and uninformed investors at time t_1.[12] At times

[12] A trader's information status (I or U) is independent of the asset value v, so $P(I|v) = Pr(I)$ and $Pr(U|v) = Pr(U)$. Furthermore, uninformed traders have no private information about v, so the probability $Pr(Q_{t_1}|U)$ with which they take a trading action Q_{t_1} does not depend on v.

$t_j > t_1$, the history-conditional probabilities are given recursively by[13]

$$\pi_{t_j}^{v} = \frac{Pr(v,\mathcal{L}_{t_j})}{Pr(\mathcal{L}_{t_j})} = \frac{Pr(v,Q_{t_j},\mathcal{L}_{t_{j-1}})}{Pr(Q_{t_j},\mathcal{L}_{t_{j-1}})} = \frac{\begin{pmatrix} Pr(Q_{t_j}|v,\mathcal{L}_{t_{j-1}},I)Pr(I|\mathcal{L}_{t_{j-1}})Pr(v|\mathcal{L}_{t_{j-1}}) \\ +Pr(Q_{t_j}|v,\mathcal{L}_{t_{j-1}},U)Pr(U|\mathcal{L}_{t_{j-1}})Pr(v|\mathcal{L}_{t_{j-1}}) \end{pmatrix}}{Pr(Q_{t_j}|\mathcal{L}_{t_{j-1}})}$$
$$= \frac{E^{\beta}[\varphi^{I}_{t_j}(x_{t_j}|\beta^{I}_{t_j},v,\mathcal{L}_{t_{j-1}})|v,\mathcal{L}_{t_{j-1}}]\ \alpha + E^{\beta}[\varphi^{U}_{t_j}(x_{t_j}|\beta^{U}_{t_j},\mathcal{L}_{t_{j-1}})|\mathcal{L}_{t_{j-1}}]\ (1-\alpha)}{Pr(Q_{t_j}|\mathcal{L}_{t_{j-1}})}\pi^{v}_{t_{j-1}} \quad (1.9)$$

Given these probabilities, the expected asset value conditional on the order history is

$$E[\tilde{v}|\mathcal{L}_{t_{j-1}}] = \pi^{\bar{v}}_{t_{j-1}}\,\bar{v} + \pi^{v_0}_{t_{j-1}}\,v_0 + \pi^{\underline{v}}_{t_{j-1}}\underline{v} \quad (1.10)$$

- Step 2: The conditional probabilities $\pi^{v}_{t_j}$ given a "feasible but not possible in equilibrium" order history $\mathcal{L}_{t_j} \in \ell^{off,\Phi}$ in which a limit order book change Q_{t_j} that is inconsistent with the strategies Φ at time t_j are set as follows:
  1. If the priors are fully revealing in that $\pi^{v}_{t_{j-1}} = 1$ for some v, then $\pi^{v}_{t_j} = \pi^{v}_{t_{j-1}}$ for all v.
  2. If the priors are not fully revealing at time t_j, then $\pi^{v}_{t_j} = 0$ for any v for which $\pi^{v}_{t_{j-1}} = 0$ and the probabilities $\pi^{v}_{t_j}$ for the remaining v's can be any non-negative numbers such that $\pi^{\bar{v}}_{t_j} + \pi^{v_0}_{t_j} + \pi^{\underline{v}}_{t_j} = 1$.
  3. Thereafter, until any next unexpected trading event, the subsequent probabilities $\pi^{v}_{t_{j'}}$ for $j' > j$ are updated according to Bayes' Rule as in (1.9).
- Step 3: The execution-contingent conditional probabilities $\hat{\pi}^{v}_{t_j} = Pr(v|\mathcal{L}_{t_{j-1}},\theta^{x}_{t_j})$ of a final asset value v conditional on a prior path $\mathcal{L}_{t_{j-1}}$ and on execution of a limit order x submitted at time t_j is

$$\hat{\pi}^{v}_{t_j} = \frac{Pr(\mathcal{L}_{t_{j-1}})Pr(v|\mathcal{L}_{t_{j-1}})\,\mathrm{Pr}(\theta^{x}_{t_{j-1}}|v,\mathcal{L}_{t_{j-1}})}{Pr(\theta^{x_{t_j}},\mathcal{L}_{t_{j-1}})} \quad (1.11)$$
$$= \frac{\mathrm{Pr}(\theta^{x}_{t_j}|v,\mathcal{L}_{t_{j-1}})}{Pr(\theta^{x}_{t_j}|\mathcal{L}_{t_{j-1}})}\pi^{v}_{t_{j-1}}$$

This holds when adjusting for a future execution contingency both when the probabilities $\pi^{v}_{t_{j-1}}$ given the prior history $\mathcal{L}_{t_{j-1}}$ are for possible paths in $\ell^{in,\Phi}$ (from (1.8) and (1.9) in Step 1) and also for feasible but not possible paths in $\ell^{off,\Phi}$ (from Step 2). These execution-contingent probabilities $\hat{\pi}^{v}_{t_j}$ are used to compute the execution-contingent conditional expected value

$$E[\tilde{v}|\mathcal{L}_{t_{j-1}},\theta^{x}_{t_j}] = \hat{\pi}^{\bar{v}}_{t_j}\,\bar{v} + \hat{\pi}^{v_0}_{t_j}\,v_0 + \hat{\pi}^{\underline{v}}_{t_j}\,\underline{v} \quad (1.12)$$

[13] A trader's information status is again independent of v, and it is also independent of the past trading history $\mathcal{L}_{t_1}$. While the probability with which an uninformed trader takes a trading action Q_{t_1} may depend on the past order history $\mathcal{L}_{t_j}$, it does not depend directly on v which uninformed traders do not know.

used by uninformed traders to compute expected payoffs for limit orders. In particular, the probabilities in (1.12) are the execution-contingent probabilities $\hat{\pi}^v_{t_j}$ from (1.11) rather than the probabilities $\pi^v_{t_j}$ from (1.9) that just condition on the prior trading history but not on the future states in which the limit order is executed.

Given these updating dynamics, we can now define an equilibrium.

Definition. A *Perfect Bayesian Nash Equilibrium* of the trading game in our model is a collection $\{\varphi^{U,*}_{t_j}(x|\beta_{t_j}, \mathcal{L}_{t_{j-1}}), \varphi^{I,*}_{t_j}(x|\beta_{t_j}, v, \mathcal{L}_{t_{j-1}}), Pr^*(\theta^x_{t_j}|\, v, \mathcal{L}_{t_{j-1}}), Pr^*(\theta^x_{t_j}|\, \mathcal{L}_{t_{j-1}}), E^*[\tilde{v}|\mathcal{L}_{t_{j-1}}, \theta^x_{t_j}]\}_{j=1}^N$ of order-submission strategies, execution-probability functions, and execution-contingent conditional expected asset-value functions such that:

- The equilibrium execution probabilities $Pr^*(\theta^x_{t_j}|\, v, \mathcal{L}_{t_{j-1}})$ and $Pr^*(\theta^x_{t_j}|\, \mathcal{L}_{t_{j-1}})$ are consistent with the equilibrium order-submission strategies $\{\varphi^{U,*}_{t_{j+1}}(x|\beta_{t_{j+1}}, \mathcal{L}_{t_j}), \ldots, \varphi^{U,*}_{t_5}(x|\beta_{t_5}, \mathcal{L}_{t_4})\}$ and $\{\varphi^{I,*}_{t_{j+1}}(x|\beta_{t_{j+1}}, v, \mathcal{L}_{t_j}), \ldots, \varphi^{I,*}_{t_5}(x|\beta_{t_5}, v, \mathcal{L}_{t_4})\}$ after time t_j.
- The execution-contingent conditional expected asset values $E^*[\tilde{v}|\mathcal{L}_{t_{j-1}}, \theta^x_{t_j}]\}$ agree with Bayesian updating equations (1.8), (1.9), (1.11), and (1.12) in Steps 1 and 3 when the order x is consistent with the equilibrium strategies $\varphi^{U,*}_{t_j}(x|\beta_{t_j}, \mathcal{L}_{t_{j-1}})$ and $\varphi^{I,*}_{t_j}(x|\beta_{t_j}, v, \mathcal{L}_{t_{j-1}})$ at date t_j and, when x is an off-equilibrium action inconsistent with the equilibrium strategies, with the off-equilibrium updating in Step 2.
- The positive-probability supports of the equilibrium strategy functions $\varphi^{U,*}_{t_j}(x|\beta_{t_j}, \mathcal{L}_{t_{j-1}})$ and $\varphi^{I,*}_{t_j}(x|\beta_{t_j}, v, \mathcal{L}_{t_{j-1}})$ (i.e., the orders with positive probability in equilibrium) are subsets of the sets of optimal orders for uninformed and informed investors computed from their optimization problems (1.6) and (1.7) and the equilibrium execution probabilities and outcome-contingent conditional asset-value expectation functions $Pr^*(\theta^x_{t_j}|\, v, \mathcal{L}_{t_{j-1}})$, $Pr^*(\theta^x_{t_j}|\, \mathcal{L}_{t_{j-1}})$, and $E^*[\tilde{v}|\mathcal{L}_{t_{j-1}}, \theta^x_{t_j}]$.

Our equilibrium concept differs from the Markov Perfect Bayesian Equilibrium used in Goettler, Parlour, and Rajan 2009. Beliefs and strategies in our model are path-dependent; that is to say, traders use Bayes Rule given the full prior order history when they arrive in the market. In contrast, Goettler, Parlour, and Rajan 2009 restricts Bayesian updating to the current state of the limit order book and does not allow for conditioning on the previous order history. Roşu 2016b also assumes a Markov Perfect Bayesian Equilibrium. The quantitative importance of the order history is considered when we discuss our results in Section 1.3.

To help with intuition, Appendix A walks through the order-submission and Bayesian updating mechanics for a particular realized equilibrium path in the extensive form of the trading game. Appendix B explains the algorithm used to compute equilibria in our model.

1.3 Results

This section presents results about how liquidity supply and demand decisions of informed and uninformed traders and the learning process of uninformed traders affect market liquidity, price discovery, and investor welfare. Section 1.3.1 first considers a model specification in which only uninformed investors have random private-value trading motives. Section 1.3.2 considers a second specification that generalizes the analysis and shows the robustness of our findings and extends them when informed investors also have private-value motives. Throughout the numerical illustrations, the number of trading rounds is $N = 5$, and the private-value dispersion σ is 15.

We focus on two time windows. The first is when the market opens at time t_1. The second is over the middle of the trading day from times t_2 through t_4. We look at these two windows because our model is non-stationary over the trading day. Much like actual trading days, our model has start-up effects at the beginning of the day and terminal horizon effects at the market close. When the market opens at time t_1, there are time-dependent incentives to provide, rather than to take, liquidity: The opening book is thin (with limit orders only from the crowd), and there is the maximum time for future investors to arrive to hit limit orders from t_1. There are also time-dependent disincentives for limit orders. Information asymmetries are maximal at time t_1, since there has been no learning through the trading process. Also, there is the maximal time for early less aggressive limit orders (at A_2 and B_2) to be undercut by more aggressive later limit orders (at A_1 and B_1). Over the day, information is revealed (lessening adverse selection costs), but the book can also become fuller (i.e., there is competition in liquidity provision from earlier limit orders with time priority at their respective limit prices), and the remaining time for market orders to arrive and execute limit orders becomes shorter. Comparing these two time windows shows how market dynamics change over the day. The market close at t_5 is also important, but trading then is straightforward. At the end of the day, investors only submit market orders (or do not trade), because the execution probability for new limit orders at t_5 is zero given our assumption that unfilled limit orders are canceled once the market closes. Our choice of $N = 5$ trading rounds in a day is computationally tractable while still allowing time for relatively less constrained endogenous choices between market and limit orders at times t_2 through t_4 away from the immediate mechanical effects of the relatively thin book at the market open at t_1 and the end-of-day market orders at t_5.

We use our model to investigate three questions: First, who provides and takes liquidity, and how does the amount of adverse selection affect investor decisions to take and provide liquidity? Second, how does market liquidity vary with different amounts of adverse selection? Third, how does the information content of different types of orders depend on an order's direction, aggressiveness, and on the prior order history?

The amount of adverse selection can change in two ways: The proportion α of informed traders can change, and the magnitude δ of the asset-value shocks can change. We present comparative statics using four different combinations of parameters with high and low informed-investor arrival probabilities ($\alpha = 0.8$ and 0.2) and high and low value-shock volatilities ($\delta =$ 1.6 and 0.2). We call markets with $\delta = 0.02$ *low-volatility* markets and markets with $\delta = 1.6$ *high-volatility* markets, because the arriving information is small relative to the $\kappa = 1$ tick size in the former parameterization and larger relative to the tick size in the later. In high-volatility markets, the final asset value v given good or bad news is beyond the outside quotes A_2 or B_2, and so even market orders at the outside prices are profitable for informed traders. However, in low-volatility markets, v is always within the inside quotes A_1 and B_1, and so market orders are never profitable for informed investors. A real-world example are markets for individual stocks where heteroskedastic fluctuation in the daily volatility of arriving information can flip the market for a given stock with a fixed one-penny tick size over time between being a high and low volatility market. Another example is that futures contracts on different underlyings have customized price grids that can be large or small relative to their underlying information flow.

1.3.1 Uninformed traders with random private-value motives

In our first model specification, only uninformed U traders have random private values β_{t_j}. Informed I traders have fixed neutral private-value factors $\beta_{t_j} = 0$. Thus, as in Kyle 1985, there is a clear differentiation between investors who speculate on private information and those who trade for purely non-informational reasons. Unlike Kyle 1985, informed and uninformed investors here can choose to trade using limit or market orders rather than being restricted to just market orders.

Trading strategies

We begin by investigating who supplies and takes liquidity and how these decisions change with the amount of adverse selection. Our starting point establishes from first principles that different forms of adverse selection affect investors' trading decisions differently.

Proposition 1 Trading strategies are affected differently by changes in adverse selection due to changes in the value-shock size δ vs. changes in the informed-investor arrival probability α.

Proof: Consider first the effect of changes in the value-shock δ on informed-investor order submissions given any fixed α. If the value-shock δ is sufficiently close to zero, then directionally informed $I_{\bar{v}}$ and $I_{\underline{v}}$ investors with good or bad news never use market orders, since the terminal asset value v is always between the inside bid and ask prices A_1 and B_1 given a discrete tick size κ. However, once δ is sufficiently large, investors with good and bad news start to use market orders for their guaranteed execution. Thus, the set of orders used by directionally informed investors can change when δ changes. This is true independently of the informed-investor arrival probability α. In contrast, consider the effect of the informed-investor arrival probability α on informed-investor order submission given a fixed δ. If the value-shocks δ are close to zero, informed investors with good or bad news never use market orders for any informed-investor arrival probability α. They are unwilling to pay a large tick size to trade on their small information. Instead act as liquidity providers using limit orders to supply liquidity asymmetrically depending on the direction of their information. Thus, the set of orders used by directionally informed investors in low-volatility markets never changes to include market orders when α changes. □

Numerical results illustrate other facets of how adverse selection affects investor trading. Table 1.1 reports results about trading early in the day at time t_1 using a 2×2 format. Each of the four cells corresponds to a different combination of parameters. Comparing cells horizontally shows the effect of a change in the value-shock size δ while holding the arrival probability α for informed traders fixed. Comparing cells vertically shows the effect of a change in the informed-investor arrival probability while holding the value-shock size fixed. In each cell corresponding to a set of parameters, there are four columns reporting conditional results for informed investors with good news, neutral news, and bad news about the asset ($I_{\overline{v}}$, I_{v_0}, $I_{\underline{v}}$) and for an uninformed investor (U) and a fifth column with the unconditional market results ($Uncond$). The table reports the order-submission probabilities and several market-quality metrics. Specifically, we report expected bid-ask spreads conditioning on the three informed-investor types $E[Spread\,|I_v]$ and on the uninformed trader $E[Spread\,|U]$, the unconditional expected market spread $E[Spread]$, and expected depths at the inside prices (A_1 and B_1) and the total at both prices ($A_1 + A_2$ and $B_1 + B_2$) on each side of the market. As we shall see, our results are symmetric for the directionally informed investors $I_{\overline{v}}$ and $I_{\underline{v}}$ on the buy and sell sides of the market. In addition, we report the probability-weighted contributions to the different investors' welfare (i.e., expected gains-from-trade) from limit and market orders respectively, and their total expected welfare.[14] Table B1 in Appendix B provides additional results about conditional and unconditional future execution probabilities for the different orders ($P^{EX}(x_{t_1})$) and also the uninformed investor's updated expected asset value $E[v|x_{t_1}]$ given different types of buy orders x_{t_1} at time t_1.

Table 1.2 shows average results for times t_2 through t_4 during the day using a similar 2×2 format. The averages are across time and trading histories. Comparing results for time t_1 with the averages for t_2 through t_4 shows intraday variation in the trading process. There is no table for time t_5, because only market orders are used at the market close.

One order-submission property that is important for market-quality and order-informativenes results below is that directionally informed investors $I_{\overline{v}}$ and $I_{\underline{v}}$ tend to trade more aggressively in a high-volatility markets in which value shocks are large relative to the tick size. This is intuitive since larger potential trading gains-from-trade make price improvement less important relative to trade execution. This property can be seen in Table 1.1 where $I_{\overline{v}}$ and $I_{\underline{v}}$ investors at time t_1 only post limit orders at the less-aggressive outsides quotes A_2 and B_2 in the two low-volatility parameterizations on the right (with $\delta = 0.2$ and $\alpha = 0.2$ or 0.8) but use limit orders with positive probability at both the aggressive inside quotes A_1 and B_1 as well as at the outside quotes in the two high-volatility parameterization cells on the left (with $\delta = 1.6$ and the same two respective αs). This trading-aggressiveness property can also be seen in different ways in the average order-submission probabilities at times t_2 through t_4 in Table 1.2. In the low-volatility parameterizatons on the right, informed $I_{\overline{v}}$ and $I_{\underline{v}}$ investors supply liquidity via limit orders on both sides of the market with order-submission probabilities that are somewhat skewed at the inside quote in the direction of their small amount of private information. Moving to the high-volatility parameterizations on the left, we see that, when the informed-investor arrival probability α is low (0.2), directionally informed investors increase the probability of using aggressive limit orders at the inside prices to trade in the direction of their information. However, when the informed-investor arrival probability α is high (0.8), the

[14] Let $W^U(\beta_{t_1})$ and $W^I(v, \beta_{t_1})$ denote the value functions when (1.6) and (1.7) are evaluated at time t_1 using the optimal strategies for the uninformed and informed investors respectively. The total ex ante welfare gain is $E[W^U(\beta_{t_1})]$ for the uninformed investor where the expectation is taken over β_{t_1} and $E[W^I(v, \beta_{t_1})]$ for the informed investor where the expectation is taken over v and β_{t_1}.

increased trading aggressiveness by informed investors in the high-volatility market takes a different form. Informed $I_{\bar{v}}$ and $I_{\underline{v}}$ investors reduce their use of all types of limit orders and increase their use of market orders at times t_2 to t_4.

Next, consider the neutrally informed I_{v_0} investors and uninformed U investors. These investors respond differently to adverse selection because the informed I_{w_0} investors have an advantage over uninformed U investors: There is no adverse selection risk for the I_{v_0} investors. They know the value shock Δ is 0 and, thus, that the unconditional valuation v_0 is correct. Tables 1.1 and 1.2 show that as adverse selection increases (via both larger δs and larger αs), liquidity-provision by the I_{v_0} investors is unchanged at time t_1 and becomes somewhat more aggressive on average in the use of limit orders at the inside prices at times t_2 through t_4. These results are qualitatively consistent with the intuition of Bloomfield, O'Hara and Saar (2005), who find in laboratory experiments that informed investors provide liquidity via limit orders when mispricing is small in a market. In contrast, uninformed U investors become less willing to provide liquidity via aggressive limit orders at the inside quotes as adverse selection worsens. Rather, they increasingly take liquidity via market orders or supply liquidity via less aggressive limit orders at the outside quotes. This reduction in liquidity provision at the inside quotes by uninformed U investors happens at time t_1 (Table 1.1) and at times t_2 through t_4 (Table 1.2) and for both larger value shocks δ and higher informed-investor arrival probabilities α.

An equilibrium interaction in investor trading behavior is noteworthy in this context. Uninformed U investors are unwilling to use aggressive limit orders at the inside quotes when the adverse selection risk is sufficiently high as in the upper-left parametrization ($\alpha = 0.8$ and $\delta = 1.6$). This explains the fact that informed $I_{\bar{v}}$ and $I_{\underline{v}}$ investors use aggressive limit orders at the inside quotes with a higher probability at time t_1 in the lower-left intermediate adverse-selection parametrization (0.930 with $\alpha = 0.2$ and $\delta = 1.6$) than in the upper-left high adverse-selection parameterization (0.360). At first glance this might seem counterintuitive since competition from future informed investors (and the possibility of being undercut by later limit orders) is greater when the informed-investor arrival probability is large ($\alpha = 0.8$) than when α is smaller. However, in equilibrium there is camouflage from the uninformed U investor limit orders at the inside quotes in the lower-left parametization, whereas limit orders at the inside quotes are fully revealing in the upper-left parametrization. Table B1 in Appendix B shows that, as a result, the execution probabilities for the fully revealing limit orders at prices that are revealed to be far from the asset's actual value are much lower (0.078) relative to the non-fully revealing limit orders (0.713).

TABLE 1.1: **Trading Strategies, Liquidity, and Welfare at Time t_1 in an Equilibrium with Informed Traders with $\beta = 0$ and Uninformed Traders with $\beta \sim Tr[\mathcal{N}(\mu, \sigma^2)]$.** This table reports results for two different informed-investor arrival probabilities α (0.8 and 0.2) and two different value-shock volatilities δ (1.6 and 0.2). The private-value factor parameters are $\mu = 0$ and $\sigma = 15$, and the tick size is $\kappa = 1$. Each cell corresponding to a set of parameters reports the equilibrium order-submission probabilities, the expected bid-ask spreads and expected depths at the inside prices (A_1 and B_1) and total depths on each side of the market after order submissions at time t_1, and expected welfare of the market participants. The first four columns in each parameter cell are for informed traders with positive, neutral and negative signals, ($I_{\bar{v}}$,I_{v_0},$I_{\underline{v}}$) and for uninformed traders (U). The fifth column ($Uncond.$) reports unconditional results for the market.

		$\delta = 1.6$					$\delta = 0.2$				
		$I_{\bar{v}}$	I_{v_0}	$I_{\underline{v}}$	U	$Uncond.$	$I_{\bar{v}}$	I_{v_0}	$I_{\underline{v}}$	U	$Uncond.$
	LSA_2	0	0.500	0.640	0.145	0.333	0	0.500	1.000	0.052	0.410
	LSA_1	0	0	0.360	0	0.096	0	0	0	0.079	0.016
	LBB_1	0.360	0	0	0	0.096	0	0	0	0.079	0.016
	LBB_2	0.640	0.500	0	0.145	0.333	1.000	0.500	0	0.052	0.410
	MBA_2	0	0	0	0.355	0.071	0	0	0	0.369	0.074
	MBA_1	0	0	0	0	0	0	0	0	0	0
	MSB_1	0	0	0	0	0	0	0	0	0	0
	MSB_2	0	0	0	0.355	0.071	0	0	0	0.369	0.074
	NT	0	0	0	0	0	0	0	0	0	0
$\alpha = 0.8$											
	E[Spread $\vert\cdot$]	2.640	3.000	2.640	3.000	2.808	3.000	3.000	3.000	2.842	2.968
	E[Depth A_2+A_1 $\vert\cdot$]	1.000	1.500	2.000	1.145	1.429	1.000	1.500	2.000	1.131	1.426
	E[Depth A_1 $\vert\cdot$]	0	0	0.360	0	0.096	0	0	0	0.079	0.016
	E[Depth B_1 $\vert\cdot$]	0.360	0	0	0	0.096	0	0	0	0.079	0.016
	E[Depth B_1+B_2 $\vert\cdot$]	2.000	1.500	1.000	1.145	1.429	2.000	1.500	1.000	1.131	1.426
	E[Welfare LO $\vert\cdot$]	0.347	0.596	0.347	0.194	0.383	0.288	0.688	0.288	0.153	0.368
	E[Welfare MO $\vert\cdot$]	0	0	0	3.361	0.672	0	0	0	3.390	0.678
	E[Welfare $\vert\cdot$]	0.347	0.596	0.347	3.554	1.055	0.288	0.688	0.288	3.543	1.046
	LSA_2	0	0.500	0.070	0.065	0.090	0	0.500	1.000	0.063	0.150
	LSA_1	0	0	0.930	0.368	0.356	0	0	0	0.397	0.318
	LBB_1	0.930	0	0	0.368	0.356	0	0	0	0.397	0.318
	LBB_2	0.070	0.500	0	0.065	0.090	1.000	0.500	0	0.063	0.150
	MBA_2	0	0	0	0.068	0.054	0	0	0	0.040	0.032
	MBA_1	0	0	0	0	0	0	0	0	0	0
	MSB_1	0	0	0	0	0	0	0	0	0	0
	MSB_2	0	0	0	0.068	0.054	0	0	0	0.040	0.032
	NT	0	0	0	0	0	0	0	0	0	0
$\alpha = 0.2$											
	E[Spread $\vert\cdot$]	2.070	3.000	2.070	2.265	2.288	3.000	3.000	3.000	2.206	2.365
	E[Depth A_2+A_1 $\vert\cdot$]	1.000	1.500	2.000	1.432	1.446	1.000	1.500	2.000	1.460	1.468
	E[Depth A_1 $\vert\cdot$]	0	0	0.930	0.368	0.356	0	0	0	0.397	0.318
	E[Depth B_1 $\vert\cdot$]	0.930	0	0	0.368	0.356	0	0	0	0.397	0.318
	E[Depth B_1+B_2 $\vert\cdot$]	2.000	1.500	1.000	1.432	1.446	2.000	1.500	1.000	1.460	1.468
	E[Welfare LO $\vert\cdot$]	2.726	1.471	2.726	3.094	2.937	0.809	1.497	0.809	3.595	3.084
	E[Welfare MO $\vert\cdot$]	0	0	0	1.045	0.836	0	0	0	0.642	0.514
	E[Welfare $\vert\cdot$]	2.726	1.471	2.726	4.139	3.773	0.809	1.497	0.809	4.238	3.598

TABLE 1.2: **Averages for Trading Strategies, Liquidity, and Welfare across Times t_2 through t_4 for Informed Traders with $\beta = 0$ and Uninformed Traders with $\beta \sim Tr[\mathcal{N}(\mu, \sigma^2)]$.** This table reports results for two different informed-investor arrival probabilities α (0.8 and 0.2) and for two different asset-value volatilities δ (1.6 and 0.2). The private-value factor parameters are $\mu = 0$ and $\sigma = 15$, and the tick size is $\kappa = 1$. Each cell corresponding to a set of parameters reports the equilibrium order-submission probabilities, the expected bid-ask spreads and expected depths at the inside prices (A_1 and B_1) and total depths on each side of the market after order submissions at times t_2 through t_4, and expected welfare for the market participants. The first four columns in each parameter cell are for informed traders with positive, neutral and negative signals, ($I_{\bar{v}}$,I_{v_0},$I_{\underline{v}}$) and for uninformed traders (U). The fifth column ($Uncond.$) reports unconditional results for the market.

		$\delta = 1.6$					$\delta = 0.2$				
		$I_{\bar{v}}$	I_{v_0}	$I_{\underline{v}}$	U	$Uncond.$	$I_{\bar{v}}$	I_{v_0}	$I_{\underline{v}}$	U	$Uncond.$
$\alpha = 0.8$	LSA_2	0	0.244	0.049	0.155	0.109	0.399	0.255	0.108	0.026	0.209
	LSA_1	0	0.256	0.253	0.027	0.141	0.192	0.239	0.288	0.064	0.205
	LBB_1	0.253	0.256	0	0.027	0.141	0.288	0.239	0.192	0.064	0.205
	LBB_2	0.049	0.244	0	0.155	0.109	0.108	0.255	0.399	0.026	0.209
	MBA_2	0.491	0	0	0.297	0.190	0	0	0	0.347	0.069
	MBA_1	0.001	0	0	0.018	0.004	0	0	0	0.058	0.012
	MSB_1	0	0	0.001	0.018	0.004	0	0	0	0.058	0.012
	MSB_2	0	0	0.491	0.297	0.190	0	0	0	0.347	0.069
	NT	0.206	0	0.206	0.007	0.111	0.013	0.010	0.013	0.011	0.012
	E[Spread $\|\cdot$]	2.174	2.276	2.174	2.529	2.272	2.269	2.275	2.269	2.738	2.364
	E[Depth A_2+A_1 $\|\cdot$]	1.048	2.326	2.467	1.755	1.909	2.165	2.300	2.433	1.608	2.161
	E[Depth A_1 $\|\cdot$]	0.001	0.362	0.826	0.235	0.364	0.226	0.362	0.506	0.131	0.318
	E[Depth B_1 $\|\cdot$]	0.826	0.362	0.001	0.235	0.364	0.506	0.362	0.226	0.131	0.318
	E[Depth B_1+B_2 $\|\cdot$]	2.467	2.326	1.048	1.755	1.909	2.433	2.300	2.165	1.608	2.161
	E[Welfare LO $\|\cdot$]	0.092	0.128	0.092	1.075	0.298	0.143	0.133	0.143	0.055	0.123
	E[Welfare MO $\|\cdot$]	0.093	0	0.093	2.960	0.642	0	0	0	3.538	0.708
	E[Welfare $\|\cdot$]	0.185	0.128	0.185	4.036	0.940	0.143	0.133	0.143	3.592	0.830
$\alpha = 0.2$	LSA_2	0	0.385	0.525	0.101	0.141	0.375	0.389	0.443	0.093	0.155
	LSA_1	0	0.099	0.242	0.058	0.069	0.044	0.096	0.116	0.066	0.070
	LBB_1	0.242	0.099	0	0.058	0.069	0.116	0.096	0.044	0.066	0.070
	LBB_2	0.525	0.385	0	0.101	0.141	0.443	0.389	0.375	0.093	0.155
	MBA_2	0.130	0	0	0.219	0.184	0	0	0	0.218	0.175
	MBA_1	0.093	0	0	0.118	0.101	0	0	0	0.120	0.096
	MSB_1	0	0	0.093	0.118	0.101	0	0	0	0.120	0.096
	MSB_2	0	0	0.130	0.219	0.184	0	0	0	0.218	0.175
	NT	0.010	0.031	0.010	0.006	0.009	0.022	0.030	0.022	0.005	0.009
	E[Spread $\|\cdot$]	2.160	2.154	2.160	2.402	2.353	2.212	2.173	2.212	2.478	2.422
	E[Depth A_2+A_1 $\|\cdot$]	1.299	2.094	2.513	1.585	1.662	1.932	2.091	2.257	1.576	1.680
	E[Depth A_1 $\|\cdot$]	0.190	0.423	0.727	0.304	0.332	0.346	0.414	0.442	0.262	0.290
	E[Depth B_1 $\|\cdot$]	0.727	0.423	0.190	0.304	0.332	0.442	0.414	0.346	0.262	0.290
	E[Depth B_1+B_2 $\|\cdot$]	2.513	2.094	1.299	1.585	1.662	2.257	2.091	1.932	1.576	1.680
	E[Welfare LO $\|\cdot$]	1.179	0.566	1.179	0.523	0.614	0.596	0.654	0.596	0.500	0.523
	E[Welfare MO $\|\cdot$]	0.177	0	0.177	3.419	2.759	0	0	0	3.417	2.734
	E[Welfare $\|\cdot$]	1.357	0.566	1.357	3.942	3.372	0.596	0.654	0.596	3.917	3.257

Market quality

Market liquidity changes when the amount of adverse selection in a market changes. A standard intuition, as in Kyle 1985, is that liquidity deteriorates given more adverse selection. Roşu 2016b also finds worse liquidity (a wider bid-ask spread) given higher value volatility in his limit order market. However, we show the standard intuition is not always true when informed investors endogenously choose whether to supply liquidity via limit orders or take liquidity via market orders.

Observation 1 Liquidity can sometimes improve when adverse selection increases.

In particular, markets can become more liquid when, given the tick size, increasing the value-shock volatility flips the value shock δ from being small to being large relative the price grid. In addition, we show how different measures of market liquidity — expected spreads, inside depth, and total depth — can respond differently to changes in adverse selection.

The impact of adverse selection on market liquidity follows directly from the trading strategy effects in Section 1.3.1. Three intuitions are useful in understanding our market liquidity results. First, the most aggressive way to trade (both on directional information and private values) is via market orders, which take liquidity. However, the next most aggressive way to trade is via limit orders at the inside prices. Thus, changes in market conditions (i.e., δ and α) that make directionally informed investors trade more aggressively (i.e., that reduce their use of limit orders at the outside prices A_2 and B_2) can improve liquidity if their stronger trading interest migrates to aggressive limit orders at the inside quotes (A_1 and B_1) rather than to market orders. We call this the *aggressive directional informed liquidity provision effect.* Second, informed investors have a comparative advantage in providing liquidity over uninformed investors since I_{v_0} investors know that the unconditional asset value is correct. We call this the *Bloomfield-O'Hara-Saar effect* since they were the first to discuss liquidity provision by neutrally informed investors. Third, liquidity can change due to composition effects when changes in α change the mix of informed and uninformed investors, since different types of investors affect liquidity differently. Informed I_{v_0} investors with neutral news are natural liquidity providers. Their impact on liquidity comes from whether they supply liquidity at the inside (A_1 and B_1) or outside (A_2 and B_2) prices. In contrast, informed $I_{\overline{v}}$ and $I_{\underline{v}}$ investors with directional news and uninformed U traders affect liquidity depending on whether they opportunistically take or supply liquidity. All three effects can contribute to overturning the standard intuition about adverse selection and liquidity.

Our main result in this section is that the relation between adverse selection and market liquidity depends on the relative magnitudes of asset-value shocks and the tick size. As measures of liquidity, we focus here on the expected bid-ask spread and on expected depth at the inside prices. In Table 1.1, liquidity improves at time t_1 when the value-shock volatility δ increases (comparing parameterizations horizontally so that α is kept fixed). This happens, contrary to the standard intuition, because the informed $I_{\overline{v}}$ and $I_{\underline{v}}$ traders submit limit orders at the inside quotes in these high-volatility markets, whereas they only use limit orders at the outside quotes in low-volatility markets. In contrast, liquidity at time t_1 worsens, as predicted by the standard intuition, when the informed-investor arrival probability α increases holding the value-shock size δ fixed at the high level. Thus, the standard intuition is sometimes wrong but can also hold.

The evidence against the standard adverse-selection intuition is even stronger on average at times t_2 through t_4 in Table 1.2. First, consider the effect of increased information volatility δ. For both high and low proportions α of informed investors, liquidity improves when δ is increased. However, the underlying causes are different. When α is high (0.8), most investors reduce their total use of inside limit orders (i.e., on both sides of the market). Thus, the reason that average liquidity at times t_2 through t_4 is better in the high-volatility market is a carry-over effect from the greater liquidity of the high-volatility market at time t_1. In contrast, when α is low (0.2), high-volatility markets are more liquid due to the increased use of inside limit orders by both the directionally informed investors and the neutrally informed investors (i.e., both the aggressive directional informed liquidity provision effect and the Bloomfield-O'Hara-Saar effect) as well as due to the liquidity carry-over effect from time t_1. Second, consider the effect of a higher arrival probability α for informed investors. For both values of asset-value volatility δ, a higher probability α of informed investors leads neutrally informed I_{v_0} investors to increase their total use of limit orders at the inside prices far more than the other investors reduce their use of these orders. That, together with a composition effect (i.e., with $\alpha = 0.8$ there are more informed investors and informed investors use inside limit orders more than the uninformed investors) and the liquidity carry-over from t_1, is why liquidity improves in this case.

Our results for the expected spread and inside depth are driven by limit-order submissions at the inside quotes. However, the effect of adverse selection on total depth at the inside and outside quotes combined can differ from those liquidity measures driven by inside limit orders. For example, total depth at time t_1 increases (in Table 1.1) when value-shock volatility δ increases when the informed-investor arrival probability α is high (comparing horizontally the top two parametrizations), but decreases in δ when α is low. In contrast, average total depth at times t_2 through t_4 is decreasing (in Table 1.2) in the value-shock volatility (comparing parameterizations horizontally). This is opposite the effect on the inside depth. Thus, different metrics for liquidity can give different results.

Our results show that the relation between adverse selection and market liquidity in limit order markets is more subtle than the standard intuition. In particular, it is the ability of investors to choose endogenously whether to supply or demand liquidity and at what limit prices that can overturn the standard intuition. Goettler, Parlour, and Rajan 2009 also investigate a market with informed traders with no private-value motives and uninformed having only private-value motives. In their model, when volatility increases, informed traders reduce their provision of liquidity and increase their demand of liquidity; with the opposite holding for uninformed traders. Our results are more nuanced. Increased value-shock volatility is associated with increased liquidity supply in some cases and decreased liquidity in others. This is because the tick size of the price grid constrains the prices at which liquidity can be supplied and demanded.

Welfare

Tables 1.1 and 1.2 also report results about investor welfare. Not surprisingly, the utility of directionally informed investors increases when information volatility δ is higher. Interestingly, more than half of their expected gains-from-trade come from limit-order submissions. Perhaps more surprisingly, uniformed-investor utility is also often higher when δ is larger. This is consistent with the associated increase in liquidity that allows uninformed investors to capture more of their potential gains from trade. The net effect is that total active investor welfare increases in high volatility markets. In contrast, total welfare is less when the arrival probability α of informed investors increases. This is due to the fact that in this model only the uninformed U investors have gains-from-trade.

Information content of orders

Traders in real-world markets and empirical researchers are interested in the information content of different types of orders.[15] A necessary condition for an order to be informative is that informed investors use it. However, the magnitude of order informativeness is determined by the mix of equilibrium probabilities with which informed and uninformed traders use an order. If uninformed traders use the same orders as informed investors, they add noise to the price discovery process, and orders become less informative. In our model, the mix of information– and noise–based orders depends on the underlying proportion α of informed investors and the value-shock volatility δ.

We expect different market and limit orders to have different information content. A natural conjecture is that the sign of the information revision associated with an order should agree with the direction of the order (e.g., buy market and limit orders should lead to positive valuation revisions). Another natural conjecture is that the magnitude of information revisions should be greater for more aggressive orders. However, while the order-sign conjecture is true in our first model specification, the order-aggressiveness conjecture does not always hold here.

Observation 2 Order informativeness is not always increasing in the aggressiveness of an order.

This, at-first-glance surprising, result is another consequence of how informed investors trade on their information. As a result, the relative informativeness of different market and limit orders can flip in high-volatility and low-volatility markets. The result is immediate for market orders versus (less aggressive) limit orders in low-volatility markets in which informed investors avoid market orders (see Table 1.1). However, this reversed ordering can also hold for aggressive limit orders at the inside quotes (A_1 and B_1) versus less aggressive limit orders at the outside quotes (A_2 and B_2).

Figure 1.3 shows the informativeness of different types of orders. Each row contains four plots showing the informativeness of particular types of orders submitted at different times during the day for the indicated market parameterizations. Informativeness at time t_1 is measured as the Bayesian revision $E[v|\, x_{t_1}] - E[v]$ in the uninformed investor's expectation of the terminal value v after observing different given types of orders x_{t_1} at time t_1. The analogous measure of informativeness at later dates t_2 through t_4 is the Bayesian revision $E[v|\mathcal{L}_{t_{j-1}}, x_{t_j}] - E[v|\mathcal{L}_{t_{j-1}}]$ for different given types of orders x_{t_j} at time t_j relative to the incoming expectation conditional on the preceding order-flow history $\mathcal{L}_{t_{j-1}}$. In particular, the informativeness of a given order may change over time and may differ conditional on different preceding order histories. The vertical heights of the individual dots in the plots indicate the informativeness of

[15] Fleming, Mizrach, and Nguyen 2017 extend the VAR estimation approach of Hasbrouck (1991) to estimate the price impacts of limit orders as well as market orders. See also Brogaard, Hendershott, and Riordan 2016.

given orders at particular times given specific preceding histories.[16] The associated probabilities can differ across the different dots. The rectangles show the range of our informativeness metrics across paths. The vertical height of the blue squares indicate the probability-weighted average informativeness of a given type of order. The figure reports results for market and limit buy orders. The results are symmetric for sell orders.

The results in Figure 1.3 point to a variety of properties about order informativeness. First, perhaps the most obvious point is the heterogeneity in the information content of a given order at different times during the day and conditional on different prior order-flow histories. For example, plot 3(c) shows the Bayesian revisions for a LBB_1 limit buy order at the inside quotes B_1 in a high volatility market with a high arrival probability of informed investors ($\delta = 1.6$ and $\alpha = 0.8$). At time t_1, an LBB_1 order is fully revealing (and so the Bayesian revision relative to the unconditional expectation is 1.6). This follows from the fact in Table 1.1 that only informed $I_{\bar{v}}$ investors with good news use LBB_1 orders at time t_1. However, at later dates an LBB_1 limit order has different information content depending on the prior history. For example, in equilibrium an LBB_1 at time t_2 can be preceded by one of four possible equilibrium orders at t_1. If it follows a LSA_2 at t_1 (i.e., from an uninformed U investor which partially lowered prices), then an LBB_1 at t_2, which is fully revealing, leads to a positive Bayesian revision of 2.42 (the high dot). If it is preceded by either a market buy or sell MBA_2 or MSB_2 (at the outside prices) at t_1 (which are uninformative since only uninformed investors use them), then the LBB_1 at t_2 is again fully revealing and is associated with a positive Bayesian revision of 1.6. Lastly, if the time t_1 order is a LBB_2 limit order (which raise prices somewhat), the LBB_1 order at t_2 is only partially revealing but still produces a smaller upward incremental revision of 0.75. In this context, note that the order histories associated with the different dots can have different probabilities of occurring in equilibrium. For example, in Plot 3(b), we see that a few equilibrium order histories cause a MBA_1 market order at time t_4 to have a large Bayesian revision of almost 3. One way this can happen, for example, is when the proceeding path of orders is $\{LSA_2, MSB_2, LSA_1\}$ which is possible given the right sequence of uninformed investors. Over time the number of equilibrium paths grows by definition, but, in addition, we also see that, in equilibrium, the amount of informational heterogeneity across paths also grows. Moreover, this includes an increasing number of paths with zero Bayesian revisions. One reason this happens is that the number of fully revealing prior order histories is non-decreasing over time.

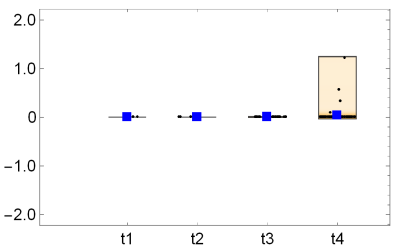
Second, Figure 1.3 shows that the aggressiveness conjecture for order informativeness can fail in a variety of ways. One way it can fail is that the average Bayesian revisions for limit orders are frequently larger than for market orders. This is follows immediately from Proposition 1 in low-volatility markets ($\delta = 0.2$). However, the conjecture also fails in high-volatility markets. For example, with $\delta = 1.6$ in the high-informed-investor proportion $\alpha = 0.8$ case, the average revisions for limit orders in Plots 3(c) and 3(d) are always larger than for market orders in Plots 3(a) and 3(b). This is also true in the low-informed investor proportion $\alpha = 0.2$ case in Plots 3(h) through 3(k). We also see that the conjecture can fail for aggressive vs. less-aggressive limit orders. Comparing Plots 3(g) and 3(h), is visually apparent that less-aggressive LBB_2 limit

[16] A given sequence of equilibrium orders might be produced by more than one investor-arrival sequence. Thus, individual dots correspond to sets of investor arrival sequences. Note here that the horizontal spacing of the dots in the plots is simply for ease of viewing.

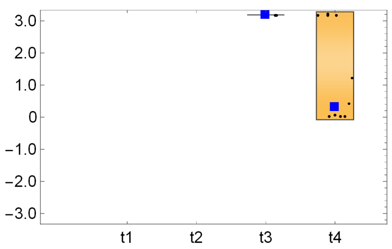
buys at t_1 have larger average Bayesian revisions than the aggressive LBB_1 limit buys. Visually, the differences are smaller in Plots 3(n) and 3(o), but the less-aggressive limit order averages are larger at all dates than for the aggressive limit orders. Having shown that the aggressiveness conjecture can fail, we also note that it does not always fail. For example, the average Bayesian revions for aggressive limit orders at times t_1 through t_3 in Plot 3(c) are larger than for the less-aggressive limit orders in Plot 3(d).

FIGURE 1.3: **Order Informativeness for the Model with Informed Traders with $\beta = 0$ and Uninformed Traders with $\beta \sim Tr[\mathcal{N}(\mu, \sigma^2)]$. for times t_1 to t_4.** This figure shows the path-contingent Bayesian value-forecast revisions $E[v|\mathcal{L}_{t_j-1}, x_{t-j}] - E[v|\mathcal{L}_{t_j-1}]$, which shows the change in the uninformed traders's expected value of the fundamental conditional on the order. We only consider orders when they are equilibrium orders for the trading periods. Each dot indicates an equilibrium revision, the plots indicate the maximum and the minimum. The plots are grouped by their respective market parameterizations (δ, α).

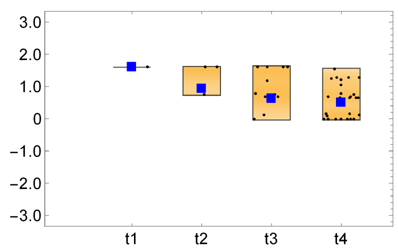
a) $\alpha = 0.8$ $\delta = 1.6$, Order:MBA_2



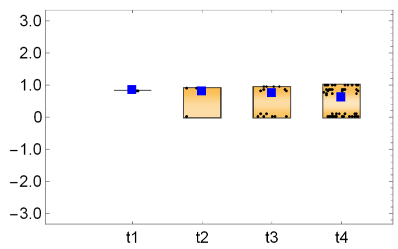
b) $\alpha = 0.8$ $\delta = 1.6$, Order:MBA_1



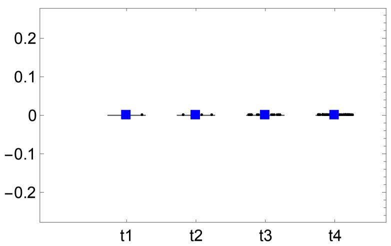
c) $\alpha = 0.8$ $\delta = 1.6$, Order:LBB_1



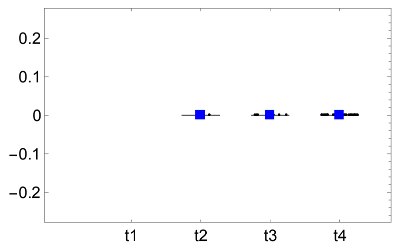
d) $\alpha = 0.8$ $\delta = 1.6$, Order:LBB_2



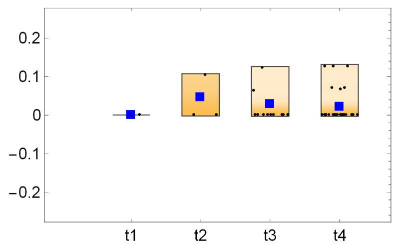
e) $\alpha = 0.8$ $\delta = 0.2$, Order:MBA_2



f) $\alpha = 0.8$ $\delta = 0.2$, Order:MBA_1



g) $\alpha = 0.8$ $\delta = 0.2$, Order:LBB_1



h) $\alpha = 0.8$ $\delta = 0.2$, Order:LBB_2

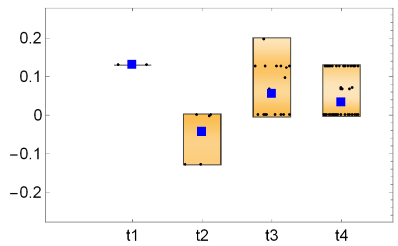
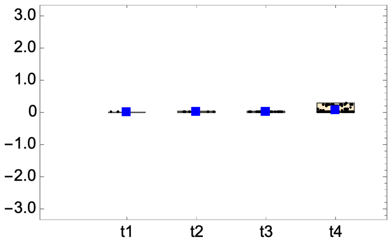
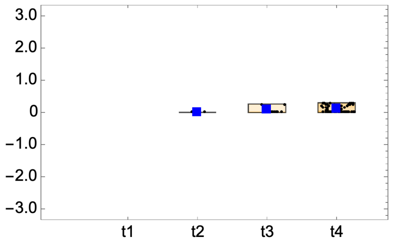


Figure 3 Continued: Order Informativeness for the Model with Informed Traders with $\beta = 0$ and Uninformed Traders with $\beta \sim Tr[\mathcal{N}(\mu, \sigma^2)]$. for times t_1 to t_4.

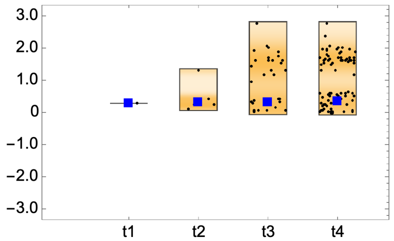
h)α = 0.2
$\delta = 1.6$,
Order:MBA_2



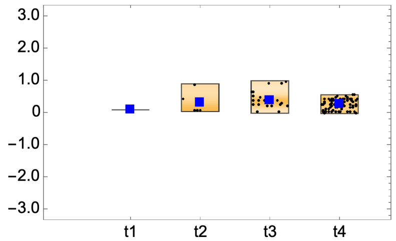
i)α = 0.2
$\delta = 1.6$,
Order:MBA_1



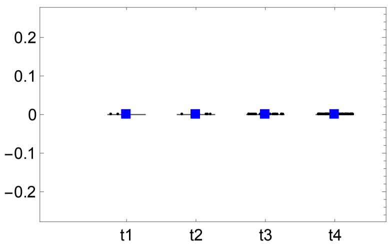
j)α = 0.2
$\delta = 1.6$,
Order:LBB_1



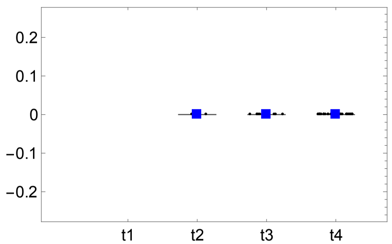
k)α = 0.2
$\delta = 1.6$,
Order:LBB_2



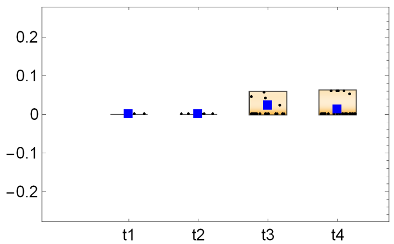
l)α = 0.2
$\delta = 0.2$,
Order:MBA_2



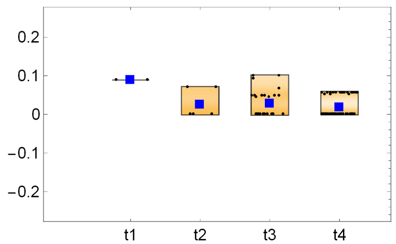
m)α = 0.2
$\delta = 0.2$,
Order:MBA_1



n)α = 0.2
$\delta = 0.2$,
Order:LBB_1



o)α = 0.2
$\delta = 0.2$,
Order:LBB_2



Non-Markovian learning

This section investigates the role of the order history on Bayesian learning. A major difference between our model and Goettler, Parlour, and Rajan 2009 and Roşu 2016b is that they assume learning is Markovian in the sense that the current limit order book L_{t_j} is a sufficient statistic at time $t_j > t_1$ for the information content of the full prior trading history $\mathcal{L}_{t_j}$. Thus, our first question here is whether the prior order history has information about the asset value v in excess of the information in the current limit order book. If it does, then learning is non-Markovian.[17]

The plots in Figure 4 measure the non-Markov information content of order histories by

$$E[v|\, \mathcal{L}_{t_j}(L_{t_j})] - E[v|\, L_{t_j}], \tag{1.13}$$

which is incremental information in the uninformed investors' expected asset value conditional on an order history path $\mathcal{L}_{t_j}(L_{t_j})$ ending with a particular limit order book L_{t_j} at time t_j net of the corresponding expectation conditional on just the ending book L_{t_j}. In particular, we are interested in books L_{t_j} that can be preceded in equilibrium by more than one different prior history. If learning is Markov, then order histories $\mathcal{L}_{t_j}(L_{t_j})$ preceding a book L_{t_j} should convey no additional information beyond L_{t_j}; in which case our metric in (1.13) should be zero. Individual dots in the plots indicate the incremental information content of particular histories preceding different orders submitted at each of the different dates. Time t_1 is included in the plot because books L_{t_1} at t_1 can potentially be produced by different sequences of investor actions x_{t_1} and crowd responses at t_1. More generally, the book L_{t_j} at each time t_j reflects information due to the path of past active investor actions, but past crowd actions can partially obscure this information (e.g., as when the crowd replenishes the book after active investors deplete the book at the outside prices). Each plot is for a different combination of adverse-selection parameters. For brevity, the plots contain all possible books, rather than having individual plots (as in Figure 1.3) for each individual order.

The main result from Figure 4 is that there is substantial incremental information in the preceding order histories after conditioning on the prior limit order book.

Observation 4 The price discovery dynamics can be significantly non-Markovian.

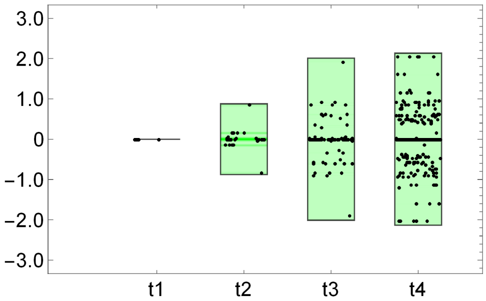
As expected, the variation in the incremental information content of the prior order history in Figure 4 is greater when the shock volatility δ is greater (note the difference in vertical scales).

Given that learning is non-Markovian, the next question is about how the size of the valuation revisions depends on the prior trading history. In Figure 5, the horizontal axis shows the valuation revision $E(v|x_{t_1}) - E(v)$ given different equilibrium actions x_{t_1} at t_1, and the vertical axis gives the corresponding cumulative valuation revision $E(v|x_{t_1}, x_{t_2}) - E(v)$ as of time t_2 given different sequences of equilibrium actions x_{t_1} at time t_1 followed by different possible equilibrium successor actions x_{t_2} at time t_2. From iterated expectations, the expectation of the two-period revision given the first period action is the first-period revision, which is denoted here by the 45° line.

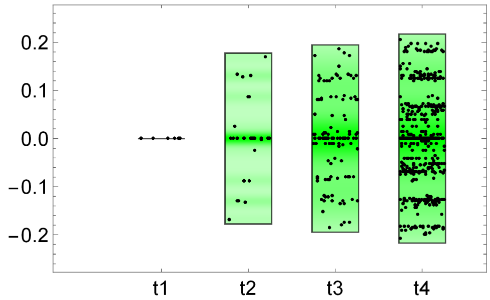
[17] The evidence of path-contingent order informativeness in Figure 1.3 by itself does not necessarily imply non-Markovian learning. Markovian learning is still possible if the incoming book L_{t_j} at time t_j summarizes the information content of the full order history $\mathcal{L}_{t_j}(L_{t_j})$ preceding book L_{t_j}.

FIGURE 4: **Informativeness of the Order History for the Model with Informed Traders with $\beta = 0$ and Uninformed Traders with $\beta \sim Tr[\mathcal{N}(\mu, \sigma^2)]$ for Times t_1 through t_4.** This figure shows the incremental information content of the past order history in excess of the information in the current limit order book observed at the end of time t_j as measured by $E[v|\mathcal{L}_{t_j}(L_{t_j})] - E[v|L_{t_j}]$ where $\mathcal{L}_{t_j}(L_{t_j})$ is a history ending in the limit order book L_{t_j}. We only consider books L_{t_j} when they occur in equilibrium in the different trading periods. The dots indicate values for particular books and paths, and the rectangles show the range of maximum and minimum values.

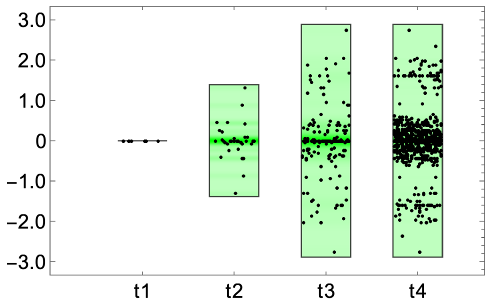
(A) Parameters: $\alpha = 0.8, \delta = 1.6$



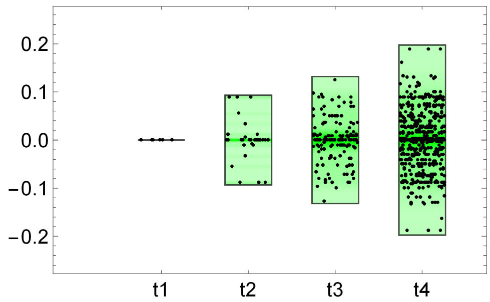
(B) Parameters: $\alpha = 0.8, \delta = 0.2$



(C) Parameters: $\alpha = 0.2, \delta = 1.6$

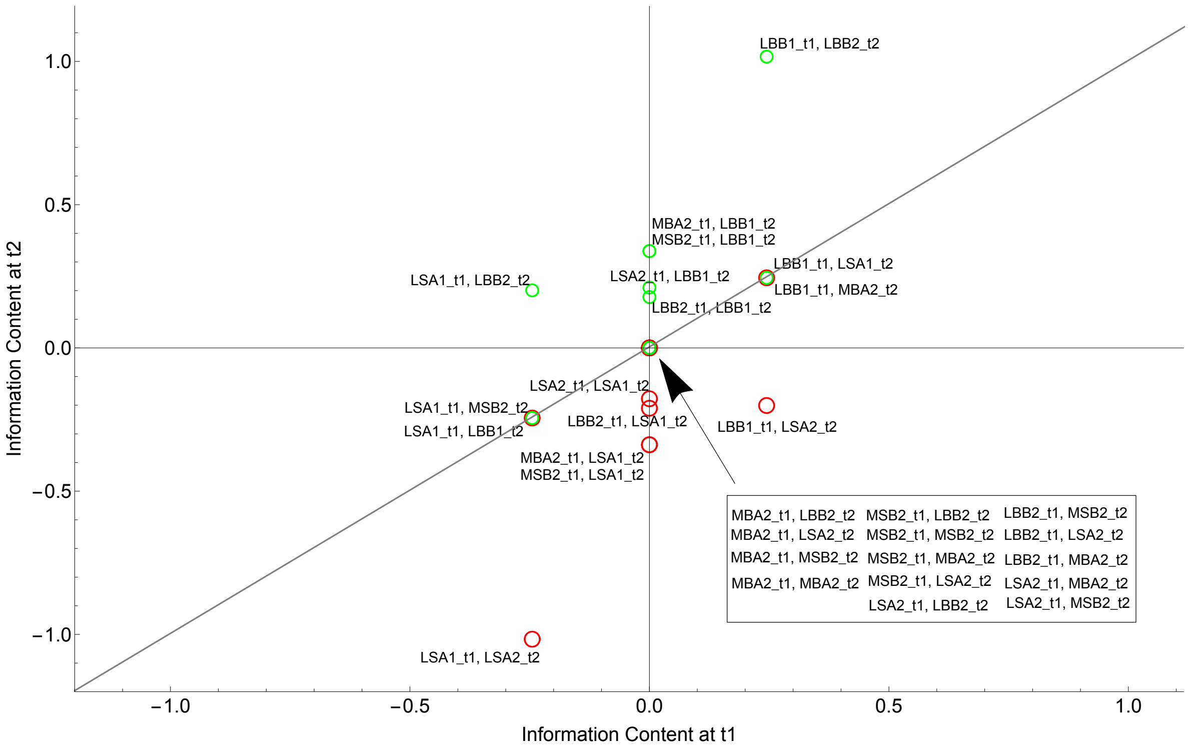


(D) Parameters: $\alpha = 0.2, \delta = 0.2$



Consistent with our previous analysis, the size of the valuation revision depends crucially on the informed investors' equilibrium strategies. As informed investors do not use market orders at t_1 (see Table 1.1), market orders have a zero price impact at t_1 and, thus, the points for pairs of time t_1 and t_2 price-impacts for sequences of a market order at t_1 and then different orders at time t_2 all line up on the vertical axis line. Interestingly, there are no observations in the first and fourth quadrants in our model, which means there are no sign reversals in the direction of the cumulative price impacts. However, there is randomness around the 45° line induced by different successor date-2 actions. The second and third quadrants (which are symmetrical) show the sequences of orders that have a positive and a negative price impact, respectively. One intuitive result about the relation between earlier orders and subsequent valuation-revision dynamics is the following: Conditional on the amount of adverse selection (i.e., the δ and α parameterization), the volatility of the incremental valuation revision at time t_2 relative to time t_1 (i.e., the vertical dispersion around the 45° line) is weakly decreasing in

FIGURE 5: **Order Informativeness for the Model with Informed Traders with $\beta = 0$ and Uninformed Traders with $\beta \sim Tr[\mathcal{N}(\mu, \sigma^2)]$ for times t_1 to t_2 and parameters $\alpha = 0.8$, $\delta = 1.6$.** The horizontal axis reports $E(v|x_{t_1}) - E(v)$ which shows how the uninformed traders' Bayesian value-forecast changes with respect to the unconditional expected value of the fundamental when uninformed traders observe at t_1 an equilibrium order x_{t_1}. The vertical axis reports $E(v|x_{t_1}, x_{t_2}) - E(v)$ which shows how the uninformed traders' Bayesian value-forecast changes with respect to the unconditional expected value of the fundamental when uninformed traders observe at x_{t_2} at t_2. We consider all the equilibrium strategies at t_1 and t_2 which are symmetrical. Green (red) circles show equilibrium buy (sell) orders at t_2.



the magnitude of the valuation revision associated with the trading action at time t_1.

Price impact of order flow

A standard empirical measure of price-discovery is the price impact of order flow. The idea is that the price impact of orders can be decomposed into two components: One measures the size of surprises in an arriving order relative to its expectation given the prior history, and the second measures the marginal (per-share) impact of order-flow surprises on the informational component of a security's valuation. Fleming, Mizrach, and Nguyen 2017 and Brogaard, Hendershott, and Riordan 2016 extend the Hasbrouck 1991 vector autoregression methodology — a standard empirical technique to estimate this decomposition — to allow for limit orders as well as market orders. Using our notation, their information innovation equation can be written as

$$E[v|x_t, \mathcal{L}_{t-1}] - E[v|\mathcal{L}_{t-1}] = \sum_k \lambda_k [Q^{x_t}_{k,t} - E[Q^{x_t}_{k,t}|\mathcal{L}_{t-1}]] \tag{1.14}$$

where $Q^{x_t}_{k,t} - E[Q^{x_t}_{k,t}|\mathcal{L}_{t-1}]$ in the innovation in the number of shares $Q^{x_t}_{k,t}$ associated with an order type k (e.g., a particular market or limit order) given the investor action x_t at time t, and λ_k is a constant marginal price impact for order type k.

Our model suggests an extension of the VAR approach that we call the *conditional price impact of order flow*. In particular, the price impact of order flow, rather than being a constant λ_k, can vary over time given different types of conditioning information. In our model, the price impact is a function $\lambda_k(t, \mathcal{L}_{t-1})$ that is conditional on the prior trading history $\mathcal{L}_{t-1}$ and on time t_j. In its most general form, our model would require machine learning techniques to deal with large amounts of transactional data and high dimensional functional relationships. Simpler empirical specifications might look at the effect of conditioning just on time via a function $\lambda_k(t)$ or conditioning just on the standing limit order book L_{t-1} at the time orders arrives via a function $\lambda_k(t, L_{t-1})$.

Figure 6 shows that even our very simple model generates substantial variation in the conditional price impact of orders. Consider an order sequence $\{\mathcal{L}_{t_{j-1}}, x_{t_j}\}$ where sequences $\{\mathcal{L}_{t_{j-1}}, x_{t_j}\}$ and $\{\mathcal{L}_{t_{j-1}}, NT\}$ both have positive probabilities. As a metric for dispersion in the conditional price impact of order flow, we compute

$$\max_{\mathcal{L}_{t_{j-1}}} E[v|\mathcal{L}_{t_{j-1}}, x_{t_j}] - E[v|\mathcal{L}_{t_{j-1}}, NT] - \min_{\mathcal{L}_{t_{j-1}}} E[v|\mathcal{L}_{t_{j-1}}, x_{t_j}] - E[v|\mathcal{L}_{t_{j-1}}, NT] \tag{1.15}$$

In words, $E[v|\mathcal{L}_{t_{j-1}}, x_{t_j}] - E[v|\mathcal{L}_{t_{j-1}}, NT]$ is the differential informational impact of a one-unit innovation in order type x_{t_j} relative to NT where differencing controls for expectations given the prior history $\mathcal{L}_{t_{j-1}}$. The metric in (1.15) is the spread between the maximal and minimum differential informational innovation across all paths $\mathcal{L}_{t_{j-1}}$ such that order x_{t_j} and NT both occur with positive probability following the different paths $\mathcal{L}_{t_{j-1}}$. As can be seen, the amount of cross-path dispersion in the conditional impact of order flow can be substantial.

Summary

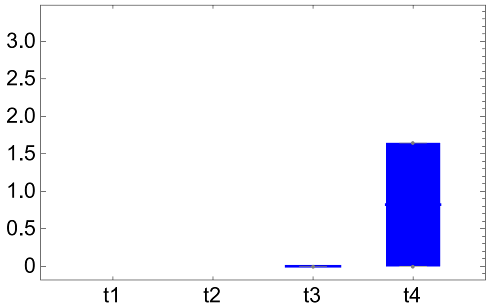
The analysis of our first model specification has identified a number of empirically testable predictions. First, liquidity and the relative information content of different orders differ in high-volatility markets (in which value shocks are large relative to the tick size) vs. in low-volatility markets. Second, it is possible for less-aggressive orders to be more informative than more aggressive orders. Third, price discovery is non-Markov, and the price impact of individual orders varies conditional on the prior order-flow history.

1.3.2 Informed and uninformed traders both have private-value motives

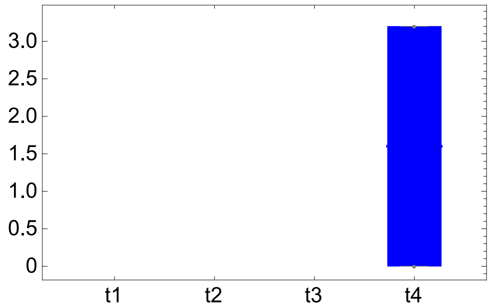
Our second model specification generalizes our earlier analysis. Now informed investors also have random private-valuation factors β_{t_j} with the same truncated-Normal distribution $\beta_{t_j} \sim Tr[\mathcal{N}(\mu, \sigma^2)]$ as the uninformed investors. Hence, informed traders not only speculate on their information, but they also have private-value motives to trade. As a result, informed investors

FIGURE 6: **Dispersion in the price impact of order flow** The plot reports $max_{\mathcal{L}_{t_{j-1}}}(E[v|\mathcal{L}_{t_{j-1}}, x_{t_j}] - E[v|\mathcal{L}_{t_{j-1}}, NT]) - min_{\mathcal{L}_{t_{j-1}}}(E[v|\mathcal{L}_{t_{j-1}}, x_{t_j}] - E[v|\mathcal{L}_{t_{j-1}}, NT])$ at different times, which shows how the prior order history affects the marginal price impact of the surprise in a given order. The parameterization is: $\alpha = 0.8$, $\delta = 1.6$

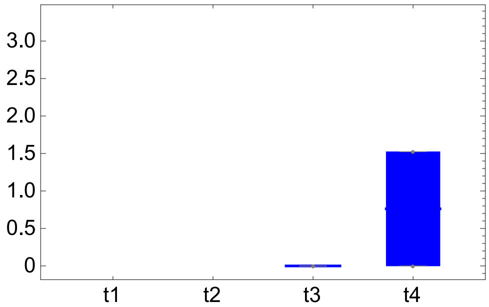
(A) Order Type: MBA_2



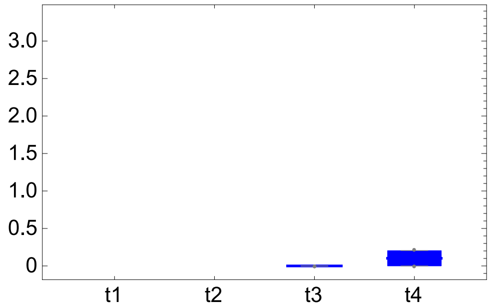
(B) Order Type: MBA_1



(C) Order Type: LBB_1



(D) Order Type: LBB_2



with the same signal may end up buying and selling from each other. This combination of trading motives has not been investigated in earlier models of dynamic limit order markets. We use our second model specification to show the robustness of the results in Section 1.3.1 and to extend them.

Trading strategies

Tables 1.3 and 1.4 report order-submission probabilities and other statistics for our second model specification for time t_1 and for averages over times t_2 through t_4. There are a few differences relative to Tables 1.1 and 1.2 for the simpler model in Section 2.1. First, since all investors have private-value motives to trade, all investors use all of the possible limit orders in both time windows. In addition, now informed investors sometimes use market orders at t_1 as well as over times t_2 through t_4 and also sometimes (during times t_2 through t_4) even when asset volatility δ is small (0.2). Second, directionally informed investors sometimes now trade opposite their asset-value information. In particular, we say an $I_{\bar{v}}$ or $I_{\underline{v}}$ investor is trading *with* their information when they are buying (selling) given good (bad) news. Trading *opposite*

TABLE 1.3: **Trading Strategies, Liquidity, and Welfare at Time t_1 in an Equilibrium with Informed and Uninformed Traders both with $\beta \sim Tr[\mathcal{N}(\mu, \sigma^2)]$.** This table reports results for two different informed-investor arrival probabilities α (0.8 and 0.2) and two different value-shock volatilities δ (1.6 and 0.2). The private-value factor parameters are $\mu = 0$ and $\sigma = 15$, and the tick size is $\kappa = 1$. Each cell corresponding to a set of parameters reports the equilibrium order-submission probabilities, the expected bid-ask spreads and expected depths at the inside prices (A_1 and B_1) and total depths on each side of the market after order submisions at time t_1, and the expected welfare of the market participants. The first four columns in each parameter cell are for informed traders with positive, neutral and negative signals, ($I_{\bar{v}}$,I_{v_0},$I_{\underline{v}}$) and for uninformed traders (U). The fifth column ($Uncond.$) reports unconditional results for the market.

		$\delta = 1.6$					$\delta = 0.2$				
		$I_{\bar{v}}$	I_{v_0}	$I_{\underline{v}}$	U	$Uncond.$	$I_{\bar{v}}$	I_{v_0}	$I_{\underline{v}}$	U	$Uncond.$
	LSA_2	0.118	0.054	0.031	0.064	0.067	0.054	0.048	0.042	0.048	0.048
	LSA_1	0.314	0.446	0.282	0.426	0.363	0.438	0.452	0.466	0.452	0.452
	LBB_1	0.282	0.446	0.314	0.426	0.363	0.466	0.452	0.438	0.452	0.452
	LBB_2	0.031	0.054	0.118	0.064	0.067	0.042	0.048	0.054	0.048	0.048
	MBA_2	0.256	0	0	0.009	0.070	0	0	0	0	0
	MBA_1	0	0	0	0	0	0	0	0	0	0
	MSB_1	0	0	0	0	0	0	0	0	0	0
	MSB_2	0	0	0.256	0.009	0.070	0	0	0	0	0
	NT	0	0	0	0	0	0	0	0	0	0
$\alpha = 0.8$											
	E[Spread $\vert\cdot$]	2.404	2.109	2.404	2.147	2.274	2.096	2.096	2.096	2.096	2.096
	E[Depth A_2+A_1 $\vert\cdot$]	1.432	1.500	1.312	1.491	1.430	1.492	1.500	1.508	1.500	1.500
	E[Depth A_1 $\vert\cdot$]	0.314	0.446	0.282	0.426	0.363	0.438	0.452	0.466	0.452	0.452
	E[Depth B_1 $\vert\cdot$]	0.282	0.446	0.314	0.426	0.363	0.466	0.452	0.438	0.452	0.452
	E[Depth B_1+B_2 $\vert\cdot$]	1.312	1.500	1.432	1.491	1.430	1.508	1.500	1.492	1.500	1.500
	E[Welfare LO $\vert\cdot$]	2.589	4.452	2.589	4.098	3.388	4.462	4.465	4.462	4.461	4.462
	E[Welfare MO $\vert\cdot$]	1.874	0	1.874	0.155	1.030	0	0	0	0	0
	E[Welfare $\vert\cdot$]	4.463	4.452	4.463	4.253	4.418	4.462	4.465	4.462	4.461	4.462
	LSA_2	0.063	0.051	0.042	0.051	0.051	0.049	0.048	0.046	0.048	0.048
	LSA_1	0.356	0.449	0.476	0.449	0.445	0.441	0.452	0.464	0.452	0.452
	LBB_1	0.476	0.449	0.356	0.449	0.445	0.464	0.452	0.441	0.452	0.452
	LBB_2	0.042	0.051	0.063	0.051	0.051	0.046	0.048	0.049	0.048	0.048
	MBA_2	0.063	0	0	0	0.004	0	0	0	0	0
	MBA_1	0	0	0	0	0	0	0	0	0	0
	MSB_1	0	0	0	0	0	0	0	0	0	0
	MSB_2	0	0	0.063	0	0.004	0	0	0	0	0
	NT	0	0	0	0	0	0	0	0	0	0
$\alpha = 0.2$											
	E[Spread $\vert\cdot$]	2.168	2.103	2.168	2.102	2.111	2.096	2.096	2.096	2.096	2.096
	E[Depth A_2+A_1 $\vert\cdot$]	1.419	1.500	1.518	1.500	1.496	1.490	1.500	1.510	1.500	1.500
	E[Depth A_1 $\vert\cdot$]	0.356	0.449	0.476	0.449	0.445	0.441	0.452	0.464	0.452	0.452
	E[Depth B_1 $\vert\cdot$]	0.476	0.449	0.356	0.449	0.445	0.464	0.452	0.441	0.452	0.452
	E[Depth B_1+B_2 $\vert\cdot$]	1.518	1.500	1.419	1.500	1.496	1.510	1.500	1.490	1.500	1.500
	E[Welfare LO $\vert\cdot$]	3.943	4.448	3.943	4.424	4.362	4.466	4.465	4.466	4.465	4.465
	E[Welfare MO $\vert\cdot$]	0.591	0	0.591	0	0.079	0	0	0	0	0
	E[Welfare $\vert\cdot$]	4.535	4.448	4.535	4.424	4.440	4.466	4.465	4.466	4.465	4.465

TABLE 1.4: **Averages for Trading Strategies, Liquidity, and Welfare across Times t_2 through t_4 for Informed and Uninformed Traders both with $\beta \sim Tr[\mathcal{N}(\mu, \sigma^2)]$.** This table reports results for two different informed-investor arrival probabilities α (0.8 and 0.2) and for two different asset-value volatilities δ (1.6 and 0.2). The private-value factor parameters are $\mu = 0$ and $\sigma = 15$, and the tick size is $\kappa = 1$. Each cell corresponding to a set of parameters reports the equilibrium order-submission probabilities, the expected bid-ask spreads and expected depths at the inside prices (A_1 and B_1) and total depths on each side of the market after order submissions at times t_2 through t_4, and the expected welfare of the market participants. The first four columns in each parameter cell are for informed traders with positive, neutral and negative signals, ($I_{\bar{v}}$,I_{v_0},$I_{\underline{v}}$) and for uninformed traders (U). The fifth column (*Uncond.*) reports unconditional results for the market.

		$\delta = 1.6$					$\delta = 0.2$				
		$I_{\bar{v}}$	I_{v_0}	$I_{\underline{v}}$	U	*Uncond.*	$I_{\bar{v}}$	I_{v_0}	$I_{\underline{v}}$	U	*Uncond.*
$\alpha = 0.8$	LSA_2	0.140	0.121	0.090	0.114	0.117	0.127	0.123	0.119	0.123	0.123
	LSA_1	0.108	0.058	0.050	0.067	0.071	0.057	0.053	0.048	0.053	0.053
	LBB_1	0.050	0.058	0.108	0.067	0.071	0.048	0.053	0.057	0.053	0.053
	LBB_2	0.090	0.121	0.140	0.114	0.117	0.119	0.123	0.127	0.123	0.123
	MBA_2	0.275	0.192	0.113	0.195	0.194	0.207	0.194	0.181	0.194	0.194
	MBA_1	0.158	0.127	0.062	0.122	0.117	0.133	0.128	0.124	0.129	0.128
	MSB_1	0.062	0.127	0.158	0.122	0.117	0.124	0.128	0.133	0.129	0.128
	MSB_2	0.113	0.192	0.275	0.195	0.194	0.181	0.194	0.207	0.194	0.194
	NT	0.003	0.003	0.003	0.005	0.004	0.004	0.003	0.004	0.004	0.004
	E[Spread $\|\cdot$]	2.365	2.325	2.365	2.375	2.356	2.336	2.337	2.336	2.337	2.336
	E[Depth A_2+A_1 $\|\cdot$]	1.599	1.600	1.537	1.563	1.576	1.590	1.593	1.596	1.593	1.593
	E[Depth A_1 $\|\cdot$]	0.301	0.339	0.338	0.314	0.324	0.324	0.333	0.344	0.333	0.334
	E[Depth B_1 $\|\cdot$]	0.338	0.339	0.301	0.314	0.324	0.344	0.333	0.324	0.333	0.334
	E[Depth B_1+B_2 $\|\cdot$]	1.537	1.600	1.599	1.563	1.576	1.596	1.593	1.590	1.593	1.593
	E[Welfare LO $\|\cdot$]	0.892	0.709	0.892	0.723	0.809	0.674	0.671	0.674	0.670	0.672
	E[Welfare MO $\|\cdot$]	3.285	3.324	3.285	3.315	3.301	3.357	3.357	3.357	3.358	3.357
	E[Welfare $\|\cdot$]	4.177	4.033	4.177	4.038	4.110	4.031	4.028	4.031	4.028	4.029
$\alpha = 0.2$	LSA_2	0.131	0.123	0.114	0.122	0.122	0.124	0.123	0.122	0.123	0.123
	LSA_1	0.059	0.054	0.049	0.053	0.054	0.053	0.053	0.052	0.053	0.053
	LBB_1	0.049	0.054	0.059	0.053	0.054	0.052	0.053	0.053	0.053	0.053
	LBB_2	0.114	0.123	0.131	0.122	0.122	0.122	0.123	0.124	0.123	0.123
	MBA_2	0.257	0.194	0.137	0.196	0.196	0.202	0.194	0.186	0.194	0.194
	MBA_1	0.160	0.127	0.090	0.127	0.127	0.133	0.128	0.124	0.128	0.128
	MSB_1	0.090	0.127	0.160	0.127	0.127	0.124	0.128	0.133	0.128	0.128
	MSB_2	0.137	0.194	0.257	0.196	0.196	0.186	0.194	0.202	0.194	0.194
	NT	0.004	0.003	0.004	0.004	0.004	0.004	0.003	0.004	0.004	0.004
	E[Spread $\|\cdot$]	2.337	2.335	2.337	2.340	2.339	2.337	2.337	2.337	2.337	2.337
	E[Depth A_2+A_1 $\|\cdot$]	1.547	1.595	1.636	1.591	1.591	1.587	1.593	1.599	1.592	1.593
	E[Depth A_1 $\|\cdot$]	0.288	0.334	0.378	0.332	0.332	0.327	0.333	0.339	0.333	0.333
	E[Depth B_1 $\|\cdot$]	0.378	0.334	0.288	0.332	0.332	0.339	0.333	0.327	0.333	0.333
	E[Depth B_1+B_2 $\|\cdot$]	1.636	1.595	1.547	1.591	1.591	1.599	1.593	1.587	1.592	1.593
	E[Welfare LO $\|\cdot$]	0.682	0.685	0.682	0.668	0.671	0.671	0.671	0.671	0.671	0.671
	E[Welfare MO $\|\cdot$]	3.481	3.345	3.481	3.355	3.371	3.359	3.357	3.359	3.357	3.358
	E[Welfare $\|\cdot$]	4.163	4.029	4.163	4.022	4.042	4.030	4.028	4.030	4.028	4.029

their information is doing the reverse. Investors trade opposite their information when their random private-value motive overwhelms their speculative motive. In particular, note that often in both tables limit orders are used more by investors trading opposite their information than with their information. That will have important implications for the information content (considered below) of such limit orders. Third, informed I_{v_0} investors with neutral news no longer just provide liquidity using limit orders. Rather, due to their private-value motive, they sometimes also take liquidity via market orders both at time t_1 when δ is large (in Table 1.3) and later at times t_2 through t_4 even when δ is small (in Table 1.4).

Consider next the impact of adverse selection on trading behavior. The effect of higher δ and higher α on the trading behavior of informed traders $I_{\bar{v}}$ and $I_{\underline{v}}$ with directional news differs when they are trading with or opposite their information. For investors trading with their information, we see the aggressiveness effect again, similar to the results in Section 1.3.1. In particular, for these investors, increased adverse selection leads to a reduction in the use of less-aggressive outside limit orders trading with directional good and bad news and an increase in the use of more aggressive orders. The net effect on aggressive limit orders at inside prices is ambiguous in these cases due to in-migration of probability from the reduced use of the outside limit orders but possible out-migration of probability to market orders. For example, comparing the upper two parameterizations in Table 1.3 shows that when δ is increased with α fixed at 0.8, the $I_{\bar{v}}$ investors with good news at time t_1 reduce the strategy probability for LBB_2 orders from 0.042 to 0.031 and increase the probability for MBA_2 orders from 0 to 0.256, and reduce the use of LBB_1 limit orders from 0.466 to 0.282.

The effect of adverse selection is different from above when $I_{\bar{v}}$ and $I_{\underline{v}}$ investors trade opposite their directional information. Increased adverse selection causes informed investors trading opposite their information to increase their use of less-aggressive limit orders at the outside prices. In particular, when δ increases, informed investors with good news $\bar{v}$ (bad news $\underline{v}$) know the security is worth more (less) and require a higher (lower) price when selling. However, when α increases, the reason is a supply/demand effect: The demand for buying (selling) increases since now more investors know the good (bad) news, and, thus, informed investors willing to sell (buy) can increase the price of the liquidity they provide.

The effects of higher volatility on uninformed U traders slightly differs at t_1 as opposed to times t_2 through t_4. At t_1 uninformed traders post slightly more aggressive orders when they demand liquidity (the strategy probabilities for MBA_2 and MSB_2 increase from 0 to 0.009), and more patient orders when they supply liquidity (the strategy probabilities for LBB_2 and LSA_2 increase slightly from 0.048 to 0.064). This change in order-submission strategies is the consequence of uninformed traders facing higher adverse selection costs. They feel safer hitting the trading crowd at A_2 and B_2 and offering liquidity at more profitable price levels to make up for the increased adverse selection costs. In later periods t_1 through t_4, as uninformed traders learn about the fundamental value of the asset, they still take liquidity at the outside quotes (the probabilities of MBA_2 and MSB_2 increase slightly to 0.195 in Table 1.4), but move to the inside quotes to supply liquidity (LSA_1 and LBB_1 increase to 0.067 for times t_2 through t_4). As they learn about the future value of the asset, uninformed traders perceive less adverse selection costs and can afford to offer liquidity at more aggressive quotes. In contrast, the effect of increased value-shock volatility on the trading behavior of I_{v_0} investors with neutral news is relatively modest both at time t_1 and at times t_2 through t_4.

Market quality

Market quality — as measured by both expected spreads and inside depth in Tables 3 and 4 — is almost always decreasing in adverse selection in this second model. This is a notable difference from our first model. However, this is not surprising given the generally greater use of market orders due to the potentially large range of private values. In particular, when the gains-from-trade are large, order execution is more important than price improvement.

Information content of orders

Figure 7 shows the distribution of Bayesian revisions for the different orders at different times and conditional on different prior order-flow paths. The format is the same as in Figure 1.3. Once again, there is heterogeneity in the information content of orders over time and conditional on the preceding history. Not surprisingly, the amount of heterogeneity is less since there is substantially less price discovery in this second model specification given that informed investor orders are now affected by noise from private values as well as information. In addition, we still see violations of the order aggressiveness conjecture. Consider, for example, the high adverse-selection parameterization with high value-shock volatility and a high informed-investor arrival probability. The most informative orders at t_1 and t_2 are the market orders. However, the less-aggressive LBB_2 limit orders are more informative than the aggressive LBB_1 limit orders at t_1 and also, less obviously visually, at t_2.

A new finding in this second model is that, surprisingly, the order-sign conjecture need not hold:

Observation 5 The Bayesian value revision can be opposite the direction of an order.

This is to say that the direction of orders is sometimes different from the sign of their information content. For example, in the high δ/high α parameterization, LBB_2 limit buys at t_1 reveal bad news (rather than good news as one might expect given that they are buy orders). The same is true of LBB_1 limit buys at t_2 through t_4. This is because, in our second model, these limit buys are used more frequently by directionally informed investor to trade opposite (rather than with) their information (i.e., due to their private-values β_{t_j}).

Non-Markovian learning

Figure 8 shows once again the variation in the incremental information $E[v|\mathcal{L}_{t_j}(L_{t_j})] - E[v|L_{t_j}]$ in the prior order histories $\mathcal{L}_{t_j}(L_{t_j})$ preceding different books L_{t_j}. The plots here confirm our earlier results about non-Markovian learning.

FIGURE 7: **Order Informativeness for the Model with Informed Traders and Uninformed Traders both with $\beta \sim Tr[\mathcal{N}(\mu, \sigma^2)]$. for times t_1 to t_4.** This figure shows the path-contingent Bayesian value-forecast revisions $E[v|\mathcal{L}_{t_j-1}, x_{t-j}] - E[v|\mathcal{L}_{t_j-1}]$, which shows the change in the uninformed traders's expected value of the fundamental conditional on the order. Plots a,c,e and g show graphs for the parametrization with $\alpha = 0.8$ and $\delta = 1.6$. Plots b,d,f and h show graphs for the parametrization with $\alpha = 0.8$ and $\delta = 0.2$. Plots i,k,m and o show graphs for the parametrization with $\alpha = 0.2$ and $\delta = 1.6$. Plots j,l,n and p show graphs for the parametrization with $\alpha = 0.2$ and $\delta = 0.2$. We only consider orders when they are equilibrium orders for the trading periods. Each dot indicates an equilibrium revision, the plots indicate the maximum and the minimum.

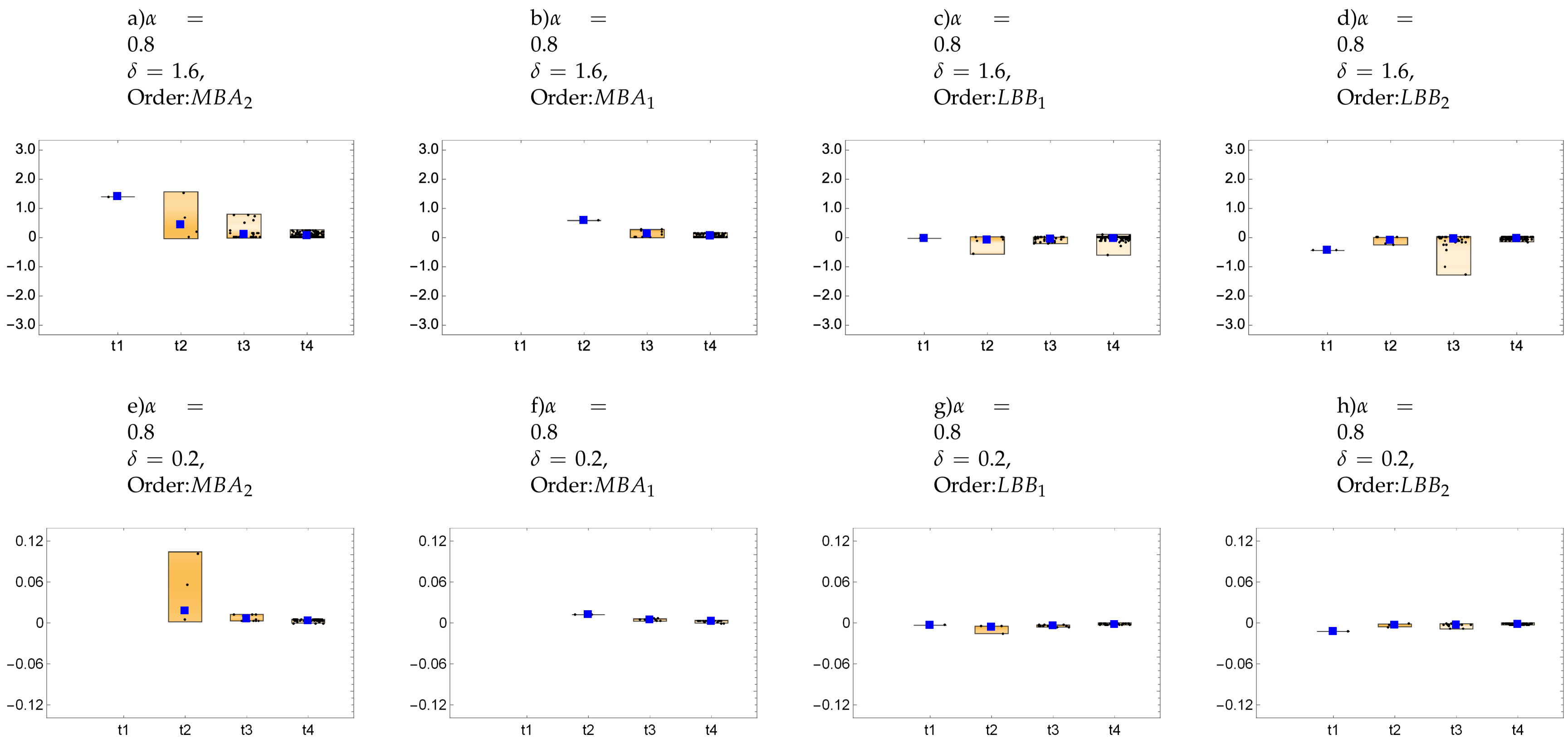
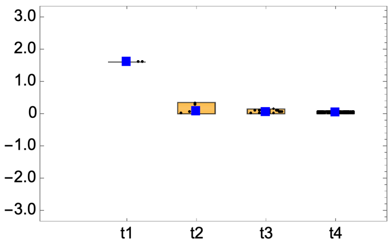
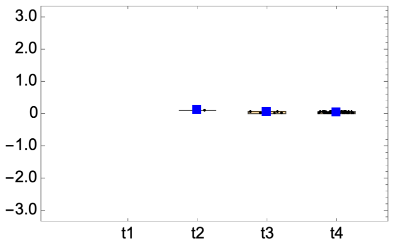


Figure 7 Continued: Order Informativeness for the Model with Informed Traders and Uninformed Traders both with $\beta \sim Tr[\mathcal{N}(\mu, \sigma^2)]$. for times t_1 to t_4.

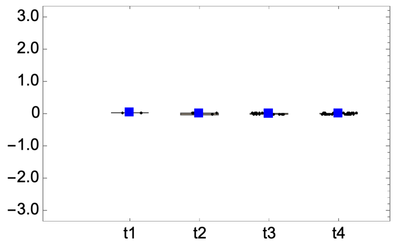
h)$\alpha = 0.2$
$\delta = 1.6$,
Order:MBA_2



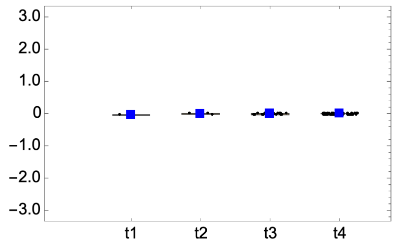
i)$\alpha = 0.2$
$\delta = 1.6$,
Order:MBA_1



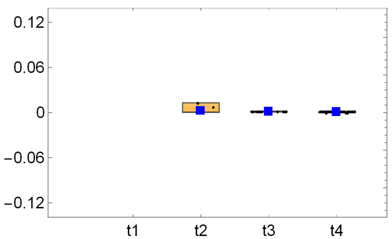
j)$\alpha = 0.2$
$\delta = 1.6$,
Order:LBB_1



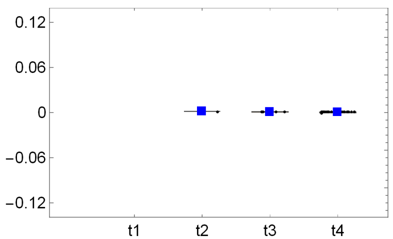
k)$\alpha = 0.2$
$\delta = 1.6$,
Order:LBB_2



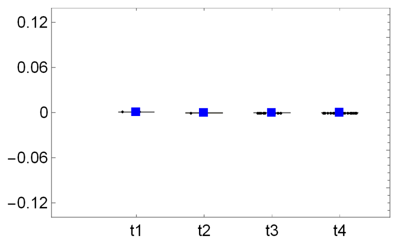
l)$\alpha = 0.2$
$\delta = 0.2$,
Order:MBA_2



m)$\alpha = 0.2$
$\delta = 0.2$,
Order:MBA_1



n)$\alpha = 0.2$
$\delta = 0.2$,
Order:LBB_1



o)$\alpha = 0.2$
$\delta = 0.2$,
Order:LBB_2

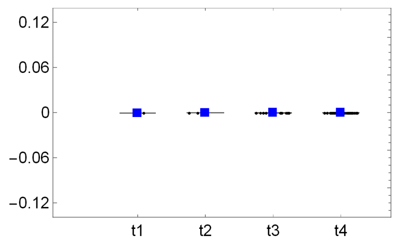


Figure 9 plots the cumulative valuation revisions up though time t_2 against the corresponding revisions along that path through time t_1 for the high adverse-selection (high δ and high α) parameterization. The relationship is more complicated than in Figure 5 due to the violation of the order-sign property in our second model. In particular, a LSA_2 sell limit order at time t_1 is associated with good news (rather than bad news) due to the opposite-side effect. The volatility of the incremental revision at time t_2 is large due to the possibility of a market buy order MBA_2 at t_2 (which would reveal further good news) or a market sell order MSB_2 (which would reveal bad news resulting in a negative cumulative revision up through time t_2). Note also that the distribution of the incremental revision at time t_2 is very skewed following a market buy order MBA_2 at time t_1. Most of the revisions are clustered near the 45° line, but there is a small equilibrium probability of a market sell order MSB_2 leading to a very negative downward revision in the lower-right quadrant.

1.3.3 Summary

The results for our second model specification — with the richer specification of the informed investors' trading motives — confirm and extend the analysis from our first model specification. First, increased adverse selection affects informed-investor trading behavior differently when directionally informed investors trade with their information versus (because of private-value shocks) against their information. Second, it is again possible for the informativeness of orders to be opposite the order aggressiveness and now also opposite the order direction. Third, information content of arriving orders is again history-dependent.

1.4 Robustness

Our analysis makes a number of simplifying assumptions for tractability, but we conjecture that our qualitative results are robust to relaxing these assumptions. We consider two of these assumptions here. First, our model of the trading day only has five periods. Relatedly, our analysis abstracts from limit orders being carried over from one day to the next. However, our results about the impact of adverse selection on investor trading strategies and about order informativeness are driven in large part by the relative size of information shocks and the tick size rather than by the number of rounds of trading. In addition, increasing the trading horizon leads to longer histories that are potentially even more informative. Second, arriving investors are only allowed to submit single orders that cannot be cancelled or modified subsequently. However, it seems likely that order-flow histories will still be informative if orders at different points in time are correlated due to correlated actions of returning investors.

1.5 Conclusions

This paper has studied information aggregation and liquidity provision in dynamic limit order markets. We show a number of notable theoretical properties in our model. First, informed investors switch between endogenously demanding liquidity via market orders and supplying liquidity via limit orders. Second, the information content/price impact of orders can be non-monotone in the direction of the order and in the aggressiveness of their orders. Third, the

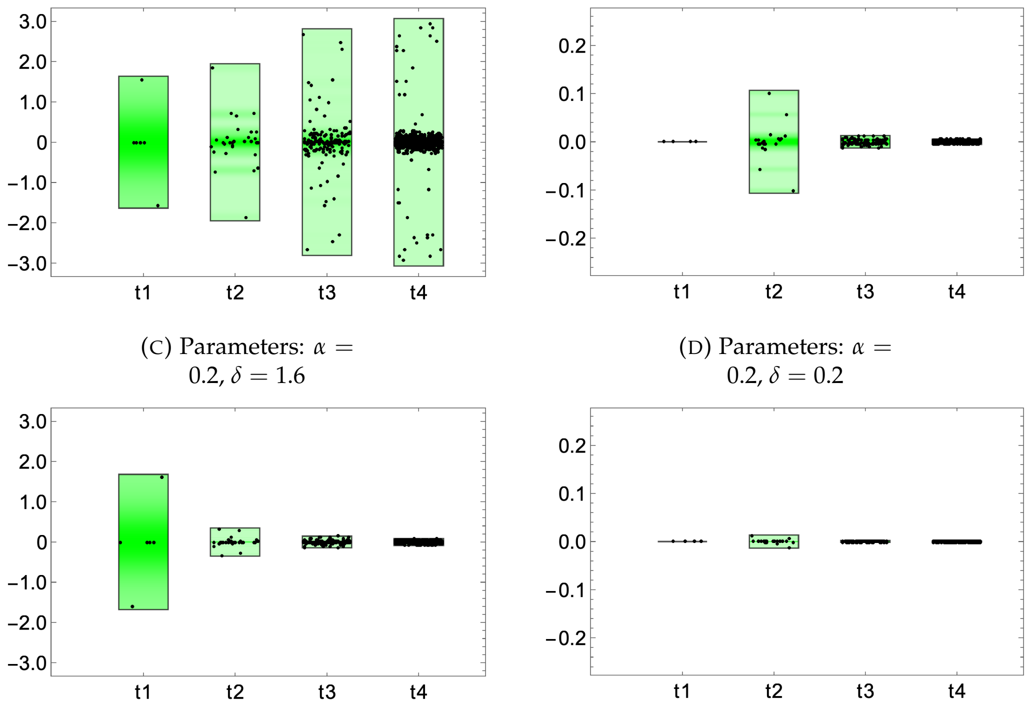
FIGURE 8: **History Informativeness for Informed and Uninformed Traders both with $\beta \sim Tr[\mathcal{N}(\mu, \sigma^2)]$ for times t_1 through t_4.** This Figure shows the incremental information content of the past order history in excess of the information in the current limit order book observed at the end of time t_j as measured by $E[v|\mathcal{L}_{t_j}(L_{t_j})] - E[v|L_{t_j}]$ where $\mathcal{L}_{t_j}(L_{t_j})$ is a history ending in the limit order book L_{t_j}. We only consider books L_{t_j} when they occur in equilibrium in the different trading periods. The candlesticks indicate for each of these two metrics the maximum, the minimum, the median and the 75^{th} (and 25^{th}) percentile respectively as the top (bottom) of the bar.

(A) Parameters: $\alpha = 0.8, \delta = 1.6$

(B) Parameters: $\alpha = 0.8, \delta = 0.2$

(C) Parameters: $\alpha = 0.2, \delta = 1.6$

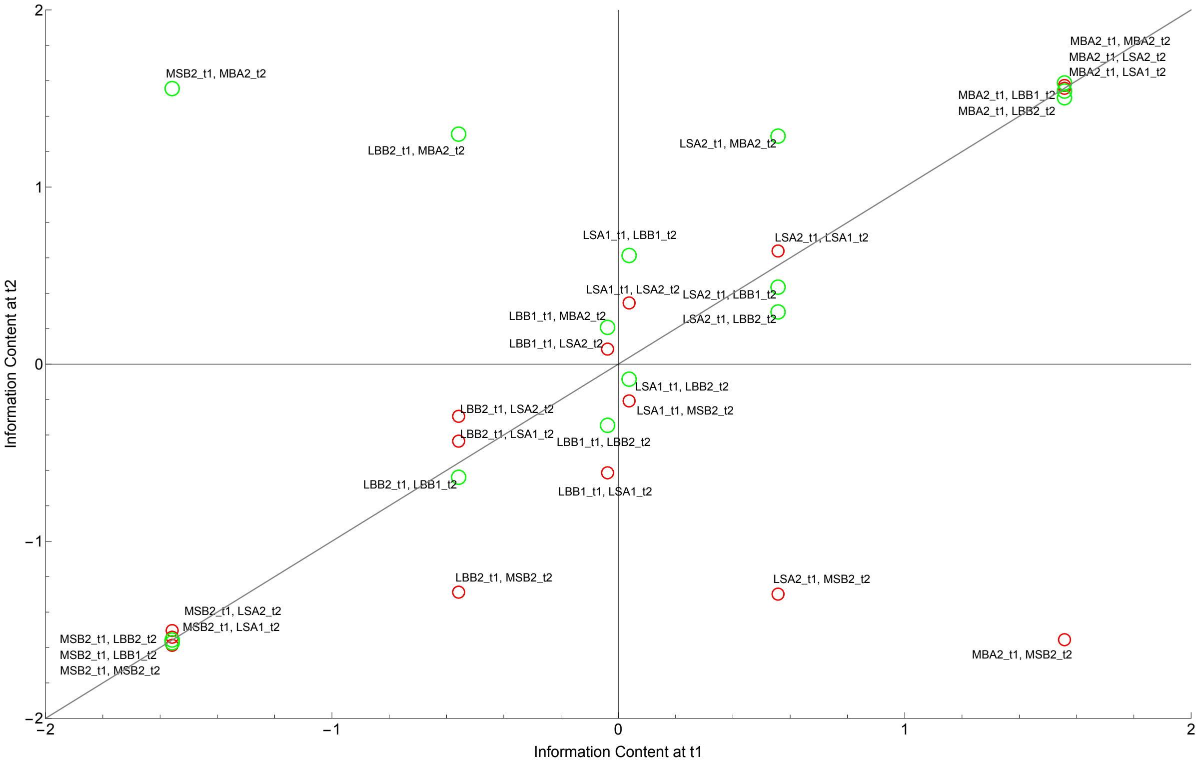
(D) Parameters: $\alpha = 0.2, \delta = 0.2$



information aggregation process is non-Markovian. In particular, the prior order history has information content beyond that in the current limit order book.

Our model suggests several directions for future research. Most importantly, our analysis provides a framework for empirical research about the changing price impacts of order flow conditional on order-flow history and time of day. There are also promising directions for future theory. First, the model can be enriched by allowing investors to trade dynamically over time (rather than just submitting an order one time) and to face quantity decisions and to use multiple orders. Second, the model could be extended to allow for trading in multiple fragmented limit order markets. This would be a realistic representation of current equity trading in the US. Third, the model could be used to study high frequency trading in limit order markets and the effect of different investors being able to process and trade on different types of information at different latencies.

FIGURE 9: **Order Informativeness for Informed and Uninformed both with $\beta \sim Tr[\mathcal{N}(\mu, \sigma^2)]$ for times t_1 to t_2 and parameters $\alpha = 0.8$, $\delta = 1.6$.** The horizontal axis reports $E(v|x_{t_1}) - E(v)$ which shows how the uninformed traders' Bayesian value-forecast changes with respect to the unconditional expected value of the fundamental when uninformed traders observe at t_1 an equilibrium order x_{t_1}. The vertical axis reports $E(v|x_{t_2}, x_{t_1}) - E(v)$ which shows how the uninformed traders' Bayesian value-forecast changes with respect to the unconditional expected value of the fundamental when uninformed traders observe at x_{t_2} at t_2. We consider all the equilibrium strategies at t_1 and t_2 which are symmetrical. Green (red) circles show equilibrium buy (sell) orders at t_2.



1.6 Appendix A: Illustration of order paths and Bayesian updating

This appendix uses an excerpt of the extensive form of the trading game in our model to illustrate order-submission and trading dynamics and the associated Bayesian updating process. The particular trading history path in Figure 10 is from the equilibrium for a model specification in which informed and uninformed investors both have random private-value motives. The model is considered in detail in Section 1.3.2. There are $N = 5$ rounds of trade, and the parameter values are $\kappa = 1$, $\sigma = 15$, $\alpha = 0.8$, and $\delta = 1.6$. This is a market with a relatively high informed-investor arrival probability and large value shocks. In this example, Nature has chosen an economic state in which there is good news ($\overline{v}$) about the asset, and the realized sequence of arriving traders over time is $\{I, U, U, I, I\}$. At each node shown here, Figure 10 reports the total book L_{t_j} of limit orders from both arriving investors and the crowd. Trading starts at t_1 with a book $[1, 0, 0, 1]$ consisting of no orders from informed and uninformed investors (since none have arrived yet) plus the additional limit orders from the trading crowd (i.e., 1 each at the outside prices A_2 and B_2). For simplicity, our discussion here only reports a few nodes of the trading game with their associated equilibrium strategies. For example, we do not include NT at the end of t_1, since Section 1.3.2 will show that NT is not an equilibrium action at t_1 for these parameters.

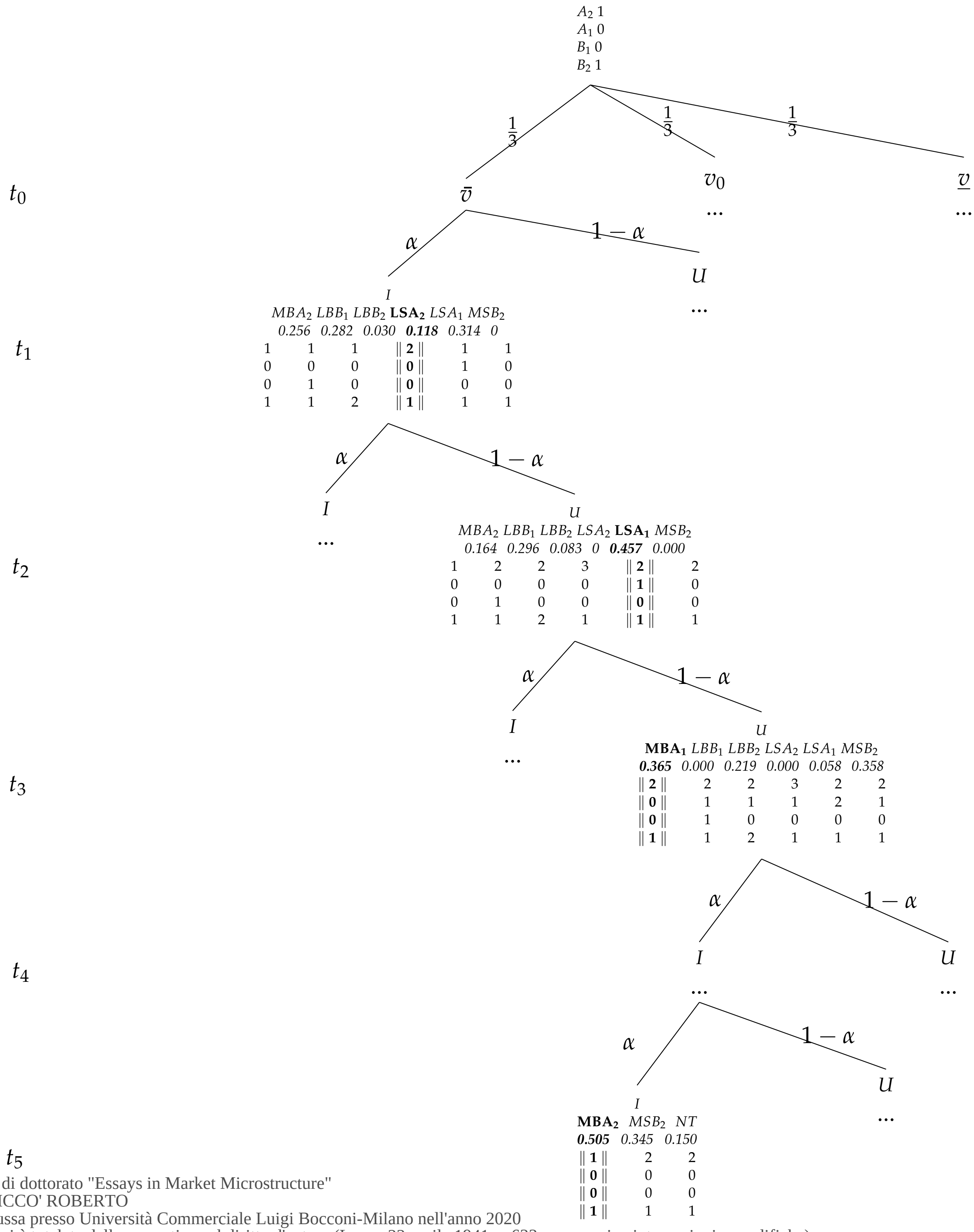
Investors in our equilibrium choose from a discrete number of possible orders given their respective information and any private-value trading motives. Along the particular equilibrium path considered in this example, the optimal strategies do not involve any randomization. Optimal orders are unique given the inputs. However, orders are random after conditioning on the arriving investor's informational type (I_v or U) due to randomness in investors' private factor β. Figure 10 shows below each order type at each time the probabilities with which the different orders are submitted by the trader who arrived. For example, if an informed investor $I_{\overline{v}}$ arrives at t_1, she chooses a limit order LSA_2 to sell at A_2 with probability 0.118. Each of these unique optimal orders is associated with a different range of β types (for both informed and uninformed investors) and value signals (for informed investors). Figure 1.2 in the main body of the paper shows an example of how order-submission probabilities are determined. At each trading time, as the trading game progresses along this path, traders submit orders (or do not trade) following their equilibrium order-submission strategies. The equilibrium execution probabilities of their orders depend on the order-submission decisions of future traders, which, in turn, depend on their trading strategies and the input information (i.e., their β realizations, any private knowledge about v, and the order history path when they arrive). At time t_1, the initial trader has rational-expectation beliefs that the execution probability of her LSA_2 order posted at t_1 is 0.644.[18] This equilibrium execution probability depends on all of the possible future trading paths proceeding from submission time t_1 up through time t_5. For example, one possibility is that the LSA_2 order will be hit by an investor arriving at time t_2 who submits a market order. Another possibility (which is what happens along this particular path) is that an uninformed trader will arrive at t_2 and post a limit order LSA_1 to sell at A_1, thereby undercutting the earlier LSA_2 order — so that the book at the end of t_2 is $[2,1,0,1]$). In this scenario, the initial LSA_2 order from t_1 will only be executed provided that the LSA_1 order submitted at t_2 is executed first. For example, the probability of a market order MBA_1 hitting the limit order at A_1 at t_3 is 0.365, and then the probability of another market order hitting the initial limit sell at A_2 is 0.423 at t_4 and 0.505 at t_5.[19] Therefore, there is a chance that the LSA_2 order from t_1 will still be executed even after it is undercut by the order LSA_1 at t_2.

The path in Figure 10 also illustrates Bayesian updating in the model. After the investor at t_1 has been observed submitting a limit order LSA_2, the uninformed trader who arrives in this example at time t_2 — who just knows the submitted order at time t_1 but not the identity or information of the trader at time t_1 — updates his equilibrium conditional valuation to be $E[\tilde{v}|LSA_2] = 10.558$ and his execution-contingent expectation given his limit order LSA_1 at time t_2 to be $E[\tilde{v}|LSA_2, \theta_{t_2}^{LSA_1}] = 10.639$. In subsequent periods, later investors observe additional realized orders and then further update their beliefs.

[18] Some of the numerical values discussed here are from equilibrium calculations reported in more detail in Tables 1.3 and 1.4 and Table B2 in Appendix B. Others are unreported calculations available from the authors upon request.

[19] Due to space constraints, we do not include the t_4 node in Figure 10.

FIGURE 10: **Excerpt of the Extensive Form of the Trading Game.** This figure shows one possible trading path of the trading game with parameters $\alpha = 0.8$, $\delta = 1.6$, $\mu = 10$, $\sigma = 15$, $\kappa = 1$, and 5 time periods. Before trading starts at time t_1, the incoming book $[1, 0, 0, 1]$ from time t_0 consists of just the initial limit orders from the crowd at A_2 and B_2. Nature selects a realized final value $v = \{\bar{v}, v_0, \underline{v}\}$ with probabilities $\{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\}$. At each trading period nature also selects an informed trader (I) with probability α and an uninformed trader (U) with probability $1 - \alpha$. Arriving traders choose the optimal order at each period which may potentially include limit orders LSA_i (LBB_i) or market orders at the best ask, $MBA_{i,t}$, or at the best bid, $MSB_{i,t}$. Below each optimal trading strategy we report in italics its equilibrium order-submission probability. Boldfaced equilibrium strategies and associated states of the book (within double vertical bar) indicate the states of the book that we consider at each node of the chosen trading path.



1.7 Appendix B: Algorithm for computing equilibrium

The computational problem to solve for a Perfect Bayesian Nash equilibrium in our model (as defined in Section 1.2.1) is complex. Given investors' equilibrium beliefs, the optimal order-submission problems in (1.6) and (1.7) require computing limit-order execution probabilities and stock-value expectations that are conditional on both the past order history and on future state-contingent limit-order execution at each time t_j at each node of the trading game. For an informed trader (who knows the asset value v), there is no uncertainty about the payoff of a market order. In contrast, the payoff of a market order for an uninformed trader entails uncertainty about the future asset value and, therefore, computing the optimal order requires computing the expected stock value $E[v|\mathcal{L}_{t_{j-1}}]$ conditional on the prior trading history up to time t_j. For limit orders, the expected payoff depends on the future limit-order execution probabilities, $Pr(\theta^x_{t_j}|v, \mathcal{L}_{t_{j-1}})$ and $Pr(\theta^x_{t_j}|\mathcal{L}_{t_{j-1}})$, for informed and uninformed investors, which depend, in turn, on the optimal order-submission probabilities of future informed and uninformed investors. In addition, the uninformed investors' learning problem for limit orders requires uninformed investors to extract information about the expected future stock value $E[v|\mathcal{L}_{t_{j-1}}, \theta^x_{t_j}]$ from both the past trading history and also from state-contingent future order execution given that the future states in which limit orders are executed are correlated with the stock value. Thus, optimal actions at each time t_j depend on past information and future order-flow contingencies where future orders also depend on the then-prior histories at future dates (which include the action at time t_j) as traders dynamically update their equilibrium beliefs as the trading process unfolds. Thus, the learning problem for limit order beliefs is both backward- and forward-looking. Lastly, rational expectations (RE) involves finding a fixed point so that the equilibrium beliefs underlying the optimal order-submission strategies are consistent with the execution probabilities and value expectations that the endogenous optimal strategies produce in equilibrium.

Our numerical algorithm uses backward induction to solve for optimal order strategies given a set of asset-value beliefs for all dates and nodes in the trading game and uses an iterative recursion to solve for RE equilibrium asset-value and order-execution beliefs. The backward induction makes order-execution probabilities consistent with optimal future behavior by later-arriving investors. It also takes future state-contingent execution into account in the uninformed investors' beliefs. Given a set of history-contingent asset-value probability beliefs, we start at time t_5 — when traders only use market orders which allows us to compute the execution probabilities of limit orders at t_4 — and recursively solve the model for optimal order strategies back to time t_1. We then embed the optimal order strategy calculation in an iterative recursion to solve for a fixed point for the RE asset-value beliefs. For a generic round r in this recursion, the outgoing asset-value probabilities $\pi^{v,r-1}_{t_j}$ from round $r-1$ are used iteratively as incoming asset-value beliefs in round r. In particular, these beliefs are used in the learning problem of the uninformed investor to extract information about the ending asset value v from the prior trading histories. They also affect the behavior of informed investors whose order-execution probability beliefs depend in part on the behavior of uninformed traders. Thus, the recursion for a generic round r involves solving by backward induction for optimal strategies for buyers

$$\max_{x\in X_{t_j}} w^{I,r}(x\,|\,v,\mathcal{L}_{t_{j-1}}) \quad = \quad [v_0+\Delta\,+\beta_{t_j}-p(x)]\,Pr^r(\theta^x_{t_j}|\,v,\mathcal{L}_{t_{j-1}}) \tag{1.16}$$

and

$$\max_{x\in X_{t_j}} w^{U,r}(x\,|\mathcal{L}_{t_{j-1}}) \quad = \quad [v_0+E^r[\Delta\,|\mathcal{L}_{t_{j-1}},\theta^x_{t_j}]\,+\beta_{t_j}-p(x)]\,Pr^r(\theta^x_{t_j}|\mathcal{L}_{t_{j-1}}) \tag{1.17}$$

where

$$E^r[\Delta|\mathcal{L}_{t_{j-1}},\theta^x_{t_j}] = (\hat{\pi}^{\bar{v},r}_{t_j}\,\bar{v}+\hat{\pi}^{v_0,r}_{t_j}\,v_0+\hat{\pi}^{\underline{v},r}_{t_j}\,\underline{v})-v_0 \tag{1.18}$$

$$\hat{\pi}^{v,r}_{t_j} \quad = \quad \frac{\Pr^r(\theta^x_{t_j}|v,\mathcal{L}_{t_j})}{Pr^r(\theta^x_{t_j}|\mathcal{L}_{t_j})}\pi^{v,r-1}_{t_j} \tag{1.19}$$

and where the calculations for sellers are symmetric. Note that at each time t_j the backward induction has already determined the future contingencies $\theta^x_{t_j}$ for limit order executions at times $t > t_j$. Thus, the order-execution probabilities $Pr^r(\theta^x_{t_j}|\,v,\mathcal{L}_{t_{j-1}})$ and $Pr^r(\theta^x_{t_j}|\mathcal{L}_{t_{j-1}})$, and the history-and execution-contingent probabilities $\hat{\pi}^{v,r}_{t_j}$ and associated asset-value expectations $E^r[\Delta|\mathcal{L}_{t_{j-1}},\theta^x_{t_j}]$ are "mongrel" moments in that they are computed using the outgoing history-contingent asset value beliefs $\pi^{v,r-1}_{t_j}$ from round $r-1$ and then updated given the order-execution contingencies computed by backward induction in round r using the round $r-1$ asset-value beliefs. At the end of round r, we then compute updated outgoing asset-value beliefs $\pi^{v,r}_{t_j}$ for round r, which are used as incoming beliefs for the next round $r+1$. The recursion is iterated to find a RE fixed point $\pi^v_{t_j}$ in the uninformed investor beliefs.

The fixed-point recursion is started in round $r=1$ by setting the initial asset-value beliefs $\pi^{v,0}_{t_j}$ of uninformed traders at each time t_j in the backward induction to be the unconditional priors $Pr(v)$ in (1.1). In particular, the algorithm starts by ignoring conditioning on history in the initial round $r=1$. Hence the traders' optimization problems in (1.17) and (1.16) in round $r=1$ simplify to:

$$\max_{x\in X_{t_j}} w^{I,r=1}(x\,|\,v,\mathcal{L}_{t_{j-1}}) \quad = \quad [v_0+\Delta\,+\beta_{t_j}-p(x)]\,Pr^1(\theta^x_{t_j}|\,v) \tag{1.20}$$

$$\max_{x\in X_{t_j}} w^{U,r=1}(x\,|\mathcal{L}_{t_{j-1}}) \quad = \quad [v_0+E^1[\Delta|\theta^x_{t_j}]\,+\beta_{t_j}-p(x)]\,Pr^1(\theta^x_{t_j}) \tag{1.21}$$

The order-execution contingencies in round r are modeled as follows: In each round r given the asset-value beliefs $\pi^{v,r-1}_{t_j}$ in that round, we solve for investors' optimal trading strategies by backward induction. Starting at t_5, the execution probability for new limit orders is zero, and therefore optimal order-submission strategies only use market orders. Given the linearity of the expected payoffs in the private-value factor β in (1.16) and (1.17), the optimal orders for an informed trader at t_5 are[20]

$$x_{t_5}^{I,r}(\beta|\mathcal{L}_{t_4}, v) = \begin{cases} MSB_{i,t_5} & if\ \beta \in [0, \beta^{MSB_{i,t_5}^{I,r}, NT_{t_5}^{I,r}}) \\ NT & if\ \beta \in [\beta^{MOB_{i,t_5}^{I,r}, NT_{t_5}^{I,r}}, \beta^{NT_{t_5}^{I,r}, MBA_{i,t_5}^{I,r}}) \\ MBA_{i,t_5} & if\ \beta \in [\beta^{NT_{t_5}^{I,r}, MBA_{i,t_5}^{I,r}}, 2] \end{cases} \tag{1.22}$$

where for each possible combination of $MSB_{i,t_5} = MSB_1, MSB_2$ and $MBA_{i,t_5} = MBA_1, MBA_2$

$$\begin{aligned} \beta^{MSB_{i,t_5}^{I,r}, NT_{t_5}^{I,r}} &= \frac{B_{i,t_5} - \Delta}{v} \\ \beta^{NT_{t_5}^{I,r}, MBA_{i,t_5}^{I,r}} &= \frac{A_{i,t_5} - \Delta}{v} \end{aligned} \tag{1.23}$$

are the critical thresholds that solve $w^{I,r}(MSB_{i,t_5}|v, \mathcal{L}_{t_4}) = w^{I,r}(NT|v, \mathcal{L}_{t_4})$ and $w^{I,r}(NT|v, \mathcal{L}_{t_4}) = w^{I,r}(MBA_{i,t_5}|v, \mathcal{L}_{t_4})$, respectively. Our notation here for market orders differs slightly from the notation in the body of the paper because we need to denote both different possible price levels and the time at which different possible orders are being compared. The optimal trading strategies and β thresholds for an uninformed traders are similar but the conditioning set does not include the asset value v:

$$x_{t_5}^{U,r}(\beta|\mathcal{L}_{t_4}) = \begin{cases} MSB_{i,t_5} & if\ \beta \in [0, \beta^{MSB_{i,t_5}^{U,r}, NT_{t_5}^{U,r}}) \\ NT & if\ \beta \in [\beta^{MSB_{i,t_5}^{U,r}, NT_{t_5}^{U,r}}, \beta^{NT_{t_5}^{U,r}, MBA_{i,t_5}^{U,r}}) \\ MBA_{i,t_5} & if\ \beta \in [\beta^{NT_{t_5}^{U,r}, MBA_{i,t_5}^{U,r}}, 2] \end{cases} \tag{1.24}$$

where

$$\begin{aligned} \beta^{MSB_{i,t_5}^{U,r}, NT_{t_5}^{U,r}} &= \frac{B_{i,t_5} - E^{r-1}[\Delta|\mathcal{L}_{t_4}]}{v} \\ \beta^{NT_{t_5}^{U,r}, MBA_{i,t_5}^{U,r}} &= \frac{A_{i,t_5} - E^{r-1}[\Delta|\mathcal{L}_{t_4}]}{v} \end{aligned} \tag{1.25}$$

Given the β ranges associated with each possible action at t_5, we compute the submission probabilities associated with each optimal order at t_5 using the truncated-Normal density $\mathfrak{n}(\cdot)$ for the private factor β.[21] At time t_4 these are the execution probabilities for new limit orders by an informed investor at the different possible best bids and asks, B_{i,t_4} and A_{i,t_4} respectively at time t_5:

$$Pr^r(\theta_{t_4}^{LBB_i}|\mathcal{L}_{t_3}, v) = \begin{cases} \alpha\Big[\int_0^{\beta^{MSB_{i,t_5}^{I,r}, NT_{t_5}^{I,r}}} \mathfrak{n}(\beta)\, d\beta\Big] + \Big(1-\alpha\Big)\Big[\int_0^{\beta^{MSB_{i,t_5}^{U,r}, NT_{t_5}^{U,r}}} \mathfrak{n}(\beta) d\beta\Big] \\ 0 \qquad\qquad\qquad\qquad\qquad\qquad \text{otherwise} \end{cases} \tag{1.26}$$

[20] For instance, an informed trader would post a MBA_1 only if the payoff is positive and thus outperforms the NT payoff of zero, i.e, $\beta v + \Delta - A_1 > 0$ or $\beta > \frac{A_1 - \Delta}{v}$.

[21] The discussion here is for the case where both informed and uninformed investors have random private factors β.

$$Pr^r(\theta_{t_4}^{LSA_i}|\mathcal{L}_{t_3}, v) = \begin{cases} \alpha\Big[\int_{\beta^{NT_{t_5}^{I,r}, MBA_{i,t_5}^{I,r}}}^{2} \mathfrak{n}(\beta)\, d\beta\Big] + \Big(1-\alpha\Big)\Big[\int_{\beta^{NT_{t_5}^{U,r}, MBA_{i,t_5}^{U,r}}}^{2} \mathfrak{n}(\beta)\, d\beta\Big] \\ 0 \qquad \text{otherwise} \end{cases} \tag{1.27}$$

where the book is either empty at A_1 and/or B_1 (but may have non-crowd limit orders at the outside prices) or is empty except for just crowd orders at A_2 and B_2. The analogous execution probabilities for an uninformed investor arriving at time t_4 are:

$$Pr^r(\theta_{t_4}^{LBB_i}|\mathcal{L}_{t_3}) = \begin{cases} \alpha\Big[\sum_{v\in\{\overline{v}, v_0, \underline{v}\}} \hat{\pi}_{t_4}^{v,r} \int_0^{\beta_{t_5}^{MSB_{i,t_5}^{Iv,r}, NT_{t_5}^{Iv,r}}} \mathfrak{n}(\beta)\, d\beta\Big] + \Big(1-\alpha\Big)\Big[\int_0^{\beta_{t_5}^{MSB_{i,t_5}^{U,r}, NT_{t_5}^{U,r}}} \mathfrak{n}(\beta)\, d\beta\Big] \\ 0 \qquad \text{otherwise} \end{cases} \tag{1.28}$$

$$Pr^r(\theta_{t_4}^{LSA_i}|\mathcal{L}_{t_3}) = \begin{cases} \alpha\Big[\sum_{v\in\{\overline{v}, v_0, \underline{v}\}} \hat{\pi}_{t_4}^{v,r} \int_{\beta^{NT_{t_5}^{Iv,r}, MBA_{i,t_5}^{Iv,r}}}^{2} \mathfrak{n}(\beta)\, d\beta\Big] + \Big(1-\alpha\Big)\Big[\int_{\beta^{NT_{t_5}^{U,r}, MBA_{i,t_5}^{U,r}}}^{2} \mathfrak{n}(\beta)\, d\beta\Big] \\ 0 \qquad \text{otherwise} \end{cases} \tag{1.29}$$

At t_4 there is only one period before the end of the trading game. Thus, the execution probability of a limit order is positive if and only if the order is posted at the best price on its own side of the market (A_{i,t_4} or B_{i,t_4}), and if there are no non-crowd limit orders already standing in the limit order book at that price at the time the new limit order is posted.

Having obtained the execution probabilities in (1.26) – (1.29) for the different limit orders at t_4, we next derive the optimal order-submission strategies at t_4. The incoming book can be configured in many different ways at t_4 depending on the different possible prior order paths $\mathcal{L}_{t_3}$ in the trading game up through time t_3. As the payoffs of both limit and market orders are functions of β, we rank all the payoffs of adjacent optimal strategies in terms of β and equate them to determine the β thresholds at time t_4.[22] Consider, for example, an order path such that t_4 has only crowd orders in the book, so that new limit and market orders are both potentially optimal orders at t_4. For an informed trader, the the optimal orders are given by:

$$x_{t_4}^{I,r}(\beta|\mathcal{L}_{t_3}, v) = \begin{cases} MSB_2 & if\ \beta \in [0,\ \beta^{MSB_{2,t_4}^{I,r}, LSA_{1,t_4}^{I,r}}) \\ LSA_1 & if\ \beta \in [\beta^{MSB_{2,t_4}^{I,r}, LSA_{1,t_4}^{I,r}},\ \beta^{LSA_{1,t_4}^{I,r}, LSA_{2,t_4}^{I,r}}) \\ LSA_2 & if\ \beta \in [\beta^{LSA_{1,t_4}^{I,r}, LSA_{2,t_4}^{I,r}},\ \beta^{LSA_{2,t_4}^{I,r}, NT_{t_4}^{I,r}}) \\ NT & if\ \beta \in [\beta^{LSA_{2,t_4}^{I,r}, NT_{t_4}^{I,r}},\ \beta^{NT_{t_4}^{I,r}, LBB_{2,t_4}^{I,r}}) \\ LBB_2 & if\ \beta \in [\beta^{NT_{t_4}^{I,r}, LBB_{2,t_4}^{I,r}},\ \beta^{LBB_{2,t_4}^{I,r}, LBB_{1,t_4}^{I,r}}) \\ LBB_1 & if\ \beta \in [\beta^{LBB_{2,t_4}^{I,r}, LBB_{1,t_4}^{I,r}},\ \beta^{LBB_1^{I,r}, MBA_{2,t_4}^{I,r}}) \\ MBA_2 & if\ \beta \in [\beta^{LBB_{1,t_4}^{I,r}, MBA_{2,t_4}^{I,r}},\ 2] \end{cases} \tag{1.30}$$

and for an uninformed trader the optimal strategies are qualitatively similar but with different

[22] Recall that the upper envelope only includes strategies that are optimal.

values for the β thresholds given the uninformed investor's different information.[23] As the payoffs of both limit and market orders are functions of β, we can rank all the payoffs of adjacent optimal strategies in terms of β and equate them to determine the β thresholds at t_4. For example, for the first β threshold we have:

$$\beta_{t_4}^{MSB_{2,t_4}^{I,r},LSA_{1,t_4}^{I,r}} = \beta \in \mathbb{R} \text{ s.t. } w_{t_4}^{I,r}\left(MSB_2 \,|\, v,\beta,\mathcal{L}_{t_3}\right) = w_{t_4}^{I,r}\left(LSA_1 \,|\, v,\beta,\mathcal{L}_{t_3}\right) \tag{1.31}$$

and we obtain the other thresholds similarly.

The next step is to use the β thresholds together with the truncated Normal cumulative distribution $\mathbb{N}()$ for β to derive the probabilities of the optimal order-submission strategies at each possible node of the extensive form of the game at t_4. For example, the submission probability of $LSA_1^{I,r}$ is:

$$Pr^r[LSA_1^{I,r} \,|\, \mathcal{L}_{t_3},v] = \mathbb{N}(\beta^{LSA_1^{I,r},LSA_2^{I,r}} \,|\, \mathcal{L}_{t_3},v) - \mathbb{N}(\beta^{MSB_2^{I,r},LSA_1^{I,r}} \,|\, \mathcal{L}_{t_3},v) \tag{1.32}$$

and the submission probabilities of the equilibrium strategies can be obtained in a similar way. Next, given the market-order submission probabilities at t_4 — which together with the execution probabilities at t_5 determine the execution probabilities for new limit orders at time t_3 — we can solve the optimal orders at t_3 and then recursively continue to solve the model by backward induction in this fashion back to time t_1.

Off-equilibrium beliefs: At each time t_j, round r of the recursion needs history-contingent asset-value beliefs $\pi_{t_j}^{v,r-1} = Pr^{r-1}(v|\mathcal{L}_{t_j})$ from round $r-1$ for all feasible paths that traders may use. Beliefs for paths that occur with positive probability in round $r-1$ are computed using Bayes' rule to update the probability $Pr^{r-1}(v|\mathcal{L}_{t_{j-1}})$ of the time-t_{j-1} sub-path $\mathcal{L}_{t_{j-1}}$ that path $\mathcal{L}_{t_j}$ extends. In contrast, Bayes' Rule cannot be used to update probabilities of paths that involve orders that are not used with positive probability in round $r-1$. Our algorithm deals with this by setting $Pr^{r-1}(v|\mathcal{L}_{t_j})$ to be $Pr^{r-1}(v|\mathcal{L}_t)$ where $\mathcal{L}_t$ is the longest positive-probability sub-path from t_0 to some time $t < t_k$ in round $r-1$ that is contained in path $\mathcal{L}_{t_j}$. For example, consider a path $\{MBA_2, MSB_2, LSA_1\}$ at time t_3 where orders $\{MBA_2, MSB_2\}$ are used with positive probability at times t_1 and t_2 in round $r-1$, but LSA_1 is not used at time t_3 after the first two orders in round $r-1$. Our recursion algorithm sets the round $r-1$ belief uninformed traders use for path $\{MBA_2, MSB_2, LSA_1\}$ to be their round $r-1$ belief for the positive-probability sub-path $\{MBA_2, MSB_2\}$. If instead MSB_2 is not a positive-probability order at t_2 in round $r-1$, then we assume that uninformed traders use their belief at t_1 conditional on the shorter sub-path $\{MBA_2\}$. Finally, if MBA_2 is also not a positive-probability order at t_1 in round $r-1$, then we assume that traders use their unconditional prior belief $Pr(v)$.

Mixed strategies: We allow for both pure and mixed strategies in our Perfect Bayesian Nash equilibrium. When different orders have equal expected payoffs, we assume that traders randomize with equal probabilities across all such optimal orders. By construction, the expected payoffs of two different strategies are the same in correspondence of the β thresholds; however

[23] If the incoming book from t_3 has non-crowd orders on any level of the book, the equilibrium strategies would be different. For example, if the book has a LSA_1 limit order, then new limit orders on the ask side cannot be equilibrium orders since their execution probability would be zero.

because we are considering single points in the support of the β distribution, the probability associated with any strategy that corresponds to those specific points is equal to zero. This means that mixed strategies that emerge in correspondence of the β thresholds, although feasible, have zero probability. Mixed strategies may also emerge in the framework in which informed traders have a fixed neutral private-value factor $\beta = 0$ (section 1.3.1). More specifically it may happen that the payoffs of two perfectly symmetrical strategies of I_{v_0} are the same, and in this case I_{v_0} randomizes between these two strategies.

In the setting of our model where informed traders have fixed neutral private-value factors $\beta = 0$, it may happen that both informed and uninformed traders switch their strategies back and forth from one round to the next. When this happens, to reach an equilibrium we assume that the informed traders play mixed strategies and at each subsequent round strategically reduce the probability with which they choose the most profitable strategy until the equilibrium is reached. As an example at t_1 informed traders with positive news, $I_{\bar{v}}$, play LBB_2 in round $r = 1$. However, in round $r = 2$ in the subsequent periods uninformed traders do not send market orders to sell at B_2 and in round $r = 3$, informed traders react by changing their strategy to LBB_1. However, in the subsequent periods uninformed traders do not send market orders to sell, this time at B_1. To find an equilibrium, we assume that at each round informed traders play mixed strategies and assign a greater weight to the most profitable strategy. In this case we assume they start playing LBB_2 with probability 0.99 and LBB_1 with probability 0.01. If these mixed strategies do not lead to an equilibrium outcome, in the subsequent round we assume that the informed traders play LBB_2 with probability 0.98 and LBB_1 with probability 0.02. We proceed by lowering the probability with which informed traders choose the most profitable strategy until we reach an equilibrium set of strategies.

Convergence: RE beliefs for a Perfect Bayesian Nash equilibrium are obtained by solving the model recursively for multiple rounds. In particular, the asset-value probabilities $\pi_{t_j}^{v,1}$ from round $r = 1$ from above are used as the priors to solve the model in round $r = 2$ (i.e., the round 1 probabilities are used in place of the unconditional priors used in round 1).[24] The asset-value probabilities $\pi_{t_j}^{v,2}$ from round $r = 2$ are then used as the priors in round $r = 3$ and so on. The recursive iteration is continued until the updating process converges to a fixed point, which are the RE beliefs. In particular, the recursive process has converged to the RE beliefs when uninformed traders no longer revise their asset-value beliefs. Operationally, we consider convergence to the RE beliefs to have occurred when the probabilities $\pi_{t_j}^{\bar{v},r}$, $\pi_{t_j}^{v_0,r}$ and $\pi_{t_j}^{\underline{v},r}$ in round r are "close enough" to the corresponding probabilities from round $r - 1$:

$$\begin{aligned} &\pi_{t_j}^{\bar{v},*} \text{ when } \left|\pi_{t_j}^{\bar{v},r} - \pi_{t_j}^{\bar{v},r-1}\right| < 10^{-7} \\ &\pi_{t_j}^{v_0,*} \text{ when } \left|\pi_{t_j}^{v_0,r} - \pi_{t_j}^{v_0,r-1}\right| < 10^{-7} \\ &\pi_{t_j}^{\underline{v},*} \text{ when } \left|\pi_{t_j}^{\underline{v},r} - \pi_{t_j}^{\underline{v},r-1}\right| < 10^{-7} \end{aligned} \tag{1.33}$$

A fixed-point solution to this recursive algorithm is an equilibrium in our model.

[24] In the second round of solutions we again solve the full 5-period model.

1.8 Appendix C: Additional numerical results

The tables is this section provide additional information on the execution probabilities of limit orders for informed investor with positive, neutral and negative signals, ($I_{\bar{v}}$,I_{v_0},$I_{\underline{v}}$) and for uninformed traders. The tables also report the asset value expectations of the uninformed investor at time t_2 after observing all the possible buy orders submissions at time t_1. The expectations for sell orders are symmetric with respect to 1. Table B1 reports results for our first model specification in which only uninformed traders have a random private value factor. Table B2 reports results for our second model in which both the informed and uniformed traders have private-value motives.

Table B1: Order Execution Probabilities and Asset-Value Expectation for Informed Traders with $\beta = 0$ and Uninformed Traders with $\beta \sim Tr[\mathcal{N}(\mu, \sigma^2)]$. This table reports results for two different values of the informed-investor arrival probability α (0.8 and 0.2) and for two different values of the asset-value volatility δ (1.6 and 0.2). $\sigma = 15$. For each set of parameters, the first four columns report the equilibrium limit order probabilities of executions for informed traders with positive, neutral and negative signals, ($I_{\bar{v}}$,I_{v_0},$I_{\underline{v}}$) and for uninformed traders (U). The fifth column (*Uncond.*) reports the unconditional order-execution probabilities in the market. Next, the columns report conditional and unconditional future order execution probabilities and the asset-value expectations of an uniformed investor at time t_2 after observing different order submissions at time t_1.

		$\delta = 1.6$					$\delta = 0.2$				
		$I_{\bar{v}}$	I_{v_0}	$I_{\underline{v}}$	U	*Uncond.*	$I_{\bar{v}}$	I_{v_0}	$I_{\underline{v}}$	U	*Uncond.*
	$P^{EX}(LSA_2\|\cdot)$	0.940	0.199	0.059	0.399	0.399	0.180	0.229	0.170	0.193	0.193
	$P^{EX}(LSA_1\|\cdot)$	0.988	0.134	0.078	0.400	0.400	0.323	0.323	0.323	0.323	0.323
	$P^{EX}(LBB_1\|\cdot)$	0.078	0.134	0.988	0.400	0.400	0.323	0.323	0.323	0.323	0.323
	$P^{EX}(LBB_2\|\cdot)$	0.059	0.199	0.940	0.399	0.399	0.170	0.229	0.180	0.193	0.193
$\alpha = 0.8$											
	$E[v\|LBB_1\ \|\cdot]$					11.600					10.000
	$E[v\|LBB_2\ \|\cdot]$					10.820					10.130
	$E[v\|MBA_1\ \|\cdot]$										
	$E[v\|MBA_2\ \|\cdot]$					10.000					10.000
	$P^{EX}(LSA_2\|\cdot)$	0.656	0.490	0.396	0.514	0.514	0.514	0.499	0.476	0.496	0.496
	$P^{EX}(LSA_1\|\cdot)$	0.886	0.763	0.713	0.787	0.787	0.792	0.792	0.790	0.791	0.791
	$P^{EX}(LBB_1\|\cdot)$	0.713	0.763	0.886	0.787	0.787	0.790	0.792	0.792	0.791	0.791
	$P^{EX}(LBB_2\|\cdot)$	0.396	0.490	0.656	0.514	0.514	0.476	0.499	0.514	0.496	0.496
$\alpha = 0.2$											
	$E[v\|LBB_1\ \|\cdot]$					10.278					10.000
	$E[v\|LBB_2\ \|\cdot]$					10.083					10.089
	$E[v\|MBA_1\ \|\cdot]$										
	$E[v\|MBA_2\ \|\cdot]$					10.000					10.000

Table B2: Order Execution Probabilities and Asset-Value Expectation for Informed and Uninformed Traders both with $\beta \sim Tr[\mathcal{N}(\mu, \sigma^2)]$. This table reports results for two different values of the informed-investor arrival probability α (0.8 and 0.2) and for two different values of the asset-value volatility δ (1.6 and 0.2). $\sigma = 15$. For each set of parameters, the first four columns report the equilibrium limit order probabilities of executions for informed traders with positive, neutral and negative signals, ($I_{\bar{v}}$,I_{v_0},$I_{\underline{v}}$) and for uninformed traders (U). The fifth column ($Uncond.$) reports the unconditional order-execution probabilities in the market. Next, the columns report conditional and unconditional future order execution probabilities and the asset-value expectations of an uniformed investor at time t_2 after observing different order submissions at time t_1.

		$\delta = 1.6$					$\delta = 0.2$				
		$I_{\bar{v}}$	I_{v_0}	$I_{\underline{v}}$	U	$Uncond.$	$I_{\bar{v}}$	I_{v_0}	$I_{\underline{v}}$	U	$Uncond.$
	$P^{EX}(LSA_2\|\cdot)$	0.644	0.502	0.410	0.519	0.519	0.502	0.487	0.472	0.487	0.487
	$P^{EX}(LSA_1\|\cdot)$	0.913	0.834	0.702	0.817	0.817	0.849	0.837	0.824	0.836	0.836
	$P^{EX}(LBB_1\|\cdot)$	0.702	0.834	0.913	0.817	0.817	0.824	0.837	0.849	0.836	0.836
	$P^{EX}(LBB_2\|\cdot)$	0.410	0.502	0.644	0.519	0.519	0.472	0.487	0.502	0.487	0.487
$\alpha = 0.8$											
	E[v|LBB_1 |·]					9.962					10.003
	E[v|LBB_2 |·]					9.442					9.988
	E[v|MBA_1 |·]										
	E[v|MBA_2 |·]					11.558					
	$P^{EX}(LSA_1\|\cdot)$	0.525	0.494	0.470	0.496	0.496	0.490	0.487	0.483	0.487	0.487
	$P^{EX}(LSA_1\|\cdot)$	0.853	0.833	0.813	0.833	0.833	0.839	0.837	0.834	0.837	0.837
	$P^{EX}(LBB_1\|\cdot)$	0.813	0.833	0.853	0.833	0.833	0.834	0.837	0.839	0.837	0.837
	$P^{EX}(LBB_2\|\cdot)$	0.470	0.494	0.525	0.496	0.496	0.483	0.487	0.490	0.487	0.487
$\alpha = 0.2$											
	E[v|LBB_1 |·]					10.029					10.001
	E[v|LBB_2 |·]					9.957					9.999
	E[v|MBA_1 |·]										
	E[v|MBA_2 |·]					11.600					

TABLE 1.5: **Trading Strategies, Liquidity, and Welfare at Time t_1 in an Equilibrium with Informed Traders with $\beta = 0$ and Uninformed Traders with $\beta \sim Tr[\mathcal{N}(\mu, \sigma^2)]$.** This table reports results for two different informed-investor arrival probabilities α (0.8 and 0.2) and two different value-shock volatilities δ (1.6 and 0.2). The private-value factor parameters are $\mu = 10$ and $\sigma = 10$, and the tick size is $\kappa = 1$. Each cell corresponding to a set of parameters reports the equilibrium order-submission probabilities, the expected bid-ask spreads and expected depths at the inside prices (A_1 and B_1) and total depths on each side of the market after order submissions at time t_1, and expected welfare of the market participants. The first four columns in each parameter cell are for informed traders with positive, neutral and negative signals, ($I_{\bar{v}}$,I_{v_0},$I_{\underline{v}}$) and for uninformed traders (U). The fifth column (*Uncond.*) reports unconditional results for the market.

		$\delta = 1.6$					$\delta = 0.2$				
		$I_{\bar{v}}$	I_{v_0}	$I_{\underline{v}}$	U	*Uncond.*	$I_{\bar{v}}$	I_{v_0}	$I_{\underline{v}}$	U	*Uncond.*
	LSA_2	0	0.500	0.620	0.155	0.330	0.010	0.500	0.990	0.054	0.411
	LSA_1	0	0	0.380	0	0.101	0	0	0	0.088	0.018
	LBB_1	0.380	0	0	0	0.101	0	0	0	0.088	0.018
	LBB_2	0.620	0.500	0	0.155	0.330	0.990	0.500	0.010	0.054	0.411
	MBA_2	0	0	0	0.345	0.069	0	0	0	0.358	0.072
	MBA_1	0	0	0	0	0	0	0	0	0	0
	MSB_1	0	0	0	0	0	0	0	0	0	0
	MSB_2	0	0	0	0.345	0.069	0	0	0	0.358	0.072
	NT	0	0	0	0	0	0	0	0	0	0
$\alpha = 0.8$											
	E[Spread $\vert\cdot$]	2.620	3.000	2.620	3.000	2.797	3.000	3.000	3.000	2.824	2.965
	E[Depth A_2+A_1 $\vert\cdot$]	1.000	1.500	2.000	1.155	1.431	1.010	1.500	1.990	1.142	1.428
	E[Depth A_1 $\vert\cdot$]	0	0	0.380	0	0.101	0	0	0	0.088	0.018
	E[Depth B_1 $\vert\cdot$]	0.380	0	0	0	0.101	0	0	0	0.088	0.018
	E[Depth B_1+B_2 $\vert\cdot$]	2.000	1.500	1.000	1.155	1.431	1.990	1.500	1.010	1.142	1.428
	E[Welfare LO $\vert\cdot$]	0.338	0.581	0.338	0.204	0.376	0.513	0.672	0.513	0.163	0.485
	E[Welfare MO $\vert\cdot$]	0	0	0	3.149	0.630	0	0	0	3.179	0.636
	E[Welfare $\vert\cdot$]	0.338	0.581	0.338	3.353	1.006	0.513	0.672	0.513	3.342	1.121
	LSA_2	0	0.500	0.180	0.058	0.091	0	0.500	1.000	0.061	0.149
	LSA_1	0	0	0.820	0.383	0.361	0	0	0	0.395	0.316
	LBB_1	0.820	0	0	0.383	0.361	0	0	0	0.395	0.316
	LBB_2	0.180	0.500	0	0.058	0.091	1.000	0.500	0	0.061	0.149
	MBA_2	0	0	0	0.060	0.048	0	0	0	0.044	0.035
	MBA_1	0	0	0	0	0	0	0	0	0	0
	MSB_1	0	0	0	0	0	0	0	0	0	0
	MSB_2	0	0	0	0.060	0.048	0	0	0	0.044	0.035
	NT	0	0	0	0	0	0	0	0	0	0
$\alpha = 0.2$											
	E[Spread $\vert\cdot$]	2.180	3.000	2.180	2.235	2.278	3.000	3.000	3.000	2.211	2.369
	E[Depth A_2+A_1 $\vert\cdot$]	1.000	1.500	2.000	1.440	1.452	1.000	1.500	2.000	1.456	1.465
	E[Depth A_1 $\vert\cdot$]	0	0	0.820	0.383	0.361	0	0	0	0.395	0.316
	E[Depth B_1 $\vert\cdot$]	0.820	0	0	0.383	0.361	0	0	0	0.395	0.316
	E[Depth B_1+B_2 $\vert\cdot$]	2.000	1.500	1.000	1.440	1.452	2.000	1.500	1.000	1.456	1.465
	E[Welfare LO $\vert\cdot$]	2.659	1.373	2.659	3.034	2.873	0.780	1.444	0.780	3.341	2.873
	E[Welfare MO $\vert\cdot$]	0	0	0	0.920	0.736	0	0	0.000	0.695	0.556
	E[Welfare $\vert\cdot$]	2.659	1.373	2.659	3.954	3.610	0.780	1.444	0.780	4.036	3.429

TABLE 1.6: **Averages for Trading Strategies, Liquidity, and Welfare across Times t_2 through t_4 for Informed Traders with $\beta = 0$ and Uninformed Traders with $\beta \sim Tr[\mathcal{N}(\mu, \sigma^2)]$.** This table reports results for two different informed-investor arrival probabilities α (0.8 and 0.2) and for two different asset-value volatilities δ (1.6 and 0.2). The private-value factor parameters are $\mu = 10$ and $\sigma = 10$, and the tick size is $\kappa = 1$. Each cell corresponding to a set of parameters reports the equilibrium order-submission probabilities, the expected bid-ask spreads and expected depths at the inside prices (A_1 and B_1) and total depths on each side of the market after order submissions at times t_2 through t_4, and expected welfare for the market participants. The first four columns in each parameter cell are for informed traders with positive, neutral and negative signals, ($I_{\bar{v}}$,I_{v_0},$I_{\underline{v}}$) and for uninformed traders (U). The fifth column ($Uncond.$) reports unconditional results for the market.

		$\delta = 1.6$					$\delta = 0.2$				
		$I_{\bar{v}}$	I_{v_0}	$I_{\underline{v}}$	U	$Uncond.$	$I_{\bar{v}}$	I_{v_0}	$I_{\underline{v}}$	U	$Uncond.$
$\alpha = 0.8$	LSA_2	0	0.243	0.047	0.157	0.109	0.394	0.254	0.106	0.027	0.206
	LSA_1	0	0.257	0.244	0.029	0.139	0.194	0.240	0.289	0.069	0.207
	LBB_1	0.244	0.257	0	0.029	0.139	0.289	0.240	0.194	0.069	0.207
	LBB_2	0.047	0.243	0	0.157	0.109	0.106	0.254	0.394	0.027	0.206
	MBA_2	0.488	0	0	0.293	0.189	0	0	0	0.338	0.068
	MBA_1	0.001	0	0	0.018	0.004	0	0	0	0.060	0.012
	MSB_1	0	0	0.001	0.018	0.004	0	0	0	0.060	0.012
	MSB_2	0	0	0.488	0.293	0.189	0	0	0	0.338	0.068
	NT	0.220	0	0.220	0.007	0.119	0.017	0.011	0.017	0.012	0.015
	E[Spread $\vert\cdot$]	2.177	2.270	2.177	2.521	2.271	2.252	2.266	2.252	2.720	2.349
	E[Depth A_2+A_1 $\vert\cdot$]	1.050	2.331	2.446	1.760	1.906	2.168	2.304	2.434	1.623	2.166
	E[Depth A_1 $\vert\cdot$]	0.001	0.365	0.823	0.239	0.365	0.234	0.367	0.514	0.140	0.325
	E[Depth B_1 $\vert\cdot$]	0.823	0.365	0.001	0.239	0.365	0.514	0.367	0.234	0.140	0.325
	E[Depth B_1+B_2 $\vert\cdot$]	2.446	2.331	1.050	1.760	1.906	2.434	2.304	2.168	1.623	2.166
	E[Welfare LO $\vert\cdot$]	0.089	0.125	0.089	1.038	0.288	0.235	0.130	0.235	0.058	0.172
	E[Welfare MO $\vert\cdot$]	0.094	0	0.094	2.796	0.609	0	0	0	3.333	0.667
	E[Welfare $\vert\cdot$]	0.183	0.125	0.183	3.834	0.898	0.235	0.130	0.235	3.390	0.838
$\alpha = 0.2$	LSA_2	0	0.380	0.472	0.106	0.142	0.374	0.387	0.439	0.095	0.156
	LSA_1	0	0.104	0.275	0.058	0.072	0.045	0.097	0.118	0.072	0.075
	LBB_1	0.275	0.104	0	0.058	0.072	0.118	0.097	0.045	0.072	0.075
	LBB_2	0.472	0.380	0	0.106	0.142	0.439	0.387	0.374	0.095	0.156
	MBA_2	0.133	0	0	0.214	0.180	0	0	0	0.210	0.168
	MBA_1	0.093	0	0	0.119	0.101	0	0	0	0.120	0.096
	MSB_1	0	0	0.093	0.119	0.101	0	0	0	0.120	0.096
	MSB_2	0	0	0.133	0.214	0.180	0	0	0	0.210	0.168
	NT	0.027	0.033	0.027	0.007	0.011	0.023	0.031	0.023	0.006	0.010
	E[Spread $\vert\cdot$]	2.151	2.131	2.151	2.400	2.349	2.197	2.157	2.197	2.454	2.400
	E[Depth A_2+A_1 $\vert\cdot$]	1.307	2.103	2.500	1.599	1.673	1.940	2.098	2.264	1.594	1.695
	E[Depth A_1 $\vert\cdot$]	0.194	0.435	0.759	0.307	0.338	0.354	0.422	0.451	0.274	0.301
	E[Depth B_1 $\vert\cdot$]	0.759	0.435	0.194	0.307	0.338	0.451	0.422	0.354	0.274	0.301
	E[Depth B_1+B_2 $\vert\cdot$]	2.500	2.103	1.307	1.599	1.673	2.264	2.098	1.940	1.594	1.695
	E[Welfare LO $\vert\cdot$]	1.116	0.551	1.116	0.507	0.592	0.581	0.637	0.581	0.505	0.524
	E[Welfare MO $\vert\cdot$]	0.230	0	0.230	3.229	2.614	0	0	0	3.213	2.570
	E[Welfare $\vert\cdot$]	1.346	0.551	1.346	3.736	3.205	0.581	0.637	0.581	3.718	3.094

Chapter 2

Optimal Market Access Pricing

2.1 Introduction

Access fees and rebates in securities markets are at the top of the agenda of financial regulators and market operators around the world. Following Reg NMS (2007) in the US (and related regulation in Europe), market access pricing became a strategic tool for trading platforms and exchanges to attract trading volume. In particular, rebates are used to incentivize investors to submit certain types of orders, while investors using other order types are charged fees. For example, under maker-taker pricing, investors receive rebates when their limit orders (making liquidity) are executed and pay fees on market orders (taking liquidity). However, rebate–based access pricing has been criticized by some practitioners as well as by Angel, L. E. Harris, and Spatt 2013 and Spatt (2019).[1] Regulators are now taking actions to study and possibly limit rebate-based pricing. Most notably, on March 14, 2018, the SEC released a proposal for a two–year Transaction Fee Pilot to experiment with reduced access fees and rebates.

The objective of this paper is to study optimal market access pricing by means of a theoretical model of a limit order market with discrete prices and strategic traders. Our approach follows seminal theoretical research by Colliard and Foucault 2012; Foucault, Kadan, and Kandel 2013; and Chao, Yao, and Ye 2018 showing how fees and rebates for taking and making liquidity via market and limit orders can alleviate trading frictions due to price discreteness. Like Chao et al. (2018), we consider the optimal access fees and rebates for a profit-maximizing exchange. Whereas Chao et al. (2018) investigates access pricing and intermarket competition, our paper is the first to consider access pricing in a multiperiod setting and also to allow for flash orders from high-frequency (HFT) traders, who are continuously present in the market. In addition, we provide new insights into the relation between access pricing, the amount of heterogeneity in investor valuations, regulatory constraints on fees, and welfare.

We study optimal access pricing in an equilibrium model of a dynamic limit order market with a discrete price tick-size in which traders arrive sequentially with heterogeneous private asset valuations. As in Foucault, Kadan, and Kandel 2013 and Chao, Yao, and Ye 2018, price discreteness creates trading frictions by limiting the prices at which investors can transact. Access fees and rebates enable side-transfers between buyers and sellers to adjust the rewards and costs for liquidity supply and demand. Our paper is most closely related to Chao, Yao, and Ye

[1] Angel, L. E. Harris, and Spatt 2013 and Spatt (2019) argue maker-taker pricing reduces the transparency of the true economic spread, negatively impacts agency problems in broker order routing decisions, and puts venues that do not make use of such fees at a competitive disadvantage. L. Harris 2015 further points out that negative fees allow for intra-tick trading, thus by-passing Reg NMS trade-through rule.

2018, which is the first model of optimal access fees for profit-maximizing exchange operators. Our analysis leads to four results that extend previous research on optimal access pricing:

- The optimal access-pricing structure depends on the distribution of gains-from-trade in the population of investors arriving in the market. When potential gains-from-trade are small, optimal access pricing involves a mix of rebates and fees (maker-taker or taker-maker), whereas strictly positive fees are optimal when potential gains-from-trade are large ex ante.

- When the trading frequency increases, the incentive for the exchange to use rebate-based pricing decreases. Furthermore, in a multiperiod market with three rounds of trader-arrival (rather than two), maker-taker and taker-maker pricing are again optimal when trader valuations are not too dispersed, but now they are no longer symmetric. The ability of traders at intermediate times to submit their own limit orders reduces the market power of limit orders posted in the first period.

- The mechanics of liquidity supply and demand change significantly with HFT trading. This is because HFT traders use flash orders to react immediately to orders submitted by slower traders over time. As a result, the range of market parameterizations in which the exchange optimally uses rebate-based access pricing becomes larger relative to the no-HFT market. The increased use of rebate-based pricing is perhaps somewhat surprising because the HFTs simply augment the set of potential counter-parties in the market. However, rebate-based access pricing allows liquidity-demanders to reduce the compensation paid to the HFTs for liquidity provision.

- The welfare effect of profit-maximizing rebate-based taker-maker and maker-taker pricing by an exchange are parameter-dependent. When ex ante investor valuations are concentrated, rebate-based access pricing improves welfare. However, when the support of investor valuations is somewhat larger, then without HFTs rebate-based pricing can still maximize exchange profits but can lower overall welfare relative to a zero-fee/zero-rebate pricing. In contrast, with HFT, total welfare increases over the entire parameter region for which rebate-based access pricing is optimal for an exchange.

Taken together, our analysis demonstrates a connection between access pricing and HFT trading. In particular, the presence of HFT traders induces exchanges to use rebate-based access pricing even in active markets (with frequent investor arrival) with large gains-from-trade. In addition, the presence of HFTs induces a substantial redistribution of welfare from slow investors to the exchange.

A sizable empirical literature investigates different aspects of access fees and rebates.[2] Malinova and Park 2015 find evidence following changes in access fees and rebates on the Toronto Stock Exchange (TSX) that appears to support the Colliard and Foucault 2012 irrelevance prediction provided that the TSX price tick-size can be interpreted as being economically small. However, other research finds evidence against Colliard-Foucault irrelevance once there is a

[2] In addition to the research discussed here, see also Skjeltorp, Sojli, and Tham 2012, He, Jarnecic, and Y. Liu 2015, Clapham et al. 2017, Anand, Hua, and T. McCormick 2016, Comerton-Forde, Grégoire, and Zhong 2019 and Lin, Swan, et al. 2017

discrete tick-size. Panayides, Rindi, and Werner 2017 find that quoted and cum-fee spreads are affected by fees and rebates on the BATS European platforms, CXE and BXE. Using Rule 605 data, O'Donoghue 2015 finds that changes in the split of trading fees between liquidity suppliers and demanders affect order choice and execution quality as predicted by Foucault, Kadan, and Kandel 2013. Cardella, Hao, and Kalcheva 2015 investigate 108 instances of fee changes for U.S. exchanges in 2008-2010 and find that changes in take fees have a larger impact on trading activity than changes in make fees. Battalio, Corwin, Jennings (2016) find that access fees and rebates appear to affect broker order-routing decisions.

Empirical research finds that rebate-based access pricing is related to HFTs, but no theory shows the relationship between HFT firms and optimal access pricing chosen by trading platform owners. Menkveld 2013 shows that access rebates are a significant part of HFT profits. We show that in equilibrium the rebate–based fee structure is consistent with HFT market participation. This is also consistent with evidence in Cardella, Hao, and Kalcheva 2015 that Reg NMS was followed by the adoption of rebate–based access pricing by most trading platforms in U.S. markets and by a sharp increase in HFT firm trading. O'Hara 2015 also links HFT trading activity and the increased use of rebate–based access pricing structures around the world.

Angel et al. (2013) and Spatt (2019) emphasize that access fees and rebates have important potential effects in terms of the transparency of economic prices (price + access pricing) vs. quoted prices, the efficacy of regulatory protections based on quoted prices, agency issues when brokers do not pass through fees and rebates to their clients, and impeding intermarket competition. In contrast, our analysis is based on the idea that constraining trade to a discrete price grid creates frictions in the trading process and that access pricing potentially reduces those frictions. Both sets of considerations are likely to be important. Moreover, a complete understanding of access pricing is likely to involve interactions between these various effects.

2.2 Background information and prior research

U.S. Regulation National Market System (Reg NMS) established the regulatory foundation for the current architecture of US equity markets. This regulation includes an explicit limit on the cost of accessing (i.e., posting and trading on) quotes displayed by U.S. equity trading platforms. Rule 610 caps access fees to no more than $0.003 per share for stocks priced over $1, and to no more that 0.3% of the quoted price for stocks priced below $1. In addition, the Sub-Penny Rule 612 of Reg NMS prohibits exchanges, market makers, and electronic platforms from displaying, ranking or accepting quotes on NMS securities in sub-penny increments unless a stock is priced less than $1 per share. Thus, under Reg NMS, access fees cannot exceed one third of the tick size.[3]

[3] According to the more recent S.E.C. (2018) Release No.34-82873 on Transaction Fee Pilot for NMS Stocks "For maker-taker exchanges, the amount of the taker fee is bounded by the cap imposed by Rule 610(c) on the fees the exchange can charge to access its best bid/offer for NMS stocks. This cap applies to the fees assessed on an incoming order that executes against a resting order or quote, but does not directly limit rebates paid. The Rule 610(c) cap on fees also typically indirectly limits the amount of the rebates that an exchange offers to less than $0.003 per share in order to maintain net positive transaction revenues. For taker-maker exchanges, the amount of the maker fee charged to the provider of liquidity is not bounded by the Rule 610(c) cap, but such fees typically are no more than $0.003, and the taker of liquidity earns a rebate." If the price of a protected quotation is less than $1.00, the access fee is no more than 0.3% of the quotation price per share SEC 2009.

The proposed two-year Transaction Fee Pilot would substantially change the current U.S. equity cap to access fees for 2 groups of NMS stocks with average daily trading volumes $\geq$ 30,000 shares and with a share price $\geq$ \$2. The pilot would apply to all equities exchanges but not dark pools and other alternative trading structures. Test Group 1 would lower the access fee to \$0.0010 and would still allow rebate-based pricing; whereas Test Group 2 would prohibit all exchange rebates and linked pricing while maintaining the existing \$0.003 per share fee cap. Test Group 3, the control group, would maintain the current access fee cap. By lowering the access fee for Test group 1 stocks and banning rebates for Test group 2 stocks, the Transaction Fee Pilot should facilitate an informed, data-driven discussion about the effects of access fees and rebates and their impact on order-routing behaviour, and market quality (SEC Release No. 34-82873).

In Europe, MiFID II (Directive 2014/65/EU) and MiFIR (Regulation 600/2014/EU) mandates a reduction in the tick size for European stocks and thereby implicitly reduced the maximum access fees given that the standard practice on European exchanges is to cap fees relative to the tick size.[4] MiFID II also sharpened the regulation of access fees by requiring new incentives on market making agreements under Stress Market Conditions (RTS 8), a maximum Order-To-Trade ratio for each instrument (RTS 9), and a periodic disclosure by exchanges of the percentage of fees and rebates on total turnover (RTS 27). It also bans "cliff-edge" pricing structures in which customer-specific fees are reduced retroactively for market participants that reach a trading volume threshold (RTS 10).

Trading fees have been investigated in a small number of theoretical papers. Colliard and Foucault 2012 show in a competitive market with continuous prices that the breakdown between make and take fees has no effects on the cum-fee-spread (net of fees spread) as traders can neutralize changes in fees by making offsetting changes in the aggressiveness of their orders. However, Foucault, Kadan, and Kandel 2013 show in a single market with a discrete tick size that the make-take breakdown matters for market quality. Panayides, Rindi, and Werner 2017 show how a change in trading fees affects market quality when two trading platforms compete for the provision of liquidity. Chao, Yao, and Ye 2018 models optimal access pricing both in a single-market setting and also with competition between multiple markets.

Our analysis builds on this previous research, and particularly on Chao, Yao, and Ye 2018, in several ways: First, we show that optimal access pricing changes qualitatively with the amount of ex ante dispersion in trader valuations. In the absence of regulation, a two-period market has two equilibria, one with maker-taker and one with symmetric taker-maker pricing. However, when regulation caps the maximum access fee (which thereby limits rebates), this no longer is true. When valuation dispersion is low, maker-taker and symmetric taker-maker pricing is still optimal, but when investor valuation dispersion is sufficiently high, then exchanges optimally charge positive fees to both limit-order submitters and market-order submitters.

Second, we show that the market power of liquidity providers changes with the number of trading rounds, which is a proxy for the rate of trading activity. Longer trading games have more opportunities for arriving investors to trade and, thus, have increased potential liquidity. As a result, the optimal trading strategy of an investor arriving at the market in the first period of the three-period trading game differs from in a two-period game. In a three-period market, the investor at time t_1 is no longer a monopolist in the provision of liquidity (as in a two-period

[4] See Article 49 of MiFID II and the following Regulatory Technical Standard 11 (RTS 11, ESMA 2017). ESMA n.d.

market) and therefore must take into account the fact that, at time t_2, the incoming trader may decide to compete and supply liquidity rather than only taking liquidity.[5] In addition, if the maximum fee is capped by regulation, as in real markets, then moving from a two-period to a three-period market, the take rebate needs to be larger than the make rebate. Therefore when a regulatory cap reduces access fees, the taker-maker pricing is optimal less of the time than maker-taker pricing. This asymmetry between maker-taker and taker-maker is consistent with the empirical observation that the maker-taker pricing structure is more common in current financial markets.

Third, we are the first to model optimal access pricing in a market in which HFT firms using flash orders are present. In this context, we confirm our earlier intuition about investor valuation heterogeneity and access pricing. The HFTs in our model have no private gains-from-trade. Thus, an exchange with HFT firms has more incentive to use a rebate-based pricing structure. In particular, HFTs in our model use flash market orders to provide liquidity to regular investors. However, given that HFTs do not have private value reasons to want to trade, rebates are needed to induce trading when regular investor gains-from-trade are concentrated.

Fourth, we show that optimal access pricing depends on both the absolute tick size and on the relative magnitude of investor valuation dispersion relative to the tick size. Moreover, when fees are capped relative to the absolute tick size, then a larger tick size is favoured by exchanges because it enlarges the degrees of freedom they have to offer rebates.

Fifth, we show that profit-maximizing rebated-based access pricing by exchanges is Pareto improving when investor valuation dispersion is low. However, without HFTs, once investor valuation dispersion is sufficiently large, rebates may still maximize exchange profits, but they can reduce overall welfare relative to a market with no fees and rebates. In contrast, with HFTs, total welfare improves for the whole parameter region in which an exchange optimally uses rebate-based access pricing.

2.3 Model

This section describes a model of access fees and rebates in a single limit order market. Traders in the model arrive sequentially over a trading day. In general, there are N periods with arrival times denoted as $t_z \in \{t_1, \ldots, t_N\}$. Section 2.4 considers a specification with two periods, $t_z \in \{t_1, t_2\}$, and then Section 2.5 extends the analysis to a trading day of three periods, $t_z \in \{t_1, t_2, t_3\}$. This allows us to investigate the relation between access pricing and investor-arrival frequency. The arriving traders are risk-neutral and are each characterized by a private valuation equal to β_{t_z} for the trader arriving at time t_z, where each β_{t_z} is an i.i.d draw from a uniform distribution, $U[\overline{\beta}, \underline{\beta}]$, where $\overline{\beta}$ and $\underline{\beta}$ are the limits of the trader valuation supports. We denote the mean of the valuation support by v, which is constant over time, and call this the ex ante *asset value*. The *support width* is denoted $\Delta = \bar{\beta} - \underline{\beta}$. Traders with extreme β_{t_z} values are more eager to trade by taking liquidity, whereas traders with β_{t_z} values close to v are more willing to supply liquidity. The wider the support, $[\underline{\beta}, \overline{\beta}]$, the higher is the probability that arriving traders will have strong heterogeneous directional demands to trade, such as, e.g., long–term asset managers. The smaller the support $[\underline{\beta}, \overline{\beta}]$, the higher is the probability that

[5]In a 2-period market, the investor arriving in the second period has no choice either than trading at the offered limit order posted by the liquidity supplier in the first period or leaving the market and not trade.

arriving traders will prefer to profit as passive liquidity providers. Later, Section 2.6 extends the model to allow for high frequency traders who have neutral private values for the asset, but who react to limit order book changes faster than regular arriving investors.

Prices are quoted on a discrete price grid $\{\ldots, P_{-k}, \ldots, P_{-1}, P_1, \ldots, P_k, \ldots\}$ centered around the mean of investor valuation v with a fixed tick size τ. The state of the limit order book at time t_z is a vector:

$$L_{t_z} = [D_{t_z}^{P_k}] \tag{2.1}$$

where $D_{t_z}^{P_k}$ indicates the total limit order depth at price P_k at time t_z. An investor arriving in the market at time t_z and facing a standing limit order book $L_{t_{z-1}}$ can take one of several different possible actions, x_{t_z}: Post a limit buy or sell order LBP_k or LSP_k at one of the available price levels P_k on the price grid, submit a market buy or sell order $MBP_{k(L_{t_{z-1}},MB)}$ or $MSP_{k(L_{t_{z-1}},MS)}$ that is then executed immediately at the best standing ask price $P_{k(L_{t_{z-1}},MB)}$ or bid price $P_{k(L_{t_{z-1}},MS)}$ on the opposite side of the market where the indices $k(L_{t_{z-1}}, MB)$ and $k(L_{t_{z-1}}, MS)$ denote the best standing quotes given a market buy or sell given the incoming book $L_{t_{z-1}}$ at time t_z, or not trade by submitting no order (NT).[6] An investor opts not to trade when the payoffs on all available actions are negative. Marketable limit orders that cross with the best available bid/ask on the opposite side of the standing book $L_{t_{z-1}}$ are treated as market orders in terms of both order execution and exchange access pricing. The investor action set at time t_z given a standing book $L_{t_{z-1}}$ is denoted as X_{t_z}. In addition, let $X^L \subset X_{t_z}$ denote the set of possible limit orders at time t_z. For tractability, we assume that limit orders cannot be modified or cancelled after submission and that investors can only send one order of unitary size at a time.[7]

The arrival of new limit and market orders augments or reduces the depth of the limit order book respectively, leading to dynamics:

$$L_{t_z} = L_{t_{z-1}} + Q_{t_z} \quad z = 1, \ldots, N \tag{2.2}$$

where $Q_{t_z} = [Q_{t_z}^{P_k}]$ is a vector of changes in the limit order book due to an arriving investor's action $x_{t_z} \in X_{t_z}$ at t_z. The change $Q_{t_z}^{P_k}$ in depth at price P_k is "+1" when an arriving limit order LOP_{k,t_z} adds an additional share and "−1" when a market order executes a limit order when P_k is the best bid or offer (BBO), and otherwise is zero (at other prices unaffected by the arriving order). The changes $Q_{t_z}^{P_k}$ are all zero if no order is submitted.

Consistent with common practice in today's financial markets, the trading platform may set different access fees $\xi(x)$ for different order types x. An investor offering liquidity by posting a limit order faces a *make fee* (MF). An investor taking liquidity via a market order (or via a marketable limit order) pays a *take fee* (TF). The set of fees is denoted as $\Xi = \{MF, TF\}$. Some fees may be negative (i.e., a rebate), in which case it is a cost for the trading platform and a reward for the investor receiving it. Under a *maker-taker* structure, investors submitting market orders pay a take fee ($TF > 0$) to the trading platform, and investors posting limit

[6] LO and MO indicate generic limit and market orders, and LB (LS) and MB (MS) indicate the buy (sell) trade direction. For simplicity we will refer to $MBP_{k(L_{t_{z-1}},MB)}$ and $MSP_{k(L_{t_{z-1}},MS)}$ as MBP_k and MSP_k, and we will refer to $P_{k(L_{t_{z-1}},MS)}$ and $P_{k(L_{t_{z-1}},MB)}$ as $P_{k,MS}$ and $P_{k,MB}$.

[7] As noted in Parlour and Seppi 2008, such limit orders are essentially "take it or leave it" offers of liquidity.

orders receive a make rebate equal to $-MF > 0$ whenever their limit order executes. In a *taker-maker* structure, the fees and rebates are reverse so that now limit-order submitters pay make fees ($MF > 0$) and market-order submitters receive take rebates ($-TF > 0$). Consistent with current practice, access fees and rebates in our model are subject to regulation. For notational simplicity, we assume the maximum allowable fee (whether take or make) is one tick. Thus, this regulatory constraint on fees is more binding for smaller tick sizes. Appendix 2.11 shows how our results depend on such regulatory restrictions and how this accounts for some of the differences between our model and Chao, Yao, and Ye 2018.

Investor order-submission behaviour depend on both quoted prices and exchange fees and rebates. Given a quoted price P_k, the total amount paid or received by an investor net of fees TF and MF is called the *cum-fee price*. Accordingly, let $P_k^{cum,MS} = P_k - TF$ denote the cum-fee price received net of take fees paid to the exchange when using a market order to sell at the quoted price P_k, and let $P_k^{cum,MB} = P_k + TF$ be the cum-fee price paid including take fees paid to the exchange when using a market order to buy at P_k. Similarly, $P_k^{cum,LS} = P_k - MF$ is the cum-fee price for a limit order to sell and $P_k^{cum,LB} = P_k + MF$ is the cum-fee price for a limit order to buy.

Liquidity supply is endogenous in our model. The limit order book opens empty at the first time period t_1, and so an investor arriving at t_1 can only post limit orders to trade. Similarly, in the final round of order submission t_N, investors can only submit market orders to trade (since new limit orders would be unexecuted). In intermediate periods (e.g., t_2 of a three-period trading game), investors can choose between market and limit orders. As in Chao, Yao, and Ye 2018, the tick size τ and trader valuation support $S = [\overline{\beta}, \underline{\beta}]$ are exogenous input parameters in our analysis.

Consider now the investor order-choice problem. An investor arriving at time t_z chooses his order x_{t_z} to maximize his expected payoff:

$$\max_{x_{t_z} \in X_{t_z}} w(x_{t_z}|S,\tau,\Xi,\beta_{t_z},L_{t_{z-1}}) = \begin{cases} [\beta_{t_z} - P(x_{t_z}) - \xi(x_{t_z})]\, Pr(\theta_{t_z}^{x_{t_z}}|S,\tau,\Xi,L_{t_{z-1}}) & x_{t_z} \text{ buy} \\ [P(x_{t_z}) - \beta_{t_z} - \xi(x_{t_z})]\, Pr(\theta_{t_z}^{x_{t_z}}|S,\tau,\Xi,L_{t_{z-1}}) & x_{t_z} \text{ sell} \\ 0 & x_{t_z}\ NT \end{cases} \quad (2.3)$$

where $\Xi = \{TF, MF\}$ is the set of market access fees $\xi(x_{t_z})$, and $P(x_{t_z})$ is the price at which order x_{t_z} trades if it is executed. The notation $\theta_{t_z}^{x_{t_z}}$ denotes the set of future trading states in which an order x_{t_z} submitted at time t_z is executed, and $Pr(\theta_{t_z}^{x_{t_z}}|S,\tau,\Xi,L_{t_{z-1}})$ is the associated probability of execution. For example, if x_{t_z} is a market order, then $P(x_{t_z})$ is the best standing quote on the other side of the market at time t_z, and $Pr(\theta_{t_z}^{x_{t_z}}|S,\tau,\Xi,L_{t_{z-1}}) = 1$, since market orders are executed immediately at the standing bid or ask (if that side of the book is non-empty). If x_{t_z} is a non-marketable limit order, then the execution price $P(x_{t_z})$ is its limit price, and the execution probability $Pr(\theta_{t_z}^{x_{t_z}}|S,\tau,\Xi,L_{t_{z-1}})$ is between 0 and 1. Table 2.5 in the Appendix shows explicitly the actions available to traders and their associated payoffs.

The optimal order-submission strategy over time is determined — given market access fees Ξ, an incoming book $L_{t_{z-1}}$, and subsequent order-execution probabilities $Pr(\theta_{t_z}^{x_{t_z}}|S,\tau,\Xi,L_{t_{z-1}})$ at a time t_z — by the upper envelope of the collection of linear functions of the investor valuations β_{t_z} corresponding to the expected investor payoffs for the different possible actions in X_{t_z} in (2.3). The associated optimal order-submission probabilities $Pr(x_{t_z}|S,\tau,\Xi,L_{t_{z-1}})$ are then the probabilities of investor valuations β_{t_z} in between threshold valuations equating expected payoffs for the different profit-maximizing orders along the support $[\underline{\beta}, \bar{\beta}]$ of investor valuations.

An exchange chooses its fees, Ξ, to maximize its expected payoff from completed transactions:

$$\max_{MF,TF} \pi(MF, TF\,|S,\tau) = \left[\sum_{t_z \in \{t_1,\dots,t_{N-1}\}} \sum_{x_{t_z} \in X^L} Pr(x_{t_z}, \theta_{t_z}^{x_{t_z}}\,|S,\tau,\Xi) \right] (MF + TF) \quad (2.4)$$
$$s.t.: \quad -\tau < MF, TF < +\tau$$

given the transaction probabilities $Pr(x_{t_z}, \theta_{t_z}^{x_{t_z}}\,|S,\tau,\Xi)$ induced by their fees and the equilibrium investor order–submission strategies that maximizes (2.3), where the transaction probabilities are the product of the probabilities of different limit orders $x_{t_z} \in X^L$ being submitted and their execution probabilities

$$Pr(x_{t_z}, \theta_{t_z}^{x_{t_z}}\,|S,\tau,\Xi) = \sum_{L_{t_{z-1}}} Pr(L_{t_{z-1}}|S,\tau,\Xi)\, Pr(x_{t_z}|S,\tau,\Xi,L_{t_{z-1}})\, Pr(\theta_{t_z}^{x_{t_z}}|S,\tau,\Xi,L_{t_{z-1}}). \quad (2.5)$$

The formula in (2.5) reflects the fact that, in a limit order market, transactions only occur when limit orders are submitted and then executed. The regulatory constraint in (2.4) guarantees traders cannot neutralize the trading fee. In particular, investors cannot adjust the prices at which limit orders are posted on a discrete price grid to exactly offset the impact of small changes in access fees and rebates on their net transaction prices. Note that the exchange has non-negative profits since $TF = MF = 0$ is feasible and gives zero profits. Given the optimization problems solved by investors and the exchange, we can now define an equilibrium:

Definition. A *Subgame Perfect Nash Equilibrium* of the trading game is a collection $\{Pr(x_{t_z}|S,\tau,\Xi^*,\beta_{t_z},L_{t_{z-1}}),\ Pr(\theta_{t_z}^{x_{t_z}}\,|S,\tau,\Xi^*,L_{t_{z-1}}),\ \Xi^*\}$ of order-submission probabilities, order-execution probabilities, and access fees such that:

- The equilibrium order-submission probabilities $Pr(x_{t_z}|S,\tau,\Xi^*,\beta_{t_z},L_{t_{z-1}})$ are the probabilities of optimal orders for investors computed from their optimization problem (2.3) given the equilibrium execution probabilities $Pr(\theta_{t_z}^{x_{t_z}}\,|S,\tau,\Xi^*,L_{t_{z-1}})$.
- The order-execution probabilities $Pr(\theta_{t_z}^{x_{t_z}}\,|S,\tau,\Xi^*,L_{t_{z-1}})$ for an order x_{t_z} submitted at time t_z are consistent with the equilibrium order-submission probabilities $Pr(x_{t_{z'}}|S,\tau,\Xi^*,\beta_{t_{z'}},L_{t_{z'-1}})$ at times $t_{z'} > t_z$.
- The access fees Ξ^* are optimal for the exchange given its optimization problem (2.4) given the equilibrium order-submission probabilities $Pr(x_{t_z}|S,\tau,\Xi^*,\beta_{t_z},L_{t_{z-1}})$.

Using first principles, we have the following existence result for our model:

Theorem 1 *The equilibrium of a trading game with N periods and a price grid with a fixed number of prices exists and can be constructed analytically via backwards induction.*

Proofs for general N-period models are in Appendix 2.9. Further details for specific versions of the model are in Appendices 2.10 through 2.13. However, the functional forms can become complex as the number of periods grows and as the number of possible limit prices increases — i.e., as more limit orders become feasible a priori as larger investor valuation supports encompass more prices or as the price grid becomes finer. As a practical matter, therefore, sometimes (e.g., as in the 3-period model) the first-order conditions that give exchange's equilibrium optimal fees in the final step of the equilibrium derivation are more easily solved numerically. In these cases, rather than explicitly differentiating the analytic exchange expected profit function, we instead evaluated it numerically and use a search algorithm to solve the first-order conditions for Ξ^*.

2.4 Results for the 2-period trading game

This section examines the 2-period version of the general model ($\{t_1, t_2\}$) centered around a mean asset value normalized to $v = 10$.[8] We consider two possible price grids with different tick sizes. In a large-tick market (LTM), the tick size τ is normalized to 1, and the price grid has four possible price levels, $P_k = \{P_{-2}, P_{-1}, P_1, P_2\}$, centered around the mean investor valuation v with $P_{-2} < P_{-1} < v < P_1 < P_2$. The outside quotes are $P_2 = v - \frac{3}{2}\tau$ and $P_{-2} = v + \frac{3}{2}\tau$, and the inside quotes are $P_1 = v - \frac{1}{2}\tau$ and $P_{-1} = v + \frac{1}{2}\tau$. In a small-tick market (STM), the tick size is smaller — which we set here to $\frac{\tau}{3}$ relative to the LTM tick size — and the price grid has ten price levels $p_j = \{p_{-5}, p_{-4}, p_{-3}, p_{-2}, p_{-1}, p_1, p_2, p_3, p_4, p_5\}$, with $p_{-5} < \ldots < p_{-1} < v < p_1 < \ldots < p_5$. The outside quotes of the STM coincide with the outside quotes of the LTM with $p_{-5} = P_{-2} = v - \frac{3}{2}\tau$ and $p_5 = P_2 = v + \frac{3}{2}\tau$.[9]

Our analysis allows for a wide range of trader-valuation supports $S = [\underline{\beta}, \bar{\beta}]$. The smallest support we consider, $[9.8333, 10.1667]$, has a support width Δ of $0.33\,\tau$ and is within the inside quotes of the LTM. This is a market environment in which arriving traders are predisposed to supply liquidity since individual potential gains-from-trade are small. This support is also equal to the inside spread of the STM. The largest support we considered, $[7.50 - 12.50]$, has a width of $5\,\tau$, and corresponds to a market populated by very heterogeneous traders, some of whom have strong trading demands (and prefer to take available liquidity) and others with weaker trading demands (who tend to supply liquidity). The rationale for the specific choice for our largest support is that it is the largest support such that in equilibrium traders never want to post limit orders beyond the outside quotes $\{P_{-2}, P_2\}$ in the LTM or $\{p_{-5}, p_5\}$ in the STM. These two supports let us compare investor and exchange behavior given different ex ante investor valuation dispersion across large and small tick markets. We also consider supports $[\underline{\beta}, \bar{\beta}]$ in between these two extreme cases.

Figure 1 shows the equilibrium optimal fees MF (blue line) and TF (orange line) chosen by the exchange for both the LTM (upper plot) and the STM (lower plot) given the various trader–valuation support widths Δ on the horizontal axes. The gray regions highlight equlibria with taker-maker and maker-taker access pricing involving rebates on market or limit orders.

[8] Our results are unchanged for other values of v if the price grids and trader-valuation supports are adjusted up or down.

[9] In real markets, if trading platforms with different tick sizes coexist, then the prices on wider price grids are also on the denser narrow price grids.

Outside of the gray region, equilibrium access fees are non-negative with no rebates. Note that multiple equilibria are possible. For support widths inside the gray region, taker-maker pricing ($TF < 0$ and $MF > 0$) on the left side and maker-taker pricing ($MF < 0$ and $TF > 0$) on the right side are both optimal. Moreover, the taker-maker and maker-taker pricing structures are symmetric here. In contrast, the equilibria with access pricing with strictly positive fees (MF, $TF > 0$) outside the rebate-based (grey) region are unique — i.e., they are identical for support widths $\Delta > 3\,\tau$ on both sides of Figure 1.

Table 2.1 provides additional details about equilibrium strategies and market properties for the LTM. It reports the equilibrium trading fees and the buyer's trading strategies associated with each support considered here, the cum-fee buy and sell transaction prices $P_k^{cum,LB}$ and $P_k^{cum,MS}$, the equilibrium probabilities of the buyer's order submission ($Pr(x_{t_z}|S,\tau,\Xi,L_{t_{z-1}})$) and execution ($Pr(\theta_{t_z}^{x_{t_z}}|S,\tau,\Xi,l_{t_{z-1}})$), and the equilibrium expected total exchange profit ($\pi(MF,TF|S,\tau)$) associated with each support. When there are two rows for a particular support, they are for the respective maker-taker and taker-maker equilibria.[10] The results are symmetric for limit sells at time t_1. Table 2.2 in Section 2.4.2 below provides similar details for the STM.

A general issue explored in our analysis is the relation between the profit-maximizing access pricing for an exchange and, on the other hand, the support S of trader private valuations and the tick size τ. In particular, optimal access pricing is driven by both the relative size of the valuation support width Δ to the tick size τ and also by the absolute tick size τ by itself given that the regulatory cap on fees is tied to the absolute tick size. We explore these issues using two types of comparative statics: First, we hold the tick size τ fixed and vary the trader valuation support width Δ, which changes the amount of potential gains-from-trade. This comparative static describes the effects of the relative valuation-support/tick-size channel alone. Second, we change the tick size τ by comparing LTMs and STMs given the same range of valuation supports. This second comparative static depends on both the relative ratio channel and the absolute tick-size channel.

2.4.1 Large Tick Market

Our 2-period LTM analysis uses equilibria constructed using the analytic recursion in Theorem 1 to demonstrate various equilibrium properties of the the 2-period LTM. Figure 1 and Table 2.1 provide specific detail.

Proposition 1 *When investor valuation dispersion is low (in that the investor valuation support width is $\Delta < 3\tau$) both maker-taker and taker-maker equilibria exist in the 2-period LTM with fees and rebates that are symmetric. When investor valuation dispersion is higher (in that the valuation support width is $\Delta \in [3\tau, 5\,\tau]$) the equilibrium fees TF and MF can be jointly positive and unique.*

[10]To economize space, Table 2.1 does not report the equilibrium strategies of the seller arriving at t_2 as they can be inferred from the buyer's equilibrium strategies at t_1. For example, if a limit buy is posted at t_1, (LBP_1), the equilibrium strategy of the seller taking liquidity at t_2 will be a MSP_1 market sell. In addition, Table 2.1 does not report the probability of No Trade as it is the complement to 0.5 of the probability of order submission on one side of the market. For example, for the support $[9.8333, 10.1667]$ with the smallest width 0.33τ, the probability of No Trade at t_1 is $0.5 - 0.333 = 0.167$.

Proposition 2 *When an exchange optimally uses maker-taker or taker-maker rebate-based access pricing in the 2-period LTM, then rebates are decreasing and fees can be increasing as the trader-valuation support width Δ increases.*

Figure 1 demonstrates these results. To start, consider the LTM with a very narrow trader-valuation support width $0.33\,\tau$. In Figure 1, we see that the LTM has a pair of symmetric equilibria for this support, one with maker-taker pricing and one with taker-maker pricing. Since this valuation support is within the inside LTM quotes, P_{-1} and P_1, there are no prices at which buyers and sellers would transact in the absence of rebates. Thus, a rebate is necessary either on the liquidity-maker or -taker side for investors to be able to trade profitably. Consider a potential buyer with a high personal valuation β_{t_1} who arrives at t_1. (The case of a potential seller with a low valuation at t_1 is symmetric). With maker-taker pricing ($TF = 0.556$ and $MF = -0.444$), the exchange offers a rebate on liquidity-making via limit orders such that the buyer is willing to use an aggressive LBP_1 limit order at t_1 to offer to buy at a quoted price P_1 above his valuation ($\beta_{t_1} \leq \bar{\beta} < P_1$) to earn the make rebate. An investor with a low personal valuation β_{t_2} arriving at t_2 can then sell at P_1 above his valuation ($\beta_{t_2} \leq \bar{\beta} < P_1$) but must also pay a take fee. In this case, maker-taker pricing generates trading by subsidizing liquidity-making via limit orders at aggressive posted prices at t_1 and imposing fees on liquidity-taking via market orders at t_2 (which benefit from the aggressive limit prices). The converse logic applies to the taker-maker equilibrium pricing ($MF = 0.556$ and $TF = -0.444$). Now investors with high personal valuations at t_1 use LBP_{-1} limit orders to try to buy at P_{-1}, and investors at t_2 then either use MSP_{-1} market orders to sell at P_{-1} and receive the take rebate, or they do not trade. In each case, the reason this works is that investor trading decisions depend on the cum-fee prices they pay or received net of market access fees and rebates (rather than on just quoted prices alone), and the exchange can use its access pricing to affect the cum-fee prices.

FIGURE 1: **Make Fees and Take Fees in a 2-Period Market.** This figure reports the equilibrium make fees (MF & mf) and take fees (TF & tf) in the Large Tick Market (LTM) (upper panel) and Small Tick Market (STM) (lower panel) corresponding to different investor valuation supports with widths ranging from $0.33\,\tau$ to $5\,\tau$ on the horizontal axes (where $\tau = 1$ is the tick size in the LTM). The figure reports in blue (orange) italics the equilibrium fees MF (TF). The left (right) side of the gray region in the figure gives the equilibrium fees for taker-maker (maker-taker) access pricing. The taker-maker and maker-taker pricing structures are optimal and symmetric.

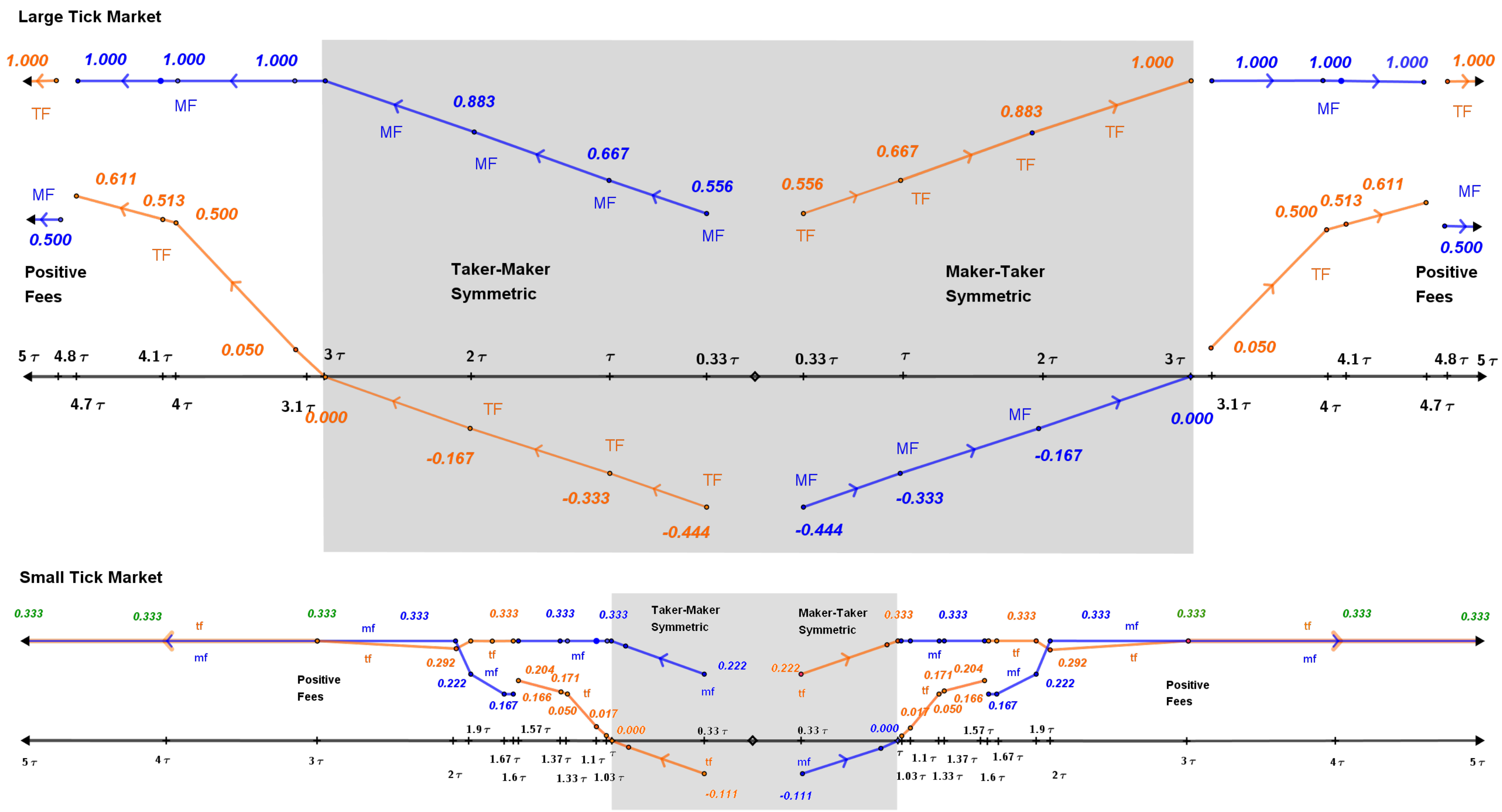
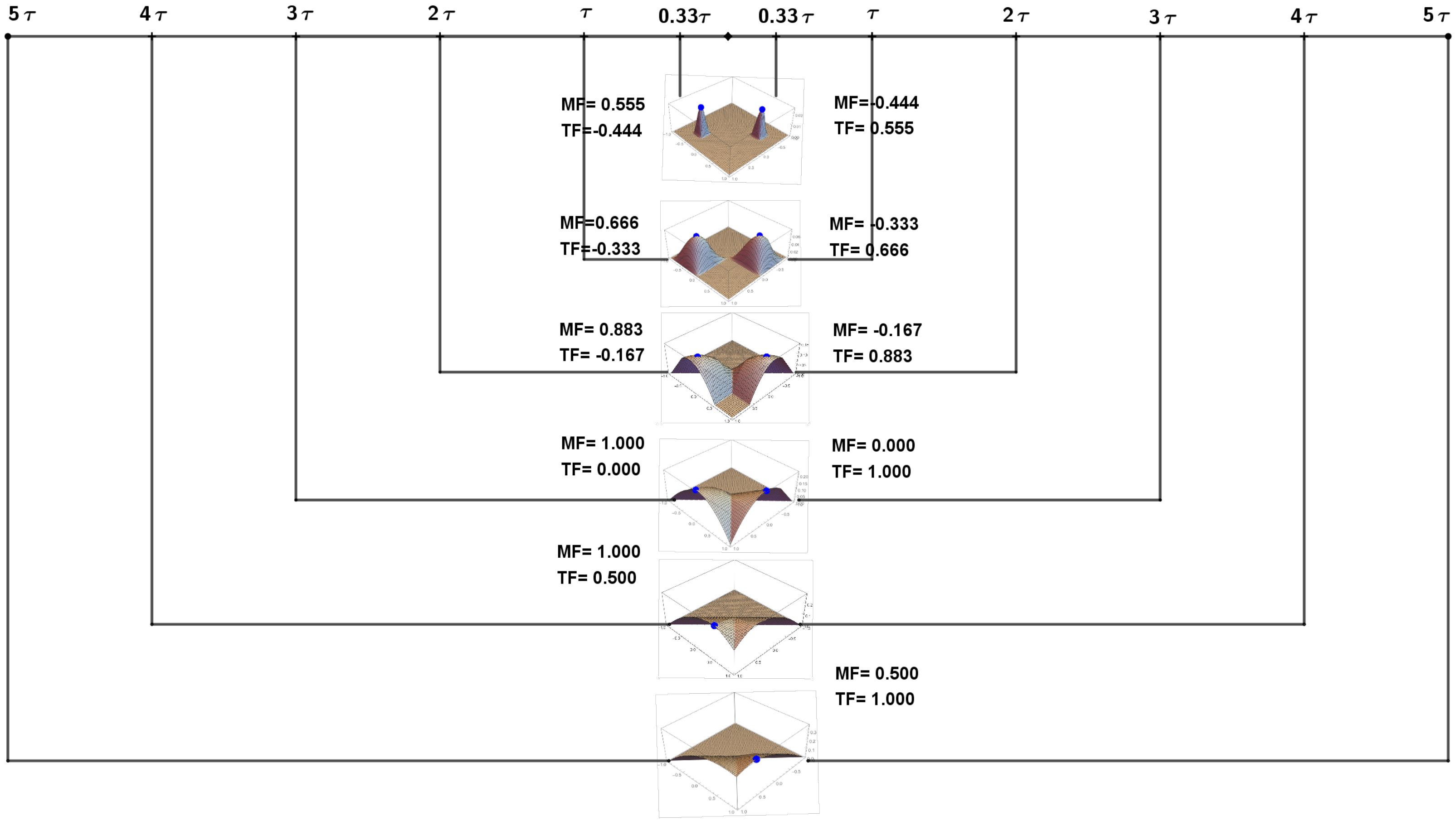


TABLE 2.1: **2-Period Large Tick Market (LTM): Equilibrium Fees and Trading Strategies.** This table reports for different investor valuation support width, $\Delta = \bar{\beta} - \underline{\beta}$ expressed in terms of the LTM tick size, τ (column 1), the extreme values of the support, $\underline{\beta}$ and $\bar{\beta}$ (column 2), the equilibrium make and take fees, MF and TF (column 3 and 4), the buyer's equilibrium trading strategies at t_1, x_{t_1} other than No Trade (column 5) and the associated probability of submission at t_1, $Pr(x_{t_1}|S, \tau, \Xi, L_{t_0})$ (column 6). The table also shows the cum-fee buy and sell prices ($P_k^{cum,LB}$ and $P_k^{cum,MS}$) (column 7 and 8), the equilibrium probability of execution of the buyer's order posted at t_1, $Pr(\theta_{t_1}^{x_{t_1}}|S, \tau, \Xi, l_{t_0})$, which correspond to the unconditional probability of MS at t_2 (column 9) and the exchange expected profit from both buyers and sellers, $\pi(MF, TF|S, \tau)$ (column 10). When the equilibrium pricing is rebate based, for each support we report first the taker-maker set of fees and then the maker-taker set of equilibrium MF and TF. Results are rounded to the third decimal.

Support width $\Delta = \bar{\beta} - \underline{\beta}$	$\underline{\beta}$, $\bar{\beta}$	MF	TF	Eq.Strategy x_{t_1} at t_1	Pr. Submission $Pr(x_{t_1}\|S, \tau, \Xi, L_{t_0})$	$P_k^{cum,LB}$	$P_k^{cum,MS}$	Pr. Execution $Pr(\theta_{t_1}^{x_{t_1}}\|S, \tau, \Xi, l_{t_0})$	Exchange E[Profit] $\pi(MF, TF\|S, \tau)$
0.33 τ	9.833, 10.167	0.556	-0.444	LBP_{-1}	0.333	10.056	9.944	0.333	0.025
0.33 τ	9.833, 10.167	-0.444	0.556	LBP_1	0.333	10.056	9.944	0.333	0.025
τ	9.500, 10.500	0.667	-0.333	LBP_{-1}	0.333	10.167	9.833	0.333	0.074
τ	9.500, 10.500	-0.333	0.667	LBP_1	0.333	10.167	9.833	0.333	0.074
2 τ	9.000, 11.000	0.833	-0.167	LBP_{-1}	0.333	10.333	9.667	0.333	0.148
2 τ	9.000, 11.000	-0.167	0.883	LBP_1	0.333	10.333	9.667	0.333	0.148
3 τ	8.500, 11.500	1.000	0.000	LBP_{-1}	0.333	10.500	9.500	0.333	0.222
3 τ	8.500, 11.500	0.000	1.000	LBP_1	0.333	10.500	9.500	0.333	0.222
3.1 τ	8.450, 11.550	1.000	0.050	LBP_{-1}	0.339	10.500	9.450	0.323	0.229
4 τ	8.000, 12.000	1.000	0.500	LBP_{-1}	0.375	10.500	9.000	0.250	0.281
4.1 τ	7.950, 12.050	1.000	0.513	LBP_{-1} , LBP_{-2}	0.369, 0.131	10.500, 9.500	8.987, 7.987	0.253, 0.009	0.286
4.7 τ	7.650, 12.350	1.000	0.611	LBP_{-1} , LBP_{-2}	0.342 , 0.157	10.500 , 9.500	8.889 , 7.889	0.264 , 0.051	0.317
4.8 τ	7.600, 12.400	0.500	1.000	LBP_1 , LBP_{-1}	0.104, 0.396	11.000, 10.000	9.500, 8.500	0.396, 0.188	0.346
5 τ	7.500, 12.500	0.500	1.000	LBP_1 , LBP_{-1}	0.100 , 0.400	11.000 , 10.000	9.500 , 8.500	0.400 , 0.200	0.360

FIGURE 2: **2-Period Large Tick Market (LTM): Exchange Expected Profit Function and Access Pricing.** This figure shows the exchange profit function and the equilibrium make fees and take fees for the LTM corresponding to different investor valuation supports with widths ranging from 0.33 τ to 5 τ (where $\tau = 1$ is the tick size in the LTM) as reported on the horizontal axis. The three-dimensional figures indicate (blue dots) the optimal make fee (MF), the optimal take fee (TF), and the associated equilibrium exchange expected profit for each support.



The profit-maximizing access-pricing structure depends on the relationship between the support of traders' evaluation and the tick size. Intuitively, as the support of investor valuations increases, the potential gains-from-trade increase, which increases investor trading demand. As a result, the exchange has less of a need to incentivize trading. Thus, the exchange, in equilibrium, exploits investors' greater ex ante gains-from-trade by increasing fees and reducing rebates. This happens with both taker-maker and maker-taker pricing. Starting from the smallest valuation support, $0.33\,\tau$, Figure 1 and Table 2.1 show that the exchange monotonically increases both MF and TF as the support width Δ increases up until the point that the regulatory cap on fees binds. For example, when the support width reaches $2\,\tau$, the buyer still buys either at P_{-1} or at P_1 and the exchange sets the symmetric taker-maker and maker-taker fee structure with a positive fee of 0.883 and rebate of -0.167. Taker-maker and maker-taker access pricing persists until, holding the LTM tick size fixed at τ, the investor valuation support width Δ reaches the outside quotes with $\Delta = 3\,\tau$.

Proposition 1 and Figure 1 show that three things happen once $\Delta > 3\,\tau$: First, investor trading demand is sufficiently strong that the exchange ceases giving rebates to incentivize trade and switches instead to a strictly positive-fee access pricing structure. Second, the regulatory cap on fees is reached on one side of the market. Third, the optimal access pricing structure becomes unique. As a result, at this point, the exchange starts charging the highest possible make fee on limit orders given the regulatory cap, $MF = 1.000$, and also charges a positive take fee for market orders. For example, when the support width is $3.1\,\tau$, the optimal take fee is $TF = 0.050$. In these parameterizations, low-valuation investors still profitably sell at the low price P_{-1}. In equilibrium, a high-valuation investor arriving at t_1 knows that, given the wide valuation support and the relatively low TF, there is a sufficiently high probability of a seller arriving in period t_2 willing to demand liquidity at the lower price P_{-1}. Strictly positive fee equilibria are new relative to Chao, Yao, and Ye 2018, who find only rebate-based access pricing. The reasons for the difference between our results and CYY are considered in Appendix 2.11.

The fact that the optimal equilibrium access pricing structure can be unique is also new. For example, numerical calculation (not reported) verifies that using hypothetical symmetric fees $MF = 0.050$ and $TF = 1.000$ leads to lower exchange expected profits than using the equilibrium fees $MF = 1.000$ and $TF = 0.050$ when $\Delta = 3.1\tau$. The reason illustrates a significant asymmetry between make and take fees. In the context of this two-period market, market orders are only used at time t_2 and, thus, take fees simply affect the willingness of the investor at time t_2 to trade with whatever limit orders happen to be in the book. In contrast, limit-order submitters at time t_1 have a decision about the price at which they optimally choose to post a limit order. As a result, make fees potentially affect a more complicated decision between multiple order-submission alternatives for limit-order submitters (i.e., as opposed to the trade/no-trade decision of market-order submitters). In the $\Delta = 3.1\tau$ example, the symmetric fees are suboptimal because, with a hypothetical make fee of 0.050, buy limit orders at P_{-1} and P_1 both have positive expected profits and, given a sufficiently low make fee (i.e., 0.050) and a wide investor valuation support (e.g., $\Delta = 3.1\tau$) — such that there is a sufficient probability of investors at t_2 with very low private valuations who would be willing to sell at a low cum-fee price of $P_{-1} - TF$, — there are some investors at t_1 (with valuations slightly above v)

who would post buy limit orders at P_{-1} rather than at P_1. Since such orders have lower execution probabilities than limit orders at P_1, this reduces exchange expected profits relative to the equilibrium fees, thereby making the hypothetical symmetric fees $MF = 0.050$ and $TF = 1.000$ suboptimal. As we will see, our three-period model in Section 2.5 shows there is a related asymmetry in make and take fees in multi-period markets.

In general, changes in the equilibrium investor strategy at t_1 coincide with changes in the exchange's optimal fee structure. We see this clearly in Table 2.1 and Figure 1. As the valuation support width Δ increases beyond $4\,\tau$ in the region with strictly positive fees, the buyers start using two possible different limit orders at t_1 — i.e., they now buy at P_{-1} or at P_{-2} — for two different intervals of β_{t_1}. While at $\Delta = 4\,\tau$ the buyer has no incentive to buy at P_{-2} (a seller with the minimum possible valuation 8 would not sell at the cum fee sell price $P_{-2}^{cum,MS} = 8.5 - 0.5 = 8$), at a wider support, e.g., $\Delta = 4.1\tau$, the buyer does have an incentive to post orders at 8.5 as the incoming seller even with the minimum valuation 7.95 would be willing to sell at $P_{-2}^{cum,MS} = 8.5 - 0.513 = 7.987$. The exchange exploits the larger gains-from-trades of the sellers by setting a higher TF, and keeps charging the buyer the maximum $MF = 1.000$ up until the support width reaches $\Delta = 4.8\,\tau$. Once $\Delta \geq 4.8\,\tau$, the buyer switches from using LBP_{-2} to using LBP_1, and the exchange halves the MF to 0.500 and increases the TF to 1.000.

Figure 2 illustrates the exchange's expected profit function for different combinations of fees and rebates given different investor valuation supports. The blue dots denote profit-maximizing combinations of make and take fees. The symmetric pairs of profit-maximizing MF and TF are clearly visible when the investor valuation supports are narrow. However, there is a unique profit-maximizing set of fees once the valuation support is large enough. The intuition for the asymmetry between maker-taker and taker-maker pricing is that additional prices (at P_{-2} and P_2) become a priori possible at time t_1 once the width is $\Delta > 3\,\tau$. The exchange can use make fees MF to directly control the expected profit on these multiple possible limit orders so as to incentivize investors at t_1 to submit orders that maximize the exchanges profits. In contrast, take fees TF only affect the order-submission behavior of investors at t_1 indirectly via its impact on the trading behavior of the investor at t_2.

Proposition 3 *The sum of the make and take fees is one third of the support width, $MF + TF = \Delta/3$ for all support widths $\Delta < 3\,\tau$ in the 2-period LTM.*

This property — which can be verified numerically in Table 2.1 — is proven analytically in Appendix 2.10. The key part of the proof is that the exchange's expected profit can be expressed as

$$\pi(MF, TF|S,\tau) = 2 \max\left\{0, \frac{\bar{\beta} - P_{-1}^{cum,LB}}{\Delta}\right\} (MF + TF) \max\left\{0, \frac{P_{-1}^{cum,MS} - \underline{\beta}}{\Delta}\right\} \quad (2.6)$$

which is the product of the relevant limit-order submission probability at time t_1 (just one possible buy limit order in equilibrium), the net fee, and the relevant market-order submission probability at time t_2 (i.e., so that the earlier limit order is executed). The specific function form of this expression follows from the uninform valuation distribution assumption and symmetry between the buy and sell sides of the market. Given this representation, we note that the three components $\bar{\beta} - P_{-1}^{cum,LB}$, $MF + TF = P_{-1}^{cum,LB} - P_{-1}^{cum,MS}$, and $P_{-1}^{cum,MS} - \underline{\beta}$ in (2.6), when they are positive, sum to the valuation support width Δ. It can then be shown that the product in (2.6) is maximized by the exchange choosing MF and TF to set these three components equal to each other, which implies that $MF + TF = \Delta/3$.

Proposition 1 states that, given a tick size τ, the optimal fee structure depends on the support of traders' evaluations and therefore on the types of traders populating the market. This leads to an empirical prediction:

Empirical Prediction 1: *Markets populated by traders with low valuation dispersion optimally have taker-maker and maker-taker access pricing. Conversely, markets populated by traders with high valuation dispersion optimally have a unique positive-fee access pricing.*

The smaller the support of the traders' evaluation, the more likely the traders will act as liquidity providers, whereas the larger the support of the traders' evaluations the more likely traders will act as hedge funds managers who do not generally trade to speculate on small price increments. Within the logic of our model, high frequency trading firms can be characterized as having asset valuations equal to the fundamental asset value (v), and these results hint at a more general conclusion that we discuss in Section 2.6 when we will extend the model to include HFT firms.

2.4.2 Small Tick Market

We next consider the effects of a smaller tick size on the optimal fee structure. The results for the STM with a tick size of $\tau/3$ are in the lower panel of Figure 1 and in Table 2.2.[11] Given a smaller tick size, as the support of traders' valuations increases, the exchange still has an incentive to increase both its make fee, MF, and take fee, TF. Thus, access pricing changes in the same direction as in the LTM. However, the pricing structure reaches the threshold when both fees are positive earlier since the regulatory cap on fees (which is tied to the tick size) binds sooner. Figure 1 shows that when the support is $[p_{-2}, p_2]$, which corresponds to a width τ in the STM and which is equivalent to the support $[P_{-1}, P_1]$ with width τ in the LTM, the optimal STM access pricing structure has positive fees on both the take and make sides ($MF = 0.333$ and $TF = 0.000$), whereas the optimal LTM access fee structure with the same valuation support $[p_{-2}, p_2]$ is still the symmetric taker-maker and maker-taker pricing.[12] Thus, the exchange's optimal access pricing Ξ^* depends on both the absolute tick size (given the regulatory restriction on fees relative to the tick size) and the relative size of the investor valuation support compared to the tick size.

Figure 1 (lower panel) and Table 2.2 show that, as for the LTM, when the investors' support widens, the incentive for the STM exchange to offer rebates decreases. All else equal, given a regulation capping fees to be smaller than the tick size (in Appendix 2.11 we relax this assumption), when the tick size is smaller the exchange may only set smaller fees (both positive and

[11] The exchange profit functions and their maximizers are qualitatively similar in the STM to the Figure 2 for the LTM.

[12] Since the STM tick size is $1/3$ of LTM tick size ($\frac{1}{3}\tau$), the STM equilibrium fees are equal to $\frac{1}{3}$ of the LTM equilibrium fee computed for a support three times larger (e.g., τ). To ease the comparison between the STM and the LTM, we provide finer numerical detail for the STM in the regions of the valuation support where there are discontinuities in optimal access pricing. These correspond to support widths in the LTM where there are discontinuities in access pricing divided by three.

negative). We show that by starting from the same smallest support as per the LTM, the region of the traders' support consistent with the exchange profitably offering the taker-maker or the maker-taker fee structures (grey regions in Figure 1) is narrower, and the fees themselves are smaller in absolute values.

We conclude that our results in the STM are qualitatively the same as in the LTM except that the STM exchange reaches the regulatory fee cap sooner. Hence, in the small tick market the exchange has fewer degrees of freedom to maximize profits by setting a taker-maker or maker-taker fee structure. The results from the STM lead to our next proposition:

Proposition 4 *When the tick size is smaller, the exchange has a smaller incentive to offer rebates in maker-taker and taker-maker fee structures, and the optimal fees can be smaller.*

Proposition 4 for the STM confirm that it is not just the absolute value of the tick size that matters when determining the optimal fee structure but rather the relation between the tick size and the width of the trader valuation support. More precisely, when the tick size is smaller, the exchange has less degree of freedom in setting the trading fees and this leads to our second empirical prediction:

Empirical Prediction 2: *When, holding the trading population constant, the tick size increases (decreases), the exchange has an incentive to offer greater (smaller) rebates.*

Our empirical prediction can be tested by investigating how a change in the tick size alters the incentive for the exchange to offer rebates. Our model predicts that when, all else equal, the tick size increases, the exchange, to attract volume, should increase the rebates offered to the same population of market participants. However, with competition, if the exchange does not adjust the rebates to the new tick size, it runs the risk of seeing orders migrating to other more profitable venues. Comerton-Forde, Grégoire, and Zhong 2019 investigate the effects of an increase in the tick size within the U.S. tick size pilot program started in October 2016 and, interestingly, find that following the increase in the U.S. tick size from 1 penny to 5 pennies a substantial amount of orders migrated from the maker-taker to the taker-maker inverted fees platforms. This finding is consistent with our model's prediction that following an increase in the tick size the exchange should offer greater rebates to ensure that volume is maximized within a trading platform.

2.5 Results for the 3-Period trading game

This section extends our model to three periods ($\{t_1, t_2, t_3\}$) and shows how the equilibrium changes relative to the 2-period equilibrium. Two key intuitions drive these changes. First, in the 2-period model in Section 2.4, investors in the first period are monopolists in supplying liquidity since there is no opportunity for later traders to compete against the first-period trader's limit orders. In particular, investors at t_2 can only accept or decline liquidity offered by the limit order posted at time t_1 since the game ends after t_2. Once there are more than two periods, the first-period liquidity supply is no longer monopolistic and some amount of intertemporal competition in liquidity supply is possible. Second and relatedly, there is a higher level of trading activity when there are more rounds of investor-arrival. The fact that more traders arrive over time increases the probability of limit order execution.

TABLE 2.2: **2-Period Small Tick Market (SMT): Equilibrium Fees and Trading Strategies.** This table reports for different investor valuation support width, $\Delta = \overline{\beta} - \underline{\beta}$ still expressed in terms of the LTM tick size τ (column 1), the extreme values of the support, $\underline{\beta}$ and $\bar{\beta}$ (column 2), the equilibrium make and take fees (MF and TF) (column 3 and 4), the equilibrium trading strategies at t_1, x_{t_1} other than No Trade (column 5) and the associated probability of submission at t_1, $Pr(x_{t_1}|S, \frac{\tau}{3}, \Xi, L_{t_0})$ (column 6). The table also shows the cum-fee buy and sell prices ($P_j^{cum,LB}$ and $P_j^{cum,MS}$) (column 7 and 8), the probability of execution of the order posted at t_1, $Pr(\theta_{t_1}^{x_{t_1}}|S, \frac{\tau}{3}, \Xi, l_{t_0})$, which correspond to the unconditional probability of MS at t_2 (column 9) and the exchange expected profit from both buyers and sellers, $\pi(MF, TF|S, \frac{\tau}{3})$ (column 10). When the equilibrium pricing is rebate based, for each support we report first taker-maker set of fees and then the maker-taker set of equilibrium MF and TF. .

Support $\Delta = \overline{\beta} - \underline{\beta}$	$\underline{\beta}, \bar{\beta}$	MF	TF	Eq. Orders x_{t_1} at t_1	Pr. Submission $Pr(x_{t_1}\|S, \frac{\tau}{3}, \Xi, L_{t_0})$	$P_j^{cum,LB}$	$P_j^{cum,MS}$	Pr. Execution $Pr(\theta_{t_1}^{x_{t_1}}\|S, \frac{\tau}{3}, \Xi, l_{t_0})$	Exchange E[Profit] $\pi(MF, TF\|S, \frac{\tau}{3})$
0.33 τ	9.833, 10.167	0.222	-0.111	LBp_{-1}	0.333	10.056	9.944	0.333	0.025
0.33 τ	9.833, 10.167	-0.111	0.222	LBp_1	0.333	10.056	9.944	0.333	0.025
τ	9.500, 10.500	0.333	0.000	LBp_{-1}	0.333	10.167	9.833	0.333	0.074
τ	9.500, 10.500	0.000	0.333	LBp_1	0.333	10.167	9.833	0.333	0.074
1.03 τ	9.485, 10.515	0.333	0.017	LBp_{-1}	0.338	10.167	9.816	0.322	0.076
1.1 τ	9.450, 10.550	0.333	0.050	LBp_{-1}	0.348	10.167	9.783	0.303	0.081
1.33 τ	9.333, 10.667	0.333	0.166	LBp_{-1}, LBp_{-2}	0.375, 0.126	10.167, 9.833	9.667, 9.334	0.251, 0.001	0.085
1.37 τ	9.315, 10.685	0.333	0.171	LBp_{-1}, LBp_{-2}	0.369, 0.131	10.167, 9.833	9.662, 9.329	0.253, 0.009	0.095
1.57 τ	9.215, 10.785	0.333	0.204	LBp_{-1}, LBp_{-2}	0.353, 0.147	10.167, 9.833	9.629, 9.296	0.263, 0.051	0.106
1.6 τ	9.200, 10.800	0.167	0.333	LBp_1, LBp_{-1}	0.104, 0.396	10.334, 10.050	9.834, 9.500	0.396, 0.188	0.115
1.67 τ	9.165, 10.835	0.167	0.333	LBp_1, LBp_{-1}	0.100, 0.400	10.334, 10.050	9.834, 9.500	0.400, 0.200	0.120
1.9 τ	9.050, 10.950	0.222	0.333	LBp_1, LBp_{-1} LBp_{-2}	0.059, 0.351 0.091	10.389, 10.055 9.722	9.834, 9.500 9.167	0.412, 0.237 0.061	0.125
2 τ	9.000, 11.000	0.333	0.292	LBp_{-1}, LBP_{-2}	0.313, 0.187	10.167, 9.833	9.541, 9.208	0.271, 0.104	0.130
3 τ	8.500, 11.500	0.333	0.333	LBp_{-1}, LBp_{-2} LBp_{-3}	0.222, 0.222 0.056	10.167, 9.833 9.500	9.500, 9.167 8.834	0.333, 0.222 0.111	0.173
4 τ	8.000, 12.000	0.333	0.333	LBp_{-1}, LBp_{-2} LBp_{-3}	0.167, 0.167 0.167	10.166, 9.833 9.500	9.500, 9.167 8.834	0.375, 0.292 0.208	0.194
5 τ	7.500, 12.500	0.333	0.333	LBp_{-1}, LBp_{-2} LBp_{-3}, LBp_{-4}	0.133, 0.133 0.133, 0.100	10.166, 9.833 9.500, 9.167	9.500, 9.167 8.834, 8.500	0.400, 0.333 0.267, 0.200	0.204

The equilibrium construction of our 3-period model is entirely analytic, based on Theorem 1, up to the final step of solving the exchange's first-order conditions. However, due to the complexity of the 3-period exchange profit function, we report numerical equilibrium fees and rebates obtained using a combination of a Simulated Annealing (SA) algorithm together with grid search to refine our results once the optimal region has been identified (Appendices C and D).

Figure 3 shows the equilibrium make fees and take fees for the 3-period model for different investor valuation supports.[13] Many of the results for the 3-period model are similar to the 2-period model. There is still a highlighted grey region of investor valuation supports with both taker-maker and maker-taker equilibria and, again, as the investor valuation support width Δ increases, the exchange increases both MF and TF subject to the regulatory cap, and eventually there is a unique equilibrium with strictly positive fees. However, there are also some differences. To help explain these differences, Table 2.3 shows the equilibrium strategies for the 3-period LTM market, together with the order-submission probabilities at t_1, $Pr(x_{t_1}$, the execution probabilities, $Pr(\theta_{t_1}^{x_{t_1}}|S, \tau, \Xi, l_{t_0})$, the equilibrium fees, MF and TF, the equilibrium cum-fee prices for buy limit orders ($P_k^{cum,LB}$) and sell market orders ($P_k^{cum,MS}$), and the exchange expected profit, $\pi(MF, TF|S, \tau)$.

Proposition 5 *The set of valuation supports associated with rebates can be smaller in the 3-period model. In addition, fees can be larger and rebates can be smaller in the 3-period model.*

Comparing Figures 1 and 3 shows that the grey region with rebate-based access pricing (maker-taker or taker-maker) is smaller in the 3-period framework. The largest support associated with the rebate-based pricing is $[8.85, 11.15]$ (with a width of $2.3\,\tau$) in the 3-period market as opposed to $[8.50, 11.50]$ (with a width of $3\,\tau$) in the 2-period framework. In addition, because trading volume is higher in the 3-period model, exchange profits are systematically higher. We also note that the levels of MF and TF in the 3-period model are smaller. The intuition for the effect of the number of trading periods on the use of rebates and the level of access pricing is the following: Holding everything fixed, the probability that limit orders are executed increases because there are more opportunities for investors with complementary reasons to trade to arrive and trade with each other. As a result, the exchange has less of an incentive to offer rebates.

Proposition 6 *Maker-taker and taker-maker pricing can be asymmetric in the 3-period model with smaller rebates in the maker-taker equilibrium than in the taker-maker equilibrium.*

This asymmetry is new and in contrast to the symmetry in our 2-period model and also Chao, Yao, and Ye 2018. The equilibrium fees are asymmetric because in the 3-period model the investor at time t_1 is no longer a monopolist in liquidity provision. An incoming investor at time t_2 may try to induce competition with the t_1 limit order by the investor at t_3.

Consider, for example, the equilibrium strategies in Row 1 of Table 2.3 for a support width $\Delta = 0.3\,\tau$. In the taker-maker equilibrium when the investor in period t_1 tries to limit buy (LBP_{-1}) at the price P_{-1}, an incoming seller in period t_2 has the option of either market selling

[13] Once again, the 3-period exchange profit functions look qualitatively similar to those for the 2-period exchange modulo the asymmetry discussed below.

(MSP_{-1}) at P_{-1} or trying to limit sell (LSP_1) at the higher price P_1 with an investor arriving at time t_3. In contrast, in the maker-taker equilibrium the investor at t_1 tries to limit buy (LBP_1) at P_1 (because of the rebate $MF = -0.428$), which consequently means that the seller arriving at t_2 has no other trading option than market selling (MSP_1) at the high price P_1 — since limit selling at P_{-1} is not allowed given the pre-existing limit buy at P_1 in order to prevent a crossed market — and therefore will be charged a positive fee $TF = 0.557$.[14]

Table 2.3 shows that in the taker-maker equilibrium the seller at t_2 opts only to market sell at P_{-1}. This choice is driven by the higher TF rebate (-0.443) that encourages transactions at t_2 in the taker-maker equilibrium. It is precisely due to the high TF rebate that the seller does not try to limit sell at t_2 in the taker-maker equilibrium. The two grey rows 3 and 4 in Table 2.3 show that if, off equilibrium, the exchange used symmetric fees — i.e., the equilibrium taker-maker TF and MF are flipped for the maker-taker MF and TF or if the equilibrium maker-taker TF and MF are flipped and used for the taker-maker MF and TF — the incoming seller would opt for either market selling (MSP_{-1}) or limit selling (LSP_1), and exchange profits would be smaller. This explains why, in equilibrium, the exchange offers a larger TF rebate than the MF rebate.

Observation: *The minimum rebate $|MF|$ in the 3-period taker-maker equilibrium is strictly positive, whereas it is 0 in the maker-taker equilibrium because the regulatory cap on the taker-maker TF binds for smaller support widths than in the maker-taker equilibrium.*

This discontinuity can be seen in Figure 3 where the minimum taker-maker rebate $|TF|$ on the left when Δ is just larger than $2.3\,\tau$ is a little less than $|-1.04|$ whereas the minimum maker-taker rebate $|MF|$ on the right is 0.

The comparison between the 2-period and 3-period frameworks also allows us to study how optimal access fees should differ for stocks with different rates of trading activity. The 3-period framework proxies for a stock with a faster rate of trading activity. This leads to the following empirical prediction:

Empirical Prediction 3: *When stocks have a higher rate of trading activity, the region of valuation supports associated with rebate-based pricing shrinks, and the exchange has an incentive to offer smaller rebates.*

A practical complication here is that exchanges generally have a single set of fees and rebates that are applied across all stocks on an exchange. Thus, access pricing optimization happens for the cross-section of traded stocks. However, actual access pricing typically involves special rules and volume-contingent pricing schedules.[15] We conjecture that this pricing complexity allows exchanges to implement some amount of access pricing customization for different types of stocks.

[14] The state of the book when the seller arrives at t_2 has a limit order at P_1, hence the seller does not compete for the provision of liquidity as a limit sell order at P_{-1} is dominated by the market sell order at P_1.

[15] See, for example, the 2018 LSE access price list at:
https://www.lseg.com/sites/default/files/content/documents/Trading%20Services%20Price%20List_effectiveOct2018.pdf

FIGURE 3: **Make Fees and Take Fees in 3-Period Market.** This figure reports the equilibrium make fees (MF) and take fees (TF) in the Large Tick Market (LTM) corresponding to different investor valuation supports ranging from 0.33 τ and 5 τ, (where τ is the tick size in the LTM) on the horizontal black axes. The left (right) part of the figure reports the equilibrium trading fees consistent with the taker-maker (maker-taker) fee structure. The figure reports in blue (orange) italic the equilibrium MF (TF) set by the exchange.

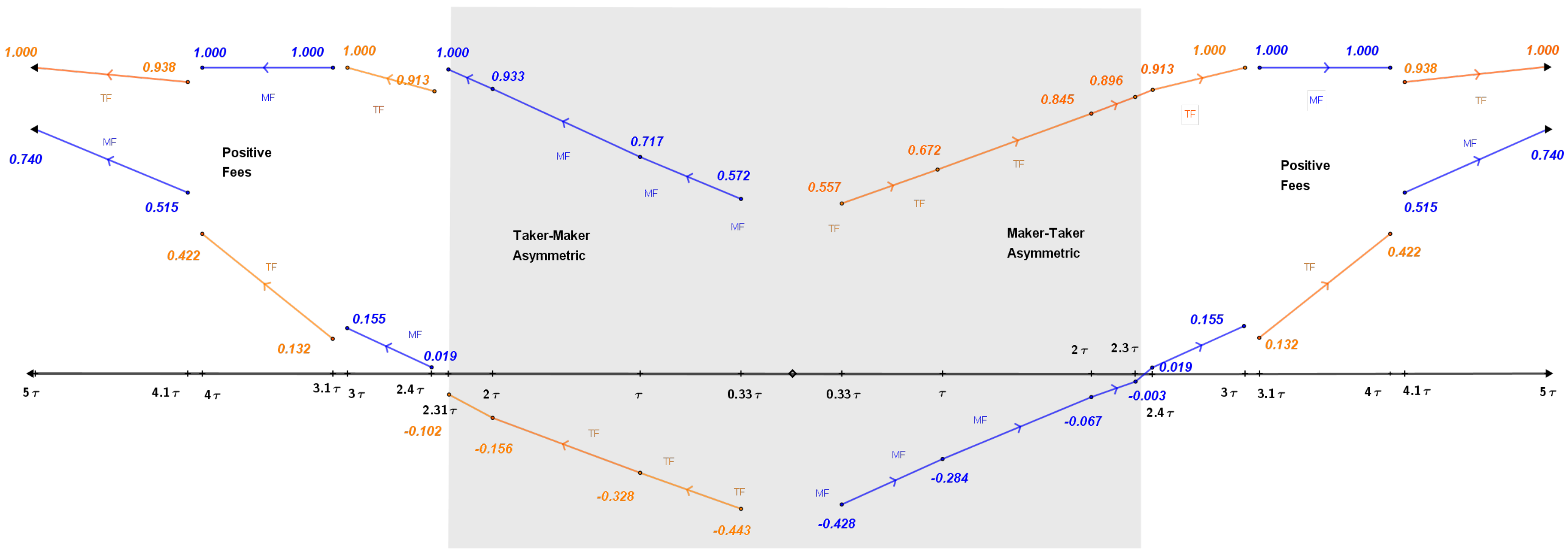


TABLE 2.3: **3-Period Large Tick Market (LTM). Equilibrium Fees and Trading Strategies.** This table reports for different investor valuation support width, $\Delta = \bar{\beta} - \underline{\beta}$ expressed in terms of the LTM tick size τ (column 1), the extreme values of the support, $\underline{\beta}$ and $\bar{\beta}$ (column 2), the equilibrium make and take fees (MF and TF) (column 3 and 4), the equilibrium orders x_{t_1} at t_1 other than No Trade (column 5) and the equilibrium orders x_{t_2} at t_2, conditional on the trading strategy indicated at t_1 (column 6). The table also shows the associated probability of submission, $Pr(x_{t_1}|S,\tau,\Xi,L_{t_0})$ and $Pr(x_{t_2}|S,\tau,\Xi,L_{t_1})$, (column 7 and 8), as well the cum-fee buy and sell prices ($P_k^{cum,LB}$ and $P_k^{cum,MS}$) (column 8 and 10), the probability of execution of the order posted at t_1, $Pr(\theta_{t_1}^{x_{t_1}}|S,\tau,\Xi,l_{t_0})$, (column 11) and the Exchange Expected Profit, $\pi(MF,TF|S,\tau)$ (column 12). The third and fourth gray rows report results (marked with a *) for off-equilibrium fees that symmetrically flip the corresponding equilibrium fees. When the equilibrium pricing is rebate based, for each support we report first maker-taker set of fees and then the taker-maker set of equilibrium MF and TF.

Support width $\Delta = \bar{\beta} - \underline{\beta}$	$\underline{\beta}, \bar{\beta}$	MF	TF	Eq.Strategies x_{t_z}		Pr. Submission $Pr(x_{t_z}\|S,\tau,\Xi,L_{t_{z-1}})$		$P_k^{cum,LB}$	$P_k^{cum,MS}$	Pr. Execution $Pr(\theta_{t_1}^{x_{t_1}}\|S,\tau,\Xi,l_{t_0})$	Exchange E[Profit] $\pi(MF,TF\|S,\tau)$
				t_1	t_2	t_1	t_2				
0.33 τ	9.833, 10.167	0.572	-0.443	LBP_{-1}	MSP_{-1}	0.284	0.328	10.072	9.943	0.548	0.051
0.33 τ	9.833, 10.167	-0.428	0.557	LBP_1	MSP_1	0.284	0.328	10.072	9.943	0.548	0.051
gray!15 gray!15 0.33 τ*	9.833, 10.167	0.557*	-0.428*	LBP_{-1}*	MSP_{-1}* & LSP_1*	0.328*	0.266* & 0.062*	10.057*	9.927*& 9.943	0.475*	0.050*
gray!15 gray!15 0.33 τ* gray!15	9.833, 10.167	-0.443*	0.572*	LBP_1*	MSP_1*	0.328*	0.284*	10.057*	9.927*	0.487*	0.050*
τ	9.500, 10.500	0.716	-0.328	LBP_{-1}	MSP_{-1}	0.284	0.328	10.216	9.828	0.548	0.152
τ	9.500, 10.500	-0.284	0.672	LBP_1	MSP_1	0.284	0.328	10.216	9.828	0.548	0.152
2 τ	9.000, 11.000	0.933	-0.156	LBP_{-1}	MSP_{-1}	0.284	0.328	10.433	9.656	0.548	0.304
2 τ	9.000, 11.000	-0.067	0.845	LBP_1	MSP_1	0.284	0.328	10.433	9.655	0.548	0.304
2.31 τ	8.850, 11.150	1.000	-0.102	LBP_{-1}	MSP_{-1}	0.284	0.328	10.500	9.602	0.548	0.351
2.31 τ	8.850, 11.150	-0.001	0.898	LBP_1	MSP_1	0.284	0.328	10.499	9.602	0.548	0.351
2.4 τ	8.800, 11.200	0.019	0.913	LBP_1	MSP_1	0.284	0.328	10.519	9.587	0.548	0.365
3 τ	8.500, 11.500	0.155	1.000	LBP_1	MSP_1	0.282	0.333	10.655	9.500	0.556	0.456
3.1 τ	8.450, 11.550	1.000	0.132	LBP_{-1}	LBP_1, MSP_{-1}, LSP_1	0.339	0.016, 0.278, 0.060	10.500, 11.500	9.368, 9.500	0.487	0.468
4 τ	8.000, 12.000	1.000	0.422	LBP_{-1}	LBP_1, MSP_{-1}, LSP_1, LSP_2	0.367	0.125, 0.231, 0.125, 0.270	10.500, 11.500	9.078, 9.500, 10.500	0.404	0.612
				LBP_{-2}	LBP_{-1}, MSP_{-2}, LSP_1, LSP_2	0.133	0.375, 0.000, 0.356, 0.270	9.500, 10.500	8.078, 10.500, 11.500	0.012	
4.1 τ	7.950, 12.050	0.515	0.938	LBP_1	LBP_2, MSP_1	0.149	0.009, 0.393	11.015, 12.015	9.562	0.628	0.627
				LBP_{-1}	LBP_1, MSP_{-1}, LSP_1	0.347	0.252, 0.088, 0,408	10.015, 11.015	8.562, 9.985	0.187	
5 τ	7.500, 12.500	0.740	1.000	LBP_1	LBP_2, MSP_1	0.108	0.052, 0.400	11.240, 12.240	9.500	0.619	0.785
				LBP_{-1}	LBP_1, MSP_{-1}, LSP_1	0.344	0.252, 0.137, 0.315	10.240, 11.240	8.500, 9.760	0.259	

2.6 High Frequency Trading and Access Pricing

Our previous results show that an exchange's optimal access pricing depend crucially on the type of investors in the market, and that the incentive to offer rebates decreases with traders' ex ante potential gains-from-trade. Thus, the more traders have personal evaluations near the asset's fundamental value, the greater is the exchange's incentive to offer rebates. In real markets, one important type of active traders whose personal valuations typically do not differ from the fundamental value of the asset are high frequency traders (HFT). This section extends our previous our previous analysis to include high frequency trading firms.

HFT firms are profit-maximizing investors that differ from regular investors (INV) in four ways in our model: First, rather than having stochastic private valuations, all HFT firm have the same non-random personal valuation, which we assume is equal to the mean INV valuation v. Second, rather than arriving sequentially, the HFTs are continually present. In particular, unlike Foucault, Kadan, and Kandel (2013), the HFTs bear no monitoring costs. Third, HFT firms can react immediately to take advantage of any profitable trading opportunities in limit orders submitted by arriving regular INVs. For example, if in period t_z an INV posts an aggressive limit buy (sell) order such that the associated cum-fee sell (buy) price for a market order is above (below) the HFT valuation v, the HFT firm can submit a sell (buy) market order within the same period t_z to take the other side of the profitable trade. We call these fast market orders *flash orders.* If more than one HFT submits a flash order, then one is randomly selected for execution, and the rest are cancelled. Fourth, Budish, Cramton, and Shim 2015 show that there is a natural bid-ask spread for HFT limit orders given endogenous picking-off costs for stale orders. Thus, to simplify our analysis, we assume that, while HFTs are willing to provide liquidity ex post to regular INVs using flash orders, they are unwilling to provide ex ante liquidity via limit orders.[16].

Based on the foregoing, the HFT action set, $X_{t_z}^{HFT} = \{MOP_{k(L_{t_z})}^{HFT}, NT\}$ consists of possible flash market orders $MOP_{k(L_{t_z})}^{HFT}$ given the current book L_{t_z} or no-trade.

In each period t_z, HFT firms choose the order $x_{t_z}^{HFT}$ to maximize their expected payoff depending on whether or not there is an aggressive limit order in the book L_{t_z} that it would be profitable to trade with

$$\max_{x_{t_z}^{HFT} \in X_{t_z}^{HFT}} w(x_{t_z}^{HFT} | \tau, \Xi, v, L_{t_z}) = \begin{cases} [v - P(x_{t_z}^{HFT}) - \xi(x_{t_z}^{HFT})] & \text{if } x_{t_z}^{HFT} \text{ buy} \\ [P(x_{t_z}^{HFT}) - v - \xi(x_{t_z}^{HFT})] & \text{if } x_{t_z}^{HFT} \text{ is sell} \\ 0 & \text{if } x_{t_z}^{HFT} \text{ is } NT. \end{cases} \quad (2.7)$$

The execution probability for a flash market order is 1 if it is submitted. Note that the current INV order Q_{t_z} is part of the current book L_{t_z} that is the conditioning information of the HFT.

Competition by the HFTs simplifies the structure of equilibrium in a market with HFTs. Since HFTs are always willing to buy and sell at v, the exchange, in equilibrium, can set the

[16]Allowing for the possibility that HFTs might sometimes use limit order when they are unwilling to use flash orders given a hypothetical exchange access pricing structure would simply complicate the analysis. In Budish, Cramton, and Shim 2015, the break-even condition in a limit order book such that HFT firms supply liquidity is that the payoff from market making is at least equal to the costs of being sniped by other competing HFT firms. Thus, our assumption of no HFT limit orders is simply a convenient reduced-form for picking-off risks for a smart trading crowd as first suggested in Seppi 1997.

fees and rebates Ξ so that in equilibrium the cum-fee prices paid and received by the HFT is their break-even valuation v.

This has two immediate implication: First, limit buys at prices below P_{-1} and limit sells at prices above P_1 are never used in equilibrium. This is because HFTs and INVs know that such limit orders would always be undercut by future HFTs who will be willing to trade via flash market orders at their break-even cum-fee prices. Second, the INVs therefore choose between submitting limit orders at P_{-1} and P_1, market orders (if there are any pre-existing limit orders in the book at P_{-1} and P_1), and NT.

Regular investors (INV) have the same formal action set $X_{t_z}^{INV}$ as before. However, there is a fundamental change in the INV order submission problem when HFTs are present and active. If the HFTs are willing to use flash orders to immediately take the other side of aggressive limit buy (sell) orders at prices above (below) v, then less aggressive standing limit buy (sell) orders at outside prices (P_2 and P_{-2}) below (above) v are never executed in equilibrium. This is because standing outside limit orders would always be undercut by HFT flash orders. If instead flash orders are not profitable for HFT firms – and therefore HFTs do not use flash orders to execute aggressive limit orders immediately, — then the market looks like the market without HFTs in that execution limit orders depends on the future arrival of later regular investors who are willing to take the other side of the limit order based on their personal gains-from-trade. As a result, the regular investor's objective function with HFTs is as follows. If an INV arrives at time t_z, he chooses his order $x_{t_z}^{INV}$ to maximize his expected payoff:

$$\max_{x_{t_z}^{INV} \in X_{t_z}^{INV}} w(x_{t_z}^{INV} | S, \tau, \Xi, \beta_{t_z}, L_{t_{z-1}})$$
$$= \begin{cases} [\beta_{t_z} - P(x_{t_z}^{INV}) - \xi(x_{t_z}^{INV})] \, Pr(\theta_{t_z}^{x_{t_z}^{INV}} | S, \tau, \Xi, L_{t_{z-1}}) & \text{if } x_{t_z}^{INV} \text{ is a buy order} \\ [P(x_{t_z}^{INV}) - \beta_{t_z} - \xi(x_{t_z}^{INV})] \; Pr(\theta_{t_z}^{x_{t_z}^{INV}} | S, \tau, \Xi, L_{t_{z-1}}) & \text{if } x_{t_z}^{INV} \text{ is a sell order} \\ 0 & \text{if } x_{t_z}^{INV} \text{ is } NT. \end{cases} \quad (2.8)$$

where now $Pr(\theta_{t_z}^{x_{t_z}^{HT}} | S, \xi, l_{t_{z-1}})$ reflects both the possibility of immediate execution of some limit orders by HFTs and possible future execution by regular INVs for other limit orders. Both HFT firms and INVs maximize their expected terminal payoff conditional on the support of the INVs evaluations, S, a set of fees, Ξ, chosen by the exchange, and the incoming state of the limit order book $L_{t_{z-1}}$. One further difference is that now limit orders are possible in equilibrium for the regular INV in the final trading date t_3 due to the possibility of execution by the HFTs.

Given the behavior of HFTs and regular investors, the exchange sets its access pricing to maximizes its expected payoff. Formally, this problem is the same as in (2.4). However, the presence of the HFTs potentially affects the behavior of the INV investors and the specific forms of the order-submission and order-execution probabilities.

A Subgame Perfect Nash Equilibrium consists of order-submission strategies $x_{t_z}^{HFT}$ and $x_{t_z}^{INV}$ that maximize expected profits for both the HFT firms and the INVs given the order-execution probabilties they induce and access fees Ξ that maximize the exchange's expected profit.

Theorem 2 *The equilibrium of an N-period model with HFTs exists and can be constructed using an analytic recursion.*

Figure 4 shows the equilibrium fees and rebates in the three-period LTM with HFTs for different INV valuation supports. Comparing these results with the previous 3-period model without HFTs (Figure 3), we see that, all else equal, the gray region characterized by an optimal pricing structure with rebates widens when HFTs are present in the market. With HFT firms present, the INV support consistent with the taker-maker or maker-taker pricing is $4\,\tau$, whereas in the 3-period protocol it was only $2.3\,\tau$. Figure 4 and Table 2.4 show that the exchange sets either the MF or the TF to attract HFT firms. Starting from the smallest support ($0.33\,\tau$), in the taker-maker regime the exchange offers a rebate on the TF slightly greater than half a tick so that the HFT firms have an incentive take liquidity at P_{-1} ($TF = -0.500^*$); in the maker-taker regime the exchange sets the TF just below half a tick ($TF = 0.499^*$) so that the HFT firms have an incentive to profitably take the limit order posted at P_1 by the INVs buying at t_1.

As the support of INVs widens and reaches $2\,\tau$, the equilibrium strategies of the liquidity suppliers arriving at t_1 does not change in the taker-maker region (LBP_{-1}) and the equilibrium MF reaches its maximun value (MF=1). The equilibrium strategies of the liquidity suppliers in the maker-taker region (LBP_1) instead changes: at t_1 the liquidity supplier no longer buys at P_1 but rather buys at P_{-1}. The reason why the equilibrium strategy of the buyer is no longer LBP_1 but rather LBP_{-1} is that the exchange exploits the now greater gains from trade and has an incentive to set the maximun MF for all market participants, and INVs anticipate that with a rebate on the TF slightly greater than half a tick ($TF = -0.500^*$), at t_2 the HFT firms will be willing to sell at P_{-1}.[17] Note that when the support becomes wider than $2\,\tau$, the exchange sets a unique taker-maker symmetric fee structure, thus inducing HFT firms to take liquidity at the inside quotes.

By widening the support even further, the fees only change when the equilibrium trading strategies also change, which is in correspondence of the support $4\,\tau$ when at t_1 the incoming buyers switch from buying either at P_{-1} or at P_1 to buying both at LBP_{-1} and at LBP_1 depending on their support. As the support reaches $4\,\tau$, the exchange finds it optimal to set $TF = 0.499^*$ to induce the HFT firms taking liquidity at P_1 (with a 0.001 profit per execution). The exchange sets $MF = 0.520$ so to make selling at P_{-1} also profitable for INVs, given their now wide support. Table 2.4 shows that beyond this threshold as the support widens, the exchange holds the TF constant at 0.499^* and gradually increases the MF to take advantage of the larger INVs' gains from trade. These results lead to our fourth empirical prediction:

Empirical Prediction 4: *Markets with HFT traders are more likely to have rebated-based access pricing.*

Our results explain the growing predominance rebate-based access pricing structures in U.S. markets since the advent of Reg NMS. In addition, they are consistent with the empirical findings of Cardella, Hao, and Kalcheva 2015 who use a three year data set (2008-2010) to show that most of the U.S. exchanges adopted after 2008 a rebate based fee structure starting right after the advent of Reg NMS.[18]

[17] Selling at 9.50 with a rebate larger than half a tick means selling net of TF at a price higher than 10.00, which allows HFTs to make some profits.

[18] Similar results hold also in the small tick STM market. In results available from the Authors upon request, we show that when, all else equal, the tick size is smaller the exchange has a a smaller incentive to offer a rebate-based fee structure.

Our results are reminiscent of the Foucault, Kadan, and Kandel 2013 findings that the fee breakdown matters when the tick size is positive. Holding the total fee constant, Foucault, Kadan, and Kandel 2013 show that when the gains-from-trade to market takers increase relative to the gains-from-trade to the market makers, the optimal trading fees become larger. Independently of the role played by HFT firms, our extension shows that exchange profits sharply increase when HFTs are active in the market and therefore exchanges set their fees to maximize the HFTs activity. As HFT firms can generate a greater amount of volume than INVs, exchanges prioritize HFT firms and set the fees to maximize HFTs volume. It is therefore not surprising that the region associated with a rebate on the take fee is larger than in the 3-period model without HFTs.

2.6.1 Results

We consider the large-tick market with the tick size equal to τ, and as before we present results for different values of the support, S. This allows us to show under which parameterization the new model collapses into the previous framework without the client-broker interaction and therefore allows us to use the principal-agent extension to better interpret our previous results.

We realistically set BC equal to $BC = 1.1$, which is greater than the tick size, and we set ORF equal to $ORF = 0.05$.[19]

We then start by solving the model for $BC = ORF = \gamma = 0$ and either $I^{PA} = 0$ or $I^{PA} = 1$. Under this parameterization, we show that the results obtained from the extended model are the same as those presented in Table 1 except for the $\tau < \Delta < 3\tau$ interval on the S^{PA} support and $I^{PA} = 0$. The intuition is straighforward: when $BC = ORF = \gamma = 0$ and $I^{PA} = 1$ the broker expected profits is zero as he neither get a commission, nor the has any reputational benefit, nor he can benefit from any rebate. When instead the broker does not pass the trading fee onto his client he can benefit from the rebate and gain positive profits. So when the support is large enough for the client to be able to trade without rebate, i.e., $\Delta > 1$, under the taker maker regime the broker will get positive profits from the rebate on the TF and will be willing to take liquidity at t_2; under the maker taker regime the broker will instead get positive profits from the rebate on the make fee and will be willing to supply liquidity at t_1.

Hence results from the PA model show that for most of the rebate based fee regime, and under the assumption that the broker does not pass trading fees onto his clients, with the Taker-Maker pricing the broker will take liquidity at t_2 in the capacity of an agent and in the Maker-Taker pricing he will provide liquidity at t_1. as the broker acts in the capacity of the client and the client does not enjoys rebates, either the probability of execution -in the Taker-Maker regime - or the probability of submission - in the Maker-Taker regime - will be smaller and the exchange will obtain smaller profits. At the same time, the broker will make less profits when acting as an Agent as opposed to acting as a principal.

[19] According to the Greenwich associates (2017) report, U.S. institutional equity high-touch bundled equity trade commission rates averaged 3.8 cents per share at the end of 2016; and electronic and algorithmic trading rates were 0.9 cents per share at the end of 2016.

FIGURE 4: **Pattern of Make Fees and Take Fees: 3-Period Model with High Frequency Traders (HFT)** This figure shows the equilibrium make fees (MF) and take fees (TF) in the Large Tick Market (LTM) corresponding to different investor valuation supports ranging from 0.33 τ to 5 τ (where τ is the tick size in the LTM) on the horizontal black axes. The left (right) part of the figure reports the equilibrium trading fees consistent with the taker-maker (maker-taker) fee structure. The figure reports in blue (orange) italic the equilibrium MF (TF) set by the exchange. Note that $-0.500^* = -0.5 - 1 \cdot 10^{-7}$ and $0.499^* = 0.5 - 1 \cdot 10^{-7}$.

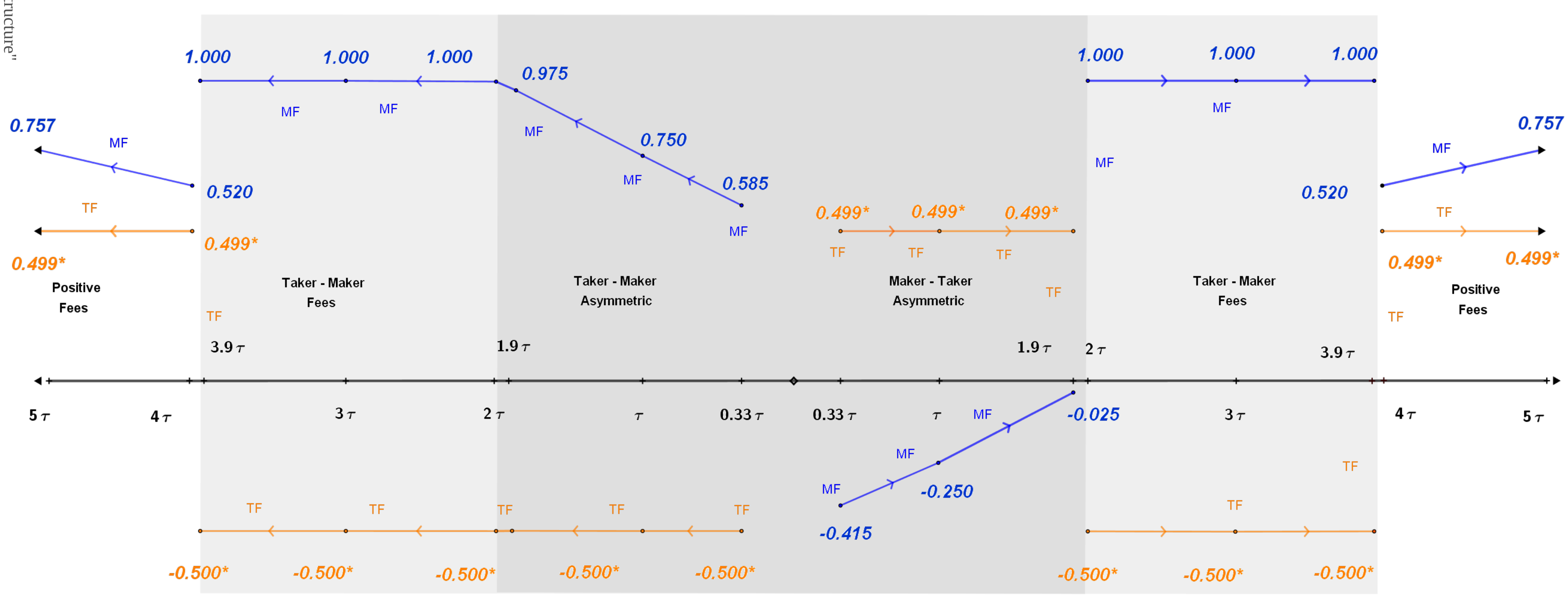


TABLE 2.4: **3-Period Large Tick Market with HFTs: Equilibrium Fees and Trading Strategies.** This table reports for different investor valuation support width $\Delta = \overline{\beta} - \underline{\beta}$ are expressed in terms of the LTM tick size τ (column 1), the extreme values of the support, $\underline{\beta}$ and $\bar{\beta}$ (column 2), the equilibrium make and take fee, MF and TF, (column 3 and 4), the equilibrium orders x_{t_1} at t_1, other than No Trade (column 5) and the equilibrium orders x_{t_2} at t_2 conditional on the trading strategy indicated at t_1 (column 6). The table also shows the associated probabilities of submission, $Pr(x_{t_1}|S,\tau,\Xi,L_{t_0})$ and $Pr(x_{t_2}|S,\tau,\Xi,L_{t_1})$ (column 7 and 8), as well the cum-fee buy and sell price, $P_k^{cum,LB}$ and $P_k^{cum,MS}$, (column 9 and 10), the probability of execution of the order posted at t_1, $Pr(\theta_{t_1}^{x_{t_1}}|S,\tau,\Xi,l_{t_0})$, (column 11) and the exchange expected profit $\pi(MF,TF|S,\tau)$ (column 12). When the equilibrium pricing is rebate-based, for each support we report first the taker-maker set of fees and then the maker-taker set of equilibrium MF and TF. We do not report the order-submission probabilities for HFT flash market orders (e.g. MSP_{-1}^{HFT}) after aggressive limit orders because in equilibrium they are always equal to 1. We only report the Pr. Execution of the limit order posted at t_1. Note that $-0.500^* = -0.5 - 1\cdot 10^{-7}$ and $0.499^* = 0.5 - 1\cdot 10^{-7}$.

Support width $\Delta = \overline{\beta} - \underline{\beta}$	$\underline{\beta}, \bar{\beta}$	MF	TF	Eq. Orders x_{t_z} t_1	t_2	Pr. Submission $Pr(x_{t_z}\|S,\tau,\Xi,L_{t_{z-1}})$ t_1	t_2	$P_k^{cum,LB}$	$P_k^{cum,MS}$	Pr. Execution $Pr(\theta_{t_1}^{x_{t_1}}\|S,\tau,\Xi,l_{t_0})$	Exchange E[Profit] $\pi(MF,TF\|S,\tau)$
0.33 τ	9.833, 10.167	0.585	-0.500*	LBP_{-1}, MSP_{-1}^{HFT}	LBP_{-1}, MSP_{-1}^{HFT}	0.245	0.245	10.085	10.000*	1	0.127
0.33 τ	9.833, 10.167	-0.415	0.499*	LBP_1, MSP_1^{HFT}	LBP_1, MSP_1^{HFT}	0.245	0.245	10.085	10.000*	1	0.127
τ	9.500, 10.500	0.750	-0.500*	LBP_{-1}, MSP_{-1}^{HFT}	LBP_{-1}, MSP_{-1}^{HFT}	0.250	0.250	10.250	10.000*	1	0.375
τ	9.500, 10.500	-0.250	0.499*	LBP_1, MSP_1^{HFT}	LBP_1, MSP_1^{HFT}	0.250	0.250	10.250	10.000*	1	0.375
1.9 τ	9.050, 10.950	0.975	-0.500*	LBP_{-1},MSP_{-1}^{HFT}	LBP_{-1}, MSP_{-1}^{HFT}	0.250	0.250	10.475	10.000*	1	0.712
1.9 τ	9.050, 10.950	-0.025	0.499*	LBP_1, MSP_1^{HFT}	LBP_1, MSP_1^{HFT}	0.250	0.250	10.475	10.000*	1	0.712
2 τ	9.000, 11.000	1.000	-0.500*	LBP_{-1}, MSP_{-1}^{HFT}	LBP_{-1}, MSP_{-1}^{HFT}	0.250	0.250	10.500	10.000*	1	0.750
3 τ	8.500, 11.500	1.000	-0.500*	LBP_{-1}, MSP_{-1}^{HFT}	LBP_{-1}, MSP_{-1}^{HFT}	0.333	0.333	10.500	10.000*	1	1.000
3.9 τ	8.050, 11.950	1.000	-0.500*	LBP_{-1}, MSP_{-1}^{HFT}	LBP_{-1}, MSP_{-1}^{HFT}	0.372	0.372	10.500	10.000*	1	1.115
4 τ	8.000, 12.000	0.520	0.499*	LBP_1, MSP_1^{HFT}	LBP_{-1}, LBP_1, MSP_1^{HFT}	0.130	0.333, 0.161	11.020, 10.020, 11.020	10.000*, 10.000*	1	1.167
				LBP_{-1}	LBP_1, MSP_{-1}, MSP_1^{HFT}, LSP_1	0.365	0.245, 0.168, 0.327	10.020, 11.020	9.000*, 9.000*, 9.980	0.315	
5 τ	7.500, 12.500	0.757	0.499*	LBP_1, MSP_1^{HFT}	LBP_{-1}, LBP_1, MSP_1^{HFT}	0.120	0.286, 0.162	11.257, 10.257, 11.257	10.000*, 10.000*	1	1.522
				LBP_{-1}	LBP_1, MSP_{-1}, MSP_1^{HFT}, LSP_1	0.328	0.259, 0.236, 0.212	10.257, 11.257	9.000*, 10.000*, 9.743	0.391	

Our first result shows that when commissions are next to zero the broker would have no incentive to pass the trading fees onto his client and at the same time, the exchange would get smaller profits. This result is consistent with Battalio, Corwin, and Jennings 2016 who show that brokers do not generally pass fees onto their institutional clients.

When the support is larger than 3τ the exchange does not offer a rebate based fee structure anymore and the absence of a broker commission makes the agent opt to trade in the capacity of principal, so the model results are the same as those from the previous model.

2.7 Welfare and Market Quality

Access pricing that maximizes exchange profits does not necessarily improve the overall welfare of other market participants. In this section we revisit the markets discussed in the previous sections and investigate how the different profit-maximizing access pricing for exchanges affect the welfare of other market participants. As a reference point for our welfare analysis, we compare equilibria with non-zero access pricing with a benchmark model with no fees or rebates. The solution for the benchmark model is analytic since it follows as a simplification of Theorem 1. Tables 2.13, 2.14, 2.15 and 2.16, 2.17, 2.18, in Appendices 2.12 and 2.13 provide specifics about how to operationalize the recursion described in Theorem 1. Tables 2.19 through 2.23 in Appendix 2.14 provide specific numerical details about welfare, market quality and other characteristics.

Figures 5 and 6 show our results about welfare. The figures show total welfare for all agents with and without optimal access pricing and also show the welfare breakdown for the various traders and the exchange. Our findings are consistent for all three model specifications: The exchange's profit-maximizing fees improve total welfare when they are small (for small valuation supports in the PIW regions); when they become larger (for larger valuation supports in the RW regions) they increase total welfare (but investors are worse off unless there are Pareto transfers from the exchange to investors); and when they are very large (in the DL regions), the optimal fees reduce total welfare relative to no access fees. For example, for the 3-period market with HFTs, optimal access pricing is Pareto improving relative to the zero-fee benchmark market with no transfers between the exchange and investors up until an investor valuation support of roughly 1.27τ. For larger valuation supports, total welfare with profit-maximizing access pricing still improves over the zero-fee market but transfers from the exchange to investors are required for investors to be better off. This region extends up until a support of roughly 3.9τ. Lastly, for still larger valuation supports, optimized access pricing actually worsens total welfare.

There are several other features to note here:

- In the 2-period market, the regions with Pareto improvement in welfare (PIW) and the region in which welfare is redistributed (RW) both become smaller in STM relative to LTM. This is intuitive, since the distortions associated with price grid discreteness are decreasing as the tick size shrinks. In particular, the PIW (RW) region extends to supports of $1.27\,\tau$ ($1.88\,\tau$) for the LTM but only to $0.42\,\tau$ ($0.63\,\tau$) for the STM.

- Going from a 2-period to a 3-period market, the Pareto improving and welfare redistribution regions get smaller and the rebates are smaller (i.e., $|MF|$ is smaller). Now the PIW

(RW) region only extends to supports of $1.2\,\tau$ ($1.62\,\tau$) for the 3-period LTM down from $1.27\,\tau$ ($1.88\,\tau$) for the 2-period LTM. This is consistent with the positive effect of higher trading activity on trade execution.

- With HFTs, the Pareto improving region increases somewhat and the redistributed welfare region of the parameter space becomes much larger. Now the PIW (RW) region extends to supports of $1.27\,\tau$ ($3.9\,\tau$) for the 3-period LTM with HFTs up from $1.2\,\tau$ ($1.62\,\tau$) without HFTs.

Two key intuitions underling our welfare results — and also the exchange's access pricing behavior more generally through the model analysis — are the roles of two different externalities. On the one hand, total welfare depends on the probability of transaction execution, whereas individual investors care about both the probability of order execution and also on price improvement on their personal payoff conditional on order-execution. Thus, in some (parametric) circumstances individual traders may submit orders with lower execution probabilities (which can reduce overall welfare) if their personal price improvement benefit dominates. However, since exchanges care about transaction execution, their access pricing can offset the individual investor price-improvement externality. This is the reason rebate-based access pricing improves overall welfare when investor valuation dispersion is small relative to the price tick size. On the other hand, there is also an externality in the exchanges behavior. In particular, exchanges care about both the transaction execution probability and also on the net fee they earn conditional on transaction execution. Thus, under other circumstances exchanges may set fees that reduce transaction execution probabilities (which reduces overall welfare) in order to increase the net fees they earn. The shaded areas reported in Figures 5 and 6 shows the DL region due to rebate-based pricing as opposed to positive pricing. The DL due to rebate-based pricing decreases when the tick size decreases and when the trading frequency increases, and it drops to zero when HFT are active in the market.

Welfare is not directly observable, but our model does provide predictions about observable measures of market quality. Tables 2.19 and 2.22 in the Appendix report measures of market quality (spread, volume and depth) and also measures of welfare for the 2-period and for the 3-period model with HFTs for a large tick size. For space reasons, the zero-fee benchmark results are reported separately in Table 2.23. The double vertical lines correspond to points where there are discontinuities in investor strategies and exchange access pricing. Analogous results for the 2-period STM and for the 3-period market without HFTs are in Tables 2.20 and 2.21 in the Appendix.

FIGURE 5: **Welfare: 2-period LTM and STM** This figure shows how the welfare of the Exchange (EXCH) - dashed line, Investors, (INV - dotted line) and Total Welfare (INV + EXCH - dashed-dotted line) change with the investors' support (S) in the large tick market (LTM) on the left and in the small tick market (STM) on the right. Both figures also report the welfare of investors under the Benchmark regime (solid line) with no trading fees (MF=TF=0). The support is expressed in large tick unit of measure (τ). Both figures show the results for three regions: Pareto Improvement Welfare (PIW), Redistribution Welfare (RW) and Deadweight Loss (DL).

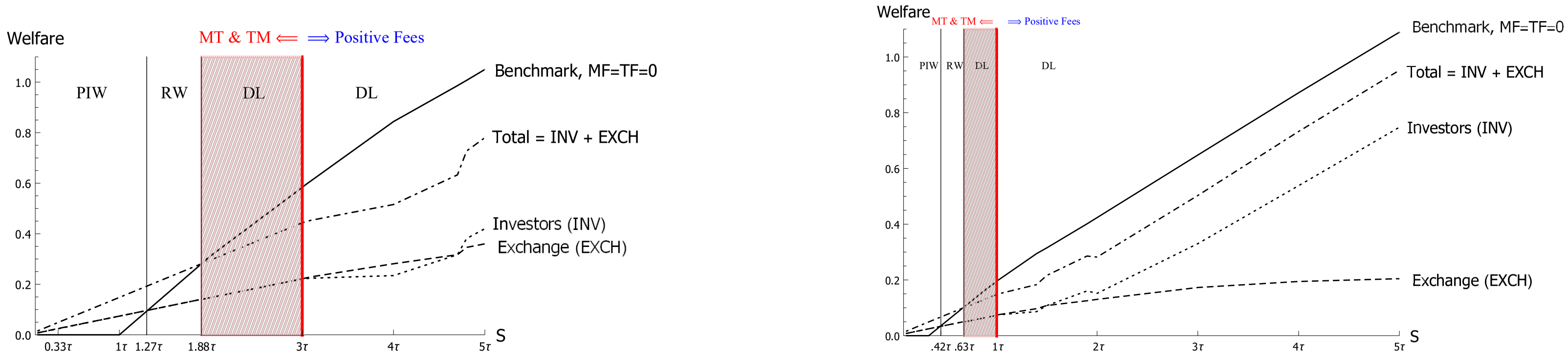
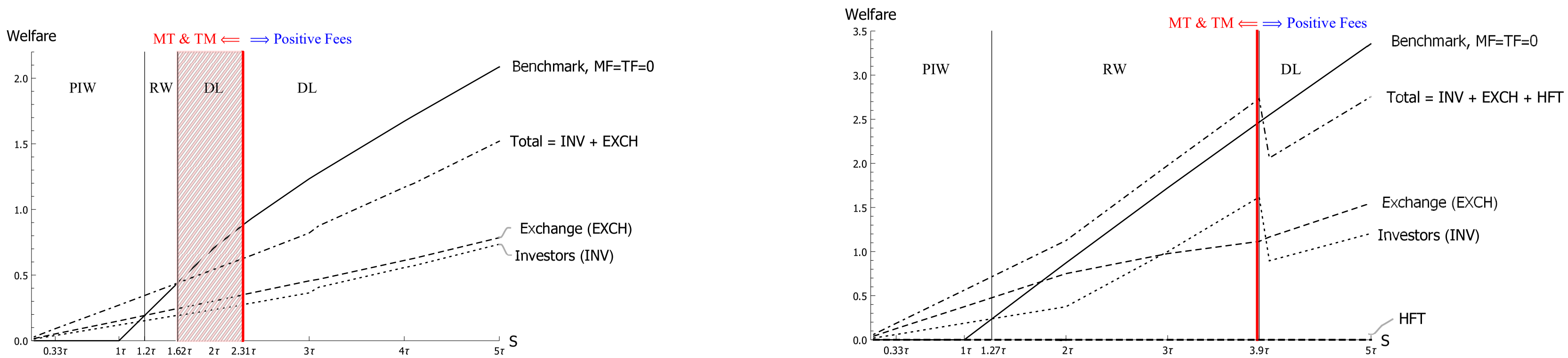


FIGURE 6: **Welfare: 3-period LTM and 3-period LTM with HFT** This figure shows on the left how the welfare of the Exchange (EXCH) - dashed line, Investors, (INV - dotted line) and Total Welfare (INV + EXCH - dashed-dotted line) change with the investors' support (S) in the large tick market (LTM); and on the right it shows how the welfare of the Exchange (EXCH) - dashed line, Investors and High Frequency Traders (INV&HFT - dotted line) and Total Welfare of Investors, HFTs and the Exchange (INV&HFT+EXCH - dashed-dotted line) change with the investors' support (S). Both figures also report the welfare of investors under the Benchmark regime (solid line) with no trading fees (MF=TF=0). The support is expressed in large tick unit measure (τ). Both figures show the results for three regions: Pareto Improvement Welfare (PIW), Redistribution Welfare (RW) and Deadweight Loss (DL). The shaded area indicates the DL region with rebate-based pricing.



2.8 Conclusions

This paper extends the existing theoretical models of optimal access pricing to allow for different populations of market participants, realistic regulatory restrictions, multiple periods, and HFT traders. Our model shows that investor valuation dispersion drives the exchanges' choice of the optimal trading fees. If the market is mainly populated by traders with valuations close to the current asset value, then in equilibrium the exchange chooses a rebate-based pricing structure. If the market is instead populated by traders with disperse valuations, then the exchange chooses jointly positive make and take fees.

Regulatory caps on access fees crucially affect the exchange choice of the optimal pricing. When there is no cap, the exchange chooses a rebate-based fee structure to maximize its total profits (volume times per-trade profit). To achieve the largest possible profits the exchange has to impose fees that induce market participants to trade only at the outside quotes. When exchange access fees are capped relative to the tick size, the exchange chooses a rebate-based pricing only when the support of the traders' evaluation is small and investors need to be subsidized to trade at the inside quotes. When traders instead have dispersed valuations, the exchange imposes positive fees on all market participants. Our model also shows how different tick size regimes affect the equilibrium pricing structure. Thus, the optimal access pricing structure depends on both the absolute tick size and the tick size relative to the dispersion. When the tick size is smaller, the region of the investors' support that is consistent with a rebate-based pricing is smaller and exchanges have less degree of freedom in setting the trading fees.

A natural question here is why the rebate-based fee structure became predominant over the positive fee structures after Reg NMS. Our answer is that technological innovations led to a sharp increase of high frequency trading. Within the context of our model, high frequency traders have valuations concentrated around the asset value. We therefore conclude that the fee structure that governs today's markets is crucially affected by the type of market participants with HFT firms driving the fee structure towards rebates-based pricing. In particular, while the observed increased in the rate of trading activity in market nowadays could be expected to induce exchanges to reduce rebate-based access pricing (based on our results for 2- and 3-period markets), it is precisely the increase in the presence of HFTs (in liquid stocks) that makes the exchange opt for rebate based fee pricings.

Importantly, we show that optimized rebate-based access pricing by exchanges can be Pareto improving, but that there is also a sizeable parameter region where rebates reduce welfare in the absence of transfer payments. In particular, our model shows the effects of different pricing structures on the welfare of different market participants. When the market is populated by investors with small gains-from-trade, the rebate-based pricing Pareto improves welfare, and hence resolves the frictions generated by discrete prices. However, in markets populated by investors with large gains-from-trade optimal access pricing generates deadweight losses. The frictions generated by discrete pricing are less severe when the tick size is smaller and therefore the positive effects of rebate-based pricing decrease when the tick size is smaller. Similarly, when trading activity increases, there is less need for the exchange to subsidise trading via rebate-based pricing and therefore the Pareto improvement in welfare generated by rebate-based pricing decreases. When the gains-from-trade increase beyond the threshold that guarantees a Pareto improvement in welfare rebate-based pricing generate a redistribution of welfare from investors to the exchange. This region decreases when the tick size decreases or the trading activity increases, but becomes overwhelming in presence of HFTs when the DL region generated by trading fees decreases substantially.

Our model is the first that includes more than two trading periods. This extension allows us to show that when the trading period is longer than two period, results may crucially change the reason being that the liquidity suppliers coming to the market in the first period are no longer monopolists of the liquidity provision. While in a 2-period model in the second period traders can only take the liquidity standing on the book or decide not to trade and leave the market, in a 3-period model he/she can also compete for the provision of liquidity. This happens in particular in the taker-maker regime when the liquidity suppliers arriving when the book is empty buy at low prices. The induced greater competition for the provision of liquidity affects the exchange optimal pricing that will try to induce the liquidity taker not to compete for the provision of liquidity, thus leading to optimal asymmetric fee structure.

Appendix

2.9 General proofs for N-period models

The proof strategies for our general N-period models are relatively standard for finite sequential games.

Proof for Theorem 1: The proof consists of three steps.

The recursion step for deriving analytic investor strategies is the following: Given access pricing fees Ξ, the order-execution probabilities $Pr(\theta_{t_z}^{x_{t_z}}|S,\tau,\Xi,L_{t_{z-1}})$ for computing the investor expected profit for each possible order $x_{t_z} \in X_{t_z}$ at any time t_z in the investor maximization problem (2.3) are either 1 for market orders at the BBO or are determined recursively for limit orders from the order-submission probabilities $Pr(x_{t_z}|S,\tau,\Xi,L_{t_{z-1}})$ at later dates. The upper envelope of the expected investor payoffs for the different possible actions at a generic time t_z determines the optimal investor actions at t_z and, given the distribution over the investor valuation β_{t_z} the associated order-submission probabilities for the optimal actions in terms of intervals on the investor valuation support S for any incoming book $L_{t_{z-1}}$. Given the assumptions of a discrete number of possible investor actions and discrete tine, the set of possible incoming books is finite.

The initiation step starts the recursion at the terminal period t_N, at which time the order-execution probabilities take a simple form: They are zero for new limit orders (since the game ends after time t_N) and one for market orders (which can only be submitted if the book is non-empty). Thus, investor expected profit for different orders, the upper envelope, the optimal orders, and the order-submission probabilities at time t_N can be derived directly.

The exchange profit optimization step is then as follows: The order-submission and order-execution probabilities from the first two steps can then be used to construct the exchange's expected profit in (2.4) analytically given arbitrary fees Ξ. Given the analytic exchange expected profit function, the profit-maximizing fees Ξ^* can then be found analytically since the set of possible fees and rebates is compact given the regulatory cap on access fees. QED

Proof of Theorem 2: The proof structure is the same as for Theorem 1 with the addition that INVs and HFTs investors arrive sequentially. First, the recursion step again involves characterizing analytic optimal order submissions and order-submission probabilities in term of intervals of valuations β_{t+z} along the support S associated with the analytic upper envelope of the payoffs of all of the possible investor actions. Again, there are a finite number of possible investor actions with linear payoff and a finite number of periods and, thus, at each point in time t_z, a finite set of possible prior histories L_{tz-1}. Analytic order-execution probabilties can then be computed from the analytic order-submission probabilities. Second, the initiation step at time N involves optimization with only market orders for which the a priori execution probability is one. Third, the exchange profit optimization step is logically similar to the same step in Theorem 1. QED

Comment: The following parts of this Appendix show how to derive the optimal trading strategies and the optimal MF and TF for both the 2-period large tick model (Appendix 2.10), and for the 3-period model (Appendix 2.12) and for the 3-period with HFTs model (Appendix 2.13). Table 2.5 shows explicitly the orders and payoffs available to investors in the LTM. They are similar in the SMT except for minor notation changes. For our proofs in the following Appendices the following Lemma 1 is relevant:

Lemma 1 *Investors with $\beta_{t_1} > v$ are potential buyers at time t_1 (i.e., they either submit limit buy orders or NT but they never submit limit sell orders). Similarly, investors with $\beta_{t_1} < v$ are potential sellers at time t_1.*

Proof of lemma 1: This result follows immediately from the fact that the investor expected profit functions from limit buy and sell orders are symmetric and increasing in the distance from the posted limit prices.

TABLE 2.5: **Trading Strategies and Payoffs** This table reports the trading strategies and associate payoffs available to investors in the LTM.

Action	Payoff
Market Order to Sell: MSP_{t_z}	$P(x_{t_z}) - \beta_{t_z} - TF$
Limit Order to Sell: LSP_{t_z}	$[P(x_{t_z}) - \beta_{t_z} - MF]\, Pr(\theta_{t_z}^{x_{t_z}} \vert S, \Xi, L_{t_{z-1}})$
No Trade: NT_{t_z}	0
Limit Order to Buy: LBP_{t_z}	$[\beta_{t_z} - P(x_{t_z}) - MF]\, Pr(\theta_{t_z}^{x_{t_z}} \vert S, \Xi, L_{t_{z-1}})$
Market Order to Buy: MBP_{t_z}	$\beta_{t_z} - P(x_{t_z}) - TF$

2.10 Equilibrium of 2-Period Model and Proofs of Propositions 1 and 2

The model is solved by backward induction. Thus, consider first the last round of trading, t_2. Investors arriving at t_2 either choose a market order or do not submit an order (NT) since new limit orders at t_2 have a zero execution probability. An investor at t_2 is willing to submit a market sell order MSP_{k,t_2} to hit a limit buy order at price P_k if his payoff $P_k^{cum,MS}(x_{t_2}) - \beta_{t_2} > 0$ is positive, where $P_k^{cum,MS}(x_{t_2}) = P_{k,MS}(x_{t_2}) - TF$ is the cum-fee market-order sell price for price P_k. Given that the investor's valuation β_{t_2} is drawn from $U[\underline{\beta}, \bar{\beta}]$, the submission probability of a market sell, x_{k,t_2}^{MS}, at t_2 is:[20]

$$Pr(x_{k,t_2}^{MS}|S, \Xi, L_{t_1}) = \max\{0, \min\{1, \frac{P_{k,MS}(x_{t_2}) - TF - \underline{\beta}}{\Delta}\}\} = Pr(\theta_{t_1}^{x_k^{LB}}|S, \Xi, L_{t_1}) \tag{2.9}$$

where the submission probability of a market sell order MSP_{k,t_2} at P_k at time t_2 is the execution probability $Pr(\theta_{t_1}^{x_k^{LB}}|S, \Xi, L_{t_1})$ of a limit buy order LBP_{k,t_1} posted at P_k at time t_1.[21] By symmetry, the submission probability of a market buy MBP_{-k,t_2} at t_2 given a cum-fee market order buy price $P_{-k}^{cum,MB}(x_{t_2}) = P_{-k,MB}(x_{t_2}) + TF$ is:

$$Pr(x_{-k,t_2}^{MB}|S, \Xi, L_{t_1}) = \max\{0, \min\{1, \frac{\bar{\beta} - P_{-k,MB}(x_{t_2}) - TF}{\Delta}\}\} = Pr(\theta_{t_1}^{x_{-k}^{LS}}|S, \Xi, L_{t_1}), \tag{2.10}$$

which is the execution probability $Pr(\theta_{t_1}^{x_{-k}^{LS}}|S, \Xi, L_{t_1})$ of a limit sell order, LSP_{-k,t_1}.

Next, consider the initial time t_1 in the 2-period market. The limit order book opens empty, and so the investor arriving at t_1 chooses between submitting limit orders and submitting no order (NT). From Lemma 1, an investor with $\beta_{t_1} > v$ is a potential buyer who only submits limit buy orders or NT. This investor optimally posts a limit buy LBP_{k,t_1} at a price P_k if two conditions hold: First, the expected payoff from LBP_{k,t_1} given a private valuation β_{t_1} is positive

$$(\beta_{t_1} - P_k^{cum.LB}(x_{t_1})) \times Pr(\theta_{t_1}^{x_k^{LB}}|S, \Xi, L_{t_1}) > 0 \tag{2.11}$$

and, second, it is greater than the expected payoff from any other alternative limit order $LBP_{\sim k,t_1}$

$$(\beta_{t_1} - P_k^{cum,LB}(x_{t_1})) \times Pr(\theta_{t_1}^{x_k^{LB}}|S, \Xi, L_{t_1}) > (\beta_{t_1} - P_{\sim k}^{cum,LB}(x_{t_1})) \times Pr(\theta_{t_1}^{x_{\sim k}^{LB}}|S, \Xi, L_{t_1}) \tag{2.12}$$

where $\sim k$ indexes other possible limit price $P_{\sim k,t_1}$, and where $P_k^{cum,LB}(x_{t_2}) = P_{k,LB}(x_{t_2}) + MF$ and $P_{\sim k}^{cum,LB}(x_{t_2}) = P_{\sim k,LB}(x_{t_2}) + MF$ are the associated cum-fee limit buy prices. Hence, the probability of submission of LBP_{k,t_1} at t_1 is the probability that conditions (2.11) and (2.12) are

[20] We extended our previous notation so that, for example, x_{k,t_2}^{MS} and MSP_{k,t_2} are used interchangeably for a market sell order at P_k at t_2. When possible, we simplify the notation to make it consistent with the notation used in the figures.

[21] The book opens empty at t_1 and therefore at t_2 the only possible order a seller can take is the one posted by the buyer at t_1

both satisfied:

$$\begin{aligned} &Pr(x^{LB}_{k,t_1}|S,\Xi,L_{t_0}) = \\ &= Pr\Big[(\beta_{t_1} - P^{cum,LB}_{k}(x_{t_1})) \times Pr(\theta^{x^{LB}_{k}}_{t_1}|S,\Xi,L_{t_1}) > 0, \\ &\quad (\beta_{t_1} - P^{cum,LB}_{k}(x_{t_1})) \times Pr(\theta^{x^{LB}_{k}}_{t_1}|S,\Xi,L_{t_1}) > (\beta_{t_1} - P^{cum,LB}_{\sim k}(x_{t_1})) \times Pr(\theta^{x^{LB}_{\sim k}}_{t_1}|S,\Xi,L_{t_1})\Big] \end{aligned} \tag{2.13}$$

By symmetry, a potential seller at t_1 with $\beta_{t_1} < v$ submits a limit sell LSP_{-k,t_1} if the analogous conditions hold:

$$(P^{cum.LS}_{-k}(x_{t_1}) - \beta_{t_1}) \times Pr(\theta^{x^{LS}_{-k}}_{t_1}|S,\Xi,L_{t_1}) > 0 \tag{2.14}$$

and

$$(P^{cum,LS}_{-k}(x_{t_1}) - \beta_{t_1}) \times Pr(\theta^{x^{LS}_{-k}}_{t_1}|S,\Xi,L_{t_1}) > (P^{cum,LS}_{\sim -k}(x_{t_1}) - \beta_{t_1}) \times Pr(\theta^{x^{LS}_{\sim -k}}_{t_1}|S,\Xi,L_{t_1}) \tag{2.15}$$

where $P^{cum,LS}_{-k}(x_{t_2}) = P_{-k,LS}(x_{t_2}) + MF$ and $P^{cum,LS}_{\sim -k}(x_{t_2}) = P_{\sim -k,LS}(x_{t_2}) + MF$ are the cum-fee limit sell prices.

Thus, the probability of submission of LSP_{-k,t_1} at t_1 is the probability that conditions (2.14) and (2.15) both hold:

$$\begin{aligned} &Pr(x^{LS}_{-k,t_1}|S,\Xi,L_{t_0}) = \\ &= Pr\Big[(P^{cum.LS}_{-k}(x_{t_1}) - \beta_{t_1}) \times Pr(\theta^{x^{LS}_{-k}}_{t_1}|S,\Xi,L_{t_1}) > 0, \\ &\quad (P^{cum,LS}_{-k}(x_{t_1}) - \beta_{t_1}) \times Pr(\theta^{x^{LS}_{-k}}_{t_1}|S,\Xi,L_{t_1}) > (P^{cum,LS}_{\sim -k}(x_{t_1}) - \beta_{t_1}) \times Pr(\theta^{x^{LS}_{\sim -k}}_{t_1}|S,\Xi,L_{t_1})\Big] \end{aligned} \tag{2.16}$$

We normalize the tick size to $\tau = 1$, and let the investor valuation support, $[\underline{\beta}, \bar{\beta}]$, vary within the outside LTM quotes so that $P_{-3} \leq \underline{\beta}$ and $\bar{\beta} \leq P_3$. Let $\Delta = \bar{\beta} - \underline{\beta} \leq 5\tau$ denote the support width. The equilibrium MF and TF for $P_{-3} \leq \underline{\beta} < \bar{\beta} \leq P_3$ are then derived in four parametric cases respectively for support widths $0 < \Delta \leq 3\tau$ (case 1), $3\tau < \Delta \leq 4\tau$ (case 2), $4\tau < \Delta \leq 4.7\tau$ (case 3), and $4.7\tau < \Delta \leq 5\tau$ (case 4).

Case 1: $0 < \Delta \leq 3\tau$

The exchange sets its access pricing MF and TF to maximize its expected profit. These fees and rebates can take one of three possible alternative forms: Taker-Maker, $\Xi_{TM} = \{0 \leq MF \leq 1, -1 \leq TF \leq 0\}$; Maker-Taker, $\Xi_{MT} = \{-1 \leq MF \leq 0, 0 \leq TF \leq 1\}$; and Positive-Fee, $\Xi_{PF} = \{0 \leq MF \leq 1, 0 \leq TF \leq 1\}$. We now show that the exchange optimization problem when $\Delta \leq 3\tau$ results in the following functional forms for the equilibrium MF and TF in the Taker-Maker regime

$$MF^* = \frac{\Delta + 3}{6} \qquad TF^* = \frac{\Delta - 3}{6} \tag{2.17}$$

and under the Maker-Taker regime

$$MF^* = \frac{\Delta - 3}{6} \qquad TF^* = \frac{\Delta + 3}{6} \tag{2.18}$$

Taker-Maker: $\Xi_{TM} = \{0 \leq MF \leq 1, -1 \leq TF \leq 0\}$

We consider first Taker-Maker pricing Ξ_{TM} with a take rebate and a positive make fee. Given $\Delta < 3$, the lower investor valuation limit in this case is $\underline{\beta} = P_{-2} + \frac{3-\Delta}{2}$, and the upper investor valuation is $\overline{\beta} = P_2 - \frac{3-\Delta}{2}$, as illustrated in Figures 7 and 8. Consider first a potential buyer arriving at t_1 with $\beta_{t_1} > v$. The logic for a potential seller arriving at t_1 is symmetric.

Order-submission probabilities for each possible market order at t_2 can be computed using (2.9) and (**??**) given the valuation-support restriction $\Delta \leq 3$ and Taker-Maker pricing. Columns 4 and 5 in Table 2.6 report the market order submission probabilities for the price levels in Column 1:

$$Pr(x^{MS}_{k,t_2}|S, \Xi, L_{t_1}) = \max\{0, \frac{P_{k,MS}(x_{t_2}) - TF - \underline{\beta}}{\Delta}\} = \max\{0, \frac{\Delta}{2} + \frac{P_k - P_{-k}}{2} - TF\} \tag{2.19}$$

$$Pr(x^{MB}_{-k,t_2}|S, \Xi, L_{t_1}) = \max\{0, \frac{\bar{\beta} - P_{-k,MB}(x_{t_2}) - TF}{\Delta}\} = \max\{0, \frac{\Delta}{2} - \frac{P_k - P_{-k}}{2} - TF\} \tag{2.20}$$

For example, Rows 3 in Column 4 and Row 4 in Column 3 in Table 2.6 gives the order-submission probability at t_2 of a market sell at P_{-1}, which is equal to the order-submission probability of a market buy at P_1

$$\begin{aligned} Pr(x^{MS}_{-1,t_2}|S, \Xi, L_{t_1}) = \max\{0, \tfrac{P^{cum,MS}_{-1}(x_{t_2}) - \underline{\beta}}{\Delta}\} = \\ Pr(x^{MB}_{1,t_2}|S, \Xi, L_{t_1}) = \max\{0, \tfrac{\bar{\beta} - P^{cum,MB}_{1}(x_{t_2})}{\Delta}\} = \max\{0, \tfrac{1}{\Delta}[\tfrac{\Delta}{2} - \tfrac{1}{2} - TF]\}. \end{aligned} \tag{2.21}$$

To understand the intuition in the last term in (2.21), note from Figure 7 that only traders with a β_{t_2} in the interval $[\underline{\beta}, P^{cum,MS}_{-1}(x_{t_2})]$ with width $\frac{\Delta}{2} - \frac{1}{2} - TF$ are willing to use a market order to sell at a posted price P_{-1}. This interval is equal to half of the support minus half the tick size, hence $\frac{1}{2}$, given $\tau = 1$, which is the distance from the fundamental asset value v to P_{-1}, minus TF (negative in the Taker-Maker regime), which increases the interval of the support including βs belonging to sellers. This interval is strictly positive for $\Delta \geq 1$, which means that $Pr(x^{MS}_{-1,t_2}|S, \Xi, L_{t_1}) > 0$ for $\Delta \geq 1$.

The market order submission probabilities at t_2 are, in turn, respectively the corresponding order-execution probabilities of limit orders posted at t_1. Thus, we can now consider the expected profits for all possible limit orders that a potential buyer and symmetrically a potential seller can post at t_1. We verify the conditions under which (2.11) and (2.12) hold — and symmetrically (2.14) and (2.15) — and finally compute the limit order submission probabilities at t_1 consistent with both (2.13) and (2.16) .

To check that conditions (2.11) and (2.14) hold, we compute $Pr(\beta_{t_1} > P^{cum,LB}_{k}(x_{t_1}))$ and $Pr(P^{cum,LS}_{-k}(x_{t_1}) > \beta_{t_1})$ for each order in Column 2 of Table 2.6. For example, for a limit order to buy at P_{-1} and to sell at P_1 we have:

$$Pr(\beta_{t_1} > P_{-1}^{cum,LB}(x_{t_1})) = \max\{0, \frac{\bar{\beta} > P_{-1}^{cum,LB}(x_{t_1})}{\Delta}\} = \quad (2.22)$$
$$Pr(P_1^{cum,LS}(x_{t_1}) > \beta_{t_1}) = \max\{0, \frac{P_1^{cum,LS}(x_{t_1}) > \underline{\beta}}{\Delta}\} = \max\{0, \frac{1}{\Delta}[\frac{\Delta}{2} + \frac{1}{2} - MF]\}.$$

To understand the intuition for the final term in (2.23), notice, for example, from Figure 7 that only traders with a β_{t_1} in the interval $[P_{-1}^{cum,LB}(x_{t_1}), \overline{\beta}_{t_1}]$ with width $\frac{\Delta}{2} + \frac{1}{2} - MF$ will be willing to buy at the quoted price P_{-1}. This interval is equal to half of the investor valuation support (consistent with Lemma 1 only traders with a personal evaluation larger than the fundamental value of the asset will be buying) plus half the tick size (the distance between the mid-point of the support/fundamental asset value v and P_{-1}) and which now increases the interval of the support including β buyers - minus MF, which instead decreases the interval of the support including βs belonging to buyers.

We next need to check whether both conditions (2.12) and (2.15) hold for each possible order at t_1:

- First, consider a limit buy at P_2 and symmetrically a limit order to sell at P_{-2}. Given the assumed investor valuation support with width $\Delta < 3$ and given the positive MF with Taker-Maker pricing, the expected payoff associated with limit orders at P_2 (P_{-2}) would be negative since the associated cum-fee buy (sell) price would be above (below) the maximum (minimum) possible trader valuation. Hence, such limit orders would never be submitted.

- Second, consider a limit buy at P_{-2} or limit sell at P_2. For these orders, the expected profit is positive:

$$(\beta_{t_1} - P_{-2}^{cum,LB}(x_{t_1})) \times Pr(\theta_{t_1}^{x_{-2}^{LB}}|S,\Xi,L_{t_1}) = (P_2^{cum,LS}(x_{t_1}) - \beta_{t_1}) \times Pr(\theta_{t_1}^{x_2^{LS}}|S,\Xi,L_{t_1}) = \quad (2.23)$$
$$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} + \frac{3}{2} - MF]\right\} \max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} - \frac{3}{2} - TF]\right\} > 0.$$

- Third, the expected profit for a limit buy at P_{-1} or a limit sell at P_1 is:

$$(\beta_{t_1} - P_{-1}^{cum,LB}(x_{t_1})) \times Pr(\theta_{t_1}^{x_{-1}^{LB}}|S,\Xi,L_{t_1}) = (P_1^{cum,LS}(x_{t_1}) - \beta_{t_1}) \times Pr(\theta_{t_1}^{x_1^{LS}}|S,\Xi,L_{t_1}) \quad (2.24)$$
$$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} + \frac{1}{2} - MF]\right\} \max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} - \frac{1}{2} - TF]\right\},$$

  which is higher than from limit buys at P_{-2} and limit sells at P_2, since the following difference is negative:

$$(\beta_{t_1} - P_{-2}^{cum,LB}(x_{t_1})) \times Pr(\theta_{t_1}^{x_{-2}^{LB}}|S,\Xi,L_{t_1}) - (\beta_{t_1} - P_{-1}^{cum,LB}(x_{t_1})) \times Pr(\theta_{t_1}^{x_{-1}^{LB}}|S,\Xi,L_{t_1}) = \quad (2.25)$$
$$= (P_2^{cum,LS}(x_{t_1}) - \beta_{t_1}) \times Pr(\theta_{t_1}^{x_2^{LS}}|S,\Xi,L_{t_1}) - (P_1^{cum,LS}(x_{t_1}) - \beta_{t_1}) \times Pr(\theta_{t_1}^{x_1^{LS}}|S,\Xi,L_{t_1}) =$$
$$= \frac{\text{MF} - \text{TF} - 2}{\Delta^2} < 0.$$

- Lastly, the expected profit from a limit buy at P_1 and limit sell at P_{-1} is positive:

$$(\beta_{t_1} - P_{1,LB}^{cum,LB}(x_{t_1})) \times Pr(\theta_{t_1}^{x_1^{LB}}|S,\Xi,L_{t_1}) = (P_{-1}^{cum,LS}(x_{t_1}) - \beta_{t_1}) \times Pr(\theta_{t_1}^{x_{-1}^{LS}}|S,\Xi,L_{t_1}) \tag{2.26}$$
$$\max\left\{0, \tfrac{1}{\Delta}[\tfrac{\Delta}{2} - \tfrac{1}{2} - MF]\right\} \max\left\{0, \tfrac{1}{\Delta}[\tfrac{\Delta}{2} + \tfrac{1}{2} - TF]\right\},$$

which is lower than the expected payoff from limit buys at P_{-1} or limit. sells at P_1, since the following difference is negative given Taker-Maker pricing:

$$\begin{aligned}(\beta_{t_1} - P_1^{cum,LB}(x_{t_1})) \times Pr(\theta_{t_1}^{x_1^{LB}}|S,\Xi,L_{t_1}) - \\ Pr(\beta_{t_1} - P_{-1}^{cum,LB}(x_{t_1})) \times Pr(\theta_{t_1}^{x_{-1}^{LB}}|S,\Xi,L_{t_1}) = \\ = (P_{-1}^{cum,LS}(x_{t_1}) - \beta_{t_1}) \times Pr(\theta_{t_1}^{x_{-1}^{LS}}|S,\Xi,L_{t_1}) - \\ (P_1^{cum,LS}(x_{t_1}) - \beta_{t_1}) \times Pr(\theta_{t_1}^{x_1^{LS}}|S,\Xi,L_{t_1}) = \frac{TF - MF}{\Delta^2} < 0\end{aligned} \tag{2.27}$$

Thus, we have shown that limit buys at P_{-1} and limit sells at P_1 are the optimal order submissions at t_1 in the $\Delta \leq 3\tau$ case with Taker-Maker pricing. In particular, we have shown that thelimit orders LBP_{-1} and LBP_1 have positive expected payoffs for the ranges in Table 8 and that they dominate all alternative orders.

To determine its optimal MF and TF, the exchange maximizes its exchange profit given the optimal strategy for potential buyers and sellers posting limit orders LBP_{-1,t_1} and LSP_{1,t_1} at t_1, which we have derived as a function of the trading fees, MF and TF, and of the investors' support, S.[22] In particular, by symmetry, the exchange's expected profit is equal to the submission probability $Pr(x_{-1,t_1}^{LB}|S,\Xi,L_{t_0})$ of LBP_{-1,t_1}, times the associated execution probability $Pr(\theta_{t_1}^{x_{-1}^{LB}}|S,\Xi,L_{t_1})$, times the per share net fee, MF+TF. Table 2.7 reports the equlibrium order submission probabilities.

$$Pr\Big[(\beta_{t_1} - P_{-1}^{cum,LB}(x_{t_1})) \times Pr(\theta_{t_1}^{x_{-1}^{LB}}|S,\Xi,L_{t_1}) > (\beta_{t_1} - P_{\sim k}^{cum,LB}(x_{t_1})) \times Pr(\theta_{t_1}^{x_{\sim k}^{LB}}|S,\Xi,L_{t_1})|S,\Xi,L_{t_1})\Big] = 1.$$

It follows that:

$$Pr(x_{-1,t_1}^{LB}|S,\Xi,L_{t_0}) = Pr[(\beta_{t_1} - P_{-1}^{cum,LB}(x_{t_1}))] = \max\left\{0, \tfrac{1}{\Delta}[\tfrac{\Delta}{2} + \tfrac{1}{2} - MF]\right\}$$

and the maximization problem of the exchange is:

[22] The case of a seller posting LSP_{1,t_1} is symmetric. As in real markets, traders arrive sequentially and, hence, either a buyer or seller may arrive at t_1.

$$\max_{MF,TF\in\Xi} \pi^{LTM}(MF,TF|S,x^{LB}_{-1,t_1},L_{t_0},x^{MS}_{-1,t_2},L_{t_1}) = Pr(x^{LB}_{-1,t_1}|S,\Xi,L_{t_0})Pr(\theta^{x^{LB}_{-1}}_{t_1}|S,\Xi,L_{t_1})(MF+TF)$$
$$= \max\left\{0,\frac{1}{\Delta}[\frac{\Delta}{2}+\frac{1}{2}-MF]\right\}(MF+TF) \quad (2.28)$$
$$\max\left\{0,\frac{1}{\Delta}[\frac{\Delta}{2}-\frac{1}{2}-TF]\right\}$$
$$s.t.:\; 0<MF<1, -1<TF<0$$
$$s.t.:\; MF+TF>0 \quad (2.29)$$
$$s.t.:\; 0<\Delta\leq 3$$

From the first-order conditions, we obtain:

$$MF^* = \frac{\Delta+3}{6} \qquad TF^* = \frac{\Delta-3}{6} \quad (2.30)$$

Computing the second and mixed derivatives, as well as the determinant, we obtain

$$\delta_{TF,TF} = -\frac{-2\text{MF}+\Delta+1}{\Delta^2} < 0 \quad (2.31)$$
$$\delta_{MF,MF} = -\frac{\Delta-2\text{TF}-1}{\Delta^2} < 0 \quad (2.32)$$
$$\delta_{MF,TF} = \frac{2\text{MF}-\Delta+2\text{TF}}{\Delta^2} \quad (2.33)$$
$$Det = (-1-4MF^2+2MF(1+\Delta-2TF)+2(-1+\Delta-2TF)TF)/\Delta^4 \quad (2.34)$$

By substituting the equilibrium fees from (2.30) into (2.34) we obtain: $Det(MF^*,TF^*) = \frac{1}{3\Delta^2} > 0$.

By substituting the desired value of Δ into MF^* and TF^* in (2.30), we obtain the equilibrium Taker-Maker fees presented in Table 2.1. QED

Maker-Taker: $\Xi_{MT} = \{-1 \leq MF \leq 0, 0 \leq TF \leq 1\}$

Now consider Maker-Taker pricing, Ξ_{MT}, with a make rebate and a positive take fee, as illustrated in Figure 8. Once again, we determine the optimal strategies for arriving investors at times t_1 and t_2 and the associated order-submission probabilities:

- First, given a positive take fee TF and an investor valuation support with a width $\Delta < 3$, the expected profit on limit buys at P_{-2} and limit sells at P_2 at t_1 is zero there will be no sellers (buyers) at t_2 willing to sell (buy) at a cum-fee price smaller (higher) than P_{-2} (P_2). Thus, such limit orders are not used in this case.
- Second, the expected profit for a limit buy at P_2 or limit sells at P_{-2} is positive

$$(\beta_{t_1} - P^{cum,LB}_2(x_{t_1})) \times Pr(\theta^{x^{LB}_2}_{t_1}|S,\Xi,L_{t_1}) = (P^{cum,LS}_{-2}(x_{t_1})-\beta_{t_1}) \times Pr(\theta^{x^{LS}_{-2}}_{t_1}|S,\Xi,L_{t_1}) = \quad (2.35)$$
$$\max\left\{0,\tfrac{1}{\Delta}[\tfrac{\Delta}{2}-\tfrac{3}{2}-MF]\right\}\max\left\{0,\tfrac{1}{\Delta}[\tfrac{\Delta}{2}+\tfrac{3}{2}-TF]\right\} > 0.$$

- Third, the expected profit from a limit buy and limit sell at P_{-1} is higher:

$$Pr(\beta_{t_1} - P_1^{cum,LB}(x_{t_1})) \times Pr(\theta_{t_1}^{x_1^{LB}}|S,\Xi,L_{t_1}) = Pr(P_{-1}^{cum,LS}(x_{t_1}) - \beta_{t_1}) \times Pr(\theta_{t_1}^{x_{-1}^{LS}}|S,\Xi,L_{t_1}) = \max\left\{0, \tfrac{1}{\Delta}[\tfrac{\Delta}{2} - \tfrac{1}{2} - MF]\right\} \max\left\{0, \tfrac{1}{\Delta}[\tfrac{\Delta}{2} + \tfrac{1}{2} - TF]\right\}, \quad (2.36)$$

since the following difference, given $\mathrm{TF} - \mathrm{MF} < 2$ with Maker-Taker pricing, is negative:

$$Pr(\beta_{t_1} - P_2^{cum,LB}(x_{t_1})) \times Pr(\theta_{t_1}^{x_2^{LB}}|S,\Xi,L_{t_1}) - Pr(\beta_{t_1} - P_1^{cum,LB}(x_{t_1})) \times Pr(\theta_{t_1}^{x_1^{LB}}|S,\Xi,L_{t_1}) =$$
$$= Pr(P_{-2}^{cum,LS}(x_{t_1}) - \beta_{t_1}) \times Pr(\theta_{t_1}^{x_{-2}^{LS}}|S,\Xi,L_{t_1}) - Pr(P_{-1}^{cum,LS}(x_{t_1}) - \beta_{t_1}) \times Pr(\theta_{t_1}^{x_{-1}^{LS}}|S,\Xi,L_{t_1}) = \frac{\mathrm{TF} - \mathrm{MF} - 2}{\Delta^2} < 0 \quad (2.37)$$

- Lastly, if the expected profit from a limit buy at P_{-1} or limit sell at P_1 is positive and equal to:

$$Pr(\beta_{t_1} - P_{-1}^{cum,LB}(x_{t_1})) \times Pr(\theta_{t_1}^{x_{-1}^{LB}}|S,\Xi,L_{t_1}) = Pr(P_1^{cum,LS}(x_{t_1}) - \beta_{t_1}) \times Pr(\theta_{t_1}^{x_1^{LS}}|S,\Xi,L_{t_1}) = \max\left\{0, \tfrac{1}{\Delta}[\tfrac{\Delta}{2} - \tfrac{1}{2} - MF]\right\} \max\left\{0, \tfrac{1}{\Delta}[\tfrac{\Delta}{2} + \tfrac{1}{2} - TF]\right\}, \quad (2.38)$$

which is lower than the expected payoff from a limit buy at P_1 or limit sell at P_{-1}, since the following difference is negative given Maker-Taker pricing with $\Xi_{MT} = \{-1 \leq MF \leq 0, 0 \leq TF \leq 1\}$:

$$Pr(\beta_{t_1} - P_{-1}^{cum,LB}(x_{t_1})) \times Pr(\theta_{t_1}^{x_{-1}^{LB}}|S,\Xi,L_{t_1}) - Pr(\beta_{t_1} - P_1^{cum,LB}(x_{t_1})) \times Pr(\theta_{t_1}^{x_1^{LB}}|S,\Xi,L_{t_1}) =$$
$$= Pr(P_1^{cum,LS}(x_{t_1}) - \beta_{t_1}) \times Pr(\theta_{t_1}^{x_1^{LS}}|S,\Xi,L_{t_1}) - Pr(P_{-1}^{cum,LS}(x_{t_1}) - \beta_{t_1}) \times Pr(\theta_{t_1}^{x_{-1}^{LS}}|S,\Xi,L_{t_1}) \quad (2.39)$$
$$= \frac{MF - TF}{\Delta^2} < 0$$

Thus, under the Maker-Taker regime the exchange will set the fees such that an investor arriving at t_1 will optimally choose either LBP_{1,t_1} or LSP_{-1,t_1}.

As for the Taker-Maker regime, the exchange anticipates that the optimal order submission strategy for the buyer (seller) is to buy at P_1 (sell at P_{-1}) and to determine the optimal fees we maximize the exchange profits conditional on the buyer now choosing LBP_{1,t_1}, the case of the seller arriving at t_1 being symmetric:

$$\begin{aligned}
\max_{MF,TF \in \Xi} \pi_{0<\Delta\leq 3\tau}^{LTM}(MF,TF|S,x_{1,t_1}^{LB},L_{t_0},x_{1,t_2}^{MS},L_{t_1}) &= Pr(x_{1,t_1}^{LB}|S,\Xi,L_{t_0}) \times Pr(\theta_{t_1}^{x_1^{LB}}|S,\Xi,L_{t_1}) \times (MF+TF) \\
&= \max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} - \frac{1}{2} - MF]\right\}(MF+TF) \times \\
\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} + \frac{1}{2} - TF]\right\} & \\
s.t.: \ -1 < MF < 0, 0 < TF < 1 & \\
s.t.: \ MF + TF > 0 &
\end{aligned}$$

From the first-order conditions, we obtain:

$$MF^* = \frac{\Delta - 3}{6} \qquad TF^* = \frac{\Delta + 3}{6} \tag{2.40}$$

Computing the second and mixed derivatives, as well as the determinant, we obtain

$$\delta_{TF,TF} = -\frac{-2\text{MF} + \Delta - 1}{\Delta^2} < 0 \tag{2.41}$$

$$\delta_{MF,MF} = -\frac{\Delta - 2\text{TF} + 1}{\Delta^2} < 0 \tag{2.42}$$

$$\delta_{MF,TF} = \frac{2\text{MF} - \Delta + 2\text{TF}}{\Delta^2} \tag{2.43}$$

$$Det = (-1 - 4MF^2 + 2MF(-1 + \Delta - 2TF) + 2(1 + \Delta - 2TF)TF)/\Delta^4 \tag{2.44}$$

By substituting the equilibrium fees from (2.40) into (2.44) we obtain: $Det(MF^*, TF^*) = \frac{1}{3\Delta^2} > 0$.

By substituting the value of Δ into MF^* and TF^* in (2.40), we obtain the equilibrium Maker-Taker fees presented in Table 2.1. QED

Interestingly, Table 2.1 shows that when the exchange opts for a Taker-Maker (or Maker-Taker) pricing Proposition (3) holds in equilibrium:

$$Pr(x^{LB}_{-1,t_1}|S, \Xi, L_{t_0}) = Pr(\theta^{x^{LB}_{-1}}_{t_1}|S, \Xi, L_{t_1}) = \frac{(MF + TF)}{\Delta} = \frac{1}{3} \tag{2.45}$$

and

$$Pr(x^{LB}_{1,t_1}|S, \Xi, L_{t_0}) = Pr(\theta^{x^{LB}_{1}}_{t_1}|S, \Xi, L_{t_1}) = \frac{(MF + TF)}{\Delta} = \frac{1}{3} \tag{2.46}$$

As Figure 7 (and 8) shows, to maximize expected profits the exchange has to maximize the product of 3 components, $\bar{\beta} - P^{cum,LB}_{-1}$, (MF+TF), $P^{cum,MS}_{-1} - \underline{\beta}$ (and , $\bar{\beta} - P^{cum,LB}_{1}$,(MF+TF), $P^{cum,MS}_{1} - \underline{\beta}$), and the sum of these three components are constrained to be equal to Δ.

TABLE 2.6: **Submission and Execution Probability.** This table reports the price levels on the LTM price grid (column 1) and the associated probabilities $Pr(\beta_{t_1} > P_k^{cum,LB}(x_{t_1})) = \max\{0, \frac{\bar{\beta}-P_k^{cum,LB}(x_{t_2})}{\Delta}\}$ and $Pr(P_k^{cum,LS}(x_{t_1}) > \beta_{t_1}) = \max\{0, \frac{P_{-k}^{cum,LS}(x_{t_2})-\underline{\beta}}{\Delta}\}$, which, in equilibrium, correspond to the submission probabilities for limit orders posted at P_k at t_1 (columns 2 and 3). In addition, the table reports the associated limit order execution probabilities, $Pr(\theta_{t_1}^{x_k^{LB}}|S,\Xi,L_{t_1}) = Pr(x_{k,t_2}^{MS}|S,\Xi,L_{t_1}) = \max\{0, \frac{P_k^{cum,MS}(x_{t_2})-\underline{\beta}}{\Delta}\}$ and $Pr(\theta_{t_1}^{x_{-k}^{LS}}|S,\Xi,L_{t_1}) = Pr(x_{-k,t_2}^{MB}|S,\Xi,L_{t_1}) = \max\{0, \frac{\bar{\beta}-P_{-k}^{cum,MB}(x_{t_2})}{\Delta}\}$ (columns 4 and 5).

P_k	$Pr(\beta_{t_1} > P_k^{cum,LB}(x_{t_1}))$	$Pr(P_{-k}^{cum,LS}(x_{t_1}) > \beta_{t_1})$	$Pr(\theta_{t_1}^{x_k^{LB}}\|S,\Xi,L_{t_1})$	$Pr(\theta_{t_1}^{x_{-k}^{LS}}\|S,\Xi,L_{t_1})$
P_{-3}	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} + \frac{5}{2} - MF]\right\}$	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} - \frac{5}{2} - MF]\right\}$	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} - \frac{5}{2} - TF]\right\}$	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} + \frac{5}{2} - TF]\right\}$
P_{-2}	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} + \frac{3}{2} - MF]\right\}$	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} - \frac{3}{2} - MF]\right\}$	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} - \frac{3}{2} - TF]\right\}$	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} + \frac{3}{2} - TF]\right\}$
P_{-1}	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} + \frac{1}{2} - MF]\right\}$	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} - \frac{1}{2} - MF]\right\}$	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} - \frac{1}{2} - TF]\right\}$	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} + \frac{1}{2} - TF]\right\}$
P_1	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} - \frac{1}{2} - MF]\right\}$	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} + \frac{1}{2} - MF]\right\}$	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} + \frac{1}{2} - TF]\right\}$	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} - \frac{1}{2} - TF]\right\}$
P_2	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} - \frac{3}{2} - MF]\right\}$	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} + \frac{3}{2} - MF]\right\}$	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} + \frac{3}{2} - TF]\right\}$	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} - \frac{3}{2} - TF]\right\}$
P_3	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} - \frac{5}{2} - MF]\right\}$	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} + \frac{5}{2} - MF]\right\}$	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} + \frac{5}{2} - TF]\right\}$	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} - \frac{5}{2} - TF]\right\}$

TABLE 2.7: **Equilibrium Submission Probability** This table reports the equilibrium submission probabilities for the buy side, $Pr(x_{k,t_1}^{LB}|S,\Xi,L_{t_0})$, conditional on the size of the support Δ. Equilibrium submission probabilities for the sell side, $Pr(x_{-k,t_1}^{LS}|S,\Xi,L_{t_0})$ are symmetric.

	$0 < \Delta \leq 4\tau$		$4 < \Delta \leq 4.7\tau$	$4.7 < \Delta \leq 5\tau$
	Taker-Maker	Maker-Taker	Positive Fees	Positive Fees
$Pr(x_{1,t_1}^{LB}\|S,\Xi,L_{t_0})$		$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} - \frac{1}{2} - MF]\right\}$		$\max\left\{0, \frac{1}{\Delta}[TF - MF]\right\}$ for $\beta > \frac{\Delta}{2} + 9.5$
$Pr(x_{-1,t_1}^{LB}\|S,\Xi,L_{t_0})$	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} + \frac{1}{2} - MF]\right\}$		$\max\left\{0, \frac{1}{\Delta}[TF + 1]\right\}$ for $\beta > \text{MF} + \frac{\Delta}{2} - \text{TF} + 8$	$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} + \frac{1}{2} - TF]\right\}$ for $\text{MF} + 9.5 < \beta < \text{MF} + \frac{\Delta}{2} + 9$
$Pr(x_{-2,t_1}^{LB}\|S,\Xi,L_{t_0})$			$\max\left\{0, \frac{1}{\Delta}[\frac{\Delta}{2} + MF - TF - 2]\right\}$ for $10 < \beta < \text{MF} + \frac{\Delta}{2} - \text{TF} + 8$	

TABLE 2.8: **Difference in expected payoff from different orders.** This table reports the difference in the expected payoffs from different orders indicated in column 1. Column 2 reports such differences as a function of Δ, whereas columns 3 to 6 reports the same differences for different values of Δ.

	Δ	$\Delta = 1$	$\Delta = 2$	$\Delta = 3$	$\Delta = 4$
$Pr(\beta_{t_1} - P_2^{cum,LB}(x_{t_1})) \times Pr(\theta_{t_1}^{x_2^{LB}}\lvert S,\Xi,L_{t_1}) - Pr(\beta_{t_1} - P_1^{cum,LB}(x_{t_1})) \times Pr(\theta_{t_1}^{x_1^{LB}}\lvert S,\Xi,L_{t_1})$ $Pr(P_1^{cum,LS}(x_{t_1}) - \beta_{t_1}) \times Pr(\theta_{t_1}^{x_1^{LS}}\lvert S,\Xi,L_{t_1}) - Pr(P_2^{cum,LS}(x_{t_1}) - \beta_{t_1}) \times Pr(\theta_{t_1}^{x_2^{LS}}\lvert S,\Xi,L_{t_1})$	$\frac{4\text{MF}-3S+2\text{TF}+5}{2S^2}$	$2\text{MF} + \text{TF} + 1$	$\frac{1}{8}(4\text{MF} + 2\text{TF} - 1)$	$\frac{1}{9}(2\text{MF} + \text{TF} - 2)$	$\frac{1}{32}(4\text{MF} + 2\text{TF} - 7)$
$Pr(\beta_{t_1} - P_2^{cum,LB}(x_{t_1})) \times Pr(\theta_{t_1}^{x_2^{LB}}\lvert S,\Xi,L_{t_1}) - Pr(\beta_{t_1} - P_{-1}^{cum,LB}(x_{t_1})) \times Pr(\theta_{t_1}^{x_{-1}^{LB}}\lvert S,\Xi,L_{t_1})$ $Pr(P_{-1}^{cum,LS}(x_{t_1}) - \beta_{t_1}) \times Pr(\theta_{t_1}^{x_{-1}^{LS}}\lvert S,\Xi,L_{t_1}) - Pr(P_2^{cum,LS}(x_{t_1}) - \beta_{t_1}) \times Pr(\theta_{t_1}^{x_2^{LS}}\lvert S,\Xi,L_{t_1})$	$\frac{2\text{MF}-3S+4\text{TF}+5}{2S^2}$	$\text{MF} + 2\text{TF} + 1$	$\frac{1}{8}(2\text{MF} + 4\text{TF} - 1)$	$\frac{1}{9}(\text{MF} + 2\text{TF} - 2)$	$\frac{1}{32}(2\text{MF} + 4\text{TF} - 7)$
$Pr(\beta_{t_1} - P_2^{cum,LB}(x_{t_1})) \times Pr(\theta_{t_1}^{x_2^{LB}}\lvert S,\Xi,L_{t_1}) - Pr(\beta_{t_1} - P_{-2}^{cum,LB}(x_{t_1})) \times Pr(\theta_{t_1}^{x_{-2}^{LB}}\lvert S,\Xi,L_{t_1})$ $Pr(P_{-2}^{cum,LS}(x_{t_1}) - \beta_{t_1}) \times Pr(\theta_{t_1}^{x_{-2}^{LS}}\lvert S,\Xi,L_{t_1}) - Pr(P_2^{cum,LS}(x_{t_1}) - \beta_{t_1}) \times Pr(\theta_{t_1}^{x_2^{LS}}\lvert S,\Xi,L_{t_1})$	$\frac{-3S+6\text{TF}+9}{2S^2}$	$3(\text{TF} + 1)$	$\frac{3}{8}(2\text{TF} + 1)$	$\frac{\text{TF}}{3}$	$\frac{3}{32}(2\text{TF} - 1)$
$Pr(\beta_{t_1} - P_1^{cum,LB}(x_{t_1})) \times Pr(\theta_{t_1}^{x_1^{LB}}\lvert S,\Xi,L_{t_1}) - Pr(\beta_{t_1} - P_{-1}^{cum,LB}(x_{t_1})) \times Pr(\theta_{t_1}^{x_{-1}^{LB}}\lvert S,\Xi,L_{t_1})$ $Pr(P_{-1}^{cum,LS}(x_{t_1}) - \beta_{t_1}) \times Pr(\theta_{t_1}^{x_{-1}^{LS}}\lvert S,\Xi,L_{t_1}) - Pr(P_1^{cum,LS}(x_{t_1}) - \beta_{t_1}) \times Pr(\theta_{t_1}^{x_1^{LS}}\lvert S,\Xi,L_{t_1})$	$\frac{\text{TF}-\text{MF}}{S^2}$	$\text{TF} - \text{MF}$	$\frac{\text{TF}-\text{MF}}{4}$	$\frac{\text{TF}-\text{MF}}{9}$	$\frac{\text{TF}-\text{MF}}{16}$
$Pr(\beta_{t_1} - P_1^{cum,LB}(x_{t_1})) \times Pr(\theta_{t_1}^{x_1^{LB}}\lvert S,\Xi,L_{t_1}) - Pr(\beta_{t_1} - P_{-2}^{cum,LB}(x_{t_1})) \times Pr(\theta_{t_1}^{x_{-2}^{LB}}\lvert S,\Xi,L_{t_1})$ $Pr(P_{-2}^{cum,LS}(x_{t_1}) - \beta_{t_1}) \times Pr(\theta_{t_1}^{x_{-2}^{LS}}\lvert S,\Xi,L_{t_1}) - Pr(P_1^{cum,LS}(x_{t_1}) - \beta_{t_1}) \times Pr(\theta_{t_1}^{x_1^{LS}}\lvert S,\Xi,L_{t_1})$	$\frac{-2\text{MF}+2\text{TF}+2}{S^2}$	$-2\text{MF} + 2\text{TF} + 2$	$\frac{1}{2}(-\text{MF} + \text{TF} + 1)$	$-\frac{2}{9}(\text{MF} - \text{TF} - 1)$	$\frac{1}{8}(-\text{MF} + \text{TF} + 1)$
$Pr(\beta_{t_1} - P_{-1}^{cum,LB}(x_{t_1})) \times Pr(\theta_{t_1}^{x_{-1}^{LB}}\lvert S,\Xi,L_{t_1}) - Pr(\beta_{t_1} - P_{-2}^{cum,LB}(x_{t_1})) \times Pr(\theta_{t_1}^{x_{-2}^{LB}}\lvert S,\Xi,L_{t_1})$ $Pr(P_{-2}^{cum,LS}(x_{t_1}) - \beta_{t_1}) \times Pr(\theta_{t_1}^{x_{-2}^{LS}}\lvert S,\Xi,L_{t_1}) - Pr(P_{-1}^{cum,LS}(x_{t_1}) - \beta_{t_1}) \times Pr(\theta_{t_1}^{x_{-1}^{LS}}\lvert S,\Xi,L_{t_1})$	$\frac{-\text{MF}+\text{TF}+2}{S^2}$	$-\text{MF} + \text{TF} + 2$	$\frac{1}{4}(-\text{MF} + \text{TF} + 2)$	$\frac{1}{9}(-\text{MF} + \text{TF} + 2)$	$\frac{1}{16}(-\text{MF} + \text{TF} + 2)$

FIGURE 7: **Taker-Maker Pricing:** $\Xi_{TM} = \{0 \leq MF \leq 1, -1 \leq TF \leq 0\}$ This Figure provides a graphical representation of how to obtain the equilibrium probabilities of order submission and execution for the Taker-Maker pricing structure and the support $\Delta \in [\underline{\beta}, \bar{\beta}]$. P_2 and P_{-2} are the outside quotes of the LTM, whereas P_1 and P_{-1} are the inside quotes of the LTM. $P_{-1}^{cum,LB}$ and $P_{-1}^{cum,MS}$ are the cum-fee buy and sell prices, respectively. LBP_{-1,t_1} is a limit buy order posted at P_{-1} at t_1, and MSP_{-1,t_2} is a market sell order posted at P_{-1} at t_2.

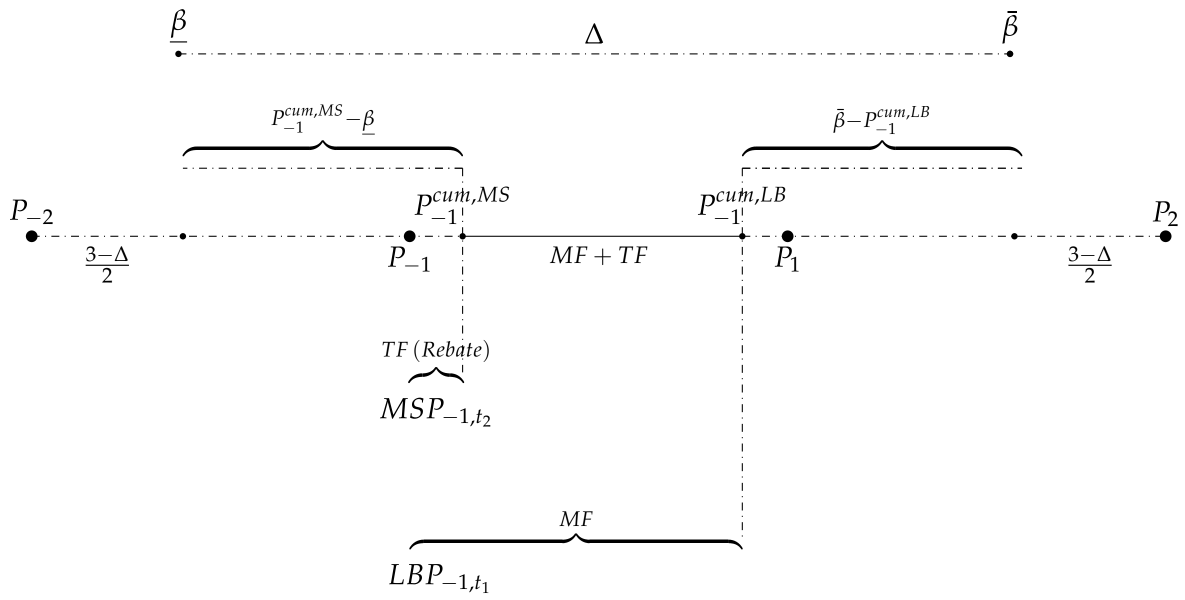
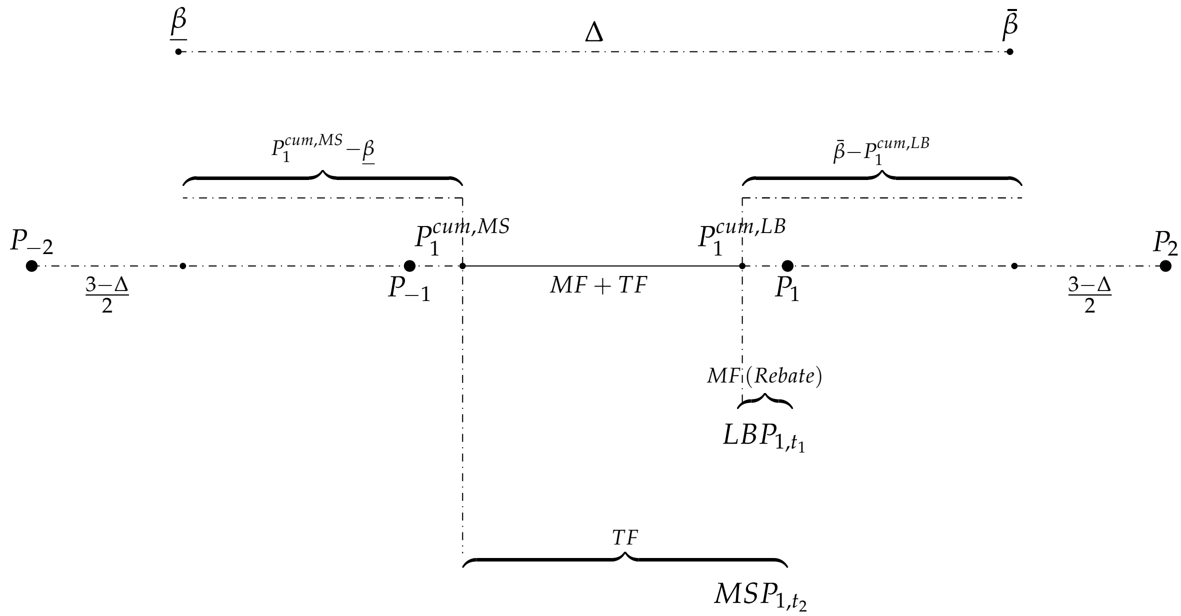


FIGURE 8: **Maker-Taker Pricing:** $\Xi_{MT} = \{-1 \leq MF \leq 0, 0 \leq TF \leq 1\}$ This Figure provides a graphical representation of how to obtain the equilibrium probabilities of order submission and execution for the Maker-Taker pricing structure and the support $\Delta \in [\underline{\beta}, \bar{\beta}]$. P_2 and P_{-2} are the outside quotes of the LTM, whereas P_1 and P_{-1} are the inside quotes of the LTM. $P_1^{cum,LB}$ and $P_1^{cum,MS}$ are the cum-fee buy and sell prices, respectively. LBP_{1,t_1} is a limit buy order posted at P_1 at t_1, and MSP_{1,t_2} is a market sell order posted P_1 at t_2.



Positive Fees: $\Xi_{PF} = \{0 \leq MF \leq 1, 0 \leq TF \leq 1\}$

Consider now the possibility of access pricing with strictly positive fees, Ξ_{PF}. The expressions in Table 2.6 (which imply zero probabilities when they are negative) show that under this pricing, the trader's expected profit is zero if he buys at P_2, LBP_{2,t_1}, or sells at P_{-2}, LSP_{-2,t_1}, as no buyers (sellers) would be willing to buy (sell) at a price net of fee higher (lower) than P_2 (P_{-2}).

The trader's expected profit would also be zero if he buys at P_{-2}, LBP_{-2,t_1}, or sells at P_2, LSP_{2,t_1}, as no sellers (buyers) would be willing to market sell (market buy) at t_2 at a price net of fee lower (higher) than P_{-2} (P_2), being $P_{-2} \leq \Delta \leq P_2$.

Table 2.6 shows that the trader's expected profit at t_1 would be positive if he buys either at P_{-1} or at P_1 (or sells at either P_1 or P_{-1}); and, considering equations (2.27) and (2.39), the difference in the expected profit would depend on the relative size of the MF and TF (Table 2.8). However, equation (2.27) shows that the trader would secure higher profits from the equilibrium strategy if the exchange set Ξ_{TM} rather than Ξ_{PF} when either a buyer buys at P_{-1} or a seller sells at P_1 at t_1; similarly equation (2.39) shows that the trader would get more profits from the equilibrium strategy if the exchange set Ξ_{MF} rather than Ξ_{PF} when either a buyer buys at P_1 or a seller sells at P_{-1} at t_1. We can therefore conclude that Ξ_{PF} is suboptimal when $P_{-2} \leq \underline{\beta} < \bar{\beta} \leq P_2$. QED

Case 2: $3\tau < \Delta \leq 4\tau$

Given that $3\tau < \Delta \leq 4\tau$, traders can choose among the same orders considered in Case 1. While the symmetry between buy and sell orders still applies, i.e., $Pr(\theta^{x^{LB}_{i,t_1}}|S, \Xi, L_{t_0}) = Pr(\theta^{x^{MS}_{\sim i,t_1}}|S, \Xi, L_{t_0})$, now $\bar{\beta} > P_2$ and $\underline{\beta} < P_{-2}$, and therefore buying at P_2 as well as selling at P_{-2} can be profitable if $Pr(x^{MS}_{-2,t_2}|S, \Xi, L_{t_1}) > 0$, i.e., if $TF < \frac{S-3}{2}$ (Table 2.6). However, Table 2.8 shows that a limit order to buy at P_2 (sell at P_{-2}) are dominated strategies.

Hence, to determine the optimal MF and TF, we maximize the exchange profits conditional on the buyer choosing LBP_{-2,t_1}, or $x^{LB}_{-1,t_1} = LBP_{-1,t_1}$ the case of the seller arriving at t_1 being symmetric:

$$\begin{aligned}
&\max_{MF,TF \in \Xi} \pi^{LTM}_{3\tau<\Delta\leq 4\tau}(MF, TF\,|S, x^{LB}_{-1,t_1}, L_{t_0}, x^{MS}_{-1,t_2}, L_{t_1}) = \qquad (2.47)\\
&= \left(Pr(x^{LB}_{-1,t_1}|S, \Xi, L_{t_0}) \times Pr(\theta^{x^{LB}_{-1}}_{t_1}|S, \Xi, L_{t_1})\right) \times (MF + TF) =\\
&= \frac{0.5(\text{TF} + 1.)(\text{MF} + \text{TF})(\Delta - 2.\text{TF} - 1.)}{\Delta^2} =\\
&s.t.: \;\; TF < \frac{\Delta - 3}{2}\\
&s.t.: \;\; -1 < MF < 1, -1 < TF < 1\\
&s.t.: \;\; MF + TF > 0\\
&s.t.: \;\; 3 < \Delta \leq 4
\end{aligned}$$

The Kuhn-Tucker Lagrangian is:

$$L(MF, TF, \Delta, \lambda_k, v_h) = \\ \pi^{LTM}_{3\tau<\Delta\leq 4\tau}(MF, TF|S, x^{LB}_{1,t_1}, L_{t_0}, x^{MS}_{1,t_2}, L_{t_1}) - \\ \lambda_1(-TF + \frac{\Delta-3}{2}) - \lambda_2(-MF+1) - \lambda_3(-\Delta+3.1) - \lambda_4(-\Delta+4) + v_1 MF + v_2 TF \tag{2.48}$$

The Kuhn-Tucker conditions are:

$$\frac{\delta\pi^{LTM}_{3\tau<\Delta\leq 4\tau}}{\delta MF} = \frac{0.5(\mathrm{TF}+1.)(\Delta-2.\mathrm{TF}-1.)}{\Delta^2} \geq 0 \ \& \ MF \times \frac{\delta\pi^{LTM}_{3\tau<\Delta\leq 4\tau}}{\delta MF} = 0 \tag{2.49}$$

$$\frac{\delta\pi^{LTM}_{3\tau<\Delta\leq 4\tau}}{\delta TF} = \frac{\mathrm{MF}(0.5\Delta-2.\mathrm{TF}-1.5)+\Delta(1.\mathrm{TF}+0.5)+(-3.\mathrm{TF}-3.)\mathrm{TF}-0.5}{\Delta^2} \geq 0 \& TF \times \frac{\delta\pi^{LTM}_{3\tau<\Delta\leq 4\tau}}{\delta TF} = 0 \tag{2.50}$$

$$\frac{\delta\pi^{LTM}_{3\tau<\Delta\leq 4\tau}}{\delta\Delta} = \frac{(\mathrm{MF}+\mathrm{TF})(\Delta(-0.5\mathrm{TF}-0.5)+\mathrm{TF}(2.\mathrm{TF}+3.)+1.)}{\Delta^3} \geq 0 \ \& \ \Delta \times \frac{\delta\pi^{LTM}_{3\tau<\Delta\leq 4\tau}}{\delta\Delta} = 0 \tag{2.51}$$

$$\frac{\delta L(MF, TF, \Delta, \lambda_k, v_h)}{\delta\lambda_1} = (-TF + \frac{\Delta-3}{2}) \geq 0 \ \& \ \lambda_1 \times \frac{\delta L(MF, TF, \Delta, \lambda_k, v_h)}{\delta\lambda_1} = 0 \tag{2.52}$$

$$\frac{\delta L(MF, TF, \Delta, \lambda_k, v_h)}{\delta\lambda_2} = (-MF+1) \geq 0 \ \& \ \lambda_2 \times \frac{\delta L(MF, TF, \Delta, \lambda_k, v_h)}{\delta\lambda_2} = 0 \tag{2.53}$$

$$\frac{\delta L(MF, TF, \Delta, \lambda_k, v_h)}{\delta\lambda_3} = (-\Delta+3.1) \geq 0 \ \& \ \lambda_3 \times \frac{\delta L(MF, TF, \Delta, \lambda_k, v_h)}{\delta\lambda_3} = 0 \tag{2.54}$$

$$\frac{\delta L(MF, TF, \Delta, \lambda_k, v_h)}{\delta\lambda_4} = (-\Delta+4) \geq 0 \ \& \ \lambda_4 \times \frac{\delta L(MF, TF, \Delta, \lambda_k, v_h)}{\delta\lambda_4} = 0 \tag{2.55}$$

The equilibrium MF^* and TF^* that satisfy these conditions are:

$$MF^* = 1 \qquad TF^* = 0.5(\Delta - 3) \tag{2.56}$$

By substituting a given value of Δ into MF^* and TF^* in (2.56), we obtain the equilibrium fees in Table 2.1.

Case 3: $4\tau < S \leq 4.7\tau$

We have shown that for investor valuation supports with widths up $\Delta = 4$, there are dominant orders for potential buyers and sellers, and so the optimal order-submission strategy can be obtained by comparing the expected payoff associated with each possible order, as shown in Tables 2.6 and 2.8; in the latter we present as an example the differences in expected payoffs conditional on different support sizes. However, for investor valuation supports with widths $\Delta > 4$, there are two possible equilibrium limit orders, and we report the outcome of (2.13) and (2.16) in Table 2.7, which shows that both a limit order at P_{-1} and at P_{-2} are sometimes optimal depending on β_{t_1}. We also report conditions on the value of β such that the equilibrium strategies hold.

To determine the optimal MF and TF, the exchange maximizes its expected profit conditional on the buyer choosing either LBP_{-2,t_1}, or LBP_{-1,t_1} the case of the seller arriving at t_1

being symmetric:

$$\max_{MF,TF\in\Xi} \pi^{LTM}_{4\tau<\Delta\leq 4.7\tau}(MF,TF\,|S,x^{LB}_{-1,t_1},x^{LB}_{-2,t_1},L_{t_0},x^{MS}_{-1,t_2},x^{MS}_{-2,t_2},L_{t_1}) = \tag{2.57}$$

$$= \Big(Pr(x^{LB}_{-1,t_1}|S,\Xi,L_{t_0}) \times Pr(\theta^{x^{LB}_{-1}}_{t_1}|S,\Xi,L_{t_1}) + Pr(x^{LB}_{-2,t_1}|S,\Xi,L_{t_0}) \times Pr(\theta^{x^{LB}_{-2}}_{t_1}|S,\Xi,L_{t_1})\Big)\times(MF+TF)$$

$$= \frac{(MF+TF)(-2.5+\Delta(1.25-0.25\Delta+0.5TF)-2TF+MF(1.5-0.5\Delta+TF))}{\Delta^2}$$

$$s.t.:\quad TF < \frac{\Delta-3}{2}$$

$$s.t.:\quad -1 < MF < 1, -1 < TF < 1$$

$$s.t.:\quad MF+TF > 0$$

$$s.t.:\quad 4 < \Delta \leq 4.7$$

The Kuhn-Tucker Lagrangian is:

$$L(MF,TF,\Delta,\lambda_k,v_h) = \tag{2.58}$$
$$\pi^{LTM}_{4\tau<\Delta\leq 4.7\tau}(MF,TF\,|S,x^{B}_{1,t_1},L_{t_0},x^{S}_{1,t_2},L_{t_1})+$$
$$\lambda_1(-TF+\frac{\Delta-3}{2})-\lambda_2(-MF+1)-\lambda_3(-\Delta+4)-\lambda_4(-\Delta+4.7)+v_1MF+v_2TF$$

The Kuhn-Tucker conditions are:

$$\frac{\delta\pi^{LTM}_{4\tau<\Delta\leq 4.7\tau}}{\delta MF} = \frac{\text{TF}(0.5\ -\text{TF})+\text{MF}(\Delta-2\text{TF}-3)+(0.25\Delta-1.25)\Delta+2.5}{\Delta^2} \geq 0 \ \&\ MF\times\frac{\delta\pi^{LTM}_{4\tau<\Delta\leq 4.7\tau}}{\delta MF} = 0 \tag{2.59}$$

$$\frac{\delta\pi^{LTM}_{4\tau<\Delta\leq 4.7\tau}}{\delta TF} = \frac{\text{MF}(0.5\ -2\text{TF})-\text{MF}^2+\Delta(0.25\Delta-\text{TF}-1.25)+4\text{TF}+2.5}{\Delta^2} \geq 0 \ \&\ TF\times\frac{\delta\pi^{LTM}_{4\tau<\Delta\leq 4.7\tau}}{\delta TF} = 0 \tag{2.60}$$

$$\frac{\delta\pi^{LTM}_{4\tau<\Delta\leq 4.7\tau}}{\delta\Delta} = \frac{\text{MF}^2(-0.5\Delta+2\text{TF}+3)+\text{MF}(1.25\Delta+\text{TF}(2\text{TF}-1)-5)+(0.5\Delta-4)\text{TF}^2+(1.25\Delta-5)\text{TF}}{\Delta^3} \geq 0$$
$$\&\ \Delta\times\frac{\delta\pi^{LTM}_{4\tau<\Delta\leq 4.7\tau}}{\delta\Delta} = 0 \tag{2.61}$$

$$\frac{\delta L(MF,TF,\Delta,\lambda_k,v_h)}{\delta\lambda_1} = (-TF+\frac{\Delta-3}{2}) \geq 0 \ \&\ \lambda_1\times\frac{\delta L(MF,TF,\Delta,\lambda_k,v_h)}{\delta\lambda_1} = 0 \tag{2.62}$$

$$\frac{\delta L(MF,TF,\Delta,\lambda_k,v_h)}{\delta\lambda_2} = (-MF+1) \geq 0 \ \&\ \lambda_2\times\frac{\delta L(MF,TF,\Delta,\lambda_k,v_h)}{\delta\lambda_2} = 0 \tag{2.63}$$

$$\frac{\delta L(MF,TF,\Delta,\lambda_k,v_h)}{\delta\lambda_3} = (-\Delta+4) \geq 0 \ \&\ \lambda_3\times\frac{\delta L(MF,TF,\Delta,\lambda_k,v_h)}{\delta\lambda_3} = 0 \tag{2.64}$$

$$\frac{\delta L(MF,TF,\Delta,\lambda_k,v_h)}{\delta\lambda_4} = (-\Delta+4.7) \geq 0 \ \&\ \lambda_4\times\frac{\delta L(MF,TF,\Delta,\lambda_k,v_h)}{\delta\lambda_4} = 0 \tag{2.65}$$

The equilibrium MF^* and TF^* that satisfy these conditions are:

$$MF^* = 1 \qquad TF^* = \frac{0.25\left(\Delta^2 - 5\Delta + 8\right)}{\Delta - 2} \tag{2.66}$$

By substituting a given value of Δ into MF^* and TF^* in (2.66), we obtain the equilibrium fees in Table 2.1.

Case 4: $4.7\tau < \Delta \leq 5\tau$

In this case, the investor valuation support width can be as large as 5τ, which is the difference between P_3 and P_{-3}. So we also consider the investor's profit conditional on orders posted at P_3 and P_{-3}. Table 2.6 shows that the investor's profit is zero if he buys at P_3 or sells at P_{-3}. Table 2.7 shows that for this interval of the support the equilibrium strategies are either $x^{LB}_{1,t_1} = LBP_{1,t_1}$, or $x^{LB}_{-1,t_1} = LBP_{-1,t_1}$. Therefore, to determine the optimal MF and TF, we maximize the exchange profits conditional on the buyer optimally using these two strategies, the case of the seller arriving at t_1 being symmetric:

$$\max_{MF,TF\in\Xi} \pi^{LTM}_{4.7\tau<\Delta\leq 5\tau}(MF,TF\,|S, x^{LB}_{1,t_1}, x^{LB}_{-1,t_1}, L_{t_0}, x^{MS}_{1,t_2}, x^{MS}_{-1,t_2}, L_{t_1}) = \tag{2.67}$$

$$= \Big(Pr(x^{LB}_{1,t_1}|S,\Xi,L_{t_0}) \times Pr(\theta^{x^{LB}_1}_{t_1}|S,\Xi,L_{t_1}) + Pr(x^{LB}_{-1,t_1}|S,\Xi,L_{t_0}) \times Pr(\theta^{x^{LB}_{-1}}_{t_1}|S,\Xi,L_{t_1})\Big) \times (MF+TF)$$

$$= \frac{0.25(-2\text{MF} + \Delta - 1)(\text{MF}+\text{TF})(\Delta - 2\text{TF} + 1)}{\Delta^2}$$

$$s.t.: \quad TF < \frac{\Delta - 3}{2}$$

$$s.t.: \quad -1 < MF < 1, -1 < TF < 1$$

$$s.t.: \quad MF + TF > 0$$

$$s.t.: \quad 4.7 < \Delta \leq 5$$

The Kuhn-Tucker Lagrangian is:

$$L(MF, TF, \Delta, \lambda_k, v_h) = \tag{2.68}$$
$$\pi^{LTM}_{4.7\tau<\Delta\leq 5\tau}(MF,TF\,|S, x^{B}_{1,t_1}, L_{t_0}, x^{S}_{1,t_2}, L_{t_1}) +$$
$$\lambda_1(-TF + \frac{\Delta - 3}{2}) - \lambda_2(-MF+1) - \lambda_3(-\Delta + 4.7) - \lambda_4(-\Delta + 5) + v_1 MF + v_2 TF$$

The Kuhn-Tucker conditions are:

$$\frac{\delta\pi^{LTM}}{\delta MF} = \frac{\mathrm{MF}(-\Delta + 2\mathrm{TF} - 1) + 0.25\Delta^2 - \Delta\mathrm{TF} + \mathrm{TF}^2 - 0.25}{\Delta^2} \geq 0 \;\&\; MF \times \frac{\delta\pi^{LTM}}{\delta MF} = 0 \tag{2.69}$$

$$\frac{\delta\pi^{LTM}}{\delta TF} = \frac{\mathrm{MF}^2 - \mathrm{MF}\Delta + 2\mathrm{MFTF} + 0.25\Delta^2 - \Delta\mathrm{TF} + \mathrm{TF} - 0.25}{\Delta^2} \geq 0 \;\&\; TF \times \frac{\delta\pi^{LTM}}{\delta TF} = 0 \tag{2.70}$$

$$\frac{\delta\pi^{LTM}}{\delta\Delta} = \frac{\mathrm{TF}^2(-2\mathrm{MF} + 0.5\Delta - 1) + \mathrm{MFTF}(\Delta - 2\mathrm{MF}) + \mathrm{MF}(\mathrm{MF}(0.5\Delta + 1) + 0.5) + 0.5\mathrm{TF}}{\Delta^3} \geq 0 \;\& \tag{2.71}$$

$$\Delta \times \frac{\delta\pi^{LTM}}{\delta\Delta} = 0$$

$$\frac{\delta L(MF, TF, \Delta, \lambda_k, v_h)}{\delta\lambda_1} = (-TF + \frac{\Delta - 3}{2}) \geq 0 \;\&\; \lambda_1 \times \frac{\delta L(MF, TF, \Delta, \lambda_k, v_h)}{\delta\lambda_1} = 0 \tag{2.72}$$

$$\frac{\delta L(MF, TF, \Delta, \lambda_k, v_h)}{\delta\lambda_2} = (-MF + 1) \geq 0 \;\&\; \lambda_2 \times \frac{\delta L(MF, TF, \Delta, \lambda_k, v_h)}{\delta\lambda_2} = 0 \tag{2.73}$$

$$\frac{\delta L(MF, TF, \Delta, \lambda_k, v_h)}{\delta\lambda_3} = (-\Delta + 4.7) \geq 0 \;\&\; \lambda_3 \times \frac{\delta L(MF, TF, \Delta, \lambda_k, v_h)}{\delta\lambda_3} = 0 \tag{2.74}$$

$$\frac{\delta L(MF, TF, \Delta, \lambda_k, v_h)}{\delta\lambda_4} = (-\Delta + 5) \geq 0 \;\&\; \lambda_4 \times \frac{\delta L(MF, TF, \Delta, \lambda_k, v_h)}{\delta\lambda_4} = 0 \tag{2.75}$$

The equilibrium MF^* and TF^* that satisfy these conditions are:

$$MF^* = 0.5 \qquad TF^* = 1 \;\; \forall \;\; 4.7 < \Delta \leq 5 \tag{2.76}$$

which are the equilibrium fees presented in Table 2.1. QED.

Comment: Proposition 1 follows from the formulas for optimal MF and TF in the parameterizations $\Delta < 4\tau$ for which rebated-based pricing is optimal. Proposition 1 also follows from the optimal fee formula when MF and TF are optionl.

2.11 Regulatory Regimes

Regulatory restrictions can have a major impact on equilibrium access pricing. In this section we disentangle the effects of three alternative regulatory specifications. Our model assumes the trading platform cannot set trading fees that (in absolute value) exceed the tick size, $-\tau \leq MF \leq \tau$ and $-\tau \leq TF \leq \tau$. We call this the *RRS Regulatory Restrictions*.[23] Our results over the $\Delta < 3\,\tau$ region agree qualitatively with Chao, Yao, and Ye 2018 (CYY) regarding the existence of symmetric maker-taker and taker-maker equilibria. However, our results differ in that we find that jointly positive fees occur when the amount of investor valuation dispersion is large ($\Delta > 3\,\tau$), whereas in CYY fees are never jointly positive.

[23]Using the exact fee cap of 0.3 of the tick size from Reg NMS would make our results stronger. In Europe — where there is no formal regulatory fee cap but possibly an informal regulatory understanding — exchanges usually access set access fees smaller than one tick.

The reason for the difference is that CYY impose different constraints on fees and rebates (footnote 16, Chao, Yao, and Ye 2018): $0 < MF < \tau$ and $-\tau < TF < 0$. We call this the *CYY Restrictions*. To show the effect of this stronger restriction, we solve our 2-period model with a support equal to $[P_{-1}, P_1] = [p_{-2}, p_2] = [9.5, 10.5]$ (with a support width of τ) under three different tick size specifications (as in CYY) for three different regulatory regimes: The RRS Regulatory Restrictions, the CYY Restrictions, and with no restrictions on access pricing ("No Restrictions").

Another difference between our analysis and CYY is the assumed investor arrival process. We assume investor valuations each period are uniformly distributed on the whole valuation support, whereas CYY assume buyers and sellers alternate each period with sellers' valuations being distributed over the lower half of the support and buyers' valuations being distributed over the upper half of the support.[24] However, we show that the qualitative differences between our results and CYY are due to the different regulatory assumptions and not the mechanical difference in investor arrival.

Table 2.9 shows that when $\Delta = \tau$ and No Restrictions are imposed on MF and TF, the equilibrium in the CYY model with the CYY investor-arrival assumption delivers the same trading fees at in the equilibrium with the RRS investor-arrival assumption, across each of the three different tick size specifications considered here (τ, $\frac{\tau}{4}$ and $\frac{\tau}{8}$). However, exchange profits in the CYY model are twice as high as in RRS model, because of the alternating buyer and seller assumption in the CYY framework. The results reported in Table 2.9 show that when No Restrictions are imposed on the trading fees, the taker-maker (shown) and symmetric maker-taker (not shown) pricing structures are both optimal in equilibrium. This holds with both the RRS and CYY investor-arrival assumptions. When, following CYY, we hold the investor valuation support constant and consider different tick sizes, we find that the equilibrium optimal make and take fees do not change. In particular, with No Restrictions on trading fees, the exchange optimally sets a positive fee of 0.667 and a rebate of -0.333 irrespective of the tick size. By doing so, the exchange forces traders to discard the tick size and trade at the outside quotes. Once again, we note the net fee is one third of the valuation support width.

Table 2.9 also shows the effects of both the RRS Regulatory Restrictions and the CYY Restrictions on the equilibrium trading fees. When we solve for equilibria under both the RRS and CYY trader-arrival models with the RRS Regulatory Restrictions, the taker-maker pricing structure and symmetrically the maker-taker pricing structure prevail only when the tick size is equal to τ. For smaller tick sizes ($\frac{\tau}{4}$ and $\frac{\tau}{8}$) both the optimal MF and TF are positive. Intuitively, under the RRS Regulatory Restrictions the exchange cannot discard the tick size rule and, not being allowed to impose extreme trading fees, maximizes profits by imposing the symmetric taker-maker or maker-taker pricing only when the support is equal to the tick size. When instead the support widens relative to the tick size, the exchange exploits the investors' increased gains from trade and imposes positive fees on both takers and makers.

Notice here that cutting the tick size to $\frac{\tau}{4}$ holding the support width constant at $\Delta = \tau$ (i.e., 1 tick) has the same impact on the support width/tick size ratio as in Table 1 where we hold the tick size equal to 1 and set $\Delta = 4\,\tau$. There we find that the equilibrium fees are positive, but

[24]The reason we do not assume alternating buyers and sellers is that, when we extend our model to three periods, the assumption that any investor may arrive at each trading period is more suited to modeling liquidity dynamics.

the results are different because changing the tick size also affects the RRS Regulatory Restriction through which fees are capped relative to the tick size. Changing the support width does not affect the fee cap, but changing the tick size does. When instead the CYY Restrictions constrain the exchange not to impose a positive TF, the taker-maker pricing is the only equilibrium trading fee structure that prevails both under the RRS and under the CYY protocol.

So far, we have shown how the optimal trading fees change when, holding the investor composition constant (i.e., holding the valuation support constant at $\Delta = \tau = 1$), we consider different markets with different tick size regimes. The natural following question is whether we obtain similar results by holding the tick size constant and changing the support of the investors' beliefs. Table 2.10 shows that under the "No Restrictions" regime, if we hold the tick size constant to 1 and gradually widen the support from one tick ($[9.50, 10.50] = \tau$), to three ticks ($[8.50, 11.50] = 3\,\tau$), to five ticks ($[7.50, 12.50] = 5\,\tau$), the taker-maker and symmetrically the maker-taker pricing structure become stronger with the (unconstrained) positive fee increasing from 0.667 to 3.333 and the rebate $|fee|$ increasing from $|-0.333|$ to $|-1.667|$. These results holds for both the RRS and the CYY investor-arrival frameworks, although as before the CYY exchange profits are twice as high in the RRS framework. In addition, notice here yet again that the net fee satifies $MF + TF = \Delta/3$. When instead we impose the RRS Regulatory Restrictions, we are back to Figure 1 and Table 2.1 that show how, when the support in the LTM reaches three ticks, the taker-maker and maker-taker are no longer equilibrium fee structures. To economize space we do not show the results obtained when running the same extensions with increasing supports for the CYY framework as they lead to the unique taker-maker equilibrium due to the restriction imposed on the TF.

TABLE 2.9: **Optimal Trading Fees and Restrictions** This table reports the equilibrium optimal make (MF) and take fee (TF), Exchange Expected Profit, equilibrium strategies, cum-fee buy and sell prices ($P_k^{LB,cum}$ and $P_k^{MS,cum}$) for a support with width $\Delta = \tau$ for markets with three different tick size specifications (τ, $\frac{\tau}{4}$ and $\frac{\tau}{8}$) and under three different regulatory regimes given both the RRS (our) and the CYY (Chao, Yao, and Ye 2018) investor-arrival frameworks. The "RRS Regulatory Restrictions" are $-\tau \leq MF, TF \leq \tau$; the "CYY Restrictions" are $0 \leq MF \leq S$ and $-\tau \leq TF \leq 0$; and the "No Restrictions" protocol imposes no restrictions on MF and TF fees.

		τ	$\frac{\tau}{4}$	$\frac{\tau}{8}$
CYY framework	MF	0.667	0.667	0.667
"No Restrictions"	TF	-0.333	-0.333	-0.333
	Exchange E[Profit]	0.148	0.148	0.148
	Eq.Strategies x_{t_1}	$LB_{9.500}$	$LB_{9.500}$	$LB_{9.500}$
$-S \leq MF \leq S$	$P_k^{LB,cum}$	10.167	10.167	10.167
$-S \leq TF \leq S$	$P_k^{MS,cum}$	9.833	9.833	9.833
RRS framework	MF	0.667	0.667	0.667
"No Restrictions"	TF	-0.333	-0.333	-0.333
	Exchange E[Profit]	0.074	0.074	0.074
	Eq.Strategies x_{t_1}	$LB_{9.500}$	$LB_{9.500}$	$LB_{9.500}$
$-S \leq MF \leq S$	$P_k^{LB,cum}$	10.167	10.167	10.167
$-S \leq TF \leq S$	$P_k^{MS,cum}$	9.833	9.833	9.833
CYY framework	MF	0.667	0.206	0.125
"RRS Regulatory	TF	-0.333	0.169	0.125
Restrictions"	Exchange E[Profit]	0.148	0.141	0.125
	Eq.Strategies x_{t_1}	$LB_{9.500}$	$LB_{9.750}$, $LB_{10.000}$	$LB_{9.750}$, $LB_{9.875}$, $LB_{10.000}$
$-\tau \leq MF \leq \tau$	$P_k^{LB,cum}$	10.167	9.956, 10.206	9.875, 10.000, 10.125
$-\tau \leq TF \leq \tau$	$P_k^{MS,cum}$	9.833	9.581, 9.831	9.625, 9.750, 9.875
RRS framework	MF	0.667	0.206	0.125
"RRS Regulatory	TF	-0.333	0.169	0.125
Restrictions"	Exchange E[Profit]	0.074	0.070	0.0625
	Eq.Strategies x_{t_1}	$LB_{9.500}$	$LB_{9.750}$, $LB_{10.000}$	$LB_{9.750}$, $LB_{9.875}$, $LB_{10.000}$
$-\tau \leq MF \leq \tau$	$P_k^{LB,cum}$	10.167	9.956, 10.206	9.875, 10.000, 10.125
$-\tau \leq TF \leq \tau$	$P_k^{MS,cum}$	9.833	9.581, 9.831	9.625, 9.750, 9.875
CYY framework	MF	0.667	0.496	0.387
"CYY Restrictions"	TF	-0.333	-0.121	-0.012
	Exchange E[Profit]	0.148	0.141	0.141
$0 \leq MF \leq S$	Eq.Strategies x_{t_1}	$LB_{9.500}$	$LB_{9.500}$, $LB_{9.750}$	$LB_{9.500}$, $LB_{9.625}$, $LB_{9.750}$
$-\tau \leq TF \leq 0$	$P_k^{LB,cum}$	10.167	9.996, 10.246	9.887, 10.012, 10.137
	$P_k^{MS,cum}$	9.833	9.621, 9.871	9.512, 9.637, 9.762
RRS framework	MF	0.667	0.496	0.387
"CYY Restrictions"	TF	-0.333	-0.121	-0.012
	Exchange E[Profit]	0.074	0.070	0.070
$0 \leq MF \leq S$	Eq.Strategies x_{t_1}	$LB_{9.500}$	$LB_{9.500}$, $LB_{9.750}$	$LB_{9.500}$, $LB_{9.625}$, $LB_{9.750}$
$-\tau \leq TF \leq 0$	$P_k^{LB,cum}$	10.167	9.996, 10.246	9.887, 10.012, 10.137
	$P_k^{MS,cum}$	9.833	9.621, 9.871	9.512, 9.637, 9.762

TABLE 2.10: **Optimal Trading Fees and No Restrictions.** This table reports results on the optimal trading fees (MF and TF), equilibrium trading strategies ($x_{t_1} = LB_{P_k}$), cum-fee prices buy and sell prices ($P_k^{LB,cum}$ and $P_k^{MS,cum}$) and Exchange Expected Profits for the "No Restrictions" protocol on access fees and for both the RRS and CYY investor-arrival frameworks. The tick size τ is equal to 1, and results are reported for three support widths, $\Delta = 1$, $\Delta = 3$ and $\Delta = 5$.

	Support width		
	$\Delta = 1$	$\Delta = 3$	$\Delta = 5$
MF	0.667	2.000	3.333
TF	-0.333	-1.000	-1.667
Eq.Strategies $x_{t_1} = LB_{P_k}$	$LB_{9.500}$	$LB_{8.500}$	$LB_{7.500}$
$P_k^{LB,cum}$	10.167	10.500	10.833
$P_k^{MS,cum}$	9.833	9.500	9.167
Exchange E[Profit] CYY	0.148	0.444	0.741
Exchange E[Profit] RRS	0.074	0.222	0.370

2.12 3-Period Model (In Progress)

To illustrate how the 3-period model works, we first use the benchmark model and show that in absence of fees, given the investors' support and the tick size, the model has a closed form solution. We present the solution for a support equal to 2τ and the large tick τ. The solution for different supports and for the small tick size can be found in a similar way. The following 3 tables show how to derive analytically the equilibrium order submission probabilities respectively at t_3, t_2 and t_1 for the benchmark model.
To obtain the optimal trading fees set by the exchange we then add the profit function of the exchange to the benchmark model without fees and we maximize the exchange profits π by using both the Simulated Annealing (SA) algorithm and the optimizing algorithm, the Fee Optimizing (FO) algorithm, that we created to refine the solutions provided by the SA algorithm. Results are shown in Table 2.3.
Table 2.21 shows market quality and welfare results for the 3-Period large tick market.

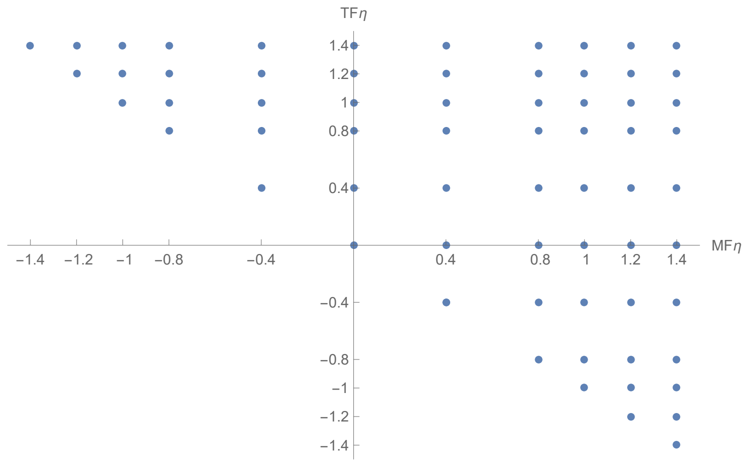
Here below we explain how we integrate the SA algorithm with the FO algorithm to maximize the exchange profits, π.

Simulated Annealing (SA) and Fee Optimizing (FO) Algorithms

We use the model described in Tables 2.13, 2.14 and 2.15 to initialize the variables that we need to compute π, i.e., the investors' support, the tick size, and the probability of order submission at each node of the trading game. We then use both the SA and the FO algorithms to determine the equilibrium MF^* and TF^* set by the exchange conditional on the support and the large tick size.[25]

The simulated annealing (SA) algorithm is an iterative procedure that starts at time η with an initial set of combinations of MF_η and TF_η, Ξ_η, with $-1.5\tau < MF_\eta, TF_\eta < 1.5\tau$, and search for the maximum profit of the exchange, π, conditional on the tick size and the support of traders' evaluations, S. Figure 9 reports the initial combinations of fees that we chose for the large tick market. The SA will then search for the maximum π within a neighborhood

FIGURE 9: **Simulated Annealing (SA) Algorithm: Large Tick Market (LTM) initial Sets of MF_η and TF_η.** This Figure reports the initial combinations of MF_η and TF_η, from which the SA algorithm starts to numerically maximize the exchange profits π.



of amplitude $2 \times \epsilon$ of each initial combination of fees. We set $\epsilon = 0.25$ so that the amplitude of the region explored around each fee is equal to half a tick. For example, given the initial set of fees, $\Xi_\eta = \{MF_\eta = 0, TF_\eta = 0.4\}$, the SA algorithm will select a value for $MF_{\eta+1}$ within the interval $\{MF_\eta = 0 - \epsilon, MF_\eta = 0 + \epsilon\}$ and a value for $TF_{\eta+1}$ within the interval $\{TF_\eta = 0.4 - \epsilon, TF_\eta = 0.4 + \epsilon\}$ with Uniform probability. Assume for example that the randomly selected set of fees is $\Xi_{\eta+1} = \{MF_{\eta+1} = 0.1, TF_{\eta+1} = 0.5\}$. If $\Xi_{\eta+1}$ is associated with an exchange profit that is higher than the exchange profit associated with the initial set of fees Ξ_η, then the SA algorithm will select the next combination of fees, $\Xi_{\eta+2}$, starting from $\Xi_{\eta+1}$, within the interval $\{MF_{\eta+2} = 0.1 - \epsilon, MF_{\eta+2} = 0.1 + \epsilon\}$ for $MF_{\eta+2}$ and $\{TF_{\eta+2} = 0.5 - \epsilon, TF_{\eta+2} = 0.5 + \epsilon\}$ for $TF_{\eta+2}$. If instead $\Xi_{\eta+1}$ is associated with an exchange profit which is lower or equal than the exchange profit associated with Ξ_η, then the algorithm will choose the new combination of fees, $\Xi_{\eta+2}$ starting from $\Xi_{\eta+1}$, within the interval $2 \times \epsilon$ with probability $\zeta_\eta = e^{\frac{\pi_\eta - \pi_{\eta-1}}{\chi_\eta}}$, whereas it will choose the new combination of fees starting from Ξ_η with probability $1 - \zeta_\eta$, where χ_η is a parameter that starts with value $\chi_\eta = 0.8$ and decreases by $0.9\chi_\eta$ at each η iteration until it reaches its minimum that we set at 0.066667. This means that as the number of iterations increases, the probability ζ_η with which the SA algorithm will explore the neighborhood of the out-of-equilibrium sets of fees will also tend to increase.

[25]Results for the STM can be obtained in a similar way and are available from the authors upon request.

Starting from the initial 66 combinations of fees, the SA algorithm explores approximately 10700 sets of fees for each support and produces a number (approximately 8) of possible equilibrium set of fees ($\Xi^{\dagger} = \{MF^{\dagger}, TF^{\dagger}\}$) for each support that differ approximately by 10^{-4} in terms of the associated π. The objective of the FO algorithm is to refine the equilibrium sets of fees generated by the SA algorithm.
We consider the 8 combinations of fees [with the highest associated π and with $-\tau < MF^{\dagger}, TF^{\dagger} < \tau$], of which 4 combinations of fees such that $MF^{\dagger} > TF^{\dagger}$ and 4 combinations of fees such that $MF^{\dagger} < TF^{\dagger}$. To illustrate how the FO algorithm works, assume that one the 8 optimal sets of SA fees chosen is $\Xi^{\dagger'} = \{MF^{\dagger'} = -0.270, TF^{\dagger'} = 0.494\}$. The FO algorithm will generate the first grid (Grid #1) with the combinations of fees that differ by 6 steps of $\Delta = 0.02$ $(-0.06, -0.04, -0.02, 0.0, 0.02, 0.04, 0.06)$ from $\Xi^{\dagger'}$. The FO algorithm will then evaluate and compare the 49 combinations of fees reported in Table 2.11 and select the set of fees with the highest associated π. Assume that the optimal set of fees generated by Grid #1 is

TABLE 2.11: **Grid**#1 This Table reports the combinations of MF and TF that differ by 6 steps of $\Delta = 0.02$ from $\Xi^{\dagger'}$.

		-0.06	-0.04	-0.02	0	0.02	0.04	0.06
		-0.33	-0.31	-0.29	-0.27	-0.25	-0.23	-0.21
-0.06	0.434	-0.33,0.434	-0.31,0.434	-0.29,0.434	-0.27,0.434	-0.25,0.434	-0.23,0.434	-0.21,0.434
-0.04	0.454	-0.33,0.454	-0.31,0.454	-0.29,0.454	-0.27,0.454	-0.25,0.454	-0.23,0.454	-0.21,0.454
-0.02	0.474	-0.33,0.474	-0.31,0.474	-0.29,0.474	-0.27,0.474	-0.25,0.474	-0.23,0.474	-0.21,0.474
0	0.494	-0.33,0.494	-0.31,0.494	-0.29,0.494	-0.27,0.494	-0.25,0.494	-0.23,0.494	-0.21,0.494
0.02	0.514	-0.33,0.514	**-0.31,0.514**	-0.29,0.514	-0.27,0.514	-0.25,0.514	-0.23,0.514	-0.21,0.514
0.04	0.534	-0.33,0.534	-0.31,0.534	-0.29,0.534	-0.27,0.534	-0.25,0.534	-0.23,0.534	-0.21,0.534
0.06	0.554	-0.33,0.554	-0.31,0.554	-0.29,0.554	-0.27,0.554	-0.25,0.554	-0.23,0.554	-0.21,0.554

$\Xi^{\dagger''} = \{MF^{\dagger''} = -0.310, TF^{\dagger''} = 0.514\}$, the FO algorithm will now generate a second grid (*Grid*#2) that differ by 6 steps of $\Delta = 0.01$ $(-0.03, -0.02, -0.01, 0.0, 0.01, 0.02, 0.03)$ from $\Xi^{\dagger''}$. The FO algorithm will then evaluate and compare the new 49 combinations of fees presented in Table 2.12, and will repeat this procedure 6 times starting from the new possible equilibrium set of fees, each time reducing Δ according to the following vector:
$\Delta \in \{0.02, 0.01, 0.005, 0.0025, 0.000125, 0.000065\}$. The set of fees associated with the highest π derived from the last grid will be finally compared with the optimal sets of fees obtained by starting from the other 7 best combinations of fees generated by the SA. The resulting set of fees associated with the highest π will be the optimal set of fees, $\xi^{*} = MF^{*}, TF^{*}$.

TABLE 2.12: **Grid**#2 This Table reports the combinations of MF and TF that differ by 6 steps of $\Delta = 0.01$ from $\Xi^{\dagger''}$.

		-0.03	-0.02	-0.01	0	0.01	0.02	0.03
		-0.34	-0.33	-0.32	-0.31	-0.3	-0.29	-0.28
-0.03	0.484	-0.33,0.484	-0.31,0.484	-0.29,0.484	-0.27,0.484	-0.25,0.484	-0.23,0.484	-0.21,0.484
-0.02	0.494	-0.33,0.494	-0.31,0.494	-0.29,0.494	-0.27,0.494	-0.25,0.494	-0.23,0.494	-0.21,0.494
-0.01	0.504	-0.33,0.504	-0.31,0.504	-0.29,0.504	-0.27,0.504	-0.25,0.504	-0.23,0.504	-0.21,0.504
0	0.514	-0.33,0.514	-0.31,0.514	-0.29,0.514	-0.27,0.514	-0.25,0.514	-0.23,0.514	-0.21,0.514
0.01	0.524	-0.33,0.524	-0.31,0.524	-0.29,0.524	-0.27,0.524	-0.25,0.524	-0.23,0.524	-0.21,0.524
0.02	0.534	-0.33,0.534	-0.31,0.534	-0.29,0.534	-0.27,0.534	-0.25,0.534	-0.23,0.534	-0.21,0.534
0.03	0.544	-0.33,0.544	-0.31,0.544	-0.29,0.544	-0.27,0.544	-0.25,0.544	-0.23,0.544	-0.21,0.544

2.13 3-Period Model With HFT (In Progress)

As for the 3-period model, we now show how to obtain the closed form solution for the benchmark model (this time with HFTs) without fees, a support equal to 2τ and the large tick size, τ.
To obtain the optimal trading fees, we use - as for the 3-period model - both the SA and the FO algorithms and we run them for different supports of the market participants. Results are shown in Table 2.4.

TABLE 2.13: **3-Period Large Tick Market (LTM). Equilibrium Strategies at t_3.** This table shows how to derive the equilibrium order submission strategies at t_3 for the benchmark model which has no trading fees ($MF = TF = 0.00$) and for an investors' support equal to 2τ. At t_1 the market opens with an empty book, [0000], where each element in the square bracket, $L_{t_z} = D^{P_i}_{t_z}$, corresponds to the depth of the book at each price level at time t_z, $[L^{P_2}_{t_z}, L^{P_1}_{t_z}, L^{P_{-1}}_{t_z}, L^{P_{-2}}_{t_z}]$. Given the chosen set of fees, four are the equilibrium strategies at t_1, LBP_1 and LBP_{-1} on the buy side and LSP_1 and LSP_{-1} on the sell side. At t_1 Table **??** presents both the buy and the sell equilibrium strategies. However, as the equilibrium strategies consistent with the states of the book derived from the buy side are symmetric to those derived from the sell side, to economize space at t_2 we only present the equilibrium strategies that are consistent with the states of the book derived from the sell equilibrium strategies at t_1. Given the equilibrium limit buy orders at t_1, the possible states of the books at the beginning of t_2 are: $[00B0]$ following a LBP_{-1} and $[0B00]$ following a LBP_1. Given the equilibrium strategies at t_2 and therefore the possible states of the books at the beginning of t_3, this table shows the equilibrium strategies at t_3 (column 1), their payoffs (column 2), the β thresholds (column 3) and the order submission probabilities (column 4).

Equilibrium Strategy	**Payoff**	**β Threshold**	**Order Submission Probability**
at t_1 the book opens empty [0000]: equilibrium strategy LBP_{-1}			
at t_2 the book opens [00B0]			
	t_2 equilibrium strategy: MSP_{-1,t_2}		
at t_3 the book opens empty [0000]			
NT_{t_3}	0	{9.000,11.000}	1
	t_2 equilibrium strategy: LSP_1		
at t_3 the book opens [0SB0]			
MSP_{-1,t_3}	$P_{-1} - \beta_{t_3} - TF = 9.500 + \beta_{t_3}$	{9.000,9.500}	0.250
NT_{t_3}	0	{9.500,10.500}	0.500
MBP_{1,t_3}	$\beta_{t_3} - P_1 - TF = -10.500 + \beta_{t_3}$	{10.500,11.000}	0.250
	t_2 equilibrium strategy: LBP_1		
at t_3 the book opens [0BB0]			
MSP_{1,t_3}	$P_1 - \beta_{t_3} - TF = 10.500 - \beta_{t_3}$	{9.000,10.500}	0.750
NT_{t_3}	0	{10.500,11.000}	0.250
at t_1 the book opens empty [0000]: equilibrium strategy LBP_1			
at t_2 the book opens [0B00]			
	t_2 equilibrium strategy: MSP_{1,t_2}		
at t_3 the book opens empty [0000]			
NT_{t_3}	0	{9.000, 11.000}	1
	t_2 equilibrium strategy: NT_{t_2}		
at t_3 the book opens [00B0]			
MSP_{1,t_3}	$P_1 - \beta_{t_3} - TF = 10.500 - \beta_{t_3}$	{9.000, 10.500}	0.750
NT_{t_3}	0	{10.500,11.000}	0.250

TABLE 2.14: **3-Period Large Tick Market (LTM). Equilibrium Strategies at t_2.** This table shows how to derive the equilibrium order submission strategies at t_2 for the benchmark model which has no trading fees ($MF = TF = 0.00$) and for a support equal to 2τ. At t_1 the market opens with an empty book, [0000], where each element in the square bracket, $L_{t_z} = D_{t_z}^{P_i}$, corresponds to the depth of the book at each price level at time t_z, $[L_{t_z}^{P_2}, L_{t_z}^{P_1}, L_{t_z}^{P_{-1}}, L_{t_z}^{P_{-2}}]$. Given the chosen set of fees, four are the equilibrium strategies at t_1: LBP_1 and LBP_{-1} on the buy side and LSP_1 and LSP_{-1} on the sell side. At t_1 we present both buy and sell the equilibrium strategies; to economize space, at t_2 we present the equilibrium strategies that are consistent with the states of the book derived from the sell equilibrium strategies at t_1, as the equilibrium strategies consistent with the states of the book derived from the buy side are perfectly symmetric. Given the equilibrium limit buy orders at t_1, the possible states of the books at the beginning of t_2 are: [00B0] following a LBP_{-1} and [0B00] following a LBP_1. Column 1 shows the Equilibrium Strategies at t_2, column 2 shows the corresponding payoffs, and columns 3 and 4 show the β thresholds and the order submission probabilities respectively.

Equilibrium Strategy	Payoff	β Threshold	Order Submission Probability
at t_1 the book opens empty [0000]: equilibrium strategy LBP_{-1}			
at t_2 the book opens [00B0]			
MSP_{-1,t_2}	$P_{-1} - \beta_{t_2} - TF = 9.500 - \beta_{t_2}$	{9.000,9.167}	0.083
LSP_1	$(P_1 - \beta_{t_2} - MF)Pr(\theta_{t_2}^{LSP_1} \mid S, \Xi, L_{t_2}) = 2.625 - 0.250\beta_{t_2}$	{9.167,10.500}	0.667
LBP_1	$(\beta_{t_2} - P_1 - MF)Pr(\theta_{t_2}^{LBP_1} \mid S, \Xi, L_{t_2}) = -7.875 + 0.750\beta_{t_2}$	{10.500,11.000}	0.250
at t_1 the book opens empty [0000]: equilibrium strategy LBP_1			
at t_2 the book opens [0B00]			
MSP_{1,t_2}	$P_1 - \beta_{t_2} - TF = 10.500 - \beta_{t_2}$	{9.000,10.500}	0.750
NT_{t_2}	0	{10.500,11.000}	0.250

TABLE 2.15: **3-Period Large Tick Market (LTM). Equilibrium Strategies at t_1.** This table shows how to derive the equilibrium order submission strategies at t_1 for the benchmark model which has no trading fees ($MF = TF = 0.00$) and for a support equal to 2τ. At t_1 the market opens with an empty book, [0000], where each element in the square bracket, $L_{t_z} = D_{t_z}^{P_i}$, corresponds to the depth of the book at each price level at time t_z, $[L_{t_z}^{P_2}, L_{t_z}^{P_1}, L_{t_z}^{P_{-1}}, L_{t_z}^{P_{-2}}]$. Given the chosen set of fees, four are the equilibrium strategies at t_1: LBP_1 and LBP_{-1} on the buy side and LSP_1 and LSP_{-1} on the sell side (column 1). Column 2 shows their payoffs, and columns 3 and 4 shows the β thresholds and the order submission probabilities respectively.

Equilibrium Strategy	Payoff	β Threshold	Order Submission Probability
at t_1 the book opens empty [0000]			
LSP_{-1}	$(P_{-1} - \beta_{t_1} - MF)Pr(\theta_{t_1}^{LSP_{-1}} \mid S, \Xi, L_{t_1}) = 8.906 - 0.938\beta_{t_1}$	{9.000,9.136}	0.068
LSP_1	$(P_1 - \beta_{t_1} - MF)Pr(\theta_{t_1}^{LSP_1} \mid S, \Xi, L_{t_1}) = 2.625 - 0.250\beta_{t_1}$	{9.136,10.000}	0.432
LBP_{-1}	$(\beta_{t_1} - P_{-1} - MF)Pr(\theta_{t_1}^{LBP_{-1}} \mid S, \Xi, L_{t_1}) = -2.375 + 0.250\beta_{t_1}$	{10.000,10.863}	0.432
LBP_1	$(\beta_{t_1} - P_1 - MF)Pr(\theta_{t_1}^{LBP_1} \mid S, \Xi, L_{t_1}) = -9.844 + 0.938\beta_{t_1}$	{10.863,11.000}	0.068

TABLE 2.16: **3-Period Large Tick Market (LTM) with HFTs. Equilibrium Strategies at t_3.** This table shows how to derive the equilibrium order submission strategies at t_3 for the benchmark model with HFTs which has no trading fees ($MF = TF = 0.00$) and for the Investors' support equal to 2τ. At t_1 the market opens with an empty book, [0000], where each element in the square bracket, $L_{t_z} = D_{t_z}^{P_i}$, corresponds to the depth of the book at each price level at time t_z, $[L_{t_z}^{P_2}, L_{t_z}^{P_1}, L_{t_z}^{P_{-1}}, L_{t_z}^{P_{-2}}]$. Given the chosen set of fees, four are the equilibrium strategies at t_1: LBP_1 (followed by a MSP_1 from an HFT firm) and LBP_{-1} on the buy side, and LSP_1 and LSP_{-1} (followed by a MBP_{-1} from an HFT firm) on the sell side. At t_1 Table 2.18 presents both the buy and the sell equilibrium strategies. However, as the equilibrium strategies consistent with the states of the book derived from the buy side are symmetric to those derived from the sell side, to economize space at t_2 we only present the equilibrium strategies that are consistent with the states of the book derived from the sell equilibrium strategies at t_1. Given the equilibrium limit buy orders at t_1, the possible states of the books at the beginning of t_2 are: $[00B0]$ following a LBP_{-1} and [0000] following a LBP_1 and a MSP_1 from an HFT firm. Given the equilibrium strategies at t_2 and therefore the possible states of the books at the beginning of t_3, this table shows the equilibrium strategies at t_3 (column 1), their payoffs (column 2), the β thresholds (column 3) and the order submission probabilities (column 4).

Equilibrium Strategy	**Payoff**	**β Threshold**	**Order Submission Probability**
at t_1 the book opens empty [0000]: equilibrium strategy LBP_{-1}			
at t_2 the book opens [00B0]			
	t_2 equilibrium strategy: MSP_{-1,t_2}		
at t_3 the book opens empty [0000]			
NT_{t_3}	0	{9.000,11.000}	1
	t_2 equilibrium strategy: LSP_1		
at t_3 the book opens [0SB0]			
MSP_{-1,t_3}	$P_{-1} - \beta_{t_3} - TF = 9.500 - \beta_{t_3}$	{9.000,9.500}	0.250
NT_{t_3}	0	{9.500,10.500}	0.500
MBP_{1,t_3}	$\beta_{t_3} - P_1 - TF = -10.500 + \beta_{t_3}$	{10.500,11.000}	0.250
	t_2 equilibrium strategy: $LBP_1 \longrightarrow$ HFT: $MSP_{1_{t_2}}$		
at t_3 the book opens [00B0]			
NT_{t_3}	0	{9.000,9.500}	0.750
MSP_{-1,t_3}	$P_{-1} - \beta_{t_3} - TF = 9.500 - \beta_{t_3}$	{9.500,11.000}	0.250
at t_1 the book opens empty [0000]: equilibrium strategy $LSP_{-1} \longrightarrow$ HFT: $MBP_{-1_{t_2}}$			
at t_2 the book opens empty [0000]			
	t_2 equilibrium strategy: $LBP_1 \longrightarrow$ HFT: $MSP_{1_{t_2}}$		
at t_3 the book opens empty [0000]			
NT_{t_3}	0	{9.000, 11.000}	1
	t_2 equilibrium strategy: LBP_{-1}		
at t_3 the book opens [00B0]			
MSP_{-1,t_3}	$P_{-1} - \beta_{t_3} - TF = 9.500 - \beta_{t_3}$	{9.000, 9.500}	0.250
NT_{t_3}	0	{9.500, 11.000}	0.750
	t_2 equilibrium strategy: LSP_1		
at t_3 the book opens [0S00]			
NT_{t_3}	0	{9.000, 10.500}	0.750
MBP_{1,t_3}	$\beta_{t_3} - P_1 - TF = -10.500 + \beta_{t_3}$	{10.500,11.000}	0.250
	t_2 equilibrium strategy: $LSP_{-1} \longrightarrow$ HFT: $MBP_{-1_{t_2}}$		
at t_3 the book opens empty [0000]			
NT_{t_3}	0	{9.000, 11.000}	1

TABLE 2.17: **3-Period Large Tick Market (LTM) with HFTs. Equilibrium Strategies at t_2.** This table shows how to derive the equilibrium order submission strategies at t_2 for the benchmark model with HFT which has no trading fees ($MF = TF = 0.00$) and for an investors' support equal to 2τ. At t_1 the market opens with an empty book, [0000], where each element in the square bracket, $L_{t_z} = D_{t_z}^{P_i}$, corresponds to the depth of the book at each price level at time t_z, $[L_{t_z}^{P_2}, L_{t_z}^{P_1}, L_{t_z}^{P_{-1}}, L_{t_z}^{P_{-2}}]$. Given the chosen set of fees, four are the equilibrium strategies at t_1: LBP_1 (followed by a MSP_1 from an HFT firm) and LBP_{-1} on the buy side, and LSP_1 and LSP_{-1} (followed by a MBP_{-1} from an HFT firm) on the sell side. At t_1 Table 2.18 presents both the buy and the sell equilibrium strategies. However, as the equilibrium strategies consistent with the states of the book derived from the buy side are symmetric to those derived from the sell side, to economize space at t_2 we only present the equilibrium strategies that are consistent with the states of the book derived from the sell equilibrium strategies at t_1. Given the equilibrium limit buy orders at t_1, the possible states of the books at the beginning of t_2 are: [00B0] following a LBP_{-1} and [0000] following a LBP_1 and a MSP_1 from an HFT firm. Column 1 shows the Equilibrium Strategies at t_2, column 2 shows the corresponding payoffs, and columns 3 and 4 show the β thresholds and the order submission probabilities respectively. We present the β Thresholds and the Order Submission Probabilities only for the regular investors; HFT firms have $\beta = 1$ and take profitable liquidity offered by aggressive orders with probability 1.

Equilibrium Strategy	**Payoff**	**β Threshold**	**Order Submission Probability**
at t_1 the book opens empty [0000]: equilibrium strategy LBP_{-1}			
at t_2 the book opens [00B0]			
MSP_{-1,t_2}	$P_{-1} - \beta_{t_2} - MF = 9.500 - \beta_{t_2}$	{9.000,9.167}	0.083
LSP_1	$(P_1 - \beta_{t_2} - MF)Pr(\theta_{t_2}^{LSP_1} \vert S, \Xi, L_{t_2}) = 2.625 - 0.250\beta_{t_2}$	{9.167,10.500}	0.667
$LBP_1 \longrightarrow HFT: \ MSP_{1,t_2}$	$(\beta_{t_2} - P_1 - TF) \times 1 = -10.500 + \beta_{t_2}$	{10.500,11.000}	0.250
at t_1 the book opens empty [0000]: equilibrium strategy $LBP_1 \longrightarrow$ HFT: MSP_{1,t_2}			
at t_2 the book opens empty [0000]			
$LSP_{-1} \longrightarrow HFT: \ MBP_{-1,t_2}$	$(P_{-1} - \beta_{t_1} - MF) \times 1 = 9.500 - \beta_{t_1}$	{9.000,9.167}	0.083
LSP_1	$(P_1 - \beta_{t_1} - MF)Pr(\theta_{t_1}^{LSP_1} \vert S, \Xi, L_{t_1}) = 2.625 - 0.250\beta_{t_1}$	{9.167,10.000}	0.417
LBP_{-1}	$(\beta_{t_1} - P_{-1} - MF)Pr(\theta_{t_1}^{LBP_{-1}} \vert S, \Xi, L_{t_1}) = -2.375 + 0.250\beta_{t_1}$	{10.000,10.833}	0.417
$LBP_1 \longrightarrow HFT: \ MSP_{1,t_2}$	$(\beta_{t_1} - P_1 - MF) \times 1 = -10.500 + \beta_{t_1}$	{10.833,11.000}	0.083

TABLE 2.18: **3-Period Large Tick Market (LTM) with HFTs. Equilibrium Strategies at t_1.** This table shows how to derive the equilibrium order submission strategies at t_1 for the benchmark model with HFTs which has no trading fees ($MF = TF = 0.00$) and for an investors' support equal to 2τ. At t_1 the market opens with an empty book, [0000], where each element in the square bracket, $L_{t_z} = D_{t_z}^{P_i}$, corresponds to the depth of the book at each price level at time t_z, $[L_{t_z}^{P_2}, L_{t_z}^{P_1}, L_{t_z}^{P_{-1}}, L_{t_z}^{P_{-2}}]$. Given the chosen set of fees, four are the equilibrium strategies at t_1: LBP_1 (followed by a MSP_1 from an HFT firm) and LBP_{-1} on the buy side, and LSP_1 and LSP_{-1} (followed by a MBP_{-1} from an HFT firm) on the sell side (column 1). Column 2 shows their payoffs, and columns 3 and 4 shows the β thresholds and the order submission probabilities respectively. We present the β Thresholds and the Order Submission Probabilities only for the regular investors; HFT firms have $\beta = 1$ and take profitable liquidity offered by aggressive orders with probability 1.

Equilibrium Strategy	**Payoff**	**β Threshold**	**Order Submission Probability**
at t_1 the book opens empty [0000]			
$LSP_{-1} \longrightarrow HFT: MBP_{-1,t_1}$	$(P_{-1} - \beta_{t_1} - MF) \times 1 = 9.500 - \beta_{t_1}$	{9.000,9.167}	0.083
LSP_1	$(P_1 - \beta_{t_1} - MF)Pr(\theta_{t_1}^{LSP_1} \vert S, \Xi, L_{t_1}) = 2.625 - 0.250\beta_{t_1}$	{9.167,10.000}	0.417
LBP_{-1}	$(\beta_{t_1} - P_{-1} - MF)Pr(\theta_{t_1}^{LBP_{-1}} \vert S, \Xi, L_{t_1}) = -2.375 + 0.250\beta_{t_1}$	{10.000,10.833}	0.417
$LBP_1 \longrightarrow HFT: MSP_{1,t_1}$	$(\beta_{t_1} - P_1 - MF) \times 1 = -10.500 + \beta_{t_1}$	{10.833,11.000}	0.083

2.14 Market Quality and Welfare

Tables 2.19, 2.20,, 2.22, 2.23 show the market quality and welfare results for the 2-period model, both large and small tick, and for the 3-period model with HFTs.

TABLE 2.19: **2-Period Large Tick Market: Market Quality and Welfare with profit-maximizing access pricing.** This table reports for each support of the investors' personal evaluation considered (row 1), our metrics of market quality and welfare. The equilibrium make and take fee (MF and TF) are reported in rows 2 and 3, and the extremes of the investors' support ($\underline{\beta}$ and $\bar{\beta}$ max) are reported in rows 4 and 5. The investors' supports are expressed in terms of the tick size of the large tick, τ. The shaded area indicates the DL region with rebate-based pricing.

Δ	**0.333 τ**		**1.000 τ**		**1.270 τ**		**1.880 τ**		**2.000 τ**		**3.000 τ**		**4.000 τ**	**5.000 τ**
MF	-0.444	0.556	-0.333	0.667	-0.288	0.712	-0.187	0.813	-0.167	0.833	0.000	1.000	1.000	0.500
TF	0.556	-0.444	0.667	-0.333	0.712	-0.288	0.813	-0.187	0.833	-0.167	1.000	0.000	0.500	1.000
$\bar{\beta}$	10.167	10.167	10.500	10.500	10.635	10.635	10.940	10.940	11.000	11.000	11.500	11.500	12.000	12.500
$\underline{\beta}$	9.833	9.833	9.500	9.500	9.365	9.365	9.060	9.060	9.000	9.000	8.500	8.500	8.000	7.500
Depth P_2 (t_1)	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
Depth P_1 (t_1)	0.333	0.033	0.333	0.333	0.333	0.333	0.333	0.333	0.333	0.333	0.333	0.333	0.375	0.500
Depth P_{-1} (t_1)	0.333	0.033	0.333	0.333	0.333	0.333	0.333	0.333	0.333	0.333	0.333	0.333	0.375	0.500
Depth P_{-2} (t_1)	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
Depth (t_1)	0.667	0.667	0.667	0.667	0.667	0.667	0.667	0.667	0.667	0.667	0.667	0.667	0.750	1.000
Volume (t_2)	0.222	0.222	0.222	0.222	0.222	0.222	0.222	0.222	0.222	0.222	0.222	0.222	0.187	0.240
Eff Spread (t_2)	-1.000	1.000	-1.000	1.000	-1.000	1.000	-1.000	1.000	-1.000	1.000	1.000	1.000	1.000	0.333
Welfare LO INV	0.012	0.012	0.037	0.037	0.047	0.047	0.070	0.070	0.074	0.074	0.111	0.111	0.141	0.260
Welfare MO INV	0.012	0.012	0.037	0.037	0.047	0.047	0.070	0.070	0.074	0.074	0.111	0.111	0.094	0.160
Welfare INV	0.025	0.025	0.074	0.074	0.094	0.094	0.139	0.139	0.148	0.148	0.222	0.222	0.234	0.420
Welfare Exchange	0.025	0.025	0.074	0.074	0.094	0.094	0.139	0.139	0.148	0.148	0.222	0.222	0.281	0.360
Welfare Tot	0.049	0.049	0.148	0.148	0.188	0.188	0.279	0.279	0.296	0.296	0.444	0.444	0.516	0.780
Depth P_2 Benchmark (t_1)	0.000		0.000		0.000		0.000		0.000		0.000		0.000	0.100
Depth P_1 Benchmark (t_1)	0.000		0.000		0.500		0.500		0.500		0.500		0.500	0.400
Depth P_{-1} Benchmark (t_1)	0.000		0.000		0.500		0.500		0.500		0.500		0.500	0.400
Depth P_{-2} Benchmark (t_1)	0.000		0.000		0.000		0.000		0.000		0.000		0.000	0.100
Depth Benchmark (t_1)	0.000		0.000		1.000		1.000		1.000		1.000		1.000	1.000
Volume Benchmark (t_2)	0.000		0.000		0.106		0.234		0.250		0.333		0.375	0.360
Eff Spread Benchmark (t_2)	0.000		1.000		1.000		1.000		1.000		1.000		1.000	1.222
W. Benchmark LO INV	0.000		0.000		0.087		0.227		0.250		0.417		0.563	0.710
W. Benchmark MO INV	0.000		0.000		0.007		0.051		0.063		0.167		0.281	0.340
W. Benchmark Tot	0.000		0.000		0.094		0.279		0.313		0.583		0.844	1.050

TABLE 2.20: **2-Period Small Tick Market: Market Quality and Welfare.** This Table reports for each support of the investors' personal evaluation considered (row 1), our metrics of market quality and welfare. The equilibrium make and take fee (MF and TF) are reported in rows 2 and 3, and the extremes of the investors' support (β min and β max) are reported in rows 4 and 5. The investors' supports are expressed in terms of the tick size of the small tick, $1/3\ \tau$.

Δ	**0.333** τ		**0.420** τ		**0.630** τ		**1.000** τ		**2.000** τ	**3.000** τ	**4.000** τ	**5.000** τ
MF	-0.111	0.222	-0.097	0.237	-0.062	0.272	0.333	0.000	0.333	0.333	0.333	0.333
TF	0.222	-0.111	0.237	-0.097	0.272	-0.062	0.000	0.333	0.292	0.333	0.333	0.333
β **min**	9.833	9.833	9.790	9.790	9.685	9.685	9.500	9.500	9.000	8.500	8.000	7.500
β **max**	10.167	10.167	10.210	10.210	10.315	10.315	10.500	10.500	11.000	11.500	12.000	12.500
Ave Depth p_5 (t_1)	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
Ave Depth p_4 (t_1)	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.100
Ave Depth p_3 (t_1)	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.056	0.167	0.133
Ave Depth p_2 (t_1)	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.187	0.222	0.167	0.133
Ave Depth p_1 (t_1)	0.333	0.333	0.333	0.333	0.333	0.333	0.333	0.333	0.313	0.222	0.167	0.133
Ave Depth p_{-1} (t_1)	0.333	0.333	0.333	0.333	0.333	0.333	0.333	0.333	0.313	0.222	0.167	0.133
Ave Depth p_{-2} (t_1)	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.187	0.222	0.167	0.133
Ave Depth p_{-3} (t_1)	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.056	0.167	0.133
Ave Depth p_{-4} (t_1)	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.100
Ave Depth p_{-5} (t_1)	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
Depth (t_1)	0.666	0.668	0.667	0.667	0.667	0.667	0.667	0.667	1.000	1.000	1.000	1.000
Volume (t_2)	0.222	0.222	0.222	0.222	0.222	0.222	0.222	0.222	0.208	0.259	0.292	0.307
Ave Eff Spread (t_2)	-0.333	0.333	-0.333	0.333	-0.333	0.333	-0.333	0.333	0.458	0.651	0.873	1.097
Welfare INV MO	0.012	0.012	0.016	0.016	0.023	0.023	0.037	0.037	0.050	0.109	0.179	0.248
Welfare INV LO	0.012	0.012	0.016	0.016	0.023	0.023	0.037	0.037	0.102	0.221	0.359	0.500
Welfare INV	0.025	0.025	0.031	0.031	0.047	0.047	0.074	0.074	0.152	0.330	0.538	0.748
Welfare Exchange	0.025	0.025	0.031	0.031	0.047	0.047	0.074	0.074	0.130	0.173	0.194	0.204
Wefare Tot	0.049	0.049	0.062	0.062	0.093	0.093	0.148	0.148	0.282	0.503	0.733	0.952
Ave Depth p_5 Benchmark (t_1)	0.000		0.000		0.000		0.000		0.000	0.000	0.000	0.000
Ave Depth p_4 Benchmark (t_1)	0.000		0.000		0.000		0.000		0.000	0.000	0.000	0.100
Ave Depth p_3 Benchmark (t_1)	0.000		0.000		0.000		0.000		0.000	0.056	0.167	0.133
Ave Depth p_2 Benchmark (t_1)	0.000		0.000		0.000		0.000		0.167	0.222	0.167	0.133
Ave Depth p_1 Benchmark (t_1)	0.000		0.500		0.500		0.500		0.333	0.222	0.167	0.133
Ave Depth p_{-1} Benchmark (t_1)	0.000		0.500		0.500		0.500		0.333	0.222	0.167	0.133
Ave Depth p_{-2} Benchmark (t_1)	0.000		0.000		0.000		0.000		0.167	0.222	0.167	0.133
Ave Depth p_{-3} Benchmark (t_1)	0.000		0.000		0.000		0.000		0.000	0.056	0.167	0.133
Ave Depth p_{-4} Benchmark (t_1)	0.000		0.000		0.000		0.000		0.000	0.000	0.000	0.100
Ave Depth p_{-5} Benchmark (t_1)	0.000		0.000		0.000		0.000		0.000	0.000	0.000	0.000
Depth Benchmark (t_1)	0.000		1.000		1.000		1.000		1.000	1.000	1.000	1.000
Volume Benchmark (t_2)	0.000		0.103		0.235		0.333		0.361	0.370	0.375	0.373
Ave Eff Spread Benchmark (t_2)	0.000		0.333		0.333		0.333		0.487	0.689	0.901	1.127
Welfare Benchmark INV LO	0.000		0.028		0.076		0.139		0.287	0.434	0.581	0.727
Welfare Benchmark INV MO	0.000		0.002		0.017		0.056		0.137	0.214	0.291	0.361
Welfare Benchmark Tot	0.000		0.030		0.094		0.194		0.424	0.648	0.872	1.089

TABLE 2.21: **3-Period Large Tick Market: Market Quality and Welfare.** This Table reports for each support of the investors' personal evaluation considered (row 1), our metrics of market quality and welfare. The equilibrium make and take fee (MF and TF) are reported in rows 2 and 3, and the extremes of the investors' support (β min and β max) are reported in rows 4 and 5. The investors' supports are expressed in terms of the tick size of the large tick, τ.

Δ	**0.333** τ		**1.000** τ		**1.200** τ		**1.620** τ		**1.667** τ		**2.000** τ		**3.000** τ	**4.000** τ	**5.000** τ
MF	0.572	-0.428	0.717	-0.284	0.760	-0.241	0.850	-0.150	0.860	-0.139	0.933	-0.068	0.983	1.000	0.740
TF	-0.443	0.557	-0.328	0.672	-0.294	0.707	-0.221	0.779	-0.213	0.786	-0.156	0.845	0.115	0.422	1.000
β **min**	9.833	9.833	9.500	9.500	9.400	9.400	9.190	9.190	9.167	9.167	9.000	9.000	8.500	8.000	7.500
β **max**	10.167	10.167	10.500	10.500	10.600	10.600	10.810	10.810	10.833	10.833	11.000	11.000	11.500	12.000	12.500
Ave Depth P_2 (t_1, t_2)	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.200	0.101
Ave Depth P_1 (t_1, t_2)	0.298	0.298	0.298	0.298	0.298	0.298	0.298	0.298	0.298	0.298	0.298	0.298	0.329	0.419	0.526
Ave Depth P_{-1} (t_1, t_2)	0.298	0.298	0.298	0.298	0.298	0.298	0.298	0.298	0.298	0.298	0.298	0.298	0.329	0.419	0.526
Ave Depth P_{-2} (t_1, t_2)	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.200	0.101
Depth (t_1, t_2)	0.597	0.597	0.597	0.597	0.597	0.597	0.597	0.597	0.597	0.597	0.597	0.597	0.659	1.239	1.254
Volume (t_2, t_3)	0.391	0.393	0.392	0.392	0.391	0.391	0.392	0.392	0.392	0.392	0.392	0.391	0.395	0.430	0.451
Ave Eff Spread (t_2, t_3)	0.392	-0.392	0.392	-0.392	0.392	-0.392	0.392	-0.392	0.392	-0.392	0.392	-0.392	0.109	0.475	0.045
Welfare INV MO	0.021	0.021	0.064	0.064	0.077	0.077	0.104	0.104	0.107	0.107	0.128	0.128	0.197	0.264	0.351
Welfare INV LO	0.019	0.019	0.056	0.056	0.067	0.067	0.090	0.090	0.093	0.093	0.111	0.111	0.167	0.293	0.384
Welfare INV	0.040	0.040	0.120	0.120	0.144	0.144	0.194	0.194	0.200	0.200	0.239	0.239	0.364	0.557	0.735
Welfare Exchange	0.051	0.051	0.152	0.152	0.183	0.183	0.246	0.246	0.253	0.253	0.304	0.304	0.451	0.612	0.785
Welfare Tot	0.091	0.091	0.272	0.272	0.326	0.326	0.440	0.440	0.453	0.453	0.544	0.544	0.815	1.169	1.520
Ave Depth P_2 Benchmark (t_1, t_2)	0.000		0.000		0.000		0.000		0.000		0.000		0.000	0.056	0.073
Ave Depth P_1 Benchmark (t_1, t_2)	0.000		0.000		0.740		0.694		0.689		0.689		0.578	0.539	0.516
Ave Depth P_{-1} Benchmark (t_1, t_2)	0.000		0.000		0.740		0.694		0.689		0.689		0.578	0.539	0.516
Ave Depth P_{-2} Benchmark (t_1, t_2)	0.000		0.000		0.000		0.000		0.000		0.000		0.000	0.056	0.073
Depth Benchmark (t_1, t_2)	0.000		0.000		1.479		1.389		1.378		1.378		1.156	1.190	1.177
Volume Benchmark (t_2, t_3)	0.000		0.000		0.246		0.528		0.548		0.548		0.767	0.767	0.769
Ave Eff Spread Benchmark (t_2, t_3)	0.000		0.000		0.136		0.070		0.064		0.064		-0.076	0.014	0.070
Welfare Benchmark INV LO	0.000		0.000		0.111		0.281		0.296		0.393		0.619	0.852	1.072
Welfare Benchmark INV MO	0.000		0.000		0.057		0.195		0.0.211		0.319		0.614	0.819	1.015
Welfare Benchmark Tot	0.000		0.000		0.168		0.476		0.507		0.712		1.233	1.672	2.087

TABLE 2.22: **3-Period Large Tick Market with HFTs: Market Quality and Welfare with access fees or rebates.** This Table reports for each support of the investors' personal evaluation considered (row 1) our metrics of market quality and welfare. The equilibrium make and take fee (MF and TF) are reported in rows 2 and 3, and the extremes of the investors support ($\underline{\beta}$ and $\bar{\beta}$) are reported in rows 4 and 5. The investors' supports are expressed in terms of the tick size of the large tick, τ. Note that $0.000^* = 1 \cdot 10^{-7}$ and $0.185^* = 0.185 + 1 \cdot 10^{-7}$.

Δ	**0.333** τ		**1.000** τ		**1.270** τ	**2.000** τ	**3.000** τ	**3.900** τ	**4.000** τ	**5.000** τ
MF	-0.415	0.585	-0.250	0.750	0.817	1.000	1.000	1.000	0.520	0.757
TF	0.500	-0.500	0.500	-0.500	-0.500	-0.500	-0.500	-0.500	0.500	0.500
$\bar{\textbf{fi}}$	10.167	10.167	10.500	10.500	10.635	11.000	11.500	11.950	12.000	12.500
$\underline{\textbf{fi}}$	9.833	9.833	9.500	9.500	9.365	9.000	8.500	8.050	8.000	7.500
Ave Depth P_2 (t_1, t_2)	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
Ave Depth P_1 (t_1, t_2)	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.439	0.373
Ave Depth P_{-1} (t_1, t_2)	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.439	0.373
Ave Depth P_{-2} (t_1, t_2)	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
Depth (t_1, t_2)	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.878	0.747
Volume (t_1, t_2)	0.980	0.980	1.000	1.000	1.001	1.000	1.333	1.487	0.649	0.671
Volume (t_2, t_3)	0.980	0.980	1.000	1.000	1.001	1.000	1.333	1.487	0.884	0.970
Volume (t_1,t_2, t_3)	1.470	1.470	1.500	1.500	1.502	1.500	2.000	2.231	1.144	1.211
Ave Quoted Spread (t_1, t_2)	5.000	5.000	5.000	5.000	5.000	5.000	5.000	5.000	3.245	3.506
Ave Eff Spread (t_1, t_2)	0.000*	1.000*	0.000*	1.000*	1.000*	1.000*	1.000*	1.000*	0.000*	0.000*
Ave Eff Spread (t_2, t_3)	0.623*	0.377*	0.625*	0.375*	0.375*	0.375*	0.333*	0.314*	0.928*	0.911*
Ave Eff Spread (t_1,t_2, t_3)	0.082*	0.585*	0.083*	0.583*	0.583*	0.583*	0.556*	0.543*	0.285*	0.274*
Ave Eff SpreadMid (t_1, t_2)	2.000*	3.000*	2.000*	3.000*	1.125*	3.000*	3.000*	3.000*	2.149*	2.160*
Ave Eff SpreadMid (t_2, t_3)	0.755*	1.132*	0.750*	1.125*	1.125*	1.125*	1.000*	0.942*	1.729*	1.645*
Ave Eff SpreadMid (t_1,t_2, t_3)	1.170*	1.755*	1.167*	1.750*	1.750*	1.750*	1.667*	1.628*	1.819*	1.764*
Welfare INV MO	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.210	0.329
Welfare INV LO	0.060	0.060	0.188	0.188	0.239	0.375	1.000	1.617	0.686	0.876
Welfare INV	0.060	0.060	0.188	0.188	0.239	0.375	1.000	1.617	0.896	1.206
Welfare HFTs	0.000*	0.000*	0.000*	0.000*	0.000*	0.000*	0.000*	0.000*	0.000*	0.000*
Welfare Exchange	0.125	0.125	0.375	0.375	0.476	0.750	1.000	1.115	1.167	1.522
Welfare Tot	0.185*	0.185*	0.562*	0.562*	0.715*	1.125*	2.000*	2.732*	2.063*	2.728*

TABLE 2.23: **3-Period Large Tick Market with HFTs: Market Quality and Welfare, with no access fees or rebates.** This Table reports for each support of the investors' personal evaluation considered (row 1) our metrics of market quality and welfare. The extremes of the investors support ($\underline{\beta}$ and $\bar{\beta}$) are reported in rows 4 and 5. The investors' supports are expressed in terms of the tick size of the large tick, τ.

Δ	**0.333 τ**	**1.000 τ**	**1.270 τ**	**2.000 τ**	**3.000 τ**	**3.900 τ**	**4.000 τ**	**5.000 τ**
$\bar{\beta}$	10.167	10.500	10.635	11.000	11.500	11.950	12.000	12.500
$\underline{\beta}$	9.833	9.500	9.365	9.000	8.500	8.050	8.000	7.500
Ave Depth P_2 (t_1, t_2)	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
Ave Depth P_1 (t_1, t_2)	0.000	0.000	0.705	0.573	0.444	0.368	0.361	0.303
Ave Depth P_{-1} (t_1, t_2)	0.000	0.000	0.705	0.573	0.444	0.368	0.361	0.303
Ave Depth P_{-2} (t_1, t_2)	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
Depth (t_1, t_2)	0.000	0.000	1.410	1.146	0.888	0.736	0.722	0.607
Volume (t_1, t_2)	0.000	0.000	0.142	0.236	0.389	0.965	0.491	0.564
Volume (t_2, t_3)	0.000	0.000	0.329	0.403	0.556	1.269	0.641	0.698
Volume (t_1,t_2, t_3)	0.000	0.000	0.354	0.324	0.481	1.709	0.577	0.643
Ave Quoted Spread (t_1, t_2)	5.000	5.000	2.179	2.708	3.222	3.528	3.556	3.787
Ave Eff Spread (t_1, t_2)	0.500	0.500	0.000	0.000	0.000	0.000	0.000	0.000
Ave Eff Spread (t_2, t_3)	0.500	0.500	0.851	0.941	0.944	0.927	0.925	0.906
Ave Eff Spread (t_1,t_2, t_3)	0.333	0.333	0.234	0.294	0.296	0.285	0.284	0.270
Ave Eff SpreadMid (t_1, t_2)	0.000	0.000	2.052	2.104	2.111	2.104	2.103	2.093
Ave Eff SpreadMid (t_2, t_3)	0.000	0.000	1.690	1.793	1.681	1.553	1.540	1.418
Ave Eff SpreadMid (t_1,t_2, t_3)	0.000	0.000	1.793	1.862	1.787	1.702	1.693	1.612
Welfare INV LO	0.000	0.000	0.148	0.482	0.972	1.472	1.531	2.140
Welfare INV MO	0.000	0.000	0.015	0.120	0.269	0.372	0.381	0.465
Welfare HFT	0.000	0.000	0.073	0.269	0.481	0.623	0.637	0.753
Welfare Exchange	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
Welfare Tot	0.000	0.000	0.236	0.872	1.722	2.467	2.549	3.358

Chapter 3

Squeezing the Shorts in Small Cap Stocks

3.1 Introduction

In this paper, we examine the short squeeze in New Concept Energy and Avalon Holdings that made the price of these stocks rise more than 500% and then fall down within days. For New Concept Energy the short squeeze was initiated the day after it was publicly announced that Realty Advisors and New Concept Energy had agreed on a deal whereby Realty Advisors would pay $1.50/share to acquire the majority of New Concept Energy[1]. Following that news, the short squeezer started buying, and within two trading days the price rose from $1.7/share to a maximum of over $12/share. As a consequence of the massive buying by the short squeezer, the vast majority of short sellers could not find the shares to deliver within the settlement period and had to close their position by buying the shares back[2]. The scarcity of shares and the strong demand by short sellers who had to close their short positions made the price surge and allowed the squeezer to sell his shares at a substantial profit.

For Avalon Holdings the short squeeze was initiated on July 24, 2018. Without any news about the company, the price started to increase from $2.20/share to a peak of $28/share on July 30, 2018[3]. The alleged manipulator acquired 60.23% of the total number of Class A outstanding shares within four days. Avalon Holdings had 3,191,100 outstanding shares of class A and 612,231 shares of Class B. Holders of class A shares had one vote per share whereas holders of class B shares had ten votes per share. Ronald Klingle, the chairman and chief executive officer, owned all the class B shares giving him around 66.74% of the voting power in Avalon Holdings. Hence there was no market for corporate control. On July 30, 2018 following the numerous inquiries regarding the trading activity in the stock, Avalon Holdings specified in a press release that Mintbroker (the alleged manipulator) was not affiliated with the company and had no information about its intentions. Ronald Klingle also stated that he had no plans to divest any of his holdings[4].

The first objective of this paper is to analyze how an alleged manipulator traded and in which market conditions. We check if the manipulator timed liquidity, past returns, volume,

[1] In the month before the deal was made public the stock price moved between $1.32 and $1.94

[2] Regulation SHO requires to close-out any failing equity security that exists on settlement day (the second business day after trade date, or "T+2")

[3] In the previous month the price moved between $2.12 and $3.21.

[4] https://www.sec.gov/Archives/edgar/data/1061069/000143774918014025/ex_119330.htm

volatility and if he used market or limit orders. We find that the manipulator used both limit and market orders, traded during periods in which the spread was lower and acted as a contrarian: bought after prices had been falling and sold after prices had been increasing.

The second objective of this paper is to study how prices reacted to the trades of the alleged manipulator. We find that on average the price intradaily did not react to the manipulator's trades: his price impact was negative, suggesting that he was quite successful in hiding his footsteps. As predicted by Allen and Gale 1992 we find that the manipulator behavior is consistent with that of informed investors in Collin-Dufresne and Fos 2015, Kacperczyk and Pagnotta 2018 and Garriott and Riordan 2019.

We contribute to the literature on market manipulation by analyzing empirically how trade-based manipulation takes place on an intraday basis and how other market participants reacted to the manipulation. In today's market dominated by high-frequency traders and algorithmic trading the intraday analysis of the trades by the manipulator is essential to understand how the manipulator was able to move the price and profit from the manipulation. As a policy implication, our evidence suggests, it might not be efficient to impose the same settlement time for all stocks. Since small-cap stocks are easier to manipulate with a relatively modest amount of capital, a longer settlement period that allows short sellers more time to find someone to borrow the shares from, might be beneficial. Another way to discourage manipulators would be to shorten the time to report changes in beneficial ownership of a company. The investing public could then identify who are the large shareholders on a timely basis. This would likely hinder the strategy of a manipulator since short sellers would react to the new information selling at higher prices if selling at all. That in turn would make it hard for manipulators to secure an average buy price lower than the average sell price, and thus to profit from the squeeze.

Preventing market manipulation is important because manipulation damages investors by obstructing the efficient allocation of resources and lowering investor confidence in the fairness of the capital markets, potentially resulting in higher risk premiums and reduced investor participation. Despite the fact that preventing manipulation was a primary motivation for U.S. securities laws after the Great Depression, the Security and Exchange Act of 1934 does not define explicitly what constitutes market manipulation[5].

Kyle and Viswanathan 2008 propose to classify a trading strategy as illegal price manipulation if it undermines economic efficiency both by decreasing price accuracy and reducing liquidity. Their definition of price manipulation encompasses both short squeezes and corners[6], in which the manipulator obtains a large position in the asset so that it becomes extremely expensive for investors with short positions to obtain the asset to deliver. In a short squeeze or a corner prices are distorted by the dominant position of the manipulator, and if investors anticipate ex ante that the risk of being cornered or squeezed has increased, this larger adverse selection will induce them to reduce liquidity. Allen and Gale 1992 classify manipulation in trade-based, information-based and action-based manipulation. Trade-based manipulation involves influencing the price of a stock through trading. Information-based manipulation is

[5]Section 9(a)(2) of the Security and Exchange Act of 1934 prohibits effecting "a series of transactions" in a security (i) that "creat[e] actual or apparent active trading" or affect its price, (ii) "for the purpose of inducing the purchase or sale of such security by others."

[6]Allen, Litov, and Mei 2006 define corner as an extreme form of short squeeze, when the buy side has almost complete control of all floating shares.

based on providing false information or spreading false rumors in order to profit from the subsequent market reaction. Action-based manipulation by an officer or director involves taking actions (such as shut down a production plant) to affect the value or the perceived value of a firm.

Our analysis studies the dynamics of a trade-based manipulation in New Concept Energy and Avalon Holdings stocks during the summer of 2018. We are able to reconstruct each transaction made by the alleged manipulator and to study how he was able to acquire and unload the shares in the open market making a substantial profit.

3.2 Literature Review

The first to study theoretically the hypotheses required for manipulation to be profitable were: Hart 1977, Hart and Kreps 1986, Vila 1989, Allen and Gorton 1991, Allen and Gale 1992, Benabou and Laroque 1992, Jarrow 1992, Jarrow 1994.

Kumar and Seppi 1992 model a manipulator that takes a substantial long position in the futures market and then bids up the spot price before the closing to profit from a more favourable futures settlement price. Pirrong 1993 shows how squeezes hamper price discovery and create deadweight losses. Bagnoli and Lipman 1996 study stock price manipulation by a bidder that earns profits by making a takeover bid solely to manipulate the target firm's stock price. Since the market cannot tell if the bid is serious, the market price of the target firm's stock rises, generating profits for a manipulator. Vitale 2000 considers manipulation in foreign exchange markets. Van Bommel 2003 shows the role of rumors in facilitating price manipulation. Goldstein and Guembel 2003 show that the signaling role of prices opens up the possibility of asset price manipulation: a speculator with a short position has an interest in depressing the stock price by short selling, since the manager in turn will think the lower price may reflect bad information and will reduce the level of investment. This in turn will reduce the true value of the firm. Chakraborty and Yılmaz 2004a and Chakraborty and Yılmaz 2004b show that in Glosten and P. R. Milgrom 1985 and Kyle 1985 models, informed traders gain by manipulating the market. Before private information is revealed, informed traders will manipulate the market by initially trading in the opposite direction to their information. This results in losses in the short run for informed traders, but the increased noise in the trading process allows them to retain their informational advantage longer and extract more profit from their information. In Chakraborty and Yilmaz 2008 when there are many competitive rational traders with coarser information than the insider but finer information than the market maker, the manipulator has an incentive to manipulate because the competitive rational traders follow the insider's trades in equilibrium. Hillion and Suominen 2004 show that closing call auctions reduce manipulation and improve price efficiency. Bernhardt and Davies 2009 suggest that fund managers have incentives to use short-term price impacts to manipulate closing prices at the end of reporting periods.

In contrast to theoretical studies, empirical studies of manipulation cases are scarcer due to data unavailability and the difficulty of ruling out all possible alternative explanations. Jegadeesh 1993 and B. D. Jordan and S. D. Jordan 1996 examine the market corner of a Treasury note auction by Salomon brothers' in 1991. Felixson and Pelli 1999 build a model to test for closing price manipulation in the Finnish stock market. Mei, Wu, and Zhou 2004 implement a model that shows that due to investors' behavioral biases and limits to arbitrage, a manipulator can profit from a "pump-and-dump" trading strategy. They also present empirical evidence from the U.S. SEC prosecution of "pump-and-dump" manipulation cases that are consistent with their model. Khwaja and Mian 2005 find evidence of price manipulation by brokers using a dataset from the Pakistani stock market. They find that brokers earn at least 8% higher returns on their trades and that neither market timing nor liquidity provision can convincingly explain this result. Merrick Jr, Naik, and Yadav 2005 examine the strategic trading behavior of market participants during an attempted delivery squeeze in a bond futures contract traded on the London International Financial Futures and Options Exchange (LIFFE). They document how market prices and market depth were distorted and estimate the profits of strategic traders during the different phases of the squeeze. Aggarwal and Wu 2006 use a dataset of SEC actions in cases of stock manipulation to show that stocks generally experience a price increase during the manipulation period, a decrease during the post-manipulation period, and an increased volatility. Comerton-Forde and Rydge 2006 study manipulation in closing call auctions. They find that traders exhibit similar behavior across markets by submitting large, unrepresentative orders in the final seconds of the call auction. Comerton-Forde and Putniņš 2011 find that closing price manipulation causes large increases in day-end returns, subsequent return reversals, increased trading volume and wider spreads. They also find that most of the abnormal returns are reversed by the following morning. Allen et al. 2019 study the impact of the Porsche-Volkswagen short squeeze on market quality and informational risk. See Putniņš 2012 for a exhaustive survey on market manipulation.

The two papers are closest to ours are Merrick Jr, Naik, and Yadav 2005 that analyze a delivery squeeze in U.K. bond futures contracts, and Allen et al. 2019 that study the short squeeze caused by Porsche when it announced the acquisition of the majority of Volkswagen stocks through an elaborate strategy employing derivatives. Instead we examine trade-based manipulation in two U.S. small-cap stocks without the confounding effects due to derivatives written on the stocks. The difference between trade-based manipulation studied in this paper and information-based manipulation analyzed in Allen et al. 2019 is that in our empirical setting stock prices were manipulated through trading whereas in Allen et al. 2019 Porsche manipulated Volkswagen's stock price by releasing information into the market.

We argue that to examine manipulation, and trade-based manipulation in particular, it is essential to look at each trade made the manipulator, since only by examining the sequence of trades we can understand how the manipulator was able to distort prices. This is even more important in today's financial markets dominated by machines, in which prices react to information in fractions of a second.

3.3 Institutional background

Mintbroker is an online broker dealer based in the Bahamas founded in 2011 by Guy Gentile. It was often cited in the news after the stocks of Avalon Holdings Corporation, New Concept Energy Inc. and MER Telemanagement Solutions Ltd. first skyrocketed and then dropped around the time Mintbroker revealed stakes in the firms[7]. The owner of Mintbroker, Guy Gentile, started working as an informant for the FBI in 2012 to avoid jail time, after he was accused of orchestrating two pump-and-dump schemes in which investors lost $17 million. The Justice Department brought charges against him in 2016, but the case was dismissed because prosecutors waited too long to file it and he is still being investigated by the SEC in what so far has been a 9 years long investigation.

On August 13, 2018 both Mintbroker and its owner Guy Gentile were sued in the Southern District of New York by Avalon Holdings to recover the short swing profits[8] realized by the defendant through the purchases and sales of Avalon Holdings common stocks. Subsequently, on September 28, 2018 Mintbroker and Guy Gentile were sued again in the Southern District of New York by New Concept Energy for the same reason: to recover the short swing profits, this time realized through the purchases and sales of New Concept Energy common stocks. Our analysis is based on data from the civil case[9] held at the Southern District of New York that to our knowledge have not been used before. The documents and the data used are retrieved from pacer.gov that provides online access to records and documents of the United States district courts, United States courts of appeals, and United States bankruptcy courts. This data allow us to reconstruct Mintbroker's accumulation and subsequent sale of both New Concept Energy and Avalon Holdings shares at high frequency. In particular the data contains the price, the size, the side and the time of each trade made by Mintbroker in New Concept Energy and Avalon Holdings stocks.

First we describe the business of New Concept Energy and show that business conditions or corporate events could not explain the price swings. Then we do the same for Avalon Holdings. New Concept Energy operates oil and gas wells as well as mineral leases in Ohio and West Virginia through the subsidiaries Mountaineer State Energy and Mountaineer State Operations. Until March 30, 2017 the company also leased and operated a retirement center in King City, Oregon with a capacity of 114 residents. The terms of the lease agreement provided that if the facility were sold to a third party the lease would be terminated. On March 30, 2017 the owners of the facility sold it and therefore the activities related to the lease of the retirement center were terminated. On March 31, 2018, the company had assets of $502,000 and liabilities of $590,000. Cash and cash equivalents were $397,000. Net cash provided by operating activities was $1,000 for the three months ended on March 31, 2018. New Concept Energy reported a net loss of $134,000 for three months ended on March 31, 2018, compared to net loss of $159,000 for the same period in 2017. By and large New Concept Energy was a small stagnant company looking for ways to grow its revenues and profits.

On June 27, 2018 Realty Advisors filed schedule 13D with the SEC "in an abundance of caution to reflect the entry into an agreement ..., the consummation of which is subject to stockholder approval as long as the Shares are listed and traded on the NYSE American."[10] The agreement between Realty Advisors and New Concept Energy, signed on May 22, 2018, stated

[7]https://www.bloomberg.com/news/articles/2018-07-30/tiny-waste-management-stock-soars-more-than-1-500-in-one-week
https://www.reuters.com/article/avalon-holdg-stock/avalon-holdings-slump-after-5-day-surge-idUSL1N1UR1S3

[8]the law defines short swing profits as profits earned within a short period of time

[9]Case 1:18-cv-07291

[10]This statement is taken directly from the original 13D schedule.

that Realty Advisors had agreed to acquire 3,000,000 newly issued shares of New Concept Energy common stock at a price of $1.50 per share in cash. Realty Advisors in the same document made clear that had no plan to acquire the entire equity interest in New Concept Energy. Both New Concept Energy and Realty Advisor agreed on the price of $1.50/share for the majority of the shares. Thus the management of New Concept Energy implicitly considered the price of $1.50/share a fair valuation for the company[11].

The day after the agreement was released publicly, on June 28, 2018, after opening higher at $3.05 from the previous closing at $1.42, the share price of New Concept Energy started to fall and align with the valuation agreed between Realty Advisors and New Concept Energy ($1.50). The closing price on that day was $1.72. Then on June 29, as shown in Figure 1, the price of New Concept Energy rose from $1.66 to $4.22 per share. On July 2, the price climbed even higher, to a maximum of $12.75 per share and closed at $8.9/share. On July 3, the opening price was $12 per share and then moments after the open, the price started to fall. The closing price on that day was $4.11/share. On July 5 the stock opened at $5.03 and closed at $4.95. On July 6 the stock opened at $4.76 per share, was fairly stable throughout the day, and closed at $4.26/share. On July 9 the opening price was $4.15 per share, the stock continued to decline throughout day and closed at $3.31 per share. Finally on July 10, New Concept Energy stock opened at $3.32 and closed at $3.88.

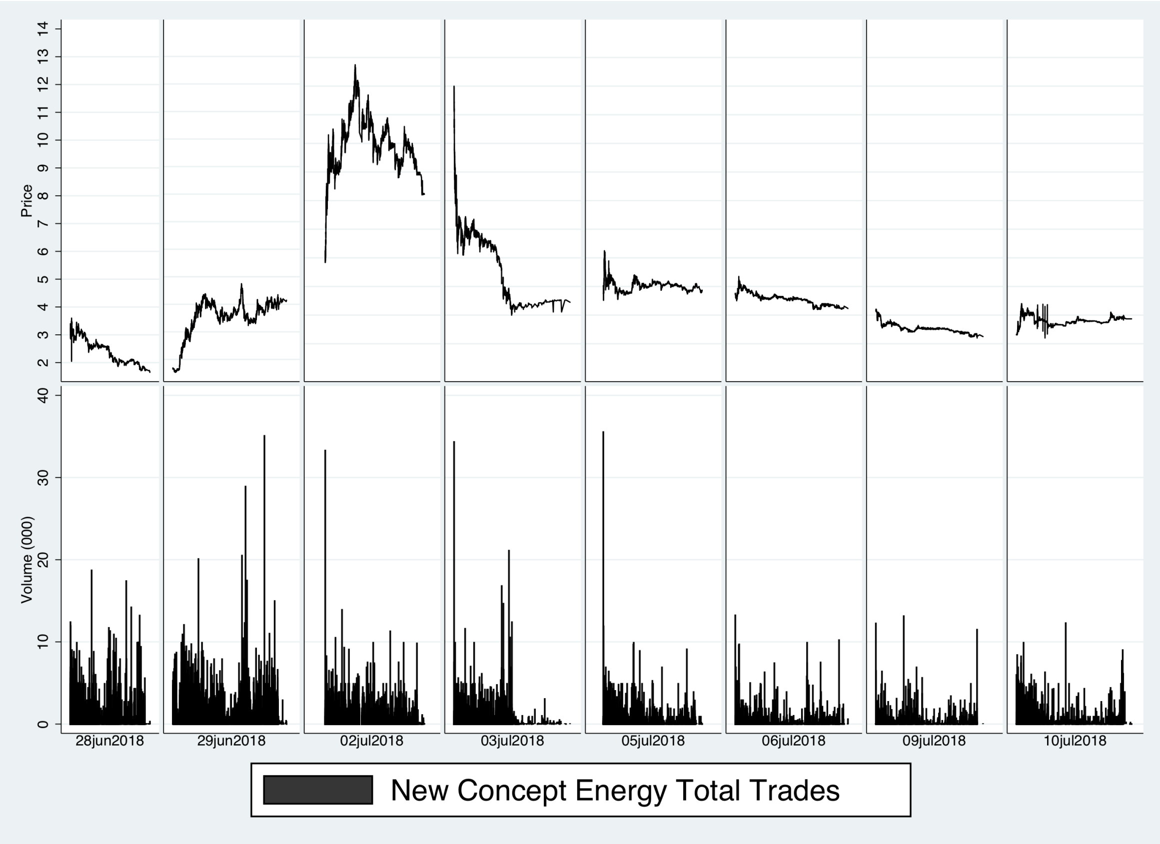
As can be seen from Figure 1, the price swings are remarkable: in 8 days the stock price went from less than $2 per share to a maximum of $12.75/share (an increase of 537.5%), and then back to less than $4 per share. Figure 2 shows Mintbroker trades and volume during the period in which it was active in New Concept Energy: on July 29, 2018 Mintbroker volume was massive, being present in almost all the trades made that day.

Two weeks after the suspicious activity in New Concept Energy shares, the shares of Avalon Holdings shown in Figure 3 started to rise. Avalon Holdings is a waste management company that operates in the Northeast and the Midwest of the United States. In addition to waste management, Avalon operates three golf courses and a hotel. Net cash provided by operating activities was $770,000 for the first three months of 2018. Avalon Holdings reported a net loss of $914,000 for three months ended March 31, 2018, compared to net loss of $1,019,000 for the same period in 2017.

On July 24, 2018, while there was no news related the company, the price of Avalon Holding rose from $2.22 to $3.35 per share. On July 25, the price kept on climbing to a maximum of $4.49 and closed at $4.32. On July 26, the opening price was $4.76 per share, again kept on rising for the entire day and closed at $5.69. On July 27 the opening price was $6.35, the intraday high was $11.21 and closed at $10.25. On this day Mintbroker filed a form 3 with the S.E.C. to reveal that it had acquired 1,922,095 Class A shares of Avalon Holdings corresponding to 60.23% of the outstanding shares. On July 30 the price opened at $16.97, reached an intraday high of $28 and then started to fall. The closing price on that day was $6.06. On July 30 Avalon Holdings issued a statement in which reiterated that Mr. Ronald Klingle, Chairman and Chief Executive Officer, held 66.74% of the voting power and had no plan to to divest any of his holdings. On

[11]New Concept Energy reported later, in schedule 14A on September 4, 2018 that "On several trading days, reliable published financial sources reported prices and volume of trading of the Company's common stock which are not capable of explanation by the Company,... the Company made inquiries of the market maker and others, reviewed the trades to the extent possible and searched (without success) for an answer to any reason for this kind of activity. The Company has no evidence of any reasonable answer."

FIGURE 1: **New Concept Energy Price and Volume** This figure reports the price and volume of New Concept Energy during the period in which Mintbroker was active in the stock: from June 28, 2018 to July 10, 2018.

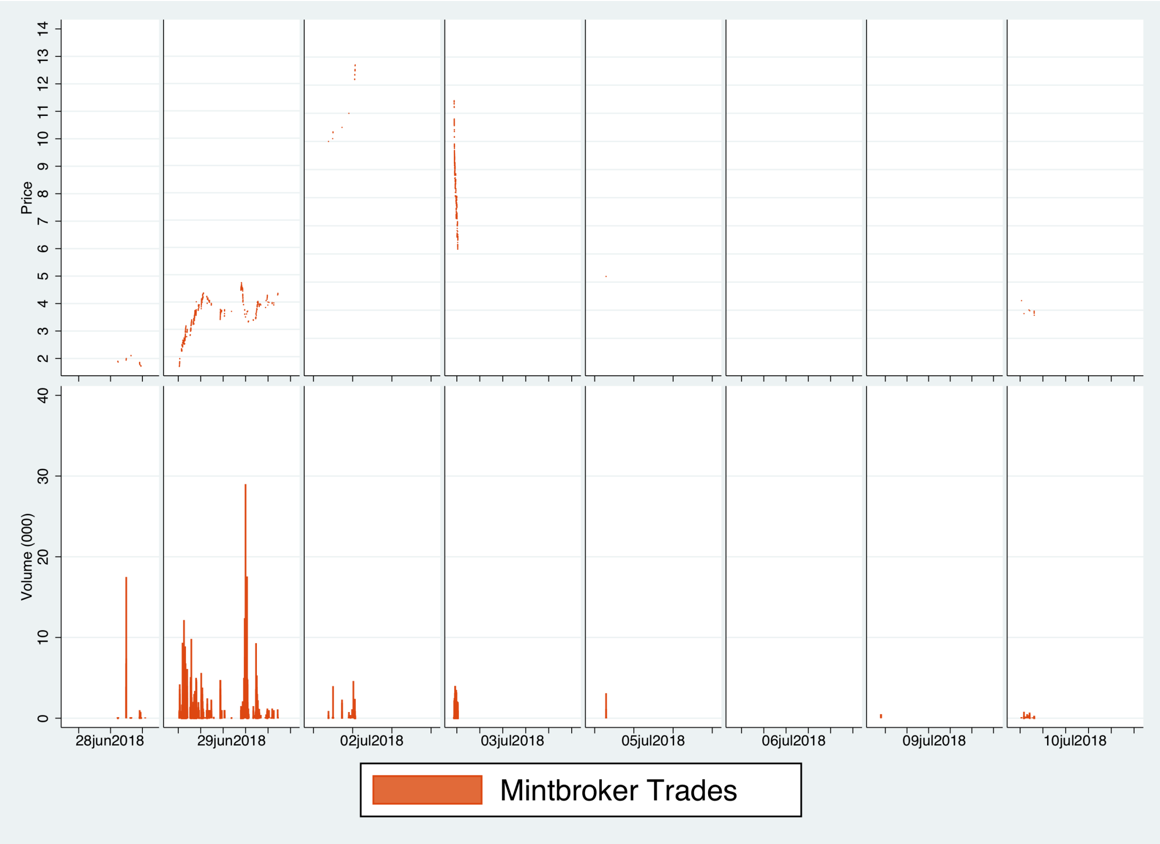


July 31, the price continued to fall, after opening at $5.80, closed at $3.67. On August 1, 2018 the last day in which Mintbroker was active, the stock opened at $4.24 and closed at $3.47.

The price swings in Avalon Holdings are even larger than those in New Concept Energy: in 7 days the stock price went from $2.22 to a maximum of $28 (an increase of 1,172.7%). Figure 4 shows Mintbroker trades and volume from July 24, 2018 to August 1, 2018. We note that Mintbroker volume is more evenly distributed throughout the period compared to its volume in New Concept Energy.

Despite the remarkable price action in New Concept Energy from June 28, 2018 to July 10, 2018 shown in Figure 1 there was no news that could justify these price movements. The market index during that period of time shown in Figure 5 was fairly stable, especially from June 28 to July 5, the period during which most of the variation in the price of New Concept Energy happened. The same is true for the CRSP U.S. Oil and Gas Index shown in Figure 7, from June 28 to July 5. In particular, there was no large increase or decrease in the index and what was gained in one day was pretty much lost in the next. The same is true for Avalon Holdings during the period in which Mintbroker was active in the stock shown in Figure 3: from July 24, 2018 to August 1, 2018 the market index in Figure 6 was stable if not declining.

FIGURE 2: **Mintbroker Trades in New Concept Energy** This figure reports Mintbroker trades in New Concept Energy during the period in which Mintbroker was active in the stock: from June 28, 2018 to July 10, 2018.

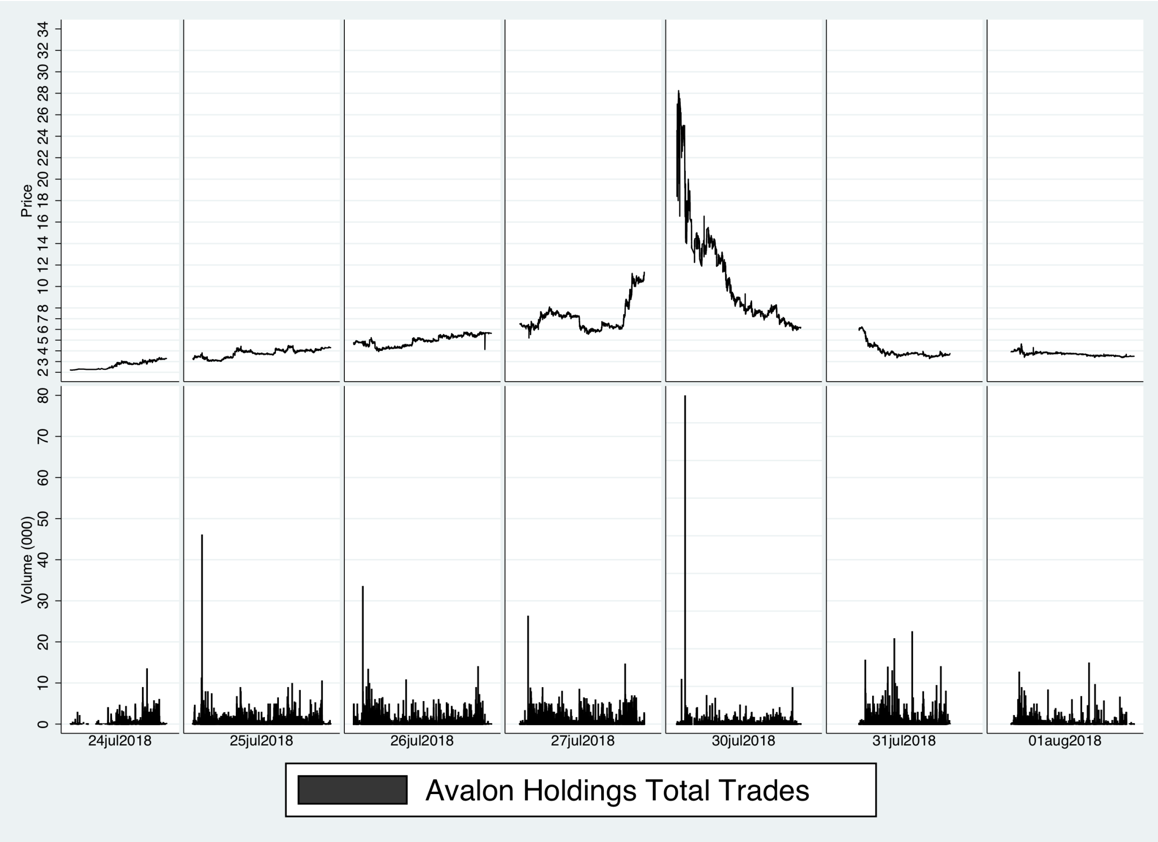


3.4 Data

Our primary data source is the evidence for Case 1:18-cv-07291 filed on August 13, 2018 in the Southern District of New York of New Concept Energy and Avalon Holdings against both Mintbroker and its owner Guy Gentile for failing to file with the Security and Exchange Commission forms 3 and 4 of Section 16(a), failing to file schedule 13D and subsequent amendments, and to recover the short-swing profits realized by the defendants. We are the first to collect data from these documents for an empirical analysis of market manipulation. This unique dataset allows us to reconstruct in detail the timeline of events as well as the sequence of Mintbroker trades.

The series of events that led to Mintbroker being sued by both New Concept Energy and Avalon Holdings is the following: on June 29, 2018 Mintbroker reported to the SEC in the initial statement of beneficial ownership of securities, that it had acquired 1,073,713 or 50.36% of the total outstanding shares of New concept Energy (the total number of shares issued was 2,131,935) common stock. Two business days later, on July 3, 2018, Mintbroker reported that it had sold all the shares it had acquired before and did not own any share of New Concept

FIGURE 3: **Avalon Holdings Price and Volume** This figure reports the price and volume of Avalon Holdings during the period in which Mintbroker was active in the stock: from July 24, 2018 to August 1, 2018.



Energy anymore. On July 27, 2018 Mintbroker communicated to the SEC that had acquired 1,922,095 or 60.23% of the outstanding shares of Avalon Holdings and on July 30, 2018 that it had sold 192,340 shares of Avalon Holdings. On July 31, 2018 Mintbroker revealed that it had sold 719,885 shares of Avalon Holdings and on August 1, 2018 that it had sold the rest of the shares. On August 13, 2018 Mintbroker and its owner Guy Gentile were sued by Avalon Holdings for failing to file with the SEC forms 3 and 4 of Section 16(a) and schedule 13D and subsequent amendments. Then on August 22, 2018 Mintbroker filed schedule 13D with the SEC for the shares acquired in Avalon Holdings. Almost one month later, on September 17, 2018 Mintbroker filed schedule 13D with the SEC for New Concept Energy. On September 28, 2018 New Concept Energy sued both Mintbroker and Guy Gentile for failing to file with SEC forms 3 and 4 of Section 16(a), schedule 13D and subsequent amendments. The purpose of the suits by both New Concept Energy and Avalon Holdings was to recover the short swing profits made by Mintbroker.

Short swing profits are addressed in section 16(b) of the Securities Exchange Act of 1934, which states that profits made by someone who is directly or indirectly the beneficial owner of more than 10% of any class of any equity security within any period of less than six months

FIGURE 4: **Mintbroker Trades in Avalon Holdings** This figure reports Mintbroker trades in Avalon Holdings during the period in which Mintbroker was active in the stock: from July 24, 2018 to August 1, 2018.

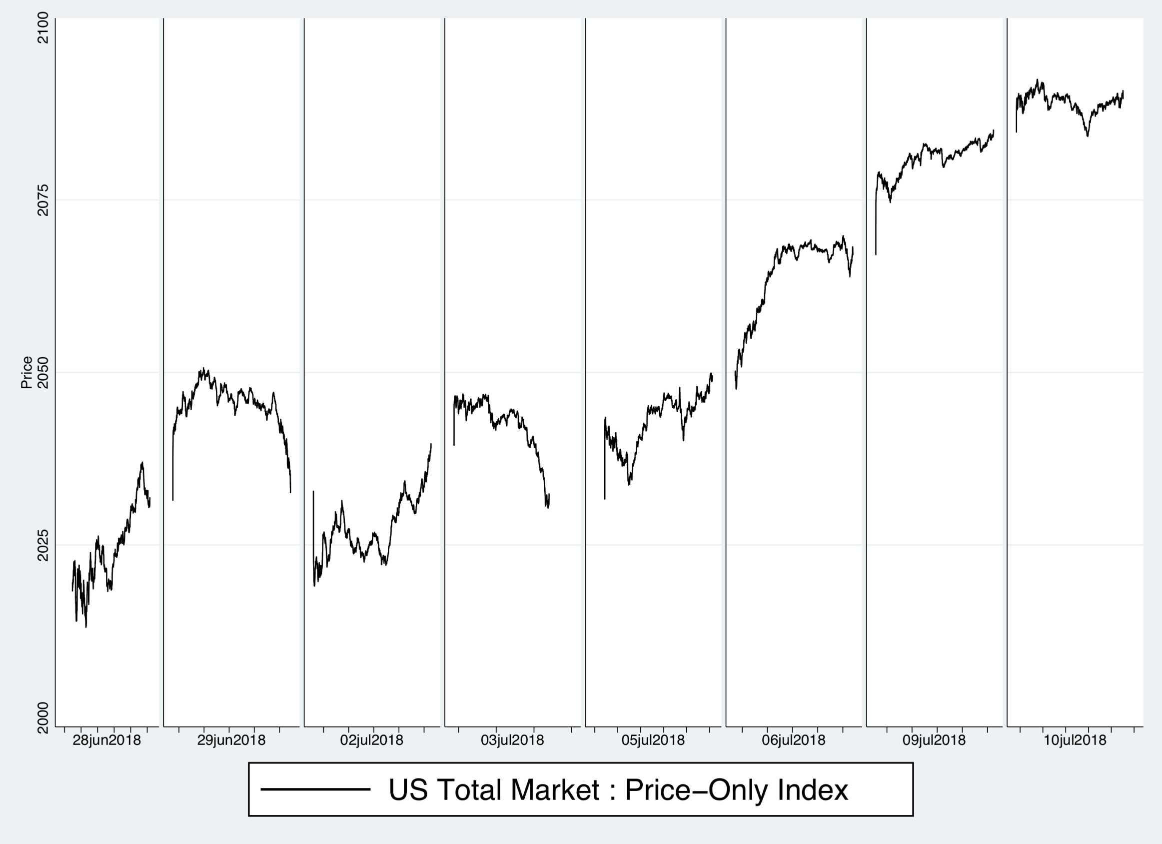
has to be recoverable by the issuer, irrespective of any intention on the part of such beneficial owner[12].

We obtained the data on Mintbroker transactions downloading the documents presented as evidence in court from PACER[13]. We then matched the court data containing each trade made by Mintbroker with the Refinitiv tick history data (formerly Thomson Reuters tick history data). For New Concept Energy, Mintbroker trading data are time-stamped to the minute and the time-ordered sequence of trades within each minute is reported. Within each minute we matched trades using volume and price according to the temporal order in which the orders

[12]Section 16(b) states that: "For the purpose of preventing the unfair use of information which may have been obtained by such beneficial owner, director, or officer by reason of his relationship to the issuer, any profit realized by him from any purchase and sale, or any sale and purchase, of any equity security of such issuer (other than an exempted security) within any period of less than six months, unless such security was acquired in good faith in connection with a debt previously contracted, shall inure to and be recoverable by the issuer, irrespective of any intention on the part of such beneficial owner, director, or officer in entering into such transaction of holding the security purchased or of not repurchasing the security sold for a period exceeding six months. Suit to recover such profit may be instituted at law or in equity in any court of competent jurisdiction by the issuer"

[13]https://www.pacer.gov

FIGURE 5: **CRSP U.S. Market Index from June 28, 2018 to July 10, 2018** This figure reports the value of CRSP U.S. Market Index from June 28, 2018 to July 10, 2018.



were submitted. This method allowed us to match 97.7% of the trades reported in the court data. For Avalon Holdings, Mintbroker trading data are time-stamped to the second and report the total volume and the volume weighted average price within each second. Within each second we matched trades that had a total volume and a volume weighted average price that corresponded with the one reported by Mintbroker. This method allowed us to match 71.39% of the trades reported by Mintbroker.

Short sales data are taken from FINRA, Nasdaq, NYSE and CBOE. To increase market transparency, these equities exchanges publish the information regarding individual short sale transactions with a delay of one month. FINRA pubblishes short sale transactions that took place off the exchanges through the alternative display facility (ADF). Short sale transaction data contain transactions time stamped to the second, and within each second the trades are reported chronologically from first to last. To match the short sale transaction data to Refinitiv tick history data we used the same algorithm used to match Mintbroker trades with the New Concept Energy transaction data. This algorithm allowed us to match at least than 95% of short sales transactions for both stocks.

FIGURE 6: **CRSP U.S. Market Index from July 24, 2018 to August 1, 2018** This figure reports the value of CRSP U.S. Market Index from July 24, 2018 to August 1, 2018.

Data about fail to deliver[14] are taken from the SEC[15] that publishes the aggregate net balance of shares that failed to be delivered as of a particular settlement date twice a month.

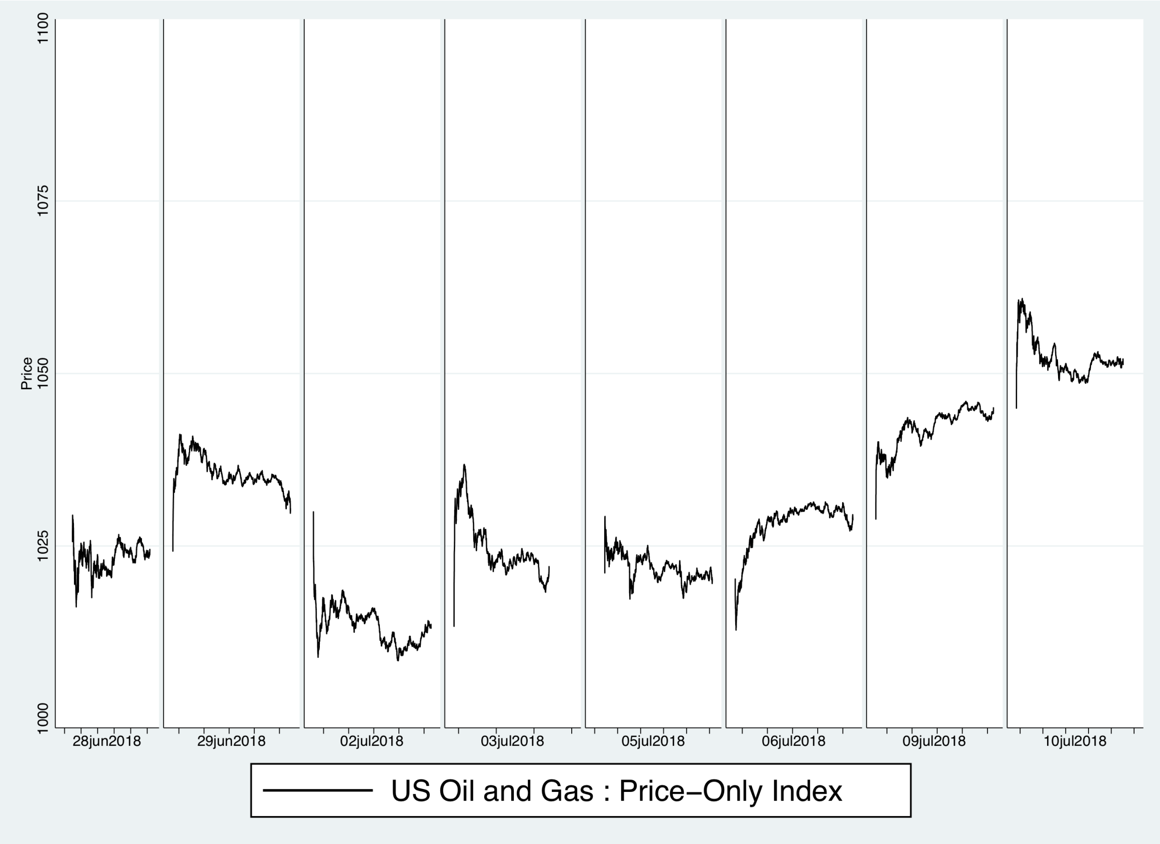
3.5 Theory Predictions

A manipulator attempting to squeeze or corner the market should try to avoid detection, since if other market participants detect his activity early on, the price would adjust making it prohibitively expensive for the manipulator to accumulate a position large enough. In order to avoid detection the manipulator could split his orders and gradually accumulate the position as informed traders in Kyle 1985. Another way to avoid detection would be to pool with informed investors as suggested by Allen and Gale 1992 and use a combination of patient limit orders and market orders to trade as well as timing liquidity. Allen and Gale 1992 show that

[14]When naked short selling occurs, an individual sells a stock without first borrowing the security or ensuring that the security can be borrowed. If the seller cannot find the security to borrow after two business days from the trading date, then a fail to deliver occurs. Fail to deliver shares represent the aggregate net balance of shares that failed to be delivered as of a particular settlement date.

[15]https://www.sec.gov/data/foiadocsfailsdatahtm

FIGURE 7: **CRSP U.S. Oil and Gas Index from June 28, 2018 to July 10, 2018** This figure reports the value of CRSP U.S. Oil and Gas Index from June 28, 2018 to July 10, 2018. The Index measures the performance of U.S companies in the oil and gas sector listed in the CRSP U.S. Total Market Index.



with incomplete information an uninformed manipulator that mimics an informed investor does make a profit because the other market participants don't know if the price movement is caused by a informed investor or by a manipulator.

The manipulator could also submit orders to sell in addition to orders to buy while building the position in the stock. That would be reminiscent of Chakraborty and Yılmaz 2004a model in which informed traders trade in the "wrong" direction for short-term losses but long-term profits. That would serve two purposes: reduce the risk by decreasing the size of the position opened within each day, and confuse the other market participants by making the signal in the order flow noisier.

3.6 Results

We first describe the summary statistics for New Concept Energy and Avalon Holdings stocks in Table 3.1, for all market participants except Mintbroker in the Pre-Event window, the Event-window and the Post-Event window. We define the Pre-Event window as the month before

Mintbroker started trading New Concept Energy and Avalon Holdings shares. The Event-window is defined as the period (8 days for New Concept Energy, 7 days for Avalon Holdings), in which Mintbroker was active in the stock. The Post-Event window is the month following the period in which Mintbroker was actively trading. Comparing Panel A and Panel B in Table 3.1 we see that the average size per trade fell from 378.84 to 279.16 shares, and the average time between trades also decreased from 55,627.79 milliseconds to 861.58 milliseconds. The average price instead increased from $1.54 to $6.37 and this sharp increase in price can also be seen in Figure 1. Comparing Panel B and Panel C yields similar conclusions. The average trade size during the event period was smaller (279.16 compared with 311.14), trading was faster (the average time between trades was 861.58 milliseconds compared with 10,186.30 milliseconds in the Post-Event period), and the average price in the event period was larger ($6.37 compared with $3.40).

Looking at summary statistics of Avalon Holdings stock in Panels D, E and F in Table 3.1, we reach similar conclusions. Comparing Panel D and Panel E in Table 3.1 we see that the average size per trade fell from 271 to 227.84 shares, and the average time between trades also decreased from 87,381 milliseconds to 867.85 milliseconds. As can be seen also in Figure 3 the average price increased substantially in the Event-window: from $2.6 to $5.83. Comparing Panel E and Panel F yields once again similar conclusions. The average trade size during the event period was smaller (227.84 compared with 259), trading was faster (the average time between trades was 867.85 milliseconds compared with 3,830 milliseconds in the Post-Event period), and the average price in the event period was larger ($5.833 compared with $4.71).

Interestingly if we look only at Mintbroker trades within the Event-window for New Concept Energy in Table 3.2, we observe that the average trade size of a Mintbroker trade was lower than the average trade size of other market participants (252.32 compared with 279.16) and the time between successive trades was much lower (96.30 milliseconds compared with 861.58 milliseconds). Hence Mintbroker, on average, traded New Concept Energy very often and its orders were relatively small compared to the ones submitted by other market participants. For Avalon Holdings Mintbroker's average trade size is larger than that of the other market participant, and there is also a considerable variation in the time between trades, with an average time between trades of 13,322.46 milliseconds for Mintbroker, much larger than 5,833 milliseconds in Table 3.1. Thus Mintbroker, on average, traded Avalon Holdings stock infrequently, submitting larger trades than other market participants.

In Table 3.3, we see that in almost all the days in which it was active in New Concept Energy Stock, Mintbroker realized a trading profit, with the exception of July 6, 2018. A profit is registered for each day in which the average buy price is lower than the average sell price and the position was at least partially closed. To compute the profit, the difference between the average buy price and the average sell price is then multiplied by the size of the position closed within that day. So a profit is registered for the days during which Mintbroker was building up its position, only if Mintbroker was also selling within those days. On June 28 and 29, in New Concept Energy and on July 24, 25, 26, 27 in Avalon Holdings, Mintbroker made a positive profit. This is reminiscent of Chakraborty and Yılmaz 2004a, since Mintbroker was trading in the "wrong" direction while building up its long position. In New Concept Energy most of the accumulation of shares by Mintbroker happened on June 29, two days after Realty Advisors disclosed that it would acquire 3,000,000 new shares of New Concept

TABLE 3.1: **Summary statistics for New Concept Energy and Avalon Holdings** This table reports the summary statistics for volume (in number of shares), price (in USD), and average time between successive trades (in milliseconds) for all the market participants except Mintbroker. Panel A statistics refer to New Concept Energy for the period from May 27, 2018 to June 27, 2018. Panel B statistics refer to to New Concept Energy for the period from June 28, 2018 to July 10, 2018 excluding Mintbroker trades. Panel C statistics refer to to New Concept Energy for the period from July 11, 2018 to August 11, 2018. Panel D statistics refer to Avalon Holdings for the period from June 23, 2018 to July 23, 2018. Panel E statistics refer to Avalon Holdings for the period from July 24, 2018 to August 1, 2018 excluding Mintbroker trades. In Panel F the statistics refer to Avalon Holdings for the period from August 2, 2018 to September 2, 2018.

		New Concept Energy			
Panel A: Pre-Event window					
	Mean	SD	Min	Max	Observations
Volume	378.84	902.20	1	40,000	9,022
Price	1.54	0.15	1.32	1.94	9,022
Time btw Trades	55,627.79	228,337.70	0	3,783,552	9,002
Panel B: Event-window					
	Mean	SD	Min	Max	Observations
Volume	279.16	650.81	1	138,821	215,378
Price	6.37	2.91	1.53	12.75	215,378
Time btw Trades	861.58	5,492.02	0	601,206	215,370
Panel C: Post-Event					
	Mean	SD	Min	Max	Observations
Volume	311.14	646.49	1	21,780	52,858
Price	3.40	0.58	2.16	4.65	52,858
Time btw Trades	10,186.30	39,150.64	0	1,138,851	52,835
		Avalon Holdings			
Panel D: Pre-Event window					
	Mean	SD	Min	Max	Observations
Volume	271	519	1	9565	5,047
Price	2.6	0.354	2.12	3.21	5,047
Time btw Trades	87,381	576,535	0	17,294,312	5,047
Panel E: Event-window					
	Mean	SD	Min	Max	Observations
Volume	227.84	520.86	1	87,299.00	188,564
Price	5.833	2.878	2.21	20.20	188,564
Time btw Trades	867.85	7,989.49	0	2,104,943.00	188,564
Panel F: Post-Event					
	Mean	SD	Min	Max	Observations
Volume	259	620	1	96,768	134,427
Price	4.71	0.951	2.71	6.79	134,427
Time btw Trades	3,830	21,287	0	904,687	134,427

Energy Common Stock at a price of $1.50 per share in cash. Most of Mintbroker's profits were realized on July 3 (5,225,639.49 out of 6,639,597.63), on that day Mintbroker sold all the shares it owned, corresponding to 44.48% of the total number of outstanding shares and then went short by 11,333 shares. In the span of 8 working days Mintbroker realized a gross profit (gross of fees and taxes) of $6,639,597.63 having invested overnight at most $3,378,390.52. These are large numbers, especially considering the short time frame in which they were realized and the relatively modest amount of capital employed.

For Avalon Holdings we see that Mintbroker realized a profit in all the days in which it was active, with the exception of August 1. In Avalon Holdings the accumulation and the successive selling of shares is more gradual. The bulk of the profits was realized in two days: on July

TABLE 3.2: **New Concept Energy and Avalon Holdings Event-window summary statistics for Mintbroker** This table reports the summary statistics for Volume (in number of shares), price (in USD), and average time between successive trades (in milliseconds) for Mintbroker during the Event-window for both New Concept Energy and Avalon Holdings. For New Concept Energy the Event-window covers the period from June 28, 2018 to July 10, 2018. For Avalon Holdings the Event-window covers the period from July 24, 2018 to August 1, 2018.

			New Concept Energy		
	Mean	SD	Min	Max	Observations
Volume	252.32	772.53	1	36,299	14,701
Price	5.93	2.99	1.64	12.75	14,701
Time btw Trades	96.30	613.19	0	46,304	14,701
			Avalon Holdings		
	Mean	SD	Min	Max	Observations
Volume	250.26	664.28	1	33,608.00	10,598
Price	4.92	2.00	2.25	20	10,598
Time btw Trades	13,322.46	101,438.49	0	4,498,018.00	10,598

27 (956,941.26 out of 3,130,179.24), and on July 30 (1,792,785.80 out of 3,130,179.24). In 7 working days Mintbroker realized a gross profit (gross of fees and taxes) of $3,130,179.24 having invested overnight at most $5,430,859.48. In Avalon Holdings the profits realized by Mintbroker are lower than those realized in New Concept Energy. Considering that the maximum investment made in Avalon Holdings was $5.5 million and the one in New Concept Energy was less than $3.5 million, the return on investment for Mintbroker had been definitely higher in New Concept Energy.

To determine if Mintbroker used limit orders, market orders or a combination of the two, we classify each trade either as buyer initiated or seller initiated using the Lee and Ready 1991 algorithm modified by Ellis, Michaely, and O'Hara 2000. Then for each transaction that involved Mintbroker we check if Mintbroker was either buying or selling and we compare that with the classification from the algorithm by Lee and Ready. We classify[16] a trade as a market order to buy if it is buyer initiated and Mintbroker was buying, as a limit order to buy if it is seller initiated and Mintbroker was buying, as a market order to sell if it is seller initiated and Mintbroker was selling and as a limit order to sell if it is buyer initiated and Mintbroker was selling.

Table 3.4 shows that Mintbroker used both market and limit orders, both to buy and to sell, that the average size of market orders to buy was larger than that of market orders to sell and generally of any other order type. Interestingly the average size of limit sell orders is considerably smaller than that of all the other types of orders, probably because Mintbroker was trying to contain as much as possible the market impact of its orders especially while selling, and couldn't reduce the price impact by selling and buying almost concomitantly as it did while accumulating the position. Reinforcing this hypothesis is the fact that both types of sell orders are substantially smaller than buy orders both for New Concept Energy and Avalon Holdings.

[16]Court documents from Mintbroker do not indicate if orders were limit or market, only the side i.e. if the order was a buy or a sell

TABLE 3.3: **Mintbroker profits and position in New Concept Energy and Avalon Holdings.** This table reports Mintbroker gross profits (in USD), overnight position (both in number of shares and market value in USD), and percentage of the outstanding shares owned.

New Concept Energy				
Date	Gross Profit	Open Volume	Dollars to acquire the opened overnight position	% of outstanding shares owned
28 Jun 18	962.15	49,228	94,217.17	2.31%
29 Jun 18	425,272.09	1,061,814	3,378,390.52	49.81%
2 Jul 18	925,673.26	948,238	3,017,023.95	44.48%
3 Jul 18	5,225,639.49	-15,950	138,647.13	-0.75%
5 Jul 18	16,125.38	-11,333	98,513.35	-0.53%
6 Jul 18	0.00	-11,333	98,513.35	-0.53%
9 Jul 18	4,887.61	-10,333	89,820.74	-0.48%
10 Jul 18	40,947.65	-2,742	17,815.28	-0.13%
Total	6,639,507.63			
Avalon Holdings				
Date	Gross Profit	Open Volume	Dollars to acquire the opened overnight position	% of outstanding shares owned
24 Jul 18	498.94	395,054	1,147,308.72	12.38%
25 Jul 18	122,045.37	747,823	2,423,819.55	23.44%
26 Jul 18	319,490.89	1,063,574	4,039,820.87	33.33%
27 Jul 18	956,941.26	1,246,570	5,430,859.48	39.07%
30 Jul 18	1,792,785.80	771,925	3,363,001.04	24.19%
31 Jul 18	6,473.86	503,175	2,192,153.44	15.77%
1 Aug 18	-68,056.88	373,951	1,629,170.71	11.72%
Total	3,130,179.24			

From Table 3.5 we see that Mintbroker submitted, for the most part, market buy orders (28.89% of the volume by Mintbroker in New Concept Energy and 33.67% of the volume by Minbroker in Avalon Holdings) for buying shares and limit sell orders (32.09% of the total volume by Minbroker in New Concept Energy and 26.20% of the volume by Minbroker in Avalon Holdings) for selling shares.

Next we look at the exchanges in which New Concept Energy and Avalon Holdings were traded during the period in which Mintbroker was active.

Table 3.6 shows that most of the volume is reported through the ADF (the alternative display facility) which is volume traded off the exchanges. 48.76% of the trades happened off the exchanges in New Concept Energy stock in the Event-window and 41.20% of the trades happened off the exchanges in Avalon Holdings in the Event-window.

Comparing Table 3.6, Table 3.7 and Table 3.8 we see that the total volume traded within the Event-window for New Concept Energy was very large compared with the one in the Pre-Event window (Table 3.7) and with the one in the Post-Event window (Table 3.8). This is remarkable since both the Pre-Event and the Post-Event windows last 1 month whereas the Event-window for New Concept Energy is only 8 days. The same is true for Avalon Holdings: the total volume was considerably higher during the Event-window (for Avalon Holdings the Event-window lasted 7 days), the total volume was 45,614,723 compared with a 1,344,607 in the Pre-Event window and 34,864,075 in the Post-Event window. This is consistent with Aggarwal and Wu 2006 that show an increase in volume for manipulated stocks during the manipulation period. The percentage of volume executed off the exchanges (ADF), dropped down to 48.76% for New Concept Energy during the Event-window and to 41.20% for Avalon Holdings compared to about 54% both before and after the Event-window for New Concept Energy and 55.41% in

TABLE 3.4: **Size of Limit Orders and Market Orders by Mintbroker** This table reports, for each order type, the average order size, in number of shares, the standard deviation of order size, and the 95% confidence interval of order size for Mintbroker in New Concept Energy and Avalon Holdings.

New Concept Energy				
	Mean	Std. Err.	[95% Conf. Interval]	
Market Buy	521.48	23.32	475.78	567.19
Limit Buy	370.29	23.30	324.63	415.96
Market Sell	269.35	22.09	226.06	312.64
Limit Sell	144.29	3.05	138.30	150.27
Avalon Holdings				
	Mean	Std. Err.	[95% Conf. Interval]	
Market Buy	333.64	10.42	313.21	354.07
Limit Buy	306.90	18.79	270.06	343.74
Market Sell	245.68	17.22	211.91	279.45
Limit Sell	195.12	6.98	181.43	208.81

TABLE 3.5: **Volume of Limit Orders and Market Orders by Mintbroker** This table reports the total volume (in number of shares) traded by Mintbroker using different order types during in New Concept Energy and Avalon Holdings.

New Concept Energy			
Order Type	Volume	% of Volume	Cumulative % of Volume
Market Buy	1,083,115	28.89%	28.89%
Limit Buy	790,203	21.08%	49.96%
Market Sell	672,836	17.95%	67.91%
Limit Sell	1,203,224	32.09%	100.00%
Total	3,749,378		
Avalon Holdings			
Order Type	Volume	% of Volume	Cumulative % of Volume
Market Buy	1,078,334	33.67%	33.67%
Limit Buy	710,160	22.17%	55.84%
Market Sell	575,136	17.96%	73.80%
Limit Sell	839,407	26.20%	100.00%
Total	3,203,037		

the Pre-Event window and 48.89% in the Post-Event window for Avalon Holdings.

We can test if the large price swings were caused by a short squeeze by looking at the short sale transactions within the Event-window. The data on short sales comes from Nasdaq, NYSE and CBOE websites. To increase market transparency, these equities exchanges publish the information regarding individual short sale transactions with a delay of one month. The data about short sale transactions executed off the exchanges comes from FINRA.

As shown in Table 3.6 we have short sale transaction data for shares traded off the exchanges reported through the alternative display facility (ADF), the CBOE BZX, the Nasdaq BX, the CBOE BYX, the CBOE EDGA, the CBOE EDGX and the Nasdaq PSX. All these facilities combined account for 71.01% and 63.98% of the volume traded during the period in which Mintbroker was active in New Concept Energy and Avalon Holdings respectively. Thus we believe that the data accurately reflect the amount and the timing of short sellers trades in New Concept Energy and Avalon Holdings.

Figure 8 shows the time, the price and the volume of New Concept Energy shares at which

TABLE 3.6: **Volume by Exchange in New Concept Energy and Avalon Holdings while Mintbroker was active** This table reports the volume traded (in number of shares) in each exchange during the period in which Mintbroker was active.
ADF identifies the alternative display facility (orders executed off the exchanges), ASE identifies the NYSE AMEX, BAT identifies BZX, BOS identifies Nasdaq BX, BTY identifies BYX, CIN identifies NYSE National, DEA identifies EDGA, DEX identifies EDGX, IEX identifies the Investor Exchange, MID identifies the Chicago Stock Exchange, NYS identifies the New York Stock Exchange, PSE identifies the NYSE Arca, THM identifies the Nasdaq intermarket, XPH identifies the Nasdaq PSX. The short sale transaction column reports the exchanges for which we have short sale transaction data.

New Concept Energy				
Exchange	Volume	% of Volume	Cumulative % of Volume	Short Sale transaction Data
ADF	31,124,188	48.76	48.76	Yes
ASE	5,778,282	9.05	57.81	No
BAT	2,558,336	4.01	61.82	Yes
BOS	521,761	0.82	62.64	Yes
BTY	1,142,958	1.79	64.43	Yes
CIN	882	0	64.43	No
DEA	581,383	0.91	65.34	Yes
DEX	9,324,328	14.61	79.95	Yes
IEX	231,929	0.36	80.31	No
MID	100	0	80.31	No
NYS	122,178	0.19	80.5	No
PSE	8,516,954	13.34	93.84	No
THM	3,861,545	6.05	99.89	No
XPH	68,483	0.11	100	Yes
Total	63,833,307			
Avalon Holdings				
Exchange	Volume	% of Volume	Cumulative % of Volume	Short Sale transaction Data
ADF	18,792,630	41.20	41.20	Yes
ASE	4,531,328	9.93	51.13	No
BAT	688,842	1.51	52.64	Yes
BOS	540,466	1.18	53.83	Yes
BTY	1,207,286	2.65	56.47	Yes
CIN	1,200	0	56.48	No
DEA	448,517	0.98	57.46	Yes
DEX	7,470,523	16.38	73.84	Yes
IEX	318,552	0.70	74.54	No
MID	0	0	74.54	No
NYS	79,228	0.17	74.71	No
PSE	7,040,478	15.43	90.14	No
THM	4,459,484	9.78	99.92	No
XPH	36,189	0.08	100	Yes
Total	45,614,723			

short sellers shorted the stock. The short volume was particularly high on June 29 (Table 3.9), the day during which Mintbroker accumulated the most shares. Table 3.3 shows that on that day, Mintbroker bought more than 47% of the total number of New Concept Energy outstanding shares.

Short sellers were betting that the share price would align with the price agreed between New Concept Energy and Realty Advisors valuing each share $1.50. Given the lack of the news they found the increase in price unjustified and kept on selling short while Mintbroker was accumulating shares.

If we look at the short volume in Avalon Holdings in Figure 9 and Table 3.9 we see that

TABLE 3.7: **Volume by Exchange in New Concept Energy and Avalon Holdings in the month before Mintbroker activity** This table reports the volume traded (in number of shares) in each exchange in the month before Mintbroker started trading New Concept Energy and Avalon Holdings.
ADF identifies the alternative display facility (orders executed in the dark), ASE identifies the NYSE AMEX, BAT identifies BZX, BOS identifies Nasdaq BX, BTY identifies BYX, CIN identifies NYSE National, DEA identifies EDGA, DEX identifies EDGX, IEX identifies the Investor Exchange, MID identifies the Chicago Stock Exchange, NYS identifies the New York Stock Exchange, PSE identifies the NYSE Arca, THM identifies the Nasdaq intermarket, XPH identifies the Nasdaq PSX.

New Concept Energy			
Exchange	Volume	% of Volume	Cumulative % of Volume
ADF	1,894,585	54.42	54.42
ASE	449,852	12.92	67.34
BAT	69,663	2	69.34
BOS	20,857	0.6	69.94
BTY	64,033	1.84	71.78
DEA	8,650	0.25	72.02
DEX	421,538	12.11	84.13
IEX	5,606	0.16	84.29
NYS	4,920	0.14	84.43
PSE	381,657	10.96	95.4
THM	150,241	4.32	99.71
XPH	10,056	0.29	100
Total	3,481,658		
Avalon Holdings			
Exchange	Volume	% of Volume	Cumulative % of Volume
ADF	745,018	55.41	55.41
ASE	121,119	9.01	64.42
BAT	9,207	0.68	65.10
BOS	7,140	0.53	65.63
BTY	12,600	0.94	66.57
DEA	1,400	0.10	66.67
DEX	181,733	13.52	80.19
IEX	3,303	0.25	80.43
NYS	1,400	0.10	80.54
PSE	142,375	10.59	91.13
THM	111,605	8.30	99.43
XPH	7,707	0.57	100
Total	1,344,607		

the short volume kept increasing each day from July 24, 2018 until July 27, 2018, and then dropped after July 27, 2018. In both New Concept Energy and Avalon Holdings, the short volume dropped on the day in which the price started to fall significantly.

Collin-Dufresne and Fos 2015 find that informed traders are more likely to trade at the beginning and at the end of the day, when uninformed volume is higher. In New Concept Energy and Avalon Holdings short transactions in Figure 8 and 9 do not appear to cluster at the opening and closing, instead are spread out during the trading day.

Mintbroker submitted most of its sell orders in New Concept Energy using relatively small limit orders as shown in Table 3.4 and Table 3.5. Since Mintbroker was selling using mostly small and frequent limit orders, the counterparts in the transactions must have bought using market orders. Moreover since short sellers were responsible for little volume within that period, market buy orders were likely submitted by short sellers that were forced to close out

TABLE 3.8: **Volume by Exchange in New Concept Energy and Avalon Holdings in the month after Mintbroker Activity** This table reports the volume traded (in number of shares) in each exchange in the month after Mintbroker stopped trading New Concept Energy and Avalon Holdings.
ADF identifies the alternative display facility (orders executed in the dark), ASE identifies the NYSE AMEX, BAT identifies BZX, BOS identifies Nasdaq BX, BTY identifies BYX, CIN identifies NYSE National, DEA identifies EDGA, DEX identifies EDGX, IEX identifies the Investor Exchange, MID identifies the Chicago Stock Exchange, NYS identifies the New York Stock Exchange, PSE identifies the NYSE Arca, THM identifies the Nasdaq intermarket, XPH identifies the Nasdaq PSX.

New Concept Energy			
Exchange	Volume	% of Volume	Cumulative % of Volume
ADF	9,364,928	54.98	54.98
ASE	1,589,049	9.33	64.3
BAT	589,639	3.46	67.77
BOS	76,555	0.45	68.22
BTY	270,705	1.59	69.8
CIN	3,310	0.02	69.82
DEA	153,819	0.9	70.73
DEX	2,024,171	11.88	82.61
IEX	48,163	0.28	82.89
MID	3,553	0.02	82.91
NYS	46,596	0.27	83.19
PSE	1,805,041	10.6	93.78
THM	1,042,180	6.12	99.9
XPH	16,907	0.1	100
Total	17,034,616		
Avalon Holdings			
Exchange	Volume	% of Volume	Cumulative % of Volume
ADF	17,046,158	48.89	48.89
ASE	3,322,315	9.53	58.42
BAT	484,519	1.39	59.81
BOS	279,720	0.80	60.61
BTY	840,705	2.41	63.03
CIN	4,641	0.01	63.04
DEA	254,176	0.73	63.77
DEX	5,054,893	14.50	78.27
IEX	155,962	0.45	78.71
MID	4,675	0.01	78.73
NYS	68,491	0.20	78.92
PSE	4,350,172	12.48	91.40
THM	2,925,618	8.39	99.79
XPH	72,030	0.21	100
Total	34,864,075		

their position.

Rule 204 of regulation SHO requires to close-out any failing equity security that exists on settlement (the second business day after trade date, or "T+2"). If the close-out does not take place on conventional settlement date, Rule 204 requires brokers to take appropriate action before the opening of trading on T+3 (in the case of short sales). This is consistent with what we observe, the drop in the short sell transactions has been exacerbated by brokers that were forced to close down their clients' short positions. Next we use the data from the SEC on failures to deliver to provide further evidence that short sellers were forced to close down their positions.

FIGURE 8: **Short Sale Transactions for New Concept Energy while Mintbroker was active in the stock** This figure reports the price and volume of short sale transactions during the period in which Mintbroker was active in New Concept Energy: from June 28, 2018 to July 10, 2018. The upper panels show the prices at which short sellers sold the stock. The lower panels show the corresponding volume traded by short sellers within each day.

Failure to deliver on a given day are computed by the SEC as

$$FailToDeliver = FailsOustanding + NewFails - FailsSettled \tag{3.1}$$

where *FailsOustanding* is the cumulative number of all fails outstanding until that day, *NewFails* is the number of new fails to that occur that day and *FailsSettled* is the number of fails that settle that day.

Looking at failures to deliver in Table 3.10 further reinforces the hypothesis that short seller were forced to close their position on July 3 for New Concept Energy and on July 27 for Avalon Holdings. The jump and subsequent drop on these two days suggest that short sellers, or their brokers, closed their short positions in those days.

FIGURE 9: **Short Sale Transactions for Avalon Holdings while Mintbroker was active in the stock** This figure reports the price and volume of short sale transactions during the period in which Mintbroker was active in Avalon Holdings: from July 24, 2018 to August 1, 2018. The upper panels show the prices at which short sellers sold the stock. The lower panels show the corresponding volume traded by short sellers within each day.

3.6.1 Impact of the Short Squeeze on Market Quality

In this section we look at how the overall market reacted to the short squeeze and whether Mintbroker's trading costs were larger, reflecting the substantial adverse selection in trading against Mintbroker. To evaluate how the market reacted to Mintbroker's short squeeze we estimate the following regressions in the spirit of Shkilko 2018:

$$EffectiveSpread_{i,t} = \alpha_i + \beta_1 Mintbroker_{i,t} + \beta_2 Volume_{i,t} + \beta_3 Momentum_{i,t-1min} + \beta_4 Volatility_{i,t-1min} + \beta_5 CumVolume_{i,t-1min} + \beta_6 ShortVolume_{i,t-1min} + \epsilon_{i,t} \quad (3.2)$$

$$PriceImpact_{i,t} = \alpha_i + \beta_1 Mintbroker_{i,t} + \beta_2 Volume_{i,t} + \beta_3 Momentum_{i,t-1min} + \beta_4 Volatility_{i,t-1min} + \beta_5 CumVolume_{i,t-1min} + \beta_6 ShortVolume_{i,t-1min} + \epsilon_{i,t} \quad (3.3)$$

Where effective spread is defined as $2 * buy * (price_{it} - mid_{it})/mid_{it}$ for stock i at time t. buy is a indicator equal to 1 for buyer initiated trades and to -1 for seller initiated trades, $price_{it}$ is

TABLE 3.9: **Daily Matched Short Sell Trades for New Concept Energy and Avalon Holdings** This table reports the total short volume traded each day (in number of shares), the percentage of the short volume that interacted directly with Mintbroker during the period in which Mintbroker was active in New Concept Energy (from June 28, 2018 to July 10, 2018) and in Avalon Holdings (from July 24, 2018 to August 1, 2018) and Mintbroker's position in number of shares at the end of each day

New Concept Energy			
Date	Short volume	% of short sold to Mintbroker directly	Mintbroker Position
28-Jun-18	791,130	2.14%	49,228
29-Jun-18	2,115,151	17.41%	1,061,814
2-Jul-18	1,222,024	0%	948,238
3-Jul-18	838,798	0%	-15,950
5-Jul-18	465,796	0%	-11,333
6-Jul-18	180,826	0%	-11,333
9-Jul-18	115,009	0.43%	-10,333
10-Jul-18	341,405	1.42%	-2,742
Avalon Holdings			
Date	Short volume	% of short sold to Mintbroker directly	Mintbroker Position
24-Jul-18	982,108	23.24%	395,054
25-Jul-18	3,629,108	0.05%	747,823
26-Jul-18	3,521,185	0.03%	1,063,574
27-Jul-18	4,214,866	0.02%	1,246,570
30-Jul-18	1,474,075	0%	771,925
31-Jul-18	1,264,795	0%	503,175
1-Aug-18	967,094	0%	373,951

TABLE 3.10: **Stocks that failed to be delivered for New Concept Energy and Avalon Holdings** This table reports the volume of New Concept Energy and Avalon Holdings (in number of shares) that failed to be delivered within the settlement time, during the period in which Mintbroker was active. The SEC calculates fails to deliver on a given day as the cumulative number of all fails outstanding until that day, plus new fails that occur that day, less fails that settle that day.

New Concept Energy	
Date	Shares failed to deliver
28-Jun-18	12,237
29-Jun-18	22,848
2-Jul-18	57,631
3-Jul-18	190,839
5-Jul-18	54,595
6-Jul-18	8,072
9-Jul-18	35,884
10-Jul-18	13,702
Avalon Holdings	
Date	Shares failed to deliver
24-Jul-18	-
25-Jun-18	-
26-Jul-18	119,078
27-Jul-18	298,858
30-Jul-18	95,656
31-Jul-18	187,010
1-Aug-18	300,949

the price of the trade and mid_{it} is the midquote of the consolidated BBO prevailing at the time of the trade.

TABLE 3.11: **Intraday Market Impact Measures for Mintbroker trades in New Concept Energy: 1 minute intervals** This table shows the relationship between Mintbroker trades and both Effective Spread and Price impact. It reports the coefficients together with the standard errors in parenthesis from the regression:$Liq\_Measure_{i,t} = \alpha_i + \beta_1 Mintbroker_{i,t} + \beta_2 Volume_{i,t} + \beta_3 Momentum_{i,t-1min} + \beta_4 Volatility_{i,t-1min} + \beta_5 CumVolume_{i,t-1min} + \beta_6 ShortVolume_{i,t-1min} + \epsilon_{i,t}$ where $Liq\_Measure_{i,t}$ is either $EffectiveSpread$ or $PriceImpact$. $EffectiveSpread$ is defined as $2 * buy * (price_{it} - mid_{it})/mid_{it}$. buy is a indicator equal to 1 for buyer initiated trades and to -1 for seller initiated trades, $price_{it}$ is the price of the trade and mid_{it} is the midquote of the consolidated BBO prevailing at the time of the trade. $PriceImpact$ is defined as $2 * buy * (mid_{it+1min} - mid_{it})/mid_{it}$ where buy is a indicator equal to 1 for buyer initiated trades and to -1 for seller initiated, $mid_{it+1min}$ is the midquote of the consolidated BBO 1 minute after the trade and mid_{it} is the midquote of the consolidated BBO at the time of the trade. $Volume_{i,t}$ is the size of the trade expressed as a multiple of 10^5, $Momentum_{i,t-1min}$ is the return in the minute before the trade, $Volatility_{i,t-1min}$ is the absolute return in the minute before the trade, $CumVolume_{i,t-1min}$ is the cumulative volume in the minute before the trade expressed as a multiple of 10^5 and $ShortVolume_{i,t-1min}$ is the cumulative short volume in the minute before the trade expressed as a multiple of 10^5. The table reports estimated coefficients and their t-statistics in parenthesis calculated using heteroskedasticity-robust standard errors controlling for day and intraday fixed effects. *, ** and *** indicate statistical significance at 10%, 5%, and 1% levels, respectively.

Panel A						
	(1) $EffSpread$	(2) $EffSpread$	(3) $EffSpread$	(4) $EffSpread$	(5) $EffSpread$	(6) $EffSpread$
Mintbroker	-0.001***	-0.001***	-0.003***	-0.005***	-0.005***	-0.005***
	(-3.505)	(-3.307)	(-7.314)	(-10.375)	(-9.270)	(-9.337)
$Volume * 10^{-5}$		0.220***	0.244***	0.246***	0.243***	0.243***
		(3.365)	(3.937)	(3.956)	(3.868)	(3.887)
Momentum			-0.114***	-0.109***	-0.106***	-0.105***
			(-15.215)	(-15.258)	(-14.921)	(-14.880)
Volatility				0.140***	0.151***	0.154***
				(12.300)	(12.897)	(12.846)
$CumVolume_{t-1min} * 10^{-5}$					-0.002***	-0.002***
					(-9.298)	(-9.345)
$ShortVolume_{t-1min} * 10^{-5}$						0.011***
						(4.174)
Intercept	0.024***	0.023***	0.022***	0.022***	0.022***	0.022***
	(25.325)	(23.544)	(22.873)	(23.068)	(22.887)	(23.021)
N	230,079	230,079	227,162	227,162	227,162	227,162
Adj. R^2	0.018	0.018	0.022	0.025	0.026	0.026
Panel B						
	(1) $PriceImpact$	(2) $PriceImpact$	(3) $PriceImpact$	(4) $PriceImpact$	(5) $PriceImpact$	(6) $PriceImpact$
Mintbroker	-0.013***	-0.013***	-0.013***	-0.013***	-0.012***	-0.012***
	(-14.189)	(-14.144)	(-13.936)	(-13.574)	(-13.083)	(-13.149)
$Volume * 10^{-5}$		0.123***	0.116***	0.116***	0.114***	0.115***
		(5.033)	(4.680)	(4.674)	(4.600)	(4.636)
Momentum			0.011**	0.011**	0.012***	0.013***
			(2.428)	(2.313)	(2.603)	(2.817)
Volatility				-0.015**	-0.010	-0.008
				(-2.370)	(-1.448)	(-1.069)
$CumVolume_{t-1min} * 10^{-5}$					-0.001***	-0.001***
					(-2.897)	(-4.028)
$ShortVolume_{t-1min} * 10^{-5}$						0.011***
						(3.186)
Intercept	0.004***	0.004***	0.004***	0.004***	0.004***	0.004***
	(7.616)	(6.644)	(6.786)	(6.775)	(6.620)	(6.851)
N	230,036	230,036	227,122	227,122	227,122	227,122
Adj. R^2	0.003	0.003	0.003	0.003	0.003	0.003

TABLE 3.12: **Intraday Market Impact Measures for Mintbroker trades in Avalon Holdings: 1 minute intervals** This table shows the relationship between Mintbroker trades and both Effective Spread and Price impact. It reports the coefficients together with the standard errors in parenthesis from the regression: $Liq\_Measure_{i,t} = \alpha_i + \beta_1 Mintbroker_{i,t} + \beta_2 Volume_{i,t} + \beta_3 Momentum_{i,t-1min} + \beta_4 Volatility_{i,t-1min} + \beta_5 CumVolume_{i,t-1min} + \beta_6 ShortVolume_{i,t-1min} + \epsilon_{i,t}$ where $Liq\_Measure_{i,t}$ is either $EffectiveSpread$ or $PriceImpact$. $EffectiveSpread$ is defined as $2 * buy * (price_{it} - mid_{it}) / mid_{it}$. buy is a indicator equal to 1 for buyer initiated trades and to -1 for seller initiated trades, $price_{it}$ is the price of the trade and mid_{it} is the midquote of the consolidated BBO prevailing at the time of the trade. $PriceImpact$ is defined as $2 * buy * (mid_{it+1min} - mid_{it}) / mid_{it}$ where buy is a indicator equal to 1 for buyer initiated trades and to -1 for seller initiated, $mid_{it+1min}$ is the midquote of the consolidated BBO 1 minute after the trade and mid_{it} is the midquote of the consolidated BBO at the time of the trade. $Volume_{i,t}$ is the size of the trade expressed as a multiple of 10^5, $Momentum_{i,t-1min}$ is the return in the minute before the trade, $Volatility_{i,t-1min}$ is the absolute return in the minute before the trade, $CumVolume_{i,t-1min}$ is the cumulative volume in the minute before the trade expressed as a multiple of 10^5 and $ShortVolume_{i,t-1min}$ is the cumulative short volume in the minute before the trade expressed as a multiple of 10^5. The table reports estimated coefficients and their t-statistics in parenthesis calculated using heteroskedasticity-robust standard errors controlling for day and intraday fixed effects. *, ** and *** indicate statistical significance at 10%, 5%, and 1% levels, respectively.

	Panel A					
	(1)	(2)	(3)	(4)	(5)	(6)
	$EffSpread$	$EffSpread$	$EffSpread$	$EffSpread$	$EffSpread$	$EffSpread$
$Mintbroker$	-0.017***	-0.018***	-0.014***	-0.013***	-0.013***	-0.013***
	(-19.330)	(-19.407)	(-17.204)	(-15.425)	(-15.443)	(-15.539)
$Volume * 10^{-5}$		0.899*	0.958**	0.934**	0.933**	0.933**
		(1.938)	(2.042)	(1.979)	(1.977)	(1.977)
$Momentum$			0.014	0.011	0.010	0.015
			(0.739)	(0.568)	(0.527)	(0.790)
$Volatility$				0.399***	0.403***	0.398***
				(15.811)	(15.716)	(15.595)
$CumVolume_{t-1min} * 10^{-5}$					-0.001**	0.011***
					(-2.545)	(5.297)
$ShortVolume_{t-1min} * 10^{-5}$						-0.041***
						(-6.928)
$Intercept$	0.033***	0.030***	0.029***	0.021***	0.022***	0.023***
	(42.150)	(18.403)	(17.327)	(12.926)	(13.115)	(13.998)
N	196,333	196,333	189,095	189,095	189,095	189,095
Adj. R^2	0.051	0.052	0.061	0.066	0.066	0.067
	Panel B					
	(1)	(2)	(3)	(4)	(5)	(6)
	$PriceImpact$	$PriceImpact$	$PriceImpact$	$PriceImpact$	$PriceImpact$	$PriceImpact$
$Mintbroker$	-0.001*	-0.001*	-0.001	-0.001	-0.001	-0.001
	(-1.657)	(-1.730)	(-1.217)	(-1.084)	(-1.126)	(-1.154)
$Volume * 10^{-5}$		0.194***	0.201***	0.200***	0.200***	0.199***
		(8.818)	(9.124)	(9.085)	(9.055)	(9.053)
$Momentum$			0.006	0.006	0.005	0.006
			(0.901)	(0.878)	(0.706)	(0.835)
$Volatility$				0.016*	0.020**	0.020**
				(1.857)	(2.412)	(2.303)
$CumVolume_{t-1min} * 10^{-5}$					-0.002***	0.001
					(-5.908)	(0.423)
$ShortVolume_{t-1min} * 10^{-5}$						-0.008**
						(-2.035)
$Intercept$	0.006***	0.003***	0.003***	0.002***	0.003***	0.003***
	(6.224)	(5.659)	(5.205)	(4.555)	(5.287)	(5.794)
N	196,335	196,333	189,095	189,095	189,095	189,095
Adj. R^2	0.001	0.001	0.001	0.001	0.001	0.001

$Mintbroker_{i,t}$ is a dummy variable equal to 1 if Mintbroker was involved in that particular trade and 0 otherwise, $Volume_{i,t}$ is the size of the trade, $Momentum_{i,t-1min}$ is the return in the minute before the trade, $Volatility_{i,t-1min}$ is the absolute return in the minute before the trade, $CumVolume_{i,t-1min}$ is the cumulative volume in the minute before the trade and $ShortVolume_{i,t-1min}$ is the cumulative short volume in the minute before the trade.

$PriceImpact_{i,t}$ for stock i at time t is defined as $2 * buy * (mid_{it+1min} - mid_{it}) / mid_{it}$ where buy is a indicator equal to 1 for buyer initiated trades and to -1 for seller initiated, $mid_{it+1min}$ is the midquote of the consolidated BBO 1 minute after the trade and mid_{it} is the midquote of the consolidated BBO at the time of the trade.

Surprisingly for both the effective spread and the price impact the coefficient of Mintbroker in Table 3.11 and 3.12 is negative. Therefore Mintbroker had effective spread and price impact lower than those of other market participants.

We control for momentum, volatility, cumulative volume in the 60 seconds before the trade and the cumulative volume of short selling 60 seconds before the trade and still we find that the effective spread and the price impact were smaller for Mintbroker trades. This implies that other market participants, on average, were not able to identify Mintbroker's trades. One explanation of why other market participants couldn't spot Mintbroker's trades is that, as shown in Table 3.5, Mintbroker executed most of its trades, especially sell orders, using limit instead of market orders. This result is consistent with Allen and Gale 1992 in which the manipulator pools with informed traders using the same trading strategies as the informed. The manipulator's behavior in Table 3.11 and Table 3.12 is consistent with informed investors' behavior in Collin-Dufresne and Fos 2015, Cornell and Sirri 1992 and Garriott and Riordan 2019 with informed investors using patient limit orders as well as aggressive market orders.

All the liquidity measures are estimated using a 1 minute horizon since Mintbroker had the technology to monitor the market and trade very frequently: the average time between successive trades was 96.3 milliseconds in Table 3.2 with a minimum value of 0 milliseconds. We checked if the results are robust to different time horizons using 5 minutes intervals in Table 3.16 and Table 3.17 and 15 minutes intervals in Table 3.18 and Table 3.19 in the appendix. We find that the coefficient of Mintroker is always negative and significant at the 1% level with the exception of the regression of $PriceImpact$ in a 1 minute interval for Avalon Holdings in Panel B of Table 3.12.

To understand how Mintbroker was able to hide in the order flow we study whether Mintbroker timed past returns, liquidity or short volumes (even though Mintbroker did not know in real time which trades involved a side that went short) by estimating the following Probit regression in a similar vein as Barclay, Hendershott, and D. T. McCormick 2003:

$$\begin{aligned} Prob(Mintbroker = 1)_{i,t} &= \alpha_i + \beta_1 Spread_{i,t-1min} + \beta_2 Momentum_{i,t-1min} \\ &+ \beta_3 PriceImpact_{i,t-1min} + \beta_4 CumVolume_{i,t-1min} + \beta_5 ShortVolume_{i,t-1min} + \epsilon_{i,t} \end{aligned} \quad (3.4)$$

where $Spread_{i,t-1min}$ is the average spread 1 minute before the trade, $Momentum_{i,t-1min}$ is the 1 minute return before the trade, $PriceImpact_{i,t-1min}$ is the average price impact in the minute before the trade and measures the losses to liquidity demanders due to adverse selection.

TABLE 3.13: **Mintbroker Market Timing in New Concept Energy: 1 minutes intervals**
This table reports a probit regression of the probability of a Mintbroker trade on *Spread*, *Momentum*, *PriceImpact*, *Cumultive_Volume* and *Short_volume* in the minute preceding the trade. *Spread* is the average difference between the ask and the bid in the minute before the trade. $Momentum_{i,t-1min}$ is the return in the minute before the trade, *PriceImpact* is defined as $2 * buy * (mid_{it+1min} - mid_{it}) / mid_{it}$ where *buy* is a indicator equal to 1 for buyer initiated trades and to -1 for seller initiated, $mid_{it+1min}$ is the midquote of the consolidated BBO 1 minute after the trade and mid_{it} is the midquote of the consolidated BBO at the time of the trade. $CumVolume_{i,t-1min}$ is the cumulative volume in the minute before the trade expressed as a multiple of 10^5 and $ShortVolume_{i,t-1min}$ is the cumulative short volume in the minute before the trade expressed as a multiple of 10^5. The table reports estimated coefficients and their *t*-statistics in parenthesis calculated using robust standard errors as well as the marginal effects at the mean in square brackets. *, ** and *** indicate statistical significance at 10%, 5%, and 1% levels, respectively.

	(1) Mintbroker	(2) Mintbroker	(3) Mintbroker	(4) Mintbroker	(5) Mintbroker
Spread	-6.416***	-9.019***	-9.152***	-10.451***	-9.328***
	(-49.808)	(-59.417)	(-58.860)	(-54.926)	(-50.902)
	[-0.516]	[-1.003]	[-1.006]	[-1.043]	[-0.919]
Momentum		-5.201***	-4.610***	-4.263***	-4.399***
		(-49.723)	(-44.761)	(-47.999)	(-48.215)
		[-0.578]	[-.507]	[-0.425]	[-0.433]
PriceImpact			-1.383***	-1.107***	-1.159***
			(-28.745)	(-26.108)	(-26.590)
			[-0.152]	[-0.111]	[-0.114]
$CumVolume * 10^{-5}$				0.242***	0.133***
				(79.683)	(28.443)
				[0.024]	[0.013]
$ShortVolume * 10^{-5}$					2.018***
					(30.437)
					[0.199]
N	375,536	227,158	227,141	227,141	227,141
pseudo R^2	0.015	0.053	0.063	0.107	0.115

$PriceImpact_{i,t-1min}$ is defined as $2 * buy * (midquote_{i,t} - midquote_{i,t-1min}) / midquote_{i,t-1min}$ where *buy* is a buy/sell indicator, 1 for buys and -1 for sells, $midquote_{i,t}$ is the midquote of the best bid and ask offer (BBO) prevailing in the last trade and $midquote_{i,t-1min}$ is the midquote prevailing in the minute before the trade. $CumVolume_{i,t-1min}$ is the cumulative volume in the minute before the trade and $ShortVolume_{i,t-1min}$ is the cumulative short volume in the minute before the trade. The negative coefficient of *Spread* in Table 3.13 and 3.14 indicates that the probability of a trade by Mintbroker is decreasing as *Spread* increases, this is coherent with the fact that most trades by Mintbroker were limit orders. The coefficient of Momentum is negative and significant in Table 3.13 but not in Table 3.14. We checked the robustness of the results taking 5 minutes and 15 minutes windows for the estimations. The results are reported in Table 3.20 and Table 3.22 in the appendix for New Concept Energy and in Table 3.21 and Table 3.23 for Avalon Holdings.

TABLE 3.14: **Mintbroker Market Timing in Avalon Holdings: 1 minute intervals** This table reports a probit regression of the probability of a Mintbroker trade on *Spread*, *Momentum*, *PriceImpact*, *Cumultive_Volume* and *Short_volume* in the minute preceding the trade. *Spread* is the average difference between the ask and the bid in the minute before the trade. $Momentum_{i,t-1min}$ is the return in the minute before the trade, *PriceImpact* is defined as $2 * buy * (mid_{it+1min} - mid_{it})/mid_{it}$ where *buy* is a indicator equal to 1 for buyer initiated trades and to -1 for seller initiated, $mid_{it+1min}$ is the midquote of the consolidated BBO 1 minute after the trade and mid_{it} is the midquote of the consolidated BBO at the time of the trade. $CumVolume_{i,t-1min}$ is the cumulative volume in the minute before the trade expressed as a multiple of 10^5 and $ShortVolume_{i,t-1min}$ is the cumulative short volume in the minute before the trade expressed as a multiple of 10^5. The table reports estimated coefficients and their *t*-statistics in parenthesis calculated using robust standard errors as well as the marginal effects at the mean in square brackets. *, ** and *** indicate statistical significance at 10%, 5%, and 1% levels, respectively.

	(1) Mintbroker	(2) Mintbroker	(3) Mintbroker	(4) Mintbroker	(5) Mintbroker
Spread	-1.853***	-1.734***	-1.734***	-1.815***	-1.911***
	(-31.328)	(-28.596)	(-28.596)	(-28.480)	(-29.339)
	[-0.193]	[-0.185]	[-0.186]	[-0.190]	[-0.199]
Momentum		-0.002	-0.002	-0.028	0.075
		(-0.016)	(-0.016)	(-0.192)	(0.522)
		[-0.000]	[-0.000]	[-0.003]	[0.008]
PriceImpact			-0.008	-0.012	0.008
			(-0.113)	(-0.161)	(0.107)
			[-0.001]	[-0.001]	[0.001]
$CumVolume * 10^{-5}$				-0.216***	0.100***
				(-20.295)	(3.875)
				[-0.023]	[0.010]
$ShortVolume * 10^{-5}$					-0.980***
					(-12.175)
					[-0.102]
N	196,335	189,097	189,097	189,097	189,097
pseudo R^2	0.013	0.011	0.011	0.018	0.020

In all the Tables the coefficient of *Spread* is always negative and significant at the 1% level, suggesting that a lower spread would increase the probability of a trade by Mintbroker. With the exception of Table 3.14 the coefficients of *Momentum* and *PriceImpact* in Tables 3.20, 3.22, 3.21 and 3.23 are always negative and significant at the 1% level suggesting that when the *PriceImpact* or the *Momentum* in the minute before the trade was high, Mintbroker would be less likely to trade. The coefficients of *CumVolume* is positive suggesting that the probability of trading by Mintbroker increased if the cumulative volume in the previous minute increased. *ShortVolume* is negative implying that a larger short volume in the minute before the trade would decrease the probability of a trade by Mintbroker. These results are consistent with Mintbroker acting as a contrarian and timing liquidity, this is once again consistent with the behavior of informed investors in Collin-Dufresne and Fos 2016 and Garriott and Riordan 2019.

3.7 Conclusions

While the SEC reacts fast to information-based manipulation, curbing trade-based manipulation is more difficult. For small-cap stocks a relatively modest amount of money might be enough to manipulate the price. We have shown that in New Concept Energy and Avalon

Holdings, Mintbroker produced two short squeezes in which short sellers lost $6 million and $3 million respectively. We have highlighted that Mintbroker trading strategy is consistent with that of informed traders in Collin-Dufresne and Fos 2015, Kacperczyk and Pagnotta 2018 and Garriott and Riordan 2019 confirming Allen and Gale 1992 model in which the manipulator pools with the informed investor. We have argued that Mintbroker used the limitation imposed to short sellers by Rule 204 of regulation SHO that imposes to close-out any failing equity security that exists on the second business day after the trade date, to unload its long position when the price of the stocks soared. For small-cap stocks the constraint imposed by regulation SHO easily binds since the low number of floating shares makes it hard to find the shares to borrow and that in turn forces short sellers to close their position, increasing the buying pressure and forcing other short sellers to close. One possible way to reduce the scope of price manipulation in small-cap stocks would be to increase the length of time between the trading date and the settlement date. Since the shares of small caps are harder to find, more time would be given to investors to locate them. Another possible way to curb the price manipulation in small-cap stocks would be to impose a more stringent regulation regarding information disclosures, with a particular emphasis on disclosures of relevant stakes.

3.8 Appendix

TABLE 3.15: **Summary Statistics for Aggarwal and Wu 2006 and for New Concept Energy and Avalon Holdings** This table reports the summary statics from Aggarwal and Wu 2006 for manipulated stocks in their sample. The table reports the sample meanfor their total sample of 78 stocks, from January 1990 to December 2001. Below the mean returns, turnover and volatility are calculated for New Concept Energy and Avalon Holdings.

	Mean
Return	0.027
Turnover	0.039
Volatility	0.573
New Concept Energy	
Return	0.238
Turnover	1.786
Volatility	2.046
Avalon Holdings	
Return	0.171
Turnover	2.041
Volatility	2.447

TABLE 3.16: **Intraday Market Impact Measures for Mintbroker trades in New Concept Energy: 5 minutes intervals** This table shows the relationship between Mintbroker trades and both Effective Spread and Price impact. It reports the coefficients together with the standard errors in parenthesis from the regression:$Liq\_Measure_{i,t} = \alpha_i + \beta_1 Mintbroker_{i,t} + \beta_2 Volume_{i,t} + \beta_3 Momentum_{i,t-5min} + \beta_4 Volatility_{i,t-5min} + \beta_5 CumVolume_{i,t-5min} + \beta_6 ShortVolume_{i,t-5min} + \epsilon_{i,t}$ where $Liq\_Measure_{i,t}$ is either $EffectiveSpread$ or $PriceImpact$. $EffectiveSpread$ is defined as $2 * buy * (price_{it} - mid_{it}) / mid_{it}$. buy is a indicator equal to 1 for buyer initiated trades and to -1 for seller initiated trades, $price_{it}$ is the price of the trade and mid_{it} is the midquote of the consolidated BBO prevailing at the time of the trade. $PriceImpact$ is defined as $2 * buy * (mid_{it+1min} - mid_{it}) / mid_{it}$ where buy is a indicator equal to 1 for buyer initiated trades and to -1 for seller initiated, $mid_{it+5min}$ is the midquote of the consolidated BBO 5 minutes after the trade and mid_{it} is the midquote of the consolidated BBO at the time of the trade. $Volume_{i,t}$ is the size of the trade expressed as a multiple of 10^5, $Momentum_{i,t-5min}$ is the return in the 5 minutes before the trade, $Volatility_{i,t-5min}$ is the absolute return in the 5 minutes before the trade, $CumVolume_{i,t-5min}$ is the cumulative volume in the 5 minutes before the trade expressed as a multiple of 10^5 and $ShortVolume_{i,t-5min}$ is the cumulative short volume in the 5 minutes before the trade expressed as a multiple of 10^5. The table reports estimated coefficients and their t-statistics in parenthesis calculated using heteroskedasticity-robust standard errors controlling for day and intraday fixed effects. *, ** and *** indicate statistical significance at 10%, 5%, and 1% levels, respectively.

Panel A

	(1) $EffSpread$	(2) $EffSpread$	(3) $EffSpread$	(4) $EffSpread$	(5) $EffSpread$	(6) $EffSpread$
$Mintbroker$	-0.003***	-0.003***	-0.004***	-0.005***	-0.005***	-0.005***
	(-8.256)	(-8.041)	(-8.295)	(-10.880)	(-10.893)	(-11.144)
$Volume * 10^{-5}$		0.232***	0.245***	0.248***	0.250***	0.250***
		(3.705)	(4.009)	(4.096)	(4.128)	(4.122)
$Momentum$			-0.002	-0.009***	-0.009***	-0.009***
			(-0.725)	(-3.386)	(-3.358)	(-3.610)
$Volatility$				0.033***	0.034***	0.034***
				(9.639)	(9.673)	(9.808)
$CumVolume_{i,t-5min} * 10^{-5}$					0.000***	-0.000***
					(3.584)	(-2.839)
$ShortVolume_{i,t-5min} * 10^{-5}$						0.007***
						(6.692)
$Intercept$	0.043***	0.042***	0.042***	0.041***	0.041***	0.041***
	(48.081)	(45.905)	(47.382)	(46.916)	(47.185)	(47.217)
N	244,089	244,089	241,886	241,886	241,886	241,886
Adj. R^2	0.022	0.022	0.023	0.023	0.024	0.024

Panel B

	(1) $PriceImpact$	(2) $PriceImpact$	(3) $PriceImpact$	(4) $PriceImpact$	(5) $PriceImpact$	(6) $PriceImpact$
$Mintbroker$	-0.052***	-0.052***	-0.051***	-0.050***	-0.050***	-0.050***
	(-28.714)	(-28.684)	(-27.860)	(-26.540)	(-26.466)	(-26.430)
$Momentum$			0.029***	0.034***	0.034***	0.034***
			(7.550)	(8.370)	(8.290)	(8.202)
$Volatility$				-0.025***	-0.028***	-0.028***
				(-4.661)	(-5.112)	(-5.080)
$CumVolume_{i,t-5min} * 10^{-5}$					-0.001***	-0.001***
					(-3.738)	(-2.721)
$ShortVolume_{i,t-5min} * 10^{-5}$						0.001
						(0.427)
$Intercept$	0.009***	0.009***	0.006***	0.006***	0.006***	0.006***
	(6.460)	(6.024)	(4.393)	(4.730)	(4.483)	(4.481)
N	244,068	244,068	241,875	241,875	241,875	241,875
Adj. R^2	0.007	0.008	0.008	0.008	0.008	0.008

TABLE 3.17: **Intraday Market Impact Measures for Mintbroker trades in Avalon Holdings: 5 minutes intervals** This table shows the relationship between Mintbroker trades and both Effective Spread and Price impact. It reports the coefficients together with the standard errors in parenthesis from the regression:$Liq\_Measure_{i,t} = \alpha_i + \beta_1 Mintbroker_{i,t} + \beta_2 Volume_{i,t} + \beta_3 Momentum_{i,t-5min} + \beta_4 Volatility_{i,t-5min} + \beta_5 CumVolume_{i,t-5min} + \beta_6 ShortVolume_{i,t-5min} + \epsilon_{i,t}$ where $Liq\_Measure_{i,t}$ is either $EffectiveSpread$ or $PriceImpact$. $EffectiveSpread$ is defined as $2 * buy * (price_{it} - mid_{it}) / mid_{it}$. buy is a indicator equal to 1 for buyer initiated trades and to -1 for seller initiated trades, $price_{it}$ is the price of the trade and mid_{it} is the midquote of the consolidated BBO prevailing at the time of the trade. $PriceImpact$ is defined as $2 * buy * (mid_{it+1min} - mid_{it}) / mid_{it}$ where buy is a indicator equal to 1 for buyer initiated trades and to -1 for seller initiated, $mid_{it+5min}$ is the midquote of the consolidated BBO 5 minutes after the trade and mid_{it} is the midquote of the consolidated BBO at the time of the trade. $Volume_{i,t}$ is the size of the trade expressed as a multiple of 10^5, $Momentum_{i,t-5min}$ is the return in the 5 minutes before the trade, $Volatility_{i,t-5min}$ is the absolute return in the 5 minutes before the trade, $CumVolume_{i,t-5min}$ is the cumulative volume in the 5 minutes before the trade expressed as a multiple of 10^5 and $ShortVolume_{i,t-5min}$ is the cumulative short volume in the 5 minutes before the trade expressed as a multiple of 10^5. The table reports estimated coefficients and their t-statistics in parenthesis calculated using heteroskedasticity-robust standard errors controlling for day and intraday fixed effects. *, ** and *** indicate statistical significance at 10%, 5%, and 1% levels, respectively.

Panel A

	(1) $EffSpread$	(2) $EffSpread$	(3) $EffSpread$	(4) $EffSpread$	(5) $EffSpread$	(6) $EffSpread$
$Mintbroker$	-0.017***	-0.018***	-0.018***	-0.016***	-0.016***	-0.016***
	(-19.330)	(-19.407)	(-19.541)	(-18.141)	(-18.130)	(-17.840)
$Volume * 10^{-5}$		0.899*	0.893*	0.892*	0.891*	0.898*
		(1.938)	(1.919)	(1.920)	(1.917)	(1.939)
$Momentum$			0.041***	0.051***	0.052***	0.063***
			(3.673)	(4.741)	(4.858)	(5.836)
$Volatility$				0.201***	0.200***	0.201***
				(13.169)	(13.145)	(13.180)
$CumVolume_{i,t-5min} * 10^{-5}$					-0.001***	0.012***
					(-3.242)	(13.614)
$ShortVolume_{i,t-5min} * 10^{-5}$						-0.038***
						(-16.279)
$Intercept$	0.033***	0.030***	0.030***	0.024***	0.024***	0.025***
	(42.150)	(18.403)	(18.069)	(14.200)	(14.410)	(15.166)
N	196,333	196,333	195,457	195,457	195,457	195,457
Adj. R^2	0.051	0.052	0.055	0.058	0.058	0.059

Panel B

	(1) $PriceImpact$	(2) $PriceImpact$	(3) $PriceImpact$	(4) $PriceImpact$	(5) $PriceImpact$	(6) $PriceImpact$
$Mintbroker$	-0.002**	-0.002**	-0.002***	-0.003***	-0.002***	-0.002***
	(-2.202)	(-2.259)	(-2.607)	(-2.944)	(-2.657)	(-2.650)
$Volume * 10^{-5}$		0.253***	0.253***	0.253***	0.250***	0.250***
		(6.774)	(6.795)	(6.796)	(6.710)	(6.714)
$Momentum$			-0.014**	-0.016***	-0.013**	-0.013**
			(-2.539)	(-2.961)	(-2.486)	(-2.435)
$Volatility$				-0.038***	-0.039***	-0.039***
				(-4.724)	(-4.848)	(-4.847)
$CumVolume_{i,t-5min} * 10^{-5}$					-0.002***	-0.001***
					(-11.622)	(-2.678)
$ShortVolume_{i,t-5min} * 10^{-5}$						-0.001
						(-0.454)
$Intercept$	-0.004*	0.000	0.001	0.002*	0.003***	0.004***
	(-1.927)	(0.181)	(0.562)	(1.754)	(3.636)	(3.640)
N	196,335	196,333	195,457	195,457	195,457	195,457
Adj. R^2	0.001	0.001	0.001	0.001	0.002	0.002

TABLE 3.18: **Intraday Market Impact Measures for Mintbroker trades in New Concept Energy: 15 minutes intervals** This table shows the relationship between Mintbroker trades and both Effective Spread and Price impact. It reports the coefficients together with the standard errors in parenthesis from the regression:$Liq\_Measure_{i,t} = \alpha_i + \beta_1 Mintbroker_{i,t} + \beta_2 Volume_{i,t} + \beta_3 Momentum_{i,t-15min} + \beta_4 Volatility_{i,t-15min} + \beta_5 CumVolume_{i,t-15min} + \beta_6 ShortVolume_{i,t-15min} + \epsilon_{i,t}$ where $Liq\_Measure_{i,t}$ is either $EffectiveSpread$ or $PriceImpact$. $EffectiveSpread$ is defined as $2 * buy * (price_{it} - mid_{it}) / mid_{it}$. buy is a indicator equal to 1 for buyer initiated trades and to -1 for seller initiated trades, $price_{it}$ is the price of the trade and mid_{it} is the midquote of the consolidated BBO prevailing at the time of the trade. $PriceImpact$ is defined as $2 * buy * (mid_{it+1min} - mid_{it}) / mid_{it}$ where buy is a indicator equal to 1 for buyer initiated trades and to -1 for seller initiated, $mid_{it+15min}$ is the midquote of the consolidated BBO 15 minutes after the trade and mid_{it} is the midquote of the consolidated BBO at the time of the trade. $Volume_{i,t}$ is the size of the trade expressed as a multiple of 10^5, $Momentum_{i,t-15min}$ is the return in the 15 minutes before the trade, $Volatility_{i,t-15min}$ is the absolute return in the 15 minutes before the trade, $CumVolume_{i,t-15min}$ is the cumulative volume in the 15 minutes before the trade expressed as a multiple of 10^5 and $ShortVolume_{i,t-15min}$ is the cumulative short volume in the 15 minutes before the trade expressed as a multiple of 10^5. The table reports estimated coefficients and their t-statistics in parenthesis calculated using heteroskedasticity-robust standard errors controlling for day and intraday fixed effects. *, ** and *** indicate statistical significance at 10%, 5%, and 1% levels, respectively.

Panel A

	(1) $EffSpread$	(2) $EffSpread$	(3) $EffSpread$	(4) $EffSpread$	(5) $EffSpread$	(6) $EffSpread$
$Mintbroker$	-0.003***	-0.003***	-0.003***	-0.003***	-0.003***	-0.003***
	(-8.256)	(-8.041)	(-7.385)	(-7.855)	(-7.369)	(-7.533)
$Volume * 10^{-5}$		0.232***	0.232***	0.233***	0.233***	0.233***
		(3.705)	(3.683)	(3.708)	(3.710)	(3.707)
$Momentum$			0.004***	0.002	0.002	0.002
			(2.755)	(1.459)	(1.414)	(1.425)
$Volatility$				0.006***	0.008***	0.008***
				(3.120)	(4.267)	(4.231)
$CumVolume_{i,t-15min} * 10^{-5}$					0.000***	-0.000***
					(4.841)	(-2.949)
$ShortVolume_{i,t-15min} * 10^{-5}$						0.003***
						(6.350)
$Intercept$	0.043***	0.042***	0.041***	0.041***	0.041***	0.041***
	(48.081)	(45.905)	(46.378)	(46.403)	(46.402)	(46.411)
N	244,089	244,089	243,481	243,481	243,481	243,481
Adj. R^2	0.022	0.022	0.022	0.022	0.022	0.023

Panel B

	(1) $PriceImpact$	(2) $PriceImpact$	(3) $PriceImpact$	(4) $PriceImpact$	(5) $PriceImpact$	(6) $PriceImpact$
$Mintbroker$	-0.082***	-0.082***	-0.081***	-0.081***	-0.082***	-0.082***
	(-28.907)	(-28.875)	(-28.492)	(-27.942)	(-28.092)	(-28.056)
$Volume * 10^{-5}$		0.296***	0.298***	0.298***	0.297***	0.297***
		(3.680)	(3.705)	(3.704)	(3.689)	(3.690)
$Momentum$			0.010***	0.010***	0.010***	0.010***
			(3.227)	(3.101)	(3.164)	(3.162)
$Volatility$				-0.001	-0.008	-0.008
				(-0.187)	(-1.366)	(-1.359)
$CumVolume_{i,t-15min} * 10^{-5}$					-0.000***	-0.000
					(-3.872)	(-0.739)
$ShortVolume_{i,t-15min} * 10^{-5}$						-0.003
						(-1.090)
$Intercept$	0.011***	0.010***	0.008***	0.008***	0.008***	0.008***
	(5.048)	(4.571)	(3.794)	(3.797)	(3.791)	(3.779)
N	244,074	244,074	243,476	243,476	243,476	243,476
Adj. R^2	0.008	0.008	0.008	0.008	0.008	0.008

TABLE 3.19: **Intraday Market Impact Measures for Mintbroker trades in Avalon Holdings: 15 minutes intervals** This table shows the relationship between Mintbroker trades and both Effective Spread and Price impact. It reports the coefficients together with the standard errors in parenthesis from the regression:$Liq\_Measure_{i,t} = \alpha_i + \beta_1 Mintbroker_{i,t} + \beta_2 Volume_{i,t} + \beta_3 Momentum_{i,t-15min} + \beta_4 Volatility_{i,t-15min} + \beta_5 CumVolume_{i,t-15min} + \beta_6 ShortVolume_{i,t-15min} + \epsilon_{i,t}$ where $Liq\_Measure_{i,t}$ is either $EffectiveSpread$ or $PriceImpact$. $EffectiveSpread$ is defined as $2 * buy * (price_{it} - mid_{it})/mid_{it}$. buy is a indicator equal to 1 for buyer initiated trades and to -1 for seller initiated trades, $price_{it}$ is the price of the trade and mid_{it} is the midquote of the consolidated BBO prevailing at the time of the trade. $PriceImpact$ is defined as $2 * buy * (mid_{it+1min} - mid_{it})/mid_{it}$ where buy is a indicator equal to 1 for buyer initiated trades and to -1 for seller initiated, $mid_{it+15min}$ is the midquote of the consolidated BBO 15 minutes after the trade and mid_{it} is the midquote of the consolidated BBO at the time of the trade. $Volume_{i,t}$ is the size of the trade expressed as a multiple of 10^5, $Momentum_{i,t-15min}$ is the return in the 15 minutes before the trade, $Volatility_{i,t-15min}$ is the absolute return in the 15 minutes before the trade, $CumVolume_{i,t-15min}$ is the cumulative volume in the 15 minutes before the trade expressed as a multiple of 10^5 and $ShortVolume_{i,t-15min}$ is the cumulative short volume in the 15 minutes before the trade expressed as a multiple of 10^5. The table reports estimated coefficients and their t-statistics in parenthesis calculated using heteroskedasticity-robust standard errors controlling for day and intraday fixed effects. *, ** and *** indicate statistical significance at 10%, 5%, and 1% levels, respectively.

Panel A

	(1) $EffSpread$	(2) $EffSpread$	(3) $EffSpread$	(4) $EffSpread$	(5) $EffSpread$	(6) $EffSpread$
$Mintbroker$	-0.017***	-0.018***	-0.018***	-0.016***	-0.016***	-0.016***
	(-19.330)	(-19.407)	(-19.464)	(-17.612)	(-17.705)	(-17.768)
$Volume * 10^{-5}$		0.899*	0.903*	0.934**	0.932**	0.930**
		(1.938)	(1.947)	(2.039)	(2.034)	(2.028)
$Momentum$			-0.027***	-0.049***	-0.048***	-0.041***
			(-4.936)	(-7.680)	(-7.579)	(-6.351)
$Volatility$				0.178***	0.176***	0.169***
				(20.036)	(20.029)	(19.114)
$CumVolume_{i,t-15min} * 10^{-5}$					-0.000***	0.005***
					(-5.626)	(17.876)
$ShortVolume_{i,t-15min} * 10^{-5}$						-0.016***
						(-21.293)
$Intercept$	0.033***	0.030***	0.032***	0.024***	0.026***	0.024***
	(42.150)	(18.403)	(19.061)	(14.688)	(15.425)	(14.589)
N	196,333	196,333	196,310	196,310	196,310	196,310
Adj. R^2	0.051	0.052	0.052	0.057	0.057	0.058

Panel B

	(1) $PriceImpact$	(2) $PriceImpact$	(3) $PriceImpact$	(4) $PriceImpact$	(5) $PriceImpact$	(6) $PriceImpact$
$Mintbroker$	-0.005***	-0.005***	-0.005***	-0.006***	-0.006***	-0.006***
	(-4.431)	(-4.477)	(-4.506)	(-5.004)	(-4.937)	(-4.945)
$Volume * 10^{-5}$		0.284***	0.284***	0.275***	0.277***	0.276***
		(5.042)	(5.053)	(4.889)	(4.916)	(4.896)
$Momentum$			-0.004	0.003	0.002	0.005
			(-0.885)	(0.568)	(0.393)	(1.074)
$Volatility$				-0.051***	-0.049***	-0.052***
				(-9.672)	(-9.530)	(-10.008)
$CumVolume_{i,t-15min} * 10^{-5}$					0.000**	0.003***
					(2.492)	(4.746)
$ShortVolume_{i,t-15min} * 10^{-5}$						-0.007***
						(-4.612)
$Intercept$	-0.003	0.003*	0.003*	0.005***	0.004**	0.003**
	(-1.138)	(1.722)	(1.782)	(3.138)	(2.546)	(2.198)
N	196,335	196,333	196,310	196,310	196,310	196,310
Adj. R^2	0.001	0.001	0.001	0.001	0.001	0.001

TABLE 3.20: **Mintborker Market Timing in New Concept Energy: 5 minutes intervals** This table reports a probit regression of the probability of a Mintbroker trade on $Spread$, $Momentum$, $PriceImpact$, $Cumultive\_Volume$ and $Short\_volume$ in the 5 minutes preceding the trade. $Spread$ is the average difference between the ask and the bid in the minute before the trade. $Momentum_{i,t-5min}$ is the return in the 5 minutes before the trade, $PriceImpact$ is defined as $2 * buy * (mid_{it+5min} - mid_{it}) / mid_{it}$ where buy is a indicator equal to 1 for buyer initiated trades and to -1 for seller initiated, $mid_{it+5min}$ is the midquote of the consolidated BBO 5 minutes after the trade and mid_{it} is the midquote of the consolidated BBO at the time of the trade. $CumVolume_{i,t-5min}$ is the cumulative volume in the 5 minutes before the trade expressed as a multiple of 10^5 and $ShortVolume_{i,t-1min}$ is the cumulative short volume in the 5 minutes before the trade expressed as a multiple of 10^5. The table reports estimated coefficients and their t-statistics in parenthesis calculated using robust standard errors as well as the marginal effects at the mean in square brackets. *, ** and *** indicate statistical significance at 10%, 5%, and 1% levels, respectively.

	(1) Mintbroker	(2) Mintbroker	(3) Mintbroker	(4) Mintbroker	(5) Mintbroker
$Spread$	-7.519***	-10.636***	-10.873***	-10.601***	-7.924***
	(-52.810)	(-52.380)	(-51.825)	(-50.292)	(-39.483)
	[-0.558]	[-1.065]	[-1.077]	[-1.040]	[-0.735]
$Momentum$		-2.963***	-2.570***	-2.473***	-3.182***
		(-58.398)	(-50.715)	(-49.112)	(-58.637)
		[-0.297]	[-0.255]	[-0.243]	[-0.295]
$PriceImpact$			-0.680***	-0.685***	-0.672***
			(-32.357)	(-32.084)	(-32.242)
			[-0.067]	[-0.067]	[-0.062]
$CumVolume * 10^{-5}$				0.027***	-0.059***
				(32.628)	(-30.663)
				[0.003]	[-0.005]
$ShortVolume * 10^{-5}$					1.383***
					(53.937)
					[0.128]
N	409,143	241,875	241,801	241,801	241,801
pseudo R^2	0.020	0.073	0.086	0.094	0.121

TABLE 3.21: **Mintborker Market Timing in Avalon Holdings: 5 minutes intervals** This table reports a probit regression of the probability of a Mintbroker trade on *Spread*, *Momentum*, *PriceImpact*, *Cumultive_Volume* and *Short_volume* in the minute preceding the trade. *Spread* is the average difference between the ask and the bid in the 5 minutes before the trade. $Momentum_{i,t-5min}$ is the return in the 5 minutes before the trade, *PriceImpact* is defined as $2 * buy * (mid_{it+5min} - mid_{it})/mid_{it}$ where *buy* is a indicator equal to 1 for buyer initiated trades and to -1 for seller initiated, $mid_{it+5min}$ is the midquote of the consolidated BBO 5 minutes after the trade and mid_{it} is the midquote of the consolidated BBO at the time of the trade. $CumVolume_{i,t-5min}$ is the cumulative volume in the 5 minutes before the trade expressed as a multiple of 10^5 and $ShortVolume_{i,t-1min}$ is the cumulative short volume in the 5 minutes before the trade expressed as a multiple of 10^5. The table reports estimated coefficients and their *t*-statistics in parenthesis calculated using robust standard errors as well as the marginal effects at the mean in square brackets. *, ** and *** indicate statistical significance at 10%, 5%, and 1% levels, respectively.

	(1) Mintbroker	(2) Mintbroker	(3) Mintbroker	(4) Mintbroker	(5) Mintbroker
Spread	-1.611***	-1.838***	-1.838***	-1.990***	-2.030***
	(-32.090)	(-34.139)	(-34.094)	(-34.793)	(-35.006)
	[-0.170]	[-0.192]	[-0.192]	[-0.205]	[-0.209]
Momentum		-1.913***	-1.910***	-1.925***	-1.903***
		(-24.673)	(-24.730)	(-23.959)	(-23.599)
		[-0.200]	[-0.200]	[-0.198]	[-0.196]
PriceImpact			-0.074**	-0.081***	-0.083***
			(-2.445)	(-2.620)	(-2.681)
			[-0.008]	[-0.008]	[-0.009]
CumVolume				-0.052***	0.015
				(-17.366)	(1.327)
				[-0.005]	[0.002]
ShortVolume					-0.189***
					(-5.885)
					[-0.019]
N	196,335	195,459	195,459	195,459	195,459
pseudo R^2	0.010	0.016	0.016	0.020	0.020

TABLE 3.22: **Mintborker Market Timing in New Concept Energy: 15 minutes intervals** This table reports a probit regression of the probability of a Mintbroker trade on $Spread$, $Momentum$, $PriceImpact$, $Cumultive\_Volume$ and $Short\_volume$ in the 15 minutes preceding the trade. $Spread$ is the average difference between the ask and the bid in the minute before the trade. $Momentum_{i,t-15min}$ is the return in the 15 minutes before the trade, $PriceImpact$ is defined as $2 * buy * (mid_{it+5min} - mid_{it})/mid_{it}$ where buy is a indicator equal to 1 for buyer initiated trades and to -1 for seller initiated, $mid_{it+15min}$ is the midquote of the consolidated BBO 15 minutes after the trade and mid_{it} is the midquote of the consolidated BBO at the time of the trade. $CumVolume_{i,t-15min}$ is the cumulative volume in the 15 minutes before the trade expressed as a multiple of 10^5 and $ShortVolume_{i,t-15min}$ is the cumulative short volume in the 15 minutes before the trade expressed as a multiple of 10^5. The table reports estimated coefficients and their t-statistics in parenthesis calculated using robust standard errors as well as the marginal effects at the mean in square brackets. *, ** and *** indicate statistical significance at 10%, 5%, and 1% levels, respectively.

	(1)	(2)	(3)	(4)	(5)
	Mintbroker	Mintbroker	Mintbroker	Mintbroker	Mintbroker
$Spread$	-9.323***	-11.981***	-12.134***	-12.141***	-8.939***
	(-52.186)	(-46.298)	(-45.579)	(-45.779)	(-35.179)
	[-0.676]	[-1.209]	[-1.205]	[-1.205]	[-0.827]
$Momentum$		-1.400***	-1.182***	-1.177***	-1.362***
		(-46.557)	(-38.395)	(-38.365)	(-40.523)
		[-0.141]	[-0.117]	[-0.117]	[-0.126]
$PriceImpact$			-0.369***	-0.368***	-0.362***
			(-34.000)	(-33.938)	(-34.051)
			[-0.037]	[-0.037]	[-0.034]
$CumVolume * 10^{-5}$				-0.001*	-0.080***
				(-1.697)	(-51.598)
				[-0.000]	[-0.007]
$ShortVolume * 10^{-5}$					0.861***
					(49.190)
					[0.080]
N	409,162	243,479	243,466	243,466	243,466
pseudo R^2	0.025	0.064	0.080	0.080	0.103

TABLE 3.23: **Mintborker Market Timing in Avalon Holdings: 15 minutes intervals** This table reports a probit regression of the probability of a Mintbroker trade on *Spread*, *Momentum*, *PriceImpact*, *Cumultive_Volume* and *Short_volume* in the 15 minutes preceding the trade. *Spread* is the average difference between the ask and the bid in the minute before the trade. $Momentum_{i,t-15min}$ is the return in the 15 minutes before the trade, *PriceImpact* is defined as $2 * buy * (mid_{it+5min} - mid_{it})/mid_{it}$ where *buy* is a indicator equal to 1 for buyer initiated trades and to -1 for seller initiated, $mid_{it+15min}$ is the midquote of the consolidated BBO 15 minutes after the trade and mid_{it} is the midquote of the consolidated BBO at the time of the trade. $CumVolume_{i,t-15min}$ is the cumulative volume in the 15 minutes before the trade expressed as a multiple of 10^5 and $ShortVolume_{i,t-15min}$ is the cumulative short volume in the 15 minutes before the trade expressed as a multiple of 10^5. The table reports estimated coefficients and their *t*-statistics in parenthesis calculated using robust standard errors as well as the marginal effects at the mean in square brackets. *, ** and *** indicate statistical significance at 10%, 5%, and 1% levels, respectively.

	(1) Mintbroker	(2) Mintbroker	(3) Mintbroker	(4) Mintbroker	(5) Mintbroker
Spread	-1.582***	-2.101***	-2.123***	-2.361***	-2.422***
	(-31.553)	(-45.406)	(-45.028)	(-48.194)	(-48.355)
	[-0.168]	[-0.214]	[-0.216]	[-0.230]	[-0.236]
Momentum		-1.899***	-1.892***	-1.736***	-1.695***
		(-35.497)	(-35.634)	(-32.173)	(-31.708)
		[-0.194]	[-0.193]	[-0.169]	[-0.165]
PriceImpact			-0.203***	-0.200***	-0.199***
			(-9.372)	(-9.093)	(-9.015)
			[-0.021]	[-0.020]	[-0.019]
$CumVolume * 10^{-5}$				-0.044***	0.024***
				(-30.824)	(3.226)
				[-0.004]	[0.002]
$ShortVolume * 10^{-5}$					-0.189***
					(-9.041)
					[-0.018]
N	196,335	196,312	196,312	196,312	196,312
pseudo R^2	0.010	0.024	0.025	0.038	0.039

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