Università Commerciale "Luigi Bocconi"
PhD School

PhD program in: Economics and Finance

Cycle: 35

Disciplinary Field (code): SECS-P/01

Thesis on Macroeconomics

Advisor: Luigi Iovino

Co-Advisor: Dmitriy Sergeyev

PhD Thesis by

Zheng Gong

ID number: 3112172

Year: 2024

Acknowledgements

I am deeply indebted to my advisors, Professors Luigi Iovino and Dmitriy Sergeyev, for their invaluable guidance and instruction throughout my PhD journey. Their exceptional mentoring was crucial to the completion of this thesis. My interactions with them have always been intellectually stimulating and deeply inspiring. I extend my sincere gratitude to Professor Morten Ravn, a committee member from the Department of Economics at University College London. He kindly hosted me at University College London from October to December 2022 and provided invaluable guidance in overcoming numerous significant technical challenges encountered in my Job Market Paper.

I would like to thank numerous professors at Bocconi University who have enriched my work with insightful comments and suggestions. In particular, I would like to thank Nicola Pavoni, Antonella Trigari, Marco Ottaviani, and Marta Prato. My special thanks also go to the Ph.D. course coordinators, Mariano Massimiliano Croce and Marco Ottaviani, and my colleagues, for their great guidance and support in navigating the PhD program.

Finally, I wish to express my deepest gratitude to my family for their unwavering support and understanding throughout my pursuit of a Ph.D. Their constant encouragement has been a cornerstone of my academic journey.

Abstract

This thesis consists of three chapters. In Chapter 1, I study the redistribution channels in heterogeneous-agent New Keynesian (HANK) models. Following a monetary policy shock, this paper analytically characterizes the redistribution channels triggered by the shock and quantitatively assesses each channel's effects. The redistribution effects amplify the responses of output and consumption and dampen the response of investment and real interest rates. On impact, redistribution effects account for 28% of consumption response and 6% of output response. All redistribution channels contribute to amplification. When considering their impact magnitude, the channels are ranked in terms of importance as follows: interest rate exposure, income exposure, liquidity, tax exposure, and asset price. In Chapter 2, I study how the heterogeneity in marginal propensities to earn (MPE) affects output's response to money supply shocks in a Lucas Island model. Compared to the benchmark case in which wealth inequality is absent, the output's response can be either amplified or dampened. Then I analyse how income cyclicity affects output's responses. In Chapter 3, I analyze the collusion behavior in an n-firm industry featuring a cross-ownership network, under Cournot competition. I find that increasing PCO can hinder tacit collusion under the uniform output distribution scheme. However, this scheme is not always feasible for collusion. I examine different subgame perfect equilibriums and conclude that tacit collusion can be facilitated when PCO increases.

Contents

Chapter 1

Decomposing HANK

Zheng Gong[†]

Abstract

This paper introduces a decomposition of the responses of macroeconomic variables to aggregate shocks in heterogeneous-agent New Keynesian (HANK) models. I decompose these responses into representative-agent (RANK) and redistribution effects. To obtain RANK effects, I introduce counterfactual transfers that counteract the redistribution triggered by the aggregate shock and ensure that all agents have the same consumption responses. In this case, the responses of the HANK model are equivalent to those of a (fictitious) RANK model. I show the existence of such transfers in various heterogeneous-agent models. Redistribution effects are derived from the HANK model's response to the redistribution shock backup from the counterfactual transfers. Further analysis of these transfers analytically breaks down the redistribution shock into five channels: income exposure, interest rate exposure, tax exposure, asset price, and liquidity. I apply this decomposition to monetary policy shocks and quantitatively assess the contribution of each redistribution channel to the differences between HANK and RANK.

[†]gong.zheng@phd.unibocconi.it, Department of Economics, Bocconi University. I would like to thank Luigi Iovino, Davide Debortoli, Dmitriy Sergeyev, Nicola Pavoni, Antonella Trigari, Morten Ravn, Vincent Sterk, Wei Cui, Alan Olivi, Jose-Victor Rios-Rull, Ricardo Reis, Luigi Bocola, Florin Bilbiie and participants at the University College London Macro Reading Group, University of Cambridge Ph.D. Macro Workshop, the Qatar Centre for Global Banking and Finance Annual Conference 2023 in King's College London, the Asia Meeting of the Econometric Society 2023 in Singapore, and the EEA-ESEM 2023 in Barcelona for their valuable discussions and suggestions.

1 Introduction

Heterogeneous-agent models have become increasingly popular in macroeconomics. By introducing nominal rigidity, heterogeneous-agent New Keynesian (HANK) models can help us understand how household heterogeneity affects aggregate demand and economic fluctuations. HANK models can amplify/dampen the general equilibrium effects of aggregate shocks, relative to representative-agent New Keynesian (RANK) models. Consider the response of consumption to an expansionary shock. If households who benefit from the shock have higher marginal propensities to consume (MPCs) than those who lose, the consumption response will be amplified; otherwise, the response will be dampened.

There is already an extensive literature on the heterogeneity in MPCs. However, how model specifications and parameterizations affect the redistribution among households, and how the redistribution is correlated with households' MPCs, are not fully understood. HANK models can feature multiple channels of redistribution. Is the consumption response always amplified? Which redistribution channel is amplifying and which channel is dampening? In terms of amplification, which channel is more important, and which channel plays a minor role? Quantitative models require sophisticated numerical methods to solve, making it difficult to interpret model results and answer these questions. The difficulty in identifying the (effects of) redistribution channels presents challenges in building, calibrating, and applying HANK models to address policy-relevant questions in a world characterized by agent heterogeneity.

This paper answers the above questions by analytically characterizing the redistribution channels inherent to a HANK model and quantitatively assessing each channel's effects. To understand the HANK model's response to an aggregate shock, I follow a two-step approach. In the first step, I consider the model's response with all redistribution channels muted, such that the consumption of agents is equally affected by the aggregate shock. Importantly, in this case, the HANK model's response is equivalent to that of a (fictitious) RANK model. In the second step, I unmute those redistribution channels one by one. This two-step approach distinguishes the impact of a 'pure' aggregate shock, characterized by homogeneous responses across agents, from the impact of the redistribution triggered by the aggregate shock.

Formally, I introduce counterfactual lump-sum transfers that counteract the redistribution implied by the aggregate shock, and ensure that all agents have the same consumption responses. These transfers are designed to be purely redistributive and sum to zero cross-sectionally. With these transfers in place, an aggregation result akin to Werning (2015) arises, and the equilibrium can be characterized with the aggregate conditions of a RANK model. Then I use the transfers to back up the redistribution shock. Consequently, the general equilibrium impulse responses in HANK can be decomposed into two components: the responses of the fictitious RANK model to the aggregate shock (RANK effects), and the HANK model's responses to the redistribution shock (redistribution effects). Essentially, the transfers are introduced to make the implicit redistribution in the model explicit. Further analysis of these transfers permits an analytical breakdown of the redistribution channels inherent to the HANK model. This breakdown also enables a quantitative assessment of each channel's role.

I prove the existence of such transfers in various heterogeneous-agent models. These models include a two-agent model with a fraction of permanent-income households and a fraction of hand-to-mouth households, the standard Bewley-Aiyagari-Huggett model of incomplete markets, and standard incomplete market models that incorporate ex-ante heterogeneity in household discount factors or illiquid assets.[1] Transfers are exogenous to household decisions. In the standard incomplete-market model and its variants, transfers depend on the history of household idiosyncratic shocks.

Consider an unexpected interest-rate cut, the two-step approach implies that to predict the policy's effects, the policymaker needs to know (i) the RANK model's response to the interest rate cut; and (ii) the HANK model's response to the redistribution shock triggered by the interest-rate cut. The RANK model's response is well-established in the literature. The HANK model's response to the redistribution shock generally requires numerically solving a full heterogeneous-agent model. This paper shows that we can obtain insights from simple partial equilibrium analysis. Since the transfers are purely redistributive, the first-order consumption response to a transitory redistribution shock in partial equilibrium is the covariance between household MPCs and the redistribution terms received. I derive the model moments of the partial-equilibrium consumption response that can be estimated from data, following Auclert (2019) and Patterson (2023). Combined with the responses of aggregates in the RANK model, the estimable moments can help identify important redistribution channels and predict the total effects of an interest-rate cut.

I first characterize the redistribution channels in a canonical HANK model in the style of McKay, Nakamura and Steinsson (2016). With counterfactual transfers, the equilibrium is equivalent to that of a textbook RANK model (Galí, 2015). In the baseline HANK model, I identify three sources of redistribution: income exposure, interest rate exposure, and tax exposure channels. Each of these channels represents a different mechanism of redistribution among households. The income exposure channel reflects the redistribution among

[1]These features allow the standard incomplete-market model to reproduce the large aggregate MPC observed in data.

households that exhibit different income elasticities to aggregate income. The interest rate exposure channel captures the redistribution between creditors and debtors. Finally, the tax exposure channel focuses on the redistribution among households that benefit unequally from the change in tax payments. Moreover, if the bond supply is cyclical, a fourth channel, the liquidity channel is also present. Due to the failure of Ricardian equivalence, the cyclical bond supply has real effects on the economy, which is captured by the liquidity channel. This paper demonstrates that, under uniform taxation, the bond supply shocks are equivalent to borrowing constraint shocks: the economy's response to increasing bond supply is equivalent to its response to a shock relaxing households' borrowing constraints, and decreasing bond supply essentially tightens borrowing constraints.

I then add investment to the baseline model and show that it affects redistribution through two channels. I assume firms own capital and make investment decisions. They issue equity and pay dividends to households. The first channel through which investment affects redistribution is the familiar income exposure channel. Investment responses are negatively correlated with dividend responses. When the share of dividends in aggregate income fluctuates, the income elasticities of dividend-income receivers and labor-income receivers are affected in opposite ways. In addition to the income exposure channel, the change in equity price leads to a newly identified asset price channel, which reflects the redistribution between asset buyers and asset sellers.

The characterization of redistribution channels allows me to discuss literature in a unified framework. One example is the role of fiscal policy in quantitative HANK models. Previous studies found that the fiscal policy response is crucial for determining the effects of aggregate shocks. As discussed in the liquidity channel, the time-varying bond supply affects aggregate demand as it changes the borrowing conditions of households. Guerrieri and Lorenzoni (2017) analyze the tightened borrowing constraint shock, a negative demand shock forcing constrained households to cut spending. When discussing the role of fiscal policy, Kaplan, Moll and Violante (2018) let government debt absorb the majority of the fiscal imbalance in the short run and find that the economy's responses to the monetary policy shock are much smaller. The borrowing constraint shock implied by the decreased bond supply is exactly the deleveraging shock in Guerrieri and Lorenzoni (2017). The same argument also applies to the analysis of fiscal multipliers. Auclert, Rognlie and Straub (2018) and Hagedorn, Manovskii and Mitman (2019) find that the deficit-financed fiscal multiplier is larger than the tax-financed fiscal multiplier. This result is due to the relaxed borrowing constraint induced by the increasing bond supply. Following a government spending shock, when the government postpones raising taxes and increases public debt, more liquidity is injected into the economy. The increasing bond supply allows constrained households to borrow, weakening the precautionary saving motive and stimulating aggregate consumption. Similarly, Wolf (2021*a*) and Wolf (2021*b*) study the role of deficit-financed lump-sum fiscal transfer as a stimulating policy tool, the effects of which are equivalent to relaxing borrowing constraints.

Finally, I use the decomposition to study the quantitative relevance of the redistribution effects in general equilibrium. In particular, I consider the model's response to an expansionary 25 basis points monetary policy shock with a quarterly persistence of 0.61. The redistribution effects amplify the responses of output and consumption and dampen the responses of investment and the real interest rate. On impact, the consumption in HANK rises by 0.5 percent. Regarding the decomposition, the RANK effects account for 72 percent of the consumption increase, the interest exposure channel for 9.7 percent, the income exposure channel for 8.6 percent, the liquidity channel for 5.9 percent, and the tax exposure channel for 2 percent. The effects of the asset price channel are close to zero.

Following an interest-rate cut, creditors lose and debtors benefit. Debtors have higher MPCs than creditors so the interest exposure channel amplifies consumption responses. When investment is more responsive than consumption, the share of dividends in aggregate income decreases and dividend-income receivers lose relative to labor-income receivers. And when investment is less responsive than consumption, the redistribution goes in the other direction. For typical calibrations, these are the cases in the short run and the long run, respectively. Since rich households receive relatively more dividend income and poor households receive relatively more labor income, the responses of investment amplify consumption responses by taxing rich households and subsidizing poor households. The path of public debt is calibrated to match the estimated increases in household loans in liquid accounts following the expansionary shock. The asset supply increases and relaxes the borrowing conditions of households. The liquidity channel stimulates aggregate consumption. When output and aggregate labor tax increase, high-labor-income households are hurt relative to low-labor-income households because the former pay a larger share of aggregate tax. When the government reduces lump-sum taxes to balance the budget, all households benefit equally. As a result, the tax exposure channel benefits low-labor-income households overall and amplifies consumption responses.

Related Literature. This paper builds on and contributes to the quantitative HANK literature. This body of work integrates nominal rigidity into incomplete-market general equilibrium models to study various macroeconomic questions. These include fiscal transfers (Oh and Reis (2012)), automatic fiscal stabilizers (McKay and Reis (2016)), monetary policy transmission (Gornemann, Kuester and Nakajima (2016), McKay, Nakamura and Steinsson (2016), Kaplan, Moll and Violante (2018), Luetticke (2021), Auclert, Rognlie and Straub (2020)), endogenous income risk (Ravn and Sterk (2017)), de-leveraging (Guerrieri and Lorenzoni (2017)), fiscal multipliers (Auclert, Rognlie and Straub (2018), Hagedorn, Manovskii and Mitman (2019)), inequality and income risk shocks (Auclert and Rognlie (2018), Bayer et al. (2019)), and business cycles (Bayer, Born and Luetticke (2020), Berger, Bocola and Dovis (2019), Bilbiie, Primiceri and Tambalotti (2023)). Instead of using a HANK model to study a specific question, this paper provides an analytical characterization of the redistribution channels in a relatively general environment, making quantitative models easier to interpret.[2]

This paper also contributes to the strand of the literature that makes simplifying assumptions to analytically study how heterogeneity changes aggregate outcomes (Werning (2015), Auclert (2019), Bilbiie (2020), Bilbiie (2018), Bilbiie, Känzig and Surico (2022), Ravn and Sterk (2021), Acharya and Dogra (2020), Debortoli and Galí (2017), Debortoli and Galí (2022)). The decomposition approach proposed in this paper is closely related to Werning (2015). Werning (2015) analyzes cases where the incomplete-market economy can be aggregated as an 'as if' representative agent economy, corresponding to the RANK effects defined in this paper. For more general cases where the 'as if' result does not hold, I introduce counterfactual transfers to preserve the aggregation. I then study how the heterogenous-agent economy deviates from the 'as if' representative agent economy without these transfers. In a theoretical framework, Auclert (2019) underscores the role of the covariance between households' MPCs and the sensitivity of their incomes in amplifying monetary policy shocks, and decomposes the aggregate consumption response into substitution and income effects (referred to as 'direct' and 'indirect' effects in Kaplan, Moll and Violante (2018)). Unlike the decomposition in Auclert (2019) and Kaplan, Moll and Violante (2018) — which combines the effects of MPCs heterogeneity and the effects of the correlation between MPCs and exposures — the decomposition in this paper isolates the correlation between MPCs and exposures to uncover the amplification mechanism.[3] Bilbiie, Känzig and Surico (2022) study the role of investment in amplification in a tractable TANK model. This paper extends their analysis of investment to a standard incomplete-market model. Debortoli and Galí (2017) use a TANK model to approximate HANK models. This paper studies amplification in HANK models and also provides insights regarding the different redistribution mechanisms in HANK and TANK

[2]The proposed decomposition abstracts from time-varying idiosyncratic risk, so it does not apply to models featuring endogenous unemployment risk such as Gornemann, Kuester and Nakajima (2016) and Ravn and Sterk (2017), or exogenous income risk shocks as in Bayer et al. (2019). In an extension I am currently exploring, I allow for a time-varying discount factor of the fictitious representative agent to incorporate the effects of time-varying idiosyncratic risks.

[3]I will provide a more detailed comparison between their approach and the approach of paper in a two-agent model.

models.

Patterson (2023) estimates the covariance between MPCs and unequal exposures in the labor market. This paper, in contrast, discusses the mechanism of unequal exposures in a fully quantitative model. The idea that counterfactual transfers can be used to construct an 'as if' representative agent is also present in the works of Hagedorn et al. (2019) and Hagedorn (2021). Hagedorn et al. (2019) use counterfactual transfers to assess the imbalance between aggregate demand and aggregate supply off-the-equilibrium path in explaining the forward guidance puzzle. Hagedorn (2021) uses a similar conceptual framework to characterize local determinacy in incomplete-market models. This paper has a different objective: I apply the counterfactual transfer approach to decompose the effects of aggregate shocks in general equilibrium. For this aim, I prove the existence of the transfers and show how to construct them.

The rest of the paper is organized as follows. Section 2 defines the decomposition of impulse responses to an aggregate shock in a general heterogeneous-agent economy. Section 3 shows the existence of transfers in a canonical HANK model and discusses the redistribution channels. Section 4 adds investment to the model. Section 5 derives the estimable moments for partial equilibrium responses to the redistribution shock. Section 6 implements the decomposition quantitatively. In the Appendix, I consider several alternative models for illustration, which include a tractable TANK model, a canonical HANK model without investment and its responses to real-rate shocks, and a HANK model with illiquid assets.

2 Aggregate Shock Decomposition

Consider a heterogeneous-agent economy. The specifics of heterogeneity will become detailed in future sections. For the current discourse, a reduced form is employed to generally outline the decomposition. Time is discrete and extends indefinitely $t = 0, 1, \cdots$. There is no aggregate risk and the perfect-foresight economy starts from its stationary equilibrium. At time $t = 0$, there is a one-time unexpected aggregate shock (MIT shock) following a mean-reverting process $\boldsymbol{\epsilon} = \{\epsilon_t\}_{t=0}^{\infty}$. In the infinite horizon, the economy is back to its initial equilibrium. I study the transition path following the aggregate shock. I first define the impulse responses and then discuss the decomposition of the shock and the impulse responses.

For any aggregate variable Y, its value in the stationary equilibrium is denoted as Y^*, which is constant across time. Following the shock $\boldsymbol{\epsilon}$, Y's value at time t along the transition path is denoted as Y_t^{ϵ}, and the entire time path is denoted as $\mathbf{Y}^{\epsilon} = \{Y_t^{\epsilon}\}_{t=0}^{\infty}$. Then we can define Y's impulse responses to the monetary policy shock $\boldsymbol{\epsilon}$ as

$$\tilde{\mathbf{Y}}^{\boldsymbol{\epsilon}} \equiv \mathbf{Y}^{\boldsymbol{\epsilon}} - Y^* \cdot \mathbf{1},$$

where $\mathbf{1}$ is the identity vector with all elements equal to one. For an individual variable y_i with individual index i, let y_{it}^* denote its value at time t in the stationary equilibrium and $\mathbf{y}_i^* = \{y_{it}^*\}_{t=0}^{\infty}$ denotes the entire time path. Along the transition path, the path of variable y_i is denoted as $\mathbf{y}_i^{\boldsymbol{\epsilon}} = \{y_{it}^{\boldsymbol{\epsilon}}\}_{t=0}^{\infty}$. The impulse responses of the individual variable y_i are defined as

$$\tilde{\mathbf{y}}_i^{\boldsymbol{\epsilon}} \equiv \mathbf{y}_i^{\boldsymbol{\epsilon}} - \mathbf{y}_i^*.$$

In the standard Bewley-Aiyagari-Huggett model of incomplete markets, the individual outcome is a function of the path of the individual's idiosyncratic shocks. In this case, the individual impulse responses are defined conditional on the path of idiosyncratic shocks.

The impulse responses of aggregate variables may deviate from that of a representative agent model, due to the heterogeneous responses of individuals. The idea of the decomposition is to introduce counterfactual transfers, to ensure agents have the same (consumption) responses. Then the heterogeneous-agent economy can be aggregated and the response is equivalent to that of a representative-agent model. Denote the set of individuals in the economy as I. Consider the transfer scheme: $\boldsymbol{\omega} = \{\boldsymbol{\omega}_i\}_{i \in I}$, where $\boldsymbol{\omega}_i = \{\omega_{it}\}_{t=0}^{\infty}$ and ω_{it} is the transfer received by individual i at time t. The aggregate shock $\boldsymbol{\epsilon}$ can be written as

$$\begin{pmatrix} \boldsymbol{\epsilon} \\ \mathbf{0} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\epsilon} \\ \boldsymbol{\omega} \end{pmatrix} + \begin{pmatrix} \mathbf{0} \\ -\boldsymbol{\omega} \end{pmatrix}.$$

The aggregate shock $\boldsymbol{\epsilon}$ is decomposed as a sum of two sets of shocks. The first set includes the aggregate shock $\boldsymbol{\epsilon}$ and the transfer scheme $\boldsymbol{\omega}$; and the second set includes only the redistribution shock, which is defined as the negative of the transfer scheme $-\boldsymbol{\omega}$. Then to the first order, the impulse responses of outcome variable Y admit an additive decomposition. Y's responses to the sum of two sets of shocks are equal to the sum of its responses to each set of shocks:

$$\tilde{\mathbf{Y}}^{\boldsymbol{\epsilon},\mathbf{0}} = \tilde{\mathbf{Y}}^{\boldsymbol{\epsilon},\boldsymbol{\omega}} + \tilde{\mathbf{Y}}^{\mathbf{0},-\boldsymbol{\omega}}.$$

This first-order relation also holds for the impulse responses of individual variables:

$$\tilde{\mathbf{y}}_i^{\boldsymbol{\epsilon},\mathbf{0}} = \tilde{\mathbf{y}}_i^{\boldsymbol{\epsilon},\boldsymbol{\omega}} + \tilde{\mathbf{y}}_i^{\mathbf{0},-\boldsymbol{\omega}}.$$

By appropriately constructing the transfers, the redistribution **induced** by the aggregate shock can be removed, and all agents have the same consumption responses (in percentage terms) to the first set of shocks. This paper shows that such transfers exist for various heterogeneous-agent models. The transfer scheme with this property will be a function of the aggregate shock and is denoted as $\boldsymbol{\omega}(\boldsymbol{\epsilon})$. The equilibrium under the first set of shocks can be characterized by the equilibrium conditions of a representative agent model. With these transfers $\boldsymbol{\omega}(\boldsymbol{\epsilon})$, I define the decomposition.

Definition 1. The RANK effects of the aggregate shock $\boldsymbol{\epsilon}$ on variable Y are variable Y's responses to the aggregate shock $\boldsymbol{\epsilon}$ and the transfer scheme $\boldsymbol{\omega}(\boldsymbol{\epsilon})$:

$$\tilde{\mathbf{Y}}^{ra} \equiv \tilde{\mathbf{Y}}^{\boldsymbol{\epsilon},\boldsymbol{\omega}(\boldsymbol{\epsilon})}.$$

The redistribution effects of the aggregate shock $\boldsymbol{\epsilon}$ on variable Y are Y's responses to the redistribution shock, which is defined as the negative of the transfer scheme $-\boldsymbol{\omega}(\boldsymbol{\epsilon})$:

$$\tilde{\mathbf{Y}}^{re} \equiv \tilde{\mathbf{Y}}^{\mathbf{0},-\boldsymbol{\omega}(\boldsymbol{\epsilon})}.$$

For an individual variable y_i, we can define the decomposition similarly. By construction, the following property holds for individual consumption:

$$\mathbf{c}_i^{ra}/\mathbf{c}_i^{*} = \mathbf{C}^{ra}/C^{*}.$$

The RANK effects provide the benchmark for analyzing the redistribution effects of the aggregate shock. By examining the sources of the redistribution shock $-\boldsymbol{\omega}(\boldsymbol{\epsilon})$, we can analytically characterize the redistribution channels and quantitatively assess the effects of each channel.

3 Decomposing a Canonical HANK Model

This section considers a one-asset HANK model without productive assets and decomposes the economy's response to a monetary policy shock. I describe the model in sections 3.1 and 3.2. In section 3.3, I show that with counterfactual transfers, the equilibrium of the presented model is equivalent to that of a textbook RANK model (Galí, 2015). In section 3.4, I decompose the redistribution shock into four redistribution channels: interest-rate exposure, income exposure, tax exposure, and liquidity. In section 3.5, I discuss the household's problem in recursive form and how to compute the redistribution effects.

3.1 Model Description

The model is a heterogeneous-agent version of the textbook New Keynesian model similar to McKay, Nakamura and Steinsson (2016). Time is discrete and infinite. The economy is populated by households, firms, a fiscal and monetary policy authorities. In this economy, households face idiosyncratic uncertainty on incomes and have access to one-period risk-less government bonds, subject to exogenous borrowing constraints. There is price stickiness in the firm's price setting. The government collects taxes from households to pay interest on the debt. A monetary authority follows a Taylor rule. I analyze the economy's response to innovation to this Taylor rule.

Households. There is a unit continuum of households that face idiosyncratic productivity shocks $z_t \in Z_t$. Let $z^t = (z_0, z_1, \cdots, z_t)$ be a history of idiosyncratic states up to period t. For ease of notation, the initial state z_0 also indexes the initial bond holdings. At $t = 0$, the economy inherits an initial distribution over idiosyncratic states and bonds $\Phi_0(z_0)$. The stochastic process then induces a distribution $\Phi(z^t)$ over histories $z^t \in Z^t$. Households are infinitely lived and have preferences over consumption $c(z^t)$ and labor supply $n(z^t)$ given by the utility function

$$E\left[\sum_{t=0}^{\infty} \beta^t u(c(z^t), n(z^t))\right], \tag{1}$$

where β is the subjective discount factor. I also assume that the period utility function is given by

$$u(c,n) = \frac{c^{1-\sigma}}{1-\sigma} - \varphi \frac{n^{1+\nu}}{1+\nu}.$$

Households derive utility from consumption and dis-utility from working. Households face budget constraints

$$c(z^t) + b(z^t) = (1+r_t)b(z^{t-1}) + W_t z_t n(z^t) + \pi(z) - \tau(z^t), \tag{2}$$

for all $t = 0, 1, \cdots$ and histories $z^t \in Z^t$. Households face labor income risks so that if they work $n(z^t)$, they supply efficient labor $z_t n(z^t)$ to firms and receive labor income $W_t z_t n(z^t)$, where W_t is the real wage. The idiosyncratic productivity z_t evolves according to the first-order auto-regressive process $\log z_t = \rho_e \log z_{it-1} + e_{it}$ with normal innovations $e_{it} \sim \mathcal{N}(-\sigma_e^2(1-\rho_e^2)^{-1}/2, \sigma_e^2)$ so that $\int z_t d\Phi_t(z^t) = 1$. Households also receive (type-speific) profits $\pi(z)$ from intermediate firms and pay taxes $\tau(z^t)$ to government. The financial markets are incomplete. Households have access to a risk-free government bond with a real interest rate r_{t+1} between periods t and $t+1$. However, households' bond holdings are subject to the constraints

$$b(z^t) \geq \phi, \tag{3}$$

where ϕ is the exogenous borrowing limit and is strictly higher than the natural borrowing limit.

Firms. A competitive final-good firm produces a final good from intermediate goods, indexed by j, according to the production function $Y_t = (\int y_{j,t}^{1/\mu} dj)^{\mu}$. The intermediate goods are produced by monopolistic competitive firms using labor as the only input with linear technology $y_{j,t} = An_{j,t}$, where $n_{j,t}$ denotes the labor hired by firm j in period t.

Each intermediate firm sets its price to maximize profits subject to quadratic price adjustment costs as in Rotemberg (1982)

$$\Theta_t(P_{j,t}, P_{j,t-1}) = \frac{\mu}{\mu - 1} \frac{1}{2\kappa} [\log(P_{j,t}/P_{j,t-1})]^2 Y_t$$

where $\kappa > 0$. The corresponding Philips curve can be derived as

$$\log(1 + \pi_t^P) = \kappa \left(\frac{W_t}{A} - \frac{1}{\mu} \right) + \frac{1}{1 + r_{t+1}} \frac{Y_{t+1}}{Y_t} \log(1 + \pi_{t+1}^P),$$

where π_t^P is the inflation. The price adjustment creates real costs Θ_t, and profits equal output net of labor expenditure and price adjustment costs $\Pi_t = Y_t - W_t N_t - \Theta_t$.

Fiscal Policy. The government collects taxes from households to pay interest on the debt, giving the budget constraint

$$T_t + B = (1 + r_t)B,$$

where T_t is the aggregate tax. Currently, I assume that the government maintains a constant level of debt and adjusts taxes to balance its budget. Later I will allow the government to adjust the level of outstanding debt and document the 'liquidity' channel of monetary policy shocks.

Monetary policy. The monetary authority sets the nominal interest rates on government bonds i_t according to a Taylor rule $i_t = r^* + \phi_\pi \pi_t^P + \epsilon_t$. The ex-post real interest rates satisfy Fisher equation $1 + r_t = (1 + i_{t-1})/(1 + \pi_t^P)$.

Equilibrium. Given a sequence of exogenous monetary policy shocks $\{\epsilon_t\}_{t=0}^{\infty}$, an equilibrium consists of the path for aggregates $\{r_t, W_t, C_t, Y_t, \pi_t^P, \Pi_t, T_t\}$, profits distribution and tax payment rules $\{\pi(z), \tau(z^t)\}$, and households choices $\{c(z^t), n(z^t), b(z^t)\}$ such that:

(i) households optimization: given initial bond holdings, the path of aggregates, and profits distribution and tax payment rules, households choose $\{c(z^t), n(z^t), b(z^t)\}$ to maximize their utility function (1) subject to the budget constraints (2) and borrowing constraints (3); The Philips Curve holds; government budget constraint holds; nominal interest rates evolve according to the Taylor rule;

(ii) market clearing: for $t = 0, 1, \cdots$ the good, labor and bond markets clear:

$$C_t + \Theta_t = Y_t,$$
$$N_t = L_t,$$
$$B_t^d = B;$$

(iii) aggregation: the aggregate quantities are consistent with household quantities,

$$\int z_t n(z^t) d\Phi_t(z^t) = N_t,$$
$$\int b(z^t) d\Phi_t(z^t) = B_t^d,$$
$$\int c(z^t) d\Phi_t(z^t) = C_t,$$
$$\int \tau(z^t) d\Phi_t(z^t) = T_t,$$
$$\int \pi(z) d\Phi_t(z^t) = \Pi_t.$$

In the economy's stationary equilibrium, aggregate quantities and prices are constant, and inflation is zero. An outcome variable Y's stationary equilibrium value is denoted as Y^*, and Y's deviation from its stationary equilibrium value is denoted as $\tilde{Y}$. The percentage deviation is denoted as $\hat{Y}$.

3.2 Transition Dynamics and Counterfactual Transfers

Assume the economy starts from the stationary equilibrium and consider the economy's response to one-time unexpected monetary policy shocks $\boldsymbol{\epsilon} = \{\epsilon_t\}_{t=0}^{\infty}$. I decompose the impulse responses of outcome variables into RANK and redistribution effects. To do this, I construct a transfer scheme $\boldsymbol{\omega} = \{\omega(z^t), \forall z^t \in Z^t\}_{t=0}^{\infty}$ where $\omega(z^t)$ is the lump-sum transfer received by the household conditional on the productivity path z^t. The household's budget constraints with the counterfactual transfers then read

$$c(z^t) + b(z^t) = (1 + r_t) b(z^{t-1}) + W_t z_t n(z^t) + \pi(z) - \tau(z^t) + \omega(z^t).$$

3.3 RANK Effects

The key for the decomposition is the construction of the transfers $\boldsymbol{\omega}$. Proposition 1 shows that, for a given monetary policy shock $\boldsymbol{\epsilon}$, there exist counterfactual transfers $\boldsymbol{\omega}$ such that the heterogeneous-agent model is 'as if' a representative-agent model.

Proposition 1. For a given monetary policy shock $\boldsymbol{\epsilon}$, there exist counterfactual transfers $\boldsymbol{\omega}$ such that:

(i) The equilibrium can be characterized with only aggregate conditions:

- Aggregate Euler equation

$$(C_t^{\epsilon,\omega})^{-\sigma} = \beta^{ra}(1+r_{t+1}^{\epsilon,\omega})(C_{t+1}^{\epsilon,\omega})^{-\sigma}, \text{ where } \beta^{ra} \equiv 1/(1+r^*);$$

- Aggregate labor supply condition

$$W_t^{\epsilon,\omega}(C_t^{\epsilon,\omega})^{-\sigma} = \varphi^{ra}(N_t^{\epsilon,\omega})^{\nu}, \text{ where } \varphi^{ra} \equiv W^*(C^*)^{-\sigma}(N^*)^{-\nu};$$

- The Philips curve; government budget constraint; Taylor rule; and market clearing conditions.

(ii) The individual consumption and labor supply satisfy:

$$c^{\epsilon,\omega}(z^t)/c^*(z^t) = C_t^{\epsilon,\omega}/C^*;$$
$$n^{\epsilon,\omega}(z^t)/n^*(z^t) = N_t^{\epsilon,\omega}/N^*.$$

(iii) The transfers sum to zero crosssectionally $\int \omega(z^t)d\Phi_t(z^t) = 0$.

Proof. See Appendix.

Such transfers are a function of the monetary policy shock $\boldsymbol{\omega}(\boldsymbol{\epsilon})$. With such transfers, we can define the decomposition as in section 2. I will use the terminology 'RANK' equilibrium to refer to the equilibrium in Proposition 1. All the variables in the 'RANK' equilibrium are denoted with superscript 'ra'. The fictitious representative agent's subjective discount factor is the steady-state real discount rate $1/(1+r^*)$. Aggregate labor supply is the sum of individual labor supply given individual consumption $c^{ra}(z^t)$. With frictionless labor markets, the aggregate labor supply condition coincides with the representative-agent case.

From the aggregate conditions in Proposition 1, we can obtain the path of aggregates $\{r_t^{ra}, W_t^{ra}, C_t^{ra}, Y_t^{ra}, \pi_t^{P,ra}, \Pi_t^{ra}, T_t^{ra}\}$ given the monetary policy shock $\boldsymbol{\epsilon}$. The path of aggregates determines the household's consumption $c^{ra}(z^t)$, labor income $W_t^{ra}z_t n^{ra}(z^t)$, profits income $\pi^{ra}(z)$, and tax payment $\tau^{ra}(z^t)$. To recover the transfer term $\omega(z^t)$ from the household budget constraint, we also need to know the bond demand $b^{ra}(z^t)$. In the proof of Proposition 1, I impose the bond demand function $b^{ra}(z^t) = b^*(z^t)$. As shown below, the bond demand function $b^{ra}(z^t)$ is not unique. The intuition is similar to the Ricardian equivalence of a representative agent model. In Ricardian equivalence, the timing of taxes does not affect the equilibrium. In our case, the timing of transfers does not affect agents' consumption decisions and the economy's equilibrium. The income loss at time t can be compensated by future or past income, and households with access to financial markets will use bonds to move income across time.

The next proposition formalizes this intuition and shows that the bond demand function $\{b^{ra}(z^t), \forall z^t \in Z^t\}$ and the corresponding transfer scheme $\boldsymbol{\omega}$ is indeterminate.

Proposition 2. For bond demand function $b^{ra}(z^t)$ satisfying $\forall z^t \in Z^t$,

(i) The borrowing constraint and complementary slackness condtion: $b^{ra}(z^t) \geq \phi, =$ if $u'(c^*(z^t)) > \beta(1+r^*)E[u'(c^*(z^{t+1}))|z^t]$;

(ii) The transversality condition: $\lim_{t\to\infty} \beta^t E_0 b^{ra}(z^t) u'(c^{ra}(z^t)) = 0$;

(iii) Bond market clearing: $\int b^{ra}(z^t) d\Phi_t(z^t) = B$,

the transfer $\omega(z^t)$ is given by

$$\omega(z^t) = c^{ra}(z^t) + b^{ra}(z^t) - (1+r_t^{ra})b^{ra}(z^{t-1}) - W_t^{ra} z_t n^{ra}(z^t) - \pi^{ra}(z) + \tau^{ra}(z^t). \tag{4}$$

Proof. See Appendix.

Unconstrained households have access to financial markets and can use bonds to implement the consumption and labor supply plan given by Proposition 1. However, to satisfy the complementary slackness condition, constrained households have a fixed bond demand at the borrowing limit ϕ.

Extensions. Proposition 1 also holds with permanent heterogeneity in discount factors and can be extended to include frictional labor supply. The restriction imposed on labor supply is that households have the same consumption responses. Consider a simple case for illustration. Assuming a fixed cost of working in the style of Broer et al. (2020): $u(c,n) = c^{1-\sigma}/(1-\sigma) - n^{1+\varphi}/(1+\varphi) - \theta \mathbb{1}_{n>0}$. Some households optimally choose not to work because of the fixed cost of working and their high consumption or low productivity levels. Let n' be the labor supply implied by the first-order condition. In this case, for given aggregate consumption C_t^{ra} and wage W_t^{ra}, the aggregate labor supply N_t^{ra} is

$$N_t^{ra} \equiv \int z_t n^{ra}(z^t) d\Phi_t(z^t)$$
$$\text{where } n^{ra}(z^t) = \begin{cases} n', \text{ if } u(c^{ra}(z^t), n') \geq u(c^{ra}(z^t) - W_t^{ra} z_t n', 0) \\ 0, \text{ otherwise} \end{cases}$$

Households supply n' if and only if $u(c^{ra}(z^t), n') \geq u(c^{ra}(z^t) - z_t W_t n', 0)$. Otherwise, the household's labor supply is zero. The aggregate labor supply is non-linear because of the non-linear individual's labor supply. This example shows that household heterogeneity in labor supply implies a deviation from representative-agent models on the supply side.

In the case of linear labor income taxes, the aggregate labor supply condition is $(1 - \Gamma_t^{ra})W_t^{ra}(C_t^{ra})^{-\sigma} = \varphi^{ra}(N_t^{ra})^{\nu}$, where $\varphi^{ra} \equiv (1-\Gamma^*)W^*(C^*)^{-\sigma}(N^*)^{-\nu}$ and Γ is the tax rate. Aggregate tax is $T_t^{ra} = \Gamma_t^{ra}W_t^{ra}N_t^{ra}$ and individual tax payments are $\tau^{ra}(z^t) = \Gamma_t^{ra}z_tW_t^{ra}n^{ra}(z^t)$.

3.4 Redistribution Channels

The appendix shows that the redistribution shock can be decomposed as follows

$$-\omega(z^t) = \underbrace{(\hat{y}^{ra}(z^t) - \hat{Y}_t^{ra})y^*(z^t)}_{\text{income exposure}} + \underbrace{(b^*(z^{t-1}) - B)(r_t^{ra} - r^*)}_{\text{interest rate exposure}} + \underbrace{(T_t^{ra} - T^*) - (\tau^{ra}(z^t) - \tau^*(z^t))}_{\text{tax exposure}}$$
$$+ \underbrace{\hat{C}_t^{ra}(y^*(z^t) - c^*(z^t))}_{\text{scaling of net saving}} + \underbrace{(b^*(z^t) - b^{ra}(z^t)) - (1 + r_t^{ra})(b^*(z^{t-1}) - b^{ra}(z^{t-1}))}_{\text{undetermined bond demand}},$$

where I define $y \equiv Wzn + \pi$ as the household's income. From the above expression, I define three sources of redistribution: the income exposure channel, the interest rate exposure channel, and the tax exposure channel. There are also two residual terms. I define $y(z^t) - c(z^t)$ as net saving. The term 'scaling of net saving' is not zero because even if the transfers compensate for the redistribution from the previous channels, households' budget constraints can not be scaled. In the stationary equilibrium, the net saving $y^*(z^t) - c^*(z^t)$ is generally not zero, and the first residual term is used to compensate for the scaling of the net saving. The effects of this term are negligible quantitatively. The last term is due to the undetermined bond demand function. Note that after imposing the bond demand function $b^{ra}(z^t) = b^*(z^t)$, the last term is zero. Since $b^*(z^t)$ is the bond demand function in the stationary equilibrium, it satisfies Proposition 2.

The income exposure channel is defined as

$$(\hat{y}^{ra}(z^t) - \hat{Y}_t^{ra})y^*(z^t), \tag{5}$$

which captures the redistribution among households with different income elasticities to aggregate income. The interest exposure channel is defined as

$$(b^*(z^{t-1}) - B)(r_t^{ra} - r^*). \tag{6}$$

which captures the redistribution among creditors and debtors. Note that for bondholders, the net bond position $b^*(z^{t-1}) - B$, rather than the gross position $b^*(z^{t-1})$, determines their exposure to the interest rate shock. This is a result of consolidating the government budget constraint into the household budget constraint. The change in aggregate tax payment $T_t -$

T^* counteracts the change in aggregate interest income $B(r_t - r^*)$, which can be seen from the government's budget constraint.

The tax exposure channel is defined as

$$(T_t^{ra} - T^*) - (\tau^{ra}(z^t) - \tau^*(z^t)), \tag{7}$$

which captures the different exposures to the change in taxes. In the case of uniform taxation $\tau^{ra}(z^t) - \tau^*(z^t) = T_t^{ra} - T^*$, the tax exposure channel is muted because all households benefit equally from the tax reduction. For more general taxing schemes, households may benefit or lose from the tax change.

3.4.1 Liquidity Channel

In the baseline model, I assume that the government maintains a constant level of debt. Previous studies in the quantitative HANK literature found that the fiscal policy response is crucial for determining the effects of aggregate shocks.[4] Following the aggregate shock, the government can also adjust the outstanding debt to balance its budget. In this section, I attribute the effects induced by the time-varying paths of government debt to the liquidity channel.

Due to the failure of Ricardian equivalence, changing asset supply through fiscal policy response has real effects: the timing of taxes directly affects the consumption of non-Ricardian households. Let $\bar{b}^{ra}(z^t)$, $\bar{\tau}^{ra}(z^t)$ and $\bar{T}_t^{ra}$ denote the bond demand function, individual tax payment, and aggregate tax, respectively, when the government debt is constant. The redistribution shock $-\omega(z^t)$ can be decomposed as follows in this case:

$$\begin{aligned}
-\omega(z^t) = &\underbrace{(\hat{y}^{ra}(z^t) - \hat{Y}_t^{ra})y^*(z^t)}_{\text{income exposure}} + \underbrace{(b^*(z^{t-1}) - B)(r_t^{ra} - r^*)}_{\text{interest rate exposure}} + \underbrace{(\bar{T}_t^{ra} - T^*) - (\bar{\tau}^{ra}(z^t) - \tau^*(z^t))}_{\text{tax exposure}} \\
&+ \underbrace{(\bar{b}^{ra}(z^t) - b^{ra}(z^t)) - (1 + r_t^{ra})(\bar{b}^{ra}(z^{t-1}) - b^{ra}(z^{t-1})) + (\bar{\tau}^{ra}(z^t) - \tau^{ra}(z^t))}_{\text{liquidity}} \\
&+ \underbrace{\hat{C}_t^{ra}(y^*(z^t) - c^*(z^t))}_{\text{scaling of net saving}} + \underbrace{(b^*(z^t) - \bar{b}^{ra}(z^t)) - (1 + r_t^{ra})(b^*(z^{t-1}) - \bar{b}^{ra}(z^{t-1}))}_{\text{undermined bond demand}}.
\end{aligned}$$

The income exposure, interest exposure, and tax exposure channels are defined as before and are independent of the path of government debt.

The liquidity channel is defined as

$$(\bar{b}^{ra}(z^t) - b^{ra}(z^t)) - (1 + r_t^{ra})(\bar{b}^{ra}(z^{t-1}) - b^{ra}(z^{t-1})) + (\bar{\tau}^{ra}(z^t) - \tau^{ra}(z^t)). \tag{8}$$

[4] See Kaplan, Moll and Violante (2018), Alves et al. (2020), Auclert, Rognlie and Straub (2018), Hagedorn, Manovskii and Mitman (2019), Wolf (2021*a*), Wolf (2021*b*).

As in the previous section, after imposing the bond demand function $\bar{b}^{ra}(z^t) = b^*(z^t)$, the last term is zero.

The liquidity channel may seem obscure at first. To understand it better, consider the subgroup of households that remain constrained $\bar{b}^{ra}(z^t) = b^{ra}(z^t) = \phi$ and uniform taxation $\bar{\tau}^{ra}(z^t) - \tau^{ra}(z^t) = \bar{T}_t^{ra} - T_t^{ra}$. For these households, the term (8) is $\bar{T}_t^{ra} - T_t^{ra}$. The liquidity channel captures the effects of altering the timing of taxes. When the government shifts the timing of taxes by deficit financing, it transfers income across time for households. In partial equilibrium, the consumption of unconstrained households hardly changes because the net present value of tax change is zero; and the consumption of constrained households responds one-to-one to the change in their tax payment. Then, in general equilibrium, the interest rate will adjust to clear the bond markets, and the unconstrained households will absorb the change in government debt.

To link the above mechanism more closely with the concept of 'liquidity', I show that in the case of uniform taxation, the liquidity channel can be proxied by counterfactual shocks to the borrowing constraint ϕ.

Proposition 3. Off the constant-debt path, assume (i)uniform taxation $\tau^{ra}(z^t) - \bar{\tau}^{ra}(z^t) = T_t^{ra} - \bar{T}_t^{ra}$; (ii) counterfactual borrowing constraint $\phi_t^{ra} = \phi + B_t^{ra} - B^*$. For the bond demand function $\bar{b}^{ra}(z^t)$ satisfying the conditions in Proposition 2, the shifted bond demand function $b^{ra}(z^t) \equiv \bar{b}^{ra}(z^t) + B_t^{ra} - B^*$ satisfies $\forall z^t \in Z^t$

(i) The borrowing constraint and complementary slackness condition: $b^{ra}(z^t) \geq \phi_t^{ra}, =$ if $u'(c^*(z^t)) > \beta(1+r^*)E[u'(c^*(z^{t+1}))|z^t]$;

(ii) The transversality condition: $\lim_{t\to\infty} \beta^t E_0 b^{ra}(z^t) u'(c^{ra}(z^t)) = 0$;

(iii) Bond market clearing: $\int b^{ra}(z^t) d\Phi_t(z^t) = B_t^{ra}$,

and the transfers $\omega(z^t)$ are invariant to the path of government debt.

Proof. See Appendix.

The idea is similar to the argument in Aiyagari (1994) and Bhandari et al. (2017), which can be viewed as a heterogeneous-agent version of Ricardian equivalence. Suppose government debt increases by δ. For the same consumption choice $c^{ra}(z^t)$, the household now holds more bonds by δ units to the next period, implying that the wealth distribution is shifted for each household in all states. To satisfy the complementary slackness condition of constrained households, the borrowing limit is also shifted by the same amount δ.

Proposition 3 demonstrates that in the case of uniform taxation, we can also use counterfactual shocks to the borrowing constraint to proxy the liquidity channel. In this case, the term (8) is zero, and the effects of the liquidity channel are the economy's response to the (negative) of borrowing constraint shocks $-\Delta\boldsymbol{\phi} \equiv -\{B_t^{ra} - B^*\}_{t=0}^{\infty}$.

A common specification of fiscal policy in the quantitative HANK literature is to use government debt to offset the fiscal imbalance in the short run and use taxes to restore the debt in the long run. This fiscal rule implies that, after a decrease in interest rates, the government debt drops on impact and gradually returns to its steady-state level. When assessing the effects of the liquidity channel, $-\Delta\boldsymbol{\phi}$ is exactly the deleveraging shock in Guerrieri and Lorenzoni (2017). The binding borrowing constraint compels poor households to deleverage, even though they may benefit from other channels. The deleveraging shock lowers equilibrium real interest rates and dampens the consumption response.

The liquidity channel can be of interest independent of monetary policy shocks. Wolf (2021*a*) and Wolf (2021*b*) study the role of deficit-financed lump-sum fiscal transfer as a stimulating policy tool, which is essentially the liquidity channel defined here. Auclert, Rognlie and Straub (2018) and Hagedorn, Manovskii and Mitman (2019) discuss the fiscal multiplier under different financing policies. The deficit-financed fiscal multiplier is larger than tax-financed multiplier, which is due to the cyclical asset supply following the government spending shock.

Note that the uniform-taxation condition is only required away from the constant-debt path. The tax exposure channel is not necessarily muted. This is the case when

$$\bar{\tau}^{ra}(z^t) - \tau^*(z^t) \neq \bar{T}_t^{ra} - T^* \tag{9}$$

$$\tau^{ra}(z^t) - \bar{\tau}^{ra}(z^t) = T_t^{ra} - \bar{T}_t^{ra} \tag{10}$$

The tax exposure channel functions because households benefit differently from the tax reduction. The liquidity channel alters the timing of households' tax payments.

In the case of non-uniform taxation, however, counterfactual shocks to borrowing constraints are not enough to proxy the liquidity channel defined in (8). Consider the bond demand function $b^{ra}(z^t)$ given by

$$\bar{b}^{ra}(z^t) - b^{ra}(z^t) = (1 + r_t^{ra})(\bar{b}^{ra}(z^{t-1}) - b^{ra}(z^{t-1})) - (\bar{\tau}^{ra}(z^t) - \tau^{ra}(z^t)). \tag{11}$$

When government debt deviates from the constant-debt path $\bar{\tau}^{ra}(z^t) \neq \tau^{ra}(z^t)$, households absorb the change of tax payment through bond holdings $b^{ra}(z^t)$. If the bond demand function $b^{ra}(z^t)$ given by (11) satisfies the transversality condition, we can use path-dependent counterfactual borrowing constraint shocks to proxy the liquidity channel. The path-dependent borrowing constraints $\phi^{ra}(z^t)$ satisfy

$$b^{ra}(z^t) \geq \phi^{ra}(z^t), = \text{ if } u'(c^*(z^t)) > \beta(1 + r^*)E[u'(c^*(z^{t+1}))|z^t]. \tag{12}$$

However, and more generally, the transversality condition does not hold when the net present value of the change in tax payments is not zero for some households. Consider the case that tax payments are proportional to households' productivity. Then households who receive low productivity during tax increases and high productivity during tax decreases gain from the change in tax timing. Correspondingly, households who receive high productivity during tax increases and low productivity during tax decreases lose from the change in tax timing. In this case, the bond demand function given by (11) will explode for some paths $z^t \in Z^t$, and the transversality condition does not hold.

The effects of changing the timing of taxes, of course, depend on the taxation scheme. Households may gain or lose in real terms from the change in tax timing, depending on their histories of tax payments. In the current decomposition framework, I attribute all effects, including those 'real' redistributive effects, caused by the varying path of government debt to the liquidity channel.[5] In the quantitative analysis of section 6, I assume the government adjusts uniform taxation when evaluating the liquidity channel.

3.5 Households' Problem in Recursive Form

To compute the redistribution effects with the method of policy function iteration, I write the household's problem in recursive form. First, I impose the following bond demand function $b^{ra}(z^t)$ in the 'RANK' equilibrium,

$$b^{ra}(z^t) = g_t(b^*(z^t)) \equiv \phi + \frac{b^*(z^t) - \phi}{B^* - \phi}(B_t^{ra} - \phi). \tag{13}$$

When government debt is constant, $B_t^{ra} = B^*$ and $b^{ra}(z^t) = b^*(z^t)$. When government debt changes, the function $g_t(\cdot)$ shrinks or stretches the stationary-equilibrium bond demand function, keeping the lower bound of bond demand at the borrowing limit. The bond demand function $b^{ra}(z^t)$ satisfies the transversality condition and the bond market clearing condition if the stationary bond demand function satisfies those conditions.

To compute the model's response to the (negative of) transfers, I write the household's problem with transfers in recursive form. Let $c^*(z, b^{ss})$ and $b'^*(z, b^{ss})$ be the household's consumption and bond demand policy function in the stationary equilibrium, where b^{ss} is the household's wealth in the stationary equilibrium. Note that from the path of aggregates

[5] Consider the case of productivity-based taxing and temporary tax cut financed by the future tax increases. Households with lower productivity expect themselves to mean-revert to higher productivity and overall lose in real terms from the change in tax timing. High-productivity households overall benefit in real terms from the change in tax timing. This 'real' redistribution will dampen consumption responses. Putting this real redistribution and 'pure' liquidity effects together will underestimate the 'pure' liquidity effects.

and the household's states in the stationary equilibrium, transfers are fully pinned down:

$$\omega_t(z, b^{ss}) = \frac{C_t^{ra}}{C^*} c^*(z, b^{ss}) + g_t(b'^*(z, b^{ss})) - (1 + r_t^{ra}) g_{t-1}(b^{ss}) - W_t^{ra} z \frac{N_t^{ra}}{N^*} n^*(z, b^{ss}) - \pi_t^{ra}(z) + \tau_t^{ra}(z).$$

I use the household's wealth in the stationary equilibrium b^{ss} as an exogenous state variable to summarize an individual's history relevant to determining the transfers he receives. The household's problem with state-dependent transfers $\omega_t(z, b^{ss})$ in recursive form is:

$$\begin{aligned}
V_t^{ra}(z, b, b^{ss}) = & \max_{\{c,n,b'\}} u(c, n) + E[V_{t+1}^{ra}(z', b', b'^{ss}) | z, b^{ss}], \\
s.t. \quad c + b' = & (1 + r_t) b + W_t z n + \pi_t(z) - \tau_t(z) + \omega_t(z, b^{ss}), \\
b' \geq & \phi.
\end{aligned}$$

The law of motion for the exogenous state b^{ss} is the bond demand policy function in the stationary equilibrium $b'^{ss} = b'^*(z, b^{ss})$. Then along the equilibrium path, the household's policy function satisfies, for $b = g_{t-1}(b^{ss})$,

$$\begin{aligned}
b_t'^{ra}(z, b, b^{ss}) &= g_t(b'^*(z, b^{ss})), \\
c_t^{ra}(z, b, b^{ss}) / c^*(z, b^{ss}) &= C_t^{ra} / C^*, \\
n_t^{ra}(z, b, b^{ss}) / n^*(z, b^{ss}) &= N_t^{ra} / N^*.
\end{aligned}$$

4 Decomposing a HANK Model with Investment

In this section, I add investment to the model and discuss the decomposition. I assume that the capital (equity) is liquid and a perfect substitute for bonds. In Appendix E, I consider a model with illiquid assets and discuss the implication of redistribution effects for liquidity premium.

4.1 Model Description

Households. Households can also trade in firm shares $v(z^t)$ with price p_t, which provides a dividend stream D_t each period. The household's budget constraint is

$$c(z^t) + b(z^t) + p_t v(z^t) = (1 + r_t) b(z^{t-1}) + (p_t + D_t) v(z^{t-1}) + z_t W_t n(z^t) + \pi(z^t) - \tau(z^t).$$

Households are subject to the non-borrowing constraints

$$b(z^t) + p_t v(z^t) \geq 0.$$

Non-arbitrage condition requires that $r_t = (p_t + D_t)/p_{t-1}$ from $t = 1$. Define total wealth $a(z^t) \equiv b(z^t) + p_t v(z^t)$, then from $t = 1$ the constraints faced by households can be written as

$$
\begin{aligned}
c(z^t) + a(z^t) &= (1 + r_t)a(z^{t-1}) + z_t W_t n(z^t) + \pi(z^t) - \tau(z^t), \\
a(z^t) &\geq 0.
\end{aligned}
$$

At $t = 0$, the return on bonds and equity can be different. The return on bonds is subject to unexpected inflation, and the return on equity is subject to unexpected capital gains:

$$
\begin{aligned}
c(z_0) + a(z_0) &= (1 + r_0)b_{-1} + (p_0 + D_0)v_{-1} + z_0 W_0 n(z_0) + \pi(z_0) - \tau(z_0), \\
a(z_0) &\geq 0.
\end{aligned}
$$

Firms. The intermediate goods firms have a Cobb Douglas production function $y_{j,t} = Ak_{j,t-1}^{\alpha} n_{j,t}^{1-\alpha}$. The Philips Curve is similar to the last section,

$$
\log(1 + \pi_t^P) = \kappa \left(mc_t - \frac{1}{\mu} \right) + \frac{1}{1 + r_{t+1}} \frac{Y_{t+1}}{Y_t} \log(1 + \pi_{t+1}^P).
$$

with marginal cost $mc_t = W_t N_t / (1 - \alpha) / Y_t$.

Firms own capital K_{t-1} and choose investment I_t to obtain the capital of the next period $K_t = (1 - \delta)K_{t-1} + I_t$, subject to quadratic capital adjustment cost. Dividends equal capital products plus post-tax monopolistic profits net of investment, capital adjustment cost, and price adjustment cost,

$$
D_t = r_t^K K_{t-1} + \alpha \Pi_t - I_t - \frac{\Psi}{2} \left(\frac{I_t}{K_{t-1}} - \delta^K \right)^2 - \Theta_t.
$$

Firms choose investment to maximize $p_t + D_t$. Tobin's Q and capital evolve according to the standard Q-theory of investment:

$$
\begin{aligned}
\frac{I_t}{K_{t-1}} - \delta^K &= \frac{1}{\Psi}(Q_t - 1), \\
(1 + r_{t+1})Q_t &= r_{t+1}^K - \frac{I_{t+1}}{K_t} - \frac{\Psi}{2} \left(\frac{I_{t+1}}{K_t} - \delta^K \right)^2 + \frac{K_{t+1}}{K_t} Q_{t+1}.
\end{aligned}
$$

I assume the monopolistic profits Π_t are taxed, so firms only receive an α fraction of the monopolistic profits. The remaining $1 - \alpha$ fraction is paid to households as a lump-sum transfer in proportion to household productivity. This profit distribution scheme fully neutralizes the impact of countercyclical markups and generates reasonable asset price responses.

Equilibrium. In the equilibrium, households and firms optimize, government budget constraint holds, nominal interest rates evolve according to the Taylor rule, and markets clear:

$$
\begin{aligned}
\int a(z^t) d\Phi_t(z^t) &= B_t + p_t, \\
C_t + I_t + \frac{\Psi}{2} \left(\frac{I_t}{K_{t-1}} - \delta \right)^2 + \Theta_t &= Y_t^{GDP}.
\end{aligned}
$$

4.2 Redistribution Channels with Investment

The appendix shows that the redistribution shock can be decomposed as follows

$$
\begin{aligned}
-\omega(z^t) = {} & \underbrace{(\hat{y}^{ra}(z^t) - \hat{Y}_t^{ra})y^*(z^t)}_{\text{income exposure}} + \underbrace{(b^*(z^{t-1}) - B^*)(r_t^{ra} - r^*)}_{\text{interest rate exposure}} \\
& + \underbrace{(\bar{T}_t^{ra} - T^*) - (\bar{\tau}^{ra}(z^t) - \tau^*(z^t))}_{\text{tax exposure}} + \underbrace{(p_t^{ra} - p^*)(v^*(z^{t-1}) - v^*(z^t))}_{\text{asset price}} \\
& + \underbrace{(\bar{b}^{ra}(z^t) - b^{ra}(z^t)) - (1 + r_t^{ra})(\bar{b}^{ra}(z^{t-1}) - b^{ra}(z^{t-1})) + (\bar{\tau}^{ra}(z^t) - \tau^{ra}(z^t))}_{\text{liquidity}} \\
& + \underbrace{\hat{C}_t^{ra}(y^*(z^t) - c^*(z^t))}_{\text{scaling of net saving}} + \underbrace{(b^*(z^t) - \bar{b}^{ra}(z^t)) - (1 + r_t^{ra})(b^*(z^{t-1}) - \bar{b}^{ra}(z^{t-1}))}_{\text{undetermined bond demand}} \\
& + \underbrace{p_t^{ra}(v^*(z^t) - v^{ra}(z^t)) - p_t^{ra}(v^*(z^{t-1}) - v^{ra}(z^{t-1}))}_{\text{undetermined equity demand}}
\end{aligned}
$$

where I define $y \equiv zWn + \pi + Dv_-$ as the household's income, including dividend income Dv_- and labor income $y^L \equiv zWn + \pi$.[6] On the aggregate level, aggregate income Y equals aggregate consumption C. The last residual term is due to the undetermined equity demand. After imposing $v^*(z^t) = v^{ra}(z^t)$, this term is zero. There is a new asset price channel, which is defined as

$$
(p_t^{ra} - p^*)(v^*(z^{t-1}) - v^*(z^t)). \tag{14}
$$

The change in asset prices affects traders rather than holders, also consistent with the argument made in Fagereng et al. (2022).

The appendix shows that the term of income exposure channel has two parts:

$$
(\hat{y}^{ra}(z^t) - \hat{Y}_t^{ra})y^*(z^t) = \underbrace{(\hat{y}^{L,ra}(z^t) - \hat{Y}_t^{L,ra})y^{L,*}(z^t)}_{\text{'within' income exposure}} + \underbrace{(\hat{D}_t^{ra} - \hat{C}_t^{ra})D^*\left(v^*(z^{t-1}) - \frac{y^{L,*}(z^t)}{Y^{L,*}}\right)}_{\text{'between' income exposure}}.
$$

The first part, 'within' income exposure

$$
(\hat{y}^{L,ra}(z^t) - \hat{Y}_t^{L,ra})y^{L,*}(z^t),
$$

is the same as the last section which captures the redistribution between households that have different labor income elasticities to aggregate labor income. The second part, 'between' income exposure

$$
(\hat{D}_t^{ra} - \hat{C}_t^{ra})D^*\left(v^*(z^{t-1}) - \frac{y^{L,*}(z^t)}{Y^{L,*}}\right),
$$

[6] Income from labor in the broad sense, including profit income interpreted as the bonus.

captures the redistribution between dividend income receivers (households with $v^*(z^{t-1}) > y^{L,*}(z^t)/Y^{L,*}$) and labor income receivers ($v^*(z^{t-1}) < y^{L,*}(z^t)/Y^{L,*}$). The income elasticities of dividend income receivers and labor income receivers are affected in opposite directions when the share of dividends in aggregate income fluctuates ($\hat{D}_t^{ra} \neq \hat{C}_t^{ra}$).

5 Estimable Moments for Partial Equilibrium Responses

The decomposition implies that if the policymaker lowers the nominal interest rate and she knows the representative-agent model's response, then she only needs to know the heterogeneous-agent model's response to the redistribution shock $-\boldsymbol{\omega}$ to get the full responses. The responses to the redistribution shock $-\boldsymbol{\omega}$ generally require solving a full HANK model numerically. However, we can gain insights from partial equilibrium analysis and derive estimable moments for partial equilibrium responses, as in Auclert (2019) and Patterson (2023).

The redistribution shock is persistent if the monetary policy shock is persistent or if the model features investment. To simplify the analysis, I truncate the redistribution shock from time $t = 1$ and only consider the redistribution at time $t = 0$.[7] To the first order, the aggregate consumption response in partial equilibrium is

$$\partial C_0 = \int MPC_{i0} \cdot (-\omega_{i0}) di = cov_I(MPC_{i0}, -\omega_{i0}).$$

The equation follows from the re-distributive nature of the transfers: $\int -\omega(z^t) d\Phi_t(z^t) = 0$. The consumption response in partial equilibrium is the cross-sectional covariance between households' MPCs and the (negative of the) transfers they receive. In the case of amplification, $cov_I(MPC_{i0}, -\omega_{i0}) > 0$; in the case of dampening, $cov_I(MPC_{i0}, -\omega_{i0}) < 0$. Since each redistribution channel sums to zero cross-sectionally, the above argument also applies to the evaluation of each channel.

Before I derive estimable moments at the redistribution-channel level, I first specify the functional form of household income and tax payment. I also specify the aggregate labor supply condition and fiscal policy to close the model for general equilibrium analysis in the next section.

[7] In Appendix D, I consider a model without investment and its response to a transitory shock in which the redistribution only happens at time $t = 0$. For a persistent shock, the consumption response at time $t = 0$ equals the sum of its responses to the redistribution at each period to the first order.

5.1 The Full Model

Household Income. I assume that households supply the same amount of labor and that the distribution of the profits is proportional to productivity. I also introduce the 'incidence function', following Guvenen et al. (2017), Werning (2015), Auclert and Rognlie (2018), Alves et al. (2020), e.t.c., to capture househols' different labor income elasticities to aggregate labor income fluctuations. The specific function form is the same as Alves et al. (2020). Household labor income is given by

$$y^L(z^t) = \frac{z_t(Y_t^L/Y^{L,*})^{\gamma(z_t)}}{E_I[z_t(Y_t^L/Y^{L,*})^{\gamma(z_t)}]} Y_t^L.$$

In the stationary equilibrium, the labor income is simply

$$y^{L,*}(z^t) = z_t Y^{L,*} = z_t(W^* N^* + (1-\alpha)\Pi^*).$$

Off the stationary equilibrium, imposing the normalization $E_I[z_t\gamma(z_t)] = 1$, then $\gamma(z_t)$ is the elasticity of the type z_t income to aggregate income Y_t^L evaluated at $Y^{L,*}$. In models without the incidence function, the 'within' income exposure channel is muted because $y^L(z^t) = z_t Y_t^L$ and $\hat{y}^L(z^t) = \hat{Y}_t^L$. Conditional on productivity level z_t, the labor income share is constant.

Given the above specification, the 'within' income exposure channel is

$$(\hat{y}^{L,ra}(z^t) - \hat{Y}_t^{L,ra}) y^{L,*}(z^t) = (\gamma(z_t) - 1)\hat{Y}_t^{L,ra} y^{L,*}.$$

If $\gamma(z_t) > 1$, then type z_t household's labor income is more elastic to aggregate labor income and the term of the 'within' income exposure for type z_t household is positive. The elasticity to aggregate labor income fluctuations $\gamma(z_t)$ is the target for calibration.

The 'between' income exposure channel is simply

$$(\hat{D}_t^{ra} - \hat{C}_t^{ra}) D^*(v^*(z^{t-1}) - z_t),$$

which is endogenously determined. In models without investment, the 'between' income exposure channel is muted.

Tax payment. The amount of taxes households pay to the government is

$$\tau(z^t) = \Gamma^* y^L(z^t) + T_t^{uniform},$$

where Γ^* is a common constant tax rate on labor income, and $T_t^{uniform}$ is a uniform tax (government transfer). Aggregate tax is $T_t = \Gamma^* Y_t^L + T_t^{uniform}$. In the case of constant public debt, the government adjusts the uniform tax $\bar{T}_t^{uniform}$ to balance its budget. Given the above specification, the tax exposure channel is

$$\Gamma^*(Y_t^{L,ra} - Y^{L,*})(1 - \gamma(z_t) z_t).$$

Labor Supply. Since I assume households supply the same amount of labor, I need to specify the aggregate labor supply condition. The modeling of the labor market is non-standard, borrowed from Alves et al. (2020) to simplify the labor-supply analysis. Households supply the same amount of labor $n(z^t) = N_t$ to firms, and the aggregate labor supply follows the wage schedule,

$$W_t = W^* \left(\frac{N_t}{N^*}\right)^{\epsilon_w}.$$

If $\epsilon_w = 0$, wages are perfectly rigid, and employment is determined by only labor demand. If $\epsilon_w > 0$, there is pressure on wages whenever employment is different from its steady-state level.

Fiscal Policy. The government budget constraint is

$$B_t + T_t = (1 + r_t)B_{t-1} + G^*,$$

where G^* is the constant government spending. The aggregate tax income for the government is $T_t = \Gamma^* Y_t^L + T_t^{uniform}$. I assume a non-standard fiscal policy to capture the increasing asset supply and relaxed borrowing conditions following an expansionary shock. The uniform taxes $T_t^{uniform}$ are chosen such that the path of government debt satisfies:

$$B_t - B^* = \rho_B(B_{t-1} - B^*) + \epsilon_t^B.$$

Following the monetary policy shock ϵ_t, there is also a shock to the level of government debt $\epsilon_t^B = \phi^B \epsilon_t$. When $\phi^B < 0$, the asset supply is procyclical (conditional on the monetary policy shock), and when $\phi^B > 0$, the asset supply is countercyclical.

5.2 Estimable Moments for Consumption Responses

Given the above specification, the partial equilibrium consumption responses to the redistribution shock $-\omega$ at the redistribution-channel level are summarized in Table 1. I omit the time script and instead denote b_i as individual i's initial bond holding at time $t = 0$ and b_i' as his bond holding at the beginning of the next period $t = 1$ in the steady state. Similarly, v_i is the initial equity and v_i' is the equity held at the beginning of the next period $t = 1$.

The covariance terms in Table 1 can be estimated as in Auclert (2019) and Patterson (2023). The main difference compared to the previous literature is the changes in aggregate quantities or prices in the equations. For example, the response of consumption to the interest rate exposure channel shock is the covariance $cov_I\left(MPC_i, b_i - B\right)$ times the change in real interest rate in the 'RANK' equilibrium, $\tilde{r}^{ra}$, rather than the change in interest rate in the HANK

Table 1: Consumption responses to the redistribution shock in partial equilibrium

Redistribution channel	**Consumption response**
Interest rate exposure	$\tilde{r}^{ra} \cdot cov_I\left(MPC_i, b_i - B\right)$
'Between' income exposure	$(\hat{D}^{ra} - \hat{C}^{ra})D^* \cdot cov_I\left(MPC_i, v_i - z_i\right)$
'Within' income exposure	$\tilde{Y}^{L,ra} \cdot cov_I\left(MPC_i, (\gamma(z_i) - 1)z_i\right)$
Tax exposure	$\Gamma\tilde{Y}^{L,ra} \cdot cov_I\left(MPC_i, 1 - \gamma(z_i)z_i\right)$
Liquidity	$\tilde{B}^{ra}/(B - \phi) \cdot cov_I\left(MPC_i, B - b_i'\right)$
Asset price	$\tilde{p}^{ra} \cdot cov_I\left(MPC_i, v_i - v_i'\right)$

Notes: Partial-equilibrium consumption response to a transitory redistribution shock. MPC_i is the marginal propensity of consumption of individual i. b_i, v_i, z_i, $\gamma(z_i)$ denote individual i's bond position, dividend income share, labor income share, and labor income elasticities, respectively.

economy used in Auclert (2019), which can only be solved or observed ex-post. To predict the effects of aggregate shocks, policymakers only need to know the responses of aggregates in the representative-agent equilibrium and the above moments.

Below I discuss briefly the standard model's prediction about these moments as well as the effects of these redistribution channels following an expansionary monetary policy shock.

Interest rate exposure. The incomplete-market model predicts that

$$cov_I\left(MPC_i, b_i - B\right) < 0.$$

Creditors ($b_i > B$) have lower MPCs than debtors ($b_i < B$), which implies a negative correlation between MPC and the exposure to interest-rate change. Following an interest rate cut $\tilde{r}^{ra} < 0$, the interest exposure channel amplifies consumption responses. The interest rate cut taxes creditors and subsidizes debtors.

'Between' income exposure. Since on average rich households receive relatively more dividend income ($v_i > z_i$) and poor households receive relatively more labor income ($v_i < z_i$), we have

$$cov_I\left(MPC_i, v_i - z_i\right) < 0.$$

If dividends are less responsive than consumption $\hat{D}^{ra} < \hat{C}^{ra}$, dividend income receivers lose, and labor income receivers gain. If dividends are more responsive than consumption, the redistribution goes the other direction.

With the presence of investment, dividend responses and investment responses are negatively correlated. From the expression of the dividend,

$$D_t = r_t^K K_{t-1} + \alpha \Pi_t - I_t - \frac{\Psi}{2}\left(\frac{I_t}{K_{t-1}} - \delta\right)^2 - \Theta_t.$$

Notice $r_t^K K_{t-1} + \alpha \Pi_t = \alpha Y_t$. Omitting the capital adjustment cost and price adjustment cost, we have $D_t = \alpha Y_t^{GDP} - I_t = \alpha C_t - (1-\alpha) I_t$. Dividends are less responsive than consumption $\hat{D}_t^{ra} < \hat{C}_t^{ra}$ if and only if the investment is more responsive than consumption $\hat{I}_t^{ra} > \hat{C}_t^{ra}$. For typical calibrations, investment is more responsive than consumption in the short run and less responsive than consumption in the long run, which implies $\hat{D}_t^{ra} < \hat{C}_t^{ra}$ in the short run and $\hat{D}_t^{ra} > \hat{C}_t^{ra}$ in the long run. The redistribution induced by the investment response amplifies the consumption response.[8]

'Within' income exposure. Empirical evidence is mixed. Patterson et al. (2019) documents a positive covariance between workers' MPCs and their elasticities of earnings to GDP in the US: $cov_I\left(MPC_i, \gamma(z_i) - 1\right) > 0$. Broer, Kramer and Mitman (2020) uses German data and finds workers at the bottom of the income distribution are more exposed to aggregate earnings risk in general and monetary policy shocks specifically. Amberg et al. (2022) documents a similar pattern in Swedish administrative individual data: there is a higher sensitivity of labor incomes to monetary shocks at the bottom than elsewhere in the income distribution. In contrast, Coibion et al. (2017) and Andersen et al. (2022) find that monetary policy has little or no effect on inequality in earnings (for the US and Denmark, respectively). Guvenen et al. (2017) estimate 'workers' betas' (i.e. systematic risk exposure) with respect to GDP using data from the US Social Security Administration's Master Earnings file and find a U-shaped elasticity, that is, exposure is high both at the bottom of the distribution and the top ($\gamma(z_i) > 1$ for both low and high z_i). As will be shown in the next section, using estimates from Guvenen et al. (2017) implies the net effects are positive:

$$cov_I\left(MPC_i, (\gamma(z_i) - 1) z_i\right) > 0.$$

[8] When unconstrained households accumulate capital for future consumption, constrained households consume additional income (from producing capital) in the current period. In the future, constrained households will have to cut their consumption when the economy de-invests and consumes the accumulated capital. Essentially, the redistribution allows constrained households to move their future consumption to today, which has a similar flavor to the liquidity channel discussed in the last section. From this perspective, the 'between' income exposure channel can also be interpreted as the liquidity channel of productive assets.

Tax exposure. Low-labor-income housheolds ($\gamma(z_i)z_i < 1$) have higher MPCs than high-labor-income housheolds ($\gamma(z_i)z_i > 1$) so

$$cov_I\left(MPC_i, 1 - \gamma(z_i)z_i\right) > 0.$$

Low-labor-income households benefit from the tax reduction overall. On the one hand, when the aggregate tax on labor income $\Gamma^* Y_t^{L,ra}$ increases, low-income households' payment increases less due to their smaller share of the tax burden. On the other hand, everyone benefits equally when the uniform tax adjusts to balance the government budget.

Liquidity. Similar to the interest-rate exposure channel

$$cov_I\left(MPC_i, B - b_i'\right) > 0.$$

When the government shifts the timing of taxes by deficit financing, it transfers income across time. For constrained households, the change in income will affect their consumption one-to-one. For unconstrained households, the change in income will be used to absorb the changes in public debt in general equilibrium. The change in the 'effective' income that can be used for consumption is smaller than that of constrained households. For the bond demand function imposed in (13), the 'effective' income change is proportional to the household's distance being constrained so we get the covariance term above. In the calibration of fiscal policy, $\phi^B < 0$ so the asset supply is procyclical $\tilde{B}^{ra} > 0$. The liquidity channel relaxes borrowing conditions of constrained households and amplifies consumption responses.

Asset price. Theoretically, sellers are households that experience a negative income shock, and buyers are those who experience a positive income shock. On average, sellers should have a higher MPC than buyers so

$$cov_I\left(MPC_i, v_i - v_i'\right) > 0.$$

Following an expansionary shock, asset prices increase $\tilde{p}^{ra} > 0$. Asset buyers are hurt and asset sellers benefit. The asset price channel results in amplification.

6 Quantitative Analysis

In this section, I implement the decomposition quantitatively. I first calibrate the model and then consider the model's response to a one-time unexpected monetary policy shock. At time $t = 0$, there is an innovation to the Taylor rule of $\epsilon_0 = -0.25$ percent (-1 percent annually) with quarterly persistence of 0.61. I use the Sequence-Space approach developed in Auclert et al. (2021) and Boppart, Krusell and Mitman (2018) to solve the model. To implement the decomposition, I first solve the model's stationary equilibrium without transfers and build the law of motion of the exogenous state (z, b^{ss}) from the household's bond demand policy function. Then I input the redistribution shock into the model.

6.1 Calibration

Table 3 summarizes the parameter values and the calibration targets. I calibrate the model to the 2004 US economy, as in Kaplan, Moll and Violante (2018). The annual real interest rate is set to 5% in the stationary equilibrium, corresponding to the average real return on equity and government bonds. The coefficient of risk aversion σ is set to 1. The value of aggregate wealth to annual output is $(B+p)/Y = 3.21$, which is the sum of government debt to annual output $B/Y = 0.29$ and equity to annual output $p/Y = 2.92$. Following the categorization of Kaplan, Moll and Violante (2018), the value of government debt to annual output B/Y is the **gross** liquid assets from Survey of Consumer Finances (SCF) divided by annual GDP, and the value of equity to annual output p/Y is the **net** illiquid assets from Flow of Funds (FoF) divided by annual GDP.[9]

The capital share parameter in production function α is set to 0.33. The depreciation rate of capital is $\delta^K = 0.07$. The capital stock in steady state satisfies $rp = \alpha Y - \delta^K K$, which gives $K/Y = 2.63$. The capitalized markup over annual output is then $p/Y - K/Y = 0.29$. The steady-state markup $1 - 1/\mu$ satisfies $\alpha(1 - 1/\mu)/r = 0.29$, giving $\mu = 1.05$. The capital share parameter and markup together imply a capital share of 31% and a labor share of 64%. The slope of the Phillips Curve is $\kappa = 0.1$, and the Taylor rule coefficient ϕ is set to 1.25, both are standard values in New Keynesian literature. The proportional labor income (and profit income) tax rate is set to $\Gamma^* = 0.3$ and the value of uniform tax to output is $T^{uniform}/Y = -0.06$. The government spending is then determined from government budget constraint $G^*/Y = 0.13$.

Income process. The income process is a quarterly discretization of the process estimated in Kaplan, Moll and Violante (2018), which captures the higher-order movements of the distribution of earnings changes documented in Guvenen, Ozkan and Song (2014). This income process is a sum of two independent components and each component is close to a typical AR(1) process.

[9] I normalize the borrowing constraint to zero. So I use the gross liquid assets as the correct measure of asset supply, which equals to the net liquid assets $B^{net} = 0.26$ plus consumer loans $B^{loan} = 0.03$.

Wealth distribution. It is well-known now that one-asset HANK models have difficulty matching aggregate MPC and aggregate wealth at the same time. Models calibrated to be consistent with measures of aggregate wealth yield too small measures of constrained households, and only generate quarterly MPCs between 3% and 5% (Kaplan and Violante (2022)); and models calibrated to have reasonable MPCs abstract from most wealth in the economy. To match the US wealth distribution and aggregate net wealth, I take the approach in Carroll et al. (2017) and introduce ex-ante heterogeneity in the discount factor. Auclert, Rognlie and Straub (2020) takes a similar calibration strategy. The discount factor and measure of each group are listed below:

Household group	1	2	3	4	5	6
Population share	First 18%	Next 18%	Next 18%	Next 18%	Next 18%	Top 10%
Discount factors (p.a.)	0.912	0.92	0.928	0.936	0.944	0.95

The appendix compares the wealth distribution generated by the model and its empirical counterpart. The model replicates the distribution fairly well. The income process does not include a 'superstar' state thus the model has difficulty in matching the very top of the distribution. For the current quantitative evaluation, this should not be a serious problem as the consumption function is approximately linear at high levels of wealth. At higher levels of wealth, the homogeneous MPCs do not correlate with exposures, implying that the redistribution among high-wealth households contributes little to the responses of aggregates (although it affects individual outcomes).

Asset portfolio. The household portfolio is undetermined. The portfolio will affect both the aggregate dynamics of HANK and the decomposition. First, there is unexpected inflation and unexpected capital gains at time-0, resulting in the different ex-post returns of bonds and equity at time-0. The time-0 revaluation affects households unequally depending on their portfolio. Second, the asset portfolio will affect the decomposition if the bond holding $b(z^t)$ or shareholding $v(z^t)$ enters the expression of one channel. In our case, the 'between' income exposure, interest rate exposure, and asset price channels will be affected by the asset portfolio. I calibrate the portfolio using data from the 2004 Survey of Consumer Finances (SCF) given the categorization of liquid and illiquid assets of Kaplan, Moll and Violante (2018). Auclert and Rognlie (2018) and McKay and Wolf (2022) take a similar calibration strategy.

Income incidence function. Guvenen et al. (2017)'s estimates have the advantage that they capture the elasticity of those at the very top of the income distribution. I incorporate Guvenen et al. (2017)'s estimates and normalize to $E_I[z_t\gamma(z_t)] = 1$.[10] Auclert and Rognlie (2018) takes the same calibration strategy. Since I assume aggregate labor income and profit income (aggregate earnings in my model) is a constant share of output. The elasticity to GDP is equal to the elasticity to aggregate earnings.

Fiscal policy. To capture the expansionary shock's effects on credit expansion, I estimate the effects of monetary policy on household loans B^{loan} in liquid assets account.[11] If the household loans increase by $\hat{B}^{loan} = 1\%$ as estimated, then the equivalent increase in the public debt is $\hat{B}^{loan} \cdot B^{loan}/B = 1\% \cdot 0.03/0.29 = 0.1\%$. I calibrate the parameter ϕ^B such that the impact response of public debt is 0.1%. I show the estimated effects of monetary policy

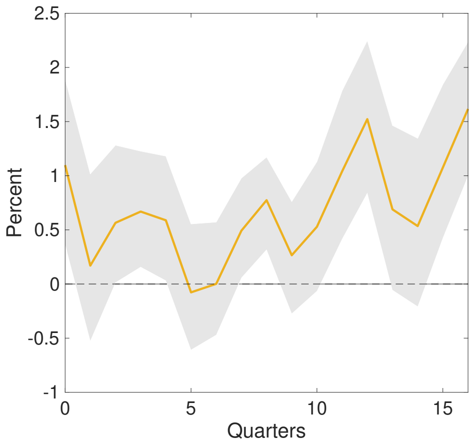

Figure 1: Real loan response to a monetary policy shock

Notes: Estimated response of real loan in liquid assets to a monetary policy shock. Monetary policy shock is normalized such that the impact decrease of the return on the 3-month treasury bill return is 25 basis points. I use quarterly data from 1988Q4 to 2016Q2 (the data on monetary policy shocks is from 1988Q4 to 2012Q2). The lagged controls are set as $\mathbf{X}_{t-1} = [i_{t-1}, \epsilon_{t-1}, U_{t-1}, Y_{t-1}, C_{t-1}, I_{t-1}, A_{t-1}, P_{t-1}]$. The shadow area represents the bootstrapped 66% confidence bounds.

shock on household loans in Figure 1. More details about the estimation are in the Appendix. The nominal loan is estimated from Flow of Funds (FoF) data as the sum of consumer credit, depository institution loans as well as other loans and advances in liability minus the loans as assets and total miscellaneous assets, then I deflate it by CPI and take the log of real loan. I estimate the responses by local projections with high-frequency identified monetary policy

[10] The estimates can be found in Table A1 of the NBER working paper version (NBER Working Paper 23163). I use the estimates for the group of males at age 36–45.

[11] The direct way to calibrate the liquidity channel is to keep the public debt constant and shock the borrowing constraint as estimated. I take a symmetric approach: I keep the borrowing constraint unchanged (normalized to zero) and shock the public debt such that the borrowing capacity $B - \phi$ follows the same path as shocking the borrowing constraint.

shocks in Gorodnichenko and Weber (2016):

$$Y_{t+h} = \beta_{h,0} + \beta_{h,1}t + \beta_{h,2}\epsilon_t + \beta_{h,3}\mathbf{X}_{t-1} + \nu_{t+h}, \quad h = 0, ..., 16$$

The aggregate real loan Y_t at the forecast horizon $h = 0, ..., 16$ is regressed on the current normalized monetary shock ϵ_t, a constant, a liner time trend, and lagged controls $\mathbf{X}_{t-1}$. To control the potential endogeneity in practice, the lagged controls are set as the federal funds rate i_{t-1}, the monetary shock ϵ_{t-1}, unemployment rate U_{t-1}, log of output Y_{t-1}, consumption C_{t-1}, investment I_{t-1}, TFP A_{t-1} and consumer price index P_{t-1}. I use quarterly data from 1988Q4 to 2016Q2. The data on monetary policy shocks is from 1988Q4 to 2012Q2.

6.2 RANK Effects

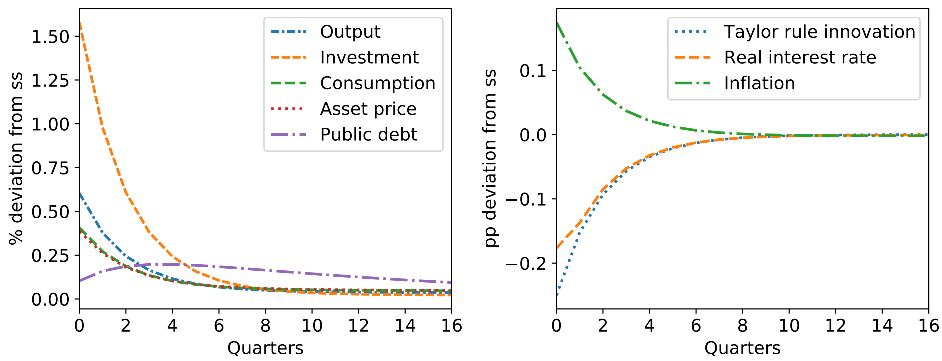

Figure 2: RANK effects

Notes: Impulse responses of the fictitious RANK model to a monetary policy shock, $\epsilon_0 = 25$ basis points.

Figure 2 shows the responses of the fictitious representative agent model. In response to an expansionary monetary policy shock, the real interest rates decrease, stimulating consumption and investment. Given the sticky price, the rising aggregate demand leads to an increase in output and inflation. The investment is more responsive than consumption in the short run and less responsive than consumption in the long run, which implies a redistribution from dividend income receivers to labor income receivers in the short run and the reverse in the long run. The rising asset price implies a redistribution from asset buyers to asset sellers. The public debt increases and more liquidity is injected into the economy after the expansionary shock.

6.3 Decomposition of Aggregates

Figure 3 shows the decomposition of output, consumption, investment, and real interest rates. The solid blue line is the response of HANK; the yellow dashed line is the response of the fictitious representative agent model, which is the RANK effects; and the green dotted line is the response of HANK to the redistribution shock $-\omega$, which is the redistribution effects. If we use RANK effects as the benchmark, then the redistribution effects amplify the responses of output and consumption and dampen the response of investment and real interest rates. On impact, redistribution effects account for 28% of consumption response and 6% of output response.

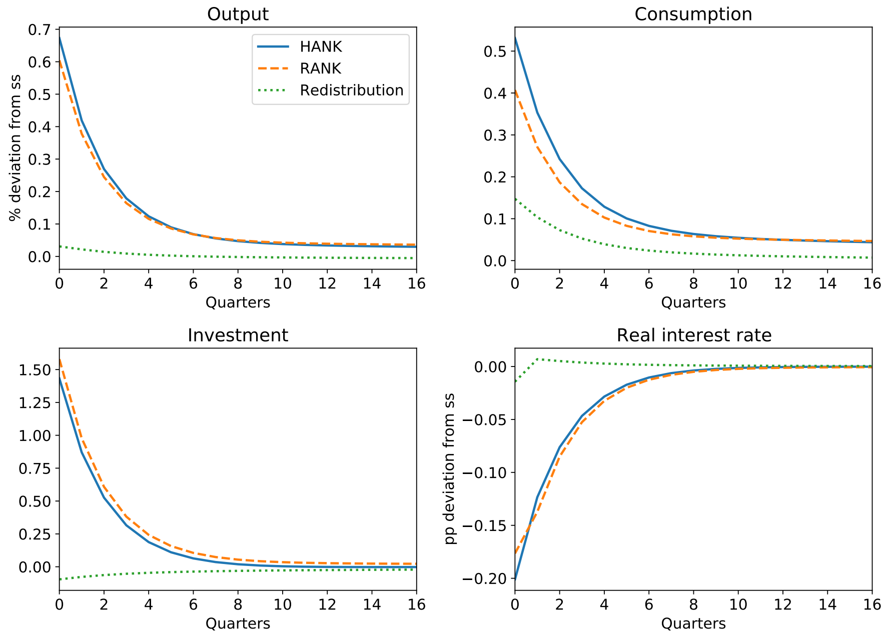

Figure 3: Decomposition of the HANK model's responses to a monetary policy shock

Notes: Decomposition of the responses of output, consumption, investment, and real interest rate to a monetary policy shock, $\epsilon_0 = 25$ basis points. The RANK effects are these variables' responses to the monetary policy shock in the fictitious RANK model, and the redistribution effects are these variables' responses to the triggered redistribution shock in the HANK model.

Figure 4 shows the channel-level decomposition for consumption. To evaluate different redistribution channels' effects on consumption, I input those redistribution channels separately into the model.

In terms of magnitude, the interest exposure channel is the largest amplifier of consumption response, accounting for around one-third of the impact amplification (0.05/0.15). A lower interest rate benefits debtors at the expense of creditors, and debtors have a higher MPC than creditors, which amplifies the consumption response.

From Figure 2 we can see that investment is more responsive than consumption before quarter 8. Due to the investment response, labor income receivers experience a higher income increase than dividend income receivers in the short run, which amplifies the consumption response through the 'between' income exposure channel. The estimates from

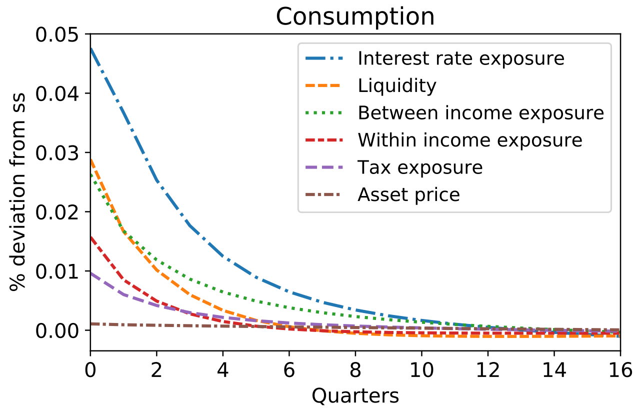

Figure 4: Decompostion of the redistribution effects (consumption)

Notes: The redistribution shock's effects on consumption are decomposed into six channels. The redistribution shock is triggered by a monetary policy shock of 25 basis points. Section 5 gives the definitions of these redistribution channels.

Guvenen et al. (2017) imply that both low-labor-income households and high-labor-income households are more exposed to business cycle fluctuations. The net effects of the 'within' income exposure channel on consumption are positive.

The government increases the asset supply through fiscal policy response, households can self-insure better, and aggregate spending increases. The liquidity channel is an amplifier rather than a dampener, in contrast to the commonly assumed stabilizing fiscal policy. When aggregate taxes on labor income increase, low-income workers benefit more from the increase as they pay a smaller share of the aggregate tax; and when the uniform tax decreases to balance the government budget, all households benefit equally. Overall the tax exposure channel benefits low-labor-income workers and amplifies consumption responses.

When asset prices increase, asset sellers gain, and asset buyers lose. In theory, sellers are households that experience a negative income shock, and buyers are those who experience a positive income shock. On average, sellers should have a higher MPC than buyers, so this channel results in amplification. But quantitively, the effects are negligible. There is a long-run view that the unexpected capital gains have large redistribution implications because wealthy households benefit much more from the rising asset prices than median and poor households. This paper contradicts this view. The reason is that asset price is an equilibrium object and it adjusts such that households are willing to hold the amount of equity that clears the market rather than consume out of the capital gains. Rising asset prices benefit sellers rather than holders.

Table 2 summarizes the direction of redistribution following the aggregate shock and the difference in MPCs between winners and losers from the redistribution.

Figure 5 further illustrates the redistribution effects at the channel level for other vari-

	Interest rate exposure	'Between' income exposure	Liquidity	Asset price	Tax exposure
Lower MPC	Creditors	Dividend income receivers	Unconstrained	Asset buyers	High labor income
	↓	S.R. ↓↑ L.R.	S.R. ↓↑ L.R.	↓	↓
Higher MPC	Debtors	Labor income receivers	Constrained	Asset sellers	Low labor income

Table 2: The direction of redistribution following an expansionary monetary policy shock

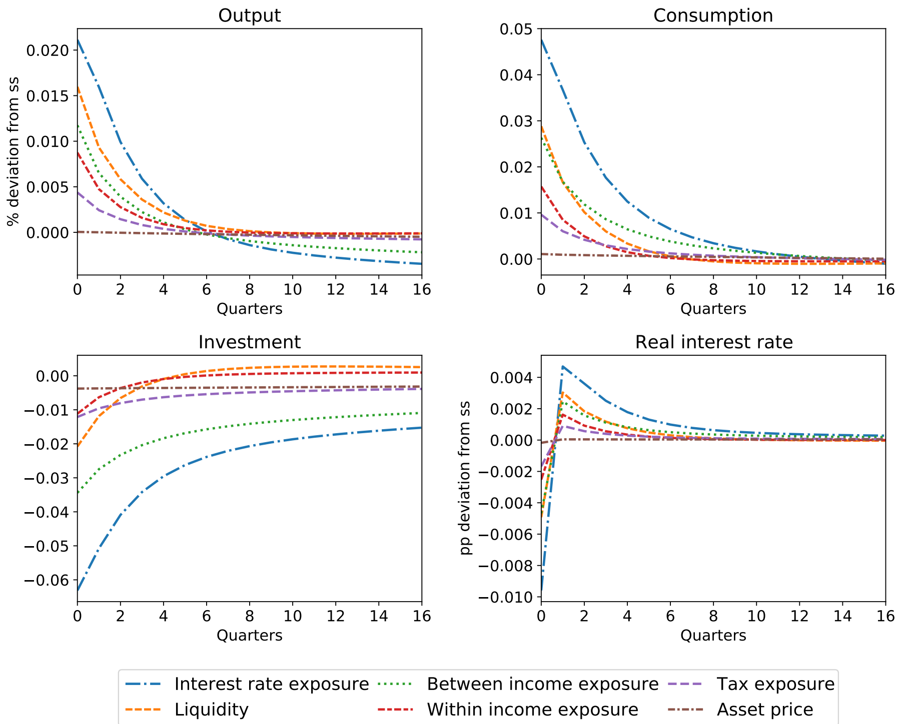

Figure 5: Decomposition of the redistribution effects

Notes: The redistribution shock's effects on output, consumption, investment, and real interest rates are decomposed into six channels. The redistribution shock is triggered by a monetary policy shock of 25 basis points. Section 5 gives the definitions of these redistribution channels.

ables. Qualitatively, the decomposition for output is similar to that for consumption, but in a smaller magnitude. This is because if one channel amplifies the consumption response, it will dampen the investment response: households who lose from the redistribution consume the capital stock. So, in the long run, the amplification of consumption and output response can be reverted because the capital stock decreases as households consume more and accumulate less. This is clear from the interest exposure channel's effects on output. From quarter 6, the output response is negative because capital stock decreases, and the economy produces less.

6.4 Individual-level Decomposition

Figure 6 shows the individual-level decomposition of consumption responses (on impact). I omit the asset price channel since the effects are close to zero. I show the effects of redistribution channels along the most relevant dimension. For example, the effects of the interest exposure channel are plotted across wealth distribution because the bond holding decides the household's interest-rate exposure. Similarly, the effects of the 'between' income exposure channel are plotted across the distribution of dividend income share v/labor income share z. The ratio v/z determines a household's income elasticity when the share of dividends in aggregate income fluctuates.

From the upper-left panel of Figure 6, we can see that poor households' average consumption responses are higher than those of rich households due to the interest-rate cut. As pointed out previously, the redistribution effects account for 28% of the consumption response on the aggregate level. On the individual level, however, the redistribution effects can account for a much larger share of the consumption responses. For households at the lowest wealth percentile, interest-rate exposure's effects on consumption are 150% of the RANK effects (0.6/0.4). For the richest households, the interest-rate exposure's effects are negative and dampen their total consumption responses.

The liquidity channel relaxes the borrowing conditions of constrained households. Unconstrained households lend to constrained households relatively homogeneously: across the distribution of bond demand b', the median and rich households exhibit similar consumption cuts. The 'between' income exposure channel allows households with a low dividend income share (low v) but a high labor income share (high z) to consume the additional income from producing capital.

Through Guvenen et al. (2017)'s estimates, the 'within' income exposure is positive both at the bottom of the labor-income distribution and the top ($\gamma(z_i) > 1$ for both low and high z_i). This is clear from the bottom-right panel of Figure 6. The median households in the

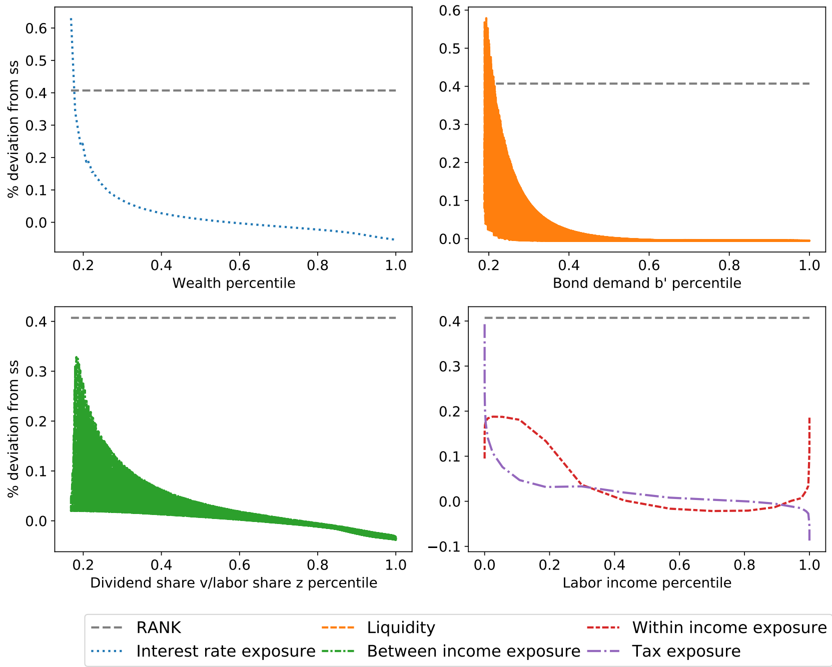

Figure 6: Individual-level decomposition of impact consumption responses

Notes: The redistribution shock's effects on individual consumption (impact) are decomposed into five channels. For comparison, I also show the RANK effects, which are homogeneous across individuals. The effects of each channel are shown across the most relevant redistribution dimension. Section 5 gives the definitions of these redistribution channels.

labor-income distribution are hurt by the 'within' income exposure channel. The tax exposure channel dampens high-labor-income households' consumption and amplifies low-labor-income households' consumption responses.

7 Conclusion

This paper decomposes the incomplete-market model's response to a monetary policy shock into two parts: the response of a fictitious representative agent to the aggregate shock and the response of the heterogeneous-agent model to a transfer scheme among agents. By further decomposing the latter, I analytically characterize the redistribution channels in the HANK model and quantitatively evaluate how different channels contribute to the deviation of HANK from RANK. The redistribution effects amplify the responses of output and consumption and dampen the response of investment and real interest rates. On impact, redistribution effects account for 28% of consumption response and 6% of output response. All redistribution channels contribute to amplification. When considering their impact magnitude, the channels are ranked in terms of importance as follows: interest rate exposure, liquidity, 'between' income exposure, 'within' income exposure, tax exposure, and asset price.

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A Proofs

Proof of Proposition 1. First, I impose the bond demand function $b^{ra}(z^t) = b^*(z^t)$ and verify the F.O.C with respect to the bond demand

$$(c^{ra}(z^t))^{-\sigma} \geq \beta(1+r^{ra}_{t+1})E[(c^{ra}(z^{t+1}))^{-\sigma}|z^t], = \text{if } b^{ra}(z^t) > \phi. \tag{15}$$

To see this

$$\frac{(c^{ra}(z^t))^{-\sigma}}{E[(c^{ra}(z^{t+1}))^{-\sigma}|z^t]} = \frac{(C^{ra}_t/C^*)^{-\sigma}(c^*(z^t))^{-\sigma}}{(C^{ra}_{t+1}/C^*)^{-\sigma}E[(c^*(z^{t+1}))^{-\sigma}|z^t]} \tag{16}$$

$$\geq \beta^{ra}(1+r^{ra}_{t+1})\beta(1+r^*) \tag{17}$$

$$= \beta(1+r^{ra}_{t+1}).$$

Equation (17) holds because in the stationary equilibrium

$$(c^*(z^t))^{-\sigma} \geq \beta(1+r^*)E[(c^*(z^{t+1})^{-\sigma}|z^t]. \tag{18}$$

In the case of $b^{ra}(z^t) > \phi$, it can only be the case that equation (18) holds with equality, so equation (17) also holds with equality.

Second, I prove the aggregate labor supply condition. The individual labor supply condition is

$$W^{ra}_t z_t(c^{ra}(z^t))^{-\sigma} = \varphi(n^{ra}(z^t))^{\nu} \tag{19}$$

Divide equation (19) by labor supply condition in the stationary equilibrium.

$$\frac{W^{ra}_t z_t(c^{ra}(z^t))^{-\sigma}}{W^* z_t(c^*(z^t))^{-\sigma}} = \frac{\varphi(n^{ra}(z^t))^{\nu}}{\varphi(n^*(z^t))^{\nu}}$$

$$\frac{W^{ra}_t}{W^*}\left(\frac{c^{ra}(z^t)}{c^*(z^t)}\right)^{-\sigma} = \left(\frac{n^{ra}(z^t)}{n^*(z^t)}\right)^{\nu}$$

$$\frac{W^{ra}_t}{W^*}\left(\frac{C^{ra}_t}{C^*}\right)^{-\sigma} = \left(\frac{N^{ra}_t}{N^*}\right)^{\nu}$$

Third, the transfer is recovered from the budget constraint:

$$\omega(z^t) = c^{ra}(z^t) + b^{ra}(z^t) - (1+r^{ra}_t)b^{ra}(z^{t-1}) - W^{ra}_t z_t n^{ra}(z^t) - \pi^{ra}(z) + \tau^{ra}(z^t).$$

Finally, aggregating over transfers $\omega(z^t)$,

$$\int \omega(z^t)d\Phi_t(z^t) = \int [c^{ra}(z^t) + b^{ra}(z^t) - (1+r^{ra}_t)b^{ra}(z^{t-1}) - W^{ra}_t z_t n^{ra}(z^t) - \pi^{ra}(z) + \tau^{ra}(z^t)]d\Phi_t(z^t)$$

$$= C^{ra}_t + B - (1+r^{ra}_t)B - W^{ra}_t N^{ra}_t - D^{ra}_t + T^{ra}_t.$$

From the market clearing condition and the government's budget constraint in the 'RANK' equilibrium, it turns out $\int \omega(z^t)d\Phi_t(z^t) = 0$.

Proof of Proposition 2. The borrowing constraint condition holds by construction. To satisfy the F.O.C (15), notice the following corollary from equation (16):

Corollary. Households are constrained in the 'RANK' equilibrium if and only if they are constrained in the stationary equilibrium.

In the case of $b^{ra}(z^t) > \phi$, it can only be the case that equation (18) holds with equality, so equation (17) also holds with equality. The transversality condition follows from the necessary condition of household optimization, and the bond market clearing condition follows from the market clearing in general equilibrium.

Proof of Proposition 3. It's easy to verify that $b^{ra}(z^t)$ satisfies the conditions in Propositition 3 if $\bar{b}^{ra}(z^t)$ satisfies the conditions in Propositition 2. To see that the transfers are invariant to the path of government debt,

$$
\begin{aligned}
&b^{ra}(z^t) - (1 + r_t^{ra})b^{ra}(z^{t-1}) + \tau^{ra}(z^t) \\
=&(\bar{b}^{ra}(z^t) + B_t^{ra} - B^*) - (1 + r_t^{ra})(\bar{b}^{ra}(z^{t-1}) + B_{t-1}^{ra} - B^*) + \bar{\tau}^{ra}(z^t) + T_t^{ra} - \bar{T}_t^{ra} \\
=&\bar{b}^{ra}(z^t) - (1 + r_t^{ra})\bar{b}^{ra}(z^{t-1}) + \bar{\tau}^{ra}(z^t).
\end{aligned}
$$

So

$$
\begin{aligned}
\omega(z^t) &= c^{ra}(z^t) + b^{ra}(z^t) - (1 + r_t^{ra})b^{ra}(z^{t-1}) - W_t^{ra} z_t n^{ra}(z^t) - \pi^{ra}(z) + \tau^{ra}(z^t) \\
&= c^{ra}(z^t) + \bar{b}^{ra}(z^t) - (1 + r_t^{ra})\bar{b}^{ra}(z^{t-1}) - W_t^{ra} z_t n^{ra}(z^t) - \pi^{ra}(z) + \bar{\tau}^{ra}(z^t).
\end{aligned}
$$

Decomposition at the channel level. Subtracting $\omega(z^t)$ from the household's budget constraint in the stationary equilibrium

$$
\begin{aligned}
-\omega(z^t) =&c^*(z^t) - c^{ra}(z^t) + b^*(z^t) - b^{ra}(z^t) - [(1 + r^*)b^*(z^{t-1}) - (1 + r_t^{ra})b^{ra}(z^{t-1})] \\
&- (y^*(z^t) - y^{ra}(z^t)) + (\tau^*(z^t) - \tau^{ra}(z^t)) \\
=&\hat{y}^{ra}(z^t)y^*(z^t) - \hat{C}_t^{ra}c^*(z^t) + (b^*(z^t) - \bar{b}^{ra}(z^t) + \bar{b}^{ra}(z^t) - b^{ra}(z^t)) \\
&- [(1 + r^*)b^*(z^{t-1}) - (1 + r_t^{ra})(b^{ra}(z^{t-1}) - b^*(z^{t-1}) + b^*(z^{t-1}) - \bar{b}^{ra}(z^{t-1}) + \bar{b}^{ra}(z^{t-1})] \\
&+ (\tau^*(z^t) - \bar{\tau}^{ra}(z^t) + \bar{\tau}^{ra}(z^t) - \tau^{ra}(z^t)) \\
=&(\hat{y}^{ra}(z^t) - \hat{Y}_t^{ra})y^*(z^t) + \hat{Y}_t^{ra}y^*(z^t) - \hat{C}_t^{ra}c^*(z^t) + b^*(z^{t-1})(r_t^{ra} - r^*) - (\bar{\tau}^{ra}(z^t) - \tau^*(z^t)) \\
&+ (\bar{b}^{ra}(z^t) - b^{ra}(z^t)) - (1 + r_t^{ra})(\bar{b}^{ra}(z^{t-1}) - b^{ra}(z^{t-1})) + (\bar{\tau}^{ra}(z^t) - \tau^{ra}(z^t)) \\
&+ (b^*(z^t) - \bar{b}^{ra}(z^t)) - (1 + r_t^{ra})(b^*(z^{t-1}) - \bar{b}^{ra}(z^{t-1})). \qquad (20)
\end{aligned}
$$

From the government budget constraint

$$
B(r_t^{ra} - r^*) = \bar{T}_t^{ra} - T^*. \qquad (21)
$$

Combing equation (20) and (21)

$$
\begin{aligned}
-\omega(z^t) =&(\hat{y}^{ra}(z^t) - \hat{Y}_t^{ra})y^*(z^t) + \hat{C}_t^{ra}(y^*(z^t) - c^*(z^t)) \\
&+ (b^*(z^{t-1}) - B)(r_t^{ra} - r^*) \\
&+ (\bar{T}_t^{ra} - T^*) - (\bar{\tau}^{ra}(z^t) - \tau^*(z^t)) \\
&+ (\bar{b}^{ra}(z^t) - b^{ra}(z^t)) - (1 + r_t^{ra})(\bar{b}^{ra}(z^{t-1}) - b^{ra}(z^{t-1})) + (\bar{\tau}^{ra}(z^t) - \tau^{ra}(z^t)) \quad (22) \\
&+ (b^*(z^t) - \bar{b}^{ra}(z^t)) - (1 + r_t^{ra})(b^*(z^{t-1}) - \bar{b}^{ra}(z^{t-1})). \quad (23)
\end{aligned}
$$

In the case that government debt is constant, we have $\bar{b}^{ra}(z^t) = b^{ra}(z^t)$ and $\bar{\tau}^{ra}(z^t) = \tau^{ra}(z^t)$, the term (22) is zero. In the case of $b^*(z^t) = \bar{b}^{ra}(z^t)$, the last term (23) is zero.

Decomposition with outside assets. Assume the budget constraints of households are

$$c(z^t) + p_t v(z^t) = (p_t + D_t)v(z^{t-1}) + zWn(z^t) + \pi_t + \omega(z^t).$$

Define $y \equiv zWn + \pi + Dv_-$ as the individual income, including labor income $zWn + \pi$ and dividend income Dv_-. Define $Y = WN + (1-\alpha)\Pi + D$ as the aggregate income, and we have $C_t = Y_t$. The negative of the transfer is

$$-\omega(z^t) = p_t^{ra} v^{ra}(z^{t-1}) + y^{ra}(z^t) - p_t^{ra} v^{ra}(z^t) - c^{ra}(z^t). \quad (24)$$

Subtracting the budget constraint in the stationary equilibrium from equation (24)

$$
\begin{aligned}
-\omega(z^t) =&p_t^{ra}(v^*(z^{t-1}) + v^{ra}(z^{t-1}) - v^*(z^{t-1})) - p^* v^*(z^{t-1}) + \hat{y}^{ra}(z^t) y^*(z^t) \\
&- (p_t^{ra}(v^*(z^t) + v^{ra}(z^t) - v^*(z^t)) - p^* v^*(z^t)) - \hat{C}_t^{ra} c^*(z^t) \\
=&(p_t^{ra} - p^*)v^*(z^{t-1}) + (\hat{y}^{ra}(z^t) - \hat{Y}_t^{ra})y^*(z^t) - (p_t^{ra} - p^*)v^*(z^t) \\
&+ p_t^{ra}(v^{ra}(z^{t-1}) - v^*(z^{t-1})) - p_t^{ra}(v^{ra}(z^t) - v^*(z^t)) + \hat{C}_t^{ra}(y^*(z^t) - c^*(z^t)) \\
=&(\hat{y}^{ra}(z^t) - \hat{Y}_t^{ra})y^*(z^t) + (p_t^{ra} - p^*)(v^*(z^{t-1}) - v^*(z^t)) \\
&+ \hat{C}_t^{ra}(y^*(z^t) - c^*(z^t)) \\
&+ p_t^{ra}(v^{ra}(z^{t-1}) - v^*(z^{t-1})) - p_t^{ra}(v^{ra}(z^t) - v^*(z^t)) \quad (25)
\end{aligned}
$$

In the case of $v^{ra}(z^t) = v^*(z^t)$, the last term (25) is zero.

Define $y^L \equiv zWn + \pi$ as the individual labor income and $Y^L \equiv WN + (1-\alpha)\Pi$ as the aggregate labor income then

$$
\begin{aligned}
&(\hat{y}^{ra}(z^t) - \hat{Y}_t^{ra})y^*(z^t) \\
=&\hat{D}_t^{ra} D^* v^*(z^{t-1}) + (\hat{y}^{L,ra}(z^t) - \hat{Y}_t^{L,ra})y^{L,*}(z^t) + \hat{Y}_t^{L,ra} y^{L,*}(z^t) - \hat{Y}_t^{ra} y^*(z^t) \\
=&(\hat{D}_t^{ra} - \hat{C}_t^{ra})D^* v^*(z^{t-1}) + (\hat{y}^{L,ra}(z^t) - \hat{Y}_t^{L,ra})y^{L,*}(z^t) + (\hat{Y}_t^{L,ra} - \hat{C}_t^{ra})y^{L,*}(z^t) \\
=&(\hat{y}^{L,ra}(z^t) - \hat{Y}_t^{L,ra})y^{L,*}(z^t) + (\hat{D}_t^{ra} - \hat{C}_t^{ra})D^* v^*(z^{t-1}) + (\hat{Y}_t^{L,ra} - \hat{C}_t^{ra})y^{L,*}(z^t). \quad (26)
\end{aligned}
$$

From $C_t = Y_t^L + D_t$ we have $\hat{C}_t C^* = \hat{Y}_t^L Y^{L,*} + \hat{D}_t D^*$. Then

$$(\hat{D}_t^{ra} - \hat{C}_t^{ra}) D^* + (\hat{Y}_t^{L,ra} - \hat{C}_t^{ra}) Y^{L,*} = 0,$$

and

$$(\hat{Y}_t^{L,ra} - \hat{C}_t^{ra}) y^{L,*}(z^t) = -(\hat{D}_t^{ra} - \hat{C}_t^{ra}) D^* \frac{y^{L,*}(z^t)}{Y^{L,*}}. \tag{27}$$

Substituting equation (27) into (26) we get the equation in the main text.

B Quantitative Results

Parameter	Description	Value	Target
r^*	real interest rate (p.a.)	0.05	
σ	Risk aversion	1	
A	TFP	0.46	Unit quarterly output
α	Capital share	0.33	
Ψ	Capital adjustment cost	13	Christiano, Eichenbaum and Trabandt (2016)
δ^K	Depreciation of capital (p.a.)	0.07	Kaplan, Moll and Violante (2018)
K/Y	Capital to GDP (p.a.)	2.4	Internally calibrated
B/Y	Government debt to GDP (p.a.)	0.29	2004 SCF gross liquid assets
p/Y	Equity to GDP (p.a.)	2.92	2004 FoF net illiquid assets
$\mu - 1$	markup	0.05	Interally calibrated
κ	Slope of Phillips curve	0.1	Christiano, Eichenbaum and Rebelo (2011)
ϵ^w	Wage elasticity	0.5	Christiano, Eichenbaum and Trabandt (2016)
ϕ_π	Coefficient on inflation	1.25	
ρ_B	Debt Persistence	0.93	Auclert and Rognlie (2018)
ϕ^B	Coefficient of shock to debt level	-0.43	IRF of real loan in liquid assets account
Γ^*	Labor income tax rate	0.3	
$T^{uniform}$	Uniform tax	-0.06	Kaplan, Moll and Violante (2018)
G^*	Government spending to GDP	0.13	Internally calibrated

Table 3: Calibration of the HANK model in the main text

Figure 7 shows the wealth distribution (Lorenz Curve) in the steady state, together with the wealth distribution from the 2004 Survey of Consumer Finances. The model replicates the wealth distribution relatively well.

Calibration of fiscal policy. Figure 8 plots the estimated responses of output Y_t, consumption C_t, investment I_t, nominal rate i_t^b (the return on the three-month treasury bill), and household loans in liquid assets L_t to a monetary shock. All variables except the nominal

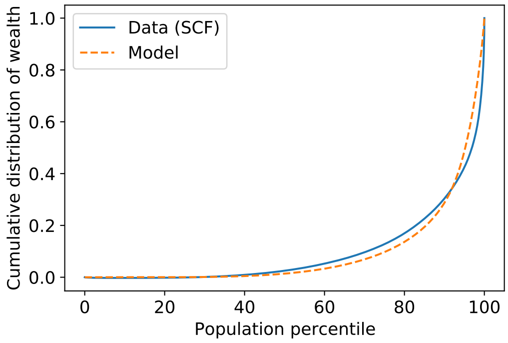


Figure 7: Distribution of net worth (Lorenz Curve)

Notes: The Data curve shows the distribution of net worth in the 2004 Survey of Consumer Finances.

rate are in real terms. The monetary policy shock is normalized such that the nominal rate i_t^b decreases by 25 basis points on impact. the lagged controls are set as the federal funds rate i_{t-1}, the monetary shock ϵ_{t-1}, unemployment rate U_{t-1}, log of output Y_{t-1}, consumption C_{t-1}, investment I_{t-1}, TFP A_{t-1} and consumer price index P_{t-1}.

Figure 9 shows the decomposition of asset price and inflation responses. Figure 10 shows the corresponding channel-level decomposition. Redistribution effects dampen the asset price responses. This is because, on the one hand, redistribution effects dampen the response of real interest rates; on the other hand, redistribution effects dampen the response of dividends. The channel-level decomposition of asset price responses is close to the channel-level decomposition of investment responses. The redistribution effects amplify the response of inflation since it amplifies the response of output.

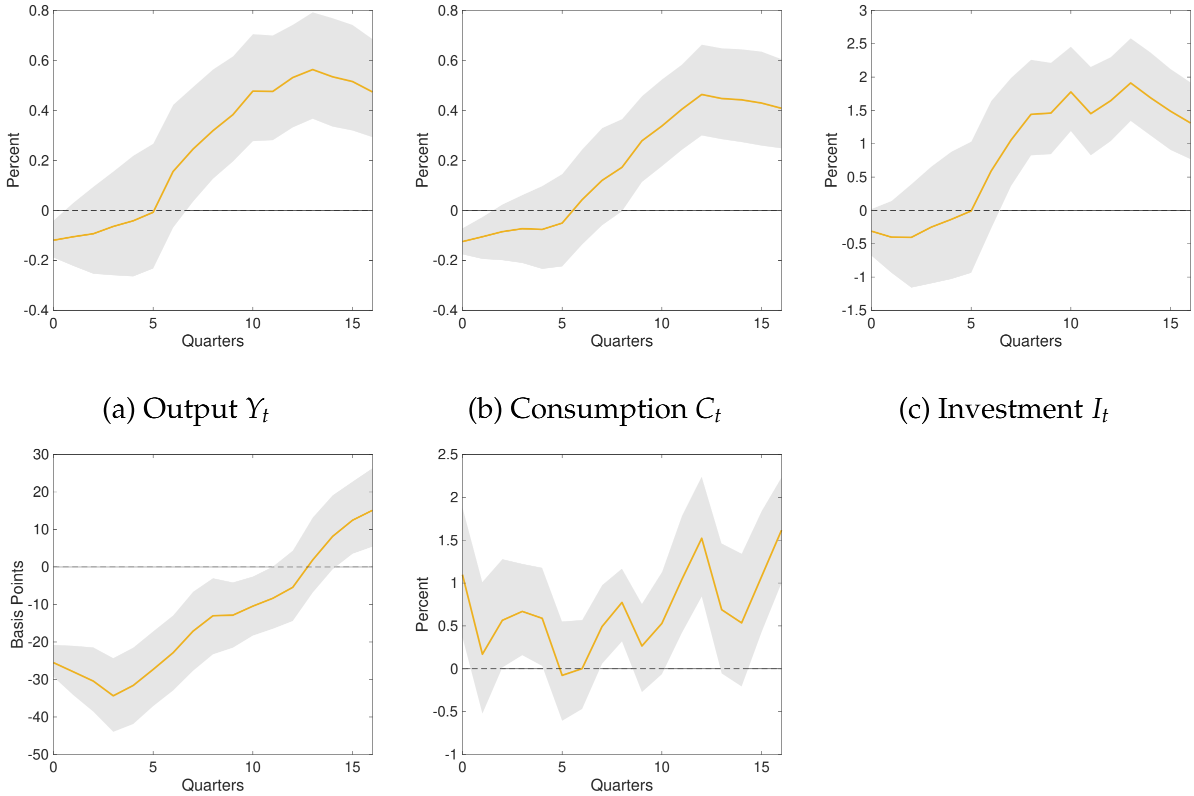
C Decomposing TANK

The decomposition can be analytically implemented in the Two-Agent New Keynesian (TANK) model. For comparison, the TANK model used here is kept identical to Bilbiie (2020).[12] I briefly describe the environment and characterize the equilibrium conditions. Details of the model can be found in Bilbiie (2020).

C.1 Model Description

There are two types of households with total unit mass. A fraction of λ households is hand-to-mouth H, who are excluded from financial markets and consume their current in-

[12] Bilbiie (2020) has aggregate uncertainty and log-linearize the model. The solution is equivalent to the linearized perfect-foresight transition path (see Boppart, Krusell and Mitman (2018)).



(a) Output Y_t (b) Consumption C_t (c) Investment I_t

(d) Nominal Rate i_t^b (e) Loans L_t

Figure 8: Aggrregate responses to a monetary shock

Notes: Estimated response of output, consumption, investment, nominal rates, and real loans to a monetary policy shock. Monetary policy shock is normalized such that the impact decrease of the return on the 3-month treasury bill return is 25 basis points. I estimate the responses by local projections with high-frequency identified monetary policy shocks in Gorodnichenko and Weber (2016). I use quarterly data from 1988Q4 to 2016Q2 (the data on monetary policy shocks is from 1988Q4 to 2012Q2). The lagged controls are set as $\mathbf{X}_{t-1} = [i_{t-1}, \epsilon_{t-1}, U_{t-1}, Y_{t-1}, C_{t-1}, I_{t-1}, A_{t-1}, P_{t-1}]$. The shadow area represents the bootstrapped 66% confidence bounds.

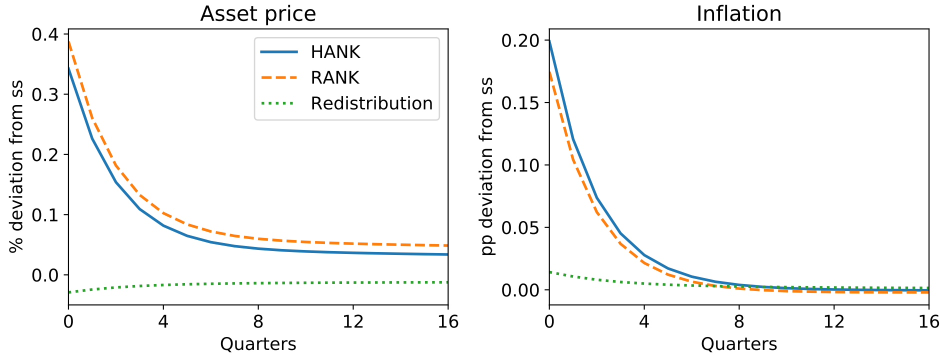


Figure 9: Asset price and inflation decomposition

Notes: Decomposition of the responses of asset price and inflation to a monetary policy shock, $\epsilon_0 = 25$ basis points. The RANK effects are these variables' responses to the monetary policy shock in a fictitious RANK model, and the redistribution effects are these variables' responses to the triggered redistribution shock in the HANK model.

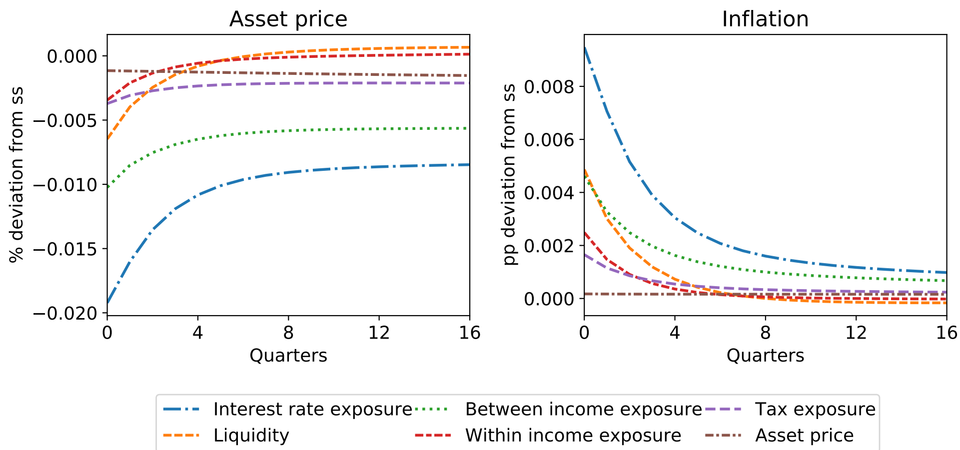


Figure 10: Asset price and inflation decomposition at channel level

Notes: The redistribution shock's effects on asset price and inflation are decomposed into six channels. The redistribution shock is triggered by a monetary policy shock of 25 basis points. Section 5 gives the definitions of these redistribution channels.

come. The budget constraint of H is given by

$$C_t^H = W_t N_t^H + D_t^H$$

where W_t is real wage, H_t is H's labor supply, and D_t^H is the firm's profits received by H. The remaining fraction $1 - \lambda$ of households are savers S, trading one-period riskless real bonds. The budget constraint of S is given by

$$C_t^S + \frac{B_{t+1}}{1 + r_t} = B_t + W_t N_t^S + D_t^S$$

where N_t^S is S's labor supply and D_t^S is the firm's profits received by S. All households maximize their discounted utility $E_0 \sum_{t=0}^{\infty} \beta^t U(C_t, N_t)$ subject to the sequence of their budget constraints. The utility function takes the form $U(C, N) = C^{1-1/\sigma}/(1 - \sigma) - N^{1+\varphi}/(1 + \varphi)$.

The supply side is standard. There is a continuum of firms, and each firm produces a differentiated good with linear technology $Y_t(i) = A_t N_t(i)$. In each period, firms have the possibility of θ to reset the price. The demand for each good is $Y_t(i) = (P_t(i)/P_t)^{-\epsilon} Y_t$ where $P_t = (\int_0^1 P_t(i)^{1-\epsilon} di)^{1/(1-\epsilon)}$ is the aggregate price index and Y_t is the aggregate output. The standard supply-side implies the canonical representation of the log linearized Philips Curve: $\pi_t = \beta E_t \pi_{t+1} + \kappa y_t$ where y_t is the log deviation of output from steady state.

The government implements standard NK optimal subsidy inducing marginal cost pricing financed by a lump-sum tax on the firms' profits. The profit function is $D_t(i) = (1 + \tau) P_t(i) Y_t(i)/P_t - W_t N_t(i) - T_t^F$. With the optimal subsidy, $\tau = 1/(\epsilon - 1)$, firms' steady-state profits are zero. in the stationary equilibrium, households have the same income and consumption. The central bank conducts monetary policy in the form of the Taylor rule: $i_t = r^* + \phi_\pi \pi_t + \epsilon_t$ where r^* is the steady state real interest rate, and ϵ_t is an exogenous monetary policy shock.

The key assumption in TANK is the distribution rule of the firm's profits. The government redistributes τ^D share of profits to H: $D_t^H = \tau^D D_t/\lambda$, and $1 - \tau^D$ share of profits to S: $D_t^S = (1 - \tau^D) D_t/(1 - \lambda)$. When $\tau^D = \lambda$, H and S receive the same profits, and their income and consumption have the same responses in equilibrium. When $\tau^D \neq \lambda$, TANK deviates from this representative-agent benchmark.

Denote log deviations of variables from their steady-state values except for interest rates by small letters. After imposing the market clearing condition, the aggregate Euler equation of TANK is derived as

$$c_t = E_t c_{t+1} - \delta^{-1} \sigma (r_t - r^*) \tag{28}$$

where $\delta^{-1} = (1 - \lambda)/(1 - \lambda\chi)$ and $\chi = 1 + \varphi(1 - \tau^D/\lambda)$. Though H have no access to financial markets and their consumption does not price the bond, one can infer the quantitative relation between their consumption and interest rates from the relation between H and S's equilibrium consumption.

From the aggregate Euler equation (28), we can see the amplifying/dampening mechanism in TANK. As already mentioned, if $\tau^D = \lambda$, it follows $\chi = 1$ and $\delta^{-1} = 1$. The elasticity of contemporaneous aggregate consumption to interest rates is the same as RANK. In equilibrium, the income and consumption responses of H and S are the same. If $\tau^D < \lambda$, H receives a smaller amount of profits than S. With counter-cyclical profits, it implies that H's consumption responds more than S's consumption. As a weighted sum, aggregate consumption also responds more than S's consumption, and its elasticity to interest rates is larger than the consumption elasticity of S: $\delta^{-1}\sigma > \sigma$. For a given change in real interest rates, the aggregate consumption response in TANK is amplified relative to RANK.

With the full characterization of the equilibrium, I now consider the output response to an exogenous monetary policy shock. For illustration, here I consider a monetary policy shock that lasts only one period: $E_t\epsilon_{t+1} = 0$. Given a monetary policy shock ϵ_t, the output response of TANK is

$$y_t = -\frac{\delta^{-1}\sigma}{1 + \delta^{-1}\sigma\phi_\pi\kappa}\epsilon_t. \tag{29}$$

In the case of amplifying, $\delta^{-1} > 1$, and the output response is larger (in abstract value) than that in RANK. In the case of dampening, $\delta^{-1} < 1$, the output is less responsive to monetary policy shocks relative to RANK.

C.2 Decomposition

I decompose the output response y_t into **RANK effects** y_t^{ra} and **redistribution effects** y_t^{re} such that $y_t = y_t^{ra} + y_t^{re}$. This decomposition is based on the observation that monetary policy shocks in TANK induce a redistribution between H and S due to their unequal exposure to the countercyclical profits, which affects their income elasticities to aggregate income. In a counterfactual scenario where this redistribution is eliminated, TANK behaves the same as RANK. To achieve this scenario, I construct lump-sum transfers to households. The difference between TANK and RANK is then attributed to the absence of these transfers.

Let ω_t^H and ω_t^S be the counterfactual transfers to H and S, respectively, that eliminate the redistribution effects of a monetary policy shock $\{\epsilon_t\}$. The RANK effects of the shock on output y_t^{ra} are the response of output to the shock and the transfers $\{\epsilon_t, \omega_t^H, \omega_t^S\}$; and the redistribution effects of the shock on output y_t^{re} are the response of output to the opposite of the transfers $\{-\omega_t^H, -\omega_t^S\}$. The counterfactual transfers are purely redistributive: $\lambda\omega_t^H + (1-\lambda)\omega_t^S = 0$, where λ is the fraction of H in the population.

RANK effects. The RANK effects on output y_t^{ra} are the output responses of a representative agent model:

$$y_t^{ra} = -\frac{\sigma}{1 + \sigma\phi_\pi\kappa}\epsilon_t. \tag{30}$$

In RANK effects, S and H have the same consumption responses and it is easy to verify the consumption of Savers $c_t^{S,ra}$ satisfies the Euler equation with interest rates $\{r_t^{ra}\}$. However, these consumption responses do not satisfy households' budget constraints without transfers. To satisfy the budget constraints, I construct lump-sum transfers $\{\omega_t^H, \omega_t^S\}$ to H and S. With lump-sum transfers, the budget constraints of households are

$$\begin{aligned} c_t^{H,ra} &= w_t^{ra} + n_t^{H,ra} + \frac{\tau^D}{\lambda} d_t^{ra} + \omega_t^H, \\ c_t^{S,ra} &= w_t^{ra} + n_t^{S,ra} + \frac{1-\tau^D}{1-\lambda} d_t^{ra} + \omega_t^S, \end{aligned} \tag{31}$$

where ω_t^S and ω_t^H are the transfers (as a percentage of steady state output Y^*) to S and H, respectively. Assuming that both households satisfy their optimal labor supply condition in equilibrium, so $c_t^{S,ra} = c_t^{H,ra}$ implies $n_t^{S,ra} = n_t^{H,ra}$, the budget constraints require:

$$\begin{aligned} \omega_t^H &= \left(1 - \frac{\tau^D}{\lambda}\right) d_t^{ra}, \\ \omega_t^S &= \left(1 - \frac{1-\tau^D}{1-\lambda}\right) d_t^{ra}. \end{aligned} \tag{32}$$

With this transfer scheme, S and H have the same consumption response, and the aggregate Euler equation holds.

Redistribution effects. Consider an exogenous transfer shock such that $\lambda T_t^H + (1-\lambda) T_t^S = 0$ where T_t^H and T_t^S are the transfers (as the percentage of steady-state output Y^*) to H and S, respectively. The proof below shows that the output response of TANK to a transfer shock is

$$y_t = -\frac{1}{\sigma\phi_\pi\kappa + \delta} \cdot \frac{1}{1 + (\sigma\varphi)^{-1}} T_t^S \tag{33}$$

To obtain the redistribution effects, I input the negative of the transfers $\{-\omega_t^H, -\omega_t^S\}$ into the model. Letting $T_t^S = -\omega_t^S$,

$$y_t^{re} = \frac{1-\delta}{\sigma\phi_\pi\kappa + \delta} y_t^{ra}.$$

Discussion. Expressing the output response (29) of TANK y_t in terms of RANK effects (30) y_t^{ra}:

$$y_t = \frac{1 + \sigma\phi_\pi\kappa}{\delta + \sigma\phi_\pi\kappa} y_t^{ra}.$$

In the case of amplification ($\tau^D < \lambda$, $\chi > 1$ and $\delta < 1$), the redistribution effects act in the same direction as RANK effects, and the total effects are greater than RANK effects (in absolute value). The endogenous redistribution through firms' profit distribution τ^D/λ in TANK amplifies the output response. To see this, consider an expansionary monetary policy shock $\epsilon_t < 0$, from (32) it follows $\omega_t^H < 0$ and $\omega_t^S > 0$. The negative of the transfers $\{-\omega_t^H, -\omega_t^S\}$ subsidize H by taxing S. In TANK, fiscal stimulus in the form of transfers from S to H is itself a policy instrument that stimulates the economy (see Bilbiie, Monacelli and Perotti (2013)). In the case of dampening ($\tau^D > \lambda$, $\chi < 1$ and $\delta > 1$), the negative of the transfers tax H and subsidize S, which will dampen the economy's response. In TANK, the RANK effects are a natural benchmark to evaluate the amplifying/dampening mechanism. When extending this decomposition approach to HANK, I also use the RANK effects as the benchmark.

Another way to decompose the response of output y_t is to decompose it into substitution and income effects, as discussed in Auclert (2019) and referred to as 'direct effects' and 'indirect effects' in Kaplan, Moll and Violante (2018). The substitution effects are the response of aggregate consumption keeping the income of households unchanged. When interest rates fall, households save less for the future and consume more today due to intertemporal substitution. The income effects are the response of aggregate consumption keeping the interest rates unchanged.[13] After some algebra, it can be shown that

$$
\begin{aligned}
c_t^{sub} &= \beta(1 - \lambda\chi)y_t, \\
c_t^{inc} &= [1 - \beta(1 - \lambda\chi)]y_t,
\end{aligned}
$$

the sizes of substitution effect c_t^{sub} and income effect c_t^{inc} depend on H's measure λ and the amplifying/dampening parameter χ. One can easily see the difference between this paper's decomposition and the decomposition in Kaplan, Moll and Violante (2018) and Auclert (2019) in the case of proportional distribution of firm profits ($\tau^D = \lambda$, $\chi = 1$ and $\delta = 1$). In this case, the economy's response is equivalent to RANK. This paper's decomposition implies zero redistribution effects $y_t^{re} = 0$. All output response is due to RANK effects regardless of the mass of hand-to-mouth households because, in equilibrium, S and H are equally exposed to the aggregate shock. But the size of substitution and income effects simply varies with H's measure λ. This is because the decomposition in Auclert (2019) and Kaplan, Moll and Violante (2018) captures both the heterogeneous MPCs across households (parameter λ) and the correlation between households' MPCs and income exposures (parameter χ). Only conditional on H's measure λ, we can infer the parameter χ from the decomposition result.

[13] Auclert (2019) further decompose those effects into an aggregate and a redistribution component, respectively. For income effects c_t^{inc}, the aggregate component is the consumption response of an average household (whose MPC is the weighted average of S and H's MPCs) to the shock y_t, and the redistribution component is the weighted sum of S's consumption response to $y_t^S - y_t$ and H's consumption resposne to $y_t^H - y_t$.

Proof. The equilibrium of TANK can be characterized by the following equations

$$c_t = E_t c_{t+1} - \delta^{-1}\sigma(r_t - r^*)$$
$$\pi_t = \beta E_t \pi_{t+1} + \kappa c_t$$
$$i_t = r^* + \phi_\pi \pi_t + \epsilon_t$$

For a transient shock $E_t\epsilon_{t+1} = 0$ and $E_t c_{t+1} = E_t\pi_{t+1} = 0$. The solution is simply

$$y_t = -\frac{\delta^{-1}\sigma}{1+\delta^{-1}\sigma\phi_\pi\kappa}\epsilon_t$$

The RANK effects are obtained by letting $\delta = 1$,

$$y_t^{ra} = -\frac{\sigma}{1+\sigma\phi_\pi\kappa}\epsilon_t$$

Expressing y_t in terms of y_t^{ra},

$$y_t/y_t^{ra} = \frac{\delta^{-1}(1+\sigma\phi_\pi\kappa)}{1+\delta^{-1}\sigma\phi_\pi\kappa} = \frac{1+\sigma\phi_\pi\kappa}{\delta+\sigma\phi_\pi\kappa}$$

Consider a transfer shock such that $\lambda T_t^H + (1-\lambda)T_t^S = 0$ where T_t^H and T_t^S are the amount of transfers (measured as the percentage of steady-state output Y^*) H and S receive, respectively. From the budget constraint of S, we could derive the relation between the S's consumption c_t^S, output y_t, and T_t^S

$$\begin{aligned}
c_t^S &= w_t + n_t^S + \frac{1-\tau^D}{1-\lambda}d_t + T_t^S \\
&= (1 - \frac{1-\tau^D}{1-\lambda})w_t + \varphi^{-1}(w_t - \sigma^{-1}c_t^S) + T_t^S \\
[1+(\sigma\varphi)^{-1}]c_t^S &= (1 - \frac{1-\tau^D}{1-\lambda} + \varphi^{-1})w_t + T_t^S \\
c_t^S &= \delta y_t + \frac{1}{1+(\sigma\varphi)^{-1}}T_t^S
\end{aligned} \tag{34}$$

From S's Euler equation, Philips Curve and Taylor rule it follows $c_t^S = -\sigma(r_t - r^*) = -\sigma\phi_\pi\kappa y_t$. Substituting into (34), the output response to the transfer shock is

$$y_t = -\frac{1}{\sigma\phi_\pi\kappa+\delta}\frac{1}{1+(\sigma\varphi)^{-1}}T_t^S \tag{35}$$

To obtain redistribution effects, note that the transfer S receive is

$$T_t^S = -\omega_t^S = -(1-\frac{1-\tau^D}{1-\lambda})d_t^{ra} = (1-\frac{1-\tau^D}{1-\lambda})(\sigma^{-1}+\varphi)y_t^{ra} \tag{36}$$

substituting (36) into (35) it follows

$$\begin{aligned}
y_t^{re} &= -\frac{1}{\sigma\phi_\pi\kappa+\delta}\frac{1}{1+(\sigma\varphi)^{-1}}(1-\frac{1-\tau^D}{1-\lambda})(\sigma^{-1}+\varphi)y_t^{ra} \\
&= \frac{1-\delta}{\delta+\sigma\phi_\pi\kappa}y_t^{ra}
\end{aligned}$$

We can verify that $y_t = y_t^{ra} + y_t^{re}$.

D Decomposition Without Investment

I implement the decomposition on the model presented in Section 3, where there is no productive capital and investment. I make a small modification to the budget constraints of households:

$$c(z^t) + \frac{b(z^t)}{1+r_t} = b(z^{t-1}) + z_t W_t n(z^t) + \pi_t(z) - \tau_t(z),$$

and we don't need to make a distinction between ex-ante interest rates and ex-post interest rates. The channel level decomposition is, instead,

$$\begin{aligned}
-\omega(z^t) = {} & \underbrace{(\hat{y}^{ra}(z^t) - \hat{Y}_t^{ra})y^*(z^t)}_{\text{income exposure}} + \underbrace{(b^*(z^t) - B)(\frac{1}{1+r^*} - \frac{1}{1+r_t^{ra}})}_{\text{interest rate exposure}} + \underbrace{(\bar{T}_t^{ra} - T^*) - (\bar{\tau}^{ra}(z^t) - \tau^*(z^t))}_{\text{tax exposure}} \\
& + \underbrace{\frac{\bar{b}^{ra}(z^t) - b^{ra}(z^t)}{1+r_t^{ra}} - (\bar{b}^{ra}(z^{t-1}) - b^{ra}(z^{t-1})) + (\bar{\tau}^{ra}(z^t) - \tau^{ra}(z^t))}_{\text{liquidity}} \\
& + \underbrace{\hat{C}_t^{ra}(y^*(z^t) - c^*(z^t))}_{\text{scaling of net saving}} + \underbrace{\frac{b^*(z^t) - \bar{b}^{ra}(z^t)}{1+r_t^{ra}} - (b^*(z^{t-1}) - \bar{b}^{ra}(z^{t-1}))}_{\text{undetermined bond demand}}, \qquad (37)
\end{aligned}$$

Where $y \equiv zWn + \pi$ is household income, including labor income zWn and profit income π.

To make the exercise more transparent, I assume that the central bank directly controls the real interest rate. At time $t = 0$ there is a quarterly real rate shock $\tilde{r}_0 = -0.25$ percent with the persistence of 0.61. Then by construction, the output response in the 'RANK' equilibrium is given by the aggregate Euler equation:

$$(C_t^{ra})^{-\sigma} = \beta^{ra}(1+r_t)(C_{t+1}^{ra})^{-\sigma}.$$

The redistribution effects are the economy's response to the redistribution shock keeping the real interest rate at the steady state level.

In the first two exercises, I assume a balanced budget fiscal policy. In the third exercise, I let the government adjust the outstanding debt to illustrate the liquidity channel.

D.1 Calibration

I consider a model with an annual real interest rate of 2% in the stationary equilibrium. The coefficient of risk aversion σ is set to 2. The Frisch elasticity of labor supply is $1/\nu = 0.5$, following Chetty (2012). For the idiosyncratic income process, I use $\rho_e = 0.966$ and $\sigma_e^2 = 0.017$, as in McKay, Nakamura and Steinsson (2016) and Guerrieri and Lorenzoni (2017). The supply of government bonds B is set to match the ratio of aggregate liquid assets to output $B/Y = 5.6$, as in McKay, Nakamura and Steinsson (2016). The borrowing constraint is zero

$\phi = 0$. The discount factor $\beta = 0.98$ and disutility from labor $\varphi = 0.933$ are calibrated to deliver the values of annual real interest and unit quarterly output. On the supply side, the slope of the Phillips Curve is $\kappa = 0.1$ and the parameter of the markup of intermediate firms is $\mu = 1.2$. The Taylor rule coefficient ϕ is set to 1.25. In the baseline calibration, I assume that household tax payments are uniform. The firm dividends are distributed to households proportional to their productivity $d(z) \sim z$, as in Kaplan, Moll and Violante (2018). Table 4 summarizes the parameter values.

Parameter	Description	Value	Target
β	Discount factor (p.q.)	0.98	2 percent annual interest rate
σ	Risk aversion	2	
$1/\nu$	Frisch elasticity	1/2	Chetty (2012)
φ	Disutility of labor	0.933	Output
ρ_e	Autocorrelation of earnings	0.966	McKay, Nakamura and Steinsson (2016)
σ_e^2	Innovation variance	0.017	McKay, Nakamura and Steinsson (2016)
B	Supply of assets (p.q.)	5.6	Aggregate liquid assets
μ	Markup of intermediate firms	1.2	Christiano, Eichenbaum and Rebelo (2011)
κ	Slope of Phillips curve	0.1	Christiano, Eichenbaum and Rebelo (2011)
ϕ	Coefficient on inflation	1.25	
$\pi(z)$	Profits distribution		Proportional to productivity
$\tau(z)$	Tax payment		Uniform across households
ρ_B	Debt reverting rate	0.1	

Table 4: Calibrated Parameter Values

D.2 Purely Transient Shocks

To begin, consider a real rate shock that lasts only one period (the persistence $\rho = 0$), in the same spirit as the thought experiments in Auclert (2019). The result is shown in Figure 11. The real interest rates decrease and stimulate consumption. Given the sticky price, the rising aggregate demand leads to an increase in both output and inflation. Regarding decomposition, redistribution effects amplify the output response. Under transient monetary policy shocks, RANK effects last for only one period, the same as in a representative-agent model. In contrast, the redistribution effects affect the economy for a long time, and all the economy's responses after time 0 are due to redistribution effects.

Figure 12 shows the transfers ω_{i0} as a function of the household's wealth and productivity. The left panel of Figure 12 shows the transfers ω_{i0} as a function of wealth at four different productivity levels. The right panel of Figure 12 shows ω_{i0} as a function of household productivity level at the wealth distribution's 20th, 40th, 60th, and 80th percentiles. The

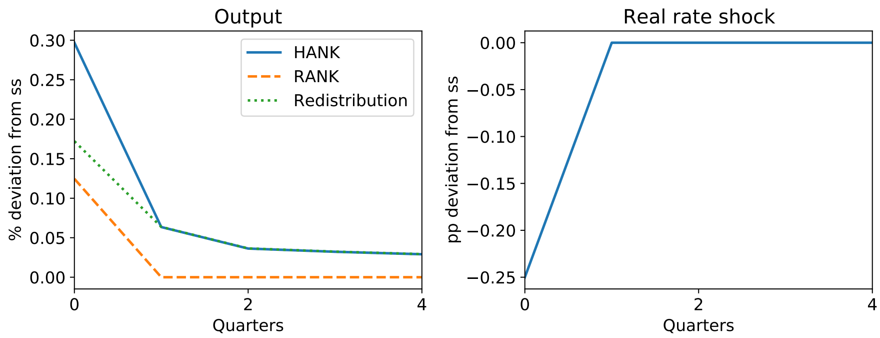


Figure 11: Decomposition of transient real rate shocks

Notes: Decomposition of the output's response to a transient real-rate shock, $\tilde{r}_0 = -0.25\%$.

transfers ω_{i0} increase with the household's wealth and (weakly) with productivity. Transfers increase with wealth because to eliminate the exposure to the interest rate cut, creditors need positive transfers, and debtors need negative transfers. The transfers increase with productivity because profits are countercyclical. The income of the household is $y = zWn + zD = z(WNn/N + D)$. Due to labor supply heterogeneity, high-income households have a higher share of profit income, which is countercyclical. High-income households' income increases less and needs positive transfers.

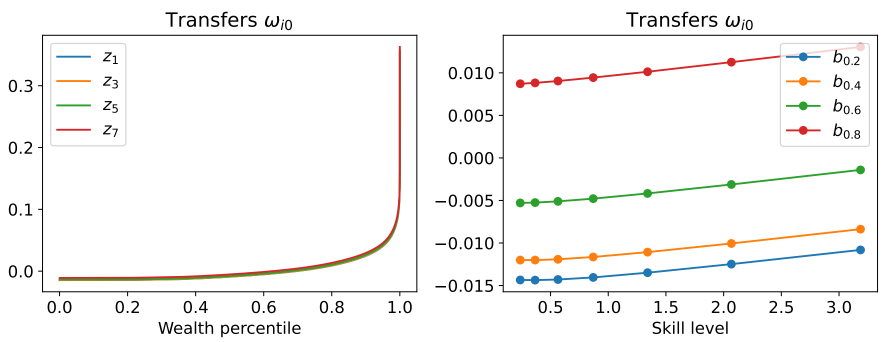


Figure 12: Transfers as a function of households characteristics

Notes: The left panel shows the transfers ω_{i0} as a function of wealth at four different productivity levels. The right panel shows ω_{i0} as a function of household productivity level at the wealth distribution's 20th, 40th, 60th, and 80th percentiles.

Overall, the redistribution shock $-\omega$ is making positive transfers to poor households by taxing rich households, similar to TANK. Since poor households have higher MPCs, it follows $cov_I(MPC_{i0}, -\omega_{i0}) > 0$. The redistribution effects stimulate aggregate consumption.

D.3 Persistent shocks

Consider the economy's response to the persistent real rate shocks. I apply the decomposition, and the result is shown in Figure 13. Output increases by 0.6% on impact. The decomposition result is qualitatively similar to the decomposition of the transient shock in Figure 11. Redistribution effects amplify the output's response to real rate shocks. On impact, RANK effects increase output by 0.31%, and redistribution effects increase output by 0.29%. The redistribution effects amplify the elasticity of output to real interest rates.

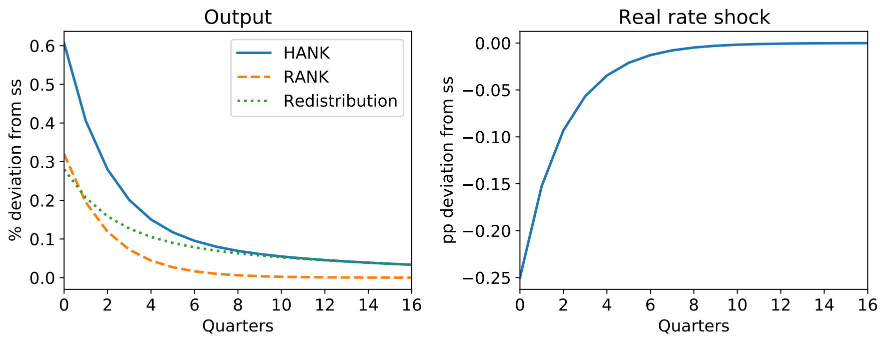


Figure 13: Decomposition of persistent real rate shocks

Notes: Decomposition of the output's response to a persistent real-rate shock, $\tilde{r}_0 = -0.25\%$. The government keeps a constant debt and adjusts the uniform tax to balance its budget.

Figure 14 further decomposes the redistribution effects into different channels for output. Quantitatively, the interest exposure channel accounts for most of the redistribution effects. On impact, the interest exposure channel increases consumption by 0.25%. The interest rate cuts tax creditors and subsidizes debtors. Given that debtors have higher MPCs, the interest rate exposure channel stimulates the economy. The income exposure channel slightly contributes to the output amplification. Since I assume uniform taxation, all households benefit equally from the tax reduction, and the tax exposure channel is muted.

D.4 Including Liquidity Channel

Assuming the fiscal policy takes the following rule:

$$T_t = T^* + \rho_B * (B_{t-1} - B^*). \tag{38}$$

The government uses debt to absorb most of the fiscal imbalance in the short run. In the long run, the government uses taxes to bring the debt back to its initial level. Similar fiscal policy specifications are assumed in Kaplan, Moll and Violante (2018), Alves et al. (2020), and Auclert, Rognlie and Straub (2018).

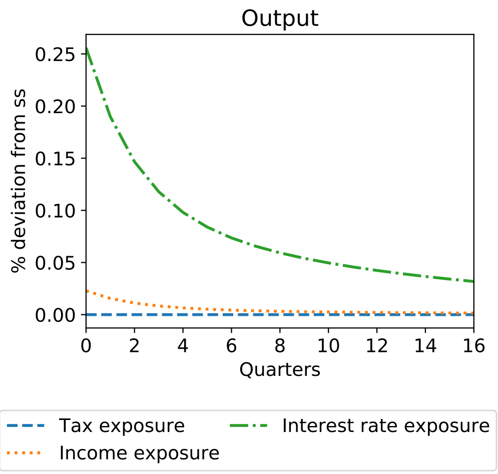


Figure 14: Channel-level decomposition

Notes: The redistribution shock's effects on output are decomposed into three channels. The government keeps a constant debt and adjusts the uniform tax after the shock. Equation (37) gives the definitions of these redistribution channels.

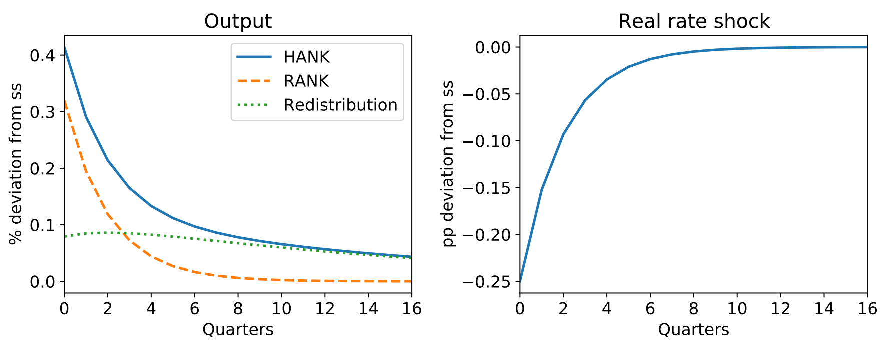


Figure 15: Decomposition with liquidity channel

Notes: Decomposition of the output's response to a persistent real-rate shock, $\tilde{r}_0 = -0.25\%$, with fiscal policy $T_t = T^* + \rho_B * (B_{t-1} - B^*)$. The government uses debt to absorb most of the fiscal imbalance in the short run. In the long run, the government uses uniform taxes to bring the debt back to its initial level.

The decomposition result is shown in Figure 15. The redistribution effects are smaller than Figure 13. On impact, redistribution effects increase output by less than 0.1%, rather than close to 0.3% under a balanced fiscal policy.

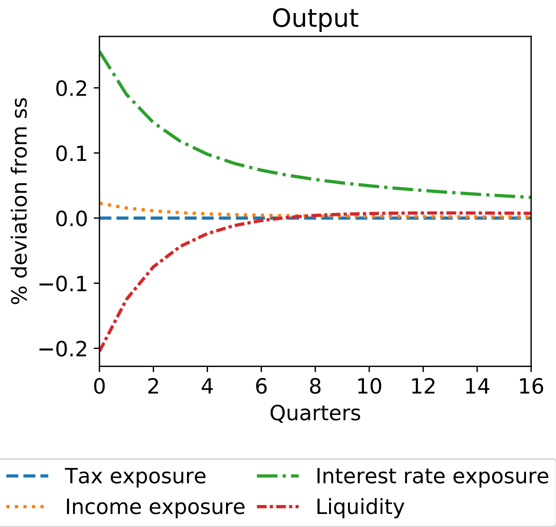


Figure 16: Channel-level Decomposition with liquidity channel

Notes: The redistribution shock's effects on output are decomposed into four channels. The government uses debt to absorb most of the fiscal imbalance in the short run. In the long run, the government uses uniform taxes to bring the debt back to its initial level. Equation (37) gives the definitions of these redistribution channels.

Figure 16 further decomposes the redistribution effects into different channels. The interest exposure, income exposure, and tax exposure channels are invariant to the path of government debt, so their effects in Figure 16 are identical to those in Figure 14. However, the liquidity channel decreases output by 0.2% on impact. The liquidity channel explains why the output response with the fiscal policy of (38) is smaller than a balanced budget.

The fiscal rule (38) implies a countercyclical asset supply. As proved in section 3.4.1, the liquidity channel is equivalent to a deleveraging shock in the case of uniform taxation. The output needs to decrease to clear the bond market. I show the household-level decomposition in Figure 17 to support this argument. Figure 17 shows the decomposition of the households' on-impact consumption responses. The interest rate exposure channel increases the consumption of poor households and decreases the consumption of rich households. However, the liquidity channel forces the constrained households to hold the additional income from other channels. As a result, the redistribution effects on poor households' consumption are smaller.

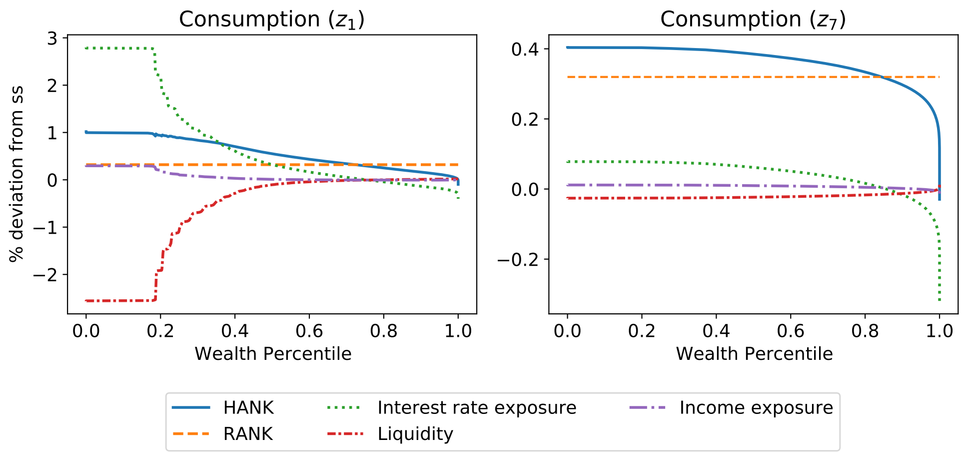


Figure 17: Household-level decomposition (on impact)

Notes: The redistribution shock's effects on individual consumption (impact) are decomposed into three channels. For comparison, I also show the RANK effects and the HANK model's responses. Equation (37) gives the definitions of these redistribution channels.

E Model with Illiquid Assets

In this section, I introduce illiquid assets to the model as in Kaplan, Moll and Violante (2018), Bayer et al. (2019), Luetticke (2021), Auclert, Rognlie and Straub (2018), Auclert, Rognlie and Straub (2020) and Kaplan and Violante (2022). I show that Proposition 1 holds with the presence of illiquid assets when modeling illiquidity a la Calvo as in Bayer et al. (2019), Luetticke (2021), Auclert, Rognlie and Straub (2018) and Kaplan and Violante (2022).

Households. Households have access to two assets: (i) liquid assets b with return r^b; (ii) illiquid assets a with return $r^a > r^b$. Households maximize subject to the following budget, adjustment, and borrowing constraints:

$$\begin{aligned} c(h^t) + b(h^t) &= (1 + r^b_t) b(h^{t-1}) - d(h^t) + z_t W_t n(h^t) + \pi(h^t) - \tau(h^t). \\ a(z^t) &= (1 + r^a_t) a(h^{t-1}) + d(h^t) \\ b(h^t) &\geq \phi, \quad a(h^t) \geq 0 \end{aligned}$$

where $h^t \equiv ((b_{-1}, a_{-1}), (z_0, s_0), (z_1, s_1), \cdots, (z_t, s_t))$ is the individual's history of idiosyncratic shocks up to time t, including both productivity shock z and adjustment shock s. Households can only adjust their holdings on illiquid assets at period t when $s_t = 1$, which occurs with iid probability λ. So, in each period, a randomly selected λ fraction of households can adjust their holdings of illiquid assets. When $s_t = 0$, the illiquid assets accumulate in the background:

$$d(h^t) = 0, \text{ if } s_t = 0.$$

The illiquid assets are invested in firm equity. Firms issue equity to households, the price of each share is p_t, and each share provides dividends D_t. The amount of equity held by households is given by $v(z^t) \equiv a(z^t)/p_t$. Denote $R_t^a \equiv 1 + r_t^a$ as the gross return on illiquid assets, then asset price and dividends satisfies $R_t^a = (p_t + D_t)/p_{t-1}$.

Firms. Firms own capital K_{t-1} and choose investment I_t to obtain the capital of the next period $K_t = (1-\delta)K_{t-1} + I_t$, subject to quadratic capital adjustment cost. Dividends equal capital products plus post-tax monopolistic profits net of investment, capital adjustment cost, and price adjustment cost,

$$D_t = r_t^K K_{t-1} + \alpha \Pi_t - I_t - \frac{\Psi}{2}(\frac{I_t}{K_{t-1}} - \delta^K)^2 - \Theta_t.$$

Firms choose investment to maximize $p_t + D_t$. Tobin's Q and capital evolve according to the standard Q-theory of investment:

$$\frac{I_t}{K_{t-1}} - \delta^K = \frac{1}{\Psi}(Q_t - 1),$$

$$(1 + r_{t+1}^a)Q_t = r_{t+1}^K - \frac{I_{t+1}}{K_t} - \frac{\Psi}{2}(\frac{I_{t+1}}{K_t} - \delta^K)^2 + \frac{K_{t+1}}{K_t}Q_{t+1}.$$

Equilibrium. The other sectors of the economy are the same as section 4. In the equilibrium, households and firms optimize, government budget constraint holds, nominal interest rates evolve according to the Taylor rule, and markets clear:

$$\int b(z^t) d\Phi_t(z^t) = B_t,$$

$$\int a(z^t) d\Phi_t(z^t) = p_t,$$

$$C_t + I_t + \frac{\Psi}{2}(\frac{I_t}{K_{t-1}} - \delta)^2 + \Theta_t = Y_t^{GDP}.$$

In the following, I show that proposition 1 holds with the presence of illiquid assets, and there is an aggregate Euler equation governing the return on illiquid assets given the path of aggregate consumption.

Proposition E.1. For a given monetary policy shock $\boldsymbol{\epsilon}$, there exist counterfactual transfers $\boldsymbol{\omega}$ such that:

(i) The equilibrium can be characterized with only aggregate conditions:

- Aggregate Euler equation with respect to liquid assets $(C_t^{ra})^{-\sigma} = \beta^{b,ra}(1 + r_{t+1}^{b,ra})(C_{t+1}^{ra})^{-\sigma}$, where $\beta^{b,ra} \equiv 1/(1 + r^{b,*})$;

- Aggregate Euler equation with respect to illiquid assets $(C_t^{ra})^{-\sigma} = \beta^{a,ra}(1+r_{t+1}^{a,ra})(C_{t+1}^{ra})^{-\sigma}$, where $\beta^{a,ra} \equiv 1/(1+r^{a,*})$;
- The aggregate labor supply condition, Philips curve, Q theory of investment, government budget constraint, Taylor rule, and market clearing conditions.

(ii) The individual consumption satisfies:

$$c^{ra}(h^t)/c^*(h^t) = C_t^{ra}/C^*;$$

(iii) The transfers sum to zero crosssectionally $\int \omega(h^t)d\Phi_t(h^t) = 0$.

Proof. The proof of the first-order condition (F.O.C) with respect to liquid assets is the same as the baseline model. I prove the F.O.C with respect to illiquid assets below. In the case of adjustment ($s_t = 1$), the F.O.C with respect to illiquid assets is:

$$\begin{aligned}
&(c(h^{t-1},(z_t,1)))^{-\sigma} \geq \\
&\Big\{\beta\lambda R_{t+1}^a E_z[(c(h^t,(z_{t+1},1)))^{-\sigma}|h^t, s_{t+1}=1]+ \\
&\beta^2\lambda(1-\lambda)R_{t+1}^a R_{t+2}^a E_z[(c(h^t,(z_{t+1},0),(z_{t+2},1)))^{-\sigma}|h^t, s_{t+1}=0, s_{t+2}=1]+ \\
&\beta^3\lambda(1-\lambda)^2 R_{t+1}^a R_{t+2}^a R_{t+3}^a E_z[(c(h^t,(z_{t+1},0),(z_{t+2},0)),(z_{t+3},1)))^{-\sigma}|h^t, s_{t+1}=s_{t+2}=0, s_{t+3}=1]+ \\
&\cdots\Big\}, = \text{ if } a(h^t) > 0.
\end{aligned} \tag{39}$$

Consider households save one additional unit of illiquid assets at time t; then, with probability λ, the (accumulated) one unit of illiquid assets can be used for consumption at time $t+1$, generating expected marginal utility

$$R_{t+1}^a E_z[(c(h^t,(z_{t+1},1)))^{-\sigma}|h^t, s_{t+1}=1]$$

at time $t+1$; with probability $\lambda(1-\lambda)$, the (accumulated) one unit illiquid assets can be used for consumption at time $t+2$, generating expected marginal utility

$$R_{t+1}^a R_{t+2}^a E_z[(c(h^t,(z_{t+1},0),(z_{t+2},1)))^{-\sigma}|h^t, s_{t+1}=0, s_{t+2}=1]$$

at time $t+2$; with probability $\lambda(1-\lambda)^2$, the (accumulated) one unit illiquid assets can be used for consumption at time $t+3$, generating expected marginal utility

$$R_{t+1}^a R_{t+2}^a R_{t+3}^a E_z[(c(h^t,(z_{t+1},0),(z_{t+2},0)),(z_{t+3},1)))^{-\sigma}|h^t, s_{t+1}=s_{t+2}=0, s_{t+3}=1]$$

at time $t+3$, etc.. Then the marginal value of the one additional unit of illiquid assets is the expected value of the (discounted) utility flows.

In the stationary equilibrium, the F.O.C with respect to illiquid assets

$$
\begin{aligned}
&(c^*(h^{t-1},(z_t,1)))^{-\sigma} \geq \\
&\Big\{\beta\lambda R^{a,*}E_z[(c^*(h^t,(z_{t+1},1)))^{-\sigma}|h^t,s_{t+1}=1]+ \\
&\beta^2\lambda(1-\lambda)(R^{a,*})^2E_z[(c^*(h^t,(z_{t+1},0),(z_{t+2},1)))^{-\sigma}|h^t,s_{t+1}=0,s_{t+2}=1]+ \\
&\beta^3\lambda(1-\lambda)^2(R^{a,*})^3E_z[(c^*(h^t,(z_{t+1},0),(z_{t+2},0)),(z_{t+3},1)))^{-\sigma}|h^t,s_{t+1}=s_{t+2}=0,s_{t+3}=1]+ \\
&\cdots\Big\}, = \text{ if } a^*(h^t)>0. \qquad (40)
\end{aligned}
$$

Given (40) holds, we verify the consumption allocation $\{c^{ra}(h^t)\}$ satisfies (39) with interest rate path $\{R^{a,ra}_{t+1}\}$. Subsituting $\{c^{ra}(h^t)\}$ into the F.O.C (39). First,

$$
(c^{ra}(h^{t-1},(z_t,1)))^{-\sigma}=(C^{ra}_t/C^*)^{-\sigma}(c^*(h^{t-1},(z_t,1)))^{-\sigma}, \qquad (41)
$$

And

$$
\begin{aligned}
&\beta\lambda R^{a,ra}_{t+1}E_z[(c^{ra}(h^t,(z_{t+1},1)))^{-\sigma}|h^t,s_{t+1}=1]+ \\
&\beta^2\lambda(1-\lambda)R^{a,ra}_{t+1}R^{a,ra}_{t+2}E_z[(c^{ra}(h^t,(z_{t+1},0),(z_{t+2},1)))^{-\sigma}|h^t,s_{t+1}=0,s_{t+2}=1]+ \\
&\beta^3\lambda(1-\lambda)^2R^{a,ra}_{t+1}R^{a,ra}_{t+2}R^{a,ra}_{t+3}E_z[(c^{ra}(h^t,(z_{t+1},0),(z_{t+2},0)),(z_{t+3},1)))^{-\sigma}|h^t,s_{t+1}=s_{t+2}=0,s_{t+3}=1]+ \\
&\cdots \qquad (42) \\
=&\beta\lambda R^{a,ra}_{t+1}(\frac{C^{ra}_{t+1}}{C^*})^{-\sigma}E_z[(c^*(h^t,(z_{t+1},1)))^{-\sigma}|h^t,s_{t+1}=1]+ \\
&\beta^2\lambda(1-\lambda)R^{a,ra}_{t+1}R^{a,ra}_{t+2}(\frac{C^{ra}_{t+2}}{C^*})^{-\sigma}E_z[(c^*(h^t,(z_{t+1},0),(z_{t+2},1)))^{-\sigma}|h^t,s_{t+1}=0,s_{t+2}=1]+ \\
&\beta^3\lambda(1-\lambda)^2R^{a,ra}_{t+1}R^{a,ra}_{t+2}R^{a,ra}_{t+3}(\frac{C^{ra}_{t+3}}{C^*})^{-\sigma}E_z[(c^*(h^t,(z_{t+1},0),(z_{t+2},0)),(z_{t+3},1)))^{-\sigma}|h^t,s_{t+1}=s_{t+2}=0,s_{t+3}=1]+ \\
&\cdots \qquad (43) \\
=&\beta\lambda\frac{R^{a,ra}_{t+1}}{R^{a,*}}(\frac{C^{ra}_{t+1}}{C^*})^{-\sigma}R^{a,*}E_z[(c^*(h^t,(z_{t+1},1)))^{-\sigma}|h^t,s_{t+1}=1]+ \\
&\beta^2\lambda(1-\lambda)\frac{R^{a,ra}_{t+1}}{R^{a,*}}\frac{R^{a,ra}_{t+2}}{R^{a,*}}(\frac{C^{ra}_{t+2}}{C^*})^{-\sigma}(R^{a,*})^2E_z[(c^*(h^t,(z_{t+1},0),(z_{t+2},1)))^{-\sigma}|h^t,s_{t+1}=0,s_{t+2}=1]+ \\
&\beta^3\lambda(1-\lambda)^2\frac{R^{a,ra}_{t+3}}{R^{a,*}}\frac{R^{a,ra}_{t+2}}{R^{a,*}}\frac{R^{a,ra}_{t+3}}{R^{a,*}}(\frac{C^{ra}_{t+3}}{C^*})^{-\sigma}(R^{a,*})^3E_z[(c^*(h^t,(z_{t+1},0),(z_{t+2},0)),(z_{t+3},1)))^{-\sigma}|h^t,s_{t+1}=s_{t+2}=0,s_{t+3}=1]+ \\
&\cdots. \qquad (44)
\end{aligned}
$$

Given $(C^{ra}_t)^{-\sigma}=\beta^{a,ra}R^{a,ra}_{t+1}(C^{ra}_{t+1})^{-\sigma}$, where $\beta^{a,ra}\equiv 1/R^{a,*}$, we have

$$
\begin{aligned}
\frac{R^{a,ra}_{t+1}}{R^{a,*}}(C^{ra}_{t+1})^{-\sigma}&=\beta^{a,ra}R^{a,ra}_{t+1}(C^{ra}_{t+1})^{-\sigma}=(C^{ra}_t)^{-\sigma}, \\
\frac{R^{a,ra}_{t+1}}{R^{a,*}}\frac{R^{a,ra}_{t+2}}{R^{a,*}}(C^{ra}_{t+2})^{-\sigma}=\beta^{a,ra}R^{a,ra}_{t+1}\beta^{a,ra}R^{a,ra}_{t+2}(C^{ra}_{t+2})^{-\sigma}&=\beta^{a,ra}R^{a,ra}_{t+1}(C^{ra}_{t+1})^{-\sigma}=(C^{ra}_t)^{-\sigma}, \\
\frac{R^{a,ra}_{t+1}}{R^{a,*}}\frac{R^{a,ra}_{t+2}}{R^{a,*}}\frac{R^{a,ra}_{t+2}}{R^{a,*}}(C^{ra}_{t+3})^{-\sigma}=\beta^{a,ra}R^{a,ra}_{t+1}\beta^{a,ra}R^{a,ra}_{t+2}\beta^{a,ra}R^{a,ra}_{t+3}(C^{ra}_{t+3})^{-\sigma}&=\beta^{a,ra}R^{a,ra}_{t+1}\beta^{a,ra}R^{a,ra}_{t+2}(C^{ra}_{t+2})^{-\sigma} \\
&=\beta^{a,ra}R^{a,ra}_{t+1}(C^{ra}_{t+1})^{-\sigma}=(C^{ra}_t)^{-\sigma}, \\
&\cdots.
\end{aligned}
$$

So equation (44) is simplified to

$$
\begin{aligned}
&(C_t^{ra}/C^*)^{-\sigma}\Big\{\beta\lambda R^{a,*}E_z[(c^*(h^t,(z_{t+1},1)))^{-\sigma}|h^t,s_{t+1}=1]+\\
&\beta^2\lambda(1-\lambda)(R^{a,*})^2E_z[(c^*(h^t,(z_{t+1},0),(z_{t+2},1)))^{-\sigma}|h^t,s_{t+1}=0,s_{t+2}=1]+\\
&\beta^3\lambda(1-\lambda)^2(R^{a,*})^3E_z[(c^*(h^t,(z_{t+1},0),(z_{t+2},0)),(z_{t+3},1)))^{-\sigma}|h^t,s_{t+1}=s_{t+2}=0,s_{t+3}=1]+\\
&\cdots\Big\},
\end{aligned}
\tag{45}
$$

which is the marginal value of illiquid assets in the stationary equilibrium scaled by $(C_t^{ra}/C^*)^{-\sigma}$. Combined with (41), we can see that given the F.O.C in the stationary equilibrium (40) holds, $\{c^{ra}(h^t)\}$ satisfies the F.O.C (39) with interest rate path $\{R_{t+1}^{a,ra}\}$.

An important implication of Proposition E.1 is that the liquidity premium is acyclical in the 'RANK' equilibrium. All the responses of the liquidity premium are due to redistribution effects.

Chapter 2

Inequality and Monetary Policy in a Lucas Island Model

Zheng Gong

Abstract

This paper studies how the heterogeneity in marginal propensities to earn (MPE) affects output's response to money supply shocks in a Lucas Island model. The simplicity of the Lucas Island model allows to obtain an analytical solution and solely focus on the role of MPE. I find that rich households are more responsive to money supply shocks. Compared to the benchmark case in which wealth inequality is absent, the output's response can be either amplified or dampened. Then I analyse how income cyclicity affects output's responses.

1 Introduction

The effect of inequality on aggregate demand and economy's response to aggregate shocks has drawn attention during and after Great Recession. In this paper, we analyse the economy's response to an unexpected money supply shocks in a Lucas island model, with the presence of wealth inequality and income cyclicity. The simplicity of Lucas island model allows us to obtain an analytical solution and analyse the channel through which inequality affects the economy's response. We find that households with less steady state labor supply exhibit larger response to money supply shocks. However, it is ambiguous whether the aggregate labor supply is more responsive compared to the case without inequality. And the size of output's response also depends on the size of steady state aggregate output.

In Lucas island model, monetary policy is non-neutral because households cannot observe aggregate money supply and have to infer the aggregate nominal price through their idiosyncratic nominal prices. Following a money supply shock, the increase in nominal prices will be identified partly as productivity shocks. Households will increase their labor supply as a response to the (mis-identified) productivity shocks. This mechanism is also the way inequality affects the economy's response to money supply shocks: the presence of wealth changes the way labor supply responds to productivity shocks. We find that households with less steady state labor supply are more responsive (measured as the percentage deviation from steady state) to productivity shocks. As a result, following an unexpected money expansion, wealthy households increase their labor supply more.

In terms of the output's response, the effect of inequality is unclear. Wealthy households supply less labor supply in steady state and are more responsive to money supply shocks; poor households are less responsive to money supply shocks but supply more labor in steady state. The weighted labor supply response is not monotonic and it achieves the maximum for 'median' households. As a result, the response of median households provides an upper bond of the output's response. Denote the economy's steady state output as Y_0 without the presence of wealth inequality (all households can only consume out of their labor income), and as Y^* with the presence of wealth inequality. We find that if $Y_0 < Y^*$, the response of output to money supply shocks is smaller than the benchmark case in which wealth inequality is absent. And a nature conjecture is that with the increase of wealth inequality, the response of output to money supply shocks decreases.

In the literature, papers have studied the macroeconomics effects of increasing inequality over time. Favilukis [2013] and Kaymak and Poschke [2016] study the effects of inequality on interest rates. Iacoviello [2008] studies inequality's effects on house-debt and Heathcote, Storesletten, and Violante [2010] explores the implications of rising wage inequality on welfare. In terms of steady state output, our interest is close to Athreya, Owens, and Schwartzman [2017]. Athreya et al. [2017] evaluate the output effects of inequality and emphasises the role of heterogeneity in marginal propensities to work. However, empirical evidence finds little evidence for this heterogeneity (Cesarini, Lindqvist, Notowidigdo, and Östling [2017]). In line with this evidence, Auclert and Rognlie [2018] and Bayer, Lütticke, Pham-Dao, and Tjaden [2019] shut down the channel of marginal propensities to work and focus instead on heterogeneity in marginal propensities and aggregate demand. Bayer et al. [2019] shut down the channel of elastic labor supply by using Greenwood, Hercowitz, and Huffman [1988] preferences. Auclert and Rognlie [2018] does so by labor market rationing. A more complex alternative is to microfound the inelastic labor supply by a search and matching model of the labor market such as the formulation in Ravn and Sterk [2017], Challe, Matheron, Ragot, and Rubio-Ramirez [2017] or Den Haan, Rendahl, and Riegler [2018]. Our approach is 'neo-classical' and close to Athreya et al. [2017] which features elastic labor supply. So the deciding force of output and its response to money supply shocks are the heterogeneity in marginal propensities to work.

A rapidly growing literature adds nominal rigidities to heterogeneous-agent models to account for the sizable observed heterogeneity among households and its significance for the transmission of monetary policy[1]. HANK models have several appealing features in comparison with standard Representative Agent New Keynesian (RANK) models. In RANK frameworks, consumption-saving behavior is generally closely in line with the permanent income hypothesis (Kaplan and Violante (2014); Bilbiie, 2019). This implies that the marginal propensity to consume (MPC) associated with temporary income changes is very small, a

[1]See Gornemann, Kuester, and Nakajima [2016], McKay, Nakamura, and Steinsson [2016], Kaplan, Moll, and Violante [2018], Werning [2015] and many other.

feature that is at odds with empirical estimates. In contrast, HANK models are successful at generating MPCs at the aggregate level that are closer to the values estimated from empirical data. In a HANK model, the monetary authority must rely on equilibrium feedbacks that boost household income in order to influence aggregate consumption. It is worth note that, however, the above dynamics claimed by HANK literature is difficult to illustrate in friction-less labor market. Auclert, Bardóczy, and Rognlie [2020] show that New Keynesian models with friction-less labor supply face a challenge: given standard parameters, they cannot simultaneously match plausible estimates of marginal propensities to consume (MPCs), marginal propensities to work (MPEs), and fiscal multipliers. In this paper we still focus on the heterogeneity in marginal propensities to work.

The remainder of the paper is organized as follows. Section 2 provides the model. Section 3 contains the equilibrium characterization and analysis. Section 4 discusses the effect of income cyclicity. Section 5 concludes.

2 Lucas Island Model with Inequality

This section incorporates Lucas island model (Lucas [1973]) with wealth inequality to discuss the economy's response to unexpected money supply shocks. We inherit the information friction about money aggregate in Lucas [1973]. Specifically, households face unobserveable idiosyncratic productivity shocks. To decide optimal labor supply they have to infer the size of productivity shock from nominal variables. For this purpose we keep two elements (i) households face idiosyncratic productivity shocks and aggregate money supply shock; (ii) households cannot directly observe the idiosyncratic productivity shocks and the money supply shock; instead, they only observe their own nominal wage and form expectation about the average nominal wage in the economy. With respect to inequality, we add real bonds to households' budget constraint. So households consume out of labor income and wealth.

2.1 Households

Consider an economy populated by a continuum of households who face idiosyncratic productivity shocks and have no access to financial markets. The problem solved by households i at time t is

$$\max_{C_{it},N_{it}} E[\frac{C_{it}^{1-\sigma}}{1-\sigma} - \frac{N_{it}^{1+\varphi}}{1+\varphi}] \tag{1}$$

$$s.t. \quad P_t C_{it} = W_{it} N_{it} + P_t b_i \tag{2}$$

where C_{it} and N_{it} are the consumption and labor supply of household i. W_{it} is the nominal wage of households i and specifically,

$$W_{it} = W_t e^{z_{it}} \tag{3}$$

where W_t is the aggregate nominal wage across the economy and z_{it} is the $i.i.d$ idiosyncratic shock to the labor productivity of household i drawn from $\Phi(z) \sim N(0, \sigma_z^2)$.

We assume that households cannot directly observe z_{it}. They only observe W_{it} and form expectation about W_t to infer the realization of productivity shocks. With perfect information about money aggregate which decides nominal GDP $W_t N_t$, this assumption is harmless and money supply shocks have no effects on real variables. But as we will see, if we impose incomplete information about money supply, the failure to precisely identify the productivity shocks will affect household's labor supply.

2.2 Firms

A large number of firms are assumed to operate in the economy, producing a homogeneous good. The representative firm's technology is described by the linear production function

$$Y_t = A N_t \tag{4}$$

where $N_t = \int e^{z_{it}} N_{it} di$ is the efficiency labor supply. Here we assume there are no aggregate technology shocks. The profit of the firm is

$$D_t = P_t Y_t - W_t N_t = 0 \tag{5}$$

In equilibrium $AP_t = W_t$.

2.3 Monetary policy

We assume that monetary authority controls nominal GDP $M_t = P_t Y_t = W_t N_t$ where M_t is the aggregate money supply and evolves exogenously

$$M_t = M_{t-1} e^{\epsilon_t} \tag{6}$$

where $\epsilon_t \sim N(0, \sigma_\epsilon^2)$ is an aggregate money supply shock and the money supply of last period M_{t-1} is common knowledge.

2.4 Equilibrium

The (rational expectation) equilibrium of the model is defined as

- Conditional on observing nominal wage W_{it}, labor supply are optimal for households:

$$N_{it}(W_{it}) = \text{argmax}_{N_{it}} E[U(C_{it}, N_{it})|W_{it}] \tag{7}$$

- the aggregate nominal wage W_t clears money market:

$$M_t = W_t N_t \tag{8}$$

- Expectations are rational.

3 Equilibrium Characterization

Denote variable X_t's steady state value as X^* and its percentage deviation from the steady state value as x_t.

Proposition 1. The equilibrium can be characterized as following:

$$y_t = \frac{\bar{k}\theta}{\bar{k}\theta + 1}\epsilon_t \tag{9}$$

$$w_t = m_{t-1} + \frac{1}{\bar{k}\theta + 1}\epsilon_t \tag{10}$$

where

$$\bar{k} = \int \frac{1 - \sigma\frac{Y_i^*}{C_i^*}}{\sigma\frac{Y_i^*}{C_i^*} + \varphi}\frac{Y_i^*}{Y^*}di \tag{11}$$

is the (weighted) average labor supply response to one percentage productivity shocks with the weight equaling to Y_i^*/Y^*. The signal extraction parameter θ solves

$$1 - \theta = \frac{\sigma_\epsilon^2}{\sigma_\epsilon^2 + (1 + \bar{k}\theta)^2\sigma_z^2} \tag{12}$$

Here $Y_i^* = AN_i^*$ is household i's steady state labor income, $C_i^* = Y_i^* + b_i$ is household i's steady state consumption and $Y^* = \int Y_i^* di$ is the economy's steady state aggregate output.

Proof. See Appendix

In Lucas [1973], the mechanism through which money supply shock affects real economy is households' confusion about money supply and productivity shocks. The wealth inequality will affect the economy's response to unexpected money supply shocks by its distortion on households' labor supply response to productivity shocks. To compare the output response with wealth inequality to the case without wealth inequality, we note that if $b_i = 0 \forall i$,

$$\bar{k} = k = \frac{1 - \sigma}{\sigma + \varphi} \tag{13}$$

$$Y^* = Y_0 = A^{\frac{1+\varphi}{\sigma+\varphi}} \tag{14}$$

which nails down to Lucas [1973]. Given the wealth level b_i, individual household i's labor supply response to one percentage unexpected money supply shock (which is identified as its productivity shock)

$$k_i = \frac{1-\sigma\frac{Y_i^*}{C_i^*}}{\sigma\frac{Y_i^*}{C_i^*}+\varphi} \tag{15}$$

is affected by its steady state labor income - consumption ratio. For households with a positive wealth ($b_i > 0$), the steady state labor income $Y_i^* < Y_0$ because of wealth effect and $Y_i^*/C_i^* < 1$. As a result, wealthy households with a smaller steady state labor supply are more responsive to unexpected money supply shocks: $k_i > k$.

At the first glance, the result that wealthy households are more responsive to monetary policy shocks is not intuitive because low-income workers are documented to have a higher labor income and labor supply pro-cyclicity (Patterson et al. [2019]). But if we consider the households' budget and the source of consumption in real world, this result is easier to reconcile: poor households have a larger positive fraction of non-labor income such as transfer and benefits. For rich households, the labor income - consumption ratio is more likely larger than 1 because of taxing and saving behavior. Hence in terms of the heterogeneous labor supply response to money supply shocks, the heterogeneity in labor income - consumption ratio is the deciding force. A larger wealth inequality corresponds to a larger dispersion of labor income - consumption ratio and labor supply response.

Now we turn to the (weighted) average of households' labor supply response to money supply shocks. Since households' steady state labor supply is affected by the wealth level, the average labor supply response is weighted by Y_i^*/Y^*:

$$\bar{k} = \int \frac{1-\sigma\frac{Y_i^*}{C_i^*}}{\sigma\frac{Y_i^*}{C_i^*}+\varphi}\frac{Y_i^*}{Y^*}di \tag{16}$$

In general, it is not clear whether $\bar{k}$ is greater or smaller than $k = (1-\sigma)/(\sigma+\varphi)$. But we can find the upper bound of $\bar{k}$.

Proposition 2.

$$\bar{k} = \int \frac{1-\sigma\frac{Y_i^*}{C_i^*}}{\sigma\frac{Y_i^*}{C_i^*}+\varphi}\frac{Y_i^*}{Y^*}di \leq \frac{1-\sigma}{\sigma+\varphi}\frac{Y_0}{Y^*} \tag{17}$$

Proof. See Appendix.

The implication of Proposition 2 is that if $Y^* \geq Y_0$ then $\bar{k} < (1-\sigma)/(1-\varphi)$: wealth inequality actually decreases output's response to money supply shocks[2]. The intuition behind Proposition 2 is that, the weighted individual response to money supply shocks $k_i Y_i^*/Y^*$ increases with Y_i^* for $Y_i^* < Y_0$ and decreases with Y_i^* for $Y_i^* > Y_0$. The 'median' households have greatest weighted labor supply response towards money supply shocks and provide the upper bound to compare with the case without inequality.

From proposition 2 we can find that whether wealth inequality increases or decreases output's response to money supply shocks is sensitive to model calibration. This conclusion should not be surprising as we focus on the channel of heterogeneous marginal propensities of working.

4 Cyclicity of Income Inequality

In section 3 we assume that the idiosyncratic productivity shocks are $i.i.d$ and not correlated to steady state labor income. In this section we analyse the effect of cyclicity of income inequality on output's response.

Proposition 3. If productivity shocks z_{it} are correlated with steady state labor income, the equilibrium can be characterized as following:

$$y_t = cov((k_i\theta+1)\frac{Y_i^*}{Y^*}, z_{it}) + \frac{\bar{k}\theta}{1+\bar{k}\theta}\epsilon_t \tag{18}$$

$$w_t = m_{t-1} - cov((k_i\theta+1)\frac{Y_i^*}{Y^*}, z_{it}) + \frac{1}{1+\bar{k}\theta}\epsilon_t \tag{19}$$

[2] Athreya et al. [2017] find that, for separable utility function, if leisure is a luxury good and consumption is a necessity, decreasing wealth inequality increases steady state output. This relation is also argued in Pijoan-Mas [2006]. The application to our case requires leisure to be a necessity and consumption to be a luxury good for $Y^* \geq Y_0$.

The definition of k_i, $\bar{k}$ and θ are the same with proposition 2.

Proof. See Appendix.

If the productivity shocks z_{it} are correlated with steady state labor supply Y_i^*, the correlation term will directly appear as the intercept term of output's response. Given the size of money supply shocks, the correlation between Y_i^* and z_{it} will add another dimension to the economy's response. The monotonicity of $(k_i\theta + 1)\frac{Y_i^*}{Y^*}$ depends on parameter specification, but from the proof of proposition 2 we know that $(k_i\theta+1)\frac{Y_i^*}{Y^*}$ increases with Y_i^* for $Y_i^* < Y_0$. The implication is that, at least inside the group of households with $Y_i^* < Y_0$, the positive correlation between steady state labor income and productivity shocks (pro-cyclical income inequality) will amplify output's response. The amplification of money supply shocks have two channels. The first channel is the 'stock' channel which depends on the covariance term $cov(\frac{Y_i^*}{Y^*}, z_{it})$: given the size of productivity shock, households with greater steady state labor supply have a greater contribution to aggregate output. The second channel is the 'increment' channel which depends on the covariance term $cov(k_i\frac{Y_i^*}{Y^*}, z_{it})$: given the size of productivity shock, households with a greater weighted labor supply response contribute more to aggregate output. Inside the group of households with $Y_i^* < Y_0$, both steady state labor supply and weighted labor supply response increases with steady state labor income Y_i^*.

Patterson et al. [2019] discuss a mechanism that increases the aggregate MPC - if high MPC workers' income is most affected by the aggregate shock, they will become disproportionately important in determining the response. Bilbiie [2018] terms this channel "cyclical inequality" and demonstrates that the covariance between household MPCs and earnings inequality is a sufficient statistic for whether household heterogeneity amplifies or dampens output's response. In our model, the mechanism of income inequality cyclicity affecting output is the labor market equivalent of heterogeneity in MPC from demand side. The effect of income inequality cyclicity on output arises from the 'matching' between productivity and labor supply response. Consider the 'increment' channel which depends on the covariance term $cov(k_i\frac{Y_i^*}{Y^*}, z_{it})$. Following a money supply shock, if households with greater labor supply response are realized with positive productivity shocks, the output response is larger.

5 Conclusion

We analytically analyse the effect of wealth inequality on output's response to monetary policy shocks in Lucas island model. The presence of wealth inequality distorts households' labor supply response to (mis-identified) money supply shocks by wealth effect. We find that households with less steady state labor supply are more responsive to unexpected money supply shocks. In our model, those households are wealthy households with positive wealth endowments. However, it is not clear whether the average response $\bar{k}$ is greater or smaller, compared to the response k in case without the presence of inequality. Re-scaling the labor supply response by weight, the 'median' households have greatest labor supply response. Combined with the calibration such that the presence of inequality itself increases steady state output, the average response is actually smaller: $\bar{k} \leq k$. We also analyse the effect of income cyclicity. We find that, following a money supply shock, if households with greater labor supply response are realized with positive productivity shocks, the output response is amplified.

Overall, our approach focuses on the heterogeneity in marginal propensities to work, as a labor market equivalent of the heterogeneity in marginal propensities to consumption. Future research can shut down the connection between MPC and MPE and focus on the heterogeneity in MPC by introducing labor market frictions in DSGE models.

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Appendix

Proof of Proposition 1. Households' F.O.C w.r.t labor supply is

$$E[\frac{W_{it}}{W_t}C_{it}^{-\sigma}|W_{it}] = N_{it}^{\varphi} \tag{20}$$

From the budget constraint, the percentage deviation of consumption c_{it} can be written as

$$c_{it} = \frac{C_{it} - C_i^*}{C_i^*} = \frac{Y_{it} + b_i - (Y_i^* + b_i)}{Y_i^*}\frac{Y_i^*}{C_i^*} = y_{it}\frac{Y_i^*}{C_i^*} \tag{21}$$

where y_{it} is the percentage deviation of labor income and $y_{it} = w_{it} - w_t + n_{it}$. The linearization of Households' F.O.C follows

$$w_{it} - E_{it}w_t = \sigma(w_{it} - E_{it}w_t + n_{it})\frac{Y_i^*}{C_i^*} + \varphi n_{it} \tag{22}$$

$$n_{it} = \frac{1 - \sigma\frac{Y_i^*}{C_i^*}}{\sigma\frac{Y_i^*}{C_i^*} + \varphi}(w_{it} - E_{it}w_t) = k_i(w_{it} - E_{it}w_t) \tag{23}$$

Similar to Lucas [1973], we assume households update the expectation about aggregate nominal wage w_t through rational expectation $E_{it}w_t = (1-\theta)w_{it} + \theta E_{t-1}w_t$ where θ is the signal extraction parameter to be solved in equilibrium. Substituting $E_{it}w_{it}$ into equation (23) we have

$$n_{it} = k_i\theta(w_{it} - E_{t-1}w_t) \tag{24}$$

Aggregating equation (24) over the economy (by weight) we have

$$n_t = \int n_{it}\frac{Y_i^*}{Y^*}di = \theta(w_t - E_{t-1}w_t)\int k_i\frac{Y_i^*}{Y^*}di = \bar{k}\theta(w_t - E_{t-1}w_t) \tag{25}$$

Money market clearing requires

$$m_t - w_t = n_t = \bar{k}\theta(w_t - E_{t-1}w_t) \tag{26}$$

According to rational expectation, the time t forecast error of aggregate nominal wage $w_t - E_{t-1}w_t$ is unpredictable at $t-1$: $E_{t-1}w_t = E_{t-1}m_t = m_{t-1}$. From (38) we have

$$w_t = \frac{1}{\bar{k}\theta + 1}\epsilon_t \tag{27}$$

$$y_t = n_t = \frac{\bar{k}\theta}{\bar{k}\theta + 1}\epsilon_t \tag{28}$$

Proof of Proposition 2. From the linear production technology and F.O.C w.r.t labor supply,

$$AC_i^{*-\sigma} = N_i^{*\varphi} \tag{29}$$

$$A^{1+\varphi}C_i^{*-\sigma} = Y_i^{*\varphi} \tag{30}$$

$$\frac{Y_i^*}{C_i^*} = \frac{Y_i^{*(1+\varphi/\sigma)}}{A^{(1+\varphi)/\sigma}} = (\frac{Y_i^*}{Y_0})^{1+\varphi/\sigma} \tag{31}$$

where $Y_0 = A^{\frac{1+\varphi}{\sigma+\varphi}}$. Letting $x_i = Y_i^*/Y_0$,

$$\bar{k} = \int \frac{1-\sigma\frac{Y_i^*}{C_i^*}}{\sigma\frac{Y_i^*}{C_i^*}+\varphi}\frac{Y_i^*}{Y^*}di = \frac{Y_0}{Y^*}\int \frac{1-\sigma x_i^{1+\varphi/\sigma}}{\sigma x_i^{1+\varphi/\sigma}+\varphi}x_i di \tag{32}$$

To keep the labor supply response to productivity shocks k_i positive we assume $Y_i^*/C_i^* < 1/\sigma$. Letting

$$F(x_i) = \frac{1-\sigma x_i^{1+\varphi/\sigma}}{\sigma x_i^{1+\varphi/\sigma}+\varphi}x_i \tag{33}$$

After some algebra we can show that

$$F'(x) = \frac{(1+\varphi)\varphi(1-x^{1+\varphi/\sigma})}{(\sigma x^{1+\varphi/\sigma}+\varphi)^2} \tag{34}$$

F(x) increases with $x<1$ and decreases with $x>1$. So we have

$$\bar{k} \leq \frac{Y_0}{Y^*}\frac{1-\sigma}{\sigma+\varphi} \tag{35}$$

Proof of Proposition 3. Since $cov(Y_i^*, z_{it}) \neq 0$, aggregating equation (24) we have

$$n_t = \int (n_{it} + z_{it}) \frac{Y_i^*}{Y^*} di = \int k_i\theta(w_t - E_{t-1}w_t) \frac{Y_i^*}{Y^*} di + \int (k_i\theta + 1) z_{it} \frac{Y_i^*}{Y^*} di \tag{36}$$

$$= \bar{k}\theta(w_t - E_{t-1}w_t) + cov((k_i\theta + 1)\frac{Y_i^*}{Y^*}, z_{it}) \tag{37}$$

Money market clearing requires

$$m_t - w_t = n_t = \bar{k}\theta(w_t - E_{t-1}w_t) + cov((k_i\theta + 1)\frac{Y_i^*}{Y^*}, z_{it}) \tag{38}$$

Then by guess and verify we can prove that the constant term $cov((k_i\theta + 1)\frac{Y_i^*}{Y^*}, z_{it})$ directly enters the output and wage's response function:

$$y_t = cov((k_i\theta + 1)\frac{Y_i^*}{Y^*}, z_{it}) + \frac{\bar{k}\theta}{1 + \bar{k}\theta}\epsilon_t \tag{39}$$

$$w_t = m_{t-1} - cov((k_i\theta + 1)\frac{Y_i^*}{Y^*}, z_{it}) + \frac{1}{1 + \bar{k}\theta}\epsilon_t \tag{40}$$

Chapter 3

Tacit collusion of partial cross ownership under Cournot competition

Zheng Gong*

Abstract

Partial cross ownership (PCO) among firms affects their incentives to engage in tacit collusion. We analyze collusion behavior in an n-firm industry which allows asymmetric cross ownership, under Cournot competition. We find that in some ways increasing PCO hinders tacit collusion under the traditional uniform output distribution scheme. However, this scheme is not always feasible for collusion. For a greater variety of situations, we examine different subgame perfect equilibriums and conclude that, tacit collusion can be facilitated when PCO increases.

JEL classification: G34, L41.

Keywords: cross ownership; collusion; Cournot model; passive investment; repeated game; renegotiation

*Department of Economics, Bocconi University. Email: gong.zheng@phd.unibocconi.it. This paper has greatly benefited from discussions with Dawen Meng, as well as useful comments and feedback from Jac Thomas.

1. Introduction

Passive partial cross ownership (PCO) refers to shareholdings among firms which claim a nonvoting share of profits. When a firm holds passive stakes in its rivals, it shares its rivals' profits but does not affect their operating decisions. Extensive cross-shareholdings within an industry would form a network through which firms are connected with each other. Firms would take account of the effects of its behaviors on its rivals. From the perspective of anti-trust agencies, PCO induces two effects: unilateral effects and coordinated effects. The unilateral effects refer to PCO's impacts on competitive equilibrium of markets, that is, the Nash equilibrium in stage game. The coordinated effects are concerned with the possibility for firms to engage in tacit collusion when they interact repeatedly.

One may expect that increasing cross ownership would yield greater common interests, which in turn would facilitate collusion among rivals. However, Malueg (1992) firstly showed that increasing PCO has two conflicting effects on tacit collusion by adopting a symmetric duopoly model. First, a firm is less encouraged to deviate from collusion because it receives more profits from its rivals and the deviation behavior is less profitable; Second, following the deviation, the deviant's loss from being punished would be borne more by its rivals, and so they would not punish the deviant as severely as before. These two conflicting effects make it unclear whether increasing PCO facilitates or hinders tacit collusion. Malueg (1992)'s seminal contribution brought attention to the discussion of PCO's conflicting effects. Gilo, Moshe, and Spiegel (2006) gave a thorough analysis on the first effects by adopting a Bertrand model. They proved that increasing PCO never hinders tacit collusion and gave the sufficient condition under which it facilitates collusion.

Our paper contributes to this discussion by adopting an asymmetric Cournot model. Within this setting we could separate the two conflicting effects in different types of firms: the firm which increases its stakes in one of its rivals, the firm whose stakes are held by one of its rivals, the firm who has a direct or indirect stake in the stake-increasing firm, and the firm who does not have such a stake. We firstly take Gilo et al. (2006)'s main conclusion one step further by using the notion of maverick firms (a maverick firm is a firm with the strongest incentive to deviate from collusion). We find that the investment in a maverick firm will strengthen its incentive to deviate, which thus hinders tacit collusion. This conclusion also holds if the maverick firm does not have either a direct or indirect stake in the stake-increasing firm. For the firm which does not have such a stake, the first effects do not exist and only the punishment-alleviating effects work, and so these firms have a higher inventive to deviate from collusion.

We then study the effects of PCO under different collusion strategies and subgame perfect equilibriums. We find that, under Cournot competition, the output distribution when colluding is indeed a problem which has not been noticed before. Under certain conditions, the traditional uniform output distribution is unprofitable for some participants: the collusion even lowers their profits. This arises from the asymmetric cross-ownership structure, and we introduce a simple symmetrization method. The asymmetric cross ownership represents firms' in-coordinate investments leading to different operating decisions. How should they distribute the profit when coming to a collusive agreement? We use the profit distribution in Cournot equilibrium as a benchmark and find that under feasible schemes (profitable for all participants) the effects of PCO are more definite: there exists a critical level of PCO, below which increasing PCO always facilitates collusion.

The extensive shareholdings among firms form a network in which not only the profits are shared, but also the losses would be borne together. In repeated games, PCO structure has an effect on the payoff set: firms could minmax theirs rivals to a negative profit. This means the collusive agreement can set a more threatening punishment for deviation behavior. We use a strategy with a more severe punishment phase than trigger strategy and observe the effects of PCO. Since the punishment is more severe, given the level of PCO, the incentive of firms to collude is strengthened. But with increasing PCO, we find that the tradeoff between the two conflicting effects still exists and in some ways the punishment-alleviating effect is more dominating. At the same time, the implementation of a punishment phase raises a concern of renegotiation, in which both the deviant and the punisher don't have an incentive to punish each other. To eliminate this concern, we adopt a strongly renegotiation-proof (SRP) collusion strategy in a duopoly game. In a sense, the SRP equilibrium is something like the equilibrium under Bertrand competition: the punishment does not change with increasing PCO. The loss from being punished is reduced only because the total profit at the punishment phase is distributed more uniformly. We conclude that the incentive of the firm whose stakes are held remains unchanged, and the firm who increases its stake in its rival has a stronger incentive to engage in collusion. Further, we know that the maverick firm is the firm with a smaller stake in its rival.

The unilateral effects which focus on the competitive behaviors of firms have been studied in a static model by Reynolds and Snapp (1986), Farrell and Shapiro (1990), Flath (1991, 1992), Reitman (1994), and Dietzenbacher, Smid, and Volkerink (2000). There is a main difference on the shareholding setting in these articles. Reynolds and Snapp (1986), Farrell and Shapiro (1990), and Reitman (1994) assume direct shareholdings between firms. Flath (1991, 1992) and Dietzenbacher et al. (2000) extend this by assuming indirect shareholdings. When firm A has a direct stake in firm B, and firm B has a direct stake in firm C, then firm A has an indirect stake in firm C and will share firm C's profits. This is a more general case in which all firms in an industry could be linked together without widely direct shareholdings and it raises a concern about a virtue monopoly. In this paper we also allow indirect shareholdings among n firms by which we could discuss the complexity arising from the shareholding network.

The coordinated effects have been studied by Malueg (1992), Gilo et al. (2006), Gilo, Spiegel, and Temurshoev (2009), and de Haas and Paha (2016). Besides Gilo et al. (2006), Gilo et al. (2009) examines a Bertrand model where firms have asymmetric costs and generalize conditions under which increasing PCO facilitates collusion. This is a more general case for most industries feature cost asymmetries among firms. de Haas and Paha (2016) consider a comprehensive duopoly model and conclude that PCO would destabilize collusion under a greater variety of situations indicated by the earlier literature. They introduce an antitrust authority responsible for detecting and sanctioning tacit collusion.

Our paper differs from earlier work in several ways. First, we consider an asymmetric cross shareholding in an n-firm industry. Malueg (1992) examined a symmetric duopoly model in which the incentives of two firms to collude change in sync. de Haas and Paha (2016) relax this assumption by allowing asymmetric shareholdings but what they focus on is still a duopoly game. Second, we adopt a Cournot framework. Gilo et al. (2006) and Gilo et al. (2009) eliminate the punishment-alleviating effect of PCO by considering a Bertrand model. Under Cournot competition, we could separate the two conflicting effects and analyze the situations under which certain effects dominate the other. Third, previous literature does not consider the output distribution when firms collude. In a symmetric case, uniform output distribution is a benchmark. However, with an asymmetric structure, we prove that there does not exist a fixed output distribution profitable for all firms when PCO increases. Instead, we adopt a proportional output distribution and on this basis discuss the effects of PCO under different subgame perfect equilibriums.

The paper proceeds as follows. In Section 2 we consider a general Cournot model in which n firms hold stakes of others and discuss the unilateral and coordinated effects of increasing PCO. Section 3 examines a strong renegotiation-proof collusion strategy. Section 4 is our conclusion. The Appendix consists of proofs and extensions.

2. The Model

Notations of this section are mostly in line with Gilo et al. (2006) and Dietzenbacher et al. (2000) which also consider an asymmetric PCO between n firms.

2.1. Unilateral effects

Considering an industry with n firms and each firm produces a homogeneous product. There is a passive cross ownership structure in this industry through which firms hold stakes of others and share profits of others but cannot influence other firms' decisions. A firm's quantity decision is made by its controller. This PCO structure is described by a $n * n$ direct shareholding matrix A.

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

Element a_{ij} represents firm j's stake held by firm i. We impose several restrictions on matrix A.

(i) $0 \leq a_{ij} < 1$ for all $i \neq j$, and $a_{ii} = 0$ for all i.

(ii) $c_j = 1 - \sum_{i \neq j} a_{ij} > 0$ for all j.

We assume that a firm cannot hold stake in itself so the diagonal elements of matrix A are zero. Limitation (ii) means that the stake held by firm j's outside controller is larger than zero.

The profit of firm i can be written as

$$\pi_i = \hat{\pi}_i + \sum_{k \neq i} a_{ik} \pi_k \tag{1}$$

The first term on the right-hand indicates firm i's operating income from market and the second term is firm i's sharing from other firms' profits. Let $\pi = (\pi_1, \pi_2, \cdots, \pi_n)^\mathsf{T}$ and $\hat{\pi} = (\hat{\pi}_1, \hat{\pi}_2, \cdots, \hat{\pi}_n)^\mathsf{T}$. The profit vector in the industry is

$$\pi = \hat{\pi} + A\pi \tag{2}$$

In input-output analysis, matrix $I - A$ is actually the Leontief matrix. With limitations on matrix A, this equation has a unique solution (see, Takayama (1985))

$$\pi = (I - A)^{-1}\hat{\pi} = B\hat{\pi} \tag{3}$$

Matrix B is the inverse Leontief matrix connecting final profit of each firm in the industry with all firms' operating income. We can also see the effects of indirect shareholding from the equation: a change of an element in matrix A is likely to cause the whole matrix B's change.

When firm i changes its direct stake in firm j, all those who have a direct stake in firm i will be affected with their indirect shareholding of firm j by firm i's conduction. Likewise those having a direct stake in firms who are affected in the first round will be affected in the next round with their indirect shareholding of firm j. Considering firm j has direct or indirect stakes in other firms, the shareholding within the industry will be affected and the profit distribution pattern changes.

Before we turn to the effects of PCO on Cournot equilibrium, we make assumptions about the production cost and the inverse demand function $p(Q)$:

Assumption 1. N firms produce a homogeneous product at a constant marginal cost which is taken to be zero.

Assumption 2. $\hat{\pi}_i = p(Q)q_i$ is concave $\forall i$.

Assumption 2 ensures that $\hat{\pi}_i = p(Q)q_i$ has a unique global maximizer. Denote as b_{ij} the ith row and jth column element of matrix B. The profit of firm i is $\pi_i = (B\hat{\pi})_i = \sum_{k=1}^{n} b_{ik}\hat{\pi}_k = \sum_{k=1}^{n} b_{ik}q_k p(Q)$. The F.O.C of the profit maximization problem is

$$\frac{\partial \pi_i}{\partial q_i} = p'(Q)\sum_{k=1}^{n} b_{ik}q_k + b_{ii}p(Q) = 0 \tag{4}$$

Under trigger strategy, if anyone deviates from collusion, the collusion breaks up and reverses to Cournot equilibrium, which is the Nash equilibrium in stage game. We firstly discuss the effects of increasing PCO on Cournot equilibrium. Considering the PCO matrix A increases to A' that firm r increases its stake in firm s, i.e., $a'_{rs} = a_{rs} + w$. And the matrix A' still conforms to the restrictions we impose on PCO matrix. We are interested in the change of the total output, and further, the bound of it.

Proposition 1. When PCO increases, that is, there exist a firm increasing its stake in another firm, the total output Q^{co} decreases monotonously within the range between the monopoly output Q^m and the classic Cournot output Q^{cn} without PCO:

(i) $\frac{\partial Q^{co}}{\partial a_{rs}} < 0$, for $r \neq s$

(ii) $Q^m < Q^{co} \leq Q^{cn}$.

Proof. See Appendix.

This conclusion is simple and accords with the intuition. The upper bound is obtained with the absence of PCO where matrix A is a zero matrix and matrix B is an identity matrix with $c_i = 1, b_{ii} = 1$ for all i. This is a classic Cournot model with n firms. The lower bound, with the restrictions imposed on matrix A, is obtained when $a_{ij} \to \frac{1}{n-1}$ for all $i \neq j$. If all stakes of a firm are uniformly held by its rivals, the industry is essentially an oligopoly market. The profit function of firm i is $\pi_i = \sum_{k=1}^{n} b_{ik}\hat{\pi}_k$, from the proof of proposition 1 we know $b_{ii} > b_{ik}$ for all $k \neq i$, the weight on firm i's operating income $\hat{\pi}_i$ is largest. Firm i always exerts externalities on other firms when maximizing its final profit. With increasing PCO, the weights of other firms' operating incomes increase and firm i tends to reduce its own output. When it comes to the limiting case where firm i has equivalent weight on each firm's operating income ($b_{ii} = b_{ik}$ for all k), its profit is in accordance with the industry's total profit.

We also obtain some useful implications from proposition 1 for further analysis.

Corollary 1.

(i) The output distribution is $\frac{q^{co}}{Q^{co}} = \frac{B^{-1}diag(B)}{\sum_i c_i b_{ii}}$, where $q^{co} = (q_1^{co}, q_2^{co}, \cdots, q_n^{co})^{\mathsf{T}}$.

(ii) The profit share of firm i is $\frac{\pi_i^{co}}{\Pi^{co}} = \frac{b_{ii}}{\sum_i c_i b_{ii}}$ and the profit share of controller i is $\frac{\tilde{\pi}_i^{co}}{\Pi^{co}} = \frac{c_i b_{ii}}{\sum_i c_i b_{ii}}$.

Proof. See Appendix.

Dietzenbacher et al. (2000) points out the calculated output maybe negative without constraints. This happens on firm i if the *ith* element of $B^{-1}diag(B)$ is negative. When firm i holds large stakes in its rivals unilaterally, it will find it more profitable to halt production. We think that this is more common in parent-subsidiary firms in which the operating decisions are not independent. In following analysis we assume the passive investment among rivals is mutual or the wide-range unilateral passive investment is relatively small.

Corollary 1(ii) describe the profit distribution pattern of the industry which will be useful in the discussion of coordinated effects. With the presence of PCO, the accounting profits of firms are overstated: $c_i \leq 1$ so $\sum_i \frac{\pi_i^{co}}{\Pi^{co}} \geq 1$. This is because when firms hold stakes reciprocally, their operating incomes will be calculated repeatedly. But this does not happen when we aggregate the total profits of the outside controllers: $\sum_i \frac{\tilde{\pi}_i^{co}}{\Pi^{co}} = 1$. Also, we can see a free rider problem from this pattern. When there exists firm r increasing its stake in s, the total output decreases and total profits increases. From the proof of proposition 1 we can see the profit share $\frac{c_i b_{ii}}{\sum_i c_i b_{ii}}$ increases for all controllers except controller s. Other than the controllers whose firm has a direct or indirect shareholding in firm r, bystander controllers also benefit from increasing PCO.

2.2. *Coordinated effects*

In this section we discuss the effects which PCO has on the possibility for firms to engage in tacit collusion when they interact repeatedly. We know that under PCO firms produce q^{co} is a Nash equilibrium in stage game (we call it PCO competition). When firms collude, they choose an output Q^m (assumed as the fully monopoly output when not specified) and an output distribution scheme q^m specifying the output of each firm. Profit distribution among firms depends both on the output distribution scheme and the PCO structure. Collusion adopts a trigger strategy that if there exists firms deviating, all firms return to stage Nash equilibrium.

2.2.1. *Uniform output distribution*

First we assume a uniform output distribution, i.e., each firm produces a Q^m/n in the collusion phase. Considering the incentive for firms to deviate when they engage in collusion. The profit of firm i under collusion is $\pi_i^m = \sum_k b_{ik} \frac{Q^m}{n} p(Q^m)$. The profit of deviating is π_i^d, that is, given other firms' outputs as Q^m/n , maximizing one period profit

$$\pi_i^d = max_{q_i}(b_{ii}q_i + \sum_{k \neq i} b_{ik} \frac{Q^m}{n})p(q_i + \frac{n-1}{n}Q^m) \tag{5}$$

After that, the collusion breaks down and all firms return to produce q^{co} and firm i obtain π_i^{co}. The collusion can sustained as a subgame perfect equilibrium if and only if

$$\frac{\pi_i^m}{1-\delta} \geq \pi_i^d + \frac{\delta}{1-\delta}\pi_i^{co} \qquad \forall\, i \tag{6}$$

Let $\delta_i^c = \frac{\pi_i^d - \pi_i^m}{\pi_i^d - \pi_i^{co}}$, this is equivalent to

$$\delta \geq \delta^c \equiv max\{\delta_i^c\} \tag{7}$$

When the discount factor δ is no less than firm i's critical discount factor δ_i^c, firm i has no incentive to deviate. The collusion can be sustained when all firms don't deviate. This needs $\delta \geq \delta_c$, which is the supremum of $\{\delta_i^c\}$. We measure the effects of PCO on tacit collusion from the perspective that how the possibility of $\delta \geq \delta_c$ changes when PCO increases. For example, if PCO increases and δ_c decreases, the possibility of $\delta \geq \delta_c$ increases, we say the increasing PCO facilitates tacit collusion.

Generally speaking, the effects of increasing PCO can be decomposed into two conflicting effects. When a firm receives more profit from its rivals and transfers more of its profit to them, the benefit from deviation $\pi_i^d - \pi_i^m$ decreases. On the other hand, increasing PCO affects the loss from being punished $\pi_i^d - \pi_i^{co}$. As proposition 1 proves, with increasing PCO, the stage Nash equilibrium Q^{co} decreases towards monopoly output which means market competition weakens. This will soften the punishment after deviation and the loss from being punished also decreases. Decreasing the benefit of deviation facilitates collusion but softening the punishment hinders it. The net effects depend on which one dominates another.

Theorem 1. Assume firm r increases its stake in firm s, for (i) firm who neither has a direct nor indirect stake in firm r (ii) firm s, the critical discount factor δ_i^c increases.
Proof. See Appendix

The complexity of PCO structure stems from the profit transmission among firms. If firm i has a stake in firm j, firm j has a stake in k, though firm i does not have a direct stake in k, it has an indirect stake in firm k and shares k's profit. And if firm k has a stake in i, these firms form a circle and each member will affect others. When PCO increases, a firm's profit and incentive to collude will be affected in many ways. In this complex pattern, however, we find that for firm s as well as these firms who does not have a stake, direct or indirect, in firm r, their critical discount factors have a definite increase. For these firms, increasing PCO does not affect the profit from deviating and the profit of collusion. But market competition weakens for all firms which means the punishment for deviating behavior alleviates. Thus they have a higher incentive to deviate.

For firm r or the firms who have a direct or indirect stake in firm r, situation is more complicated. The punishment alleviates but their deviating profits also decreases. The net effects depends on the PCO level as well as the market demand.

We use the concept of maverick firms to refer to firms who have a higher incentive to deviate from collusion, that is, the critical discount factor of a maverick firm δ_m^c is highest among all firms. This concept is also used in Gilo et al. (2006), for detailed discussions, see Baker (2002). A simple conclusion derive from theorem 1.

Proposition 2. Assume firm r increase its stake in firm s. If the maverick firm does not have a direct nor indirect stake in firm r, or firm s is the maverick firm, increasing PCO hinders tacit collusion.

Proposition 2 provide evidence for the lenient approach of antitrust authorities towards the passive investment in the maverick firm. Gilo et al. (2006) proves that a passive investment in the maverick firm does not facilitate tacit collusion, which contradicts popular belief. We carry this conclusion forward: not only does this investment not facilitate tacit collusion, but it hinders tacit collusion.

2.2.2. *Proportional output distribution*

A necessary condition to realize the collusion is that the collusion is profitable for all participants. If we adopt a uniform output distribution, this means $\sum_k b_{ik}\frac{Q^m}{n}p(Q^m) \geq \frac{b_{ii}}{\sum_i c_i b_{ii}}\Pi^{co}, \forall i$. However, this condition does not always hold.

Example. Assume $n = 2$, $p = 1 - Q$, $Q \in (0, 1)$. Under uniform output distribution the collusion profit $\pi_i^m = \sum_k b_{ik}\frac{Q^m}{2}p(Q^m)$. The profit of PCO competition is $\pi_i^{co} = \frac{b_{ii}}{\sum_i c_i b_{ii}}Q^{co}p(Q^{co})$. By Calculating we have $Q^m = 1/2$ and $Q^{co} = \frac{\sum_i c_i b_{ii}}{1+\sum_i c_i b_{ii}}$. Then $\frac{\pi_1^{co}}{\pi_1^m} = \frac{8(1-a_{12}a_{21})^2}{(3-a_{12}a_{21}-a_{12}-a_{21})^2(1+a_{12})}$. With $a_{12} = 0$, $\frac{\pi_1^{co}}{\pi_1^m} = \frac{8}{(3-a_{21})^2}$. If $a_{21} > 3 - 2\sqrt{2} \approx 0.18$, $\frac{\pi_1^{co}}{\pi_1^m} > 1$. The collusion is unprofitable for firm 1.

The intuition behind this example is shown here. Though the industry's total profit when colluding increases, the share of some firms may decrease if the collusion allocates equal output among firms. The profit of firm i under PCO competition can be written as $\pi_i^{co} = \frac{b_{ii}}{\sum_i c_i b_{ii}}\Pi^{co}$. The profit under uniform output collusion can be written as $\pi_i^m = \sum_k b_{ik}\frac{Q^m}{n}p(Q^m) = \frac{\sum_k b_{ik}}{n}\Pi^m$. Considering a firm whose stake is held by others and doesn't hold other firms' stake ($b_{ii} = 1$ and $b_{ik} = 0$ for all $i \neq k$). $\Pi^{co} < \Pi^m$, but $\frac{1}{\sum_i c_i b_{ii}} > \frac{1}{n}$. Firms who don't hold others' stake intend to produce more than others and the uniform output distribution cuts down their outputs in a large scale. This will have a significant negative effect on the their profits because a firm's profit function has the largest weight on its own operating income. One way to avoid this is to allocate the same share of total profit to each firm it accounts for under PCO competition.

Definition 1. Under trigger strategy, an output distribution scheme q^m is feasible for collusion if $\pi_i(q^m) \geq \pi_i^{co}, \forall i$.

Theorem 2. The proportional output distribution scheme is feasible for all possible PCO structures:

$$\frac{q^m}{Q^m} = \frac{q^{co}}{Q^{co}} = \frac{B^{-1}diag(B)}{\sum_i c_i b_{ii}} \tag{8}$$

Proof. See Appendix

Earlier literature did not consider the output distribution when firms collude. Malueg (1992) consider a symmetric cross shareholding in which uniform output distribution satisfies equation (8) . Gilo et al. (2006) discuss a Bertrand model in which the payoff of Nash equilibrium in stage game is zero and arbitrary positive collusive profit is profitable. de Haas and Paha (2016) discuss an asymmetric duopoly Cournot model and ignore this possibility.

If we consider the collusion profit distribution as a bargaining problem, the objective is to distribute the extra profit from collusion $\Pi^m - \Pi^{co}$ among n controllers. The uniform output distribution is unfeasible under certain condition because some controllers obtain a negative part. A solution is the Nash bargaining solution which distributes the extra profit uniformly:

$$\tilde{\pi}_i^m = c_i \sum_k b_{ik} q_k^m p(Q^m) = \frac{c_i b_{ii}}{\sum_i c_i b_{ii}} \Pi^{co} + \frac{1}{n}(\Pi^m - \Pi^{co}) \tag{9}$$

The idea is the same as theorem 2: when PCO increases, the output distribution changes synchronously. There does not exist a fixed scheme (distributing to each firm a fixed output) feasible for all possible PCO structures. With changing schemes, it is difficult to separate and discuss the effects of PCO on collusion: the output distribution also affects the profits and incentives to collude. In theorem 2 each controller obtains $\frac{c_i b_{ii}}{\sum_i c_i b_{ii}}(\Pi^m - \Pi^{co})$ and this scheme is 'fixed' relative to the PCO structure: it distributes to a firm the output share it accounts for under PCO competition thus ensures that the collusion profit share is the same as PCO competition.

We think that, the necessity to distribute the total profit proportionally lies essentially on the investment nature of shareholding: it represents a firm's investment and a claim of the industry's profit. Firms holding more stakes receive larger shares of the total profit and firms holding fewer stakes receive fewer shares. Thus the profit share pattern in stage Nash equilibrium is a benchmark for monopoly profit distribution. This pattern does not change with an external impact such as a demand shock or an overall cost arising. From the perspective of anti-trust agencies, an output distribution different from that of PCO competition is abnormal. In the following section we discuss the effects of PCO under this scheme. We also give a basic analysis of the scheme in equation (9) in Appendix.

Consider the incentive for firm i to deviate when they engage in collusion. The deviating profit

$$\pi_i^d = max_{q_i}(b_{ii} q_i + \sum_{k \neq i} b_{ik} q_k^m) p(q_i + \sum_{k \neq i} q_k^m) \tag{10}$$

and $\delta_i^c = \frac{\pi_i^d - \pi_i^m}{\pi_i^d - \pi_i^{co}}$. Letting $\alpha = \frac{1}{\sum_i c_i b_{ii}}$.

Theorem 3. The critical discount factor of firm i δ_i^c is identical through all firms, we have the expression for δ^c

$$\delta^c(\alpha) = max\{\delta_i^c\} = \frac{(Q^d - (1-\alpha)Q^m)p(Q^d) - \alpha \Pi^m}{(Q^d - (1-\alpha)Q^m)p(Q^d) - \alpha \Pi^{co}} \tag{11}$$

where $Q^d = Q^d(\alpha)$ satisfies $Q^d + \frac{p(Q^d)}{p'(Q^d)} = Q^m(1-\alpha)$ and $Q^{co} = Q^{co}(\alpha)$ satisfies $\alpha Q^{co} + \frac{p(Q^{co})}{p'(Q^{co})} = 0$.

Proof. See Appendix

This conclusion is surprising and will simplify our analysis. Generally, different firms have different levels of critical discount factor because of their nonidentical stakes in other firms, like the Bertrand model discussed by Gilo et al. (2006) and uniform output distribution discussed above. But with a proportional output distribution, the incentive to deviate is identical through the industry. We need only to pay attention to how this critical discount factor δ^c is affected by the level of PCO. And what makes the pattern clearer is that δ^c is a function of $\sum_i c_i b_{ii}$, which changes monotonously when PCO matrix A increases (see lemma A1). $\alpha = \frac{1}{\sum_i c_i b_{ii}}$ an 'indicator ' of the level of PCO. When PCO increases, α increases. It describes the extent to which these firms are linked together, or, the market concentration.

Proposition 3. There exists a critical value $\hat{\alpha}$ such that, if $1/n < \hat{\alpha}$, for $\alpha \in (1/n, \hat{\alpha}]$, increasing PCO facilitates collusion.
Proof. See Appendix.

Proposition 3 establishes sufficient condition under which increasing PCO surly facilitates tacit collusion. We use a family of demand functions introduced by Maleug (1992) to show the effects of increasing PCO on tacit collusion.

Example. Assuming the inverse demand function is $p(Q) = (1 - Q)^x$ $(x > 0)$. Price equals zero if output exceeds 1. The critical discount factor δ^c is

$$\delta^c = \frac{(\frac{\alpha + x}{1 + x})^{1+x} - \alpha}{(\frac{\alpha + x}{1 + x})^{1+x} - (\frac{\alpha + \alpha x}{1 + \alpha x})^{1+x}} \tag{12}$$

Since $\alpha = \frac{1}{\sum_i c_i b_{ii}}$, $\alpha \in [\frac{1}{n}, 1)$.

The inverse demand function is concave for $x \leq 1$ and strictly convex for $x > 1$. Figure 1 plots δ^c as a function of α for different values of x when $n = 2$ (there are two firms). This is a extreme case like the symmetric direct cross ownership discussed by Malueg (1992). From the figure we can find that when $x \leq 1$, δ^c decreases with α so increasing PCO facilitates tacit collusion. However, when $x > 1$, the effects of increasing PCO are ambiguous. There exists an interval in which δ^c increases with α so increasing PCO on the contrary hinders collusion. What's special in the $n = 2$ case is that α has already been large thus the market concentration is high from the beginning. The condition of $1/n < \hat{\alpha}$ in Proposition 3 can't be met.

Figure 2 plots more general cases in which $n > 2$ ($n = 5$ and $n = 20$). It should be noted

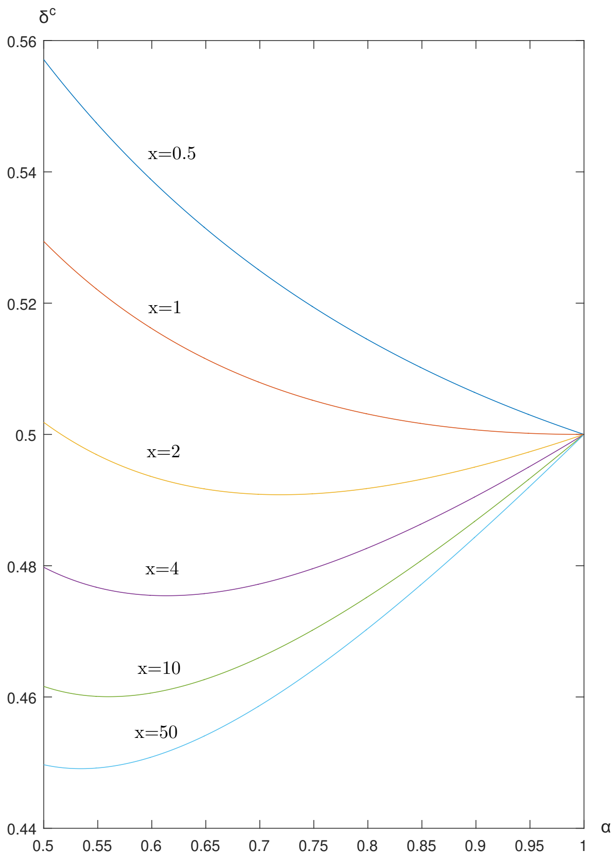

Fig. 1. Critical discount factor δ^c for different values of x when $n = 2$

that the figure of small n case is a part of the figure of large n case for the lower bound of α equals $1/n$. When there are more firms engaging in tacit collusion, before α reaches $\hat{\alpha}$, increasing PCO always facilitates collusion whatever the market demand is. And if we focus on the relative small shareholding in PCO, this conclusion applies in a wide range. This contradicts the main opinions of Malueg (1992) and de Haas and Paha (2016) in which the $n = 2$ case is discussed.

3. Renegotiation-proofness

From the perspective of repeated game, we are interested in the effects of PCO on the payoff set. The folk theorem asserts that every feasible and strictly individually rational payoff is the payoff of some subgame perfect equilibrium of the repeated game with sufficiently large discount factor δ. In our case, the key difference is that with the presence of PCO, the minmax payoffs of players may not be normalized to zero and there exist negative feasible and individually rational payoffs. To see this we assume a duopoly game and two firms produce

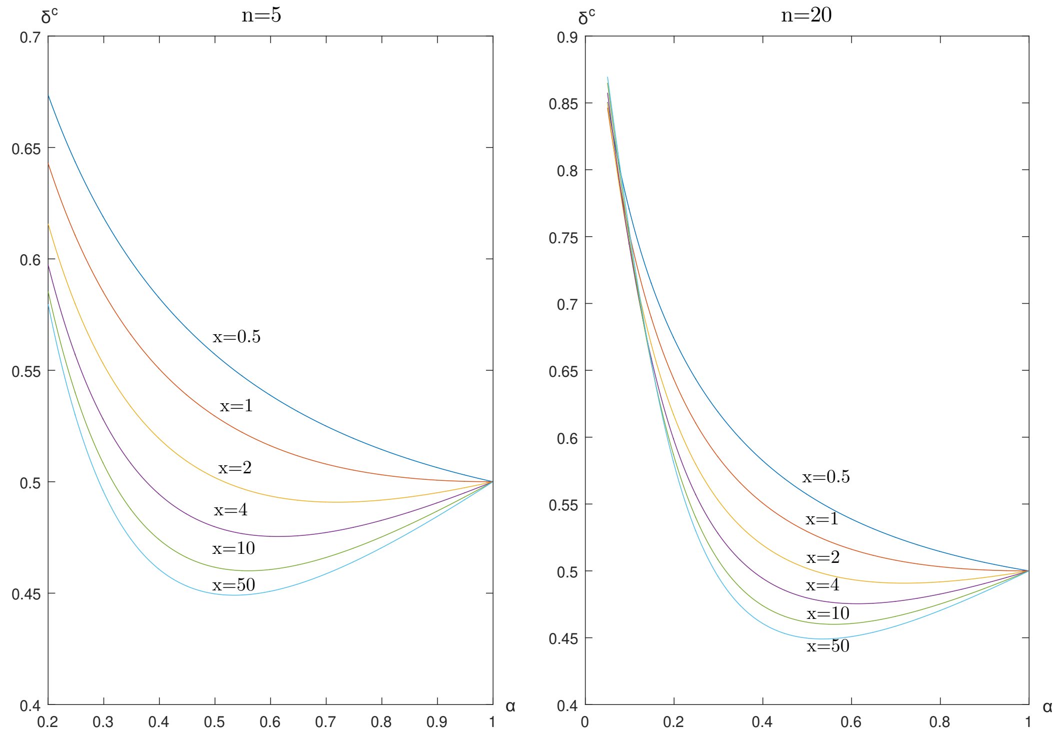

Fig. 2. Critical discount factor δ^c for different values of x when $n = 5$ and $n = 20$

in a constant marginal cost c which is not zero. The profit of firm i is $\pi_i = b_{ii}\tilde{\pi}_i + b_{ij}\tilde{\pi}_j$. Consider the minmax process, if firm j produce a large output that the price is smaller than cost c, the best response of firm i is to halt production and its payoff is $\pi_i = b_{ij}\tilde{\pi}_j$, which is negative. If we impose some restrictions on the production, such as the capacity in single period $\bar{Q}$ ($p(\bar{Q}) < c$), the minmax payoff of firm i is $\underline{\pi}_i = b_{ij}(p(\bar{Q}) - c)\bar{Q}$. Similarly, the minmax payoff of firm j is $\underline{\pi}_j = b_{ji}(p(\bar{Q}) - c)\bar{Q}$. When firms hold stakes of others, they give their rival the right to hurt them because they have to bear the losses together. With a large enough discount factor, the equilibrium in which firms obtain negative profits can be achieved.

This inspires us to construct a lower critical discount factor using a strategy different from trigger strategy. If the punishment is more severe than stage Nash equilibrium, the collusion can be sustained with a lower discount factor, i.e., can be sustained more easily. We are interested in the effects of PCO under this new strategy: if firms have the ability to threaten each other, does this threaten become more effective for preventing deviation with increasing PCO? In Appendix we adopt a stick-and-carrot strategy and proves that, similar to proposition 3, there exists a critical value $\hat{\alpha}$ and if $\alpha < \hat{\alpha}$, increasing PCO facilitates tacit collusion. However, this strategy has some constraints, such as the concavity of the inverse demand function and the strong symmetry of actions, which limits its application.

Another concern of the implemention of a more severe punishment is about renegotiation (this criticism also applies to the trigger strategy): since the punishment is both harmful to the deviant and the punisher, they have an incentive to renegotiate. In this section, we adopt a strong renegotiation-proof (SRP) strategy to eliminate this concern and discuss the effects of increasing PCO under this strategy.

Consider a duopoly game, the SRP strategy has a simple structure: assume firm i deviates from collusion, then in punishment phase firm i produces a zero output, and firm j produces the monopoly output Q^m, after which both return to collusion. If firm i deviates from punishment, the punishment restarts. If firm j deviates from punishing firm i, both switch to punish firm j, i.e., firm i produces Q^m and firm j produces a zero output. Denote the punishment action pair for i ($q_i = 0, q_j = Q^m$) as q^i. In single period, this is the most severe SRP punishment can be imposed to the deviant. Extending the length of punishment can make the punishment more severe but does not change our conclusion.

Assumption 3. $\pi_i^d(q^i) < \pi_i^m$

This assumption means the deviating profit from a zero output punishment is smaller than collusion profit thus ensures the SRP equilibrium exists. This strategy is subgame perfect:

$$\pi_i^d(q^m) + \delta\pi_i(q^i) \leq \pi_i(q^m) + \delta\pi_i(q^m) \tag{13}$$

$$\pi_i^d(q^i) + \delta\pi_i(q^i) \leq \pi_i(q^i) + \delta\pi_i(q^m) \tag{14}$$

$$\pi_j^d(q^i) + \delta\pi_j(q^j) \leq \pi_j(q^i) + \delta\pi_j(q^m) \tag{15}$$

These conditions also apply to firm j. Equation (13) means deviating from collusion is not profitable, (14) means for firm i deviating from punishing itself is not profitable, (15) means for firm j deviating from punishing firm i is not profitable.

Theorem 4.

(i) The firm who is held more stakes has a stronger incentive to engage in collusion. Specifically, if firm j increases its stake in firm i, firm i's critical discount factor δ_i^c decreases.

(ii) Assume the collusion distributes fixed outputs between firms, if firm j increases its stake in firm i, then δ_j^c decreases, δ_i^c remains unchanged.

Proof. See Appendix

Theorem 4 establish basic conclusions about the effects of PCO if collusion adopts a SRP strategy. The effects on the firm which is being held more stakes are clear: its incentive to engage in collusion is strengthen. The effects on the firm who increases stakes in the other are not that clear and need more specific assumptions. In the proof of theorem 4 we give a sufficient condition under which δ_i^c decreases if firm i increases its stake in firm j. And if this condition is satisfied, both the firm who is held more stakes and the firm who increases its stakes have a stronger incentive to engage in collusion, and the collusion is facilitated.

If we drop theorem 2 and adopt a fixed output distribution such as a uniform distribution (suppose the collusion is profitable for both firms, this means the PCO structure is not too asymmetric), the effects of increasing PCO are definite. The firm who increases its stake in the other firm has a stronger incentive to engage in collusion, and the other firm's incentive remains unaffected.

Proposition 4.

(i) A symmetric increasing PCO ($a_{ij} = a_{ji}$) facilitates tacit collusion.

(ii) If the collusion distributes equal output between firms, the firm with a fewer stake in its rival is the maverick firm. And increasing PCO launched by the maverick firm will facilitate tacit collusion. Otherwise the tacit collusion is not affected.

This conclusion is in line with the property of SRP strategy: the punishment output Q^m does not change with increasing PCO. The loss from being punished decreases only because the total profit in punishment phase is distributed more uniform between the two firms. Therefore the punishment-alleviating effects of increasing PCO are not dominating. The profit from deviating decreases more than the loss from being punished, firms are not willing to deviate from collusion. Similar to proposition 2, the investment in the maverick does not strengthen the maverick firm's incentive to collude. On the contrary, the investment by the maverick firm should raise more concern.

4. Conclusion

We discuss the unilateral effects and coordinated effects of increasing PCO. The value of α is a simple indicator describing the PCO level. The increasing PCO affects the monopoly behavior of firms in both ways. When PCO increases, the stage Nash equilibrium changes towards perfectly monopoly. Firms are linked more closely with each other and the market concentration increases. Like the classic analysis of Cournot equilibrium, if the number of firms n increases, the market behaves more like a perfectly competitive market, and if n decreases, the market changes towards a perfectly monopoly market. The PCO structure among n firms provide a way achieving the equilibrium of a market whose number of firms is fewer than n.

	Trigger strategy	Stick-and-carrot strategy	SRP strategy
Uniform distribution	Proposition 2	NA	Proposition 4
Proportional distribution	Proposition 3	Proposition A1	Theorem 4(i)

Table 1: The effects of increasing PCO in different situations

For coordinated effects, we give a more detailed analysis and establish a general framework analyzing situations with more than two firms under Cournot competition. In general, the coordinated effects are not definite and depend on the PCO level as well as the collusion strategy. The traditional uniform output distribution can in some ways hinders tacit collusion. However, the uniform output distribution is not always feasible. If we consider some other output distribution schemes and collusion strategies, we find that in most cases increasing PCO facilitates collusion. From the perspective of anti-trust agencies, these findings may have a constructive implication on the regulation of partial cross ownership. Increasing PCO not only decreases the total output and drives the market towards a perfectly monopoly market in stage game, but also in a wide range extends the scope in which the tacit collusion can be sustained when firms interact repeatedly.

It would be useful to drop the homogeneous-firm assumption in future study. With asymmetric costs or differentiated products, it is possible for firms to achieve a strategic collusive agreement in which the unequal collusion profit distribution would be an equilibrium. This would be also helpful to endogenize the PCO structure. Throughout this paper we discuss the effects of PCO as it is exogenously given. This does not explain the incentive of firms to invest in its rivals. With an equilibrium achieved by the collusive agreement, firms would decide whether to acquire a passive stake in its rivals and engage in tacit collusion.

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Appendix A. Proofs

Proof of Proposition 1. Writing the F.O.C equation for all firms in matrix form,

$$p'(Q)Bq + diag(B)p(Q) = 0 \tag{A.1}$$

where $q = (q_1, q_2, \cdots, q_n)^\intercal$ and $diag(B)$ represents the column vector with diagonal elements of matrix B. Left multiplied by $\mathbb{1}^\intercal B^{-1}$, we get

$$p'(Q)Q + \mathbb{1}^\intercal B^{-1} diag(B)p(Q) = 0 \tag{A.2}$$

Noticing that $\mathbb{1}^\intercal B^{-1} diag(B) = \sum_i c_i b_{ii}$. Indicate Q^{co} as the total output under PCO, Equation (A.1) simplifies to

$$p'(Q^{co})Q^{co} + \sum_{i=1}^{n} c_i b_{ii} p(Q^{co}) = 0 \tag{A.3}$$

Use Q^{cn} to denote the total output under classic Cournot competition without PCO and Q^m the perfectly monopoly output. Q^{cn} and Q^m satisfies[1]

$$p'(Q^{cn})Q^{cn} + np(Q^{cn}) = 0 \tag{A.4}$$

$$p'(Q^m)Q^m + p(Q^m) = 0 \tag{A.5}$$

Compare equation (A.3) and equation (A.4)(A.5), it's easy to find that the total output Q^{co} relative to Q^{cn} and Q^m depends on $\sum_i c_i b_{ii}$. We have lemma A1 for $\sum_i c_i b_{ii}$.

Lemma A1. $\sum_i c_i b_{ii}$ has following properties:

(i) $\frac{\partial \sum_i c_i b_{ii}}{\partial a_{rs}} < 0$, for $r \neq s$

(ii) $1 < \sum_i c_i b_{ii} \leq n$.

Proof of Lemma A1 (i) Let matrix *A'* differs from matrix A in the rsth element that $a'_{rs} = a_{rs} + w$. Write this in matrix form, $A' = A + w e_r e_s^\intercal$, where $e_r = (0, 0, \cdots, 1, 0, \cdots, 0)^\intercal$, the rth element is 1. According to *Sherman-Morrison formula*, $B' = (I - A')^{-1} = B + \frac{w}{1 - w b_{sr}} B e_r e_s^\intercal B$, thus $b'_{ij} = b_{ij} + \frac{w}{1 - w b_{sr}} b_{ir} b_{sj}$. Suppose $f(A) = \sum_i c_i b_{ii}$, then

$$\frac{\partial f(A)}{\partial a_{rs}} = \lim_{w \to 0} \frac{\sum_i c'_i b'_{ii} - \sum_i c_i b_{ii}}{w}$$

[1] Without PCO, matrix A is a zero matrix and $c_i = 1$, $b_{ii} = 1$ for all i, $\sum_i c_i b_{ii} = n$.

For $i \neq s$, $c'_i = c_i$, $b'_{ii} = b_{ii} + \frac{w}{1-wb_{sr}} b_{ir} b_{si}$. For $i = s$, $c_s = c'_s - w$, $b'_{ss} = \frac{b_{ss}}{1-wb_{sr}}$. We quote two useful properties of matrix B from Gilo et al. (2006), see also Dietzenbacher et al. (2000):

(i) $b_{ii} \geq 1$ for all i, and $0 \leq b_{ij} < b_{ii}$ for all $j \neq i$.

(ii) $\sum_{j=1}^{n} c_j b_{ji} = 1$ for i.

Then

$$\begin{aligned}
\frac{\partial f(A)}{\partial a_{rs}} &= \lim_{w\to 0} \frac{\sum_{i\neq s} c'_i b'_{ii} - \sum_{i\neq s} c_i b_{ii}}{w} + \lim_{w\to 0} \frac{c'_s b'_{ss} - c_s b_{ss}}{w} \\
&= \sum_{i\neq s} c_i b_{ir} b_{si} + b_{ss}(c_s b_{sr} - 1) \\
&< b_{ss} \sum_{i\neq s} c_i b_{ir} + b_{ss}(c_s b_{sr} - 1) \\
&= b_{ss}(\sum_i c_i b_{ir} - 1) \\
&= 0
\end{aligned}$$

(ii) When matrix A is a zero matrix, $\sum_i c_i b_{ii} = n$. This is the upper bound of $\sum_i c_i b_{ii}$. The lower bound is obtained by setting $a_{ij} \to \frac{1}{n-1}, \forall i \neq j$. Assume $a_{ij} = a\,(i \neq j)$, then $c_i = 1-(n-1)a$, $b_{ii} = \frac{(n-2)a-1}{(n-1)a^2+(n-2)a-1}$. And $\lim_{a\to\frac{1}{n-1}} c_i b_{ii} = \frac{1}{n}$, $\lim_{a_{ij}\to\frac{1}{n-1}} \sum_i c_i b_{ii} = 1$. $1 < \sum_i c_i b_{ii} \leq n$.

□

The total output Q^{co} decreases monotonously when PCO increases, with the upper bound Q^{cn} and lower bound Q^{m}.

□

Proof of Corollary 1. From equation (A.1) and equation (A.3) we have $\frac{q^{co}}{Q^{co}} = \frac{B^{-1}diag(B)}{\sum_i c_i b_{ii}}$. The profit of firm i is

$$\pi_i^{co} = (B\hat{\pi}^{co})_i = (Bq^{co})_i p(Q^{co}) = \frac{b_{ii}}{\sum_i c_i b_{ii}} Q^{co} p(Q^{co}) = \frac{b_{ii}}{\sum_i c_i b_{ii}} \Pi^{co}$$

And the profit of controller i is $\tilde{\pi}_i^{co} = c_i \pi_i^{co} = \frac{c_i b_{ii}}{\sum_i c_i b_{ii}} \Pi^{co}$

□

Proof of Theorem 1

$$\delta_i^c = \frac{\pi_i^d - \pi_i^m}{\pi_i^d - \pi_i^{co}} = \frac{(b_{ii}q_i^d + \sum_{k\neq i} b_{ik}\frac{Q^m}{n})p(q_i^d + \frac{n-1}{n}Q^m) - \sum_k b_{ik}\frac{Q^m}{n}p(Q^m)}{(b_{ii}q_i^d + \sum_{k\neq i} b_{ik}\frac{Q^m}{n})p(q_i^d + \frac{n-1}{n}Q^m) - \frac{b_{ii}}{\sum_i c_i b_{ii}}\Pi^{co}}$$
$$= \frac{(q_i^d + \frac{Q^m}{n}\sum_{k\neq i}\frac{b_{ik}}{b_{ii}})p(q_i^d + \frac{n-1}{n}Q^m) - \frac{\Pi^m}{n}\sum_k \frac{b_{ik}}{b_{ii}}}{(q_i^d + \frac{Q^m}{n}\sum_{k\neq i}\frac{b_{ik}}{b_{ii}})p(q_i^d + \frac{n-1}{n}Q^m) - \frac{1}{\sum_i c_i b_{ii}}\Pi^{co}}$$

where $q_i^d = \arg\max_{q_i}(b_{ii}q_i + \sum_{k\neq i} b_{ik}\frac{Q^m}{n})p(q_i + \frac{n-1}{n}Q^m)$. Denote $Q_i^d = q_i^d + \frac{n-1}{n}Q^m$. The F.O.C

$$\frac{\partial \pi_i}{\partial q_i^d} = b_{ii}(p(Q_i^d) + (q_i^d + \frac{Q^m}{n}\sum_{k\neq i}\frac{b_{ik}}{b_{ii}})p'(Q_i^d)) = 0 \tag{A.6}$$

Suppose firm r increases its stake in firm s, $a'_{rs} = a_{rs} + w$, then $b'_{ij} = b_{ij} + \frac{w}{1-wb_{sr}}b_{ir}b_{sj}$. If firm i does not have a direct nor indirect stake in firm r, $b_{ir} = 0$, $b'_{ij} = b_{ij}$, $\sum_{k\neq i}\frac{b'_{ik}}{b'_{ii}} = \sum_{k\neq i}\frac{b_{ik}}{b_{ii}}$. For firm s, $b'_{sj} = \frac{1}{1-wb_{sr}}b_{sj}$, $\sum_{k\neq s}\frac{b'_{sk}}{b'_{ss}} = \sum_{k\neq s}\frac{b_{sk}}{b_{ss}}$. Consider (A.6), this means for these firms the deviating output q_i^d does not change. And the terms in δ_i^c do not change except $\frac{1}{\sum_i c_i b_{ii}}\Pi^{co}$. As proved in lemma A1, $\sum_i c_i b_{ii}$ decreases and Π^{co} increases, so δ_i^c increases.

□

Proof of Theorem 2.

$$\pi_i^m = (Bq^m)_i p(Q^m) = \frac{b_{ii}}{\sum_i c_i b_{ii}}Q^m p(Q^m) = \frac{b_{ii}}{\sum_i c_i b_{ii}}\Pi^m > \frac{b_{ii}}{\sum_i c_i b_{ii}}\Pi^{co} = \pi_i^{co}$$

The collusion is profitable for all firms.

□

Proof of Theorem 3. We first prove that the total deviating output Q_i^d is identical among firms.

Lemma A2. Q_i^d is identical through all firms given α and is independent of the collusive output of the deviating firm. It satisfies

$$Q_i^d + \frac{p(Q_i^d)}{p'(Q_i^d)} = Q^m(1-\alpha) \tag{A.7}$$

Proof of Lemma A2. From the F.O.C we have $q_i^d + \frac{1}{b_{ii}}\sum_{k\neq i} b_{ik}q_k^m + \frac{p(Q_i^d)}{p'(Q_i^d)} = 0$, subtracting and adding q_i^m on the left side $q_i^d - q_i^m + \frac{1}{b_{ii}}\sum_k b_{ik}q_k^m + \frac{p(Q_i^d)}{p'(Q_i^d)} = 0$.

$\sum_k b_{ik} q_k^m = (Bq^m)_i = \frac{b_{ii}}{\sum_i c_i b_{ii}} Q^m$. Using $Q_i^d - Q^m$ to substitute $q_i^d - q_i^m$, we obtain

$$Q_i^d + \frac{p(Q_i^d)}{p'(Q_i^d)} = Q^m (1 - \frac{1}{\sum_i c_i b_{ii}})$$

All Q_i^d satisfies the same equation and it has a unique solution.

□

Denote this identical total output as Q^d. Similarly, $\pi_i^d = b_{ii}(q_i^d + \sum_{k \neq i} \frac{b_{ik}}{b_{ii}} q_k^m) p(Q^d) = b_{ii}(q_i^d - q_i^m + \sum_k \frac{b_{ik}}{b_{ii}} q_k^m) p(Q^d) = b_{ii}(Q^d - Q^m + \alpha Q^m) p(Q^d)$. Recalling the profit of firm i under collusion is $\pi_i^m = \frac{b_{ii}}{\sum_i c_i b_{ii}} \Pi^m$ and under PCO competition is $\pi_i^{co} = \frac{b_{ii}}{\sum_i c_i b_{ii}} \Pi^{co}$.

$$\begin{aligned}
\delta_i^c &= \frac{\pi_i^d - \pi_i^m}{\pi_i^d - \pi_i^{co}} \\
&= \frac{b_{ii}(Q^d - (1-\alpha)Q^m) p(Q^d) - \frac{b_{ii}}{\sum_i c_i b_{ii}} \Pi^m}{b_{ii}(Q^d - (1-\alpha)Q^m) p(Q^d) - \frac{b_{ii}}{\sum_i c_i b_{ii}} \Pi^{co}} \\
&= \frac{(Q^d - (1-\alpha)Q^m) p(Q^d) - \alpha \Pi^m}{(Q^d - (1-\alpha)Q^m) p(Q^d) - \alpha \Pi^{co}} \qquad \text{(A.8)}
\end{aligned}$$

□

Proof of Proposition 3.

$$\delta^c = \frac{(Q^d - (1-\alpha)Q^m) p(Q^d) - \alpha \Pi^m}{(Q^d - (1-\alpha)Q^m) p(Q^d) - \alpha \Pi^{co}} = \frac{1}{1 + \frac{\alpha \Pi^m - \alpha \Pi^{co}}{(Q^d - (1-\alpha)Q^m) p(Q^d) - \alpha \Pi^m}}$$

Denote $M(\alpha) = \alpha \Pi^m - \alpha \Pi^{co}$, $N(\alpha) = (Q^d - (1-\alpha)Q^m) p(Q^d) - \alpha \Pi^m$. According to definition, $Q^d = \arg\max_Q c_i b_{ii}(Q - (1-\alpha)Q^m) p(Q)$. Applying envelop theorem, $\frac{dN}{d\alpha} = Q^m p(Q^d) - \Pi^m < 0$. For $M(\alpha)$, $\lim_{\alpha \to 0} M(\alpha) = 0$, $\lim_{\alpha \to 1} M(\alpha) = 0$. And $M(\alpha) > 0$, there exists a $\hat{\alpha} \in (0,1)$, when $\alpha \leq \hat{\alpha}$, $\frac{dM}{d\alpha} \geq 0$. So $\frac{d\delta^c}{d\alpha} < 0$.

□

Proof of Theorem 4. Since Q^m is firm j's optimal output given $q_i^i = 0$, equation (15) is not binding. We prove (14) is more binding than (13), then we just need to pay attention to (14).

Lemma A3.

$$\frac{\pi_i^d(q^i) - \pi_i(q^i)}{\pi_i(q^m) - \pi_i(q^i)} > \frac{\pi_i^d(q^m) - \pi_i(q^m)}{\pi_i(q^m) - \pi_i(q^i)}$$

Proof of Lemma A3. We need to prove

$$\pi_i^d(q^i) - \pi_i(q^i) > \pi_i^d(q^m) - \pi_i(q^m) \tag{A.9}$$

Assume $f(q_j) = \pi_i^d(q_j) - \pi_i(Q^m - q_j, q_j)$, $0 \leq q_j \leq Q^m$, equation (A.9) equals to $f(Q^m) > f(q_j^m)$. We prove $f'(q_j) > 0$.

$$\frac{df(q_j)}{dq_j} = \frac{d\pi_i^d(q_j)}{dq_j} - \frac{d\pi_i(Q^m - q_j, q_j)}{dq_j}$$

According to definition, $\pi_i^d(q_j) = \max_{q_i}(b_{ii}q_i + b_{ij}q_j)p(q_i + q_j)$. Applying envelope theorem, $\frac{d\pi_i^d}{dq_j} = b_{ij}p(Q_i^d) + (b_{ii}q_i^d + b_{ij}q_j)p'(Q_i^d)$ where q_i^d satisfies F.O.C $\frac{d\pi_i}{dq_i^d} = b_{ii}p(Q_i^d) + (b_{ii}q_i^d + b_{ij}q_j)p'(Q_i^d) = 0$. Substituting F.O.C to $\frac{d\pi_i^d}{dq_j}$ we obtain $\frac{d\pi_i^d}{dq_j} = (b_{ij} - b_{ii})p(Q_i^d)$.
$\pi_i(Q^m - q_j, q_j) = (b_{ii}(Q^m - q_j) + b_{ij}q_j)p(Q^m) = ((b_{ij} - b_{ii})q_j + b_{ii}Q^m)p(Q^m)$. $\frac{d\pi_i(Q^m - q_j, q_j)}{dq_j} = (b_{ij} - b_{ii})p(Q^m)$. Then we prove $Q_i^d > Q^m$. In the proof of lemma 1 we know $b_{ij} < b_{ii}$. At $q_i^d = Q^m - q_j$, the F.O.C $\frac{d\pi_i^d}{dq_i} = b_{ii}p(Q^m) + (b_{ii}q_i^d + b_{ij}q_j)p'(Q^m) > b_{ii}(p(Q^m) + Q^m p'(Q^m)) = 0$, increasing q_i^d can improve π_i^d, so $q_i^d + q_j > Q^m$. Then

$$\begin{aligned}\frac{df(q_j)}{dq_j} &= \frac{d\pi_i^d(q_j)}{dq_j} - \frac{d\pi_i(Q^m - q_j, q_j)}{dq_j} \\ &= (b_{ij} - b_{ii})(p(q_i^d + q_j) - p(Q^m)) > 0\end{aligned}$$

And $Q^m > q_j^m$, $f(Q^m) > f(q_j^m)$.

□

Let $\delta_i^c = \frac{\pi_i^d(q^i) - \pi_i(q^i)}{\pi_i(q^m) - \pi_i(q^i)}$, we discuss the effects of PCO on δ_i^c.
(i) $\pi_i^d(q^i) = \max_{q_i}(b_{ii}q_i + b_{ij}Q^m)p(q_i + Q^m)$. The F.O.C

$$\frac{d\pi_i}{dq_i^d} = b_{ii}(p(Q_i^d) + (q_i^d + \frac{b_{ij}}{b_{ii}}Q^m)p'(Q_i^d)) = 0 \tag{A.10}$$

where $Q_i^d = q_i^d + Q^m$. $\pi_i(q^i) = b_{ij}\Pi^m$, $\pi_i(q^m) = \frac{b_{ii}}{\sum_i c_i b_{ii}}\Pi^m$. Then

$$\begin{aligned}
\delta_i^c &= \frac{\pi_i^d(q^i) - \pi_i(q^i)}{\pi_i(q^m) - \pi_i(q^i)} \\
&= \frac{(b_{ii}q_i^d + b_{ij}Q^m)p(Q_i^d) - b_{ij}\Pi^m}{\frac{b_{ii}}{\sum_i c_i b_{ii}}\Pi^m - b_{ij}\Pi^m} \\
&= \frac{(q_i^d + \frac{b_{ij}}{b_{ii}}Q^m)p(Q_i^d) - \frac{b_{ij}}{b_{ii}}\Pi^m}{\alpha\Pi^m - \frac{b_{ij}}{b_{ii}}\Pi^m}
\end{aligned} \tag{A.11}$$

Consider firm j increases its stake in firm i $a'_{ji} = a_{ji} + w$. By calculating the entries in B we know $\frac{b'_{ij}}{b'_{ii}} = \frac{b_{ij}}{b_{ii}} = a_{ij}$. Consider equation (A.10), q_i^d does not change. In (A.11) only $\alpha\Pi^m$ increases so δ_i^c decreases.
If a_{ij} increases,

$$\begin{aligned}
\frac{d\delta_i^c}{da_{ij}} =& \frac{(Q^m p(Q_i^d) - \Pi^m)(\pi_i(q^m) - \pi_i(q^i)) - (\pi_i^d(q^i) - \pi_i(q^i))(\frac{d\alpha}{da_{ij}}\Pi^m - \Pi^m)}{b_{ii}(\pi_i(q^m) - \pi_i(q^i))^2} \\
<& \frac{(\pi_i^d(q^i) - \pi_i(q^i))(Q^m p(Q_i^d) - \frac{d\alpha}{da_{ij}}\Pi^m)}{b_{ii}(\pi_i(q^m) - \pi_i(q^i))^2}
\end{aligned}$$

if $Q^m p(Q_i^d) - \frac{d\alpha}{da_{ij}}\Pi^m < 0$, $\frac{d\delta_i^c}{da_{ij}} < 0$. This equals $\frac{p(Q_i^d)}{p(Q^m)} < (\frac{c_i}{c_i + c_j})^2$.
(ii)

$$\begin{aligned}
\delta_i^c &= \frac{(b_{ii}q_i^d + b_{ij}Q^m)p(Q_i^d) - b_{ij}Q^m p(Q^m)}{(b_{ii}q_i^m + b_{ij}q_j^m)p(Q^m) - b_{ij}Q^m p(Q^m)} \\
&= \frac{(q_i^d + \frac{b_{ij}}{b_{ii}}Q^m)p(Q_i^d) - \frac{b_{ij}}{b_{ii}}Q^m p(Q^m)}{p(Q^m)q_i^m(1 - \frac{b_{ij}}{b_{ii}})}
\end{aligned} \tag{A.12}$$

If the output distribution is fixed, q_i^m does not change with a_{ij} or a_{ji}. Consider PCO increases, if a_{ji} increases, no term in equation (A.12) changes so δ_i^c keeps unchanged. If a_{ij} increases, applying envelope theorem

$$\begin{aligned}
\frac{d\delta_i^c}{da_{ij}} &= \frac{(1 - a_{ij})(Q^m p(Q_i^d) - Q^m p(Q^m) + ((q_i^d + a_{ij}Q^m)p(Q_i^d) - a_{ij}Q^m p(Q^m))}{q_i^m p(Q^m)(1 - a_{ij})^2} \\
&= \frac{Q^m(p(Q_i^d) - p(Q^m)) + q_i^d p(Q_i^d)}{q_i^m p(Q^m)(1 - a_{ij})^2} = \frac{\Pi^d - \Pi^m}{q_i^m p(Q^m)(1 - a_{ij})^2} < 0
\end{aligned}$$

□

Appendix B. Extensions

B.1. *Uniform extra profit distribution*

Assume the collusion distribute the extra profit from collusion $\Pi^m - \Pi^{co}$ uniformly to n controllers. The output distribution scheme satisfies

$$\tilde{\pi}_i^m = c_i \sum_k b_{ik} q_k^m p(Q^m) = \frac{c_i b_{ii}}{\sum_i c_i b_{ii}} \Pi^{co} + \frac{1}{n}(\Pi^m - \Pi^{co}) \tag{B.1}$$

Writing (B.1) in matrix form and left multiplied by B^{-1}

$$q^m = q^{co} \frac{p(Q^{co})}{p(Q^m)} + B^{-1}(\frac{1}{c_i}) \frac{\Pi^m - \Pi^{co}}{np(Q^m)}$$

where $(\frac{1}{c_i})$ is the column vector with the ith entry $\frac{1}{c_i}$. The first term in the right-hand side is the output distribution proportional to PCO competition. The second term is the adjustment to uniformly distribute the extra profit $\Pi^m - \Pi^{co}$. Denote s_i^m as the collusion output share of firm i and s_i^{co} as the output share under PCO competition.

$$s_i^m = s_i^{co} \frac{\Pi^{co}}{\Pi^m} + (\frac{1}{c_i} - \sum_{k \neq i} \frac{a_{ik}}{c_k}) \frac{\Pi^m - \Pi^{co}}{n \Pi^m}$$

Consider a duopoly game, if a_{ji} increases, we can see that s_j^m decreases and s_i^m increases. The critical discount factor

$$\delta_i^c = \frac{\pi_i^d - \pi_i^m}{\pi_i^d - \pi_i^{co}} = \frac{1}{1 + \frac{\pi_i^m - \pi_i^{co}}{\pi_i^d - \pi_i^m}}$$

With increasing PCO, $\pi_i^m - \pi_i^{co} = \frac{1}{n}(\Pi^m - \Pi^{co})$ decreases. Since q_j^m decreases, recall lemma A3, we know for firm i the benefit from deviation $\pi_i^d - \pi_i^m$ also decreases. The net effect is ambiguous. For firm j, similar to lemma A3, we have

$$\frac{d\pi_j^d}{da_{ji}} - \frac{d\pi_j^m}{da_{ji}} = b_{jj}(q_i^m + \frac{dq_i^m}{da_{ji}}(a_{ji} - 1))(p(Q_i^d) - p(Q^m)) \tag{B.2}$$

And we know $\frac{dq_i^m}{da_{ji}} > 0$. If $\frac{s_i^m}{c_i} < \frac{ds_i^m}{da_{ji}}$ is satisfied, $\pi_j^d - \pi_j^m$ increases and δ_j^c increases. After increasing its stake in firm i, the incentive for firm j to engage in collusion is weakened. This needs more specific assumptions. Generally speaking, for the scheme in equation (B.1), the effects of increasing PCO depends on the PCO level as well as the market demand.

B.2. More severe punishment

The punishment adopts a stick-and-carrot structure raised in Abreu (1986)[2], for a consice version, see Mailath and Samuelson (2006). If anyone deviates from the collusive output, firms produce a punishment output q^p, and then return to collusion; if anyone deviates from the punishment, that is, don't produce q_i^p, the punishment restarts, until all firms produce according to q^p. To ensure the punishment is indeed a 'punishment' for the deviant, the output distribution of q^p is in line with theorem 2, if not, the profit in punishment phase may be higher for some deviant. First step is determining the optimal punishment output Q^p. This strategy needs to be subgame perfect:

$$\pi_i^d(q^m) + \delta\pi_i(q^p) \leq \pi_i(q^m) + \delta\pi_i(q^m) \tag{B.3}$$

$$\pi_i^d(q^p) + \delta\pi_i(q^p) \leq \pi_i(q^p) + \delta\pi_i(q^m) \tag{B.4}$$

These conditions apply to all i. Equation (B.3) means deviation from the collusion and a following punishment is not profitable. Equation (B.4) means deviation from the punishment and a following restart of the punishment is not profitable, so the punishment threat is credible. We have

$$\delta \geq \delta_i^c \equiv max\{\frac{\pi_i^d(q^m) - \pi_i(q^m)}{\pi_i(q^m) - \pi_i(q^p)}, \frac{\pi_i^d(q^p) - \pi_i(q^p)}{\pi_i(q^m) - \pi_i(q^p)}\} \, \forall i$$

That is, the punishment is severe enough that the collusion can be sustained with δ and at the same time, can not be too severe that the threat to punishment is not credible. To construct the lowest δ_i^c, we have $\pi_i^d(q^m) - \pi_i(q^m) = \pi_i^d(q^p) - \pi_i(q^p)$. This means the benefit from deviation is identical in collusion phase and in punishment phase. It can be seen that the two conflicting effects of increasing PCO still exists. Assume players cooperate at an output Q^c, if $Q^c < Q^{co}$, the benefit from deviation $\pi_i^d(q^c) - \pi_i(q^c)$ decreases with Q^c; if $Q^c > Q^{co}$ the benefit from deviation increases with Q^c. The benefit from deviation is zero at $Q^c = Q^{co}$: q^{co} is a Nash equilibrium and no player has an incentive to deviate. As section 2 shows, the benefit of deviating from Q^m decreases as α increases, to obtain an optimal punishment Q^p, we need also decreases Q^p. This means the punishment alleviates[3].

[2] In Abreu (1986) the stick-and-strategy is optimal strongly symmetric strategy where same quantity is chosen by every firm, however, this also holds to our case as is shown in theorem 3: firms produce a share of total output appointed in theorem 2 and have the same incentive to collude or deviate.

[3] If the inverse demand function is concave or not too convex, this tradeoff exists. However, if the inverse demand function is convex to the extent that the benefit from deviation decreases at Q^p where Q^p satisfies $\pi_i^d(q^m) - \pi_i(q^m) = \pi_i^d(q^p) - \pi_i(q^p)$, this tradeoff does not exist: increasing q^p would make equation (B.3) more binding than (B.4). To obtain the optimal punishment Q^p, we should set Q^p to the maximum value of output, which means the punishment does not change with increasing PCO thus tacit collusion is facilitated.

If the punishment output Q^p is not severe to the extent that the deviation from punishment is to halt production (this needs $p(Q^{dp}) \geq c$, if not, firm i's operating income $\hat{\pi}_i = (p(Q^{dp}) - c)q_i^{dp}$ is negative and a better deviation is to halt production), lemma A2 also applies to Q^{dp}: $Q^{dp} + \frac{p(Q^{dp})-c}{p'(Q^{dp})} = Q^p(1-\alpha)$. Calculating profits along different paths similar to section 2:

$$\begin{aligned}
\pi_i^d(q^m) &= b_{ii}(Q^{dm} - (1-\alpha)Q^m)(p(Q^{dm}) - c) \\
\pi_i(q^m) &= \frac{b_{ii}}{\sum_i c_i b_{ii}} \Pi^m \\
\pi_i^d(q^p) &= b_{ii}(Q^{dp} - (1-\alpha)Q^p)(p(Q^{dp}) - c) \\
\pi_i(q^p) &= \frac{b_{ii}}{\sum_i c_i b_{ii}} \Pi^p
\end{aligned}$$

Where Q^{dm} is the total output when deviating from the collusive output Q^m and Q^{dp} is the total output when deviating from the punishment output Q^p. The optimal punishment output Q^p satisfies $\pi_i^d(q^m) - \pi_i(q^m) = \pi_i^d(q^p) - \pi_i(q^p)$. The critical discount factor δ_i^c is

$$\begin{aligned}
\delta_i^c &= \frac{\pi_i^d(q^m) - \pi_i(q^m)}{\pi_i(q^m) - \pi_i(q^p)} \\
&= \frac{(Q^{dm} - (1-\alpha)Q^m)(p(Q^{dm}) - c) - \alpha\Pi^m}{\alpha\Pi^m - \alpha\Pi^p} \qquad \text{(B.5)}
\end{aligned}$$

Also, the critical discount factor δ^c is identical through all firms. Using $p = 1 - Q$ as example, figure 3 describes δ^c as a function of α, compared with the curve of trigger strategy discussed in section 2 (the cost c is set to zero). Since the punishment is more severe than trigger strategy, given the value of α, the critical discount factor is always lower than that of trigger strategy. Figure 4 plots a family of demand functions $p = (1-Q)^x$. Conditions to use lemma A2 are satisfied. Similar to trigger strategy, proposition 3 also applies to stick-and-carrot strategy

Proposition A1. For stick-and-carrot strategy, there exists a critical value $\hat{\alpha}$ such that, if $1/n < \hat{\alpha}$, for $\alpha \in (1/n, \hat{\alpha}]$, increasing PCO facilitates collusion.

Proof. The proof is the same as proof of proposition 3 substituting Π^{co} with Π^p.

From the example of $p = (1-Q)^x$, we may find the punishment-alleviating effects of increasing PCO may be more obvious than trigger strategy: as α increases, Q^p decreases more quickly than Q^{co} (Q^m and Q^p is in a sense 'symmetric' around Q^{co}).

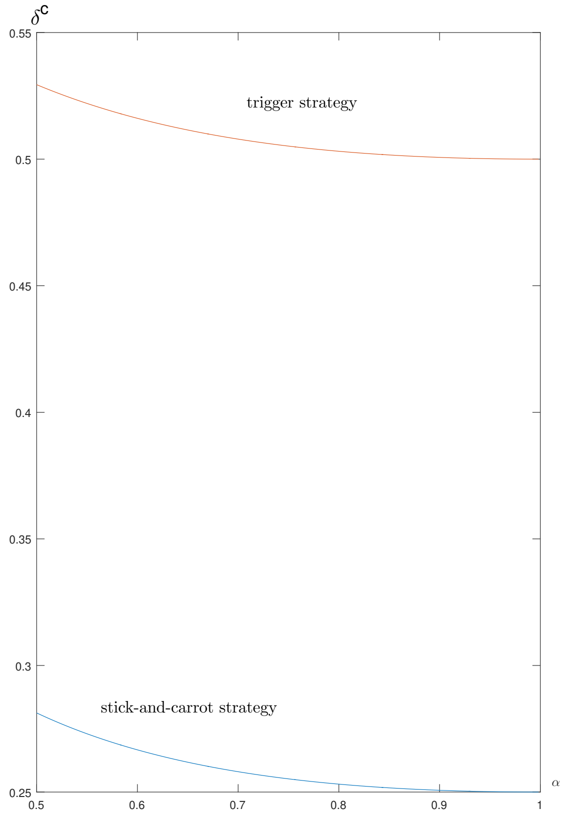

Fig. 3. Critical discount factor δ^c for different strategies in liner demand

Situations will be much more complicated if the best deviating behavior is to halt production in the punishment phase. If the punishment is so severe that the Q^{dp} given by lemma 2 is large to $p(Q^{dp}) < c$, a better deviation is to produce nothing hence lemma A2 fails and instead $Q_i^{dp} = Q^p - q_i^p$. Different firms correspond to different total deviating outputs, so are the critical discount factors. It can be shown that in the punishment phase the firm with the largest punishment output has a largest critical discount factor[4]. In this case, stick-and-carrot strategy doesn't apply anymore. This is possible if the number of firms engaging in collusion is large and at the beginning of cross ownership (α is small), because the stage Nash equilibrium Q^{co} is near to perfect competition. To obtain a severe enough punishment Q^p has to be large enough than Q^{co} that $p(Q^p)$ is much lower than cost c. In this case to construct an optimal punishment is intractable[5]. For the example of $p = 1 - Q$ and $c = 0$,

[4] $\pi_i^d(q^p) = c_i \sum_{k \neq i} b_{ik} q_k^p (p(Q_i^d) - c) = c_i b_{ii}(\alpha Q^p - q_i^p)(p(Q^p - q_i^p) - c)$, since $p < c$, larger q_i^p corresponds to larger π_i^d.

[5] Abreu (1986) prove that there exists an asymmetric dynamic punishment phase which is more optimal than symmetric stick-and-carrot strategy.

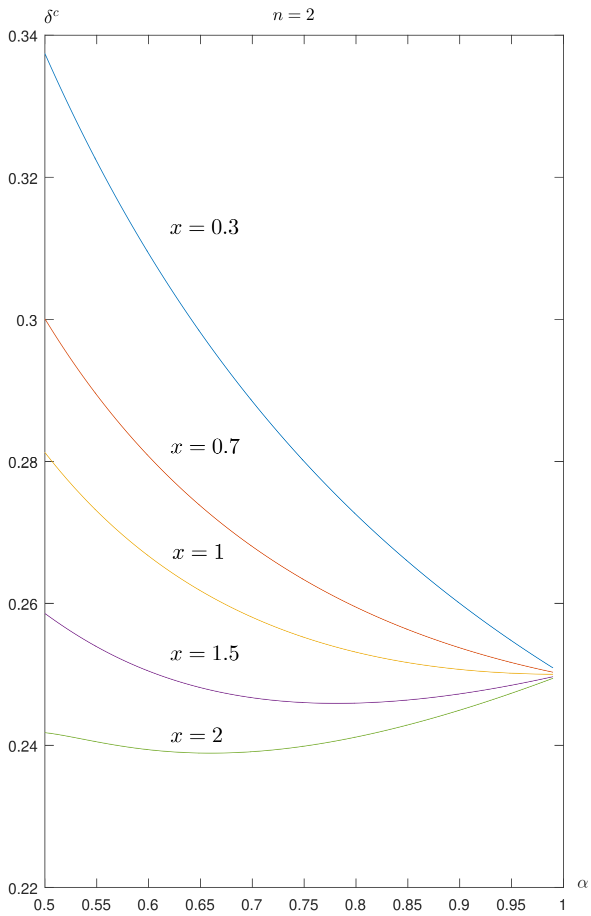

Fig. 4. Critical discount factor δ^c for different x under stick-and-carrot strategy

if $n \geq 5$, the stick-and-carrot strategy fails unless the PCO is symmetric.