Tamar Kavtaradze

A Bayesian - Martingale Approach to the General Disorder Problem

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A Bayesian - Martingale Approach to the General Disorder Problem

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The thesis "**A Bayesian - Martingale Approach to the General Disorder Problem**" by **Tamar Kavtaradze** is recommended for acceptance by the members of the delegated committee, as stated by the enclosed reports, in partial fulfillment of the requirements for the degree of **Doctor of Philosophy**.

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To my parents

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Introduction

In this thesis we present a Bayesian - martingale approach to the general disorder problem.

Classical disorder problems consider the earliest detection of a random instant of time θ at which the mean (or other probabilistic characteristics) of the observable stochastic process X_t changes. The random time θ at which the disruption occurs is called the change-point. A sequential change-point detection procedure is identified with a stopping time τ with respect to the filtration F_t^X of observable events (interpreted as the time at which the "alarm signal" is given), at which it is declared that a change has occurred. The aim of the prob em is to find a stopping time τ, based on the observations X_t, which is "as close as possible" to the change-point θ. More exactly, the design of the quickest change-point detection procedures involves optimizing the tradeoff between two kinds of performance measures, one being a measure of detection delay and the other being a measure of the frequency of false alarm.

Disorder problems have originally arisen and still play a prominent role in quality control where one observes the output of a production line and wishes to detect deviation from an acceptable level. After the introduction of the original control charts by Shewhart (1931) various modifications of the problem have been recognized and implemented in a number of applied sciences (see Kolmogorov at.al.(1990)). These problems include: epidemiology (where one tests whether the incidence of a disease has remained constant over time and wishes to estimate the time of a change in order to suggest possible causes); rhythm analysis in electrocardiograms (where the use of change detection methods constitutes a part of a pattern recognition analysis); changes of the critical models in electric-energy systems; the appearance of a target in radio/radar location; the appearance of 'breaks' in geological data; the beginning of earthquakes or tsunamis; seismic signal processing ; the appearance of a shock wave front; the study of historical texts or manuscripts; the study of archeological sites, etc. Specific applications described in Carstein at.al.(1994) include: statistical image processing and edge detection in noisy images; change-points in economic regression models (split or two-phase regression); detection of discontinuities in astrophysical time series with dependent data; changes in hazard rates as shown to occur of DNA sequences; the simultaneous estimation of smoothly varying parts and discontinuities of curves and surfaces. Applications in financial data analysis (detection of arbitrage) are discussed in Shiryaev (2002).

The disorder problem was first analyzed in detail by Shiryaev (1963, 1967, 1978) and has since been extensively studied. In general, a disorder problem can be viewed as consisting of three distinct elements.

(1) *The probabilistic structure*

The probabilistic structure specifies the distribution of the observation before and after the disorder occurs. Shiryaev (1963,1978) deals with a change in distribution of a sequence of independent and identically distributed random variables and also in the sudden change in the drift of the Wiener process. Davis (1976) and Peskir and Shiryaev (2002) have dealt with the Poisson process. Davis (1975), Balakrishnan

(1981) and Mazumdar (1983) consider continuous and discrete time Markov processes. In our setting the change-point represents a random time of bifurcation of two arbitrary probability measures given on the space of right-continuous functions and the Wiener and the Poisson disorder problems are considered as particular cases.

(2) *The a priori distribution*

The second element in the disorder problem is the a priori distribution of the change-point. In general it is assumed that the a priori distribution is exponential for continuous time and geometric for discrete time. Shiryaev (1963) defines the stationary version of the disorder problem as the limiting case, when the mean of the exponential prior goes to infinity. In the Bayesian formulation of the problem (see Shiryaev, 1967) it is assumed that the change-point θ admits a known a priori distribution, although the variable θ itself is unknown for us, since it cannot be observed directly. Alternatively a minimax approach may be used, in which no priori assumptions are made on the change-point (disorder time) (see Pollak, 1985). In our general setting the change-point θ a priori is supposed to have an arbitrary distribution function $\psi = \psi(x)$ concentrated on $[0, \infty)$.

(3) *Criterion for measuring the performance*

The third element is the criterion for measuring the performance of the stopping time detector τ. Two Bayesian formulations of the problem was proposed by Shiryaev (1978). In the "free" formulation (below referred to as the "Bayesian approach") one minimizes a linear combination of the probability of a 'false alarm' and the expectation of a 'delay' in detecting change-point correctly with no constraint on the former. In the 'fixed false-alarm' formulation, known as the 'variational problem' the same linear combination is minimized under constraint that the probability of a 'false alarm' cannot exceed a given value. The most common criterion is to minimize the average detection delay time, for a given probability of false alarm. Bojdecki (1979) and Bojdecki and Hosza (1984) deal with the "probability maximizing approach", whereby one attempts to maximize the probability that the difference between the alarm time and the true disorder time is at most some specified amount. In the minimax approach the goal is to minimize the mean detection delay time for the least favorable disorder time and a specified mean false alarm time. In the present work the Bayesian approach to the problem is considered, where an average delay is measured by an F_t predictable increasing process K, which is more natural for models involving martingales with non-absolutely-continuous characteristics.

Among all processes considered in the context of disorder problem, the Wiener process and the Poisson process take a central place. Nowadays these problems are understood sufficiently well, as complete solutions are found for both of them. Shiryaev (1967) (see also Shiryaev (1978)) derived an explicit solution of a Wiener disorder problem, reducing the initial optimal stopping problem to a free-boundary problem for a parabolic differential operator. The Poisson disorder problem was first studied by Gal'chuk and Rozovskii (1971), where the problem was solved in some particular cases. Their results have been extended by Davis (1976), who required less restrictions on the model parameters. The complete solution of the Poisson disorder problem was given in Peskir and Shiryaev (2002) by reducing the initial optimal stopping problem to a free-boundary problem for a differential- difference operator. Note that in all these papers the case of infinite time horizon is considered. A solution of the Bayesian and variational problem for a Wiener process with finite horizon was recently derived by Gapeev and Peskir (2003).

In this thesis we present a Bayesian-martingale approach to the general disorder problem with infinite time horizon where the change-point represents a random time of bifurcation of two probabilistic measures given on the space of right-continuous functions. We derive a martingale stochastic differential equation for the a posteriori probability process π_t of the change-point θ, which plays, as it is well known, a crucial rôle in reducing the disorder problem to the optimal stopping problem and to determine the value process and the optimal stopping rule. We introduce the value process of the related optimal stopping problem and show that this process uniquely solves a suitable reflecting backward stochastic differential equation (RBSDE). The value functions related to disorder problems(or to an optimal stopping problem in general) of Markov processes are usually solutions of suitable free boundary problems. So the RBSDE for the value processes and the free boundary problems for the value functions should be equivalent in some sense, at least in simpler cases when the a posteriori probability process π_t is a sufficient statistics and the value process V_t of the problem is related with the value function $\rho(\pi)$ of the same problem by the equality $V_t = \rho(\pi_t)$. The problem is to show the differentiability properties and smooth fit conditions for the value functions, based on the properties of the process $\rho(\pi_t)$ being a solution of a RBSDE. We consider classical disorder problems for Wiener and Poisson processes and show that in these cases related RBSDEs for value processes and the corresponding free boundary problems are equivalent.

The Wiener disorder problem when its drift changes from one value to another at a random instant of time θ was solved by Shiryaev (1967) who gave an explicit expression for the value function $\rho(\pi)$ of initial stopping problem, showing that this function (together with the optimal threshold A^*) uniquely solves the corresponding free-boundary problem. By means of the obtained results former we give a probabilistic proof of this fact. We show that $\rho(\pi)$ is a solution of the free-boundary problem if and only if the process $\rho(\pi_t)$ is a solution of corresponding RBSDE. The key step here is to show that if the value process $V_t = \rho(\pi_t)$ satisfies RBSDE, then the function $\rho(\pi)$ is continuously differentiable on $(0, 1]$ and twice continuously differentiable on $(0, A^*), 0 < A^* < 1$. In particular this implies that the smooth fit condition is satisfied, but the smooth fit of the second derivative fails.

We consider the disorder problem for a Poisson process whose intensity changes from λ_0 to λ_1 at some random time θ. As mentioned above, the closed form solution of this problem was given in Peskir and Shiryaev (2002), where the problem was reduced to a free-boundary differential-difference problem. We show that this free-boundary problem is also equivalent to the well posedness of the general RDSDE. In particular, this shows that the unique solution of a free-boundary differential-difference problem coincides with the value function of the problem. Besides, we derive the smooth fit conditions for the value function (in cases when this condition is satisfied) and establish when the smooth fit condition breaks down directly from the RBSDE for the value process.

The outline of this thesis chapter by chapter is as follows: Chapter 1 is dedicated to the Bayesian formulation of the general disorder problem. The setting of the Bayesian - Martingale approach is presented in Section 1.1. We shall give a brief description of this statement:

Let $(\Omega, \mathcal{F}; F = (F_t, t \geq 0))$ be a filtered measurable space with $\mathcal{F} = F_\infty$. We assume that P^0 and P^1 are two fixed locally equivalent probability measures ($P^1 \overset{loc}{\sim} P^0$) defined on $(\Omega, \mathcal{F})$ and $\psi = \psi(x)$ is a distribution function of some non-negative random variable. Without loss of generality (e.g., taking $P = \frac{1}{2}(P^1 + P^0)$) one can assume that there is a probability measure P on $(\Omega, \mathcal{F})$ such that

$$P^1 << P, \quad P^0 << P, \quad P^1 \overset{loc}{\sim} P, \quad P^0 \overset{loc}{\sim} P.$$

For simplicity we assume that the restrictions of the measures P^0 and P^1 coincide on the σ-algebra $\mathcal{F}_0$.

Let $(Z_t^i = \frac{dP_t^i}{dP_t}, t \geq 0), i = 0, 1$, be the density process of the measure P^i relative to P, which is a uniformly integrable $P-$martingale with $Z_t^i > 0$ $P - a.s.$ for any $t \in [0, \infty[$. Then there exists a local martingale $M^i \in \mathcal{M}_{loc}(F, P)$ such that

$$Z^i = \mathcal{E}(M^i) = (\mathcal{E}_t(M^i), t \geq 0), \quad i = 0, 1,$$

where $\mathcal{E}(M)$, called the Dolean exponential of M, is the unique solution of the linear Stochastic Differential Equation (SDE)

$$Z_t = 1 + \int_0^t Z_{s-} dM_s. \tag{0.0.1}$$

(see, *e.g.* Liptser and Shiryaev, 1986, or Jacod, 1979).

For the statement of the problem in a general martingale setting we extend the initial filtered measurable space as follows:

$\overline{\Omega} = \Omega \otimes R^+$, $\overline{F} = \mathcal{F} \otimes \mathcal{B}(R^+)$, $\overline{F}_t = F_t \otimes \mathcal{B}(R^+)$, where $\mathcal{B}(R^+)$ is the Borel $\sigma-$algebra on $R^+ = [0, \infty)$.

The measure $\overline{P}^\psi$ on $\mathcal{F} \otimes \mathcal{B}(R^+)$ is defined in a following way: let for every $A \in \mathcal{F}$ and $B \in \mathcal{B}(R^+)$

$$\overline{P}^\psi(A \times B) = \int_A \int_B \mathcal{E}_\infty(M^x) \psi(dx) P(d\omega), \tag{0.0.2}$$

where

$$M_t^x = \int_0^t I_{\{x \leq s\}} dM_s^1 + \int_0^t I_{\{x > s\}} dM_s^0. \tag{0.0.3}$$

We denote by P^ψ the restriction of the measure $\overline{P}^\psi$ on the $\sigma-$algebra $\mathcal{F} \otimes R^+$ and respectively by P_t^ψ we denote the restriction of the measure P^ψ on the σ-algebra $\mathcal{F}_t \equiv F_t \otimes R^+$. For every $A \in F_t$

$$P_t^\psi = P^\psi(A \times R^+) = (1-\psi(t))P^0(A) + \int_A \int_{[0,t]} \frac{\mathcal{E}_{s-}(M^0)}{\mathcal{E}_{s-}(M^1)} \psi(ds) dP^1. \qquad (0.0.4)$$

where we assume that $\frac{\mathcal{E}_{0-}(M^0)}{\mathcal{E}_{0-}(M^1)} = 1$;

Thus, the measures $\overline{P}_t^\psi$ (and P_t^ψ) do not depend on the choice of the dominating measure P. It is easy to see that $P^\psi << P$ and

$$Z_t^\psi \equiv \frac{dP_t^\psi}{dP_t} = (1-\psi(t))\mathcal{E}_t(M^0) + \mathcal{E}_t(M^1) \int_{[0,t]} \frac{\mathcal{E}_{s-}(M^0)}{\mathcal{E}_{s-}(M^1)} \psi(ds). \qquad (0.0.5)$$

We define the random variable θ on the space $(\overline{\Omega}, \overline{F})$

$$\theta = \theta(\overline{\omega}) = \theta(\omega, x) = x.$$

It is easy to check that

$$\overline{P}^\psi(\theta \le x) = \overline{P}^\psi(\Omega \times [0,x]) = \psi(x).$$

This means that the distribution function $\psi = \psi(x)$ by means of which we have defined the new measure $\overline{P}^\psi$ on the extended measurable space $(\overline{\Omega}, \overline{F})$ comes to be *the a priori distribution function of the variable* θ, associated with the random time of 'disorder', which can not be observed directly. We shall also note that θ represents a random instant time of bifurcation of two probability measures given on the measurable space $(\Omega, \mathcal{F})$ endowed with the filtration $F = (F_t, t \ge 0))$ of the observable events. Indeed, by Fubini's theorem the measure $\overline{P}^\psi$ can be rewritten as follows:

$$\overline{P}^\psi(A \times B) = \int_B \Big(\int_A \mathcal{E}_\infty(M^x) P(d\omega) \Big) \psi(dx), \qquad (0.0.6)$$

which implies that for any $A \in \mathcal{F}$ the measure P^{ψ}, which is the restriction of the measure $\overline{P}^{\psi}$ on the $\sigma-$algebra $\mathcal{F} \otimes R^{+}$, takes the form

$$P^{\psi}(A) = \int_{R^{+}} P^{x}(A)\psi(dx),$$

where the measure

$$P^{x}(A) = \int_{A} \mathcal{E}_{\infty}(M^{x})P(d\omega)$$

defined on the measurable space $(\Omega, \mathcal{F})$ can be interpreted as the conditional measure given that the bifurcation of the measures P^0 and P^1 has occurred at the time $\theta = x$.

As we have already mentioned above the aim of the problem is to find a stopping time τ with respect to the filtration F_t of observable events (interpreted as the time at which the "alarm signal" is given) which is "as close as possible" to the change point θ. Following Shiryaev (1978) we define the cost criterion by

$$V(\tau) = \bar{P}^{\psi}(\tau < \theta) + E^{\psi} \max(K_{\tau} - K_{\theta}, 0), \tag{0.0.7}$$

where $\bar{P}^{\psi}(\tau < \theta)$ is a probability of "false alarm" and $E^{\psi} \max(K_{\tau} - K_{\theta}, 0)$ is an average delay (measured by an F_t - predictable increasing process K) of detecting the change-point correctly. Note that Shiryaev (1978) considers the case $K_t = ct$ for measuring the average delay.

The stopping time τ^* is called optimal if

$$V(\tau^*) = \inf_{\tau} V(\tau), \tag{0.0.8}$$

where inf is taken over the class of all F- stopping times.

Introducing the a posteriori probability process π_t

$$\pi_t = \overline{P}^{\psi}(\theta \leq t \mid \mathcal{F}_t),$$

similarly to Shiryaev (1978) one can reduce problem (0.0.8) to the optimal stopping problem

$$V(\tau^*) = \inf_{\tau} E^{\psi}[(1-\pi_\tau) + \int_0^\tau \pi_{s-} dK_s], \qquad (0.0.9)$$

since $\bar{P}^{\psi}(\tau < \theta) = E^{\psi}(1-\pi_\tau)$ and

$$\bar{E}^{\psi} \max(K_\tau - K_\theta, 0) = \bar{E}^{\psi} \int_0^\tau I_{(\theta \le s)} dK_s = E^{\psi} \int_0^\tau \pi_{s-} dK_s$$

by the projection theorem (see Theorem A.2.2).

We introduce the value process of the problem (0.0.9)

$$V_t = \operatorname*{essinf}_{\tau \ge t} E^{\psi}[(1-\pi_\tau) + \int_t^\tau \pi_{s-} dK_s / F_t]. \qquad (0.0.10)$$

It is well known that (see, *e.g.* El Karoui, 1981) the stopping time τ^* defined by

$$\tau^* = \inf\{t : V_t = 1 - \pi_t\} \qquad (0.0.11)$$

is optimal for the problem (0.0.9). In the case of the Wiener disorder problem considered by Shiryaev (1978) the optimal stopping time is of the following simple form

$$\tau^* = \inf\{t : \pi_t \ge A^*\}, \qquad (0.0.12)$$

where the constant A^* is a solution of a certain integral equation and the value function V is explicitly calculated as a function of $\psi(0) = \pi$ and A^*. Here the differential equation for the process π_t plays a crucial role.

In our general setting the process π_t is no longer sufficient to determine the optimal stopping rule, however equation for π_t is essential to characterize the value process V_t as a solution of the corresponding RBSDE. Therefore, in Section 1.2 we focus our attention to derivation of a stochastic differential equation for π_t.

After giving some auxiliary facts in Section 1.2 and recalling properties of Girsanov transform we derive the stochastic differential equation for the a posteriori distribution process π_t of the change-point θ, based on knowing it's a priori distribution function ψ and the local martingales $M^i \in \mathcal{M}_{loc}(F, P)$, $i = 0, 1$ (see Theorem 1.2.2). For simplicity we display here Remark 1.2.3 of this theorem:

Remark 1.2.3 *The a posteriori probability process π_t satisfies the following stochastic differential equation:*

$$\pi_t = \pi_0 - \int_0^t (1 - \pi_{s-}) \frac{1 - \psi(s)}{1 - \psi(s-)} dL_s(M^0 - M^\psi, M^\psi) + \int_0^t \frac{1 - \pi_{s-}}{1 - \psi(s-)} \psi(ds), \quad (0.0.13)$$

where

$$M_t^\psi = \int_0^t (1 - \pi_{s-}) dM_s^0 + \int_0^t \pi_{s-} dM_s^1 + \sum_{s \le t} (1 - \pi_{s-}) \frac{\Delta\psi_s}{1 - \psi(s-)} (\Delta M_s^1 - \Delta M_s^0) \quad (0.0.14)$$

(see Lemma 1.2.1) and

$$L_t(X, Y) = X_t - \int_0^t \frac{1}{1 + \Delta Y_s} d[Y, X]_s$$

is the Girsanov transform.

The main steps needed for passing from the Bayesian formulation of the disorder problem with a change in distribution of a sequence of independent and identically distributed random variables imposed by Shiryaev (1978) to the general Bayesian - martingale approach considered in this work are illustrated in Section 1.3. We also derive the equation for the a posteriori distribution $(\pi_n, n \ge 1)$ in discrete time from equation (0.0.13).

In Chapter 2 we introduce the value process V_t of the related optimal stopping problem (see 0.0.10) and show that this process uniquely solves a suitable reflecting backward stochastic differential equation (RBSDE).

We define a solution of RBSDE related to the disorder problem as a triple (Y_t, ν_t, L_t) of adapted processes satisfying:

I) L_t is a uniformly integrable martingale,

II) ν_t is a predictable process with $0 \le \nu_t \le 1$,

III) Y_t is a semimartingale from S^1,

IV) $Y_t \le 1 - \pi_t \quad \text{for all} \quad t \ge 0$,

V) $\lim_{t\to\infty} Y_t = 0, \quad P^\psi - \text{a.s.}$

$$VI) \quad Y_t = Y_0 + \int_0^t (1-\nu_s) I_{(Y_{s-}=1-\pi_{s-})} d\Big(\int_0^{\cdot} \pi_{u-} dK_u - \int_0^{\cdot} \frac{1-\pi_{u-}}{1-\psi(u-)} \psi(du) \Big)_s^+ - \\ - \int_0^t \pi_{s-} dK_s + L_t. \qquad (0.0.15)$$

One of the main theorem of this work is the following

Theorem 2.2.1 *Assume that*
A) ψ *is a distribution function concentrated on* $[0,\infty]$.
B) K *is a predictable increasing process such that* $EK_t < \infty$ *for any* $t \in [0,\infty)$.

There exists a solution of RBSDE (0.0.15) satisfying I)-VI). If a triple (Y_t, ν_t, L_t) *satisfies I)-VI) conditions, then* $Y_t = V_t$ *and* L_t *coincides with the martingale part of the value process* V.

Chapter 3 presents classical disorder problems for Wiener and Poisson processes as particular cases of the General Disorder problem introduced in this thesis. In Section 3.1 we consider the Wiener disorder problem and show that in this case the RBSDE (0.0.15) is equivalent to the free boundary problem considered by Shiryaev (1978).

In order to embed the Wiener disorder problem in our general setting we assume that Ω is the space C of continuous functions $x = (x_t, t \ge 0)$, $\mathcal{F}$ - the Borel σ-algebra $\mathcal{B}(C)$ of C and $(\mathcal{B}_t(C), t \ge 0)$ - the corresponding filtration.

We assume that P^0 is the measure on $(C, \mathcal{B}(C))$ such that $\frac{1}{\sigma}X_t$ is a standard Wiener process, where $X_t(x) = x_t$ is a coordinate process and P^1 is the measure on $(C, \mathcal{B}(C))$ such that the process

$$\frac{1}{\sigma}(X_t - rt)$$

is a Wiener process under P^1, where r is some constant. Then $P^1 \overset{loc}{\sim} P^0$ and the density process of P^1 with respect to P^0 is of the form

$$Z_t \equiv Z_t(x) = \frac{dP_t^1}{dP_t^0}(x) = \exp\{\frac{r}{\sigma}x_t - \frac{r^2}{2\sigma^2}t\}.$$

Thus, $Z_t = \mathcal{E}_t(M)$, with $M_t \equiv M_t(x) = \frac{r}{\sigma}x_t$.

Let ψ be a distribution function such that

$$\psi(0) - \psi(0-) = \pi$$

$$1 - \psi(t) = (1-\pi)\exp\{-\lambda t\}, \quad t > 0, \tag{0.0.16}$$

where λ is a known strictly positive constant and $0 \leq \pi \leq 1$.

In this case $\hat{M}_t^\psi = \frac{r}{\sigma}\int_0^t \pi_{s-}dx_s$ and

$$L_t(M, \hat{M}^\psi) = \frac{r}{\sigma}(x_t - \frac{r}{\sigma}\int_0^t \pi_{s-}ds) \tag{0.0.17}$$

where $\bar{W}_t = X_t - \frac{r}{\sigma}\int_0^t \pi_{s-}ds$ is a Wiener process with respect to the measure $\hat{P}^\psi$ which we shall denote hereafter by P^π. Note also that in this case $\frac{1}{1-\psi(s)}\psi(ds) = \lambda ds$.

Therefore, it follows from equation (0.0.13) that in this case the equation for π_t coincides with the equation derived by Shiryaev (1978)

$$\pi_t = \pi_0 + \frac{r}{\sigma}\int_0^t \pi_s(1-\pi_s)d\bar{W}_s + \lambda\int_0^t (1-\pi_s)ds. \tag{0.0.18}$$

Assuming $K_t = ct$ we consider the cost criterion of the same form as in Shiryaev (1978)

$$\rho_\tau(\pi) = P^\pi(\tau < \theta) + cE^\pi \max(\tau - \theta, 0), \tag{0.0.19}$$

and the value function of the optimal stopping problem (0.0.9) is

$$\rho(\pi) = \inf_\tau E^\pi(1 - \pi_\tau + c\int_0^\tau \pi_s ds). \tag{0.0.20}$$

Since $(\pi_t, \mathcal{F}_t, P^\pi)$ is a time-homogeneous Markov process, we have that

$$V_t = \rho(\pi_t) \quad \text{a.s. for all} \quad t \geq 0. \tag{0.0.21}$$

According to the general theory of optimal stopping the optimal stopping rule is

$$\tau^* = \inf\{t : \rho(\pi_t) = 1 - \pi_t\}. \tag{0.0.22}$$

The key step in the main theorem for the Wiener disorder problem given below is to show that if the value process $V_t = \rho(\pi_t)$ satisfies RBSDE, then the function $\rho(\pi)$ is twice differentiable. In particular this implies that the smooth fit condition is satisfied. This was done by Shiryaev (1978) first solving a suitable free boundary problem and then showing that the unique solution of this problem is the value function. Our main aim here was to show that since the value process $V_t = \rho(\pi_t)$ satisfies RBSDE (0.0.15), the value function $\rho(\pi)$ will be the solution of the free boundary problem considered by Shiryaev.

Theorem 3.1.2 *The value function $\rho(\pi)$ is a non-negative concave function and there is a constant $A^* \in (0, 1]$ such that:*

1) $\rho(\pi)$ is twice continuously differentiable on $(0, A^*)$ and satisfies the PDE

$$\frac{r^2}{2\sigma^2}\pi^2(1-\pi)^2\rho''(\pi) + \lambda(1-\pi)\rho'(\pi) = -c\pi, \quad \text{if} \quad 0 \leq \pi < A^*, \tag{0.0.23}$$

2) $\rho(\pi)$ is equal to $1 - \pi$ if $\pi \geq A^*$ and

3) satisfies the smooth fit condition

$$\rho^{'}(A^{*}) = -1;$$

Besides the value function satisfies the normal entrance condition:

$$\rho^{'}(0+) = 0.$$

Conversely, if $\tilde{\rho}(\pi)$ is a non-negative concave function satisfying 1), 2), 3) for some $B^{*} \in [0,1]$, then the triple $Y_t = \tilde{\rho}(\pi_t), \nu_t = 0$ and L_t equal to the martingale part of $\tilde{\rho}(\pi_t)$ satisfies the RBSDE I)-VI). In particular this implies that $\tilde{\rho}(\pi) = \rho(\pi)$ and $A^{*} = B^{*}$.

It should be mentioned that in this case we prove that the smooth fit of the second derivatives fails (see Remark 3.1.1).

In Section 3.2 we consider the disorder problem for a Poisson process whose intensity changes from λ_0 to λ_1 at some random time θ and show that in this case the RBSDE (0.0.15) is equivalent to the free-boundary differential-difference problem considered by Peskir and Shiryaev (2002).

In order to embed the Poisson disorder problem in our general setting assume that Ω is the space X of piecewise-constant functions $x = (x_t, t \geq 0)$ such that $x_0 = 0$ and $x_t = x_{t-} + (0 \text{ or } 1)$, $\mathcal{B} = \sigma\{x : x_s, s \geq 0\}$ and $\mathcal{B}_t = \sigma\{x : x_s, s \leq t\}$.

Note that for any $x = (x_t, t \geq 0) \in X$, x_t is expressed as

$$x_t = \sum_{i \geq 1} I_{\{\tau_i(x) \leq t\}},$$

where

$$\tau_i(x) = \begin{cases} \inf\{s \geq 0 : x_s = i\} \\ \infty \qquad \lim_{t \to \infty} x_t < i \end{cases} \tag{0.0.24}$$

Let P^0 and P^1 be two Poisson measures on $(X, \mathcal{B})$ with parameters λ_0 and λ_1 respectively. This means that under the measure P^i the compensator of the coordinate process $X_t(x) = x_t, t \geq 0$, is equal to $A_i(t,x) = \lambda_i t$, i=1,2. (Note that the family of $\sigma-$algebras $(\mathcal{B}_t, t \geq 0)$, completed by P^0 and P^1, are right continuous.)

As it is known

$$P^1 \overset{loc}{\sim} P^0 \quad \text{and} \quad \frac{dP^1_t}{dP^0_t} = \exp\{ln\frac{\lambda_1}{\lambda_0}X_t - (\lambda_1 - \lambda_0)t\}$$

(see Liptser and Shiryaev, 2001).

It is easy to see that $\frac{dP^1_t}{dP^0_t} = \mathcal{E}_t(M)$, where

$$M_t = (\frac{\lambda_1}{\lambda_0} - 1)(X_t - \lambda_0 t), \quad M \in \mathcal{M}_{\text{loc}}(F, P^0).$$

Let $\psi(0) - \psi(0-) = \pi$ and $1 - \psi(t) = (1-\pi)\exp\{-\lambda t\}$, where λ is a known constant and $0 \leq \pi \leq 1$.

By Lemma 1.2.1(see Remark 1.2.1)

$$\hat{M}^{\psi}_t = (\frac{\lambda_1}{\lambda_0} - 1)\int_0^t \pi_{s-} d(X_t - \lambda_0 t) \tag{0.0.25}$$

and, hence,

$$L_t(M, \hat{M}^{\psi}) = (\frac{\lambda_1}{\lambda_0} - 1)\int_0^t \frac{dX_s}{1 + \pi_{s-}(\frac{\lambda_1}{\lambda_0} - 1)} - (\lambda_1 - \lambda_0)t. \tag{0.0.26}$$

Since $\Delta\psi_t = 0$, it follows from Remark 1.2.2 that the a posteriori probability process π_t satisfies the same equation, as in Peskir and Shiryaev (2002):

$$d\pi_t = \lambda(1-\pi_{t-})dt + \frac{\pi_{t-}(1-\pi_{t-})(\lambda_1 - \lambda_0)}{\lambda_1\pi_{t-} + \lambda_0(1-\pi_{t-})}(dX_t - (\lambda_1\pi_{t-} + \lambda_0(1-\pi_{t-}))dt), \tag{0.0.27}$$

where the process $(X_t - \int_0^t(\lambda_1\pi_{s-} + \lambda_0(1-\pi_{s-}))ds, \mathcal{B}_t), t \geq 0$ is a martingale under P^{ψ}. Note that $(\pi_t, \mathcal{B}_t, P^{\psi})$ is a time-homogeneous (strong) Markov process and the representation (0.0.21) for the value process is also valid in this case.

As it was mentioned at the beginning in a Poisson disorder problem we show that the free-boundary problem considered by Peskir and Shiryaev (2002) is also equivalent to the well posedness of the general RBSDE (0.0.15). In particular, this shows that the unique solution of the free-boundary differential-difference problem coincides with the value function of the problem. Note that this fact is not clearly proved in Peskir and Shiryaev (2002), since there is only shown that the differential - diff rence equation admits an unique solution and it is not obvious that this solution coincides with the value function. Besides, we derive the smooth-fit conditions for the value function (in cases when this condition is satisfied) and establish when the smooth-fit condition breaks down directly from the RBSDE for the value process.

Note that the cases $\lambda_1 < \lambda_0$ and $\lambda_1 > \lambda_0$ are quite different and we consider these cases separately. e.g., a key difference between these cases is the fact that when $\lambda_1 < \lambda_0$ equation (0.0.28) has no singularity points, whereas $\hat{B} = \frac{\lambda}{\lambda_1 - \lambda_0}$ is a singularity point of (0.0.28), whenever $\lambda < \lambda_1 - \lambda_0$. We first consider the case $\lambda_1 > \lambda_0$.

Theorem 3.2.1 *Let $\lambda_1 > \lambda_0$. The value function $\rho(\pi)$ is a non-negative concave function and there exists a constant $B^* \in (0, 1]$ such that:*
1) $\rho(\pi)$ admits a continuous first derivative on $(0, B^*)$ (perhaps except the point $\hat{B} = \frac{\lambda}{\lambda_1 - \lambda_0}$) and satisfies a differential-difference equation:

$$(\lambda - \pi(\lambda_1 - \lambda_0))(1-\pi)\rho'(\pi) + (\lambda_1\pi + \lambda_0(1-\pi))[\rho(\frac{\lambda_1\pi}{\lambda_1\pi + \lambda_0(1-\pi)}) - \rho(\pi)] = -c\pi \tag{0.0.28}$$

if $\pi < B^*$.
2) it is equal to $1 - \pi$, if $\pi \geq B^*$.
3) it satisfies the continuous fit condition

$$\rho(B^*-) = 1 - B^*.$$

Moreover, if $c > \lambda_1 - \lambda_0 - \lambda$; then
3) the value function $\rho(\pi)$ satisfies the smooth fit condition:*

$$\rho'(B^*-) = -1.$$

Conversely, if $\tilde{\rho}(\pi)$ is a non-negative, concave function satisfying 1), 2), 3) *in the case $c \leq \lambda_1 - \lambda_0 - \lambda$ and* 1), 2), 3*) *in the case $c > \lambda_1 - \lambda_0 - \lambda$ for some $A^* \in (0, 1]$, then the triple $Y_t = \tilde{\rho}(\pi_t), \nu_t = 0$ and L_t equal to the martingale part of $\tilde{\rho}(\pi_t)$ where*

$$L_t \equiv \int_0^t \frac{\pi_{s-}(1-\pi_{s-})(\lambda_1 - \lambda_0)}{\lambda_1 \pi_{s-} + \lambda_0(1-\pi_{s-})} \tilde{\rho}'_-(\pi_{s-})(dX_s - (\lambda_1 \pi_{s-} + \lambda_0(1-\pi_{s-}))ds$$

satisfies the RBSDE I)-VI). In particular t is implies that $\tilde{\rho}(\pi) = \rho(\pi)$ and $A^* = B^*$.

In the next theorem we consider the case $\lambda_1 < \lambda_0$. Note that in this case, contrary to the case $\lambda_1 > \lambda_0$ the process μ_t is not equal to zero, which leads us to additional technical difficulties.

Theorem 3.2.2 *Let $\lambda_1 < \lambda_0$. The value function $\rho(\pi)$ is a non-negative concave function and there exists a constant $B^* \in (0, 1]$ such that:*
1) $\rho(\pi)$ admits a continuous first derivative on $(0, B^*)$ and satisfies a differential-difference equation:

$$(\lambda - \pi(\lambda_1 - \lambda_0))(1-\pi)\rho'(\pi) + (\lambda_1\pi + \lambda_0(1-\pi))[\rho(\frac{\lambda_1 \pi}{\lambda_1\pi + \lambda_0(1-\pi)}) - \rho(\pi)] = -c\pi, \tag{0.0.29}$$

if $\pi < B^*$.
2) it is equal to $1 - \pi$, if $\pi \geq B^*$,
3) it satisfies the smooth fit condition:*

$$\rho'(B^*-) = -1.$$

Conversely, if $\tilde{\rho}(\pi)$ is a non-negative, concave function satisfying 1), 2), 3*) *for some $A^* \in (0, 1]$, then the triple $Y_t = \tilde{\rho}(\pi_t), \nu_t = \frac{(\lambda + \pi_t(\lambda_1 - \lambda_0))(1 - S(\pi_t) - \rho(S(\pi_t)))}{c\pi_t - \lambda(1-\pi_t)} I_{(\tilde{\rho}(\pi_t) = 1 - \pi_t)}$, where $S(\pi) = \frac{\lambda_1 \pi}{\lambda_0 + (\lambda_1 - \lambda_0)\pi}$ and L_t equal to the martingale part of $\tilde{\rho}(\pi_t)$ satisfies the RBSDE I)-VI). In particular this implies that $\tilde{\rho}(\pi) = \rho(\pi)$ and $A^* = B^*$.* □

Finally, in the Conclusions we present the results obtained in this thesis, as well as the possible lines for future research. The Auxiliary facts with the basic concepts and facts used throughout this thesis is included at the end of the thesis.

Chapter 1

Bayesian Formulation of the General Disorder Problem

1.1 Statement of the problem

In this section after some preliminaries we discuss the Bayesian statement of the problem for a general martingale model.

Let $(\Omega, \mathcal{F}, F = (F_t, t \geq 0))$ be a filtered measurable space of the observable events with $\mathcal{F} = F_\infty$. Assume that P^0 and P^1 are two fixed locally equivalent probability measures ($P^1 \overset{loc}{\sim} P^0$) defined on $(\Omega, \mathcal{F})$ and let $\psi = \psi(x)$ be a distribution function of some non-negative random variable. Without loss of generality (e.g., taking $P = \frac{1}{2}(P^1 + P^0)$) one can assume that there is a probability measure P on $(\Omega, \mathcal{F})$ such that

$$P^1 << P, \quad P^0 << P, \quad P^1 \overset{loc}{\sim} P, \quad P^0 \overset{loc}{\sim} P.$$

For simplicity let us assume that the restrictions of the measures P^0 and P^1 coincide on the σ-algebra $\mathcal{F}_0$.

Let $(Z_t^i = \frac{dP_t^i}{dP_t}, t \geq 0), i = 0, 1,$ be the density process of the measure P^i relative to P, which is an uniformly integrable $P-$martingale with $Z_t^i > 0$ $P-a.s.$ for any $t \in [0, \infty[$. Then there exists a local martingale $M^i \in \mathcal{M}_{loc}(F, P)$ such that

$$Z^i = \mathcal{E}(M^i) = (\mathcal{E}_t(M^i), t \geq 0), \quad i = 0, 1,$$

where $\mathcal{E}(M)$, called the Dolean exponential of M, is the unique solution of the linear Stochastic Differential Equation (SDE)

$$Z_t = 1 + \int_0^t Z_{s-} dM_s. \tag{1.1.1}$$

(see Theorem A.4.2).

For the statement of the problem in a general martingale setting let us extend the initial probability space as follows:

$\overline{\Omega} = \Omega \otimes R^+$, $\overline{F} = \mathcal{F} \otimes \mathcal{B}(R^+)$, $\overline{F}_t = F_t \otimes \mathcal{B}(R^+)$, where $\mathcal{B}(R^+)$ is the Borel $\sigma-$algebra on $R^+ = [0, \infty)$.

The measure $\overline{P}^\psi$ on $\mathcal{F} \otimes \mathcal{B}(R^+)$ is defined in a following way: let for every $A \in \mathcal{F}$ and $B \in \mathcal{B}(R^+)$

$$\overline{P}^\psi(A \times B) = \int_A \int_B \mathcal{E}_\infty(M^x) \psi(dx) P(d\omega), \tag{1.1.2}$$

where

$$M_t^x = \int_0^t I_{\{x \leq s\}} dM_s^1 + \int_0^t I_{\{x > s\}} dM_s^0. \tag{1.1.3}$$

Note that, since

$$E\mathcal{E}_\infty(M^x) = E\mathcal{E}_{x-}(M^0) \frac{\mathcal{E}_\infty(M^1)}{\mathcal{E}_{x-}(M^1)} = E\mathcal{E}_{x-}(M^0) E(\frac{\mathcal{E}_\infty(M^1)}{\mathcal{E}_{x-}(M^1)} / F_{x-}) = 1,$$

the Fubini theorem implies that $\overline{P}^\psi$ is a probability measure.

For every $u < v$ and t we have

$$\int_{(u,v]} \mathcal{E}_t(M^x) \psi(dx) = \int_{(u,v]} I_{\{x > t\}} \mathcal{E}_t(M^0) \psi(dx) + \int_{(u,v]} I_{\{x \leq t\}} \mathcal{E}_t(M^x) \psi(dx)$$

$$= \mathcal{E}_t(M^0)(\psi(v \vee t) - \psi(u \vee t)) + \mathcal{E}_t(M^1) \int_{(u, v \wedge t]} \frac{\mathcal{E}_{x-}(M^0)}{\mathcal{E}_{x-}(M^1)} \psi(dx). \tag{1.1.4}$$

So, we could define the measure $\overline{P}^{\psi}$ just by P^0, P^1 and ψ. For every $u < v$ and $A \in \mathcal{F}_t$

$$\overline{P}^{\psi}(A \times]u, v]) = (\psi(v) - \psi((u \vee t))P^0(A) + \int_A \int_{(u, v \wedge t]} \frac{\mathcal{E}_{s-}(M^0)}{\mathcal{E}_{s-}(M^1)} \psi(ds) dP^1.$$

Let us denote by P^{ψ} the restriction of the measure $\overline{P}^{\psi}$ on the $\sigma-$algebra $\mathcal{F} \otimes R^+$ and respectively by P_t^{ψ} we denote the restriction of the measure P^{ψ} on the σ-algebra $\mathcal{F}_t \equiv F_t \otimes R^+$. Then for every $A \in F_t$ we have

$$P_t^{\psi} = P^{\psi}(A \times R^+) = (1 - \psi(t))P^0(A) + \int_A \int_{[0,t]} \frac{\mathcal{E}_{s-}(M^0)}{\mathcal{E}_{s-}(M^1)} \psi(ds) dP^1, \tag{1.1.5}$$

where we assume that $\frac{\mathcal{E}_{0-}(M^0)}{\mathcal{E}_{0-}(M^1)} = 1$;

Thus, the measures $\overline{P}_t^{\psi}$ (and P_t^{ψ}) do not depend on the choice of the dominating measure P.

It is easy to see that $P^{\psi} << P$ and

$$Z_t^{\psi} \equiv \frac{dP_t^{\psi}}{dP_t} = (1 - \psi(t))\mathcal{E}_t(M^0) + \mathcal{E}_t(M^1) \int_{[0,t]} \frac{\mathcal{E}_{s-}(M^0)}{\mathcal{E}_{s-}(M^1)} \psi(ds). \tag{1.1.6}$$

Remark 1.1.1. Since $P^1 \overset{loc}{\sim} P^0$, we have that $P^{\psi} \overset{loc}{\sim} P^0$ and one can express the density process $\hat{Z}_t^{\psi} = dP_t^{\psi}/dP_t^0$ in the form

$$\hat{Z}_t^{\psi} = \frac{dP_t^{\psi}}{dP_t^0} = (1 - \psi(t)) + \mathcal{E}_t(M) \int_{[0,t]} \mathcal{E}_{s-}^{-1}(M) \psi(ds), \tag{1.1.7}$$

where $Z_t = (\mathcal{E}_t(M), t \geq 0)$ is the density process of P^1 relative to P^0.

Let us define on the space $(\overline{\Omega}, \overline{F})$ the random variable

$$\theta = \theta(\overline{\omega}) = \theta(\omega, x) = x.$$

It is evident from (1.1.2) that

$$\overline{P}^{\psi}(\theta \leq x) = \overline{P}^{\psi}(\Omega \times [0, x]) = \psi(x). \tag{1.1.8}$$

This means that the distribution function $\psi = \psi(x)$ by means of which we have defined the new measure $\overline{P}^{\psi}$ on the extended measurable space $(\overline{\Omega}, \overline{F})$ comes to be the a priori distribution function of the variable θ, associated with the random time of 'disorder', which can not be observed directly. We shall also note that θ represents a random instant time of bifurcation of two probability measures given on the measurable space $(\Omega, \mathcal{F})$ endowed with the filtration $F = (F_t, t \geq 0))$ of the observable events. Indeed, by Fubini's theorem the measure $\overline{P}^{\psi}$ can be rewritten as follows:

$$\overline{P}^{\psi}(A \times B) = \int_B \Big(\int_A \mathcal{E}_\infty(M^x) P(d\omega) \Big) \psi(dx), \tag{1.1.9}$$

which implies that for any $A \in \mathcal{F}$ the measure P^{ψ}, which is the restriction of the measure $\overline{P}^{\psi}$ on the $\sigma-$algebra $\mathcal{F} \otimes R^+$, takes the form

$$P^{\psi}(A) = \int_{R^+} P^x(A) \psi(dx),$$

where the measure

$$P^x(A) = \int_A \mathcal{E}_\infty(M^x) P(d\omega)$$

defined on the measurable space $(\Omega, \mathcal{F})$ can be interpreted as the conditional measure given that the bifurcation of the measures P^0 and P^1 has occurred at the time $\theta = x$.

The aim of the problem is to find a stopping time τ with respect to the filtration F_t of observable events (interpreted as the time at which the "alarm signal" is given) which is "as close as possible" to the change point θ. Following Shiryaev (1978) we define the cost criterion by

$$V(\tau) = \bar{P}^{\psi}(\tau < \theta) + E^{\psi}\max(K_\tau - K_\theta, 0), \tag{1.1.10}$$

where $\bar{P}^{\psi}(\tau < \theta)$ is a probability of "false alarm" and $E^{\psi}\max(K_\tau - K_\theta, 0)$ is an average delay (measured by an F_t - predictable increasing process K) of detecting the change point correctly.

The stopping time τ^* is called optimal if

$$V(\tau^*) = \inf_{\tau} V(\tau), \tag{1.1.11}$$

where inf is taken over the class of all F- stopping times.

Introducing the a posteriori probability process π_t

$$\pi_t = \overline{P}^{\psi}(\theta \leq t \mid \mathcal{F}_t),$$

similarly to Shiryaev (1978) one can reduce problem (1.1.11) to the optimal stopping problem

$$V(\tau^*) = \inf_{\tau} E^{\psi}[(1 - \pi_\tau) + \int_0^\tau \pi_{s-} dK_s], \tag{1.1.12}$$

since $\bar{P}^{\psi}(\tau < \theta) = E^{\psi}(1 - \pi_\tau)$ and

$$\bar{E}^{\psi}\max(K_\tau - K_\theta, 0) = \bar{E}^{\psi}\int_0^\tau I_{(\theta \leq s)} dK_s = E^{\psi}\int_0^\tau \pi_{s-} dK_s$$

by the projection theorem (see Theorem A.2.2).

Let us introduce the value process of the problem (1.1.12)

$$V_t = \operatorname*{essinf}_{\tau \geq t} E^{\psi}[(1 - \pi_\tau) + \int_t^\tau \pi_{s-} dK_s / F_t]. \tag{1.1.13}$$

It is well known that (see, *e.g.* El Karoui, 1981) the stopping time τ^* defined by

$$\tau^* = \inf\{t : V_t = 1 - \pi_t\} \tag{1.1.14}$$

is optimal for the problem (1.1.12). In the case of the Wiener disorder problem considered by Shiryaev (1978) the optimal stopping time is of the following simple form

$$\tau^* = \inf\{t : \pi_t \geq A^*\}, \tag{1.1.15}$$

where the constant A^* is a solution of a certain integral equation and the value function V is explicitly calculated as a function of $\psi(0) = \pi$ and A^*. Here the differential equation for the process π_t plays a crucial role.

In our general setting the process π_t is no longer sufficient to determine the optimal stopping rule, however equation for π_t is essential to characterize the value process V_t as a solution of the corresponding RBSDE. Therefore, in the next section we focus our attention to derivation of a stochastic differential equation for π_t.

1.2 Differential equation for the a posteriori distribution process

In this section we derive the stochastic differential equation for the a posteriori distribution process of the change-point θ based on knowing it's a priori distribution function ψ and the local martingales $M^i \in \mathcal{M}_{loc}(F, P)$, $i = 0, 1$.

It follows from the generalized Bayes' Theorem (see A.5.1) that

$$\pi_t = \frac{\int_{R^+} I_{(x \leq t)} \mathcal{E}_t(M^x) \psi(dx)}{Z_t^\psi}, \tag{1.2.1}$$

where

$$Z_t^{\psi} = \int_{R^+} \mathcal{E}_t(M^x)\psi(dx). \tag{1.2.2}$$

Using the formula (1.1.4) we get

$$\pi_t = \frac{\mathcal{E}_t(M^1) \int\limits_{[0,t]} \frac{\mathcal{E}_{s-}(M^0)}{\mathcal{E}_{s-}(M^1)}\psi(ds)}{(1-\psi(t))\mathcal{E}_t(M^0) + \mathcal{E}_t(M^1) \int\limits_{[0,t]} \frac{\mathcal{E}_{s-}(M^0)}{\mathcal{E}_{s-}(M^1)}\psi(ds)}. \tag{1.2.3}$$

Dividing the numerator and the denominator of the right hand side of (1.2.3) on $\mathcal{E}_t(M^0)$, one can write π_t also in the form not depending on the dominating measure P

$$\pi_t = \frac{\mathcal{E}_t(M) \int\limits_{[0,t]} \mathcal{E}_{s-}^{-1}(M)\psi(ds)}{(1-\psi(t)) + \mathcal{E}_t(M) \int\limits_{[0,t]} \mathcal{E}_{s-}^{-1}(M)\psi(ds)}, \tag{1.2.4}$$

where, $\mathcal{E}_t(M) = dP_t^1/dP_t^0$ is the density process of P^1 relative to P^0.

Lemma 1.2.1. *The martingale Z_t^{ψ} is the Dolean exponential of the local martingale M^{ψ} (i.e., $Z_t^{\psi} = \mathcal{E}_t(M^{\psi})$), where*

$$M_t^{\psi} = \int_0^t (1-\pi_{s-})dM_s^0 + \int_0^t \pi_{s-}dM_s^1 + \sum_{s\le t}(1-\pi_{s-})\frac{\Delta\psi_s}{1-\psi(s-)}(\Delta M_s^1 - \Delta M_s^0). \tag{1.2.5}$$

Proof. Note that from (1.2.3) we have that

$$\pi_t Z_t^{\psi} = \mathcal{E}_t(M^1) \int\limits_{[0,t]} \frac{\mathcal{E}_{s-}(M^0)}{\mathcal{E}_{s-}(M^1)}\psi(ds), \tag{1.2.6}$$

$$(1-\pi_t)Z_t^{\psi} = (1-\psi(t))\mathcal{E}_t(M^0). \tag{1.2.7}$$

Therefore, an application of Itô's rule to (1.1.6) yields

$$Z_t^{\psi} = (1-\psi(t))\mathcal{E}_t(M^0) + \mathcal{E}_t(M^1) \int\limits_{[0,t]} \frac{\mathcal{E}_{s-}(M^0)}{\mathcal{E}_{s-}(M^1)}\psi(ds) =$$

$$= 1 + \int_0^t (1-\psi(s-))\mathcal{E}_{s-}(M^0)dM_s^0 + \int_0^t \int_{[0,s)} \frac{\mathcal{E}_{u-}(M^0)}{\mathcal{E}_{u-}(M^1)}\psi(du)\mathcal{E}_{s-}(M^1)dM_1^s +$$

$$\sum_{s\le t}[\Delta\mathcal{E}_s(M^1)\frac{\mathcal{E}_{s-}(M^0)}{\mathcal{E}_{s-}(M^1)} - \Delta\mathcal{E}_s(M^0)]\Delta\psi(s). \tag{1.2.8}$$

Since by the equation (1.2.7)

$$\mathcal{E}_{t-}(M^0) = \frac{Z_{t-}^{\psi}(1-\pi_{t-})}{1-\psi(t-)}, \tag{1.2.9}$$

and

$$\mathcal{E}_{t-}(M)(1+\Delta M_t) = \mathcal{E}_t(M), \tag{1.2.10}$$

we have that the last term of (1.2.8) equals to

$$\sum_{s\le t}(1-\pi_{s-})Z_{s-}^{\psi}\frac{\Delta\psi_s}{1-\psi(s-)}(\Delta M_s^1 - \Delta M_s^0). \tag{1.2.11}$$

Therefore, from (1.2.6), (1.2.7) and (1.2.8) we obtain that $Z_t^{\psi} = \mathcal{E}_t(M^{\psi})$ satisfies

$$Z_t^{\psi} = 1 + \int_0^t (1-\pi_{s-})Z_{s-}^{\psi}dM_s^0 + \int_0^t \pi_{s-}Z_{s-}^{\psi}dM_s^1$$

$$+\sum_{s\le t} Z_{s-}^{\psi}(1-\pi_{s-}))\frac{\Delta\psi_s}{1-\psi(s-)}(\Delta M_s^1 - \Delta M_s^0)$$

$$= 1 + \int_0^t Z_{s-}^{\psi}[(1-\pi_{s-})dM_s^0 + \pi_{s-}dM_s^1 + (1-\pi_{s-})\frac{\Delta\psi_s}{1-\psi(s-)}d(M_s^1 - M_s^0)]. \tag{1.2.12}$$

and the assertion of lemma follows from the uniqueness of the solution of of equation (1.1.1).

Remark 1.2.1. Similarly as above, one can show that the density process $\hat{Z}_t^{\psi}$ defined by (1.1.7) admits the representation $\hat{Z}_t^{\psi} = \mathcal{E}_t(\hat{M}^{\psi}))$, where

$$\hat{M}_t^{\psi} = \int_0^t \pi_{s-}dM_s + \sum_{s\le t}(1-\pi_{s-})\frac{\Delta\psi_s}{1-\psi(s-)}\Delta M_s. \tag{1.2.13}$$

For two semimartingales X and Y, with $\Delta Y_t \neq -1$ for all t, let us denote $L(X,Y)$ the Girsanov transform

$$L_t(X,Y) = X_t - \int_0^t \frac{1}{1+\Delta Y_s} d[Y,X]_s.$$

Note that (see Preposition A.4.4 and (A.4.6))

$$\frac{\mathcal{E}_t(X)}{\mathcal{E}_t(Y)} = \mathcal{E}_t(L(X-Y,Y)). \tag{1.2.14}$$

Since for any X-integrable predictable process H

$$L(H \cdot X, Y) = H \cdot L(X,Y),$$

from (1.2.5)

$$L_t(M^1 - M^\psi, M^\psi) = \int_0^t (1-\pi_{s-}) dL_s\big(M^1 - M^0 - \sum_{u\leq\cdot} \frac{\Delta\psi_u}{1-\psi(u-)} \Delta(M_u^1 - M_u^0), M^\psi\big). \tag{1.2.15}$$

It is also evident that

$$\Delta L(X,Y) = \frac{\Delta X}{1+\Delta Y}$$

and, in particular

$$\Delta L_t(M^1 - M^\psi, M^\psi) = (1-\pi_{t-}) \frac{\Delta(M_t^1 - M_t^0)}{1+\Delta M_t^\psi} \frac{1-\psi(t)}{1-\psi(t-)}. \tag{1.2.16}$$

Theorem 1.2.2. *The a posteriori probability process π_t satisfies the following stochastic differential equation*

$$\pi_t = \pi_0 + \int_0^t \pi_{s-}(1-\pi_{s-}) dL_s\big(M^1 - M^0 - \sum_{u\leq\cdot} \frac{\Delta\psi_u}{1-\psi(u-)} \Delta(M_u^1 - M_u^0), M^\psi\big) +$$

$$\sum_{s\leq t} (1-\pi_{s-})^2 \frac{(1-\psi(s))}{(1-\psi(s-))^2} \frac{\Delta(M_s^1 - M_s^0)}{1+\Delta M_s^\psi} \Delta\psi(s) + \int_0^t \frac{1-\pi_{s-}}{1-\psi(s-)} \psi(ds). \tag{1.2.17}$$

Proof. By virtue of (1.2.6) and (1.2.14)

$$\pi_t = \mathcal{E}_t(L(M^1 - M^\psi, M^\psi)) \int\limits_{[0,t]} \frac{\mathcal{E}_{x-}(M^0)}{\mathcal{E}_{x-}(M^1)} \psi(dx). \qquad (1.2.18)$$

From (1.2.18) using the Itò formula we have

$$\pi_t = \pi_0 + \int\limits_0^t \int\limits_{[0,s)} \frac{\mathcal{E}_{x-}(M^o)}{\mathcal{E}_{x-}(M^1)} \psi(dx) \mathcal{E}_{s-}(L(M^1 - M^\psi, M^\psi)) dL(M^1 - M^\psi, M^\psi)$$

$$+ \int\limits_{[0,t]} \frac{\mathcal{E}_{s-}(M^0)}{\mathcal{E}_{s-}(M^\psi)} \psi(ds) + \sum_{s \leq t} \Delta \mathcal{E}_s(L(M^1 - M^\psi, M^\psi) \frac{\mathcal{E}_{s-}(M^0)}{\mathcal{E}_{s-}(M^1)} \Delta\psi(s). \qquad (1.2.19)$$

Equations (1.2.15) and (1.2.18) imply that the first term of the right-hand side of (1.2.19) is equal to

$$\pi_0 + \int\limits_0^t \pi_{s-}(1 - \pi_{s-}) dL_s(M^1 - M^0 - \sum_{u \leq s} \frac{\Delta\psi_u}{1 - \psi(u-)} \Delta(M_u^1 - M_u^0), M^\psi). \qquad (1.2.20)$$

On the other hand using successively (1.2.10), (1.2.14), (1.2.16) and (1.2.9) we obtain

$$\sum_{s \leq t} \Delta \mathcal{E}_s(L(M^1 - M^\psi, M^\psi) \frac{\mathcal{E}_{s-}(M^0)}{\mathcal{E}_{s-}(M^1)} \Delta\psi(s) =$$

$$\sum_{s \leq t} \Delta L_s(M^1 - M^\psi, M^\psi) \frac{\mathcal{E}_{s-}(M^0)}{\mathcal{E}_{s-}(M^\psi)} \Delta\psi(s) =$$

$$\sum_{s \leq t} (1 - \pi_{s-})^2 \frac{(1 - \psi(s))}{(1 - \psi(s-))^2} \frac{\Delta(M_s^1 - M_s^0)}{1 + \Delta M_s^\psi} \Delta\psi(s). \qquad (1.2.21)$$

Note that (1.2.9) also implies that the second term of the right-hand-side of (1.2.19) is equal to

$$\int\limits_0^t \frac{1 - \pi_{s-}}{1 - \psi(s-)} \psi(ds). \qquad (1.2.22)$$

Therefore relations (1.2.19)-(1.2.22) imply that π_t satisfies the stochastic differential equation (1.2.17).

Remark 1.2.2. Sometimes it is more convenient to write equation (1.2.17) using the martingale $\hat{M}^\psi$ from Remark 1.2.1. Similarly to Theorem 1.2.2 one can show that π_t satisfies equation

$$\pi_t = \pi_0 + \int_0^t \pi_{s-}(1-\pi_{s-})dL_s\big(M - \sum_{u\le\cdot} \frac{\Delta\psi_u}{1-\psi(u-)}\Delta M_u, \hat{M}^\psi\big) +$$

$$\sum_{s\le t}(1-\pi_{s-})^2 \frac{(1-\psi(s))}{(1-\psi(s-))^2}\frac{\Delta M_s}{1+\Delta\hat{M}_s^\psi}\Delta\psi(s) + \int_0^t \frac{1-\pi_{s-}}{1-\psi(s-)}\psi(ds). \quad (1.2.23)$$

In particular, if $\psi(t)$ is continuous, this equation for π_t takes the form

$$\pi_t = \pi_0 + \int_0^t \pi_{s-}(1-\pi_{s-})dL_s(M, \hat{M}^\psi) + \int_0^t \frac{1-\pi_{s-}}{1-\psi(s)}\psi(ds).$$

Remark 1.2.3. a) Another form for the equation for the a posteriori distribution process $(\pi_t, t \ge 0)$ can be given by applying Itô's formula to the left-hand side of (1.2.7):

$$\pi_t = \pi_0 - \int_0^t (1-\pi_{s-})\frac{1-\psi(s)}{1-\psi(s-)}dL_s(M^0 - M^\psi, M^\psi) + \int_0^t \frac{1-\pi_{s-}}{1-\psi(s-)}\psi(ds). \quad (1.2.24)$$

b) Since $L_t(M^0 - M^\psi, M^\psi) \in \mathcal{M}_{loc}(P^\psi)$(see Theorem A.4.6), the second term of (1.2.24) $(1-\pi_-)\frac{1-\psi}{1-\psi_-} \cdot L(M^0 - M^\psi, M^\psi)$ is a martingale part of the decomposition (1.2.24) for $(\pi_t, t \ge 0)$.

1.3 From the general disorder problem to the disorder problem in discrete time

In this section we illustrate main steps needed for passing from the Bayesian formulation of the disorder problem with a change in distribution of a sequence of independent and identically distributed random variables imposed by Shiryaev (1978) to the general Bayesian - martingale approach considered in this work. We also derive the equation for the a posteriori distribution $(\pi_n, n \geq 1)$ in discrete time from the general equation for π_t derived in previous section (see 1.2.17).

For clearness we shall give at first the Bayesian formulation of the problem imposed by Shiryaev (1978):

The author assumes that on a measurable space $(\Omega, \mathcal{F})$ there are given random variable $\theta, \xi_1, \xi_2, ...$ and a probability measure P^π such that θ has a geometric distribution ψ under the P^π:

$$P^\pi(\theta = 0) = \Delta\psi(0) = \pi,$$

$$P^\pi(\theta = n) = \Delta\psi(n) = (1-\pi)(1-p)^{n-1}p, \quad n \geq 1, \tag{1.3.1}$$

where p and π are known constants with $0 < p \leq 1$ and $\pi \in [0,1]$; and for each set $A = \{\omega : \xi_1 \leq y_1, ..., \xi_n \leq y_n\}$, $\omega \in \Omega$

$$P^\pi(A) = \sum_{k=0}^{n-1} \Delta\psi(k+1) P^0\{\omega : \xi_1 \leq y_1, ..., \xi_k \leq y_k\} P^1\{\omega : \xi_{k+1} \leq y_{k+1}, ..., \xi_n \leq y_n\} +$$

$$+\Delta\psi(0) P^1(A) + (1-\psi(n)) P^0(A), \tag{1.3.2}$$

where P^1 and P^0 a e probability measures on $(\Omega, \mathcal{F}^\xi)$, $\mathcal{F}^\xi = \sigma\{\omega, \xi_1, \xi_2, ...\}$, independent of π and having the property that

$$P^j\{\omega : \xi_1 \leq y_1, ..., \xi_n \leq y_n\} = \prod_{k \leq n} P^j\{\omega : \xi_k \leq y_k\} = \prod_{k \leq n} P^j\{\omega : \xi_1 \leq y_k\} \quad j = 0, 1. \tag{1.3.3}$$

Without loss of generality it is assumed that the measures $P^j\{\omega : \xi_1 \leq y\}$ have the densities $p^j(y), j = 0, 1$, with respect to a σ-finite measure μ on $\mathcal{B}(R)$.

Recall that ψ is the a priori distribution function of θ, the obvious meaning of the conditions given by (1.3.1)-(1.3.3) is as follows:

If $\theta = 0$, we observe the sequence of independent identically distributed random variables $\xi_1, \xi_2, ...$ with the probability density $p^1(y)$.

If $\theta = k+1$ the random variables $\xi_1, ..., \xi_k, \xi_{k+1}, ...$ are mutually independent. Besides $\xi_1, ..., \xi_k$ are identically distributed with the probability density $p^0(y)$ and the random variables $\xi_{k+1}, \xi_{k+2}, ...$ are also identically distributed, but with the different probability density $p^1(y)$;

For our purposes we shall rewrite (1.3.2) in the terms of the density processes. It easy to see that the measure P^π is locally absolutely continuous with respect to the direct product of the measure μ, which we denote as $P \equiv \otimes\mu$. Therefore by denoting P_n^π and P_n respectively the restrictions of the measures P^π and P on $\mathcal{B}(R^n)$, it follows from (1.3.2) that:

$$\frac{dP_n^\pi}{dP_n}(y_1, y_2, ..., y_n) = \Delta\psi(0)\prod_{i=1}^{n} p^1(y_i) + \sum_{k=0}^{n-1}\Delta\psi(k+1)\prod_{i=1}^{k} p^0(y_i)\prod_{i=k+1}^{n} p^1(y_i)$$
$$+(1-\psi(n))\prod_{i=1}^{n} p^0(y_i), \qquad (1.3.4)$$

where the density processes $(Z_n^j = \prod_{i=1}^{n} p^j(\xi_i), n \geq 1)$ of the measures P^j relative to $\otimes\mu$ can be written as the Dolean exponential of the local martingales

$$M_n^j = \sum_{k=1}^{n}\left(p^i(\xi_k) - 1\right) \qquad j = 0, 1, \qquad (1.3.5)$$

hence $Z_n^j = \mathcal{E}_n(M^j)$ and (1.3.4) takes the following form:

$$\frac{dP_n^\pi}{dP_n} = (1-\psi(n))\mathcal{E}_n(M^0) + \mathcal{E}_n(M^1)\sum_{k=0}^{n}\frac{\mathcal{E}_{k-1}(M^0)}{\mathcal{E}_{k-1}(M^1)}\Delta\psi(k). \qquad (1.3.6)$$

Taking in mind the above facts it follows from (1.3.6) that for any $A \in \mathcal{F}_n^\xi$ the measure P_n^π can be written as

$$P_n^\pi(A) = \int_A \Big(\sum_{k\in Z^+}\mathcal{E}_n(M^k)\Delta\psi(k)\Big)P_n(d\omega) \qquad (1.3.7)$$

where

$$M_n^k = \sum_{j=1}^{n} I_{\{k\le j\}} \Delta M_j^1 + \sum_{j=1}^{n} I_{\{k>j\}} \Delta M_j^0, \tag{1.3.8}$$

and which plays the same role as the measure $\overline{P}^{\psi}$ defined in general setting(see 1.3.10).

We shall note that by Fubini's theorem the measure $P_n^{\pi}(A)$ can be written as

$$P_n^{\pi}(A) = \sum_{k\in Z+} \Big(\int_A \mathcal{E}_n(M^k) P_n(d\omega)\Big) \Delta\psi(k) = \int_{Z+} P^k(A)\psi(dk), \tag{1.3.9}$$

where

$$P^k(A) = \int_A \mathcal{E}_n(M^k) P_n(d\omega)$$

is the conditional measure defined on $(\Omega, \mathcal{F}^{\xi})$ given that the disruption has occurred at the time $\theta = k$.

After the above discussion we think that now is easier to understand the idea how it can be constructed the general setting from the Bayesian formulation of the disorder problem for i.i.d. variables considered by Shiryaev (1978). Here we give the main steps of this transformation:

Instead of one basic measurable space $(\Omega, \mathcal{F})$ (as it was in Shiryaev's Bayesian formulation) at first we need to introduce the initial measurable space with $\Omega = R^{\infty}, \mathcal{F} = \mathcal{B}(R^{\infty})$ and to define a sequence of the observable random variables $(\xi_n, n \ge 1)$ as being coordinate functions: for any $\omega = (x_1, x_2, ...) \in R^{\infty}$

$$\xi_n(\omega) = x_n, \quad n \ge 1.$$

Besides we assume that P^1 and P^0 are probability measures on $(R^{\infty}, \mathcal{B}(R^{\infty}))$ such that for any $y = (y_1, ..., y_n) \in R^n$ the condition (1.3.3) is fulfilled, which means that $(\xi_n, n \ge 1)$ is the i.i.d. sequence under P^0 and P^1. Moreover, as in Shiryaev's setting we assume that the measures $P^j\{\omega : \xi_1 \le y\}$ have the densities $p^j(y), j = 0, 1$, with respect to some measure μ on $(R, B(R))$ and P is the measure on $(R^\infty, \mathcal{B}(R^\infty)$ defined as direct product of the measure μ. As we have done before the Dolean exponential $Z_n^j = \mathcal{E}_n(M^j)$ can be considered here, where M^j is defined as in (1.3.5).

The main step in general setting (different from Shiryaev) is that after introducing the initial measurable space (space of observable random variables) we consider extended measurable space $(\overline{\Omega}, \overline{F})$, where $\overline{\Omega} = R^\infty \otimes Z^+$, $\overline{F} = \mathcal{B}(R^\infty) \otimes \mathcal{B}(Z^+)$. On this space we define together random variables $\theta, \xi_1, \xi_2, \ldots$ as

$$\theta(\overline{\omega}) = \theta(\omega, x) = x, \quad \xi_n(\overline{\omega}) = \xi_n(\omega, x) = x_n$$

and the measure $\overline{P}^\psi$

$$\overline{P}^\psi(A \times B) = \int_A \big(\sum_{x \in B} \mathcal{E}_\infty(M^x) \Delta\psi(x)\big) P(d\omega) =, \tag{1.3.10}$$

$$= \int_A \int_B \mathcal{E}_\infty(M^x)\psi(dx)P(d\omega) = \int_B P^x(A)\psi(dx) \qquad \forall A \in \mathcal{F}, \forall B \in \mathcal{B}(R^+),$$

where (see 1.3.8)

$$M_t^x = \int_0^t I_{\{x \le s\}} dM_s^1 + \int_0^t I_{\{x > s\}} dM_s^0.$$

We recall the fact that distribution function $\psi = \psi(x)$ by means of which we have defined the new measure $\overline{P}^\psi$ on the extended measurable space $(\overline{\Omega}, \overline{F})$ comes to be the a priori distribution function of the variable θ (see 1.1.8), associated with the random time of 'disorder', which can not be observed directly. We shall also note that the measure $\overline{P}^\psi$ plays the same role as the measure P^π (see 1.3.2) defined in Shiryaev's setting.

The equation for the a posteriori distribution $(\pi_n, n \geq 1)$

Now we shall derive the equation for the a posteriori distribution $(\pi_n, n \geq 1)$ in the discrete time from the general Bayesian - Martingale approach to the disorder problem.

Let $\Omega = R^\infty, \mathcal{F} = \mathcal{B}(R^\infty), F_n = \mathcal{B}(R^n)$ be the corresponding filtration. Let us define a sequence $(\xi_n, n \geq 1)$ of coordinate functions: for any $x = (x_1, x_2, ...) \in R^\infty$ let

$$\xi_n(x) = x_n, \quad n \geq 1.$$

Assume that P^1 and P^0 are probability measures on $(R^\infty, \mathcal{B}(R^\infty))$ such that for any $y = (y_1, ..., y_n) \in R^n$

$$P^j\{x : \xi_1 \leq y_1, ..., \xi_n \leq y_n\} = \prod_{k \leq n} P^j\{x : \xi_k \leq y_k\} = \prod_{k \leq n} P^j\{x : \xi_1 \leq y_k\} \quad j = 0, 1. \tag{1.3.11}$$

Thus, $(\xi_n, n \geq 1)$ is the i.i.d. sequence under P^0 and P^1.

Let us consider an extended measurable space $(\overline{\Omega}, \overline{F})$, where $\overline{\Omega} = R^\infty \otimes Z^+$, $\overline{F} = \mathcal{B}(R^\infty) \otimes \mathcal{B}(Z^+)$ and define random variables $\theta, \xi_1, \xi_2, ...$ as

$$\theta(\overline{\omega}) = \theta(x, k) = k, \quad \xi_n(\overline{\omega}) = \xi_n(x, k) = x_n,$$

where $x = (x_1, ..., x_n, ...) \in R^\infty$.

Let $\psi(n)$ be a distribution function on $Z^+ = \{0, 1, 2, ...\}$. Without loss of generality we may assume that the measures $P^j\{x : \xi_1 \leq y\}$ have the densities $p^j(y), j = 0, 1$, with respect to some measure μ on $(R, B(R))$. Let P be the measure on $(R^\infty, \mathcal{B}(R^\infty)$ defined as direct product of the measure μ. Denote by P_n^0, P_n^1 and P_n respectively restrictions of measures P^0, P^1 and P on $\mathcal{B}(R^n)$. Then in this case

$$Z_n^j = \frac{dP_n^j}{dP_n} = \prod_{k \leq n} p^j(\xi_k)$$

and

$$M_n^j = \sum_{k=1}^{n}(p^j(\xi_k) - 1), \qquad j = 0, 1. \tag{1.3.12}$$

Taking in mind the above facts from (1.1.5) we have that

$$P_n^\psi(A) = \Delta\psi(0)P^1(A) + \int_A \sum_{k=1}^{n} \frac{\mathcal{E}_{k-1}(M^0)}{\mathcal{E}_{k-1}(M^1)}\Delta\psi(k)dP^1 + (1-\psi(n))P^0(A) =$$

$$+\sum_{k=0}^{n-1}\Delta\psi(k+1)P^0\{x : \xi_1 \le y_1, ..., \xi_k \le y_k\}P^1\{x : \xi_{k+1} \le y_{k+1}, ..., \xi_n \le y_n\}+$$

$$+\Delta\psi(0)P^1(A) + (1-\psi(n))P^0(A) \quad \text{for any} \quad A = \{x : \xi_1 \le y_1, ..., \xi_n \le y_n\}. \tag{1.3.13}$$

Therefore (1.2.5) and (1.2.16)we have that

$$1 + \Delta M_{n+1}^\psi = (1-\pi_n)p^0(\xi_{n+1})\frac{1-\psi(n+1)}{1-\psi(n)}+$$

$$\pi_n p^1(\xi_{n+1}) + (1-\pi_n)p^1(\xi_{n+1})\frac{\Delta\psi(n+1)}{1-\psi(n)} \tag{1.3.14}$$

and

$$\Delta\mathcal{L}_n(M^1 - M^0 - \sum_{k\le n}\frac{\Delta\psi(k)}{1-\psi(k-1)}\Delta(M^1-M^0), M^\psi) = \frac{p^1(\xi_n) - p^0(\xi_n)}{1+\Delta M_n^\psi}\frac{1-\psi(n)}{1-\psi(n-1)}.$$

Using now (1.2.17), we get that ($P^\psi - a.s.$) the a posteriori distribution π_t satisfies the following recursive equation::

$$\pi_{n+1} = \pi_n + \pi_n(1-\pi_n)\frac{p^1(\xi_{n+1}) - p^0(\xi_{n+1})}{1+\Delta M_{n+1}^\psi}\frac{1-\psi(n+1)}{1-\psi(n)}$$

$$+(1-\pi_n)^2\frac{1-\psi(n+1)}{1-\psi(n)}\frac{\Delta\psi(n+1)}{1-\psi(n)}\frac{p^1(\xi_{n+1}) - p^0(\xi_{n+1})}{1+\Delta M_{n+1}^\psi}$$

$$+(1-\pi_n)\frac{\Delta\psi(n+1)}{1-\psi(n)}. \tag{1.3.15}$$

Substituting (1.3.14) in (1.3.15) after some simple calculations we derive that for any a priori distribution function ψ of the change-point θ the a posteriori probability process π_t in a discrete time has the following form:

$$\pi_{n+1} = \frac{\pi_n p^1(\xi_{n+1}) + (1-\pi_n)p^1(\xi_{n+1})\frac{\Delta\psi(n+1)}{1-\psi(n)}}{(1-\pi_n)p^0(\xi_{n+1})\frac{1-\psi(n+1)}{1-\psi(n)} + \pi_n p^1(\xi_{n+1}) + (1-\pi_n)p^1(\xi_{n+1})\frac{\Delta\psi(n+1)}{1-\psi(n)}} \tag{1.3.16}$$

If similarly to Shiryaev (1978) we assume that a priori θ has the geometric distribution:

$$P^\psi(\theta = 0) = \psi(0) - \psi(0-) = \pi,$$

$$P^\psi(\theta = n) = (1-\pi)(1-p)^{n-1}p, \quad n \geq 1, \tag{1.3.17}$$

where p and π are known constants with $0 < p \leq 1$ and $\pi \in [0,1]$, then it follows from equation (1.2.17) that

$$\pi_{n+1} = \frac{\pi_n p_1(\xi_{n+1}) + (1-\pi_n)pp_1(\xi_{n+1})}{\pi_n p_1(\xi_{n+1}) + (1-\pi_n)pp_1(\xi_{n+1}) + (1-\pi_n)(1-p)p_0(\xi_{n+1})}, \tag{1.3.18}$$

which coincides with the equation for the a posteriori distribution function in the discrete time derived by Shiryaev (1978).

Chapter 2

Differential characterization of the Value process related to the Disorder problem

2.1 The Value process related to the Disorder problem and some basic (known) facts from the optimal stopping theory

Let us introduce the value process of the problem (1.1.12)

$$V_t = \operatorname*{essinf}_{\tau \geq t} E^{\psi}[(1 - \pi_\tau) + \int_t^\tau \pi_{s-} dK_s / F_t],$$

where E^{ψ} is an expectation w.r.t. the measure P^{ψ}, which we consider as a reference probability measure throughout this chapter.

It is well known that (see *e.g.*, El Karoui, 1981) V_t is a RCLL process such that

i) $V_t \leq 1 - \pi_t$ for all t,

ii) the process $V_t + \int_0^t \pi_{s-} dK_s$ is a submartingale,

iii) V_t is the largest process satisfying i) and ii).

Moreover for any $t \geq 0$ the stopping time τ^* defined by

$$\tau_t^* = \inf\{s \geq t : V_s = 1 - \pi_s\}$$

is t-optimal (at least if K and ψ are continuous and F is quasi-left-continuous (see El Karoui, 1981, or Jacka, 1993)), that is

$$V_t = E^{\psi}[(1 - \pi_{\tau_t^*}) + \int_t^{\tau_t^*} \pi_{s-} dK_s / F_t].$$

Hence V_t is a special semimartingale with canonical decomposition

$$V_t = V_0 - \int_0^t \pi_{s-} dK_s + B_t + N_t, \tag{2.1.1}$$

where N is a martingale and B is a predictable increasing process with $B_0 = 0$.

It is also well-known (see *e.g.* El Karoui, 1981, Jacka, 1993 or Shashiashvili, 1993) that increasing process B_t is growing only on the set $\{V_{t-} = 1 - \pi_{t-}\}$(on the stop region) and $V_t + (\pi_- \cdot K)_t$ is a martingale on the go-region $\{V_{t-} < 1 - \pi_{t-}\}$, i.e., the process B_t satisfies relation

$$\int_0^T I_{\{V_{s-} < 1 - \pi_{s-}\}} dB_s = 0, \tag{2.1.2}$$

which implies that the process

$$\int_0^t I_{\{V_{s-} < 1 - \pi_{s-}\}} d(V_s + \int_0^s \pi_{u-} dK_u) = \int_0^t I_{\{V_{s-} < 1 - \pi_{s-}\}} dN_s$$

is a martingale.

Note that relation (2.1.2) guaranties the maximality of V and together with i) and ii) uniquely determines the value process. But the maximality of V as well, as condition (2.1.2) is difficult to verify and this leads to necessity to give a differential characterization of the value process. We shall combine the results of Chitashvili (1988), Jacka (1993), Shashiashvili (1993) and El Karoui et al.(1997) to derive a reflecting BSDE for the process V in our case.

2.2 Reflecting Backward Stochastic Differential Equation (RBSDE) for the Value process

In this section we provide the reflecting RBSDE for the value process of the optimal stopping problem (1.1.12).

Denote by S^1 the class of semimartingales X with the decomposition

$$X_t = X_0 + A_t + M_t, \quad t \geq 0,$$

where M_t is a uniformly integrable martingale and A_t is a process of integrable variation on $[0, \infty]$.

We define a solution of RBSDE related to the disorder problem as a triple (Y_t, ν_t, L_t) of adapted processes satisfying:

I) $L_t \in \mathcal{M}$,

II) $\nu_t \in \mathcal{P}$ with $0 \leq \nu_t \leq 1$,

III) Y_t is a semimartingale from S^1,

$IV)$ $Y_t \leq 1 - \pi_t$ for all $t \geq 0$,

$V)$ $\lim_{t\to\infty} Y_t = 0, \quad P^\psi - \text{a.s.}$

$$VI) \quad Y_t = Y_0 + \int_0^t (1 - \nu_s) I_{(Y_{s-}=1-\pi_{s-})} d\Big(\int_0 \pi_{u-} dK_u - \int_0 \frac{1-\pi_{u-}}{1-\psi(u-)} \psi(du)\Big)_s^+ -$$

$$-\int_0^t \pi_{s-}dK_s + L_t. \tag{2.2.1}$$

Theorem 2.2.1. *Assume that*

A) ψ is a distribution function concentrated on $[0,\infty]$.

B) K is a predictable increasing process such that $EK_t < \infty$ for any $t \in [0,\infty)$.

There exists a solution of RBSDE (2.2.1) satisfying I)-VI). If a triple (Y_t, ν_t, L_t) satisfies conditions I)-VI), then $Y_t = V_t$ and L_t coincides with the martingale part of the value process V.

Proof: Using equation (1.2.24) for π_t and decomposition (2.1.1) we have

$$1-\pi_t - V_t = 1-\pi_0 - V_0 - \int_0^t \frac{1-\pi_{s-}}{1-\psi(s-)}\psi(ds) + \int_0^t \pi_{s-}dK_s -$$

$$-B_t + \int_0^t (1-\pi_{s-})\frac{1-\psi(s)}{1-\psi(s-)}d\tilde{M}_s - N_t, \tag{2.2.2}$$

where by $\tilde{M}$ we denoted the P^ψ–martingale $\tilde{M}_t = L_t(M^0 - M^\psi, M^\psi)$.

By Tanaka's formula

$$(1-\pi_t - V_t)^+ = (1-\pi_0 - V_0)^+ + \int_0^t I_{\{1-\pi_{s-}>V_{s-}\}}d(1-\pi_s - V_s) +$$

$$\frac{1}{2}\mathcal{L}_t^0(1-\pi-V) + \sum_{0<s\le t} I_{(1-\pi_{s-}-V_{s-}=0)}(1-\pi_s - V_s), \tag{2.2.3}$$

where $\mathcal{L}_t^0(1-\pi-V)$ is the local time of the process $1-\pi_t - V_t$ at 0. Therefore, from (2.2.2) and(2.2.3)

$$(1-\pi_t - V_t)^+ = (1-\pi_0 - V_0)^+ + \int_0^t I_{(1-\pi_{s-}>V_{s-})}\pi_{s-}dK_s - \int_0^t I_{(1-\pi_{s-}>V_{s-})}\frac{1-\pi_{s-}}{1-\psi(s-)}\psi(ds) -$$

$$\int_0^t I_{(1-\pi_{s-}>V_{s-})}dB_s + \sum_{0<s\le t} I_{(1-\pi_{s-}-V_{s-}=0)}(1-\pi_s-V_s) + \frac{1}{2}\mathcal{L}_t^0(1-\pi-V)+$$

$$\int_0^t I_{(1-\pi_{s-}>V_{s-})}d\Big(\int_0^{\cdot} \pi_{u-}(1-\pi_{u-})d\tilde{M}_u - N_s\Big). \qquad (2.2.4)$$

Since $V_t \le 1-\pi_t$ and $\int_0^t I_{(1-\pi_{s-}>V_{s-})}dB_s = 0$, comparing the finite variation parts of right-hand sides of (2.2.2) and(2.2.4) we obtain that

$$\int_0^t I_{(1-\pi_{s-}=V_{s-})}\pi_{s-}dK_s - \int_0^t I_{(1-\pi_{s-}=V_{s-})}\frac{1-\pi_{s-}}{1-\psi(s-)}\psi(ds) - \Big(\frac{1}{2}\mathcal{L}_t^0(1-\pi-V)+\tilde{A}_t\Big) = B_t. \qquad (2.2.5)$$

where by $\tilde{A}_t$ we have denoted the compensator of the process $(\sum_{0<s\le t} I_{(1-\pi_{s-}-V_{s-}=0)}(1-\pi_s-V_s), t\ge 0)$.

Since B and $\mathcal{L}^0$ are increasing processes, relation (2.2.5) implies that the measures dB_t and $d(\mathcal{L}^0+\tilde{A})_t$ are absolutely continuous w.r.t. the measure dK_t. Moreover, from (2.2.5) we also have

$$\int_0^t I_{(1-\pi_{s-}=V_{s-})}d\Big(\pi_-\cdot K - \frac{1-\pi_-}{1-\psi_-}\cdot\psi\Big)_s^+ - \Big(\frac{1}{2}\mathcal{L}_t^0(1-\pi-V)+\tilde{A}_t\Big) =$$

$$= B_t + \int_0^t I_{(1-\pi_{s-}=V_{s-})}d\Big(\pi_-\cdot K - \frac{1-\pi_-}{1-\psi_-}\cdot\psi\Big)_s^- \in \mathcal{A}_{loc}^+ \qquad (2.2.6)$$

is an increasing process. Therefore

$$\frac{1}{2}\mathcal{L}_t^0(1-\pi-V)+\tilde{A}_t << I_{(1-\pi_-=V_-)}\cdot\Big(\pi_-\cdot K - \frac{1-\pi_-}{1-\psi_-}\cdot\psi\Big)^+$$

and hence, there exists a predictable process μ_t such that

$$\Big(\frac{1}{2}\mathcal{L}_t^0(1-\pi-V)+\tilde{A}_t\Big) = \int_0^t \mu_s I_{(1-\pi_{s-}=V_{s-})}d\Big(\int_0^{\cdot}\pi_{u-}dK_u - \int_0^{\cdot}\frac{1-\pi_{u-}}{1-\psi(u-)}\psi(du)\Big)_s^+, \qquad (2.2.7)$$

where $A_t = A_t^+ - A_t^-$ is a unique decomposition of a process of finite variation A as a difference of two increasing processes such that the non-negative measures induced by A^+ and A^- on $[0,t]$ have disjoint supports. The variation of such a process is given by $(VarA)_t = A_t^+ + A_t^-$.

It follows from (2.2.6) and(2.2.7) that

$$\int_0^t (1-\mu_s) I_{(1-\pi_{s-}=V_{s-})} d(\pi_- \cdot K - \frac{1-\pi_-}{1-\psi_-} \cdot \psi)_s^+ -$$

$$-\int_0^t I_{(1-\pi_{s-}=V_{s-})} d(\pi_- \cdot K - \frac{1-\pi_-}{1-\psi_-} \cdot \psi)_s^- = B_t \in \mathcal{A}_{loc}^+, \tag{2.2.8}$$

which implies that

$$0 \le \mu_s \le I_{(1-\pi_{s-}=V_{s-})} \quad d(\pi_- \cdot K - \frac{1-\pi_-}{1-\psi_-} \cdot \psi)_s^+ - \text{a.e.} \quad \text{and} \tag{2.2.9}$$

$$\{s : 1-\pi_{s-} = V_{s-}\} \subseteq \text{supp}(\pi_- \cdot K - \frac{1-\pi_-}{1-\psi_-} \cdot \psi)_s^+. \tag{2.2.10}$$

In particular, we have that

$$B_t = \int_0^t (1-\mu_s) I_{(1-\pi_{s-}=V_{s-})} d(\pi_- \cdot K - \frac{1-\pi_-}{1-\psi_-} \cdot \psi)_s^+$$

$$= \int_0^t (1-\mu_s) I_{(1-\pi_{s-}=V_{s-})} d(\pi_- \cdot K - \frac{1-\pi_-}{1-\psi_-} \cdot \psi)_s. \tag{2.2.11}$$

Therefore (2.2.11) and(2.1.1) imply that

$$V_t = V_0 + \int_0^t (1-\mu_s) I_{(V_{s-}=1-\pi_{s-})} d\Big(\int_0 \pi_{u-} dK_u - \int_0 \frac{1-\pi_{u-}}{1-\psi(u-)} \psi(du)\Big)_s^+ -$$

$$-\int_0^t \pi_{s-} dK_s + N_t, \tag{2.2.12}$$

which means that the triple (V, μ, N) satisfies equation (2.2.1).

It follows from equality (2.2.11) that the value process satisfies also equation

$$V_t = V_0 - \int_0^t (I_{(1-\pi_{s-} > V_{s-})} + \mu_s I_{(1-\pi_{s-} = V_{s-})})\pi_{s-} dK_s -$$

$$- \int_0^t (1-\mu_s) I_{(1-\pi_{s-} = V_{s-})} \frac{1-\pi_{s-}}{1-\psi(s-)} \psi(ds) + N_t, \qquad (2.2.13)$$

which implies that V_t is a supermartingale. Since V is bounded, it is a supermartingale of the class $\mathcal{D}$ (see Definition A.1.3) and by the uniqueness of the Doob-Meyer decomposition (see Theorem A.1.1) N is a uniformly integrable martingale and V is a semimartingale from the class S^1.

Since $0 \le V_t \le 1 - \pi_t$ and $\lim_{t\to\infty} \pi_t = 1$ (P^{ψ}-a.s.), we have that $\lim_{t\to\infty} V_t$ exists and is equal to zero.

Thus, the triple (V, μ, N) is a solution of I)-VI).

Uniqueness: Let a triple (Y_t, ν_t, L_t) be a solution of I)-VI). Then it follows from (2.2.1) and II) that the process $Y_t + \int_0^t \pi_{s-} dK_s$ is a submartingale. Since V_t is the largest process that satisfies i) and ii), we have $V_t \ge Y_t$.

Let us show that $Y_t \ge V_t$. Let

$$\sigma_t = \inf\{s \ge t : Y_s = 1 - \pi_s\}.$$

By condition IV) we have $Y_t < 1 - \pi_t$ on the interval $[t; \sigma_t)$. Therefore, it follows from (2.2.1)

$$Y_{\sigma_t} - Y_t = - \int_t^{\sigma_t} \pi_{s-} dK_s + L_{\sigma_t} - L_t. \qquad (2.2.14)$$

On the other hand condition V) implies that $Y_{\sigma_t} = 1 - \pi_{\sigma_t}$. Therefore taking conditional expectations in (2.2.14) we obtain that

$$Y_t = E(1 - \pi_{\sigma_t} + \int_t^{\sigma_t} \pi_{s-} dK_s / \mathcal{F}_t)$$

and by definition of the value process $Y_t \geq V_t$. Thus $Y_t = V_t$. It is evident that the martingale parts of V and Y are also indistinguishable. □

Remark 2.2.1. By (2.2.7), (2.2.10) and (2.2.13) we have that the value process also satisfies the following equation:

$$V_t = V_0 - \int_0^t I_{(1-\pi_{s-} > V_{s-})} \pi_{s-} dK_s - \int_0^t I_{(1-\pi_{s-} = V_{s-})} \frac{1 - \pi_{s-}}{1 - \psi(s-)} \psi(ds)$$

$$-(\frac{1}{2} \mathcal{L}_t^0 (1 - \pi - V) + \tilde{A}_t) + N_t. \qquad (2.2.15)$$

Remark 2.2.2. Comparing the martingale parts of (2.2.2) and(2.2.3) we have that

$$\int_0^t I_{(1-\pi_{s-} = V_{s-})} (1 - \pi_{s-}) \frac{1 - \psi(s)}{1 - \psi(s-)} d\tilde{M}_s = \int_0^t I_{(1-\pi_{s-} = V_{s-})} dN_s +$$

$$+ (\sum_{0 < s \leq t} I_{(1-\pi_{s-} - V_{s-} = 0)} (1 - \pi_s - V_s) - \tilde{A}_t). \qquad (2.2.16)$$

Let us write the a priori distribution functions in the form:

$$\psi^{\pi}(t) = \pi \delta_0(t) + (1 - \pi) \varphi(t) \qquad (2.2.17)$$

where $\delta_0(t)$ is a dirac measure having a mass at 0, and $\varphi(t)$ is any fixed distribution function of some positive random variable. From now on taking expectation with respect to the measure $\bar{P}^{\psi^\pi}$(resp. P^{ψ^π}) we will denote as $\bar{E}^\pi$(resp. E^π) ($\bar{E}^{\psi^\pi} \to \bar{E}^\pi$). Hence the value V_0 can be rewritten as a function of π (π and ω in general):

$$V_0(\pi) = \inf_\tau E^\pi[(1-\pi_\tau) + \int_0^\tau \pi_s dK_s]$$

Now we shall prove the concavity of the value function $V_0(\pi)$, which will be essentially used in the sequel. For the value function corresponding to the classical disorder problems this fact was proved in Shiryaev (1978).

Lemma 2.2.2. *The value function $V_0(\pi)$ is a concave function of π.*

Proof: We need to show that for any $\pi_1, \pi_2 \in [0,1]$ and $\alpha \in (0,1)$

$$V_0(\alpha\pi_1 + (1-\alpha)\pi_2) \geq \alpha V_0(\pi_1) + (1-\alpha)V_0(\pi_2),$$

Let $\pi = \alpha\pi_1 + (1-\alpha)\pi_2$. By (2.2.17) $\psi^\pi(t) = \alpha\psi^{\pi_1}(t) + (1-\alpha)\psi^{\pi_2}(t)$ and $\bar{P}^{\psi^\pi} = \alpha\bar{P}^{\psi^{\pi_1}} + (1-\alpha)\bar{P}^{\psi^{\pi_2}}$ by the definition of the measure $\bar{P}^\psi$ (see 1.1.2).

As $V_0(\pi) = \inf_\tau \bar{E}^\pi(I_{(\tau<\theta)} + (K_\tau - K_\theta)^+)$ the concavity of the function $V_0(\pi)$ is straightforward. □

Chapter 3

Classical Disorder Problems

3.1 The Wiener disorder problem. The equivalence between RBSDE related to the Wiener disorder problem and the free-boundary problem of the parabolic differential operator

In this section we consider the classical disorder problem for a Wiener process and show that in this case the RBSDE (2.2.1) is equivalent to the free boundary problem considered by Shiryaev (1978).

Let Ω be the space C of continuous functions $x = (x_t, t \geq 0)$, $\mathcal{F}$ the Borel σ-algebra $\mathcal{B}(C)$ of C, $(\mathcal{B}_t(C), t \geq 0)$ the corresponding filtration.

Assume that P^0 is the measure on $(C, \mathcal{B}(C))$ such that $\frac{1}{\sigma}X_t$ is a standard Wiener process, where $X_t(x) = x_t$ is a coordinate process and P^1 is the measure on $(C, \mathcal{B}(C))$ such that the process

$$\frac{1}{\sigma}(x_t - rt)$$

is a Wiener process under P^1, where r is some constant. Then $P^1 \overset{loc}{\sim} P^0$ and the density process of P^1 with respect to P^0 is of the form

$$Z_t \equiv Z_t(x) = \frac{dP_t^1}{dP_t^0}(x) = \exp\{\frac{r}{\sigma}x_t - \frac{r^2}{2\sigma^2}t\}.$$

Thus, $Z_t = \mathcal{E}_t(M)$, with $M_t \equiv M_t(x) = \frac{r}{\sigma}x_t$. We shall note that in sequel we will omit the argument x in similar cases.

Let ψ be a distribution function such that

$$\psi(0) - \psi(0-) = \pi$$

$$1 - \psi(t) = (1-\pi)\exp\{-\lambda t\}, \quad t > 0, \tag{3.1.1}$$

where λ is a known strictly positive constant and $0 \leq \pi \leq 1$.

In this case $\hat{M}_t^\psi = \frac{r}{\sigma}\int\limits_0^t \pi_{s-}dx_s$ (see Remark 1.2.1) and by (A.4.5)

$$L_t(M, \hat{M}^\psi) = \frac{r}{\sigma}(x_t - \frac{r}{\sigma}\int\limits_0^t \pi_{s-}ds) \tag{3.1.2}$$

where $\bar{W}_t = x_t - \frac{r}{\sigma}\int\limits_0^t \pi_{s-}ds$ is a Wiener process with respect to the measure $\hat{P}^\psi$ (see Theorem A.4.8), which we shall denote hereafter by P^π. Note also that in this case $\frac{1}{1-\psi(s)}\psi(ds) = \lambda ds$.

Therefore, it follows from equation (1.2.23) (see Remark 1.2.2) that in this case the equation for π_t coincides with the equation derived by Shiryaev (1978)

$$\pi_t = \pi_0 + \frac{r}{\sigma}\int\limits_0^t \pi_s(1-\pi_s)d\bar{W}_s + \lambda\int\limits_0^t (1-\pi_s)ds. \tag{3.1.3}$$

Lemma 3.1.1. *Let $a \leq \pi$, where $a, \pi \in [0,1)$. Then*

$$0 < \lambda(1-a)\int\limits_0^\infty P^\pi(\pi_s \leq a)ds \leq E^\pi \mathcal{L}_\infty^\pi(a) \leq 2(1-\pi). \tag{3.1.4}$$

Proof. By the Itô-Tanaka formula

$$|\pi_t - a| = |\pi - a| + \lambda \int_0^t (1-\pi_s) sign(\pi_s - a) ds + \mathcal{L}_t^\pi(a) +$$

$$\frac{r}{\sigma} \int_0^t \pi_s (1-\pi_s) sign(\pi_s - a) d\tilde{W}_t. \qquad (3.1.5)$$

Taking expectations with respect to the measure P^π, since the stochastic integral from (3.1.5) is a martingale, we have

$$E^\pi \mathcal{L}_t^\pi(a) = E^\pi |\pi_t - a| - |\pi - a| - \lambda E^\pi \int_0^t (1-\pi_s) sign(\pi_s - a) ds. \qquad (3.1.6)$$

Since (3.1.3) implies that

$$E^\pi (1-\pi_s) = (1-\pi) \exp\{-\lambda t\}, \qquad (3.1.7)$$

from (3.1.6) we obtain

$$E^\pi \mathcal{L}_t^\pi(a) \leq E^\pi |\pi_t - a| - |\pi - a| + \lambda \int_0^t E^\pi (1-\pi_s) ds \leq$$

$$\leq E^\pi |\pi_t - a| - |\pi - a| + (1-\pi)(1 - \exp\{-\lambda t\}).$$

Therefore, the passage to the limit as $t \to \infty$ in the last inequality, taking in mind that $\lim_{t\to\infty} \pi_t = 1$, gives the second inequality of (3.1.4)

$$E^\pi \mathcal{L}_\infty^\pi(a) \leq 1 - a - (\pi - a) + (1 - \pi) = 2(1-\pi).$$

On the other hand, from (3.1.6) we also have

$$E^\pi \mathcal{L}_t^\pi(a) = E^\pi |\pi_t - a| - |\pi - a| - \lambda E^\pi \int_0^t (1-\pi_s) I_{(\pi_s > a)} ds + \lambda E^\pi \int_0^t (1-\pi_s) I_{(\pi_s \leq a)} ds \geq$$

$$\geq E^\pi|\pi_t - a| - |\pi - a| - \lambda E^\pi \int_0^t E^\pi(1-\pi_s)ds + \lambda(1-a)E^\pi \int_0^t I_{(\pi_s \leq a)}ds. \quad (3.1.8)$$

It follows from (3.1.7) and relation $\lim_{t\to\infty} \pi_t = 1$ that for $\pi \geq a$

$$\lim_{t\to\infty} \left(E^\pi|\pi_t - a| - |\pi - a| - \lambda \int_0^t E^\pi(1-\pi_s)ds\right) = 0.$$

Therefore, passing to the limit in (3.1.8) we obtain the validity of the inequality

$$E^\pi \mathcal{L}_\infty^\pi(a) \geq \lambda(1-a) \int_0^\infty P^\pi(\pi_s \leq a)ds.$$

Finally, since

$$\int_{(\pi-\varepsilon,\pi+\varepsilon)} \frac{1+\lambda(1-x)}{x^2(1-x)^2}dx < \infty, \quad \text{for some} \quad \varepsilon > 0,$$

at every $\pi \in (0,1)$, the process π_t is regular in $(0,1)$ (see, *e.g.* Dayanik and Karatzas, 2003). This means that π_t reaches a level x with positive probability starting at π, for every x and π from $(0,1)$. Therefore $\int_0^\infty P^\pi(\pi_s \leq a)ds$ is strictly positive. □

Assume that $K_t = ct$. So, the cost criterion is of the same form as in Shiryaev (1978)

$$\rho_\tau(\pi) = P^\pi(\tau < \theta) + cE^\pi \max(\tau - \theta, 0), \quad (3.1.9)$$

and the value function of the optimal stopping problem (1.1.12) is

$$\rho(\pi) = \inf_\tau E^\pi(1 - \pi_\tau + c\int_0^\tau \pi_s ds). \quad (3.1.10)$$

Since $(\pi_t, \mathcal{F}_t, P^\pi)$ is a time-homogeneous Markov process, we have that

$$V_t = \rho(\pi_t) \quad \text{a.s. for all} \quad t \geq 0. \quad (3.1.11)$$

According to the general theory of optimal stopping the optimal stopping rule is

$$\tau^* = \inf\{t : \rho(\pi_t) = 1 - \pi_t\}. \quad (3.1.12)$$

Since $\rho(\pi)$ is concave by Lemma 2.2.2, $\rho(\pi) \leq 1-\pi$ and $\rho(\pi) = 1-\pi$ if $\pi = 1$, we have that $\rho(\pi) = 1-\pi$ for all $\pi \geq A^*$ and $\rho(\pi) < 1-\pi$ if $\pi < A^*$, where

$$A^* = \inf\{A : \rho(A) = 1 - A\}.$$

Therefore, the optimal stopping time of (1.1.12) is in this case of the form

$$\tau^*(\pi) = \inf\{t : \pi_t \geq A^*\} \tag{3.1.13}$$

and the aim is to calculate $\rho(\pi)$ and the constant A^*. This was done by Shiryaev (1978) first solving a suitable free boundary problem and then showing that the unique solution of this problem is the value function. Our main aim in this section is to show that since the value process $V_t = \rho(\pi_s)$ satisfies RBSDE (2.2.1), the value function $\rho(\pi)$ will be the solution of the free boundary problem considered by Shiryaev.

Theorem 3.1.2. *The value function $\rho(\pi)$ is a non-negative concave function and there is a constant $A^* \in (0,1]$ such that:*

1) $\rho(\pi)$ is twice continuously differentiable on $(0, A^*)$ and satisfies the PDE

$$\frac{r^2}{2\sigma^2}\pi^2(1-\pi)^2\rho''(\pi) + \lambda(1-\pi)\rho'(\pi) = -c\pi, \quad \text{if} \quad 0 \leq \pi < A^*, \tag{3.1.14}$$

2) $\rho(\pi)$ is equal to $1-\pi$ if $\pi \geq A^*$ and

3) satisfies the smooth fit condition

$$\rho'(A^*) = -1;$$

Besides the value function satisfies the normal entrance condition:

$$\rho'(0+) = 0.$$

Conversely, if $\tilde{\rho}(\pi)$ *is a non-negative concave function satisfying* 1), 2), 3) *for some* $B^* \in [0,1]$, *then the triple* $Y_t = \tilde{\rho}(\pi_t), \nu_t = 0$ *and* L_t *equal to the martingale part of* $\tilde{\rho}(\pi_t)$ *satisfies the RBSDE I)-VI). In particular this implies that* $\tilde{\rho}(\pi) = \rho(\pi)$ *and* $A^* = B^*$.

Proof. Let $D = \{\pi : \rho(\pi) < 1 - \pi\}$ and let ∂D be the boundary of this set. It is evident that $\rho(\pi) \leq 1 - \pi$ and $\rho(1) = 0$ (since $\pi_t = 1$ for all $t \geq 0$, if $\pi_0 = 1$). Therefore, the concavity of $\rho(\pi)$ implies that ∂D contains only one point (say A^*) and according to Theorem 6 of Jacka (1993) $L^0(1 - \pi - V) = 0$, which means that the process μ_t from (2.2.13) is indistinguishable from zero.

Thus (3.1.11) and (2.2.13) imply that the value process $V_t = \rho(\pi_t)$ satisfies equation

$$\rho(\pi_t) = \rho(\pi_0) - c\int_0^t \pi_s I_{(\rho(\pi_s)<1-\pi_s)}ds - \lambda\int_0^t (1-\pi_s)I_{(\rho(\pi_s)=1-\pi_s)}ds + N_t. \quad (3.1.15)$$

Since $\rho(\pi)$ is concave, $\rho(\pi) \leq 1-\pi$ and $\rho(\pi) = 1-\pi$ if $\pi = 1$, we have that $\rho(\pi) = 1-\pi$ for all $\pi \geq A^*$ and $\rho(\pi) < 1 - \pi$ if $\pi < A^*$, where

$$A^* = \inf\{A : \rho(A) = 1 - A\} = \partial D.$$

Besides, the optimal stopping rule is of the form (3.1.13) and

$$\{(\omega, s) : \rho(\pi_s) < 1 - \pi_s\} = \{(\omega, s) : \pi_s < A^*\},$$

$$\{(\omega, s) : \rho(\pi_s) = 1 - \pi_s\} = \{(\omega, s) : \pi_s \geq A^*\}.$$

Therefore, there exists $A^* \in (0,1)$ such that $\rho(\pi_t)$ satisfies equation

$$\rho(\pi_t) = \rho(\pi_0) - c\int_0^t \pi_s I_{(\pi_s<A^*)}ds - \lambda\int_0^t (1-\pi_s)I_{(\pi_s\geq A^*)}ds + \int_0^t Z_s d\tilde{W}_s, \quad (3.1.16)$$

where $N = Z \cdot \tilde{W}$ by integral representation theorem and $Z_s = -\frac{b}{\sigma}\pi_s(1-\pi_s)$ on the set $\{\pi_s \geq A^*\}$ $dP \times ds$ a.e. by Remark 2.2.2.

Since $\rho(\pi)$ is concave , by Tanaka-Meyer's formula

$$\rho(\pi_t) = \rho(\pi_0) + \lambda \int_0^t \rho'_-(\pi_s)(1-\pi_s)ds + \frac{1}{2}\int_R \mathcal{L}_t^\pi(a)\nu''(da) + \frac{r}{\sigma}\int_0^t \rho'_-(\pi_s)\pi_s(1-\pi_s)d\tilde{W}_s, \tag{3.1.17}$$

where $L_t^\pi(a)$ is the local time at the point a of the process π_t, ρ'_- is the left-hand derivative of $\rho(\pi)$ and ν'' is the measure of the second derivative of ρ.

Comparing the parts of finite variations of (3.1.17) and (3.1.16), taking in mind that $\rho'_-(\pi_s) = -1$ on the set $\{\pi_s > A^*\}$, we have

$$\frac{1}{2}\int_R \mathcal{L}_t^\pi(a)\nu''(da) = -\int_0^t [c\pi_s + \lambda(1-\pi_s)\rho'_-(\pi_s)]I_{(\pi_s < A^*)}ds. \tag{3.1.18}$$

Let $h(x), x \in R$ be a bounded measurable function. Since the measure $d\mathcal{L}_t^\pi(a)$ is a.s. carried by the set $\{t : \pi_t = a\}$, integrating the process $h(\pi_s)\pi_s^2(1-\pi_s)^2$ with respect to the both parts of equality (3.1.18) and using Fubini's theorem we get

$$\int_R \mathcal{L}_t^\pi(a)h(a)a^2(1-a)^2\nu''(da) = -\int_0^t h(\pi_s)\pi_s^2(1-\pi_s)^2[c\pi_s + \lambda(1-\pi_s)\rho'_-(\pi_s)]I_{(\pi_s<A^*)}ds. \tag{3.1.19}$$

By the occupation formula (see Theorem A.3.5)

$$\int_0^t h(\pi_s)\pi_s^2(1-\pi_s)^2[c\pi_s + \lambda(1-\pi_s)\rho'_-(\pi_s)]I_{(\pi_s<A^*)}ds =$$

$$= \frac{\sigma^2}{r^2}\int_0^t h(\pi_s)[c\pi_s + \lambda(1-\pi_s)\rho'_-(\pi_s)]I_{(\pi_s<A^*)}d < \pi >_s =$$

$$= \frac{\sigma^2}{r^2}\int_R \mathcal{L}_t^\pi(a)h(a)[ca + \lambda(1-a)\rho'_-(a)]I_{(a<A^*)}da. \tag{3.1.20}$$

Therefore,

$$\int\limits_{[0,1]} \mathcal{L}_t^\pi(a)h(a)a^2(1-a)^2\nu''(da) = -\frac{2\sigma^2}{r^2}\int\limits_{[0,1]} \mathcal{L}_t^\pi(a)h(a)[ca+\lambda(1-a)\rho'_-(a)]I_{(a<A^*)}da. \tag{3.1.21}$$

Since $\rho(\pi)$ is concave and decreasing we have that $-1 \le \rho'_- \le 0$ and we may use Fubini's theorem and the Lebesgue theorem of monotone convergence, i.e., taking mathematical expectations with respect to the measure P^π (for some $\pi < 1$) and passing to the limit as $t \to \infty$ in the last equality, we obtain that

$$\int\limits_R h(a)a^2(1-a)^2E^\pi\mathcal{L}_\infty^\pi(a)\nu''(da) = -\frac{2\sigma^2}{r^2}\int\limits_R h(a)[ca+\lambda(1-a)\rho'_-(a)]I_{(a<A^*)}E^\pi\mathcal{L}_\infty^\pi(a)da \tag{3.1.22}$$

for any bounded measurable function h.

Since by Lemma 3.1.1 we have $0 < E^\pi\mathcal{L}_\infty^\pi(a) < \infty$ for all a, π such that $0 \le a \le \pi < 1$, (3.1.22) and the arbitrariness of the function h imply that the measure $\nu''(da)$ is absolutely continuous with respect to the Lebesgue measure on $(0, 1)$ and, hence, $\rho(\pi)$ admits a second order generalized derivative. Therefore, by Sobolev's embedding theorem there exists the first derivative of $\rho(\pi)$ in the usual sense and this derivative is continuous.

If we denote by $\rho''(\pi)$ the second order generalized derivative of ρ from (3.1.22) we have that a.e. with respect to the Lebesgue measure the value function $\rho(\pi)$ satisfies the PDE

$$\frac{r^2}{2\sigma^2}\pi^2(1-\pi)^2\rho''(\pi) = -\lambda(1-\pi)\rho'(\pi) - c\pi \tag{3.1.23}$$

on the open interval $(0, A^*)$.

Since equality (3.1.23) is fulfilled on the set $(0, A^*)$ a.e. with respect to the Lebesgue measure and the right-hand-side of (3.1.23) is continuous, then there exists a modification of $\rho''(\pi)$ (for convenience we denote this modification also by $\rho''(\pi)$) which is continuous on $(0, A^*)$. It is evident that the continuous modification of $\rho''(\pi)$ coincides with the ordinary second order derivative of ρ and equation (3.1.23) is satisfied for all $\pi \in (0, A^*)$.

Since $\rho(\pi) = 1 - \pi$ for all $\pi \geq A^*$ and $\rho(\pi)$ admits a continuous derivative, we have that $\rho'(\pi) = -1$ for all $\pi \geq A^*$ and, therefore, the constant A^* one can calculate from the smooth fit condition

$$\rho'(A^*) = -1.$$

Let us show now that $\rho'(0) = 0$. We shall first show that the value function $\rho(\pi)$ is a decreasing function. Let $\pi \leq \pi' \leq A^*$ and define $\sigma = \inf\{t : \pi_t^\pi \geq \pi'\}$. It is evident that $\pi_\sigma^\pi = \pi'$ and it follows from equation (3.1.16) that

$$\rho(\pi_\sigma^\pi) = \rho(\pi) - c\int_0^\sigma \pi_s^\pi I_{(\pi_s^\pi < A^*)} ds + \int_0^\sigma Z_s d\bar{W}_s. \tag{3.1.24}$$

Since $Z \cdot \bar{W}$ is a martingale and $\rho(\pi_\sigma^\pi) = \rho(\pi')$, taking expectations in (3.1.24) we obtain that

$$\rho(\pi') - \rho(\pi) = -cE^\pi \int_0^\tau \pi_s^\pi ds \leq 0.$$

Let $(\pi_n, n \geq 1)$ be a sequence such that $\pi_n \downarrow 0$. Then from (3.1.23)

$$\frac{r^2}{2\sigma^2}\pi_n^2(1-\pi_n)^2\rho''(\pi_n) = -\lambda(1-\pi_n)\rho'(\pi_n) - c\pi_n \tag{3.1.25}$$

for each $n \geq 1$. Since $\rho'(\pi)$ is continuous, the limit as $n \to \infty$ of the right-hand side exists and is equal to $\rho'(0+)$. Therefore there exists the limit of the left-hand side and since $\rho(\pi)$ is concave, this limit is non-positive, i.e., $\rho'(0+) \leq 0$. But by Lemma 2.2.2 $\rho'(\pi_n)$ is non-positive and, hence, the limit of the right-hand side is non-negative, i.e., $\rho'(0+) \geq 0$. Thus $\rho'(0+) = 0$ and equation (3.1.23) for $\pi = 0$ is also satisfied.

Thus, we have showed that the value function $\rho(\pi)$ is a concave function admitting the second order derivative ($\rho''(\pi)$ can be discontinuous only at points $\pi = 0$ and $\pi = A^*$) and it satisfies the free boundary problem 1)-3).

Conversely, let $\tilde{\rho}(\pi)$ be a non-negative concave function satisfying 1), 2), 3). Then by Itô's formula

$$\tilde{\rho}(\pi_t) = \tilde{\rho}(\pi_0) + \lambda \int_0^t \tilde{\rho}'(\pi_s)(1-\pi_s)ds + \frac{r^2}{2\sigma^2} \int_0^t \pi_s^2 (1-\pi_s)^2 \tilde{\rho}''(\pi_s)ds +$$

$$+ \frac{r}{\sigma} \int_0^t \pi_s (1-\pi_s) \tilde{\rho}'(\pi_s) d\tilde{W}_s. \tag{3.1.26}$$

Since $\tilde{\rho}''(\pi) = 0$ and $\tilde{\rho}'(\pi) = -1$ for all $\pi > B^*$, it follows from (3.1.14) and (3.1.26) that

$$\tilde{\rho}(\pi_t) = \tilde{\rho}(\pi_0) - \lambda \int_0^t (1-\pi_s) I_{(\pi_s \geq B^*)} ds - c \int_0^t \pi_s I_{(\pi_s < B^*)} ds + \frac{r}{\sigma} \int_0^t \pi_s (1-\pi_s) \tilde{\rho}'(\pi_s) d\tilde{W}_s. \tag{3.1.27}$$

Let $\tilde{A} = \inf\{A : \tilde{\rho}(A) = 1 - A\}$. Since $\tilde{\rho}(\pi)$ is concave, the smooth fit condition $\tilde{\rho}'(B^*) = -1$ implies that $B^* \in [\tilde{A}, 1]$. On the other hand if $B^* > \tilde{A}$ then on the interval $(\tilde{A}, B^*)$ we shall have $\tilde{\rho}''(\pi) = 0, \tilde{\rho}'(\pi) = -1$ and for any $\pi \in (\tilde{A}, B^*)$ equation (3.1.14) will not be satisfied. Thus $B^* = \tilde{A}$ and

$$\{\pi_s < B^*\} = \{\tilde{\rho}(\pi_s) < 1 - \pi_s\},$$

$$\{\pi_s \geq B^*\} = \{\tilde{\rho}(\pi_s) = 1 - \pi_s\} \tag{3.1.28}$$

From (3.1.27) and (3.1.28) we obtain that

$$\tilde{\rho}(\pi_t) = \tilde{\rho}(\pi_0) - \lambda \int_0^t (1-\pi_s) I_{(\tilde{\rho}(\pi_s)=1-\pi_s)} ds -$$

$$-c\int_0^t \pi_s I_{(\tilde{\rho}(\pi_s)<1-\pi_s)}ds + \frac{r}{\sigma}\int_0^t \pi_s(1-\pi_s)\tilde{\rho}'(\pi_s)d\tilde{W}_s. \tag{3.1.29}$$

We shall show now that $\frac{\lambda}{\lambda+c} \leq B^*$. Indeed, passing to the limit in (3.1.23) as $\pi \uparrow B^*$ and using the smooth fit condition we have that

$$-\frac{r^2}{2\sigma^2}(B^*)^2(1-B^*)^2 \liminf_{\pi\uparrow B^*} \rho''(\pi) \leq cB^* - \lambda(1-B^*) \tag{3.1.30}$$

From the concavity of the function $\rho(\pi)$ we have that the left-hand side of this inequality is non-negative and hence $\frac{\lambda}{\lambda+c} \leq B^*$. This inequality implies that $c\pi_s - \lambda(1-\pi_s)$ is positive on the set $\pi_s \geq B^*$. Therefore, we can rewrite (3.1.27) in the following form:

$$\tilde{\rho}(\pi_t) = \tilde{\rho}(\pi_0) - c\int_0^t \pi_s ds + \int_0^t (c\pi_s - \lambda(1-\pi_s))^+ I_{(\tilde{\rho}(\pi_s)=1-\pi_s)}ds + \frac{r}{\sigma}\int_0^t \pi_s(1-\pi_s)\tilde{\rho}'(\pi_s)d\tilde{W}_s, \tag{3.1.31}$$

which enables us to conclude that the triple

$$Y_t = \tilde{\rho}(\pi_t), \quad \nu_t = 0 \quad \text{and} \quad L_t = \frac{r}{\sigma}\int_0^t \pi_s(1-\pi_s)\tilde{\rho}'(\pi_s)d\tilde{W}_s$$

satisfies the RBSDE 2.2.1). It is easy to see that this triple satisfies I)-V). Indeed, since $\tilde{\rho}(\pi)$ is concave, condition 2) implies that $\tilde{\rho}(\pi_t) \leq 1 - \pi_t$ for all $t \geq 0$ and $\lim_{t\to\infty} \tilde{\rho}(\pi_t) \leq \lim_{t\to\infty}(1-\pi_t) = 0$. Besides, the positivity of $\tilde{\rho}(\pi)$ implies that $\lim_{t\to\infty}\tilde{\rho}(\pi_t) = 0$ and that $\tilde{\rho}(\pi_t)$ is bounded. Therefore, it follows from (3.1.27) that $\tilde{\rho}(\pi_t)$ is a supermartingale from the class S^1. Thus condition I)-V) are satisfied and by Theorem 2.2.1 $\tilde{\rho}(\pi_t)$ coincides with the value process V_t. Hence by (3.1.11) $\tilde{\rho}(\pi_t) = \rho(\pi_t)$ and $\tilde{\rho}(\pi) = \rho(\pi)$ for all $\pi \in [0,1]$.□

Thus we have proved that the RBSDE I)-VI) and the free boundary problem 1)-3) are equivalent. The solution of the free boundary problem 1)-3) is given by Shiryaev

(1978). Following Shiryaev, if we denote $\rho'(\pi)$ by $g(\pi)$ from (3.1.14) we have that

$$g'(\pi) = -\frac{2\lambda\sigma^2}{r^2\pi^2(1-\pi)}g(\pi) - \frac{2c\sigma^2}{r^2\pi(1-\pi)^2}.$$

Since $g(0) = 0$, we find that for $\pi < A^*$

$$g(\pi) = \rho'(\pi) = -\frac{2c\sigma^2}{r^2}\int_0^{\pi} \exp\{-\frac{2\lambda\sigma^2}{r^2}(H(\pi) - H(y))\}\frac{dy}{y(1-y)^2}, \qquad (3.1.32)$$

where $H(y) = \ln\frac{y}{1-y} - \frac{1}{y}$. If we define A^* as a unique solution of equation $g(A^*) = -1$, then the value function $\rho(\pi)$ coincides with

$$\rho(\pi) = \begin{cases} 1 - A^* - \int_{\pi}^{A^*} g(x)dx, & 0 \le \pi \le A^* \\ 1 - \pi, & A^* \le \pi \le 1. \end{cases} \qquad (3.1.33)$$

Remark 3.1.1. Let us note that the smooth fit of the second derivative can not be fulfilled and the second order derivative of $\rho(\pi)$ is discontinuous at the point A^*. Indeed 1)-3) implies that $\rho''(\pi)$ can be continuous only if $A^* = \frac{\lambda}{\lambda+c}$. On the other hand by using partial integration formula

$$\int_0^{\pi} \frac{y}{1-y}\exp\{\frac{2\lambda\sigma^2}{r^2}H(y)\}dH(y) =$$

$$= \frac{r^2}{2\lambda\sigma^2}\frac{\pi}{1-\pi}\exp\{\frac{2\lambda\sigma^2}{r^2}H(\pi)\} - \frac{r^2}{2\lambda\sigma^2}\int_0^{\pi}\exp\{\frac{2\lambda\sigma^2}{r^2}H(y)\}\frac{dy}{(1-y)^2}. \qquad (3.1.34)$$

Since $g(\pi)$ defined in (3.1.32) can be written as

$$g(\pi) = \rho'(\pi) = -\frac{2c\sigma^2}{r^2}\exp\{-\frac{2\lambda\sigma^2}{r^2}H(\pi)\}\int_0^{\pi}\frac{y}{1-y}\exp\{\frac{2\lambda\sigma^2}{r^2}H(y)\}dH(y)$$

from (3.1.34) we obtain that

$$g(\pi) > -\frac{c}{\lambda}\frac{\pi}{1-\pi}. \qquad (3.1.35)$$

Therefore on the set $\{\pi : \pi \leq \frac{\lambda}{\lambda+c}\}$ we have

$$g(\pi) > -1. \tag{3.1.36}$$

In particular $\rho'(\frac{\lambda}{\lambda+c}) > -1$ and hence $A^* \neq \frac{\lambda}{\lambda+c}$. Thus, the second order derivative of the value function is discontinuous at the point A^*. □

3.2 The Poisson disorder problem. The equivalence between RBSDE related to the Poisson disorder problem and the free-boundary problem of the differential-difference operator

In this section we consider the disorder problem for a Poisson process whose intensity changes from λ_0 to λ_1 at some random time θ and show that in this case the RBSDE (2.2.1) is equivalent to the free-boundary differential-difference problem considered by Peskir and Shiryaev (2002). Besides, we derive the smooth fit conditions for the value function (in cases when this condition is satisfied) and establish when the smooth fit condition breaks down directly from the RBSDE for the value process.

Let Ω be the space X of piecewise-constant functions $x = (x_t, t \geq 0)$ such that $x_0 = 0$ and $x_t = x_{t-} + (0 \text{ or } 1)$, $\mathcal{B} = \sigma\{x : x_s, s \geq 0\}, \mathcal{B}_t = \sigma\{x : x_s, s \leq t\}$.

Note that for any $x = (x_t, t \geq 0) \in X$, x_t is expressed as

$$x_t = \sum_{i \geq 1} I_{\{\tau_i(x) \leq t\}},$$

where

$$\tau_i(x) = \begin{cases} \inf\{s \geq 0 : x_s = i\} \\ \infty \qquad \lim_{t\to\infty} x_t < i \end{cases} \tag{3.2.1}$$

Let P^0 and P^1 be two Poisson measures on $(X, \mathcal{B})$ with parameters λ_0 and λ_1 respectively. This means that under the measure P^i the compensator of the coordinate process $X_t(x) = x_t, t \geq 0$, is equal to $A_i(t, x) = \lambda_i t$, i=1,2. (Note that the family of $\sigma-$algebras $(\mathcal{B}_t, t \geq 0)$, completed by P^0 and P^1, are right continuous.)

As it is known

$$P^1 \overset{loc}{\sim} P^0 \quad \text{and} \quad \frac{dP^1_t}{dP^0_t} = \exp\{ln\frac{\lambda_1}{\lambda_0}X_t - (\lambda_1 - \lambda_0)t\}$$

(see Theorem A.4.9).

It is easy to see that $\frac{dP^1_t}{dP^0_t} = \mathcal{E}_t(M)$, where

$$M_t = (\frac{\lambda_1}{\lambda_0} - 1)(X_t - \lambda_0 t), \quad M \in \mathcal{M}_{\text{loc}}(F, P^0).$$

Let $\psi(0) - \psi(0-) = \pi$ and $1 - \psi(t) = (1 - \pi)\exp\{-\lambda t\}$, where λ is a known constant and $0 \leq \pi \leq 1$.

By Lemma 1.2.1(see Remark 1.2.1)

$$\hat{M}^\psi_t = (\frac{\lambda_1}{\lambda_0} - 1)\int_0^t \pi_{s-} d(X_t - \lambda_0 t) \tag{3.2.2}$$

and hence (see A.4.5),

$$L_t(M, \hat{M}^\psi) = M_t - \sum_{s\leq t}\frac{\Delta M_s \Delta\hat{M}^\psi_s}{1 + \Delta\hat{M}^\psi_s} = (\frac{\lambda_1}{\lambda_0} - 1)(X_t - \lambda_0 t) - \int_0^t \frac{(\frac{\lambda_1}{\lambda_0} - 1)^2 \pi_{s-}}{1 + \pi_{s-}(\frac{\lambda_1}{\lambda_0} - 1)} dX_s =$$

$$(\frac{\lambda_1}{\lambda_0} - 1)\int_0^t \frac{dX_s}{1 + \pi_{s-}(\frac{\lambda_1}{\lambda_0} - 1)} - (\lambda_1 - \lambda_0)t. \tag{3.2.3}$$

Since $\Delta\psi_t = 0$, it follows from Remark 1.2.2 that the a posteriori probability process π_t satisfies equation

$$d\pi_t = \lambda(1-\pi_{t-})dt + \frac{\pi_{t-}(1-\pi_{t-})(\lambda_1-\lambda_0)}{\lambda_1\pi_{t-}+\lambda_0(1-\pi_{t-})}\big(dX_t - (\lambda_1\pi_{t-}+\lambda_0(1-\pi_{t-}))dt\big), \quad (3.2.4)$$

which coincides with the equation derived by Peskir and Shiryaev (2002).

Remark 3.2.1. The process $(X_t - \int_0^t (\lambda_1\pi_{s-} + \lambda_0(1-\pi_{s-}))ds, \mathcal{B}_t), t \geq 0$ is a martingale under P^ψ (see Theorem A.4.7) and $(\pi_t, \mathcal{B}_t, P^\psi)$ is a time-homogeneous (strong) Markov process.

Assume that $K_t = ct$. So, the cost criterion is of the same form as in (3.1.9) and the value function of the optimal stopping problem (1.1.12) is as in (3.1.10).

Since $(\pi_t, \mathcal{F}_t, P^\pi)$ is a time-homogeneous(strong) Markov process, we have that

$$V_t = \rho(\pi_t) \quad \text{a.s. for all} \quad t \geq 0. \quad (3.2.5)$$

Note that the cases $\lambda_1 < \lambda_0$ and $\lambda_1 > \lambda_0$ are quite different. e.g., a key difference between these cases is the fact that when $\lambda_1 < \lambda_0$ equation (3.2.6) has no singularity points, whereas $\hat{B} = \frac{\lambda}{\lambda_1-\lambda_0}$ is a singularity point of (3.2.6), whenever $\lambda < \lambda_1 - \lambda_0$ (see Peskir and Shiryaev (2002) for detailed analysis of these cases). Let first consider the case $\lambda_1 > \lambda_0$.

Theorem 3.2.1. *Let $\lambda_1 > \lambda_0$. The value function $\rho(\pi)$ is a non-negative concave function and there exists a constant $B^* \in (0,1]$ such that:*

1) $\rho(\pi)$ admits a continuous first derivative on $(0, B^*)$ (perhaps except the point $\hat{B} = \frac{\lambda}{\lambda_1-\lambda_0}$) and satisfies a differential-difference equation:

$$(\lambda - \pi(\lambda_1-\lambda_0))(1-\pi)\rho'(\pi) + (\lambda_1\pi + \lambda_0(1-\pi))[\rho(\frac{\lambda_1\pi}{\lambda_1\pi+\lambda_0(1-\pi)}) - \rho(\pi)] = -c\pi, \quad (3.2.6)$$

if $\pi < B^*$.

2) it is equal to $1-\pi$, *if* $\pi \geq B^*$.

3) it satisfies the continuous fit condition

$$\rho(B^*-) = 1 - B^*.$$

Moreover, if $c > \lambda_1 - \lambda_0 - \lambda$, *then*

3^**) the value function* $\rho(\pi)$ *satisfies the smooth fit condition:*

$$\rho'(B^*-) = -1.$$

Conversely, if $\tilde{\rho}(\pi)$ *is a non-negative, concave function satisfying* 1), 2), 3) *in the case* $c \leq \lambda_1 - \lambda_0 - \lambda$ *and* 1), 2), 3*) *in the case* $c > \lambda_1 - \lambda_0 - \lambda$ *for some* $A^* \in (0,1]$, *then the triple* $Y_t = \tilde{\rho}(\pi_t), \nu_t = 0$ *and* L_t *equal to the martingale part of* $\tilde{\rho}(\pi_t)$ *satisfies the RBSDE I)-VI). In particular this implies that* $\tilde{\rho}(\pi) = \rho(\pi)$ *and* $A^* = B^*$.

Proof: Similarly to the Wiener case

$$\{(\omega, s) : \rho(\pi_s) < 1 - \pi_s\} = \{(\omega, s) : \pi_s < B^*\},$$

$$\{(\omega, s) : \rho(\pi_s) = 1 - \pi_s\} = \{(\omega, s) : \pi_s \geq B^*\},$$

where

$$B^* = \inf\{B : \rho(B) = 1 - B\}$$

and the optimal stopping rule is of the form

$$\tau^* = \inf\{t : \rho(\pi_t) = 1 - \pi_t\} = \inf\{t : \pi_t \geq B^*\}. \qquad (3.2.7)$$

Taking into account the above facts it follows from Remark 2.2.1(see 2.2.15) that the value process $\rho(\pi_t)$ satisfies the following equation

$$\rho(\pi_t) = \rho(\pi_0) - c\int_0^t \pi_s I_{(\pi_s < B^*)} ds - \lambda \int_0^t (1-\pi_s) I_{(\pi_s \geq B^*)} ds -$$

$$-\left(\frac{1}{2}\mathcal{L}_t^0(1-\pi-V)+\tilde{A}_t\right)+N_t. \qquad (3.2.8)$$

It is evident that the process $(\mathcal{L}_t^0(1-\pi-V), t \geq 0)$ is indistinguishable from zero. Indeed, since the both functions π_t and V_t have the jumps at the discontinuity points of the process X_t and the number of these points for the process X_t on each interval $(0,t]$ is finite, we will have that the following condition is fulfilled (see A.3.3):

$$\sum_{0<s\leq t} |\Delta(1-\pi_s-V_s)| < \infty \quad \text{a.s., for each} \quad t>0,$$

Besides, as the processes π_t and V_t do not have the continuous martingale parts, we have that $\mathcal{L}_t^0(1-\pi-V)=0$ (see Corollary A.3.4)..

Recall that $\tilde{A}_t$ is the compensator of the process $\sum_{0<s\leq t} I_{(1-\pi_{s-}=\rho(\pi_{s-})}(1-\pi_s-\rho(\pi_s))$. Therefore it can be written as follows:

$$-\tilde{A}_t = \int_0^t (\lambda_1\pi_s+\lambda_0(1-\pi_s))[\rho(\frac{\lambda_1\pi_s}{\lambda_1\pi_s+\lambda_0(1-\pi_s)})-\rho(\pi_s)]I_{(\pi_s\geq B^*)}ds+$$

$$+\int_0^t(\lambda_1-\lambda_0)\pi_s(1-\pi_s)I_{(\pi_s\geq B^*)}ds. \qquad (3.2.9)$$

Since $\frac{\lambda_1\pi}{\lambda_1\pi+\lambda_0(1-\pi)} > \pi$, $\quad \rho(\frac{\lambda_1\pi}{\lambda_1\pi+\lambda_0(1-\pi)}) = 1-\frac{\lambda_1\pi}{\lambda_1\pi+\lambda_0(1-\pi)}$ and $\rho(\pi)=1-\pi$ for any $\pi \geq B^*$. Therefore it follows easily from (3.2.9) that $\tilde{A}_t=0$. Hence the process μ_t from (2.2.13) is indistinguishable from zero.

Thus from (3.2.8) we have

$$\rho(\pi_t)=\rho(\pi_0)-c\int_0^t \pi_s I_{(\pi_s<B^*)}ds-\lambda\int_0^t(1-\pi_s)I_{(\pi_s\geq B^*)}ds+N_t. \qquad (3.2.10)$$

where $N=\tilde{Z}\cdot\tilde{N}$ by integral representation theorem, $\tilde{N}_t = X_t - \int_0^t(\lambda_1\pi_{s-}+\lambda_0(1-\pi_{s-}))ds$ (see Remark 3.2.1) and $\tilde{Z}_s = -\frac{\pi_{t-}(1-\pi_{t-})(\lambda_1-\lambda_0)}{\lambda_1\pi_{t-}+\lambda_0(1-\pi_{t-})}$ on the set $\{\pi_s \geq B^*\}$ $\quad dP^\pi \times ds$ a.e. by Remark 2.2.2.

Since the function $\rho(\pi)$ is concave and the martingale part of the process $(\pi_t, t \geq 0)$ is a pure-jump process and so $\mathcal{L}_t^a(\pi) = 0$, by Tanaka-Meyer's formula

$$\rho(\pi_t) = \rho(\pi_0) + \int_0^t \rho'_-(\pi_{s-})d\pi_s + \sum_{s\leq t}(\rho(\pi_s) - \rho(\pi_{s-}) - \rho'_-(\pi_{s-})\triangle\pi_s), \qquad (3.2.11)$$

where by $\rho'_-(\pi)$ we have denoted the left derivative of the function $\rho(\pi)$.

As the compensator of the last summand of equation (3.2.11) is equal to

$$\hat{A}_t \equiv \int_0^t (\lambda_1\pi_s + \lambda_0(1-\pi_s))[\rho(\frac{\lambda_1\pi_s}{\lambda_1\pi_s + \lambda_0(1-\pi_s)}) - \rho(\pi_s)]ds -$$

$$-(\lambda_1 - \lambda_0)\int_0^t \pi_s(1-\pi_s)\rho'_-(\pi_s)ds,$$

taking in mind that $\rho'_-(\pi) = -1$ for $\pi > B^*$, from (3.2.9) we obtain that

$$\hat{A}_t = \int_0^t (\lambda_1\pi_s + \lambda_0(1-\pi_s))[\rho(\frac{\lambda_1\pi_s}{\lambda_1\pi_s + \lambda_0(1-\pi_s)}) - \rho(\pi_s)]I_{(\pi_s<B^*)}ds -$$

$$-\int_0^t (\lambda_1 - \lambda_0)\pi_s(1-\pi_s)\rho'_-(\pi_s)I_{(\pi_s<B^*)}ds. \qquad (3.2.12)$$

Therefore by (3.2.11) we have

$$\rho(\pi_t) = \rho(\pi_0) + \lambda\int_0^t (1-\pi_s)\rho'_-(\pi_s)I_{(\pi_s<B^*)}ds - \lambda\int_0^t (1-\pi_s)I_{(\pi_s\geq B^*)}ds +$$

$$\int_0^t (\lambda_1\pi_s + \lambda_0(1-\pi_s))[\rho(\frac{\lambda_1\pi_s}{\lambda_1\pi_s + \lambda_0(1-\pi_s)}) - \rho(\pi_s)]I_{(\pi_s<B^*)}ds$$

$$-\int_0^t (\lambda_1 - \lambda_0)\pi_s(1-\pi_s)\rho'_-(\pi_s)I_{(\pi_s<B^*)}ds + \tilde{M}_t, \qquad (3.2.13)$$

where $\tilde{M}_t$ is a martingale.

By the uniqueness of the canonical decomposition from (3.2.9), (3.2.10) and (3.2.13) we have that

$$\int_0^t (\lambda-(\lambda_1-\lambda_0)\pi_s)(1-\pi_s)\rho'_-(\pi_s)I_{(\pi_s<B^*)}ds+\int_0^t c\pi_s I_{(\pi_s<B^*)}ds+$$

$$+\int_0^t (\lambda_1\pi_s+\lambda_0(1-\pi_s))\Big(\rho(\frac{\lambda_1\pi_s}{\lambda_1\pi_s+\lambda_0(1-\pi_s)})-\rho(\pi_s)\Big)I_{(\pi_s<B^*)}ds=0. \qquad (3.2.14)$$

Further let us define

$$\hat{B}=\frac{\lambda}{\lambda_1-\lambda_0} \qquad (3.2.15)$$

and observe that (3.2.4) can be rewritten in the following form:

$$d\pi_t=(\lambda_1-\lambda_0)(\hat{B}-\pi_{t-})(1-\pi_{t-})dt+\frac{\pi_{t-}(1-\pi_{t-})(\lambda_1-\lambda_0)}{\lambda_1\pi_{t-}+\lambda_0(1-\pi_{t-})}dX_t \qquad (3.2.16)$$

Hence, if $\pi<\hat{B}$, then $\pi_t\downarrow\pi$ as $t\to 0$ P^π-a.s. and if $\pi>\hat{B}$, then $\pi_t\uparrow\pi$ as $t\to 0$ P^π-a.s. More exactly for each $\omega\in N$, for some $N\subset\Omega$, with $P^\pi(N)=1$, there exists $t_0=t_0(\omega)$, such that $\pi_t\uparrow\pi$ for $t_0(\omega)\geq t\to 0$.

At the same time, since $\rho(\pi)$ is a concave function and $\rho'_-(\pi)$ is a non-increasing left-continuous function having right-side limits, we have that

$\lim_{t\to 0}\rho'_-(\pi_t)=\rho'_+(\pi)$ P^π-a.s, if $\pi>\hat{B}$ and

$\lim_{t\to 0}\rho'_-(\pi_t)=\rho'_-(\pi)$ P^π-a.s, if $\pi\leq\hat{B}$.

Taking into consideration these facts by dividing the right-hand-side of (3.2.14) by t, the passage to the limit as $t\to 0$ gives that the value function $\rho(\pi)$ satisfies the following differential-difference equation:

$$(\lambda-(\lambda_1-\lambda_0)\pi)(1-\pi)\tilde{\rho}'(\pi)=$$

$$=-((\lambda_1\pi+\lambda_0(1-\pi))[\rho(\frac{\lambda_1\pi}{\lambda_1\pi+\lambda_0(1-\pi)})-\rho(\pi)]+c\pi) \qquad (3.2.17)$$

for all $\pi < B^*$, where $\tilde{\rho}'(\pi) = \rho'_-(\pi)$, if $\pi < \hat{B}$ and $\tilde{\rho}'(\pi) = \rho'_+(\pi)$, if $\pi > \hat{B}$.

Since the right-hand-side of (3.2.17) is a continuous function,then $\tilde{\rho}'(\pi)$ is continuous except the point $\pi = \hat{B}$. Since $\tilde{\rho}'(\pi)$ coincides with left or right derivatives of the function $\rho(\pi)$ and $\tilde{\rho}'(\pi)$ is continuous, we obtain that $\rho(\pi)$ admits a continuous derivative and $\tilde{\rho}'(\pi) = \rho'(\pi)$ for all $\pi \in (0, B^*)$(perhaps except the point $\pi = \hat{B}$). Therefore (3.2.17) implies that $\rho(\pi)$ satisfies equation (3.2.6) for all $\pi \in (0, B^*)$.

Going to the limit as $\pi \to 0+$ in (3.2.17) we obtain that $\rho(\pi)$ satisfies the normal entrance condition

$$\rho'(0+) = 0,$$

hence $\rho(\pi)$ is a decreasing function.

Since $\rho(\pi)$ is continuous and $\rho(\pi) = 1 - \pi$ for $\pi > B^*$, the continuous fit condition

$$\rho(B^*-) = 1 - B^*.$$

is fulfilled. Thus the value function $\rho(\pi)$ satisfies conditions 1), 2), 3).

We shall show now that in the case $c > \lambda_1 - \lambda_2 - \lambda$ the smooth fit condition 3*) is satisfied. Since $\{\rho(\pi_s) = 1 - \pi_s\} = \{\pi_s \geq B^*\}$, from (2.2.10) we have that $c\pi_s - \lambda(1 - \pi_s) \geq 0$ on the set $\{\pi_s \geq B^*\}$ and hence

$$B^* \geq \frac{\lambda}{\lambda + c}. \tag{3.2.18}$$

Passing to the limit in (3.2.6), as $\pi \uparrow B^*$, we have

$$\tilde{\rho}'(B^*-) = \frac{B^*[(\lambda_1 - \lambda_0 - c) - (\lambda_1 - \lambda_0)B^*]}{(1 - B^*)[\lambda - B^*(\lambda_1 - \lambda_0)]}, \tag{3.2.19}$$

if $B^* \neq \hat{B} = \frac{\lambda}{\lambda_1 - \lambda_0}$.

Since $\rho(\pi)$ is concave and satisfies the normal entrance condition, we have that $\rho^{'}(\pi) \in [-1, 0]$. By resolving the system

$$\begin{cases} \tilde{\rho}^{'}(B^*-) \geq -1 \\ \tilde{\rho}^{'}(B^*-) \leq 0 \end{cases} \tag{3.2.20}$$

we have

$$B^* \in [\max(\frac{\lambda_1 - \lambda_0 - c}{\lambda_1 - \lambda_0}, 0), \frac{\lambda}{\lambda + c}],$$

which together with (3.2.18) implies that

$$B^* = \frac{\lambda}{\lambda + c};$$

Substituting $\frac{\lambda}{\lambda+c}$ instead of B^* in (3.2.19) we obtain that

$$\rho^{'}(B^*-) = -1.$$

Hence condition 3*) is also satisfied and the first part of the theorem is proved.

Conversely, let $\tilde{\rho}(\pi)$ be a non-negative, concave function satisfying 1), 2), 3) if $c \leq \lambda_1 - \lambda_0 - \lambda$ and conditions 1), 2), 3*) if $c > \lambda_1 - \lambda_0 - \lambda$. Then by Tanaka-Meyer's formula

$$\tilde{\rho}(\pi_t) = \tilde{\rho}(\pi_0) + \int_0^t \tilde{\rho}^{'}_{-}(\pi_{s-}) d\pi_s + \sum_{s \leq t} (\tilde{\rho}(\pi_s) - \tilde{\rho}(\pi_{s-}) - \tilde{\rho}^{'}_{-}(\pi_{s-}) \Delta \pi_s). \tag{3.2.21}$$

Since the compensator of the last summand is equal to

$$\int_0^t (\lambda_1 \pi_s + \lambda_0 (1 - \pi_s))[\tilde{\rho}(\frac{\lambda_1 \pi_s}{\lambda_1 \pi_s + \lambda_0 (1 - \pi_s)}) - \tilde{\rho}(\pi_s)] ds - (\lambda_1 - \lambda_0) \int_0^t \pi_s (1 - \pi_s) \tilde{\rho}^{'}_{-}(\pi_s) ds,$$

taking in mind the fact that $\tilde{\rho}^{'}(\pi) = -1$ for all $\pi > A^*$, it follows from (3.2.6) that

$$\tilde{\rho}(\pi_t) = \tilde{\rho}(\pi_0) - c \int_0^t \pi_s I_{(\pi_s < A^*)} ds - \lambda \int_0^t (1 - \pi_s) I_{(\pi_s \geq A^*)} ds + \tilde{L}_t, \tag{3.2.22}$$

where

$$\tilde{L}_t \equiv \int_0^t \frac{\pi_{s-}(1-\pi_{s-})(\lambda_1-\lambda_0)}{\lambda_1\pi_{s-}+\lambda_0(1-\pi_{s-})}\tilde{\rho}'_-(\pi_{s-})(dX_s-(\lambda_1\pi_{s-}+\lambda_0(1-\pi_{s-}))ds$$

is the martingale part of this decomposition.

Let $\tilde{B} = \inf\{B : \tilde{\rho}(B) = 1 - B\}$. Since $\tilde{\rho}(\pi)$ is concave, the continuous fit condition implies that $A^* \in [\tilde{B}, 1]$. On the other hand if $A^* > \tilde{B}$ then on the interval $(\tilde{B}, A^*)$ we shall have $\tilde{\rho}'(\pi) = -1$. Further $\frac{\lambda_1\pi}{\lambda_1\pi+\lambda_0(1-\pi)} > \pi$, which implies that

$$\rho(\frac{\lambda_1\pi}{\lambda_1\pi+\lambda_0(1-\pi)}) = 1 - \frac{\lambda_1\pi}{\lambda_1\pi+\lambda_0(1-\pi)} \qquad \forall\pi \in (\tilde{B}, A^*).$$

Taking into consideration these facts we obtain that equation (3.2.6) in this case can be satisfied only at the point $\pi = \frac{\lambda}{\lambda+c}$, which gives a contradiction. Thus $\tilde{B} = A^*$ and

$$\{\pi_s < A^*\} = \{\tilde{\rho}(\pi_s) < 1 - \pi_s\},$$

$$\{\pi_s \geq A^*\} = \{\tilde{\rho}(\pi_s) = 1 - \pi_s\} \tag{3.2.23}$$

Therefore from (3.2.22) and (3.2.23) we obtain that

$$\tilde{\rho}(\pi_t) = \tilde{\rho}(\pi_0) - c\int_0^t \pi_s I_{(\tilde{\rho}(\pi_s)<1-\pi_s)}ds - \lambda\int_0^t (1-\pi_s)I_{(\tilde{\rho}(\pi_s)=1-\pi_s)}ds + \tilde{L}_t, \tag{3.2.24}$$

Since $\tilde{\rho}(\pi)$ is a bounded function, $\tilde{\rho}(\pi_t)$ is a supermartingale of the class D. Henceg $\tilde{\rho}(\pi_t) \in S^1$ and $\tilde{L}_t$ is an uniformly integrable martingale.

We shall show now that $A^* \geq \frac{\lambda}{\lambda+c}$. Indeed passing to the limit in (3.2.6) when $\pi \uparrow A^*$ and using the continuous fit condition we obtain that

$$\tilde{\rho}'(A^*-) = \frac{A^*[(\lambda_1-\lambda_0-c)-(\lambda_1-\lambda_0)A^*]}{(1-A^*)[\lambda-A^*(\lambda_1-\lambda_0)]} \tag{3.2.25}$$

if $A^* \neq \hat{B} = \frac{\lambda}{\lambda_1-\lambda_0}$.

Since $\tilde{\rho}(\pi)$ is concave and satisfies the normal entrance condition, we have $\tilde{\rho}'(\pi) \in [-1,0]$.

Consider the following 3 cases:

a) If $c < \lambda_1 - \lambda_0 - \lambda$, by resolving the system

$$\begin{cases} \tilde{\rho}'(A^*-) \geq -1 \\ \tilde{\rho}'(A^*-) \leq 0 \end{cases} \tag{3.2.26}$$

we obtain that

$$A^* \in [\frac{\lambda}{\lambda + c}, \frac{\lambda_1 - \lambda_0 - c}{\lambda_1 - \lambda_0}].$$

b) If $c = \lambda_1 - \lambda_0 - \lambda$, and $A^* \neq \hat{B} = \frac{\lambda}{\lambda+c} = \frac{\lambda}{\lambda_1 - \lambda_0}$, then by (3.2.25) $\tilde{\rho}'(A^*-) = \frac{A^*}{1-A^*} > 0$, which gives a contradiction, since $\tilde{\rho}'$ is a non-positive function. Thus, $A^* = \hat{B} = \frac{\lambda}{\lambda+c}$.

c) If $c > \lambda_1 - \lambda_0 - \lambda$, then the smooth fit condition and (3.2.25) implies that

$$\frac{A^*[(\lambda_1 - \lambda_0 - c) - (\lambda_1 - \lambda_0)A^*]}{(1 - A^*)[\lambda - A^*(\lambda_1 - \lambda_0)]} = -1,$$

which gives

$$A^* = \frac{\lambda}{\lambda + c}.$$

Hence $A^* \geq \frac{\lambda}{\lambda+c}$ in all cases. This inequality implies that $c\pi_s - \lambda(1-\pi_s)$ is positive on the set $\pi_s \geq A^*$. Therefore we can rewrite (3.2.24) into the following form:

$$\tilde{\rho}(\pi_t) = \tilde{\rho}(\pi_0) - c\int_0^t \pi_s ds + \int_0^t (c\pi_s - \lambda(1 - \pi_s))^+ I_{(\tilde{\rho}(\pi_s)=1-\pi_s)} ds + \tilde{L}_t, \tag{3.2.27}$$

This implies that the triple

$Y_t = \tilde{\rho}(\pi_t)$, $\nu_t = 0, L_t = \int_0^t \frac{\pi_{s-}(1-\pi_{s-})(\lambda_1-\lambda_0)}{\lambda_1\pi_{s-}+\lambda_0(1-\pi_{s-})} \tilde{\rho}'_-(\pi_{s-})(dX_s - (\lambda_1\pi_{s-} + \lambda_0(1 - \pi_{s-}))ds$

satisfies RBSDE (2.2.1). Besides, IV) straightforwardly follows from the concavity of the function $\tilde{\rho}(\pi)$. Since $\lim_{t\to\infty} \pi_t = 1$, it gives together with IV) that condition V) is also satisfied. Therefore by Theorem 2.2.1 $\tilde{\rho}(\pi_t)$ coincides with the value process V_t. Hence by (3.2.5) $\tilde{\rho}(\pi_t) = \rho(\pi_t)$ and $\tilde{\rho}(\pi) = \rho(\pi)$ for all $\pi \in [0,1]$. □

In the next theorem we consider the case $\lambda_1 < \lambda_0$. Note that in this case, contrary to the case $\lambda_1 > \lambda_0$ the process μ_t is not equal to zero, which leads us to additional technical difficulties.

Theorem 3.2.2. *Let $\lambda_1 < \lambda_0$. The value function $\rho(\pi)$ is a non-negative concave function and there exists a constant $B^* \in (0,1]$ such that:*

1) $\rho(\pi)$ admits a continuous first derivative on $(0, B^*)$ and satisfies a differential-difference equation:

$$(\lambda - \pi(\lambda_1 - \lambda_0))(1-\pi)\rho'(\pi) + (\lambda_1\pi + \lambda_0(1-\pi))[\rho(\frac{\lambda_1\pi}{\lambda_1\pi + \lambda_0(1-\pi)}) - \rho(\pi)] = -c\pi, \tag{3.2.28}$$

if $\pi < B^*$.

2) it is equal to $1-\pi$, if $\pi \geq B^*$,

3) it satisfies the smooth fit condition:*

$$\rho'(B^*-) = -1.$$

Conversely, if $\tilde{\rho}(\pi)$ is a non-negative, concave function satisfying 1), 2), 3) for some $A^* \in (0,1]$, then the triple $Y_t = \tilde{\rho}(\pi_t)$, $\nu_t = \frac{(\lambda+\pi_t(\lambda_1-\lambda_0))(1-S(\pi_t)-\rho(S(\pi_t)))}{c\pi_t-\lambda(1-\pi_t)} I_{(\tilde{\rho}(\pi_t)=1-\pi_t)}$, where $S(\pi) = \frac{\lambda_1\pi}{\lambda_0+(\lambda_1-\lambda_0)\pi}$ and L_t equal to the martingale part of $\tilde{\rho}(\pi_t)$ satisfies the RBSDE I)-VI). In particular this implies that $\tilde{\rho}(\pi) = \rho(\pi)$ and $A^* = B^*$.*

Proof: Similarly to the Wiener case it follows from Remark 2.2.1 (see 2.2.15) that the value process $\rho(\pi_t)$ satisfies the following equation

$$\rho(\pi_t) = \rho(\pi_0) - c\int_0^t \pi_s I_{(\pi_s<B^*)}ds - \lambda\int_0^t (1-\pi_s)I_{(\pi_s\geq B^*)}ds -$$

$$-\left(\frac{1}{2}\mathcal{L}_t^0(1-\pi-V)+\tilde{A}_t\right)+N_t. \qquad (3.2.29)$$

It is evident that the process $(\mathcal{L}_t^0(1-\pi-V), t\geq 0)$ is indistinguishable from zero. Indeed, since the both functions π_t and V_t have the jumps at the discontinuity points of the process X_t and the number of these points for the process X_t on each interval $(0,t]$ is finite, we will have that the following condition is fulfilled (see A.3.3):

$$\sum_{0<s\leq t} |\triangle(1-\pi_s-V_s)| < \infty \quad \text{a.s., for each} \quad t>0,$$

Besides, as the processes π_t and V_t do not have the continuous martingale parts, we have that $\mathcal{L}_t^0(1-\pi-V)=0$ (see Corollary A.3.4).

Recall that $\tilde{A}_t$ is the compensator of the process $\sum_{0<s\leq t} I_{(1-\pi_{s-}=\rho(\pi_{s-}))}(1-\pi_s-\rho(\pi_s))$, by the condition 3) it can be written as follows:

$$-\tilde{A}_t = \int_0^t \Big((\lambda_1\pi_s+\lambda_0(1-\pi_s))[\rho(\tfrac{\lambda_1\pi_s}{\lambda_1\pi_s+\lambda_0(1-\pi_s)})-\rho(\pi_s)]+(\lambda_1-\lambda_0)\pi_s(1-\pi_s)\Big)I_{(\pi_s\geq B^*)}ds$$

$$= \int_0^t (\lambda_0+(\lambda_1-\lambda_0)\pi_s)(\rho(\frac{\lambda_1\pi_s}{\lambda_0+(\lambda_1-\lambda_0)\pi_s}) - (1-\frac{\lambda_1\pi_s}{\lambda_0+(\lambda_1-\lambda_0)\pi_s}))I_{(\pi_s\geq B^*)}ds.$$

Hence

$$\tilde{A}_t = \int_0^t -(\lambda_0+(\lambda_1-\lambda_0)\pi_s)\big(\rho(S(\pi_s))-(1-S(\pi_s))\big)I_{(\pi_s\geq B^*)}ds, \qquad (3.2.30)$$

where $S(\pi)$ is defined as

$$S(\pi) = \frac{\lambda_1\pi}{\lambda_0+(\lambda_1-\lambda_0)\pi}. \qquad (3.2.31)$$

Since $\lambda_1 < \lambda_0$ implies that $\frac{\lambda_1\pi}{\lambda_1\pi+\lambda_0(1-\pi)} < \pi$, it can happen that $S(\pi) < B^*$ for some $\pi > B^*$. Therefore in this case $\tilde{A}_t$ is not equal to zero and equation (3.2.29) takes the form

$$\rho(\pi_t) = \rho(\pi_0) - c\int_0^t \pi_s I_{(\pi_s<B^*)}ds - \lambda\int_0^t (1-\pi_s)I_{(\pi_s\geq B^*)}ds + N_t +$$

$$+\int_0^t (\lambda_0 + (\lambda_1-\lambda_0)\pi_s)\big(\rho(S(\pi_s)) - (1-S(\pi_s))\big)I_{(\pi_s\geq B^*)}ds. \qquad (3.2.32)$$

Since the function $\rho(\pi)$ is concave and the martingale part of the process $(\pi_t, t\geq 0)$ is a pure-jumped process and so $\mathcal{L}_t^a = 0$, by Tanaka - Meyer's formula (see Theorem A.3.7)

$$\rho(\pi_t) = \rho(\pi_0) + \int_0^t \rho'_-(\pi_{s-})d\pi_s + \sum_{s\leq t}(\rho(\pi_s) - \rho(\pi_{s-}) - \rho'_-(\pi_{s-})\Delta\pi_s) = \rho(\pi_0) + \qquad (3.2.33)$$

$$\int_0^t \rho'_-(\pi_{s-})d\pi_s + \int_0^t [\rho(\frac{\lambda_1\pi_{s-}}{\lambda_1\pi_{s-}+\lambda_0(1-\pi_{s-})}) - \rho(\pi_{s-}) - \rho'_-(\pi_{s-})\frac{(\lambda_1-\lambda_0)\pi_{s-}(1-\pi_{s-})}{\lambda_1\pi_{s-}+\lambda_0(1-\pi_{s-})}]dX_s,$$

where by $\rho'_-(\pi)$ we have denoted the left derivative of the function $\rho(\pi)$.

As the compensator of the last summand of equation (3.2.33) is equal to

$$\hat{A}_t \equiv \int_0^t (\lambda_1\pi_s + \lambda_0(1-\pi_s))[\rho(\frac{\lambda_1\pi_s}{\lambda_1\pi_s+\lambda_0(1-\pi_s)}) - \rho(\pi_s)]ds -$$

$$-(\lambda_1-\lambda_0)\int_0^t \pi_s(1-\pi_s)\rho'_-(\pi_s)ds$$

(see Corollary A.2.4) taking in mind that $\rho'_-(\pi) = -1$ for $\pi > B^*$, from (3.2.30) we obtain that

$$\hat{A}_t = -\tilde{A}_t + \int_0^t (\lambda_1\pi_s + \lambda_0(1-\pi_s))[\rho(\frac{\lambda_1\pi_s}{\lambda_1\pi_s+\lambda_0(1-\pi_s)}) - \rho(\pi_s)]I_{(\pi_s<B^*)}ds -$$

$$-\int_0^t (\lambda_1-\lambda_0)\pi_s(1-\pi_s)\rho'_-(\pi_s)I_{(\pi_s<B^*)}ds. \qquad (3.2.34)$$

Therefore by (3.2.33) we have that

$$\rho(\pi_t)=\rho(\pi_0)+\lambda\int_0^t(1-\pi_s)\rho'_-(\pi_s)I_{(\pi_s<B^*)}ds-\lambda\int_0^t(1-\pi_s)I_{(\pi_s\geq B^*)}ds-\tilde{A}_t+$$

$$\int_0^t(\lambda_1\pi_s+\lambda_0(1-\pi_s))[\rho(\frac{\lambda_1\pi_s}{\lambda_1\pi_s+\lambda_0(1-\pi_s)})-\rho(\pi_s)]I_{(\pi_s<B^*)}ds- \qquad (3.2.35)$$

$$\int_0^t(\lambda_1-\lambda_0)\pi_s(1-\pi_s)\rho'_-(\pi_s)I_{(\pi_s<B^*)}ds+\tilde{M}_t,$$

where by $\tilde{M}_t$ we have denoted the martingale part of this decomposition.

By the uniqueness of the canonical decomposition from (3.2.30), (3.2.32) and (3.2.35) we have that

$$\int_0^t(\lambda-(\lambda_1-\lambda_0)\pi_s)(1-\pi_s)\rho'_-(\pi_s)I_{(\pi_s<B^*)}ds+\int_0^t c\pi_s I_{(\pi_s<B^*)}ds+$$

$$+\int_0^t(\lambda_1\pi_s+\lambda_0(1-\pi_s))\Big(\rho(\frac{\lambda_1\pi_s}{\lambda_1\pi_s+\lambda_0(1-\pi_s)})-\rho(\pi_s)\Big)I_{(\pi_s<B^*)}ds=0 \qquad (3.2.36)$$

Further let us observe that (3.2.4) can be rewritten in the following form:

$$d\pi_t=(\lambda+\pi_{t-}(\lambda_0-\lambda_1))(1-\pi_{t-})dt+\frac{\pi_{t-}(1-\pi_{t-})(\lambda_1-\lambda_0)}{\lambda_1\pi_{t-}+\lambda_0(1-\pi_{t-})}dX_t, \qquad (3.2.37)$$

which implies that $\pi_t\downarrow\pi$ P^π-a.s, as $t\to 0$. More exactly for each $\omega\in N$, for some $N\subset\Omega$, with $P^\pi(N)=1$, there exists $t_0=t_0(\omega)$, such that $\pi_t\downarrow\pi$ for $t_0(\omega)\geq t\to 0$.

At the same time, since $\rho(\pi)$ is a concave function and $\rho'_-(\pi)$ is a non-increasing left-continuous function having right-side limits, we have that

$\lim_{t\to 0}\rho'_{-}(\pi_t) = \rho'_{-}(\pi)\ P^{\pi}$-a.s.

Taking into consideration these facts by dividing the right-hand-side of (3.2.36) by t, the passage to the limit as $t \to 0$ gives that the value function $\rho(\pi)$ satisfies the following differential-difference equation:

$$(\lambda - (\lambda_1 - \lambda_0)\pi)(1-\pi)\tilde{\rho}'(\pi) = -((\lambda_1\pi + \lambda_0(1-\pi))[\rho(\frac{\lambda_1\pi}{\lambda_1\pi + \lambda_0(1-\pi)}) - \rho(\pi)] + c\pi) \tag{3.2.38}$$

for all $\pi < B^*$, where $\tilde{\rho}'(\pi) = \rho'_{-}(\pi)$.

Since the right-hand-side of (3.2.38) is a continuous function,then $\tilde{\rho}'(\pi)$ is continuous for all $\pi < B^*$. Since $\tilde{\rho}'(\pi)$ coincides with left derivative of the function $\rho(\pi)$ and $\tilde{\rho}'(\pi)$ is continuous, we obtain that $\rho(\pi)$ admits a continuous derivative and $\tilde{\rho}'(\pi) = \rho'(\pi)$ for all $\pi \in (0, B^*)$. Therefore (3.2.38) implies that $\rho(\pi)$ satisfies equation (3.2.28) for all $\pi \in (0, B^*)$.

Going to the limit as $\pi \to 0+$ in (3.2.17) we obtain that $\rho(\pi)$ satisfies the normal entrance condition

$$\rho'(0+) = 0,$$

hence $\rho(\pi)$ is a decreasing function.

Since $\rho(\pi)$ is continuous and $\rho(\pi) = 1 - \pi$ for $\pi > B^*$, the continuous fit condition

$$\rho(B^*-) = 1 - B^*.$$

is fulfilled. Thus the value function $\rho(\pi)$ satisfies 1), 2) conditions.

We shall show now that the smooth fit condition 3*) is satisfied. From (2.2.7) we have that

$$\tilde{A}_t = \int_0^t \mu_s(c\pi_s - \lambda(1-\pi_s))^+ I_{(\pi_s \geq B^*)} ds,$$

where

$$0 \le \mu_s \le 1$$

by condition II). Therefore, since $B^* \ge \frac{\lambda}{\lambda+c}$ (see 3.2.18)

$$\int_0^t (\lambda_0 + (\lambda_1 - \lambda_0)\pi_s)\big(\rho(S(\pi_s)) - (1 - S(\pi_s))\big) I_{(\pi_s \ge B^*)} ds \le$$

$$\le \int_0^t (c\pi_s - \lambda(1 - \pi_s)) I_{(\pi_s \ge B^*)} ds. \tag{3.2.39}$$

Dividing both parts of (3.2.39) by t and passing to the limit as $t \to 0$,we obtain that the following inequality is satisfied for any $\pi \ge B^*$:

$$-(\lambda_0 + (\lambda_1 - \lambda_0)\pi)\big(\rho(S(\pi)) - (1 - S(\pi))\big) \le c\pi - \lambda(1 - \pi).$$

In particular the above condition is also valid for $\pi = B^*$ and hence

$$B^* \ge \frac{\lambda}{\lambda + c} + \frac{(\lambda_0 + (\lambda_1 - \lambda_0)B^*)\big((1 - S(B^*)) - \rho(S(B^*))\big)}{\lambda + c}. \tag{3.2.40}$$

From the other side passing to the limit in (3.2.28) as $\pi \uparrow B^*$, we have that

$$(\lambda - (\lambda_1 - \lambda_0)B^*)(1 - B^*)\rho'(B^*-) + (\lambda_0 + (\lambda_1 - \lambda_0)B^*)\big(\rho(S(B^*)) - (1 - B^*)\big) = -cB^*,$$

which implies that

$$(\lambda - (\lambda_1 - \lambda_0)B^*)(1 - B^*)(\rho'(B^*-) + 1) = -cB^* +$$

$$\lambda(1 - B^*) - (\lambda_0 + (\lambda_1 - \lambda_0)B^*)\big(\rho(S(B^*)) - (1 - S(B^*))\big). \tag{3.2.41}$$

Since $\rho'(B^*-) + 1 \ge 0$, the right-hand side of the above equality is also non-negative and

$$B^* \le \frac{\lambda}{\lambda + c} + \frac{(\lambda_0 + (\lambda_1 - \lambda_0)B^*)\big((1 - S(B^*)) - \rho(S(B^*))\big)}{\lambda + c}, \tag{3.2.42}$$

thus making use of (3.2.40) and (3.2.42) gives us that

$$B^* = \frac{\lambda}{\lambda + c} + \frac{(\lambda_0 + (\lambda_1 - \lambda_0)B^*)\big((1 - S(B^*)) - \rho(S(B^*))\big)}{\lambda + c}. \tag{3.2.43}$$

From (3.2.41) it is easy to see that the smooth fit condition is satisfied at this point

$$\rho'(B^*-) = -1.$$

Conversely, let $\tilde{\rho}(\pi)$ be a non-negative, concave function satisfying $1), 2), 3^*)$. Then similarly to Theorem 3.2.1 we obtain that

$$\tilde{\rho}(\pi_t) = \tilde{\rho}(\pi_0) - c\int_0^t \pi_s I_{(\pi_s < A^*)} ds - \lambda \int_0^t (1 - \pi_s) I_{(\pi_s \geq A^*)} ds +$$

$$+ \int_0^t (\lambda_0 + (\lambda_1 - \lambda_0)\pi_s)\big(\tilde{\rho}(S(\pi_s)) - (1 - S(\pi_s))\big) I_{(\pi_s \geq A^*)} ds + \tilde{L}_t, \tag{3.2.44}$$

where

$$\tilde{L}_t \equiv \int_0^t \frac{\pi_{s-}(1 - \pi_{s-})(\lambda_1 - \lambda_0)}{\lambda_1 \pi_{s-} + \lambda_0 (1 - \pi_{s-})} \tilde{\rho}'_-(\pi_{s-})(dX_s - (\lambda_1 \pi_{s-} + \lambda_0(1 - \pi_{s-}))ds$$

is the martingale part of this decomposition and $S(\pi)$ is defined in (3.2.31).

Let $\tilde{B} = \inf\{B : \tilde{\rho}(B) = 1 - B\}$. Since $\tilde{\rho}(\pi)$ is concave the continuous fit condition implies that $A^* \in [\tilde{B}, 1]$. On the other hand if $A^* > \tilde{B}$ then on the interval $(\tilde{B}, A^*)$ we shall have $\tilde{\rho}'(\pi) = -1$. Since $\frac{\lambda_1 \pi}{\lambda_1 \pi + \lambda_0(1-\pi)} < \pi$, for any $\pi \in (\tilde{B}, \frac{\lambda_0 \tilde{B}}{\lambda_1 + (\lambda_0 - \lambda_1)\tilde{B}})$, it is easy to check that $S(\pi) \in (\frac{\lambda_1 \tilde{B}}{\lambda_1 \tilde{B} + \lambda_0(1 - \tilde{B})}, \tilde{B})$. Consider the following interval

$$\Delta \equiv (\tilde{B}, \frac{\lambda_0 \tilde{B}}{\lambda_1 + (\lambda_0 - \lambda_1)\tilde{B}} \wedge A^*),$$

then $\forall \pi \in \Delta$ we have that $\tilde{\rho}(\pi) = 1 - \pi$, $\tilde{\rho}'(\pi) = -1$ and

$$S(\pi) \in (\frac{\lambda_1 \tilde{B}}{\lambda_1 \tilde{B} + \lambda_0(1 - \tilde{B})}, \tilde{B} \wedge S(A^*)).$$

By condition 1) equation (3.2.28) is satisfied on the Δ interval. Taking into consideration the above facts form (3.2.28) we derive that

$$-(\lambda - \pi(\lambda_1 - \lambda_0))(1-\pi) + (\lambda_1\pi + \lambda_0(1-\pi))[\tilde{\rho}(S(\pi)) - (1-\pi)] = -c\pi$$

which implies that

$$\tilde{\rho}(S(\pi)) = 1 - S(\pi) + \frac{\lambda - \pi(\lambda + c)}{\lambda_0 + (\lambda_1 - \lambda_0)\pi}. \tag{3.2.45}$$

Since

$$\pi = \frac{\lambda_0 S(\pi)}{\lambda_1 + (\lambda_0 - \lambda_1)S(\pi)} \quad \text{(see} \quad 3.2.31),$$

we will have that

$$\tilde{\rho}(S(\pi)) = -(\frac{c}{\lambda_1} + \frac{\lambda}{\lambda_0} + 1)S(\pi) + (\frac{\lambda}{\lambda_0} + 1).$$

Hence by assuming that $\tilde{B} < A^*$ we have got that $\tilde{\rho}(\pi)$ is a linear function on the

$$S(\Delta) = (\frac{\lambda_1 \tilde{B}}{\lambda_1 \tilde{B} + \lambda_0(1-\tilde{B})}, \tilde{B} \wedge S(A^*))$$

interval, which contradicts the fact that $\tilde{\rho}(\pi)$ is a concave function with $\tilde{\rho}(\pi) \leq 1 - \pi$ (see condition 2)).Thus $\tilde{B} = A^*$ and

$$\{\pi_s < A^*\} = \{\tilde{\rho}(\pi_s) < 1 - \pi_s\},$$

$$\{\pi_s \geq A^*\} = \{\tilde{\rho}(\pi_s) = 1 - \pi_s\}. \tag{3.2.46}$$

From (3.2.44) and (3.2.46) we obtain that

$$\tilde{\rho}(\pi_t) = \tilde{\rho}(\pi_0) - c\int_0^t \pi_s I_{(\tilde{\rho}(\pi_s) < 1-\pi_s)} ds - \lambda \int_0^t (1-\pi_s) I_{(\tilde{\rho}(\pi_s) = 1-\pi_s)} ds$$

$$+ \int_0^t (\lambda_0 + (\lambda_1 - \lambda_0)\pi_s)\big(\tilde{\rho}(S(\pi_s)) - (1 - S(\pi_s))\big) I_{(\tilde{\rho}(\pi_s) = 1-\pi_s)} ds + \tilde{L}_t, \tag{3.2.47}$$

We shall show now that $A^* \geq \frac{\lambda}{\lambda+c}$. Indeed passing to the limit in (3.2.28) when $\pi \uparrow A^*$ and using continuous and smooth fit conditions we obtain that

$$-(\lambda - (\lambda_1 - \lambda_0)A^*)(1 - A^*) + (\lambda_0 + (\lambda_1 - \lambda_0)A^*)(\tilde{\rho}(S(A^*)) - (1 - A^*)) = -cA^*,$$

which implies that

$$A^* = \frac{\lambda}{\lambda + c} + \frac{(\lambda_0 + (\lambda_1 - \lambda_0)A^*)[(1 - S(A^*)) - \tilde{\rho}(S(A^*))]}{\lambda + c} \tag{3.2.48}$$

The second term of the above equality is positive and hence

$$A^* \geq \frac{\lambda}{\lambda + c}. \tag{3.2.49}$$

This inequality implies that $c\pi_s - \lambda(1 - \pi_s)$ is positive on the set $\pi_s \geq A^*$. Therefore we can rewrite (3.2.47) in the following form:

$$\tilde{\rho}(\pi_t) = \tilde{\rho}(\pi_0) - c\int_0^t \pi_s ds + \int_0^t (c\pi_s - \lambda(1 - \pi_s))^+ I_{(\tilde{\rho}(\pi_s) = 1 - \pi_s)} ds - \breve{A}_t + \tilde{L}_t, \tag{3.2.50}$$

where

$$\breve{A}_t = \int_0^t -(\lambda_0 + (\lambda_1 - \lambda_0)\pi_s)\big(\tilde{\rho}(S(\pi_s)) - (1 - S(\pi_s))\big) I_{(\tilde{\rho}(\pi_s) = 1 - \pi_s)} ds. \tag{3.2.51}$$

From (3.2.48) it follows that for any $\pi \geq A^*$

$$(\lambda + c)\pi - \lambda \geq (\lambda_0 + (\lambda_1 - \lambda_0)A^*)[(1 - S(A^*)) - \tilde{\rho}(S(A^*))]. \tag{3.2.52}$$

Since $\tilde{\rho}'(\pi) \in [-1, 0]$ it is easy to check that the function $1 - \pi - \tilde{\rho}(\pi)$ is decreasing and hence, for any $\pi \geq A^*$

$$(\lambda_0 + (\lambda_1 - \lambda_0)\pi)[1 - S(\pi) - \tilde{\rho}(S(\pi))] \leq (\lambda_0 + (\lambda_1 - \lambda_0)A^*)[1 - S(A^*) - \tilde{\rho}(S(A^*))]. \tag{3.2.53}$$

From (3.2.52) and (3.2.53) we obtain that for any $\pi \geq A^*$

$$(\lambda + c)\pi - \lambda \geq (\lambda_0 + (\lambda_1 - \lambda_0)\pi)[1 - S(\pi) - \tilde{\rho}(S(\pi))]. \qquad (3.2.54)$$

Thus

$$\frac{(\lambda_0 + (\lambda_1 - \lambda_0)\pi)[1 - S(\pi) - \tilde{\rho}(S(\pi))]}{(\lambda + c)\pi - \lambda} \leq 1$$

for any $\pi \geq A^*$ and condition II) is valid for the density process $\nu_t = \frac{(\lambda+\pi_t(\lambda_1-\lambda_0))(1-S(\pi_t)-\rho(S(\pi_t)))}{c\pi_t-\lambda(1-\pi_t)} I_{(\tilde{\rho}(\pi_t)=1-\pi_t)}$:

$$0 \leq \nu_t \leq I_{(\tilde{\rho}(\pi_t)=1-\pi_t)}.$$

Similarly to Theorem 3.2.1, one can check that I)-V) conditions are also satisfied. This implies that the triple

$$Y_t = \tilde{\rho}(\pi_t), \qquad \nu_t = \frac{(\lambda + \pi_t(\lambda_1 - \lambda_0))(1 - S(\pi_t) - \rho(S(\pi_t)))}{c\pi_t - \lambda(1 - \pi_t)} I_{(\tilde{\rho}(\pi_t)=1-\pi_t)}$$

and

$$L_t = \int_0^t \frac{\pi_{s-}(1 - \pi_{s-})(\lambda_1 - \lambda_0)}{\lambda_1\pi_{s-} + \lambda_0(1 - \pi_{s-})} \tilde{\rho}'_-(\pi_{s-})(dX_s - (\lambda_1\pi_{s-} + \lambda_0(1 - \pi_{s-}))ds$$

satisfies RBSDE (2.2.1). Therefore by Theorem 2.2.1 $\tilde{\rho}(\pi_t)$ coincides with the value process V_t. Hence by (3.2.5) $\tilde{\rho}(\pi_t) = \rho(\pi_t)$ and $\tilde{\rho}(\pi) = \rho(\pi)$ for all $\pi \in [0, 1]$. □

Conclusions

The aim of this thesis was to develop an analyze of the General Disorder problem from the point of Bayesian - martingale approach. In particular in our setting the three distinct elements of the problem: (1) *the probabilistic structure*, (2) *the a priori distribution* and (3) *criterion for measuring the performance* were proposed respectively to be as follows:

(1) the change-point θ represents a random time of bifurcation of two arbitrary probability measures given on the space of right-continuous functions.

(2) θ a priori is supposed to have an arbitrary distribution function $\psi = \psi(x)$ concentrated on $[0, \infty)$.

(3) As a third element we considered the Bayesian approach to the problem, where an average delay is measured by an F_t predictable increasing process K, which is more natural for models involving martingales with non-absolutely-continuous characteristics.

In this thesis we presented a Bayesian-martingale approach to the general disorder problem with infinite time horizon. We first derived a martingale stochastic differential equation for the a posteriori probability process π_t of the change-point θ, which plays, as it is well known, a crucial role in reducing the disorder problem to the optimal stopping problem and then we introduced the value process V_t of the related optimal stopping problem. We showed that this process uniquely solves a suitable reflecting backward stochastic differential equation (RBSDE). We considered classical disorder problems for Wiener and Poisson processes and showed that in these cases related RBSDEs for value processes and the corresponding free boundary problems are equivalent.

The Wiener disorder problem when its drift changes from one value to another at a random instant of time θ was solved by Shiryaev (1967) who gave an explicit expression for the value function $\rho(\pi)$ of initial stopping problem, showing that this function (together with the optimal threshold A^*) uniquely solves the corresponding free-boundary problem for a parabolic differential operator. We gave a probabilistic proof of this fact. We showed that $\rho(\pi)$ is a solution of the free-boundary problem if and only if the process $\rho(\pi_t)$ is a solution of corresponding RBSDE. The key step here was to show that if the value process $V_t = \rho(\pi_t)$ satisfies RBSDE, then the function $\rho(\pi)$ is continuously differentiable on $(0, 1]$ and twice continuously differentiable on $(0, A^*), 0 < A^* < 1$. In particular this implies that the smooth fit condition is satisfied, but the smooth fit of the second derivative fails.

We considered the disorder problem for a Poisson process whose intensity changes from λ_0 to λ_1 at some random time θ. The closed form solution of this problem was given in Peskir and Shiryaev (2002), where the problem was reduced to a free-boundary differential-difference problem. We showed that this free-boundary problem is also equivalent to the well posedness of the general RDSDE. In particular, this shows that the unique solution of a free-boundary differential-difference problem coincides with the value function of the problem. Besides, we derived the smooth fit conditions for the value function (in cases when this condition is satisfied) and there was also established when the smooth fit condition breaks down directly from the RBSDE for the value process.

From the research carried out in this thesis the new problems can be viewed for future work:

1. Generalization of the results obtained in the thesis in the case of finite time horizon.

2. To prove an equivalence between RBSDE related to the disorder problem for a diffusion process with the free-boundary problem of a parabolic differential operator in the case when the observable process ξ_t is of the form

$$d\xi_t = I_{(\theta \geq t)} b(t, \xi_t)dt + \sigma(t, \xi_t)dW_t,$$

under some regularity conditions on the coefficients $b = b(t, x)$ and $\sigma = \sigma(t, x)$.

3. To find the closed form solutions of the problem 2) in the case when the coefficients b and σ depend only on time parameter.

Appendix A

Auxiliary facts

A.1 The basic notations and facts

We assume given a stochastic basic $(\Omega, \mathcal{F}, F = (\mathcal{F}_t, t \geq 0), P)$. Recall that we have adopted the usual conditions, with the convention $\mathcal{F} = \mathcal{F}_\infty, \mathcal{F}_0 = \mathcal{F}_{0-}$.

Definition A.1.1. *A stochastic process $A = (A_t, t \geq 0)$ adapted to the family $F = (\mathcal{F}_t, t \geq 0)$, with $A_0 = 0$, whose each path is a cadlag increasing (resp. of finite variation on every interval $[0, t]$) function on $[0, \infty[$ is called an increasing process (resp. process of finite variation).*

Denote

T_p - the class of predictable stopping times.

$\mathcal{M}$ - the class of uniformly integrable martingales.

$\mathcal{S}$ - the class of semimartingales

$\mathcal{P}$ - the class of predictable processes

V^+ - the class of increasing processes,

V - the class of processes of finite variation,

$\mathcal{A}^+$ - the class of integrable increasing processes,

$\mathcal{A}$ - the class of processes of integrable variation.

Definition A.1.2. *Let $\mathcal{K}$ be a class of stochastic processes. We say that X belongs to the localized class $\mathcal{K}_{loc}$ if and only if there exists an increasing sequence of stopping times(depending on X) such that $\lim_{n\to\infty}\tau_n = \infty$ a.s. and the stopped process X^{τ_n} belongs to $\mathcal{K}$ for each $n \geq 1$. The sequence $(\tau_n, n \geq 1)$ is called a localizing sequence for X.*

e.g. $\mathcal{M}_{loc}$ is a class of local martingales, ${\mathcal{A}^+}_{loc}$ - the class of locally integrable increasing processes and etc. It is obvious that $V^+_{\text{loc}} = V^+, V_{\text{loc}} = V$ and

$$\mathcal{A}^+ \subset \mathcal{A}^+_{\text{loc}} \subset V^+, \mathcal{A} \subset \mathcal{A}_{\text{loc}} \subset V.$$

Besides, if $A \in V$ (resp. V^+) is predictable then $A \in \mathcal{A}_{\text{loc}}$ (resp. $\mathcal{A}^+_{\text{loc}}$), i.e.

$$V \cap \mathcal{P} = \mathcal{A}_{\text{loc}} \cap \mathcal{P},\ V^+ \cap \mathcal{P} = \mathcal{A}^+_{\text{loc}} \cap \mathcal{P}$$

Definition A.1.3. *A stochastic process $X = (X_t, \mathcal{F}_t)_{t\geq 0}$ belongs to the class $\mathcal{D}$, if the family of random variables*

$$\{X_\tau : \tau \in T, \tau < \infty\}$$

is uniformly integrable, i.e.

$$\sup_\tau E\{|X_\tau| I(|X_\tau| > N)\} \to 0, \quad N \to \infty,$$

where sup is taken over the class of all stopping times T.

Definition A.1.4. *The process X is said to be evanescent if X is indistinguishable with the process equal to zero.*

Definition A.1.5. *The set $A \subset \Omega \times R^+$ is said to be evanescent if $X_t(\omega) = I_A(\omega, t)$ is an evanescent process.*

Theorem A.1.1. *(Doob-Meyer's theorem) (Dellasherie, 1972)*

a)If X is a submartingale of class $\mathcal{D}$, then there exists a unique (up to an evanescent set) predictable, integrable, increasing process A ,with $A_0 = 0$, such that the process $M = X - A$ is a uniformly integrable martingale ,i.e.

$$X = M + A, \qquad A \in \mathcal{A}^+ \cap \mathcal{P},\ M \in \mathcal{M}. \tag{A.1.1}$$

b) If X is a submartingale, then there exists a unique (up to an evanescent set) predictable, localy integrable, increasing process A , such that the process $M = X - A$ is a local martingale ,i.e.

$$X = M + A, \qquad A \in \mathcal{A}^+_{loc} \cap \mathcal{P},\ M \in \mathcal{M}_{loc}.$$

A.2 Projections and dual projections

Theorem A.2.1. *(Dellasherie, 1972) Let $X = (X_t, t \geq 0)$ be a measurable stochastic process. There exists a unique (up to an evanescent set) predictable process ${}^pX$ (taking its values in $]-\infty, \infty]$), called the predictable projection, such that*

$$({}^pX)_\tau = E(X_\tau/\mathcal{F}_{\tau-}) \text{ on } \{\tau < \infty\} \text{ for all } \tau \in T_p.$$

Theorem A.2.2. *(Dellasherie, 1972) Let A be a predictable process of integrable variation (or let $A \in \mathcal{A}_{loc}$). If X and Y are two positive measurable processes, whose predictable projections coincide (up to an evanescent set), then*

$$E\int_0^\infty X_s dA_s = E\int_0^\infty Y_s dA_s.$$

Theorem A.2.3. *(Dellasherie, 1972), Let $A \in \mathcal{A}^+_{loc}$. Then there exists a unique (up to an evanescent set) predictable, locally integrable increasing process $\tilde{A}$, called the dual predictable projection (or compensator) of A, which satisfies one of the equivalent conditions:*

1) $A - \tilde{A}$ is a local martingale,

2) $EA_\tau = E\tilde{A}_\tau$ for each stopping time τ,

3) $E(H \cdot A)_\infty = E(H \cdot \tilde{A})_\infty$ for every non-negative predictable process H.

Corollary A.2.4. *a) If $A \in \mathcal{A}_{loc}$ then there exists a unique (up to an evanescent set) predictable process $\tilde{A}$ of locally integrable variation such that*

$$A - \tilde{A} \in \mathcal{M}_{loc}.$$

b) If H is predictable and $H \cdot A \in \mathcal{A}_{loc}$ then

$$(\widetilde{H \cdot A}) = H \cdot \tilde{A} \in \mathcal{A}_{loc} \quad \text{and} \quad H \cdot A - H \cdot \tilde{A} \in \mathcal{M}_{loc}.$$

A.3 Local times and the Tanaka-Meyer formula

1. Convex functions and semimartingales

Let $(\Omega, \mathcal{F}, F = (\mathcal{F}_t, t \geq 0), P)$ be a stochastic basis satisfying the usual conditions.

We recall that the following conditions on the function $f = (f(x), x \in R)$ are equivalent:

(1) For every compact interval $\mathcal{K}$ there exists a pair (f^1, f^2) of convex functions on $\mathcal{K}$ such that $f = f^1 - f^2$.

(2) The function f has a left-continuous left-hand derivative (denoted by f'_-) which has finite variation on every compact interval.

(3) A weak second derivative of f may be identified (on each compact) to a signed measure ν^f such that

$$\int \psi_{xx}(x)f(x)dx = \int \psi(x)\nu^f(dx)$$

for all $\psi \in C_0^\infty$.

Theorem A.3.1. *(Dellasherie and Meyer, 1980) Let $f : R \to R$ be a convex function and let X be a semimartingale. Then the process*

$$Y_t = f(X_t) - f(X_0) - \int_0^t f'_-(X_{s-})dX_s$$

is increasing. In particular the process $(f(X_t), t \in R_+)$ is a semimartingale.

Corollary A.3.2. *Since an increasing process is a sum of its jumps and of a continuous increasing process, when f is convex then the process*

$$f(X_t) - f(X_0) - \int_0^t f'_-(X_{s-})dX_s - \sum_{s \le t}(f(X_s) - f(X_{s-}) - f'_-(X_{s-})\Delta X_s)$$

is increasing and continuous.

2. Local times

Definition A.3.1. *For each semimartingale X a continuous increasing process $\mathcal{L}_t^X(a) \equiv \mathcal{L}_t^a, t \in R_+$ defined by the equality (which is called Tanaka's formula)*

$$\mathcal{L}_t^X(a) = |X_t - a| - |X_0 - a| - \int_0^t sign(X_{s-} - a)dX_s - \tag{A.3.1}$$

$$-\sum_{s\le t}(|X_s-a|-|X_{s-}-a|-sign(X_{s-}-a)\Delta X_s),$$

where $signx = -I_{]-\infty,0]}(x) + I_{]0,\infty[}(x)$*, is called a local time of the process* X *spent at the point* $a \in R$.

Remark A.3.1. The process defined by equality (A.3.1) is continuous in t and increasing according to Corollary A.3.2 and, hence, this definition is correct.

It is easy to see that if X is a positive semimartingale then

$$\mathcal{L}_t^0 = \int_0^t I_{(X_{s-}=0)}dX_s - \sum_{s\le t} I_{(X_{s-}=0)}\Delta X_s. \tag{A.3.2}$$

Indeed, the trivial equation $|X_t| = X_t = X_0 + \int_0^t I_{(X_{s-}>0)}dX_s + \int_0^t I_{(X_{s-}=0)}dX_s$ together with (A.3.1) implies (A.3.2).

The name local time for the process $\mathcal{L}^X$ is justified by the following statement: Let $X = (X_t, t \in R_+)$ be a semimartingale such that

$$\sum_{s\le t}|\Delta X_s| < \infty \tag{A.3.3}$$

P-a.s. for each $t \in R_+$.

Theorem A.3.3. *(Protter, 1995) Let* X *be a semimartingale satisfying Hypotheses (A.3.3). Then there exists a* $\mathcal{B}(R)\otimes\mathcal{P}$ *measurable version of* $(a,t,\omega)\to\mathcal{L}_t^a(\omega)$ *which is everywhere jointly right continuous in* a *and continuous in* t*. Moreover a.s. the limits* $\mathcal{L}_t^a = \lim_{b\to a, b<a}\mathcal{L}_t^b$ *exists.*

Corollary A.3.4. *(Protter, 1995) Let* X *be a semimartingale satisfying condition (A.3.3). Then for every* (a,t) *P-a.s.*

$$\mathcal{L}_t^a = \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_0^t I_{(a\le X_s\le a+\epsilon)}d<X^c,X^c>_s \tag{A.3.4}$$

and

$$\mathcal{L}_t^{a-} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_0^t I_{(a-\epsilon \le X_s \le a)} d < X^c, X^c >_s,$$

where X^c *is a continuous martingale part of* X *and* $< X^c, X^c >$ *is a quadratic characteristic of* X^c *(see, e.g., Corollary 3 of Theorem 56, Chapter 4, Protter, 1995).*

Theorem A.3.5. *(Occupation formula) (Revuz and Yor, 1994) For every bounded measurable function* g *and every semimartingale* X

$$\int_0^t g(X_s) d < X^c, X^c >_s = \int_R \mathcal{L}_t^a g(a) da. \tag{A.3.5}$$

Corollary A.3.6. *Let* X *be a semimartingale with* $X^c = 0$. *Then* $\mathcal{L}_t^a = 0$ *for each* $a \in R$. *In particular* $\mathcal{L}_t^a = 0$ *for any process of finite variation.*

3. Tanaka-Meyer's formula

Let $f(x) = |x - a|, a \in R$ and let X be a semimartingale. Then the process $f(X_t) = |X_t - a|, t \in R_+$ is also a semimartingale and Tanaka's formula gives the following explicit decomposition

$$|X_t - a| = |X_0 - a| + \int_0^t sign(X_{s-} - a) dX_s +$$

$$+ \sum_{s \le t} (|X_s - a| - |X_{s-} - a| - sign(X_{s-} - a)\Delta X_s) + \mathcal{L}_t^a. \tag{A.3.6}$$

Now we give an extension of Tanaka's formula to an arbitrary convex function (or to difference of two convex functions) due to Brosamler (1970) and Meyer (1972).

Theorem A.3.7. *Let* $f : R \to R$ *be a function representable as a difference of two convex functions and let* X *be a semimartingale. Then the process* $f(X_t)$ *is a* *semimartingale and*

$$f(X_t) = f(X_0) + \int_0^t f'_-(X_{s-})dX_s + \frac{1}{2}\int \nu^f(da)\mathcal{L}_t^a +$$

$$+ \sum_{s\le t}[(f(X_s) - f(X_{s-})) + f'_-(X_{s-})\Delta X_s],$$

where f'_- is the left derivative of f, ν^f is a second derivative of f in a weak sense and $\Delta X_t = X_t - X_{t-}$.

A.4 Stochastic exponent and Girsanov's transform

1. Stochastic exponent

Let $X = (X_t, t \ge 0)$ be a semimartingale defined on some stochastic basic $(\Omega, \mathcal{F}, F = (\mathcal{F}_t, t \ge 0), P)$ and let Z_0 be a finite $\mathcal{F}_0$-measurable random variable. Consider the following linear stochastic equation

$$Z_t = Z_0 + \int_0^t Z_{s-}dX_s, \qquad \text{(A.4.1)}$$

which is called a stochastic differential equation of Doleans-Dade.

Theorem A.4.1. *(Jacod, 1979) In the class of semimartingales S there exists a unique solution of equation (A.4.1) and the following representation is valid*

$$Z_t = Z_0 exp\{X_t - X_0 - \frac{1}{2} < X^c, X^c >\}\prod_{s\le t}(1 + \Delta X_s)\exp\{-\Delta X_s\}, \qquad \text{(A.4.2)}$$

where the infinite product converges absolutely for each $t \in R_+$.

The unique solution of (A.4.1), when $Z_0 = 1$, is denoted by $\mathcal{E}(X)$ and is called the stochastic exponent (or the Dolean exponent). If X and Y are two semimartingales with $[X, Y] = 0$, where

$$[X,Y] = < X^c, Y^c > + \sum_{s \leq \cdot} \Delta X_s \Delta Y_s,$$

then the property $\mathcal{E}(X)\mathcal{E}(Y) = \mathcal{E}(X+Y)$ of the usual exponents is correct for stochastic exponents. In the general case the following assertion is true.

Proposition A.4.2. *(Jacod, 1979) If X and Y are two semimartingales then*

$$\mathcal{E}(X)\mathcal{E}(Y) = \mathcal{E}(X + Y + [X, Y]).$$

Proposition A.4.3. *(Jacod, 1979)*

1) If $X \in V$, then $\mathcal{E}(X) \in V$,

2) If $X \in \mathcal{M}_{loc}$, then $\mathcal{E}(X) \in \mathcal{M}_{loc}$.

3) If $X \in S_p$, then $\mathcal{E}(X) \in S_p$,

where S_p is a class of special semimartingales.

Proposition A.4.4. *(Memin, 1978) Let Y^1 and Y^2 be two semimartingales such that $Y_0^1 = Y_0^2 = 0$ and $\Delta Y_t^1 \neq -1$ for each $t \in R_+$. Then there exists a unique semimartingale X such that*

$$\mathcal{E}(X)\mathcal{E}(Y^1) = \mathcal{E}(Y^2).$$

The semimartingale X is defined by the equality

$$X_t = Y_t^2 - Y_t^1 - < (Y^2 - Y^1)^c, (Y^1)^c >_t - \sum_{s \leq t} \frac{\Delta (Y^2 - Y^1)_s \Delta Y_s^1}{1 + \Delta Y_s^1};$$

2. Girsanov's transform

The Girsanov problem is to give the canonical decomposition w.r.t. the measure Q (equivalent or absolutely continuous w.r.t. P) of a semimartingale X for which the decomposition w.r.t. P is given (or the problem consists in computation of the characteristics of a semimartingale X relative to Q, from its characteristics under the measure P).

Let the stochastic basis $(\Omega, \mathcal{F}, F = (\mathcal{F}_t, t \geq 0), P)$ is given and let Q be an another measure defined on $(\Omega, \mathcal{F})$.

Theorem A.4.5. *(Jacod, 1979) Let $Q \overset{loc}{<<} P$. Then there exists a unique (up to an P and Q evanescent set) P-local martingale Z, such that $Z_t = dQ_t/dP_t$(the Radon-Nikodym derivative) for all $t \geq 0$.*

Moreover

a) one may take $Z \geq 0$ identically.

b) for every $\tau \in T$(resp. $\tau \in T_p$)

$$Z_\tau = dQ_\tau/dP_\tau \quad (resp. Z_{\tau-} = dQ_{\tau-}/dP_{\tau-})$$

on the set $\{\tau < \infty\}$.

c) there exists the limit $Z_\infty = \lim_{t\to\infty} Z_t$ (P and Q-a.s.) which is finite $P-$a.s and $Q-$a.s. if and only if $Q \ll P$.

d) the process $(Z_t, t \in \bar{R}_+)$ is the positive supermartingale, which is a uniformly integrable martingale iff $EZ_\infty = EZ_0 = 1$, if and only if $Q \ll P$.

Definition A.4.1. *The process $Z = (Z_t, t \geq 0)$ is called the density process of Q w.r.t. the measure P.*

Here is the main result on change of laws known as the Girsanov theorem. The definitive statements is due to Van Schuppen, Wong (1974) and Jacod, Memin (1976) respectively. In complete generality this problem was solved by Langlart (1977) (see also Jacod (1979)).

Theorem A.4.6. *(Jacod, 1979) Let X be a P-local martingale and let Z be the density process $Z_t = dQ_t/dP_t$.*

a) Assume that $Q \overset{loc}{\ll} P$ and $[X, Z] \in \mathcal{A}_{loc}(P)$. Then there exists $< X, Z >$ and the process

$$X^Q = X - I_{(Z_->0)}\frac{1}{Z-} \cdot < X, Z >,$$

where $< X, Z >$ is the dual projection (or compensator) of $[X, Z]$, is a Q-local martingale. Moreover the processes $I_{(Z_->0)} \cdot < X^{Q,c} >$ and $I_{(Z_->0)} \cdot < X^c >$ are P-indistinguishable.

b) Let $Q \overset{loc}{\sim} P$. Then the process

$$X^Q = X - \frac{1}{Z} \cdot [X, Z] \qquad \text{(A.4.3)}$$

is Q-local martingale.

Now we give Girsanov's theorem in the form, which is convenient for our purposes. Let $Q \overset{loc}{\sim} P$ and Z is the density process of Q relative to P. Then the process Z_-^{-1} is locally bounded and hence the stochastic integral $M = Z_-^{-1} \cdot Z$ is defined. Therefore the equality $Z_t = Z_0 + \int_0^t Z_{s-} dM_s$ and Theorem A.4.2 implies that the density process Z can be represented as an exponential martingale

$$Z_t = Z_0 \mathcal{E}_t(M), \quad \text{with} \quad M = Z_-^{-1} \cdot Z \in \mathcal{M}_{loc}(P).$$

Using (A.4.3) we obtain that in this case

$$X^Q = X - \frac{1}{1 + \Delta M} \cdot [X, M], \tag{A.4.4}$$

where

$$X^Q \equiv L(X, M)$$

is called the Girsanov transform.

In general for two semimartingales X and Y, with $\Delta Y_t \neq -1$ for all t, $L(X, Y)$ is defined in the same way:

$$L_t(X, Y) = X_t - \int_0^t \frac{1}{1 + \Delta Y_s} d[Y, X]_s =$$

$$= X_t - < X^c, Y^c >_t - \sum_{s \le t} \frac{\Delta X_s \Delta Y_s}{1 + \Delta Y_s}. \tag{A.4.5}$$

Note that

$$\Delta L(X, Y) = \frac{\Delta X}{1 + \Delta Y}.$$

Besides it follows from Proposition A.4.4 that (see Memin, 1978)

$$\frac{\mathcal{E}_t(X)}{\mathcal{E}_t(Y)} = \mathcal{E}_t(L(X - Y, Y)) \tag{A.4.6}$$

and for any X-integrable predictable process H

$$L(H \cdot X, Y) = H \cdot L(X, Y),$$

Theorem A.4.7. *Let* $Q \overset{loc}{\sim} P$ *and* $[X, M] \in \mathcal{A}_{loc}(P)$. *Then the process*

$$X^Q = X - < X, M >$$

is a Q*-local martingale.*

Now we shall give the analogue of the Girsanov theorem A.4.6 for a Wiener process.

Theorem A.4.8. *(Girsanov)(Liptser and Shiryaev, 2001) Let* W_t, $t \in [0, T]$, $T \leq \infty$ *is a Wiener process given on the filtered probability space* $(\Omega, \mathcal{F}, F = (\mathcal{F}_t, t \geq 0), P)$ *and* $f : \Omega \times [0, T] \to R$ *is a predictable process satisfying the condition*

$$\int_0^T f_t^2 dt < \infty, \quad P - a.s.$$

Suppose that

$$E[\xi_0^T(f)] = 1,$$

where

$$\xi_0^t(f) = \exp\Big(\int_0^t f_s dW_s - \frac{1}{2}\int_0^t f_s^2 ds\Big).$$

If Q *is a probability measure on* $(\Omega, \mathcal{F})$ *defined as* $\frac{dQ}{dP} = \xi_0^T(f)$, *then the process*

$$\bar{W}_t = W_t - \int_0^t f_s ds$$

is a Wiener process on $(\Omega, \mathcal{F}, F = (\mathcal{F}_t, t \geq 0), Q)$.

Theorem A.4.9. *(Liptser and Shiryaev, 2001) Let the point process* $X = (x_t, \mathcal{B}_t, \mu)$ *has the continuous compensator* $A_t(x)$ *(or, equivalently, the process* X *is left quasi-continuous). A necessary and sufficient condition for the measure* $\tilde{\mu}$ *to be absolutely* *continuous with respect to the measure* μ *is that* ($\tilde{\mu} - a.s.$) *the following conditions be satisfied:*

1.

$$\tilde{A}_t(x) = \int_0^t \lambda_s(x) dA_s(x). \quad t < \tau_\infty;$$

2.

$$\int_0^t (1 - \sqrt{\lambda_s(x)})^2 dA_s(x) < \infty,$$

where $(\lambda_t(x), \mathcal{B}_t)$ *is some non-negative predictable process. In this case*

$$\frac{d\tilde{\mu}}{d\mu}(t, x) = \exp\Big(\int_0^t \ln \frac{d\tilde{A}_s(x)}{dA_s(x)} dx_s - [\tilde{A}_t(x) - A_t(x)] \Big).$$

A.5 Generalized Bayes theorem

Let $(\Omega, \mathcal{F}, P)$ be a probability space. We shall consider random variable $\theta = \theta(\omega)$ and functions $g = g(a)$ with $E|g(\theta)| < \infty$.

The Generalized Bayes theorem gives an analog of Bayes' theorem for conditional expectations $E[g(\theta)|\mathcal{G}]$ with respect to a $\sigma-$algebra, $\mathcal{G} \subseteq \mathcal{F}$.

Let

$$Q(B) = \int_B g(\theta) P(d\omega), \quad B \in \mathcal{G}.$$

Then by the definition of conditional expectation and Radon-Nikodym derivative

$$E[g(\theta)|\mathcal{G}] = \frac{dQ}{dP}(\omega).$$

We also consider the σ−algebra $\mathcal{G}_\theta$. Then by the definition of conditional probability

$$P(B) = \int\limits_{\Omega} P(B|\mathcal{G}_\theta)dP$$

or, by the formula for change of variables in Lebesgue integrals,

$$P(B) = \int\limits_{-\infty}^{+\infty} P(B|\theta = a)P_\theta(da).$$

Since

$$Q(B) = E[g(\theta)I_B] = E[g(\theta) \cdot E(I_B|\mathcal{G}_\theta)],$$

we have

$$Q(B) = \int\limits_{-\infty}^{+\infty} g(a)P(B|\theta = a)P_\theta(da).$$

Now suppose that the conditional probability $P(B|\theta = a)$ is regular and admits the representation

$$P(B|\theta = a) = \int\limits_{B} \rho(\omega; a)\lambda(d\omega),$$

where $\rho = \rho(\omega; a)$ is non-negative and measurable in the two variables jointly, and λ is a σ-finite measure on $(\Omega, \mathcal{G})$.

Let $E|g(\theta)| < \infty$. Then the Generalized Bayes theorem is stated as follows (see, *e.g.* Shiryaev, 1996, pp. 230-233)

$$E[g(\theta)|\mathcal{G}] = \frac{\int_{-\infty}^{+\infty} g(a)\rho(\omega; a)P_\theta(da)}{\int_{-\infty}^{+\infty} \rho(\omega; a)P_\theta(da)} \qquad (P - \text{a.s.}) \qquad \text{(A.5.1)}$$

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