



The kind of silence: managing a reputation for voluntary disclosure in financial markets

Miles B. Gietzmann¹ · Adam J. Ostaszewski² 

Received: 7 November 2022 / Accepted: 23 May 2023
© The Author(s) 2023

Abstract

We create a continuous-time setting in which to investigate how the management of a firm controls a dynamic choice between two generic voluntary disclosure decision rules (strategies) in the period between two consecutive mandatory disclosure dates: one with full and transparent disclosure termed *candid*, the other, termed *sparing*, under which values only above a dynamic threshold are disclosed. We show how parameters of the model such as news intensity, pay-for-performance and time-to-mandatory-disclosure determine the optimal choice of candid versus sparing strategies and the optimal times for management to switch between the two. The model presented develops a number of insights, based on a very simple ordinary differential equation characterizing equilibrium in a piecewise-deterministic model, derivable from the background Black–Scholes model and Poisson arrival of signals of firm value. It is shown that in equilibrium when news intensity is low a firm may employ a *candid* disclosure strategy throughout, but will otherwise switch (alternate) between periods of being *candid* and periods of being *sparing* with the truth (or the other way about). *Significantly, with constant pay-for-performance parameters, at most one switching can occur.*

Keywords Asset-price dynamics · Voluntary disclosure · Dynamic disclosure policy · Markov piecewise-deterministic modelling · Corporate transparency reputation

JEL Classification G32 · D82

List of symbols

$\alpha, \beta, \kappa = 1 - \alpha/\beta$: Pay-for-performance coefficients, Sects. 2 and 3.3

✉ Adam J. Ostaszewski
a.j.ostaszewski@lse.ac.uk

Miles B. Gietzmann
miles.gietzmann@unibocconi.it

¹ Bocconi University, Milan, Italy

² London School of Economics and Political Sciences, London, UK

π :	Probability of sparing behaviour, Sect. 2
γ :	Valuation curves: γ_t^1 sparing throughout, γ^* switching curve, Sect. 3.3
λ :	Poisson news-arrival intensity, Sect. 2
θ :	Switching time, Sect. 2
μ :	Co-state variable, Sect. 3.4
φ :	Integrating factor (associated with μ), Appendix
h :	Discounting function (depending on time), Sects. 2 and 5
$g(t) := \int_0^t h(u)du$	-
X_t :	Economic state; Y_t : its (noisy) observation, Sect. 2
σ :	Aggregate volatility, Sect. 3.2

1 Introduction

When investors value firms, they not only base their inferences upon what news signals managers make public, but also on the likelihood that management may be hiding other news. A legal environment with high penalties reduces the chances managers will hide very bad news signals, but a constant concern for investors is: do management release early warning signals of potentially less severe bad news in a timely fashion, or do they hide it in an underhand way in the hope conditions will evolve differently and only disclose when potential legal liability arises? (Marinovic and Vargas 2016).

The Dye (1985) model of voluntary disclosure addressed this issue in 1985 in a static setting and derives equilibrium conditions under which management adopts a sparing approach with a threshold-disclosure strategy, when deciding whether or not to voluntarily disclose new information ahead of a mandatory disclosure date. Institutional features, such as news-arrival rates and time to the mandatory disclosure date, cannot be modelled in such a static setting. In response Beyer and Dye 2012 develop a two-period model in which managers may make a voluntary disclosure in order to build a reputation for being *candid*—forthright (or ‘forthcoming’)—by always faithfully disclosing their updated information. By contrast managers could exploit their asymmetric endowment of information and only make voluntary disclosures if their received signal is high enough, as defined by a valuation threshold (cutoff). Such behaviour will here be termed *sparing*.¹ For the two-period model Beyer and Dye show why managers’ concerns in the first period—for how investors form second-period inferences (based on observed first-period dividend outcomes)—affect their voluntary disclosure strategy. Their model predicts a diversity in management strategies, as in equilibrium some managers will choose a strategy which leads investors to assign a high probability that they will behave sparingly, while others choose to be candid. An insight from this model is that a voluntary disclosure strategy may be used to influence future firm-value over and above the direct effect of disclosure of any idiosyncratic signals of value. That is, establishing a reputation for being candid at times shifts firm-value upwards, over and above the direct discounted present-value of the most

¹ Candid and sparing reporting strategies are termed *forthcoming* and *strategic* by Beyer and Dye.

recent signals of value, precisely because investors now assign a reduced likelihood for management hiding bad news.

The structure of the paper is as follows. Section 2 presents our main findings. Section 3 develops the theory of the optimal sparing-disclosure threshold in a continuous framework, for which the main optimization tool comes from control theory and relies on the Pontryagin Principle. Examples are given in Sect. 4. Proofs of theorems are in Sect. 5 with some technical results relegated to an Appendix². We present conclusions in Sect. 6. We note that all valuations are viewed as *discounted*.

2 Model generalities and findings

Consider a firm whose financial state X_t evolves in continuous time according to a Black–Scholes model with periodic mandatory disclosure dates and with interim intermittent capability of voluntary disclosures of the next mandatory expected financial report. This would be based on partial observation Y_t of the financial state X_t .

Consider two possible reporting behaviours executed by the firm management at any time t when Y_t is observed:

candid (faithful) reporting—reporting the observed value, as seen, i.e. unconditionally;

sparing (threshold) reporting—reporting only the value observed when above a time- t dependent threshold, i.e. conditionally.

These behaviours are both capable of being applied at any one time, i.e. leading to their use in some combination, and are assumed to be both truthful and prompt (i.e. without delay). We also assume that managers cannot *credibly* assert absence of information arriving at time t (i.e. absence of knowledge of Y_t). Furthermore, no evidence of an undisclosed observation is retained. We admit no further sources of information about Y_t ; modelling with the inclusion of further sources is touched on in Gietzmann et al. (2020).

Sparing here is used in the sense of being economical with information delivery as in ‘economical with the truth’ or ‘actuality’,³ sometimes called strategic. It is of course a *foundational question* whether it is suboptimal to withhold information. An early finding in the disclosure literature, provided by Grossman (1981) and Milgrom (1981), has become known as the *unravelling result*. It suggested that withholding information would lead investors to discount the valuations, thus incentivising a firm to make a full disclosure in order to restore the value.

The contribution of Dye (1985) was to provide, in a discrete framework with one interim date (say at some time s between two mandatory disclosure dates of 0 and 1), a rationale for why this ‘full disclosure unravelling’ result might not occur at the interim

² A survey of the relevant literature, referring also to the contributions of Bertomeu et al. (2022), Einhorn and Ziv (2008), Guttman et al. (2014), may be found in the longer <https://arxiv.org/abs/2210.11315v2> version of this paper, which also contains a number of routine calculations in a second Appendix, present only there.

³ The phrase ‘economical with the truth’, though it dates back to 1897, was not common parlance in the UK till Robert Armstrong’s reference to it—in defence of his stance during the Spycatcher trial in Australia in 1986, resurfacing in 1992 in the Arms-for-Iraq case.

date s , and to supply an equation uniquely determining the resulting market discount in value in an equilibrium framework. The market discount is an appropriate *weighted average* that combines the possibility that management lacks fresh information with the possibility that management may hide information which if disclosed would have led to an even larger discount (i.e. below the weighted average).

Dye's paradigm for valuing a non-disclosing firm may be characterized by an amended statement of the Grossman–Hart paradigm as follows

Minimum principle [Ostaszewski and Gietzmann (2008), cf. Acharya et al. (2011)—their Prop. 1] *In equilibrium the market values the firm at the least level consistent with the beliefs and information available to the market as to its productive capability.*

See also Sect. 3. This result carries some detailed implications to which we return later in Sect. 3.3. But the principle already suggests intuitively that if the management reporting behaviour is believed by the market to be at times candid, then in equilibrium the weighted average valuation may at times move upwards, by placing less weight on the chance of poor observation being withheld.

We will demonstrate the validity of such a suggestion in the continuous-time context of the firm as described above, by creating a continuous analogue of Dye's argument in which the Poisson *arrival rate* of the observation time of Y_t is λ and assuming management can report in a sparing mode (relative to an optimally generated threshold) with a probability π_t at time t when simultaneously the market believes (in equilibrium) that the selected probability is indeed π_t .

Management choice of π_t is motivated through the maximization of an appropriate objective function rewarding in proportion to a factor α_t the instantaneous firm value and penalizing in proportion to a factor β_t the instantaneous *value-differential* (value relative to that derived from sparing behaviour executed throughout all time). The parameter $\kappa_t = 1 - \alpha_t/\beta_t$ emerges as significant (see Sect. 3.3).

The aim of the penalty term is to provide a tension between sparing and candid reporting: adopting candour throughout would require more frequent disclosure of potentially bad news, which could result in larger falls in firm value, i.e. over and above falls that resulted from continued sparing silence (non-disclosure).

We discuss our findings in this section, leaving details of the optimization and proofs to the next and later section. Our first surprising finding is that an optimal disclosure behaviour is of *bang-bang* type which will always *switch* (alternate) between intervals of constancy with only $\pi = 0$ or $\pi = 1$, i.e. a mixed strategy is ruled out in the following theorem. (See the Appendix for the stronger statement in Theorem 1 S.)

Theorem 1 (Non-mixing theorem). *When α_t, β_t are constant:*

A mixing control with $\pi_t \in (0, 1)$ is non-optimal over any interval of time.

The theorem agrees with empirical findings (due to Grubb 2011) that after an announcement management is observed to follow initially either candid or sparing behaviour but not a mixture.

Accordingly, we study $\pi_t \in \{0, 1\}$. In particular, we study the possible occurrence of an *initially candid (candid-first) equilibrium* in which management at first adopt candid behaviour out of which they switch after some time θ , the *switching time*, in favour of a sparing policy, and also *initially sparing (sparing-first) equilibrium* in

which management at first adopt sparing behaviour out of which after some time they switch in favour of candid behaviour. This provides a model framework for *empirical detection of regime change* (disclosure policy change): cf. Løkka (2007). Theorems 2a and 2b identify both the *location* of the switching time of such an equilibrium policy and the attendant necessary and sufficient *existence conditions* guaranteeing an equilibrium. These theorems are followed by a clarifying discussion concerning the location conditions.

We stress that having merely a characterization of the location condition is not adequate. The technical nature of the existence conditions emerges from a Hamiltonian analysis (Sect. 3.4 below) in which the Pontryagin Principle relies on Theorem 1 (the non-mixing) in supplying a necessary and sufficient optimality condition.

Our findings refer to a decreasing discount function $h(t)$ (responsible for the rate of fall in values when continued absence of disclosure is attributed to sparing behaviour—see Sect. 3.2 equation (cont-eq)) and to its integral

$$g(t) = \int_0^t h(u) du \quad (0 \leq t \leq 1).$$

Theorem 2a (Single switch equilibrium location and existence for an initially candid strategy). *Assume that α_t, β_t are constant and $0 < \kappa < 1$ for $\kappa := 1 - \alpha_t / \beta_t$.*

In an equilibrium, if such exists, in which $\pi = 0$ on $[0, \theta) = 0$ and $\pi = 1$ on $[\theta, 1]$, the uniquely optimal switching time θ solves

$$\kappa_1 := \kappa e^{\lambda g(\theta)} = 1, \text{ i.e. } g(\theta) = (\log \kappa^{-1}) / \lambda.$$

For given λ this equation is solvable for large enough κ , in fact iff

$$1 > \kappa \geq e^{-\lambda g(1)}. \tag{cand}$$

Such an equilibrium exists iff the unique switching time θ satisfies

$$g(\theta) / (\theta h(0)) > \frac{\log \kappa^{-1}}{\kappa^{-1} - 1}.$$

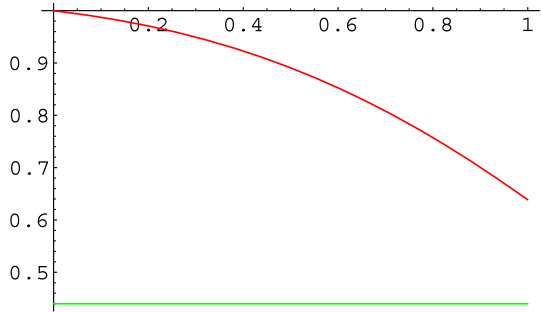
In such an equilibrium, the unique switching time θ satisfies

$$\theta'(\lambda) = \frac{\log \kappa}{\lambda^2 h(\theta(\lambda))} < 0.$$

So larger news-arrival rates λ create shorter periods of initial candid behaviour.

Remark The left-hand side term of the existence condition above is monotonically decreasing from 1 down to $g(1)/h(0)$, as in the red graph in Fig. 1 below; its lowest value is dictated by σ , a volatility measure. The right-hand side ranges monotonically from 0 to 1; thus a fixed value of κ supplies a value to the right-hand side and is illustrated in green below for a choice which allows all values θ to satisfy the inequality here (with other choices restricting the θ range).

Fig. 1 Pay-for-performance bounds (color figure online)



Matters are more complicated in an equilibrium that is initially sparing.

Theorem 2b (Single switch equilibrium location and existence for an initially sparing strategy). Assume that α_t, β_t are constant and $0 < \kappa < 1$ for $\kappa := 1 - \alpha_t/\beta_t$.

In an equilibrium, if such exists, in which $\pi = 1$ on $[0, \theta) = 0$ and $\pi = 0$ on $[\theta, 1]$, the uniquely optimal switching time θ solves the first-order condition

$$1 = \theta + \frac{\kappa^{-1} - 1}{\lambda h(\theta)}, \text{ equivalently } (1 - \theta)h(\theta) = \frac{\kappa^{-1} - 1}{\lambda}.$$

For given λ , this equation is solvable for large enough κ , in fact iff

$$1 > \kappa \geq [1 + \lambda h(0)]^{-1}. \tag{spar}$$

Such an equilibrium exists iff the unique switching time θ satisfies

$$(1 - \theta)h(\theta)(-\log h(\theta)/h(0))/g(\theta) > \kappa^{-1} - 1, \text{ (a lower bound for } \theta),$$

$$-\log h(\theta)/h(0)/g(\theta) > \lambda \text{ (a bound on } \lambda \text{ in terms of } \theta).$$

In such an equilibrium, the unique switching time θ satisfies

$$\theta'(\lambda) = \frac{(1 - \theta)h(\theta)}{\lambda[h(\theta) - (1 - \theta)h'(\theta)]} > 0.$$

So larger news-arrival rates λ create longer periods of initial sparing behaviour.

The theorems expose a fact of direct relevance to empirical study: that with the same parameter values it may happen that both equilibrium types coexist, as in Figs. 2 and 3. (Conditions on parameter values κ, λ permitting this can be derived numerically from the conditions of Theorems 2a and 2b). In particular, the two conditions (cand) and (spar) on κ with fixed λ , may both hold simultaneously: indeed, since the map $\lambda \mapsto e^{\lambda g(1)}$ is convex, there is a unique $\lambda = \lambda_{\text{crit}} > 0$ such that

$$e^{\lambda g(1)} = 1 + \lambda h(0),$$

and

$$e^{-\lambda g(1)} \leq (1 + \lambda h(0))^{-1} \text{ according as } \lambda_{\text{crit}} \leq \lambda.$$

Thus, for $\lambda < \lambda_{\text{crit}}$, both conditions are met when κ satisfies (cand):

$$(1 + \lambda h(0))^{-1} < e^{-\lambda g(1)} < \kappa.$$

In contrast to the coexistence of equilibria in which switching occurs, there does exist a constantly candid equilibrium (i.e. with no switching):

Qualitative corollary. *For small enough λ , candid (unconditional) disclosure throughout the period of silence is an equilibrium policy.*

For proof: see Corollary 2 in Sect. 5. The location of optimal switches, assuming they correspond to an equilibrium (requiring additional conditions), can be pursued in generality. We identify the consequent generalization as this leads to yet another surprising finding stated in the Corollary below. The proof of Theorems 3a and 3b is essentially the same as for Theorems 2a and 2b, hence omitted: see Appendix B in the arXiv version of this paper for details.

Theorem 3a (General switching equation). *Assume that α_t, β_t are constant and $0 < \kappa < 1$ for $\kappa := 1 - \alpha_t/\beta_t$. Assume further that the n consecutive switches located at times $0 = \theta_0 < \theta_1 < \theta_2 < \dots < \theta_n < 1 = \theta_{n+1}$ are selected optimally. Then, for $1 \leq i \leq n$,*

$$g \left(\theta_i + \frac{1 - \bar{\kappa}_{i-1} e^{\lambda g(\theta_{i-1})}}{\bar{\kappa}_{i-1} e^{\lambda g(\theta_{i-1})} \lambda h(\theta_i)} \right) = g(\theta_i) - (\log \bar{\kappa}_{i-1})/\lambda - g(\theta_{i-1}),$$

if $\pi = 1$ on $[\theta_{i-1}, \theta_i)$,

$e^{-\lambda g(\theta_i)} = \bar{\kappa}_i$, if $\pi = 0$ on $[\theta_{i-1}, \theta_i)$,

where the parameters $\bar{\kappa}_j$ defined below depend on $\{\theta_k : k \leq j - 1\}$.

$$\bar{\kappa}_j := \kappa \gamma_j \quad (\text{for } \gamma_j := \gamma_{\theta_j}).$$

Theorem 3b (Multiple switching locations). *Assume that α_t, β_t are constant and $0 < \kappa < 1$ for $\kappa := 1 - \alpha_t/\beta_t$. The sequence of solutions to the switching equation defines the switching times according to the recurrence*

$$\theta_{i+1} = \theta_i + \frac{\bar{\kappa}_{i-1}^{-1} e^{-\lambda g(\theta_{i-1})} - 1}{\lambda h(\theta_i)}, \text{ if } \pi = 1 \text{ on } [\theta_{i-1}, \theta_i),$$

$$g(\theta_i) = (\log \bar{\kappa}_i^{-1})/\lambda, \text{ if } \pi = 0 \text{ on } [\theta_{i-1}, \theta_i),$$

unless $i = n$, so that $\theta_{i+1} = 1$, in which case θ_i is determined by the equation

$$\theta_{i+1} = 1 = \theta_i + \frac{\bar{\kappa}_{i-1}^{-1} e^{-\lambda g(\theta_{i-1})} - 1}{\lambda h(\theta_i)}.$$

If θ_1 is a right endpoint of an interval where $\pi = 0$, then

$$g(\theta_1) = (\log \kappa^{-1})/\lambda.$$

Furthermore, the sequence $\bar{\kappa}_i$ is (weakly) decreasing with alternate members strictly decreasing.

Corollary (Candid-first single switching). Assume that α_t, β_t are constant and $0 < \kappa < 1$ for $\kappa := 1 - \alpha_t/\beta_t$.

If $\pi = 0$ on $[0, \theta_1)$, so that $\gamma_{\theta_1} = \gamma_0 = 1$, then $\pi = 1$ on $[\theta_1, \theta_2)$ and so for $i = 2$, as $\bar{\kappa}_1 = \kappa = e^{-\lambda g(\theta_1)}$,

$$\theta_3 = \theta_2 + \frac{1 - \bar{\kappa}_1 e^{\lambda g(\theta_1)}}{\bar{\kappa}_1 e^{\lambda g(\theta_1)} \lambda h(\theta_2)} = \theta_2,$$

a contradiction to $\theta_2 < \theta_3$. Consequently, there cannot be a further switching from sparing to candid mode.

A similar result appears to be supported by numerical analysis for a sparing-first equilibrium policy, albeit Theorem 3b (on its own, i.e. without invoking equilibrium conditions) implies by a similar argument that if $\pi = 1$ on $[0, \theta_1)$, then $\theta_4 = \theta_3$, i.e. at most two switchings can occur.

In summary, this section has characterized how tractable single-switching conditions can be derived. The issue of equilibrium selection (which of the sparing-first or the candid-first) must rest on the underlying assumption that the market has found its way to one or other of the two by some evolutionary game-theoretic mechanism; for a standard textbook view of the latter, see e.g. Weibull (1997).

Remark Optimal multi-switching becomes possible when a time-varying κ_t replaces the constant κ ; this is particularly easy to arrange in the case of a piecewise-constant κ_t with constancy on each inter-switching interval (interval between successive switching points): see Example 3 in Sect. 3.

We have seen above that single-switching should be regarded as natural in the constant κ context and not just a stylized model choice. Moreover, single-switching provides the pragmatic, preferred equilibrium choice by an appeal to *focal-point* (Schelling-point) equilibrium selection—for a standard textbook view of which, see e.g. Fudenberg and Tirole (1991).

3 The sparing disclosure threshold

Here for the moment, as in Dye (1985) we work with three dates: 0, s , 1.268 (*time*) \rightarrow (*time* | $t = 0$). For X_s the financial state and Y_s its observation assume the regression function $m_s(y) := \mathbb{E}_{t=0}[X_1 | Y_s = y]$ is increasing. Then the optimal threshold γ_s is uniquely determined and has three properties, the first of which in (i) below implies the Minimum Principle of Sect. 2.

(i) *Minimum Principle* (Ostaszewski and Gietzmann (2008)): The valuation function

$$W(\gamma) := \mathbb{E}_0[X_1 | ND_s(\gamma)]$$

has a unique *minimum* at $\gamma = \gamma_s$;

(ii) *Risk-neutral consistency property*: γ_s is the unique value γ such that

$$\mathbb{E}_0[X_1 | Y_0] = \tau_D \mathbb{E}_0[X_1 | Y_t \geq \gamma \text{ disclosed}] + \tau_{ND} m_s(\gamma),$$

where τ_D is the (time t) probability of disclosure occurring at s . This is highly significant, in that the valuation at time 0 anticipates the potential effects of a voluntary disclosure at the future interim date s . In brief, the approach is consistent with the principles of *risk-neutral valuation*; for background see Bingham and Kiesel (1998), Chap. 6. In particular, the risk-neutral valuation is a *martingale*, constructed via iterated expectations from $\tilde{\gamma}_s := \mathbb{E}_s[X_1 | Y_1] = \mathbb{E}_s[\tilde{\gamma}_1]$ (as $\tilde{\gamma}_1 := \mathbb{E}_1[X_1 | Y_1]$)—see Gietzmann and Ostaszewski (2016).

(iii) Interim discount: From the perspective of time 0, in a model with only three dates: 0, s , 1, the Dye equation at time s may be written:

$$\frac{1 - q_s}{q_s} (\gamma_0 - \gamma_s) = \int_0^{\gamma_s} F_0(u) \, du = \int_0^{\gamma_s} (\gamma_s - u)^+ \, dF_0(u),$$

and has the interpretation of a *protective put-option* with strike γ_s against a fall in value at t . Here q_s is the probability that Y_s is observed, and $F_0(u) = \mathbb{Q}(X_1 \leq u | Y_0)$.

The argument leading to these results is also sketched in the next section.

3.1 Derivation of the dye threshold equation

Relocating the dates to $t < s < 1$, the interim discount γ_s , which is also the threshold for announcements in equilibrium at time s , is the value $\gamma = \gamma_s$ which satisfies

$$\gamma = \mathbb{E}^{\mathbb{Q}}(X_s | ND_s(\gamma), \mathcal{F}_t), \tag{indif}$$

with \mathbb{Q} = the market's probability measure for all relevant events,

\mathcal{F}_t = market information at time t ;

$ND_s(\gamma)$ = event at time s that no disclosure occurs when the information Y_s is below γ ;

RHS = the market's expectation of value conditional on no-disclosure $ND_s(\gamma_s)$.

If management observe a value γ_s at time s , then they are indifferent between disclosing the valuation as γ_s and withholding said information.

If the probability of information reaching management at time s is $q = q_s$, assumed exogenous and independent of the state of the firm, then for $p := 1 - q$

$$\gamma_s = \frac{p\gamma_t + q \int_0^{\gamma_s} x d\mathbb{Q}_t(x)}{p + q\mathbb{Q}_t(\gamma_s)}, \tag{cond}$$

as $\gamma_t = \mathbb{E}_t[\tilde{\gamma}_1]$, where the subscript on \mathbb{Q}_t indicates conditioning on \mathcal{F}_t . Equivalently, we have

$$p(\gamma_t - \gamma_s) = q \int_0^{\gamma_s} (\gamma_s - x) d\mathbb{Q}_t(x), \tag{put}$$

$$\mathbb{Q}_t((\gamma_s - X_s)^+) = \text{protective put.}$$

Here we may routinely evaluate this put using the Black–Scholes formula.

As above, rearrangement will show incorporation of future information:

$$\gamma_t = \tau_D^t \cdot \mathbb{E}[x|D_s(\gamma_s), \mathcal{F}_t] + (1 - \tau_D^t)\gamma_s, \tag{(r-n val)}$$

as in risk-neutral valuation, where $D_s(\gamma) =$ event of time s when values above γ are disclosed,

τ_D^t = the market’s evaluation at time t of the disclosure probability at time s .

The presumption this far precluded the use of a candid strategy. If management restricts application of the sparing (threshold-generated) strategy to act with probability π and the market likewise believes (in equilibrium) that this probability is π , then in a period of silence:

$$\gamma_s = \frac{p\gamma_t + \pi q \int_0^{\gamma_s} x d\mathbb{Q}_t(x)}{p + \pi q \mathbb{Q}_t(\gamma_s)}, \tag{(cond-\pi)}$$

For $\pi = 1$ (sparing) this reduces to the Dye equation.

For $\pi = 0$ (candid) this acknowledges that $\gamma_s = \gamma_t$.

3.2 Equilibrium condition: continuous-time analogue

We embed the three dates $t < s < 1$ of the Dye model into the unit interval to provide a continuous-time framework in which any future date $s > t$ can be interpreted as a time at which the management have the opportunity to disclose a forecast of value to the market. As in the Dye model, key here is the creation of an *ambiguity* at time s , so that the market knows that absence of a disclosure is caused either by absence of fresh endowment of private information or by a management decision to withhold the private information arriving at moment s . With this aim we introduce a Poisson process with intensity λ whose jump at time s , when privately observed by management, determines that an observation of Y_s occurs. The market does not observe the jumps. Thus every moment now takes on the character of an interim disclosure date and, depending on the disclosure policy believed by the market to be implemented by management, absence of a disclosure can mean no new observation or a withheld observation.

With the Poisson process in place, for $t < s$ take $q = q_{ts} = \lambda(s - t) + o(s - t)$, employing the Landau little-o notation. Passage to the limit as $s \searrow t$ yields:

$$(1 - q_{ts})(\gamma_t - \gamma_s) + o(s - t) = q_{ts} \int_{z_s \leq \gamma_s} (z_s - \gamma_s) d\mathbb{Q}_t(z_s)$$

$$= -\lambda(s - t) \int_{z_s \leq \gamma_s} \mathbb{Q}_t(z_s) dz_s. \text{ (By parts.)}$$

Dividing by $-\lambda(s - t)$:

$$-(1 - q_{ts}) \frac{\gamma_s - \gamma_t}{(s - t)} = \lambda \int_{z_s \leq \gamma_s} \mathbb{Q}_t(z_s) dz_s,$$

ignoring errors of order $o(s - t)/(s - t)$. With economic activity and the noisy observation in a standard Black–Scholes setting, this yields

$$-\gamma'_t = \lambda \gamma_t h(t) \text{ with } h(t) = 2\Phi((\sigma/2)\sqrt{1 - t}) - 1, \tag{cont-eq}$$

with σ the (*aggregate*) volatility (aggregating productive and observation vols.); for the proof see Gietzmann and Ostaszewski (2016).

This ODE is our continuous-time-disclosure equilibrium condition in any period of silence (i.e. when the management make no disclosures). It equilibrates in a period of silence between the market’s ability to downgrade the valuation below γ_t and the management’s potential ability to upgrade the valuation were they to observe a value of Y_t greater than γ_t (cf. the weighted average discussed in Sect. 2). We refer to this as the *equilibrium ODE*.

Denoting successive public disclosures (voluntary or mandatory) generated stochastically by $\tau_0 = 0 < \tau_1 < \tau_2 < \dots < 1$, and writing N for their number so that $\tau_{N+1} = 1$, one has

$$\begin{aligned} \gamma'_t &= -\lambda \gamma_t h(t - \tau_i) \text{ for } \tau_i < t < \tau_{i+1} \\ \text{s.t. } \gamma_{\tau_i} &= \text{disclosed value at time } t = \tau_i, \end{aligned}$$

with $h(t - \tau_i)$ the (per-unit of value, γ) *firm-specific, instantaneous protective put* at times t , for $\tau_i < t < \tau_{i+1}$.

The market valuation of the firm γ_t is thus a *piecewise-deterministic Markov* process in the sense of Davis (1984) and Davis (1993).

3.3 Probabilistic strategy optimization

The governing equation of our continuous-time analogue of the Dye model, the equilibrium ODE, is based on the assumption that the manager’s objective is to achieve the highest possible valuation at all times t preceding the subsequent mandatory disclosure date. However, if management follow the sparing threshold rule with probability π_t and otherwise disclose the observation candidly with probability $1 - \pi_t$, then, as in (cond- π), for an equilibrium strategy π the corresponding valuation $\gamma_t = \gamma_t^\pi$ satisfies:

$$\gamma'_t = -\lambda \gamma_t \pi_t h(t) \text{ with } h(t) = 2\Phi((\sigma/2)\sqrt{1 - t}) - 1, \tag{cont-eq- π }$$

where $t = 0$ corresponds to the last public disclosure (after a change of origin here, *mutatis mutandis*). We rescale the valuation so that $\gamma_0 = 1$.

Consistently with this last equation, we will employ the notation: γ_t^1 for its solution when $\pi \equiv 1$ (sparing policy applied throughout), so that

$$\gamma_t = \gamma_t^1 \text{ denotes the solution of } \gamma_t' = -\lambda\gamma_t h(t) \text{ with } \gamma_0 = 1.$$

With comparison against this solution in mind, the manager is now induced to maximize an objective in selecting π so as to yield

$$\max_{\pi} \mathbb{E} \int_0^1 (1 - \pi_t)(\alpha_t \gamma_t - \beta_t(\gamma_t - \gamma_t^1)) \, dt. \quad (\text{obj-1})$$

As before, $t = 0$ denotes the most recent time of disclosure and unit time is left to the mandatory disclosure (*time to expiry*).

This objective includes a *penalty* proportional to $(\gamma_t - \gamma_t^1)$. The *amended unravelling principle* of Sect. 2 implies that introduction in a market equilibrium of candour (candid reporting) will cause the valuation γ_t to exceed γ_t^1 and the aim of the penalty term is to motivate management into protecting the value of the firm from potential falls were a candid strategy followed for too long (i.e. from excessive use of a candid position).

In equivalent form, the objective may be rewritten as

$$\max_{\pi} \mathbb{E} \int_0^1 (1 - \pi_t)\beta_t[\gamma_t^1 - \kappa_t \gamma_t] \, dt \text{ for } \kappa_t := 1 - \alpha_t/\beta_t. \quad (\text{obj-2})$$

It is thus natural to demand that for some proper interval of time

$$\gamma_t^1 > \kappa_t \gamma_t$$

holds, so we make the *blanket assumption*

$$0 < \alpha_t/\beta_t < 1, \text{ i.e. } 0 < \kappa_t < 1,$$

which enables discounting of γ_t by κ_t to a level below γ_t^1 .

3.4 Hamiltonian analysis: Pontryagin principle

We approach the maximization problem via the *Pontryagin Maximum Principle*, PMP, for which see Bressan and Piccoli 2007 (esp. Ch. 7 on sufficiency conditions for PMP), or the more concise textbook sketches in Liberzone (2012); Sasane (2016), or Troutman (1996). It is also possible to establish the results below by solving the *Bellman equation* along the lines of Davis (1993) (p. 165), a matter we hope to return to elsewhere.

In a period of silence, the valuation is deterministic and so we formulate optimisation in Hamiltonian terms. We apply a standard Hamiltonian approach from control theory to maximizing the objective of the preceding section by treating γ_t as a state variable

and π_t as a control variable. Denoting the co-state variable by μ_t , the Hamiltonian is

$$\mathcal{H}(\gamma_t, \pi_t, \mu_t) = [(1 - \pi_t)\beta_t(\gamma_t^1 - \kappa_t\gamma_t)] + \mu_t[-\lambda\gamma_t\pi_t h(t)],$$

by construction linear in π_t . So with μ_t continuous and piecewise smooth:

$$\mu_t' = -\frac{\partial \mathcal{H}}{\partial \gamma_t} = \beta_t\kappa_t(1 - \pi_t) + \mu_t\pi_t\lambda h(t), \text{ with } \mu_1 = 0,$$

where we follow the *càdlàg convention* that π_t is right-continuous with left limits and satisfies $0 \leq \pi_t \leq 1$. Thus

$$\mu_t' - \mu_t\pi_t\lambda h(t) = \beta_t\kappa_t(1 - \pi_t).$$

We now apply the Pontryagin Principle. Evidently, concentrating only on terms involving π_t below, the Hamiltonian

$$\mathcal{H}(\gamma_t, \pi_t, \mu_t) = \dots - \pi_t \cdot \beta_t[\gamma_t^1 - \kappa_t\gamma_t + \mu_t\lambda\gamma_t h(t)/\beta_t]$$

is maximized by setting π_t at 0 or 1 according as

$$\gamma_t[\kappa_t - \lambda h(t)\mu_t/\beta_t] < \gamma_t^1 \text{ or } > \gamma_t^1, \text{ resp.}$$

It emerges that $\mu_t \leq 0$ (see Appendix, Proposition 3), consistently with its being interpreted as a penalty term in \mathcal{H} , so

$$[\kappa_t - \mu_t\lambda h(t)/\beta_t] > 0.$$

This gives rise to an *optimal switching curve* and associated optimality rule:

$$\gamma_t^* := \frac{\gamma_t^1}{\kappa_t - \lambda h(t)\mu_t/\beta_t} > 0.$$

Proposition 1 (Optimality Rule) *A necessary and sufficient for π to be optimal is given by the rule:*

$$\pi_t = \begin{cases} 1, & \text{suppress } X_t \text{ unless } X_t \geq \gamma_t \text{ if } \gamma_t \geq \gamma_t^*, \\ 0, & \text{reveal } X_t \text{ if } \gamma_t < \gamma_t^*. \end{cases}$$

Proof By the Non-mixing Theorem, π_t can only take the values 0 and 1 and so by the Pontryagin Principle the optimality condition above is *necessary and sufficient*: the strong form of Theorem 1 (see the Appendix) asserts that if $\gamma_t = \gamma_t^*$ on an interval of time, then $\pi_t = 1$ on that interval. \square

A corollary of the above form of γ_t^* now follows.

Qualitative Corollary. *A large enough valuation γ_t allows sparing reporting, a small enough valuation γ_t encourages candid revelation.*

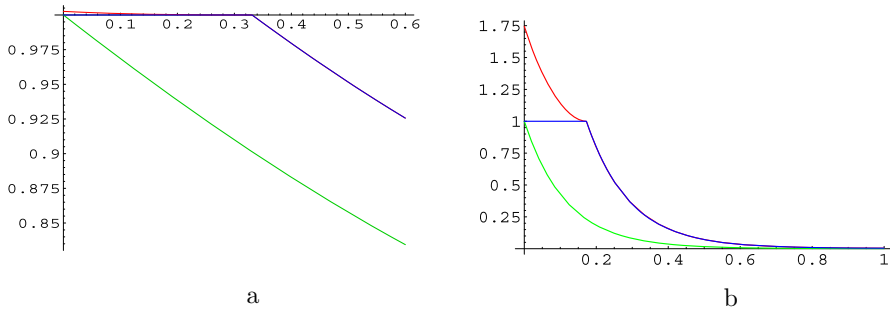


Fig. 2 **a** Switching curve γ_t^* red; sparing-first valuation γ_t blue, coalescing after θ with red; **b** γ_t^* red; γ_t blue; sparing-throughout valuation γ_t^1 green (color figure online)

Remark 1 Evidently, the value of π_t is not instantly observable, so management may at any instance of bad news (however defined) hide it and so deviate from their prescribed equilibrium strategy. However, systematic deviation of this sort is statistically observable and so deviation leads to loss of reputation, removing the very means by which the firm maintains a higher valuation, which in turn hurts the deviating agent. We therefore assume that managers hold themselves to their prescribed equilibrium strategy. For further background on the *Bayesian persuasion* aspect here, see Kamenica and Gentzkow (2011).

Remark 2 The optimality rule of Proposition 1 above is based on the ODE dynamics of a single firm. In principle, the analysis of reputation management may be extended to a natural multi-firm environment such as a sector of covarying firms, all of which influence market expectations and some of which may be candid. Key to the approach would be a disaggregation of dependences, as in Gietzmann and Ostaszewski (2014), along lines also followed in the context of Gietzmann and Ostaszewski (2016).

4 Examples of equilibrium behaviour

In this section we give three examples of different equilibrium behaviour in the form of graphs which include the switching curve derived in the preceding section. The role of the switching curve is to confirm, by Proposition 1, the optimality of the equilibrium valuation.

4.1 Example 1: Candid first

Here $\pi = 0$ initially (candid). Figure 2 above and Fig. 3 below share the same parameters: $\kappa = 0.799432$, $\lambda = 0.940489$, $\sigma = 4$; here $\theta = 0.260229$. Figure 2b illustrates a more pronounced switching curve with $\kappa = 3/13$, $\lambda = 9$, $\sigma = 4$ and $\theta = 0.175$.

Commentary to Example 1. Here under silence, initially the switching curve γ_t^* (shown in red) is above the starting firm value $\gamma_0 = 1$ and so candour ($\pi = 0$) is initially

optimal; it is rational to infer that silence here means managers have received no information, hence the valuation remains unchanged until the switching time is reached, as signified by the stationarity of the switching curve at the switching time. Thereafter, γ_t^* falls below $\gamma = 1$ and so the optimal strategy yields a superior valuation to that given by γ_t^1 (which would have resulted from a policy of being sparing-throughout, i.e. $\pi \equiv 1$); here the valuation ignores the kind of silence that hides bad news. In this time interval γ_t^* and γ_t coalesce, as predicted by the non-mixing result (Theorem 1 S). With a higher intensity-value λ of the private managerial news-arrival, the switching time would come earlier, thus absorbing the higher chances of ensuing privately received bad news, which strategically wants to be withheld. The figures above graphically depict the reputational benefits to a firm following a candid-first strategy. Starting at $t = 0$ investors do not downgrade the value of the firm when they see no disclosure, since they infer this follows from non-observation of updated information. The blue curve in both figures remains flat. This is in contrast to how investors treat a firm applying a sparing strategy, for which they continually downgrade firm value in the presence of silence. Thus the distance between the blue and green lines reflects the reputational benefit of following a candid strategy at times. In summary, the reputational benefit is in the fact that investors do not downgrade firm value quite so heavily in the presence of continuing silence.

As shown in Theorem 2a below, here the switching and value curves take the form

$$\gamma_t^* = \begin{cases} \gamma_t^1/\kappa[1 + \lambda(\theta - t)h(t)], & t \leq \theta; \\ \gamma_t^1/\kappa = e^{-\lambda g(t)}/\kappa, & t > \theta; \end{cases} \quad \gamma_t = \begin{cases} \gamma_0 = 1 < \gamma_t^*, & t \leq \theta; \\ e^{-\lambda[g(t)-g(\theta)]} = \gamma_t^1/\kappa, & t > \theta. \end{cases}$$

Intersection at $t = \theta$ follows from $e^{-\lambda g(\theta)} = \kappa$, so that for $\lambda > \log(\kappa^{-1})$

$$g(\theta) = \log(\kappa^{-1})/\lambda < 1, \text{ where } g(\theta) = \int_0^\theta h(t) \, dt \text{ depends on } \sigma,$$

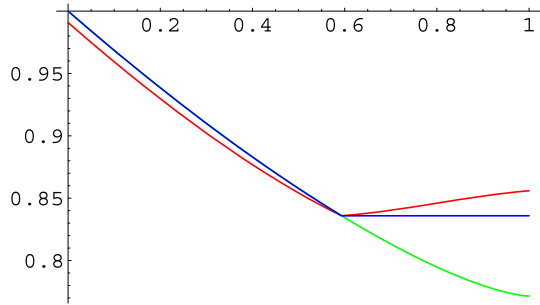
and the optimal time to switch θ thus depends on the variability σ , the pay-for-performance κ and the intensity λ . By contrast, candour throughout corresponds to $\lambda < \log(\kappa^{-1})$.

It will be seen that on the interval $0 < t \leq \theta$ (before the switch), it is the case that $\gamma_t = \gamma_0 = 1 < \gamma_t^*$, implying candour first; it emerges that (afterwards), as above, $\gamma_t = \gamma_t^*$ for $t > \theta$, and sparing behaviour is optimal. These curves all stay above

$$\gamma_t^1 = e^{-\lambda g(t)} \text{ for } 0 < t < 1,$$

so demonstrating the reputational effect throughout. In the two examples shown above, Fig. 2b uses round numbers, but Fig. 2 has been derived so that a sparing-first equilibrium also exists for these same parameters: see Fig. 3.

Fig. 3 λ and κ values as in Fig. 2, but initially sparing (color figure online)



4.2 Example 2: Sparing first

Here $\pi = 1$ initially (initially sparing behaviour). In Fig. 3, with κ , λ and σ values as in Fig. 2a, the switching time is $\theta = 0.565997$.

Commentary to Example 2. Here, under silence, initially the switching curve γ_t^* (red) is below the firm valuation γ_t , and $\gamma_t = \gamma_t^1$ is consequently the optimal dynamic disclosure-threshold (sparing policy threshold) curve. Thereafter, γ_t^* is above the γ_t^1 (green) curve, so it is optimal to switch to candour, which yields a constant equilibrium valuation under silence (shown in blue). The valuation subsequently omits to account for the kind of silence that hides bad news. After the disclosure policy switch from $\pi = 1$ to $\pi = 0$, it is rational to infer that silence means managers have received no new information. With a higher intensity λ of private managerial news-arrival, the switching time would come later, thus absorbing the higher chances of bad news arrival needing strategically to be withheld.

In brief, again as shown in Theorem 2b below, here the switching and value curves take the form

$$\gamma_t^* = \begin{cases} \gamma_t^1 / \kappa [1 + \lambda(1 - \theta)h(t)e^{-\lambda(g(\theta) - g(t))}], & t < \theta, \\ \gamma_t^1 / \kappa [1 + \lambda(1 - t)h(t)], & t \geq \theta, \end{cases} \quad \gamma_t = \begin{cases} \gamma_t^1 = e^{-\lambda g(t)}, & t \leq \theta; \\ \gamma_\theta = e^{-\lambda g(\theta)}, & t > \theta. \end{cases}$$

Here the optimal timing θ satisfies $\gamma_\theta^* = \gamma_\theta$ so that

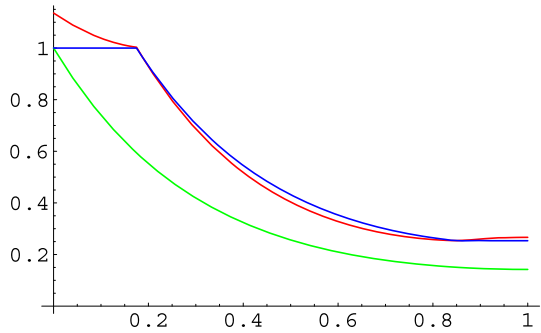
$$\lambda(1 - \theta)h(\theta) = \kappa^{-1} - 1.$$

On the interval $0 < t \leq \theta$, it is the case that $\gamma_t = \gamma_t^1 > \gamma_t^*$, implying sparing behaviour first; thereafter $\gamma_t < \gamma_t^*$ for $t > \theta$ (after the switch), where candid behaviour is optimal. After $t > \theta$, the γ_t curve is constant at γ_θ and now stays above

$$\gamma_t^1 = e^{-\lambda g(t)} \text{ for } \theta < t < 1,$$

demonstrating the reputational effect after switching.

Fig. 4 Switching curve γ_t^* red; γ_t^1 green; γ_t blue (color figure online)



4.3 Example 3: Piecewise-constant

Here $\pi = 0$ on $[0, \theta_1)$ with $\theta_1 = 0.175$ and $\kappa = 0.594$ (candid) and is followed by $\pi = 1$ on $[\theta_1, \theta_2)$ with $\theta_2 = 0.85$ and $\kappa = 0.533$ (sparing), and finally by $\pi = 0$ on $[\theta_2, 1]$ (candid again). Throughout $\lambda = 3.2$ and $\sigma = 2$ (see Fig. 4).

A choice of *piecewise constancy* may at first sight seem specious. However, this is an arrangement of a pre-determined managerial reward capable of being agreed by the shareholders, as the switching times are not dynamically selected. Moreover, our analysis with constant κ can be adapted (by reference to the first mean-value theorem of integration) to the general case of continuous κ_t by replacing within any inter-switching interval a proposed varying κ_t by some appropriate constant value (along the lines of a ‘certainty equivalent’ relative to the Poisson jumps), that value being intermediate between those taken by κ_t on that interval.

Here similar equations are satisfied with $\gamma_t^* > \gamma_t$ on $[0, \theta_1)$, then with $\gamma_t^* < \gamma_t$ on (θ_1, θ_2) and finally with $\gamma_t^* > \gamma_t$ on $(\theta_2, 1]$. Thus $\gamma_t = 1$ for $t < \theta_1$ where

$$\theta_1 = 0.175 = \arg_{\theta} \{ \log(\kappa_1^{-1}) / \lambda = g(\theta) \} \text{ with } \kappa_1 = 0.5938\dots \text{ and } \lambda = 3.2.$$

Since $\gamma_{\theta_1} = 1$ and $\pi = 1$ on $[\theta_1, \theta_2)$ we have,

$$\gamma_{\theta_2} = e^{-\lambda(g(\theta_2) - g(\theta_1))} = \gamma_{\theta_2}^* = \frac{\gamma_{\theta_2}^1}{\kappa_2 [1 + \lambda(1 - \theta_2)h(\theta_2)]} \text{ with } \gamma_{\theta_2}^1 = e^{-\lambda g(\theta_2)},$$

as in Theorem 2b. So, cancelling $g(\theta_2)$,

$$\theta_2 = 0.85 = \arg_{\theta} \{ 1 + \lambda(1 - \theta)h(\theta) = e^{-\lambda g(\theta_1)} / \kappa_2 \} \text{ with } \kappa_2 = 0.533\dots \text{ and } \lambda = 3.2,$$

and $\gamma_t = 1$ again for $t \geq \theta_2$.

5 Proofs

The proof of Theorem 1 (actually in a stronger form) is in the Appendix. Here we consider Theorems 2a and 2b. We recall that below α_t and β_t are assumed constant. We begin with some preliminary observations.

Proposition 2 (i) *On any interval $[\theta', \theta]$ where $\pi = 1$, the co-state equation and solution take the form:*

$$\begin{aligned}\frac{\mu'_t}{\mu_t} &= \lambda h(t) > 0, \text{ for } \theta' < t < \theta, \\ \mu_t &= K \exp[-\lambda(g(\theta) - g(t))],\end{aligned}$$

so that μ_t , being negative, is decreasing with $K < 0$ a constant and

$$\partial \mu_t / \partial \lambda = (g(\theta) - g(t))(-\mu_t) > 0, \text{ for } \theta' < t < \theta.$$

(ii) *On any interval where $\pi = 0$,*

$$\mu'_t = (\beta_t - \alpha_t) > 0 : \mu_t = -(\beta - \alpha)(K - t) \text{ if } \beta_t \equiv \beta \text{ and } \alpha_t \equiv \alpha \text{ with } K \leq 1.$$

Thus here μ_t is increasing.

In particular, in both cases μ_t is either non-constant or zero.

Proof Since the co-state equation asserts that

$$\mu'_t - \mu_t \pi_t \lambda h(t) = (\beta_t - \alpha_t)(1 - \pi_t),$$

the conclusions are immediate from the form of the differential equation. \square

Remark 3 (Behaviour of γ). If switches occur at the three times $\theta_1 < \theta_2 < \theta_3$ with $\pi = 1$ on (θ_1, θ_2) , then $\gamma_t = e^{\lambda g(\theta_1)} e^{-\lambda g(t)} \gamma_{\theta_1}$ for $t \in [\theta_1, \theta_2]$, so

$$\gamma_{\theta_3} = \gamma_{\theta_2} = \gamma_{\theta_1} e^{\lambda g(\theta_1)} e^{-\lambda g(\theta_2)}.$$

Applying the formula inductively, if $\pi = 0$ near $t = 0$, so that $\gamma_{\theta_1} = 1$, then

$$\gamma_{\theta_{2n}} = e^{\lambda g(\theta_1)} e^{-\lambda g(\theta_2)} e^{\lambda g(\theta_3)} e^{-\lambda g(\theta_4)} \dots e^{\lambda g(\theta_{n-1})} e^{-\lambda g(\theta_n)};$$

likewise if $\pi = 1$ near $t = 0$, so that $\gamma_{\theta_1} = \gamma_{\theta_2} = e^{-\lambda g(\theta_1)}$, then

$$\gamma_{\theta_{2n}} = e^{-\lambda g(\theta_1)} e^{\lambda g(\theta_2)} e^{-\lambda g(\theta_3)} e^{\lambda g(\theta_4)} \dots e^{\lambda g(\theta_{n-1})} e^{-\lambda g(\theta_n)}.$$

Corollary 1 (Final switching conditions). *If the last two intervals of π constancy are given by $\pi = 0$ switching at θ to $\pi = 1$, then $\mu_t = 0$ on $[\theta, 1]$ and near and to the left of θ :*

$$\mu_t = (\beta - \alpha)(t - \theta).$$

For the reversed strategy, if the last two aforementioned intervals are given by $\pi = 1$ switching at θ to $\pi = 0$, then on $[\theta, 1]$

$$\begin{aligned} \mu_t &= (\beta - \alpha)(t - 1), \text{ so that } \mu_\theta = (\beta - \alpha)(\theta - 1), \\ \mu_t &= (\beta - \alpha)(\theta - 1)e^{-\lambda[g(\theta) - g(t)]} \text{ for } t < \theta \text{ near } \theta. \end{aligned}$$

Corollary 2 Being candid at all times ($\pi \equiv 0$) is optimal for λ sufficiently low, i.e. below a threshold depending on κ (equivalently, depending on α/β).

Proof The assumption $\pi \equiv 0$ implies $\gamma_t \equiv 1$ and $\gamma_t < \gamma_t^*$. From Cor. 1, since $\mu_1 = 0$, we have $\mu_t = -(\beta - \alpha)(1 - t)$, and so

$$\gamma_t^* = \frac{e^{-\lambda g(t)}}{\kappa(1 + \lambda h(t)(1 - t))} > 1 \text{ iff } e^{-\lambda g(t)} > \kappa(1 + \lambda h(t)(1 - t)).$$

This holds for all λ small enough (depending on κ); indeed,

$$\lim_{\lambda \rightarrow 0} \frac{e^{-\lambda g(1)}}{1 + \lambda} = 1 > \kappa,$$

so for all λ small enough

$$e^{-\lambda g(1)} > \kappa(1 + \lambda),$$

and, since $h(0) < 1$ and $h(t)(1 - t)$ decreases on $[0, 1]$,

$$e^{-\lambda g(t)} > e^{-\lambda g(1)} > \kappa(1 + \lambda) > \kappa(1 + \lambda h(t)(1 - t)),$$

as $g(t)$ is increasing. □

We need some details about the function h :

Lemma 1 The function h is decreasing with $h(1) = 0$ and $h(t) < 1$ for $t \in [0, 1]$.

Proof Recall that $h(t) = [2\Phi(\hat{\sigma}/2) - 1]$, where $\hat{\sigma} = \sigma\sqrt{1 - t}$. So h is decreasing (to 0), since, for $0 \leq t < 1$,

$$h'(t) = -\frac{1}{2}\sigma^2\varphi(\hat{\sigma}/2)/\hat{\sigma} < 0$$

with φ the standard normal density. Note that $h(1) = 0$, as $\Phi(0) = 1/2$, and further that $h(0) > 0$ as $\Phi(\sigma/2) > 1/2$ for $\sigma > 0$. Finally note that $h(t) < 1$, since $h(0) < 1$, the latter because

$$2\Phi(\sigma/2) - 1 < 1 \text{ as } \Phi(\sigma/2) < 1.$$

□

If in equilibrium there is just one optimal switching θ , it is straightforward to derive its location property using a first-order condition on θ , as given in Theorems 2a and 2b above. The existence conditions for the equilibria are derived from the general Hamiltonian formulation; this requires the explicit derivation of the switching curve, which is the content of a technical lemma and carries all the work for Theorems 2a and likewise 2b.

Lemma 2a *If $\pi_t = 0$ for $t < \theta$, and $\pi_t = 1$ for $t > \theta$, is optimal, then*

$$\mu_t = \begin{cases} \beta\kappa(t - \theta) = (\beta - \alpha)(t - \theta), & t < \theta, \\ 0, & t \geq \theta. \end{cases}$$

Here the switching curve is given by

$$\gamma_t^* = \begin{cases} \gamma_t^1 / \kappa [1 + \lambda(\theta - t)h(t)], & t < \theta, \\ \gamma_t^1 / \kappa, & t \geq \theta. \end{cases}$$

Further,

$$\gamma_t^* = \gamma_t, \text{ for } t \geq \theta.$$

Remark The graph of γ_t^* is stationary at $t = \theta$, for, writing ∂_t for the time derivative,

$$\kappa \partial_t \gamma_t^* = \frac{-\lambda h(t)e^{-\lambda g(t)}}{1 + \lambda(\theta - t)h(t)} - \frac{-\lambda h(t) + \lambda(\theta - t)h'(t)}{[1 + \lambda(\theta - t)h(t)]^2} e^{-\lambda g(t)} \Big|_{t=\theta} = 0.$$

Proof of Lemma 2a. We will deduce μ_t directly from the formal solution (Appendix) via the integrating factor φ there, which reduces to

$$\varphi(t) = \begin{cases} 1, & t \leq \theta, \\ e^{-\lambda[g(t)-g(\theta)]}, & \theta < t < 1. \end{cases}$$

For $t > \theta$, $\mu_t = 0$. For $t < \theta$, we have

$$\int_t^\theta \varphi_s \varphi_t^{-1} (\beta - \alpha)(1 - \pi_s)_{=1} \, ds + \int_\theta^1 \varphi_s \varphi_t^{-1} (\beta - \alpha)(1 - \pi_s)_{=0} \, ds = -\mu_t,$$

so

$$(\beta - \alpha)(\theta - t) = -\mu_t.$$

For consistency we need at $t = \theta = \theta(\lambda)$ that

$$\gamma_\theta^* = \frac{\gamma_\theta^1}{\kappa} = 1 : \kappa = \gamma_\theta^1 = e^{-\lambda g(\theta)} = \exp \left(-\lambda \int_0^{\theta(\lambda)} h(s) \, ds \right).$$

As regards the coalescence, note that if θ solves $e^{-\lambda g(\theta)} = \kappa$, as above, then, for $t \geq \theta$,

$$\gamma_t = e^{-\lambda[g(t)-g(\theta)]} = \gamma_t^1 / \kappa = e^{-\lambda g(t)} / \kappa,$$

rather than the expected inequality ($>$); so as in Theorem 1 S (Appendix),

$$\gamma_t^* = \gamma_t,$$

and since $\gamma_t' = -\lambda h(t)\gamma_t$, it follows that $\pi = 1$.

Armed with Lemma 2a we proceed to

Proof of Theorem 2a. We begin with the location condition. For $t \in [0, \theta)$, we have $\pi_t = 0$ so $\gamma_t \equiv 1$ and $\gamma_t^1 = e^{-g(t)}$ on this interval and so the objective function reduces to

$$\int_0^\theta e^{-\lambda g(t)} dt - \kappa\theta,$$

since $1 - \pi_t = 0$ on $[\theta, 1]$, which yields a zero contribution. Differentiation w.r.t. θ yields the first-order condition

$$e^{-\lambda g(\theta)} - \kappa = 0,$$

which on re-arrangement yields the claim.

We turn to the existence condition. To ensure that $\gamma_0^* > \gamma_0$ requires by Lemma 2a that

$$1/\kappa[1 + \lambda\theta h(0)] > 1, \text{ equivalently } \lambda < (\kappa^{-1} - 1)/(\theta h(0)).$$

(The upper bound on λ is illustrated by the green curve in Fig. 5 below.) This also guarantees that $\gamma_t^* > \gamma_t$ for $t < \theta$, since $(\theta - t)h(t)$ is decreasing in t . It is also required that $\gamma_\theta^* = 1$, i.e. $e^{-\lambda g(\theta)} = \kappa$, and so this holds iff

$$-\log \kappa / g(\theta) = \lambda < (\kappa^{-1} - 1) / (\theta h(0)).$$

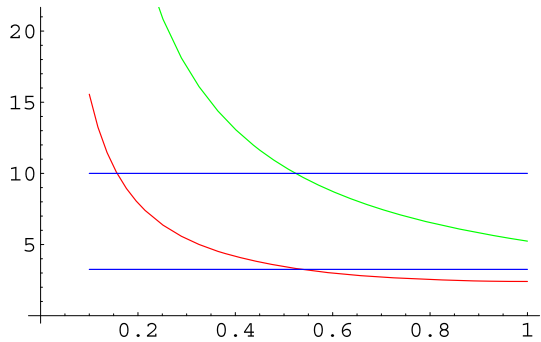
(The form on the left side here is illustrated by the red curve in Fig. 5.) By Lemma 2a, $\gamma_t^* = \gamma_t$ for $t > \theta$. The displayed inequality is feasible iff

$$\log(\kappa^{-1}) / (\kappa^{-1} - 1) < g(\theta) / (\theta h(0)).$$

Finally, we compute the rate of change of θ w.r.t. λ from the location condition expressed as

$$\lambda \int_0^{\theta(\lambda)} h(s) ds = -\log \kappa > 0 :$$

Fig. 5 λ delimited above by $(\kappa^{-1} - 1)/(\theta h(0))$ and below by $\log \kappa^{-1}/g(\theta)$ against θ (color figure online)



$$\int_0^{\theta(\lambda)} h(s) \, ds + \lambda h(\theta(\lambda))\theta'(\lambda) = 0, \text{ so } \theta'(\lambda) = \frac{\log \kappa}{\lambda^2 h(\theta(\lambda))} < 0.$$

So the larger is λ , the smaller is θ . □

From where the horizontal blue line $\lambda = 10$ in the figure intersects the green curve one may drop vertically to the red curve to obtain a value of λ which lies below the green and on the red curve and lower blue line.

We again begin with a technical Lemma.

Lemma 2b Assume that α_t, β_t are constant.

In an equilibrium where $\pi_t = 1$ for $t < \theta$ and $\pi_t = 0$ for $t > \theta$, one has

$$\mu_t = \begin{cases} (\beta - \alpha)(\theta - 1)[\exp -\lambda \int_t^\theta h(u)du], & t < \theta, \\ (\beta - \alpha)(t - 1), & \theta \leq t. \end{cases}$$

Here

$$\gamma_t^* = \begin{cases} \gamma_t^1 / \kappa [1 + \lambda(1 - \theta)h(t) \exp[-\lambda(g(\theta) - g(t))]], & t < \theta, \\ \gamma_t^1 / \kappa [1 + \lambda(1 - t)h(t)], & t \geq \theta. \end{cases}$$

This curve defines the switching time θ as the intersection time $t = \theta$ of γ_t^* with γ_t^1 when

$$\lambda(1 - \theta)h(\theta) = \kappa^{-1} - 1, \text{ i.e. } 1 = \theta + \frac{\kappa^{-1} - 1}{\lambda h(\theta)}.$$

Proof As in Lemma 2a, we again deduce μ_t directly from the integrating factor (see Appendix), which here is

$$\varphi(t) = \begin{cases} e^{-\lambda g(t)}, & t < \theta, \\ e^{-\lambda g(\theta)}, & \theta \leq t < 1. \end{cases}$$

For $t \geq \theta$, as $s \geq \theta$ and $\pi = 0$ below,

$$\int_t^1 \varphi_s \varphi_t^{-1} (\beta_s - \alpha_s) (1 - \pi_s) \, ds = -\mu_t, \text{ i.e. } (\beta - \alpha)(1 - t) = -\mu_t.$$

For $t < \theta$ we have

$$\begin{aligned} -\mu_t &= \int_t^\theta \varphi_s \varphi_t^{-1} (\beta_s - \alpha_s) (1 - \pi_s)_{=0} \, ds + \int_\theta^1 \varphi_s \varphi_t^{-1} (\beta_s - \alpha_s) (1 - \pi_s)_{=1} \, ds \\ &= [\exp \lambda \int_0^t h(u) \, dsu] [\exp -\lambda \int_0^\theta h(u) \, du] (\beta - \alpha)(1 - \theta), \\ &= (\beta - \alpha)(1 - \theta) [\exp -\lambda \int_t^\theta h(u) \, du]. \end{aligned}$$

□

We may now prove Theorem 2b.

Proof of Theorem 2b. We begin with the location condition. Since $\pi_t = 1$ on $[0, \theta)$, $\gamma_t = \gamma_t^1 \equiv e^{-g(t)}$ on $[0, \theta)$, here the objective function reduces to

$$\int_\theta^1 e^{-\lambda g(t)} \, dt - \kappa e^{-\lambda g(\theta)} (1 - \theta),$$

since there is a zero contribution to the objective function on $[0, \theta]$. Differentiation w.r.t. θ yields the first-order condition

$$-e^{-\lambda g(\theta)} + \kappa e^{-\lambda g(\theta)} + \kappa e^{-\lambda g(\theta)} \lambda h(\theta) (1 - \theta) = 0,$$

which on re-arrangement yields the claim.

We turn to the existence condition. By Lemma 2b, intersection at $t = \theta$ of γ_t^* with γ_t^1 occurs iff

$$\lambda(1 - \theta)h(\theta) \exp \left[-\lambda \int_\theta^\theta h(u) \, du \right] = \lambda(1 - \theta)h(\theta) = \kappa^{-1} - 1.$$

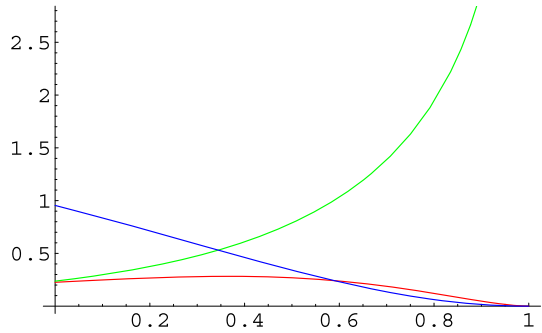
Combining this with the requirement that $\gamma_0^* < 1$ yields

$$\begin{aligned} 1/\kappa [1 + \lambda(1 - \theta)h(0) \exp[-\lambda g(\theta)]] &< 1, \\ \lambda(1 - \theta)h(\theta) = \kappa^{-1} - 1 &< \lambda(1 - \theta)h(0) \exp[-\lambda g(\theta)]. \end{aligned}$$

This holds for some κ iff

$$h(\theta)/h(0) < \exp[-\lambda g(\theta)], \text{ i.e. } \lambda < -[\log h(\theta)/h(0)]/g(\theta), \tag{1}$$

Fig. 6 Bounds for selecting λ . Graph in blue of $h(\theta)$ against θ (color figure online)



yielding a bound on λ in terms of the switching time θ (as illustrated by a green graph in Fig. 6). From here we obtain

$$\kappa^{-1} - 1 = \lambda(1 - \theta)h(\theta) < (1 - \theta)h(\theta)(-\lceil \log h(\theta)/h(0) \rceil / g(\theta)), \quad (2)$$

in turn a lower bound on θ (as illustrated by a red graph in Fig. 6).

Finally we compute the rate of change of θ w.r.t. λ from the location condition. Here for $\theta = \theta(\lambda)$ we have, as $h'(\theta) < 0$,

$$\lambda(1 - \theta)h(\theta) = \frac{\alpha}{\beta - \alpha}, \text{ so } \theta'(\lambda) = \frac{(1 - \theta)h(\theta)}{\lambda[h(\theta) - (1 - \theta)h'(\theta)]} > 0.$$

Remark In Fig. 6 above the red curve traces possible values of the function in the first condition (1) above (and in Theorem 2b earlier); the green curve corresponds to the function in the second display (2) above. The blue curve identifies the θ value given a horizontal λ value. Thus λ must lie on the portion of the blue curve lying in between the red and green.

6 Conclusions

The disclosure model of Dye (1985) alerts us to consider the implications of firms remaining silent between mandatory disclosure dates. The model developed here shows why management of a firm may benefit from establishing a reputation for being candid on some time intervals, voluntarily disclosing all news, good or bad. At issue is when would one expect to see such behaviour in an equilibrium and whether it is likely to be time-invariant, once established. Corollary 2 (Sect. 5) proves that if the news intensity-arrival rate is sufficiently low an equilibrium exists in which managers are always candid. The comparative statics of $\theta(\lambda)$ in Theorem 2a establishes that, as the news-arrival rate rises, eventually the optimal policy for management is to switch to a sparing disclosure policy; a similar effect is caused by the remaining factors in the model, namely of time-to-expiry (to the next mandatory disclosure time) and pay-for-performance ratio κ . In the model, reputation for adoption of a candid disclosure strategy is derived endogenously and we see that for higher levels of

news-arrival such a strategy will not be time-invariant—managers will start to ‘burn their reputation’ (switching from candid reporting to sparing disclosure) the closer they get to a mandatory disclosure date. If the aim is to understand asset pricing in a continuous-time setting, then this model provides insights into how firm management will voluntarily disclose information to update markets in between mandatory disclosure dates. Litigation concerns may ensure that very negative news is always disclosed; nevertheless, as this model shows, once management switch out of candid disclosure into a sparing policy they will tend to hide “slightly” negative news, both when close to a mandatory disclosure time and when their private news-arrival rate is higher.

Acknowledgements We are grateful to the Referee for recommendations concerning presentation and to Nick Bingham and Amol Sasane (LSE) and Kevin Smith (Stanford) for very helpful comments.

Data availability Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article’s Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article’s Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

Appendix

Proposition 3 $\mu_t \leq 0$ for $t \in [0, 1]$.

Proof This follows from a formal solution of the co-state equation. With π_t piecewise-constant and h as in Sect. 3.3, take

$$\varphi(t) = \exp\left(-\int_0^t \pi_u \lambda h(u) du\right) \geq 0$$

(the integrating factor for the co-state equation), a decreasing function so that $\varphi(t) \leq \varphi(s)$ for $s \geq t$, as indeed,

$$\varphi(s)/\varphi(t) = \exp\left(-\int_t^s \pi_u \lambda h(u) du\right) \leq 1.$$

With μ_t piece-wise smooth and $\mu_1 = 0$ (see Sect. 3.4), integration from t to 1 leads from

$$\frac{d}{dt} \mu_t \varphi_t = \varphi_t (\beta_t - \alpha_t) (1 - \pi_t) \geq 0 \text{ as } \beta_t \geq \alpha_t$$

to: (see Sect. 3.3)

$$\begin{aligned}
 0 - \mu_t \varphi_t &= \int_t^1 \varphi_s (\beta_s - \alpha_s) (1 - \pi_s) \, ds, \\
 -\mu_t &= \varphi_t^{-1} \int_t^1 \varphi_s (\beta_s - \alpha_s) (1 - \pi_s) \, ds \geq 0, \\
 -\mu_t &= \int_t^1 \varphi_s \varphi_t^{-1} (\beta_s - \alpha_s) (1 - \pi_s) \, ds \leq \int_t^1 (\beta_s - \alpha_s) (1 - \pi_s) \, ds.
 \end{aligned}$$

Above we used the blanket assumption that $\alpha_t \leq \beta_t$, so we conclude that $\mu_t \leq 0$. \square

Corollary 4. $|\mu_t|$ is bounded on $[0, 1]$ when α_t, β_t are constant.

Proof This follows again from the blanket assumption and from

$$\begin{aligned}
 0 \leq -\mu_t &= \int_t^1 \varphi_s \varphi_t^{-1} (\beta_s - \alpha_s) (1 - \pi_s) \, ds \leq \int_t^1 (\beta_s - \alpha_s) (1 - \pi_s) \, ds \\
 &\leq (\beta - \alpha) \int_t^1 (1 - \pi_s) \, ds \text{ if } \alpha, \beta \text{ are constant } \leq (\beta - \alpha)(1 - t).
 \end{aligned}$$

\square

Theorem S1 (Non-mixing Theorem—Strong Form). Assume α_t, β_t are constant.

(i) If the state trajectory and switching curve coalesce on an interval I , then $\pi \equiv 1$ on I .

(ii) A mixing control with $\pi_t \in (0, 1)$ is non-optimal over any interval of time.

Proof If a mixing control occurs, then, from the Hamiltonian maximisation, it follows that $\gamma_t = \gamma_t^*$ on an interval of time; hence (ii) follows from (i), by contradiction. To prove (i), we compute in Step 1 the corresponding control π_t from the equilibrium equation and then, in Step 2, show that this control does not satisfy the co-state equation unless $\mu_t \equiv 0$, in which case $\pi_t \equiv 1$.

Step 1. Since γ_t and so also γ_t^* satisfies the equation (cont-eq- π) on I ,

$$(\gamma_t^*)' = -\pi_t \lambda \gamma_t^* h(t).$$

For ease of calculations, write

$$\gamma_t^* = \psi \gamma_t^1 \text{ with } \psi = \psi_t := \frac{1}{[\kappa - \mu_t \lambda h(t) / \beta]} > 0.$$

Then, as $(\gamma_t^1)' = -\lambda \gamma_t^1 h$,

$$(\gamma_t^*)' = \psi' \gamma_t^1 + \psi \gamma_t^{1'} = \psi' \gamma_t^1 + \psi [-\lambda \gamma_t^1 h].$$

So, from the equation (cont-eq- π),

$$-\pi_t \lambda h = \frac{\gamma_t'^*}{\gamma_t^*} = \frac{\psi' \gamma_t^1 - \psi \lambda h \gamma_t^1}{\psi \gamma_t^1} = \frac{\psi' - \psi \lambda h}{\psi},$$

$$\pi_t = \frac{\psi \lambda h - \psi'}{\psi \lambda h} = 1 - \frac{\psi'}{\psi \lambda h}.$$

Substituting for ψ and ψ' and writing h_t for $h(t)$ yields

$$\pi_t = 1 - \frac{[\kappa - \mu_t \lambda h_t / \beta]}{\lambda h_t} \frac{(\mu_t h_t)' \lambda / \beta}{[\kappa - \mu_t \lambda h_t / \beta]^2} = 1 - \frac{(\mu_t h_t)' / \beta}{h_t [\kappa - \mu_t \lambda h_t / \beta]}$$

$$= 1 - \frac{\psi}{\beta} \frac{(\mu_t h_t)'}{h_t} = 1 - \frac{\psi}{\beta} \left(\frac{\mu_t' h_t}{h_t} + \frac{\mu_t h_t'}{h_t} \right) = 1 - \frac{\psi}{\beta} \left(\mu_t' + \mu_t \frac{h'(t)}{h(t)} \right).$$

Step 2. We now substitute this value for π_t into the co-state equation,

$$-\mu_t' + \mu_t (\lambda \pi_t h(t)) = -\beta \kappa (1 - \pi_t),$$

(as $\kappa = 1 - \alpha / \beta$). We compute π_t from the co-state equation to be

$$\pi_t = \frac{\mu_t' - \beta \kappa}{[\mu_t \lambda h(t) - \beta \kappa]},$$

the division being valid, since $\mu_t \lambda h(t) \leq 0$ and $\beta \kappa > 0$. So,

$$\pi_t = \frac{\mu_t' / \beta - \kappa}{[\mu_t \lambda h(t) / \beta - \kappa]} = -\psi_t [\mu_t' / \beta - \kappa] = \psi_t [\kappa - \mu_t' / \beta].$$

Consistency of this and the formula from Step 1 requires that

$$\psi_t [\kappa - \mu_t' / \beta] = 1 - \frac{\psi_t}{\beta} \left(\mu_t' + \mu_t \frac{h'(t)}{h(t)} \right).$$

Since the μ_t' terms cancel on each side, this last holds iff

$$\psi_t \kappa = 1 - \frac{\psi_t}{\beta} \mu_t \frac{h'(t)}{h(t)}, \text{ so } -1 + \psi_t \kappa = -\frac{\psi_t}{\beta} \mu_t \frac{h'(t)}{h(t)}.$$

Computing the left-hand side, using the definition of ψ_t , gives

$$-1 + \psi_t \kappa = -1 + \frac{\kappa}{\kappa - \mu_t \lambda h(t) / \beta} = \frac{\mu_t \lambda h(t)}{\kappa - \mu_t \lambda h(t) / \beta}.$$

So, again using ψ_t ,

$$\psi_t \mu_t \lambda h(t) = -\psi_t \mu_t \frac{h'(t)}{h(t)},$$

implying, as $\psi_t > 0$, either $\mu_t = 0$ or $\lambda h(t)^2 = -h'(t)$. The latter gives

$$\frac{dh}{h^2} = -\lambda dt : \quad h^{-1} = \lambda t + \text{const.}$$

But on I this contradicts

$$h(t) := [2\Phi(\hat{\sigma}/2) - 1], \quad \text{where } \hat{\sigma} = \sigma(1 - t).$$

So consistency requires that $\mu_t = 0$ on I . But in this case the co-state equation,

$$\pi_t[\mu_t \lambda h(t) - (\beta - \alpha)] = \mu_t' - (\beta - \alpha),$$

implies that $\pi_t = 1$ on the interval I . □

References

- Acharya, V., DeMarzo, P., Kremer, I.: Endogenous information flows and the clustering of announcements. *Am. Econ. Rev.* **101**(7), 2955–79 (2011)
- Bertomeu, J., Marinovic, I., Terry, S.J., Varas, F.: The dynamics of concealment. *J. Financ. Econ.* **143**(1), 227–246 (2022)
- Beyer, A., Dye, R.A.: Reputation management and the disclosure of earnings forecasts. *Rev. Account. Stud.* **17**, 877–912 (2012)
- Bingham, N.H., Kiesel, R.: *Risk-Neutral Valuation*. Springer, Berlin (1998)
- Bressan, A., Piccoli, B.: *Introduction to Mathematical Control Theory*. AIMS (2007)
- Davis, M.H.A.: Piecewise-deterministic Markov processes: a general class of non-diffusion stochastic models. *J. R. Stat. Soc. Ser. B*, **46**(3), 353–388 (1984)
- Davis, M.H.A.: *Markov models and optimization*. Monographs on Statistics and Applied Probability 49. Chapman & Hall (1993)
- Dye, R.A.: Disclosure of nonproprietary information. *J. Account. Res.* **23**, 123–145 (1985)
- Einhorn, E., Ziv, A.: Intertemporal dynamics of corporate voluntary disclosures. *J. Acc. Res.* **46**(3), 567–589 (2008)
- Fudenberg, D., Tirole, J.: *Game Theory*. MIT (1991)
- Gietzmann, M.B., Ostaszewski, A.J.: Multi-firm voluntary disclosures for correlated operations. *Ann. Finance* **10**, 1–45 (2014)
- Gietzmann, M.B., Ostaszewski, A.J.: The sound of silence: equilibrium filtering and optimal censoring in financial markets. *Adv. Appl. Probab.* **48A**, 119–144 (2016)
- Gietzmann, M.B., Ostaszewski, A.J., Schröder, M.H.G.: Guiding the guiders: foundations of a market-driven theory of disclosure. *Stochastic modeling and control* (2020), pp. 107–132, Banach Center Publ., 122 pp
- Grossman, S.J.: The informational role of warranties and private disclosure about product quality. *J. Law Econ.* **24**, 461–483 (1981)
- Grubb, M.D.: Developing a reputation for reticence. *J. Econ. Man. Strat.* **20**(1), 225–268 (2011)
- Guttman, I., Kremer, I., Skrzypacz, A.: Not only what but also when: a theory of dynamic voluntary disclosure. *Am. Econ. Rev.* **104**(8), 2400–2420 (2014)
- Kamenica, E., Gentzkow, M.: *Bayesian Persuasion*. *Am. Econ. Rev.* **101**(6), 2590–2615 (2011)
- Liberzone, D.: *Calculus of Variations and Optimal Control Theory: A Concise Introduction*, Princeton (2012)
- Løkka, A.: Detection of disorder before an observable event. *Stochastics* **79**, 219–231 (2007)
- Marinovic, I., Vargas, F.: No news is good news: voluntary disclosure in the face of litigation. *RAND J. Econ.* **47**(4), 822–856 (2016)
- Milgrom, P.R.: Good news and bad news: representation theorems and applications. *Bell J. Econ.* **12**, 380–391 (1981)

Ostaszewski, A.J., Gietzmann, M.B.: Value creation with Dye's disclosure option: optimal risk-shielding with an upper tailed disclosure strategy. *Rev. Quant. Finance Account.* **31**, 1–27 (2008)

Sasane, A.: *Optimization in Function Spaces*, Dover (2016)

Troutman, J.L.: *Variational Calculus and Optimal Control*. Springer, Berlin (1996)

Weibull, J.W.: *Evolutionary Game Theory*. MIT (1997)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.