PhD THESIS DECLARATION

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Abstract

This thesis deals with two problems of sequential analysis for continuous time, strong Markov processes: sequential detection of a random disorder time and sequential testing of two simple hypotheses. The associated optimal stopping problems can be solved through their reduction to free-boundary problems, that, depending on the path structure of the observed process, can be characterized at the stopping boundaries by the principles of smooth and /or continuous fit.

The sequential detection of the drift of a diffusion process with constant signal-to-noise ratio function, under exponential penalty for the delay, is analyzed: we show that the martingale and free-boundary approaches used in the literature for solving this problem are strictly related. By means of the free-boundary approach, we prove that the reward function can be decomposed inside the continuation region into the product between the two key elements of the martingale approach: a gain function of the weighted likelihood ratio process and a positive martingale.

Problems of sequential testing are analyzed for a wide class of Lévy processes. We study the Bayesian and fixed error probability formulations of the sequential testing of two simple hypotheses for the Lévy-Khintchine triplet of pure increasing jump Lévy processes. In the Bayesian problem, an important ingredient is the principle of continuous fit, which uniquely identifies a boundary point of the two sided stopping region. In the fixed error probability formulation, we employ the general theory of Markov processes, in order to determine the stopping boundaries, the expected length and the region of admissible error probabilities of the associated sequential probability ratio test. In both the formulations, the explicit solutions of the sequential testing for the parameter $p \in (0,1)$ of a negative binomial process are derived.

The Bayesian problem of sequential testing of two simple hypotheses about the parameter $\alpha > 0$ of a gamma process is subsequently analyzed. To the best of our knowledge, it is the first time that a process with infinite jump activity is considered in this context. The original optimal stopping problem is reduced to a free-boundary problem, where, again, the principle of continuous fit plays a key role. The high complexity of the integro-differential equation solved by the value function on the continuation set makes extremely hard the explicit derivation of the solution. Thus, we propose a numerical scheme for computing the solution of the free-boundary problem: our method extends the well known collocation method for boundary value problems and allows us to get solutions with any degree of accuracy. Its performances will be evaluated in problems where closed form solutions are available.

In the last part, we study the Bayesian problem of sequential testing of two simple hypotheses for the Lévy-Khintchine triplet of a jump-diffusion Lévy process, where the jump component is assumed to be of finite variation. We prove that the original optimal stopping problem can be turned into a free-boundary problem, where the unknown value function solves on the continuation set a second order integro-differential equation and at the boundary points satisfies the principle of the smooth fit. The numerical method devised for the sequential testing of a gamma process is here retrieved and properly modified, in order to obtain precise numerical solutions of the constructed free-boundary problem.

Chapter 1

Introduction

In this introductive chapter, we illustrate the motivations that led us to study optimal stopping problems in mathematical statistics, we give a general glance to the steps of the development of the optimal stopping theory and we recall its main results. Then, we summarize the main contributions of this thesis.

1.1 Motivations

In the real life, we are often required to unconsciously solve optimal control or stopping problems, that is, we have to decide if it is convenient at a certain time to perform a specific action or to postpone it at a later moment, where we are more aware of the problem, since a bigger amount of information is available. This natural mechanism is realized so that (expected) rewards and losses are maximized and minimized, respectively. In other words, decisions are usually sequentially made. Here are some examples.

Suppose that after a working day, we decide to drive back home using the freeway, which allows us to avoid the traffic of the city. If during our trip we listen to the radio that a crash between two cars is causing long queues along our direction, we probably decide to exit the freeway and to drive through urban streets. It means that we updated our information and, on the basis of this, we performed the best action to save time for getting home.

Bayesian statistics is another familiar framework where the sequential character of decisions is present. Indeed, our prior knowledge about a certain unknown parameter, defining the distribution of a random variable, is adjusted by the information we recover through an observed sample. The posterior distribution of the parameter, incorporating the new data, is thus obtained and its mean can be used as reasonable estimate: in this way, the mean square error is minimized.

The outputs of a production system can be sequentially monitored, in order to find out as soon as possible the moment at which the system goes “out of control”. For instance, this could happen because of the break of some mechanical components. A rapid detection of the problem allows to immediately make the required repairs and to put back the system “in control”.

The following example is due to Wald [79]. Suppose that in a testing procedure the rejection region based on N observations is defined by $W_N = \{(x_1, \dots, x_N) : x_1^2 + \dots + x_N^2) \geq c\}$. If for some $n < N$, we find that $x_1^2 + \dots + x_n^2 \geq c$, we can stop the observation at the stage n and save therefore $N - n$ observations.

From the above examples, we observe that sequential procedures could lead to save time and, hence, money. Motivated by these arguments, in this thesis we mainly study two problems of mathematical statistics, known as problems of *sequential detection* and *sequential testing*. They arise in many applied disciplines, like quality control, finance, seismology and signal theory.

The *sequential detection* (or *disorder*) problem can be explained as follows: we start the observation of a stochastic process, that at an unknown and random time, called disorder time, changes its characteristics. The goal is to detect as soon as possible the time at which the disorder has occurred, through a stopping rule which must return the best trade-off between the probability of having a false alarm, that is, the probability that we sound the alarm before the disorder happens, and the measure of the delay for the correct identification, that is, the delay since the disorder occurred until we stop the process. This problem is analyzed in the Bayesian formulation, in the sense that an exponential prior distribution for the disorder time is given.

The problem of *sequential testing* of two simple hypotheses about the distributional features of an observed stochastic process aims to determine as soon and as accurate as possible which of the two hypotheses is true. This problem admits two different formulations. In the fixed error probability formulation, the first and second type error probabilities are given and the goal is to find a stopping rule with the smallest expected length under the two hypotheses, within the class of decision rules, whose error probabilities are lower than the fixed ones. In the Bayesian formulation of the problem, a prior distribution on the two hypotheses is given and the scope is to determine a stopping rule, which minimizes the total risk, given by the expected loss arising from choosing the incorrect hypothesis and the expected cost of sampling.

We will face the above two problems for continuous time, strong Markov stochastic processes. Hence, the associated optimal stopping problems can be solved through the powerful technique of the free-boundary problems.

1.2 Origin and development of the optimal stopping theory

Optimal stopping theory has its roots in two disciplines: calculus of variation and sequential analysis.

The problems of Lagrange, Mayer and Bolza in the classical calculus of variation were inherited by Bellman [12], within the theory of stochastic optimal control. This led to the "dynamic programming principle", that in optimal stopping takes the form of the method of backward induction. Let us explain its mechanism. Let $G = (G_n)_{n \geq 0}$ be a sequence of random variables defined on the filtered probability space $(\Omega, (\mathscr{F}_n)_{n \geq 0}, P)$, where G_n can be interpreted as the gain we have, if we stop the observation of G at time n. The following optimal stopping problem is considered:

$$V^N = \sup_{0\leq\tau\leq N} E[G_\tau] \tag{1.2.1}$$

where N is a fixed integer number. V^N is the value function if we start the observation of G at time $n = 0$. Next to V, we define the sequence of value functions $(V_n^N)_{1\leq n\leq N}$, given by

$$V_n^N = \sup_{n\leq\tau\leq N} E[G_\tau], \quad n = 1, \dots, N \tag{1.2.2}$$

Consider the problem (1.2.2) with $n = N$: of course, $V_N^N = G_N$; for $n = N - 1$, $V_{N-1}^N = \max\{G_{N-1}, E[V_N^N|\mathscr{F}_{N-1}]\}$. In general,

$$\begin{aligned} V_n^N &= G_N, \quad n = N, \\ V_n^N &= \max\{G_n, E[V_{n+1}|\mathscr{F}_n]\}, \quad n = 0, \dots, N-1 \end{aligned}$$

The backward induction method suggests that for the problem (1.2.1) we should consider the stopping time

$$\tau^N = \inf\{0 \leq n \leq N : V_n^N = G_n\}. \tag{1.2.3}$$

Indeed, we have that this stopping time is optimal for V^N, that is, $V^N = E[G_{\tau^N}]$; further, $(V_k^N)_{0\leq k\leq N}$, with $V_0^N = V^N$, is the smallest supermartingale which dominates $(G_k)_{0\leq k\leq N}$.

The expression "sequential analysis" refers to those statistical inference procedures, like estimation and hypotheses testing, where the number of observation is not fixed in advance. This discipline can be considered one of the main sources which have been inspiring the evolution of the optimal stopping theory during the years. Sequential analysis formally began with the works of Wald [79, 80], where the so called sequential probability ratio test (SPRT) for a sequence of i.i.d. observations was devised. Only later, Wald and Wolfowitz [81] formally showed its optimality character: among all the other tests, the SPRT requires on the average fewest observations. In their work, Wald and Wolfowitz formulated an optimal stopping problem, whose solution was used as technical device for achieving the proof of the desired result. Based on this, the works of Arrow et al. [4] and Wald and Wolfowitz [82] gave a vigorous impulse to the optimal stopping theory and led Snell [76] to formulate a more general optimal stopping problem, with infinite horizon:

$$V = \sup_{\tau} E[G_\tau], \tag{1.2.4}$$

where the supremum is taken over all the stopping time with values in the set of natural numbers and the process $G = (G_n)_{n\geq 0}$ is interpreted as before. Let $(V_n)_{n\geq 0}$, with $V_0 = V$, be the sequence of value functions defined by

$$V_n = \sup_{\tau\geq n} E[G_\tau]. \tag{1.2.5}$$

Snell [76] succeeded in proving that the sequence of value functions $(V_n)_{n\geq 0}$ (known as Snell's envelope) is the smallest supermartingale dominating $G = (G_n)_{n\geq 0}$, satisfying

$$V_n = \operatorname{ess\,sup}_{\tau\geq n} E[G_\tau|\mathscr{F}_n], \quad n = 0, 1, \dots \tag{1.2.6}$$

as well as

$$V_n = \max\{G_n, E[V_{n+1}|\mathscr{F}_n]\}, \quad n = 0, 1, \dots \tag{1.2.7}$$

Further,

$$\tau^\star = \inf\{n \geq 0 : V_n = G_n\} \tag{1.2.8}$$

is the smallest optimal stopping time in (1.2.4). For obvious reasons, this method for solving optimal stopping problems is known as the method of essential supremum.

We refer to the book of Chow et al. [20] and Peskir and Shiryaev [67, Chap. 1] for a deep treatment of optimal stopping problems in discrete time.

One of the first articles in sequential analysis dealing with continuous time stochastic processes is due to Dvoretzky et al. [28], who analyzed the Wald's SPRT for the drift of a Wiener process and the intensity of a Poisson process.

The 1960s represented a decisive period for the development of optimal stopping theory. Dynkin [29] began the study of optimal stopping problems with Markovian structure in discrete time and he characterized the value function (depending now on the starting point of the Markov process, see Chapter 2) as the smallest superharmonic function, which dominates the gain function. The connection between optimal stopping problems for continuous time Markov processes and free-boundary problems with smooth fit condition appeared in the works of Mikhalevich [57], Chernoff [19], Lindley [52], McKean [56] and Shiryaev [72, Chap. 4].

Again, a huge stimulus to the optimal stopping theory came from sequential analysis: Peskir and Shiryaev [65, 66] introduced the continuous fit principle, as variational principle, alike the smooth fit, for characterizing the stopping region of an optimal stopping problem. This finding is due to the jump structure of the involved Markov process and has been further investigated by Alili and Kyprianou [2], for the case of Lévy processes.

In recent years, a big improvement to the theory of optimal stopping has been given by optimal prediction problems: a sequentially observed stochastic process over a finite time interval must be stopped as close as possible to its ultimate maximum. The latter is a quantity that the observer knows just at the end of the period: from here the name of optimal prediction problems. We refer to Graversen et al. [41], Du Toit and Peskir [27], Bernyk et al. [13] and Glover et al. [40].

Problems of financial mathematics, like pricing an American put option, contributed in a decisive manner to the development of optimal stopping. We refer to Peskir and Shiryaev [67, Chap. 7] and the references there included.

We also refer to Lai [49], who gave a general overview on several classical problems of sequential analysis and the possible directions of research. Problems of sequential analysis in Bayesian non-parametric statistics were studied by Ferguson [31] and Ghosh and Mukherjee [39].

1.3 Main contributions of this thesis

The main aim of this thesis is to solve optimal stopping problems arising in mathematical statistics. In particular, the problems of sequential detection and sequential testing are analyzed for some diffusion and Lévy processes. Their Markovian structure allows us to reduce the initial optimal stopping problems to free-boundary problems.

This thesis is organized as follows:

Chapter 2: the basic elements on the theory of optimal stopping for Markov processes and their relationship with free-boundary problems are provided. In particular, we recall: the definition of a strong Markov process and its main properties, some elements of optimal stopping theory for continuous time Markov processes, the definition of infinitesimal generator for a continuous time Markov process and the method of reduction of an optimal stopping problem to a free-boundary problem, the variational principles of smooth and continuous fit. We conclude the chapter with a brief analysis of the results contained in Peskir and Shiryaev [65], as application of the previous theory.

Chapter 3: the procedures used by Shiryaev [72, Chap. 4] in the Bayesian problems of sequential detection and sequential testing for the drift of a Brownian motion are revisited for diffusion processes with constant signal-to-noise ratio function. We show, by means of stochastic calculus, that the filtering equations of the disorder time and the hypotheses to be tested have the same structure of the ones obtained by Shiryaev. This allows us to exploit his techniques for the solution of the associated free-boundary problems.

Chapter 4: we investigate on the connection between two approaches which have been used in the literature, in order to sequentially detect a shift in the drift of a diffusion process, when an exponential penalty for the delay is used. These approaches are the martingale approach, devised by Beibel and Lerche [10, 11], and the free-boundary approach, used by Gapeev and Shiryaev [38]. Under the assumption that the observed diffusion process has constant signal-to-noise ratio function, we show that the decomposition of the reward function into the product between a gain function of the weighted likelihood ratio process and a positive martingale inside the continuation region, obtained by Beibel [9], is naturally entailed by the free-boundary approach.

Chapter 5: the Bayesian problem of sequential testing of two simple hypotheses concerning the Lévy-Khintchine triplet of a wide class of Lévy processes is studied. The main results on Lévy processes that will be used in the chapter are recalled. After the formal statement of the problem, we revise the sequential testing for a Wiener process. Then we concentrate on the sequential testing for pure increasing jump Lévy processes: we reduce the original optimal stopping problem to a free-boundary problem. The two sided stopping region is characterized by the smooth and continuous fit principles, which hold on one boundary point, and by the continuous fit principle only, which holds on the other one. This is due to the presence of jumps in the paths of the observed process. Our results extend those of Peskir and Shiryaev [65, Sec. 2]. We finally provide the explicit solution of the sequential testing for a negative binomial process.

Chapter 6: the Wald's sequential probability ratio test (SPRT) for the Lévy-Khintchine triplet of a large class of Lévy processes is studied. In particular, the strong Markov property of a Lévy process allows us to employ a general method for determining the stopping boundaries and the expected length of the SPRT, for a given admissible pair of first and second type error probabilities. We revise the results obtained by Shiryaev [72, Sec. 4.2] for the drift of a Wiener process and we extend the results of Dvoretzky et al. [28] and Peskir and Shiryaev [65, Sec. 3] to pure increasing jump Lévy processes. Our procedure is exploited for the explicit derivation of the SPRT for a negative binomial process.

Chapter 7: the Bayesian problem of sequentially testing two simple hypotheses concerning the parameter $\alpha > 0$ of a gamma process is studied. The novelty of this chapter is represented by the fact that for the first time a process with infinite jump activity on any finite time interval is analyzed in the context of sequential testing. The original optimal stopping problem is reduced to a free-boundary problem, characterized by an integro-differential operator; we show that the smooth fit principle characterizes only one of the two optimal stopping boundaries, since the other one is uniquely characterized by the continuous fit. We also provide a simple and efficient numerical scheme, based on the well known collocation method, in order to solve the free-boundary problem, which does not seem to be explicitly solvable. We apply our numerical procedure to problems of sequential testing, for which explicit solutions are known. This allows us to evaluate the performances of our method.

Chapter 8: we study the Bayesian problem of sequential testing of two simple hypotheses about the Lévy-Khintchine triplet of a Lévy process, showing diffusion component and jump component of finite variation. As usual, the initial optimal stopping problem is turned into a free-boundary problem, characterized by a second order integro-differential operator and the smooth fit principle, which holds on both the two boundary points. Since solving the free-boundary problem is extremely complex, we apply an extended version of the collocation method, devised in the previous chapter. We illustrate our technique through several examples.

Conclusions: we summarize the results presented in the previous chapters and we discuss some possible lines of research.

1.4 Publication Details

From this thesis the following research papers have been extracted:

- Buonaguidi, B. and Muliere, P. (2012). A Note on some Sequential Problems for the Equilibrium Value of a Vasicek Process, *Pioneer Journal of Theoretical and Applied Statistics* 4: 101–116;

- Buonaguidi, B. and Muliere, P. (2013). Sequential Testing Problems for Lévy Processes, *Sequential Analysis* 32: 47-70;

- Buonaguidi, B. and Muliere, P. (2013). On the Wald's Sequential Probability Ratio Test for Lévy Processes, *Sequential Analysis* 32: 267–287

- Buonaguidi, B. and Muliere, P. On the Martingale and Free-Boundary Approaches in Sequential Detection Problems with Exponential Penalty for Delay, accepted in *Stochastics An International Journal of Probability and Stochastic Processes*.

- Buonaguidi, B. and Muliere, P. A Collocation Method for the Sequential Testing of a Gamma process. Submitted.

- Buonaguidi, B. and Muliere, P. On the Sequential Testing for Lévy Processes with Diffusion and Jump Components. Submitted.

Chapter 2

Optimal Stopping Theory and Free-Boundary Problems

In this chapter, we analyze the connection between optimal stopping and free-boundary problems. Since the problems we will deal with in this thesis involve the observation of a time homogeneous strong Markov process, we begin by recalling its definition and some important properties. Then, we illustrate the main theory underlying an optimal stopping problem and we concentrate on the relationship between optimal stopping and free-boundary problems. The latter are indeed the tool we use for determining the solution of the optimal stopping problems that will be faced in the next chapters. Then, we recall the fundamental concepts of smooth and continuous fit principles, which play a key role for the correct formulation of a free-boundary problem. We conclude the chapter applying the previous theory to an optimal stopping problem, solved by Peskir and Shiryaev [65].

The results we are going to present are based on the monograph of Peskir and Shiryaev [67, chap. 1-4].

2.1 Basic elements on Markov processes

Because of the nature of problems that will be subsequently studied, we concentrate on continuous time, time homogeneous Markov processes.

Let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, P)$ be a filtered probability space, where $\mathscr{F}_t$ is a σ-algebra of subsets of Ω and $\mathscr{F}_s \subseteq \mathscr{F}_t \subseteq \mathscr{F}$, for $s \leq t$. On this space, the stochastic process $X = (X_t)_{t\geq 0}$, with values in the measurable space $(S, \mathscr{S})$, is defined. We say that X is a Markov process if

$$P(X_t \in B | \mathscr{F}_s) = P(X_t \in B | X_s), \quad P\text{-a.s.}, \tag{2.1.1}$$

for all $B \in \mathscr{S}$ and $s \leq t$. Let $\{P_x\}_{x\in S}$ be a family of probability measure on $(\Omega, \mathscr{F})$, where P_x is the probability measure under which X starts at $x \in S$. Then, we say that $X = (X_t)_{t\geq 0}$ is a time homogeneous Markov process on $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, P_x)$, if

$$P_x(X_{s+t} \in B | \mathscr{F}_s) = P_{X_s}(X_t \in B), \tag{2.1.2}$$

for all $x \in S$ and $B \in \mathscr{S}$. In other words, X is a time homogeneous Markov process, if the future X_{s+t} does not depend on the whole past $\mathscr{F}_s$, but just on the last observation X_s and, conditional on X_s, the stochastic process $(X_{s+t})_{t\geq 0}$ starts afresh from X_s.

As we will see, the strong Markov property is the main element allowing us to transform an optimal stopping problem into a free-boundary problem. Before defining it, let us recall the concept of stopping time.

Definition 2.1.1 A random variable $\tau : \Omega \to [0, \infty)$ is a stopping time if τ is $\mathscr{F}_t$-measurable, $t \geq 0$, that is, if the event $\{\tau \leq t\} \in \mathscr{F}_t$, $t \geq 0$.

We will come back on this concept in Section 2.2. We are now in the position to define the strong Markov property. We say that $X = (X_t)_{t\geq 0}$ is a strong Markov process, if for any stopping time τ

$$P_x(X_{\tau+t} \in B|\mathscr{F}_\tau) = P_{X_\tau}(X_t \in B), \tag{2.1.3}$$

being $\mathscr{F}_\tau = \{A \in \mathscr{F} : A \cap \{\tau \leq t\} \in \mathscr{F}_t, \forall t \geq 0\}$. The intuitive explanation of the strong Markov property is similar to the one we gave before for the Markov property: conditional on X_τ, the process $(X_{\tau+s})_{s\geq 0}$ is independent on $(X_s)_{0\leq s<\tau}$ and starts afresh from X_τ. Obviously, the strong Markov property implies the Markov property, by setting $\tau = s$.

For the purposes of Section 2.3, we believe it is convenient to introduce the following notation. Let the sample space Ω be the canonical space, that is, the space consisting of the functions $\omega = (x_s)_{s\geq 0}$, $x_s \in S$. So, each $\omega \in \Omega$ is a trajectory of X and it is assumed that $X_t(\omega) = x_t$. Let θ_t, $t \geq 0$, be the shift operator, defined by $\theta_s(\omega) = (x_{s+t})_{t\geq 0}$, which shifts the trajectory ω from x_0 to the new starting point x_s. With this notation, (2.1.2) and (2.1.3) also read as:

$$P_x(X_t \circ \theta_s(\omega) \in B|\mathscr{F}_s) = P_{X_s}(X_t \in B), \tag{2.1.4}$$

$$P_x(X_t \circ \theta_\tau(\omega) \in B|\mathscr{F}_\tau) = P_{X_\tau}(X_t \in B), \tag{2.1.5}$$

for $x \in S$ and $B \in \mathscr{S}$, where $\theta_\tau(\omega) = \theta_{\tau(\omega)}(\omega)$. One can easily show that (2.1.5) holds if and only if

$$E_x(H \circ \theta_\tau|\mathscr{F}_\tau) = E_{X_\tau}(H), \tag{2.1.6}$$

for any stopping time τ and any non-negative and $\mathscr{F}$-measurable function $H = H(\omega)$.

2.2 Theory of optimal stopping

In an optimal stopping problem a decision-maker observes a random quantity evolving in the time and on the basis of the collected information must decide when he/she can interrupt the observation of the process, in order to maximize an expected reward or minimize an expected loss. Formally, an optimal stopping problem is defined as

$$V(x) = \sup_\tau E_x[G(X_\tau)], \tag{2.2.1}$$

where it is assumed that $X = (X_t)_{t\geq 0}$ is a continuous time, time homogeneous strong Markov process defined on $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0})$ with values in some measurable space $(S, \mathscr{S})$ and the expectation is under P_x, the measure under which X starts at $x \in S$. Usually, the filtration $(\mathscr{F}_t)_{t\geq 0}$

coincides with $(\mathscr{F}_t^X)_{t\geq 0}$, where $\mathscr{F}_t^X = \sigma(X_s : 0 \leq s \leq t)$, but might be, in general, larger. V and G are known as value function and gain function, respectively. If G is interpreted as loss function, then the previous problem becomes

$$V(x) = \inf_{\tau} E_x[G(X_\tau)]. \tag{2.2.2}$$

The supremum and infimum in (2.2.1) and (2.2.2), respectively, are taken over stopping times τ of X, that is, over $\mathscr{F}_t^X$-measurable functions, $t \geq 0$. Then, the choice of interrupting the process at a certain time must depend only on the information generated by X so far.

Solving an optimal stopping problem means two things: determining the time at which the supremum or infimum in the above expressions are attained and making as much as possible explicit the associated value function V. The theory we are going to analyze refers to (2.2.2), but is of course immediately applicable to (2.2.1).

The decision-maker observes a sample path of $t \mapsto G(X_t)$; the Markov property of X, and hence the fact that conditional on the last observation the process starts afresh, allows him/her to establish at any time if it is convenient to stop or continue the observation. It means that the state space S of X can be partitioned into the so called *stopping* and *continuation regions*, denoted by D and C, respectively. In particular, we have

$$C = \{x \in S : V(x) < G(x)\}, \tag{2.2.3}$$

$$D = \{x \in S : V(x) = G(x)\}. \tag{2.2.4}$$

It follows that the first time the process X enters D, that is,

$$\tau_D = \inf\{t \geq 0 : X_t \in D\}, \tag{2.2.5}$$

is optimal in (2.2.2). The problem boils down to determining D and C. We observe that if V is upper semicontinuous and G is lower semicontinuous, then D is closed and C is open.

We define now the concept of *subharmonic* function, which turns out to be fundamental for the next results.

Definition 2.2.1 *A measurable function $f : S \to \mathbb{R}$ is said subharmonic if for any stopping time τ and all $x \in S$*

$$f(x) \leq E_x[f(X_\tau)]. \tag{2.2.6}$$

The next theorem characterizes the function V, defined in (2.2.2).

Theorem 2.2.1 *Let $\tau^\star$ be an optimal stopping time in* (2.2.2)*, that is,*

$$V(x) = E_x[G(X_{\tau^\star})], \quad x \in S. \tag{2.2.7}$$

Then, the value function V is the biggest subharmonic function, dominated by the loss function G on S. Further, if V is upper semicontinuous and G is lower semicontinuous, the stopping time τ_D, given by (2.2.5)*, is optimal in* (2.2.2) *and $\tau_D \leq \tau^\star$ P_x-a.s., for all $x \in S$.*

The following theorem establishes a sufficient condition for the existence of an optimal stopping time in the problem (2.2.2).

Theorem 2.2.2 *In the problem* (2.2.2)*, assume that* $V^\star$ *is the biggest subharmonic function dominated by* G *on* S *and let* $V^\star$ *be upper semicontinuous and* G *be lower semicontinuous. Let* $D = \{x \in S : V^\star = G\}$*. Then:*
i) if $P_x(\tau_D < \infty) = 1$ *for all* $x \in S$*,* $V^\star = V$ *and* τ_D *is optimal in* (2.2.2)*;*
ii) if $P_x(\tau_D < \infty) < 1$ *for at least one* $x \in S$*, there is no optimal stopping time in* (2.2.2)*.*

2.3 From optimal stopping to free-boundary problems

The results of the previous section state that the optimal stopping problem (2.2.2) reduces to determining the biggest subharmonic function V dominated by G on S and the stopping set D, which defines the optimal stopping time τ_D. In this case, the expression (2.2.2) takes the form

$$V(x) = E_x[G(X_{\tau_D})], \quad x \in S. \tag{2.3.1}$$

It is immediate to notice from (2.3.1) the existence of a relationship between V and the equation which rules the deterministic/expected evolution of the process $(G(X_t))_{t\geq 0}$.

This connection leads to functional equations that V solves on the continuation set C, defined as in (2.2.3). Depending on the properties of the Markov process X, these equations are differential or integro-differential equations. In addition to V, the problem (2.3.1) also requires determining the stopping and continuation sets D and C. This can be realized by imposing some conditions on the unknown boundaries of the continuation region that V naturally satisfies. In this way, we constructed a free-boundary problem, that is, a problem where a functional equation must be solved and the unknown (free) boundary must be determined. Before analyzing the way to reduce an optimal stopping problem to a free-boundary problem, we need to recall some well known facts.

Let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{\geq 0}, P_x)$ be a filtered probability space hosting a strong Markov process $X = (X_t)_{t\geq 0}$, taking values in the measurable space $(S, \mathscr{S})$, where, for simplicity, we assume X being one dimensional, $S = \mathbb{R}$ and $\mathscr{S} = \mathscr{B}(\mathbb{R})$. Then, we know that property (2.1.6) holds, as well as the following relationships:

$$\tau(\omega) = \sigma(\omega) + \tau \circ \theta_\sigma(\omega), \quad \text{for } \sigma \leq \tau, \tag{2.3.2}$$

where σ and τ are two stopping times and the shift operator has been defined at the end of Section 2.1, and

$$X_\tau \circ \theta_\sigma = X_{\sigma + \tau\circ\theta_\sigma}, \tag{2.3.3}$$

for all stopping times σ and τ.

Let $f : S \to \mathbb{R}$; the expected motion of the process $(f(X_t))_{t\geq 0}$ is given by the *characteristic operator* $\mathbb{L}$ of X, that, applied to f, is defined by

$$(\mathbb{L}f)(x) = \lim_{U \downarrow x} \frac{E_x[f(X_{\tau_{U^c}})] - f(x)}{E_x[\tau_{U^c}]}, \tag{2.3.4}$$

where $x \in E$, the limit is taken over a family of open neighborhoods U of x shrinking to x and τ_{U^c} is the first exit time of X from U. Since the above expression usually coincides with the so called infinitesimal operator of X, acting on f through

$$(\mathbb{L}f)(x) = \lim_{t \downarrow 0} \frac{E_x[f(X_t)] - f(x)}{t}, \tag{2.3.5}$$

we will not distinguish between the two quantities and we refer to them as *infinitesimal generator* of X.

According to Dynkin [29], one can show that

$$\begin{aligned}(\mathbb{L}f)(x) =& b(x)f'(x) + \frac{1}{2}\sigma^2(x)f''(x) \\ &+ \int_{\mathbb{R}\setminus\{0\}} \big(f(y) - f(x) - (y-x)f'(x)\big)\, v(dy),\end{aligned} \tag{2.3.6}$$

where $b : E \to \mathbb{R}$ is the drift coefficient, $\sigma^2 : E \to \mathbb{R}_+$ is the diffusion coefficient and $v(\cdot)$ is the compensator of the measure μ of the jumps of X, that is, $v(dx)dt = E[\mu(dt, dx)]$, being $\mu((0,t], A) = \sum_{0<s\le t} \mathbf{1}(\Delta X_s \in A)$, $A \in \mathscr{B}(\mathbb{R} \setminus \{0\})$.

In order to be consistent with the optimal stopping problems that in the subsequent chapters will be studied, we assume that the problem (2.2.2) takes the following form:

$$V(x) = \inf_{\tau} E_x \left[M(X_\tau) + \int_0^\tau L(X_t)\, dt \right], \tag{2.3.7}$$

where M, $L : S \to \mathbb{R}$ are measurable and continuous functions. They are due to Mayer and Lagrange in calculus variation theory and provide two different measures of the final loss: for example, in a sequential statistical procedure, M could represent the loss entailed by a final wrong choice (e.g., a certain distance between the estimate and the true value of the parameter to be estimated or the loss caused by the rejection of a true hypothesis), while L could express the cost of sampling.

From (2.3.1), we see that the problem (2.3.7) admits the representation

$$V(x) = E_x \left[M(X_{\tau_D}) + \int_0^{\tau_D} L(X_t)\, dt \right]. \tag{2.3.8}$$

The next theorem relates optimal stopping and free-boundary problems. Due to its importance, the proof is shown.

Theorem 2.3.1 *Let the stopping region D be closed and ∂C denote the set of boundary points of the continuation region C; then, V from* (2.3.8) *solves the free-boundary problem*

$$(\mathbb{L}V)(x) = -L(x), \quad x \in C, \tag{2.3.9}$$

$$V(x) = M(x), \quad x \in \partial C. \tag{2.3.10}$$

Proof. We put together the steps illustrated by Peskir and Shiryaev [67, pp. 130-132]. Let $U \subseteq C$ be an open neighborhood of x and observe that $\tau_{U^c} \le \tau_D$. According to (2.1.6), (2.3.2) and (2.3.3), we have

$$\begin{aligned}
E_x[V(X_{\tau_{U^c}})] &= E_x E_{X_{\tau_{U^c}}} \left[M(X_{\tau_D}) + \int_0^{\tau_D} L(X_t)\, dt \right] \\
&= E_x E_x \left[M(X_{\tau_D}) \circ \theta_{\tau_{U^c}} + \int_0^{\tau_D} L(X_t)\, dt \circ \theta_{\tau_{U^c}} \middle| \mathscr{F}_{\tau_{U^c}} \right] \\
&= E_x \left[M(X_{\tau_{U^c} + \tau_D \circ \tau_{U^c}}) + \int_0^{\tau_D \circ \tau_{U^c}} L(X_{\tau_{U^c}+t})\, dt \right] \\
&= E_x \left[M(X_{\tau_D}) + \int_0^{\tau_D - \tau_{U^c}} L(X_{\tau_{U^c}+t})\, dt \right] \\
&= E_x \left[M(X_{\tau_D}) + \int_{\tau_{U^c}}^{\tau_D} L(X_s)\, ds \right] \\
&= E_x \left[M(X_{\tau_D}) + \int_0^{\tau_D} L(X_s)\, ds \right] - E_x \left[\int_0^{\tau_{U^c}} L(X_s)\, ds \right]. \qquad (2.3.11)
\end{aligned}$$

From (2.3.8) and (2.3.11), we obtain

$$\lim_{U \downarrow x} \frac{E_x[V(X_{\tau_{U^c}})] - V(x)}{E_x[\tau_{U^c}]} = \lim_{U \downarrow x} - \frac{E_x \left[\int_0^{\tau_{U^c}} L(X_s)\, ds \right]}{E_x[\tau_{U^c}]}. \qquad (2.3.12)$$

The expressions (2.3.4) and (2.3.12), as well as the continuity of L, give (2.3.9).

Due to the closeness of D, $\partial C \subseteq D$, implying that $\tau_D = 0$, if the process X starts at $x \in \partial C$. Then, (2.3.10) follows. ∎

2.4 The principles of smooth and continuous fit

In the previous section, we saw that when the problem (2.2.2) can be more explicitly expressed as (2.3.7), then V solves the free-boundary problem (2.3.9) and (2.3.10). It is pretty intuitive that the properties of "biggest" and "subharmonic" possessed by V, according to Theorems 2.2.1 and 2.2.2, can be ensured only by special and unique boundary ∂C, that is, only by special and unique C and D.

Sometimes, the boundary condition (2.3.10), also known as *continuous fit principle*, is not sufficient to guarantee the unique identification of C, that, instead, can be accomplished through the so called *smooth fit principle*; other times, the smooth fit breaks down at the boundary and the continuous fit becomes the only criterion one can use to determine C. Quite surprisingly, the fact that the continuos fit condition can play a decisive role for the identification of the optimal boundary, on equal footing as the smooth fit principle, has been recently discovered by Peskir and Shiryaev [65, 66].

Let us recall that the smooth fit principle states that the value function from (2.2.2) must be smooth on the boundary of C, that is,

$$V'(x) = G'(x), \quad x \in \partial C, \qquad (2.4.1)$$

or, using the formulation (2.3.7),

$$V'(x) = M'(x), \quad x \in \partial C. \qquad (2.4.2)$$

In this case, the optimal stopping problem (2.3.7) is reduced to the free-boundary problem (2.3.9), (2.3.10) and (2.4.2).

The continuous fit principle affirms that the value function from (2.2.2) must be only continuous on ∂C, that is,

$$V(x) = G(x), \quad x \in \partial C, \tag{2.4.3}$$

or, if we uses (2.3.7), it means that only (2.3.10) must hold, without any further boundary condition. Hence, while the smooth fit states that the optimal boundary points must be selected so that the value function is differentiable at those points, the continuous fit states that the optimal boundaries must be computed so that the value function is just continuous and not differentiable at those points.

For the correct formulation of the free-boundary problem, it is of paramount importance to understand when the smooth fit principle holds. It turns out the latter is intimately related with the concept of regularity of the boundary ∂C for the stopping region D.

Definition 2.4.1 Let

$$\sigma_D = \inf\{t \geq 0 : X_t \in \text{int}(D)\}, \tag{2.4.4}$$

being $\text{int}(D)$ the set of the interior points of D. Then, a point $x \in \partial C$ is said to be regular for D if σ_D satisfies

$$P_x(\sigma_D = 0) = 1. \tag{2.4.5}$$

The boundary ∂C is regular for D if x is regular for D, for all $x \in \partial C$.

Generally speaking, ∂C is regular for D if X, starting from ∂C, immediately enters $\text{int}(D)$. As conjectured by Alili and Kyprianou [2], regularity of a point can be used as reasonable rule for predicting when the smooth fit principle holds. The following rule of thumb is valid.

If $x \in \partial C$ is regular for D, then the smooth fit principle holds and (2.4.1) applies. This occurs for example when X has diffusion component, or when X has unbounded variation or when X is of bounded variation but its drift is "D oriented". Instead, if $x \in \partial C$ is not regular for D, the continuous fit only holds and (2.4.3) applies.

2.5 Example: sequential testing of a Poisson process

Peskir and Shiryaev [65] solved the following optimal stopping problem

$$V(\pi) = \inf_{\tau} E_\pi \left[\tau + g_{a,b}(\pi_\tau)\right], \tag{2.5.1}$$

where $(\pi_t)_{t\geq 0}$ is a stochastic process with state space $[0, 1]$, starting under P_π at $\pi_0 = \pi$, and $g_{a,b}(x) = \min\{ax, b(1-x)\}$, $a, b > 0$. The expression of π_t is given by

$$\pi_t = \left(\frac{\pi}{1-\pi}\varphi_t\right) \bigg/ \left(1 + \frac{\pi}{1-\pi}\varphi_t\right), \tag{2.5.2}$$

where

$$\varphi_t = \exp\left(X_t \log\frac{\lambda_1}{\lambda_0} - t(\lambda_1 - \lambda_0)\right), \tag{2.5.3}$$

and $X = (X_t)_{t\geq 0}$ is a Poisson process with intensities λ_0 and λ_1, with probabilities $1-\pi$ and π, respectively, under P_π. It was assumed $\lambda_1 > \lambda_0$.

The optimal stopping problem (2.5.1) arises from the problem of sequentially testing two simple hypotheses about the intensity λ of an observed Poisson process $X = (X_t)_{t\geq 0}$, that is,

$$H_0 : \lambda = \lambda_0 \quad \text{and} \quad H_1 : \lambda = \lambda_1, \tag{2.5.4}$$

whose prior probabilities are $1-\pi$ and π. The goal is to find a rule which allows to interrupt the observations, so that the best trade-off between the sampling cost and the loss for a final wrong decision is achieved. In particular, it is assumed a unitary sampling cost per unit of time and a (resp. b) is the loss one incurs if H_0 (resp. H_1) is chosen, but the true hypothesis is H_1 (resp. H_0).

By means of standard arguments, one can verify that:

- $(\pi_t)_{t\geq 0}$ is a time homogeneous, strong Markov process under P_π, for any $\pi \in [0,1]$;
- the stopping set D takes the form $D = [0, A] \cup [B, 1]$, where $0 < A \leq c \leq B < 1$, being $c = b/(a+b)$;
- from the above point and according to the optimal stopping theory, $\tau_D = \inf\{t \geq 0 : \pi_t \in D\}$ is optimal in (2.5.1).

We observe that, setting $M(x) = g_{a,b}(x)$ and $L(x) = 1$, (2.3.7) and (2.5.1) coincide; then, using the theory recalled in Sections 2.2 and 2.3, the following free-boundary problem for the unknown value function V and the unknown boundaries A and B can be formulated:

$$(\mathbb{L}V)(\pi) = -1, \quad \pi \in (A, B) \tag{2.5.5}$$

$$V(\pi) = g_{a,b}(\pi), \quad \pi \in [0, A] \cup [B, 1], \tag{2.5.6}$$

$$V(\pi) < g_{a,b}(\pi), \quad \pi \in (A, B), \tag{2.5.7}$$

$$V(A_+) = aA, \quad \textit{(continuous fit)}, \tag{2.5.8}$$

$$V(B_-) = b(1-B), \quad \textit{(continuous fit)}, \tag{2.5.9}$$

where the infinitesimal generator $\mathbb{L}$ of $(\pi_t)_{t\geq 0}$ is given by

$$\begin{aligned}\big(\mathbb{L}f\big)(\pi) = &-f'(\pi)\pi(1-\pi)(\lambda_1 - \lambda_0) \\ &+ \big(\lambda_1\pi + \lambda_0(1-\pi)\big)\left(f\left(\frac{\lambda_1 \pi}{\lambda_1\pi + \lambda_0(1-\pi)}\right) - f(\pi)\right).\end{aligned} \tag{2.5.10}$$

As we said in Section 2.4, the correct formulation of a free-boundary problem requires understanding if the smooth fit principle holds at the optimal boundaries. In other words, we should verify if A and B are regular for $D = [0, A] \cup [B, 1]$. To this aim, define

$$\tau_x^- = \inf\{t \geq 0 : \pi_t < x\} \quad \text{and} \quad \tau_x^+ = \inf\{t \geq 0 : \pi_t > x\}. \tag{2.5.11}$$

Denote by

$$Y_t = X_t - t\frac{\lambda_1 - \lambda_0}{\log(\lambda_1/\lambda_0)}. \tag{2.5.12}$$

We observe from (2.5.2) and (2.5.3) that under P_A

$$\tau_A^- = \inf\{t \geq 0 : Y_t < 0\}, \tag{2.5.13}$$

while under P_B

$$\tau_B^+ = \inf\{t \geq 0 : Y_t > 0\}. \tag{2.5.14}$$

Because of the assumption $\lambda_1 > \lambda_0$, we notice that (2.5.12) is a Poisson process with negative drift, under P_π, for any $\pi \in [0,1]$. Then, $(Y_t)_{t\geq 0}$ starts at 0 and before the first jump occurs it creeps linearly downwards. This consideration, (2.5.13) and (2.5.14) show that

$$P_A(\tau_A^- = 0) = 1 \quad \text{and} \quad P_B(\tau_B^+ = 0) = 0, \tag{2.5.15}$$

that is, A is a regular point for D, while B is not regular for D (formally, this result is stated in Sato [71, Th. 43.21, case 2, p. 324]).

It follows that the smooth fit principle holds at A only, so that, according to (2.4.2), we add to the free-boundary problem (2.5.5)-(2.5.9) the following boundary condition:

$$V'(A) = a, \quad \textit{(smooth fit)}. \tag{2.5.16}$$

We refer to Peskir and Shiryaev [65] for the explicit solution of (2.5.5)-(2.5.9) and (2.5.16). It is worthwhile to mention that the fact that the continuous fit was used as unique variational principle for characterizing the (upper) boundary of the continuation region is one of the main findings of the aforementioned work.

2.6 Concluding remarks

Once the formulated free-boundary problem has been solved, one needs to verify that its solution coincides with the one of the initial optimal stopping problem. This step is performed by means of a verification theorem, which exploits some results of stochastic calculus, like Itô's formula and the optional sampling theorem.

In summary, when we deal with an optimal stopping problem of the type we described before, we initially reduce the optimal stopping problem to a free-boundary problem. In this way we get a candidate solution of the optimal stopping problem. Then, we come back to the initial problem by proving that the solutions of the free-boundary and optimal stopping problems coincide.

The free-boundary problem technique represents one possible way to approach optimal stopping problems, concerning stochastic processes in continuous time. Another approach was developed by Beibel and Lerche [10, 11] and applied to sequential statistical problems by Beibel [9]. The connection between the two approaches in sequential detection problems with exponential penalty for delay will be discussed in Chapter 4.

Chapter 3

Sequential Problems for Some Diffusion Processes

We apply the Shiryaev's sequential procedures to time-homogeneous diffusion processes, characterized by a constant signal-to-noise ratio function. The problems of the sequential testing of two simple hypotheses and of the quickest detection of an abrupt change, both concerning the drift of the process, are faced. The solutions to these optimal stopping problems coincide with those of the associated free-boundary problems, solved through the principle of the smooth fit.

3.1 Introduction

The purpose of this chapter is to illustrate how the Shiryaev's sequential procedures for the solution of sequential detection and sequential testing problems can be naturally applied to a certain class of time-homogeneous diffusion processes. Let $X = (X_t)_{t\geq 0}$ be a time-homogeneous diffusion process, described by

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t, \quad X_0 = x, \tag{3.1.1}$$

where $\mu(\cdot)$ and $\sigma^2(\cdot)$ are the drift and diffusion coefficients of the stochastic differential equation (3.1.1) and $W = (W_t)_{t\geq 0}$ is a standard Wiener process. A trajectory of the process is begun to be observed at time $t = 0$. Two distinct problems are considered: 1) the drift $\mu(\cdot)$ of the process changes at an unknown time ϑ from $\mu_0(\cdot)$ to $\mu_1(\cdot)$; 2) $\mu(\cdot)$ can be either $\mu_1(\cdot)$ or $\mu_0(\cdot)$. It is assumed that the signal-to-noise ratio function $\rho^2(\cdot)$ is constant, independent of the current state of the observations, that is,

$$\rho^2(x) := \left(\frac{\mu_1(x) - \mu_0(x)}{\sigma(x)}\right)^2 = \rho^2 \in \mathbb{R}. \tag{3.1.2}$$

Both the situations will be approached in a Bayesian way, in the sense that prior distributions for the random time ϑ and for the two hypotheses regarding $\mu(\cdot)$ are given. Through the sequential observation of the process, we want, in the first case, to declare the "disorder" at a time as close as possible to ϑ, while, in the latter, we want to decide optimally if the true drift is either $\mu_1(\cdot)$ or $\mu_0(\cdot)$. The properties of "closeness" and "optimality" must be read as minimization of suitable loss functions.

The original Bayesian formulations of the disorder problem and the sequential testing for the drift of a Wiener process are due to Shiryaev [72, Chap. 4]. Not long ago, the same problems were solved by Peskir and Shiryaev [65, 66] for the intensity of a Poisson process. An application to financial data for detecting arbitrage is contained in Shiryaev [73].

Our main aim is to show that, by means of stochastic calculus, the filtering equations of the posterior probability processes, for the change point ϑ and for the correct identification of $\mu(\cdot)$, have the same structure of the ones obtained by Shiryaev [72, Chap. 4], for the disorder problem and the sequential testing of the drift of a Brownian motion. This allows us to exploit his well known techniques for the derivation of the solution of the associated optimal stopping problems.

3.2 Disorder problem

Let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, P^\pi)$, $\pi \in [0,1]$, be a filtered probability space, where:

$$P^\pi = \pi P_0 + (1-\pi)\int_0^\infty \lambda e^{-\lambda s} P_s \, ds \tag{3.2.1}$$

is the probability measure, formalizing the prior believes about the realization of a non-negative random variable ϑ, with $P_s(\vartheta = s) = 1$, $s \geq 0$. Hence, under P^π, ϑ is 0, with probability π, and, given that is greater than 0, exponentially distributed, with parameter $\lambda > 0$. Let $W = (W_t)_{t\geq 0}$ be a standard Wiener process, supposed to be independent of ϑ. At time $t = 0$ we start observing the process $X = (X_t)_{t\geq 0}$, satisfying the following stochastic differential equation:

$$dX_t = \Big[\mu_0(X_t) + \Big(\mathbf{1}_{\{t\geq\vartheta\}}\big(\mu_1(X_t) - \mu_0(X_t)\big)\Big)\Big]dt + \sigma(X_t)dW_t, \tag{3.2.2}$$

with $X_0 = x$. Thus, X is a time-homogeneous diffusion process with drift $\mu_0(\cdot)$, up to the time of the disorder ϑ, and $\mu_1(\cdot)$, after ϑ.

Let $\mathscr{F}_t^X = \sigma\{X_s : 0 \leq s \leq t\}$ be the natural filtration generated by X until the time $t \geq 0$ and let τ be a stopping time of X, i.e., an $\mathscr{F}_t^X$-measurable random variable. By continuously updating the available information, through the sequential observation of the process, the aim is to determine the optimal time, at which one can declare that the disorder has occurred. Especially, we want to compute:

$$V(\pi) = \inf_\tau \Big(P^\pi(\tau < \vartheta) + cE^\pi(\tau - \vartheta)^+\Big), \tag{3.2.3}$$

where $P^\pi(\tau < \vartheta)$ is the probability of a false alarm, $E^\pi(\tau - \vartheta)^+$ is the average delay in detecting the disorder correctly and $c > 0$ is the cost "per unit of time". By means of standard arguments, (see Shiryaev [72, pp. 195-96]), (3.2.3) can be rewritten as

$$V(\pi) = \inf_\tau E^\pi\Big(1 - \pi_\tau + c\int_0^\tau \pi_t \, dt\Big), \tag{3.2.4}$$

where $\Pi = (\pi_t)_{t\geq 0}$, defined by $\pi_t = P^\pi\big(\vartheta \leq t | \mathscr{F}_t^X\big)$, is the *a posteriori probability process*, with $\pi_0 = \pi$.

According to (3.2.1), one can easily verify that the map $\pi \mapsto V(\pi)$ is concave and decreasing; further, from (3.2.4), one can straightforwardly observe that the closer $(\pi_t)_{t\geq 0}$ to 1, the smaller the possibility to have a decrease in the loss, if one continues the observation. This suggests the existence of a point A, such that $\tau_A = \inf\{t \geq 0 : \pi_t \geq A\}$ is optimal in (3.2.4).

Applying the Bayes formula,

$$\pi_t = \pi \frac{d\left(P_0|\mathscr{F}_t^X\right)}{d\left(P^\pi|\mathscr{F}_t^X\right)} + (1-\pi)\int_0^t \lambda e^{-\lambda s}\frac{d\left(P_s|\mathscr{F}_t^X\right)}{d\left(P^\pi|\mathscr{F}_t^X\right)}\,ds, \tag{3.2.5}$$

$$1-\pi_t = (1-\pi)e^{-\lambda t}\frac{d\left(P_\infty|\mathscr{F}_t^X\right)}{d\left(P^\pi|\mathscr{F}_t^X\right)}, \tag{3.2.6}$$

where P_∞ is the probability measure under which X has drift $\mu_0(\cdot)$, $P_s|\mathscr{F}_t^X$ is the restriction of P_s to $\mathscr{F}_t^X$ and $d\left(P_s|\mathscr{F}_t^X\right)/d\left(P^\pi|\mathscr{F}_t^X\right)$ is the Radon-Nikodým derivative of $P_s|\mathscr{F}_t^X$ with respect to $P^\pi|\mathscr{F}_t^X$, for $s \geq 0$.

Define by

$$\varphi_t = \frac{\pi_t}{1-\pi_t} \tag{3.2.7}$$

the *likelihood ratio process* $(\varphi_t)_{t\geq 0}$; then by (3.2.6) and (3.2.7) we can rewrite the previous expression as

$$\varphi_t = e^{\lambda t} Z_t \left(\frac{\pi}{1-\pi} + \lambda \int_0^t \frac{e^{-\lambda s}}{Z_s}\,ds\right), \tag{3.2.8}$$

where $Z_t = d\left(P_0|\mathscr{F}_t^X\right)/d\left(P_\infty|\mathscr{F}_t^X\right)$ is, by virtue of Girsanov theorem (see, e.g., Liptser and Shiryaev [54, corollary of Th. 7.18, p.275]):

$$Z_t = \exp\left\{\int_0^t \frac{\mu_1(X_s)-\mu_0(X_s)}{\sigma^2(X_s)}\,dX_s - \frac{1}{2}\int_0^t \frac{\mu_1^2(X_s)-\mu_0^2(X_s)}{\sigma^2(X_s)}\,ds\right\}. \tag{3.2.9}$$

Making use of Itô's formula (see, e.g., Liptser and Shiryaev [54, th. 4.4, p. 118]) and noticing from (3.2.7) that

$$\pi_t = \frac{\varphi_t}{1+\varphi_t}, \tag{3.2.10}$$

we obtain the following stochastic differential equations:

$$dZ_t = \frac{\mu_1(X_t)-\mu_0(X_t)}{\sigma^2(X_t)} Z_t\big(dX_t - \mu_0(X_t)\,dt\big); \tag{3.2.11}$$

$$d\varphi_t = \left(\lambda(1+\varphi_t) - \varphi_t\frac{\mu_1(X_t)-\mu_0(X_t)}{\sigma^2(X_t)}\mu_0(X_t)\right)dt + \frac{\mu_1(X_t)-\mu_0(X_t)}{\sigma^2(X_t)}\varphi_t dX_t, \tag{3.2.12}$$

with $Z_0 = 1$ and $\varphi_0 = \pi/(1-\pi)$. Thus,

$$d\pi_t = \lambda(1-\pi_t)dt + \rho\pi_t(1-\pi_t)d\widetilde{W}_t, \quad \pi_0 = \pi, \tag{3.2.13}$$

where ρ is defined through (3.1.2) and $\widetilde{W} = (\widetilde{W}_t)_{t\geq 0}$, defined by

$$\widetilde{W}_t = \int_0^t \frac{dX_s}{\sigma(X_s)} - \int_0^t \frac{\mu_0(X_s)(1-\pi_s)+\mu_1(X_s)\pi_s}{\sigma(X_s)}\,ds, \tag{3.2.14}$$

is a standard Wiener process under P^π (see Liptser and Shiryaev [54, Th. 7.12, p. 258]).

By means of Itô's formula, the expression of the infinitesimal generator of $\Pi = (\pi_t)_{t\geq 0}$, acting on $f = f(\pi) \in C^2[0,1]$, is:

$$\big(\mathbb{L}f\big)(\pi) = \lambda(1-\pi)f'(\pi) + \frac{1}{2}\rho^2\pi^2(1-\pi)^2 f''(\pi). \tag{3.2.15}$$

From (3.2.13), it is evident that Π evolves continuously in the time: this provides the intuition to guess that the smooth fit condition holds at the boundary A. Further, (3.2.13) highlights that Π is a time-homogeneous (strong) Markov process under P^π, $\forall \pi \in [0,1]$. Then, standard arguments based on the strong Markov property (see Chapter 2 or, e.g., Peskir and Shiryaev [67, Chap. 3 and 4] or Shiryaev [72, Chap. 3]) lead to the formulation of the following free-boundary problem, for the unknown function V and the unknown point A:

$$\mathbb{L}V = -c\pi, \qquad \text{for} \quad \pi \in [0,A), \tag{3.2.16}$$

$$V < 1-\pi, \qquad \text{for} \quad \pi \in [0,A), \tag{3.2.17}$$

$$V = 1-\pi, \qquad \text{for} \quad \pi \in [A,1], \tag{3.2.18}$$

$$V(A) = 1-A, \tag{3.2.19}$$

$$V'(A) = -1 \qquad (\textit{smooth fit}), \tag{3.2.20}$$

$$V'(0^+) = 0 \qquad (\textit{normal entrance condition}). \tag{3.2.21}$$

The equation (3.2.16), together with the expression (3.2.15), is a first order linear differential equation in V', whose general solution is:

$$V'(\pi) = e^{-\lambda k H(\pi)}\left(C - ck\int_0^\pi \frac{e^{\lambda k H(u)}}{u(1-u)^2}\,du\right), \tag{3.2.22}$$

where $k = 2/\rho^2$, $H(\pi) = \log(\frac{\pi}{1-\pi}) - \frac{1}{\pi}$ and C is a constant, that must be equal to 0, to satisfy (3.2.21). Hence, set $C = 0$. In order to comply with (3.2.18) and (3.2.19), we set

$$V(\pi) = \begin{cases} 1 - A^\star - \int_\pi^{A^\star} V'(x)\,dx, & \pi \in [0, A^\star), \\ 1-\pi, & \pi \in [A^\star, 1], \end{cases} \tag{3.2.23}$$

where $A^\star$ is the optimal boundary point solving the equation (3.2.20), defined through (3.2.22) (see Figure 3.1 below).

Theorem 3.2.1 *The pay-off function in* (3.2.4) *is explicitly given by* (3.2.23)*. The stopping time* $\tau_{A^\star} = \inf\{t \geq 0 : \pi_t \geq A^\star\}$ *is optimal in* (3.2.4)*, where* $A^\star$ *is the unique solution of the equation given by* (3.2.20) *and* (3.2.22)*, (with* $C = 0$*).*

Proof. First we prove the uniqueness of $A^\star$. A simple application of de L'Hôpital theorem to (3.2.22) implies that $\lim_{\pi\to 0^+} V'(\pi) = 0$ and $\lim_{\pi \to 1^-} V'(\pi) = -\infty$; so it is sufficient to verify that V' is a decreasing function. By integration by parts,

$$\int_0^\pi \frac{e^{\lambda k H(u)}}{u(1-u)^2}\,du = \frac{1}{\lambda k}\left(e^{\lambda k H(\pi)}\frac{\pi}{1-\pi} - \int_0^\pi \frac{e^{\lambda k H(u)}}{u(1-u)^2}\,du\right). \tag{3.2.24}$$

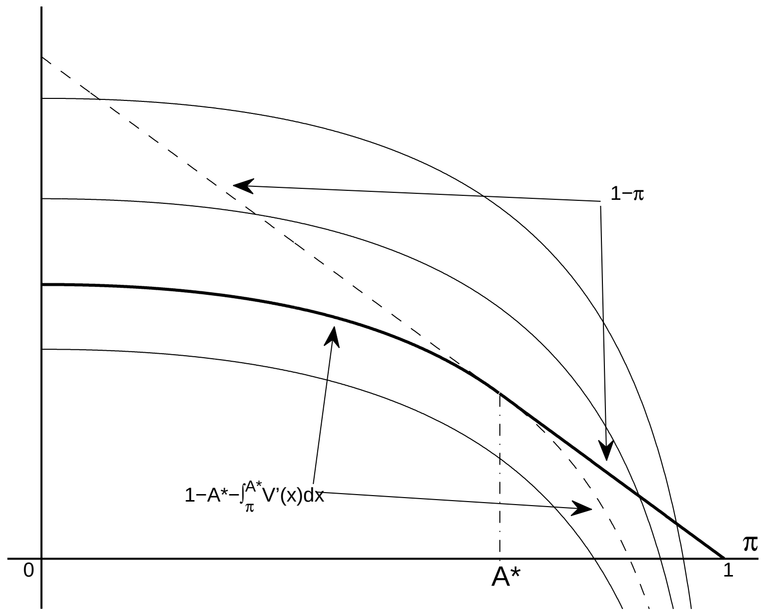

Figure 3.1: A computer drawing of the map $\pi \mapsto V(\pi)$ (bold curve), as expressed by (3.2.23), coinciding with the final payoff, given by (3.2.4). We set $c = 0.5$, $\rho = 0.24$ and $\lambda = 1$. The optimal boundary $A^\star$ is 0.6709.. The set $\{\pi \in [0,1] : V(\pi) = 1 - \pi\} = [A^\star, 1]$ is the stopping region, while the interval $[0, A^\star)$ is the set of continued observation of the posterior probability process $(\pi_t)_{t\geq 0}$. The other curves represent the map $\pi \mapsto 1 - A - \int_\pi^A V'(x)\,dx$, with A=0.3, 0.5 and 0.8 and show the uniqueness of $A^\star$.

Replacing (3.2.24) in (3.2.22), one has

$$V'(\pi) > -\frac{c}{\lambda}\frac{\pi}{1-\pi}, \tag{3.2.25}$$

so that, from (3.2.15), (3.2.16) and (3.2.25),

$$V''(\pi) = k\left(\frac{-\lambda}{\pi^2(1-\pi)}V'(\pi) - \frac{c}{\pi(1-\pi)^2}\right) < 0. \tag{3.2.26}$$

Therefore, V' is decreasing and V is concave (an alternative proof of the uniqueness of $A^\star$ can be found in Shiryaev [72, p. 204]). From (3.2.20) and (3.2.25), replacing π with $A^\star$, we notice that

$$A^\star > \frac{\lambda}{\lambda + c}. \tag{3.2.27}$$

It remains to verify that the solution to the free-boundary problem (3.2.16)-(3.2.21), expressed by (3.2.23), coincides with (3.2.4), hereafter denoted for convenience by $V^\star(\pi)$. Since $V(\pi)$ is two-times continuously differentiable on $[0, A^\star) \cup (A^\star, 1]$, one-time continuously differentiable at $A^\star$, because of the smooth fit (3.2.20), and the Lebesgue measure of those $t > 0$ for which $\pi_t = A^\star$ is zero, Itô's formula can be applied to $V(\pi)$ in its standard form:

$$V(\pi_t) = V(\pi) + \int_0^t (\mathbb{L}V)(\pi_s)\mathbf{1}_{\{\pi_s \neq A^\star\}}\,ds + M_t, \tag{3.2.28}$$

where $M = (M_t)_{t\geq 0}$, defined by

$$M_t = \int_0^t \rho\pi_s(1-\pi_s)V'(\pi_s)d\widetilde{W}_s, \tag{3.2.29}$$

is a continuous martingale under P^π, because $V'(\pi)$ is bounded for any π. Since $\mathbb{L}(1-\pi) = -\lambda(1-\pi) \geq -c\pi$ for $\pi \in [\lambda/(\lambda+c), 1]$, from (3.2.16), (3.2.18) and (3.2.27), it results that

$$(LV)(\pi) \geq -c\pi, \quad \forall \pi \in [0,1]. \tag{3.2.30}$$

From (3.2.17), (3.2.18), (3.2.28) and (3.2.30), we easily get that

$$1 - \pi_t + c\int_0^t \pi_s\, ds \geq V(\pi) + M_t. \tag{3.2.31}$$

Hence, for any stopping time τ with finite expectation, so that $E^\pi[M_\tau] = 0$ for the optional sampling theorem, we obtain

$$E^\pi\left(1 - \pi_\tau + c\int_0^\tau \pi_t\, dt\right) \geq V(\pi), \tag{3.2.32}$$

showing that $V^\star(\pi) \geq V(\pi)$, for any $\pi \in [0,1]$. We notice that

$$P^\pi(\tau_{A^\star} > t) \leq P^\pi(1-\pi_t > 1-A^\star) < \frac{E^\pi[1-\pi_t]}{1-A^\star} = \frac{(1-\pi)e^{-\lambda t}}{1-A^\star}, \tag{3.2.33}$$

implying that $E^\pi[\tau_{A^\star}] = \int_0^\infty P^\pi(\tau_{A^\star} > t)\, dt$ is finite. Then, the definition of $\tau_{A^\star}$, (3.2.16), (3.2.19) and (3.2.28) lead to

$$1 - \pi_{\tau_{A^\star}} = V(\pi_{\tau_{A^\star}}) = V(\pi) - c\int_0^{\tau_{A^\star}} \pi_t\, dt + M_{\tau_{A^\star}}. \tag{3.2.34}$$

Hence, taking the P^π expectation in the above equality and applying the optional sampling theorem, we obtain

$$E^\pi\left(1 - \pi_{\tau_{A^\star}} + c\int_0^{\tau_{A^\star}} \pi_t\, dt\right) = V(\pi). \tag{3.2.35}$$

The expressions (3.2.32) and (3.2.35) show that $V^\star = V$ and that $\tau_{A^\star}$ is optimal in (3.2.3). ∎

3.3 Sequential testing

On the filtered probability space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, P^\pi)$, with

$$P^\pi = \pi P_1 + (1-\pi)P_0, \qquad \pi \in [0,1], \tag{3.3.1}$$

a trajectory of the process $X = (X_t)_{t\geq 0}$, defined by

$$dX_t = \big[\mu_0(X_t) + \vartheta\big(\mu_1(X_t) - \mu_0(X_t)\big)\big]\, dt + \sigma(X_t)dW_t, \quad X_0 = x, \tag{3.3.2}$$

is begun to be observed. Now, ϑ is a dichotomous and $\mathscr{F}_0$-measurable random variable, with $P^\pi(\vartheta = 1) = \pi$ and $P^\pi(\vartheta = 0) = 1 - \pi$. Hence, X is a diffusion process, starting from x, whose drift $\mu(\cdot)$ depends on the realization of ϑ at time 0: $\mu(\cdot) = \mu_0(\cdot)$, if $\vartheta = 0$, while $\mu(\cdot) = \mu_1(\cdot)$, if $\vartheta = 1$. Notice that P_i is the probability measure under which the diffusion process X has drift $\mu_i(\cdot)$, $i = 0, 1$. Through the sequential observation of the process, the aim is to test the two simple hypotheses

$$H_0 : \vartheta = 0 \quad Vs \quad H_1 : \vartheta = 1. \tag{3.3.3}$$

Let τ be a stopping time with respect to the filtration $\mathscr{F}_t^X = \sigma\{X_s : 0 \leq s \leq t\}$ and let d be an $\mathscr{F}_\tau^X$-measurable random variable, taking on value 0, if H_0 is accepted, or 1, otherwise. Each pair (τ, d), called sequential decision rule, implies a loss due to the cost of observing the process and due to a possible wrong final decision between H_0 and H_1. So, we want to evaluate

$$V(\pi) = \inf_{(\tau,d)} E^\pi \left(c\tau + a\mathbf{1}_{(d=0,\vartheta=1)} + b\mathbf{1}_{(d=1,\vartheta=0)} \right), \quad c, a, b > 0, \tag{3.3.4}$$

and choose the rule $(\tau_\pi^\star, d_\pi^\star)$, called π-*Bayes decision rule*, for which the infimum in (3.3.4) is attained. Here, c is the cost "per unit of time", while a and b are the losses in case of wrong decision. It is easily verified that the map $\pi \mapsto V(\pi)$ is concave on [0,1].

Standard arguments, (see Shiryaev [72, Lemma 1, pp. 166-67]), show that (3.3.4) is equivalent to

$$V(\pi) = \inf_\tau E^\pi \left(c\tau + g_{a,b}(\pi_\tau) \right), \tag{3.3.5}$$

where $\Pi = (\pi_t)_{t\geq 0}$, defined by $\pi_t = P^\pi(\vartheta = 1|\mathscr{F}_t^X)$, is the *a posteriori probability process* and $g_{a,b}(\pi) = \min\{a\pi, b(1-\pi)\}$. Further, the optimal decision function $d_\pi^\star$ has the following expression:

$$d_\pi^\star = \begin{cases} 0 & \text{if} \quad \pi_{\tau_\pi^\star} < m \\ 1 & \text{if} \quad \pi_{\tau_\pi^\star} \geq m \end{cases}, \tag{3.3.6}$$

being $m = b/(a+b)$.

From (3.3.5), one can notice that the closer $(\pi_t)_{t\geq 0}$ to 0 or 1, the smaller the chance to decrease the loss, if the observation continues. This intuitively means that there exist two points A and B, such that $\tau_{A,B} = \inf\{t \geq 0 : \pi_t \notin (A, B)\}$ is optimal in (3.3.5).

Denote by $\mathscr{L} = (\mathscr{L}_t)_{t\geq 0}$ the *likelihood ratio process*

$$\mathscr{L}_t = \frac{d\left(P_1|\mathscr{F}_t^X\right)}{d\left(P_0|\mathscr{F}_t^X\right)}, \tag{3.3.7}$$

where $d\left(P_1|\mathscr{F}_t^X\right)/d\left(P_0|\mathscr{F}_t^X\right)$, $t \geq 0$, is the Radon-Nikodým derivative of $P_1|\mathscr{F}_t^X$ with respect to $P_0|\mathscr{F}_t^X$. Its expression is given by (3.2.9) and therefore solves the stochastic differential equation (3.2.11). Defining $P^\pi|\mathscr{F}_t^X = \pi P_1|\mathscr{F}_t^X + (1-\pi)P_0|\mathscr{F}_t^X$, the Bayes' formula and the expression (3.3.7) allow us to write

$$\pi_t = \pi \frac{d\left(P_1|\mathscr{F}_t^X\right)}{d\left(P^\pi|\mathscr{F}_t^X\right)} = \left(\frac{\pi}{1-\pi} \mathscr{L}_t \right) \Big/ \left(1 + \frac{\pi}{1-\pi} \mathscr{L}_t \right). \tag{3.3.8}$$

Applying the Itô's formula, one has that Π evolves according to

$$d\pi_t = \rho\pi_t(1-\pi_t)d\widetilde{W}_t, \quad \pi_0 = \pi, \tag{3.3.9}$$

being ρ given by (3.1.2) and $\widetilde{W} = (\widetilde{W}_t)_{t\geq 0}$, given by (3.2.14), a standard Wiener process under P^π; hence, the infinitesimal generator of Π, acting on $f = f(\pi) \in C^2[0,1]$, is:

$$\big(\mathbb{L}f\big)(\pi) = \frac{1}{2}\rho^2\pi^2(1-\pi)^2 f''(\pi). \tag{3.3.10}$$

From (3.3.9), we observe that $\Pi = (\pi_t)_{t\geq 0}$ is a time-homogeneous (strong) Markov process; besides, it evolves continuously in the time, so that we may guess that the smooth fit condition holds at the boundary points A and B. Standard arguments based on the strong Markov property (see Chapter 2 or, e.g., Peskir and Shiryaev [67, Chap. 3 and 4] or Shiryaev [72, Chap. 3]) naturally entail the formulation of the following free-boundary problem, for the unknown function V and the unknown points A and B:

$$\mathbb{L}V = -c \qquad \text{for} \quad \pi \in (A,B), \tag{3.3.11}$$
$$V = g_{a,b} \qquad \text{for} \quad \pi \notin (A,B), \tag{3.3.12}$$
$$V < g_{a,b} \qquad \text{for} \quad \pi \in (A,B), \tag{3.3.13}$$
$$V(A) = aA, \tag{3.3.14}$$
$$V'(A) = a \qquad (\textit{smooth fit}), \tag{3.3.15}$$
$$V(B) = b(1-B), \tag{3.3.16}$$
$$V'(B) = -b \qquad (\textit{smooth fit}). \tag{3.3.17}$$

For a fixed $B \in (m,1)$, the second order differential equation (3.3.10) and (3.3.11), together with the initial-boundary conditions (3.3.16) and (3.3.17), has the unique solution:

$$V(\pi,B) = b(1-B) + (B-\pi)\big(b + h\psi'(B)\big) + h\big(\psi(\pi) - \psi(B)\big), \tag{3.3.18}$$

for any $\pi \in [0,B)$, where we set $h = 2c/\rho^2$ and $\psi(\cdot)$ and $\psi'(\cdot)$ are given by

$$\psi(\pi) = (1-2\pi)\log\left(\frac{\pi}{1-\pi}\right), \tag{3.3.19}$$
$$\psi'(\pi) = \frac{1}{\pi} - \frac{1}{1-\pi} + 2\log\left(\frac{1-\pi}{\pi}\right). \tag{3.3.20}$$

From (3.3.14) and (3.3.15), the optimal boundaries $A^\star$ and $B^\star$ can be now determined, solving the system of transcendental equations

$$V(A^\star, B^\star) = aA^\star, \tag{3.3.21}$$
$$V'(A^\star, B^\star) = a. \tag{3.3.22}$$

Theorem 3.3.1 *The π-Bayes decision rule $(\tau^\star_\pi, d^\star_\pi)$ for the problem* (3.3.5) *is explicitly given by*

$$\tau^\star_\pi = \inf\{t \geq 0 :\ \pi_t \notin (A^\star, B^\star)\}, \tag{3.3.23}$$
$$d^\star_\pi = \begin{cases} 0 & (\textit{accept } H_0) \quad \textit{if} \quad \pi_{\tau^\star_\pi} \leq A^\star \\ 1 & (\textit{accept } H_1) \quad \textit{if} \quad \pi_{\tau^\star_\pi} \geq B^\star \end{cases}, \tag{3.3.24}$$

where the optimal boundaries $A^\star$ and $B^\star$, with $0 < A^\star < m < B^\star < 1$, are obtained as unique solution of the system of transcendental equations (3.3.21) *and* (3.3.22). *The explicit expression of the value function in* (3.3.5) *is given by*

$$V(\pi) = \begin{cases} V(\pi, B^\star) & \textit{for} \quad \pi \in (A^\star, B^\star) \\ g_{a,b}(\pi) & \textit{for} \quad \pi \in [0, A^\star] \cup [B^\star, 1] \end{cases}, \tag{3.3.25}$$

where $\pi \mapsto V(\pi, B)$ is given by (3.3.18).

Proof. We begin by proving that the system (3.3.21) and (3.3.22) admits a unique solution. It easily verified that for any $B \in (m, 1)$ the map $\pi \mapsto V(\pi, B)$, expressed by (3.3.18), is concave on $[0, B)$, $V(\pi, B) \to -\infty$, as $\pi \downarrow 0$, and $V(\pi, B) < 0$, as $B \uparrow 1$, for any $\pi \in (0, 1)$. Further, by construction of the map $\pi \mapsto V(\pi, B)$ (or by a direct verification on (3.3.18)), $\lim_{B \downarrow m} V'(B^-, B) = -b < a$: geometrically, it means that for some $B \in (m, 1)$, there exists a point $A \in (0, m)$, such that $\pi \mapsto V(\pi, B)$ intersects $\pi \mapsto a\pi$ at A. Since the curves $\pi \mapsto V(\pi, B')$ and $\pi \mapsto V(\pi, B'')$, with $B' < B''$, do not intersect on $(0, B']$ (see Peskir and Shiryaev [65, Remark 2.2, p.850]), we conclude that, moving B on the interval $(m, 1)$, there exists a unique pair of points $(A^\star, B^\star)$, satisfying (3.3.21) and (3.3.22) (see Figure 3.2).

The proof of the optimality of $\tau^\star_\pi$ and of the equality between (3.3.5) and (3.3.25) follows the same line of arguments used in the proof of Theorem 2.1 (see also Shiryaev [72, pp. 184-185] or Peskir and Shiryaev [67, pp. 290-292]). Anyway, their application requires to show that $E^\pi[\tau^\star_\pi]$ is finite: if $\pi \notin (A^\star, B^\star)$, obviously $E^\pi[\tau^\star_\pi] = 0$, so assume that $\pi \in (A^\star, B^\star)$. According to well known facts about one-dimensional diffusions (see, e.g., Peskir and Shiryaev [67, pp. 200-201]), the speed measure and the Green function on $[A^\star, B^\star]$ of Π are respectively given by

$$s(d\pi) = \frac{2}{\rho^2 \pi^2 (1-\pi)^2}\, d\pi, \tag{3.3.26}$$

$$G_{A^\star, B^\star}(\pi, y) = \begin{cases} \frac{(B^\star - \pi)(y - A^\star)}{B^\star - A^\star} & \text{if } A^\star \le y \le \pi, \\ \\ \frac{(B^\star - y)(\pi - A^\star)}{B^\star - A^\star} & \text{if } \pi \le y \le B^\star. \end{cases} \tag{3.3.27}$$

Therefore, denoted by

$$\xi(\pi) = \log\left(\frac{\pi}{1-\pi}\right) + \frac{1}{1-\pi}, \tag{3.3.28}$$

it results

$$E^\pi[\tau^\star_\pi] = \int_{A^\star}^{B^\star} G_{A^\star, B^\star}(\pi, y)\, s(dy) = \frac{2}{\rho^2 (B^\star - A^\star)} \times \tag{3.3.29}$$
$$\Big((B^\star - \pi)\Big(\xi(\pi) - \xi(A^\star) - A^\star\big(\psi'(A^\star) - \psi'(\pi)\big)\Big) +$$
$$(\pi - A^\star)\Big(B^\star\big(\psi'(\pi) - \psi'(B^\star)\big) - \xi(B^\star) + \xi(\pi)\Big)\Big).$$

Hence, from the above expression, we can conclude that $E^\pi[\tau^\star_\pi]$ is finite for any pair $(A^\star, B^\star)$. ∎

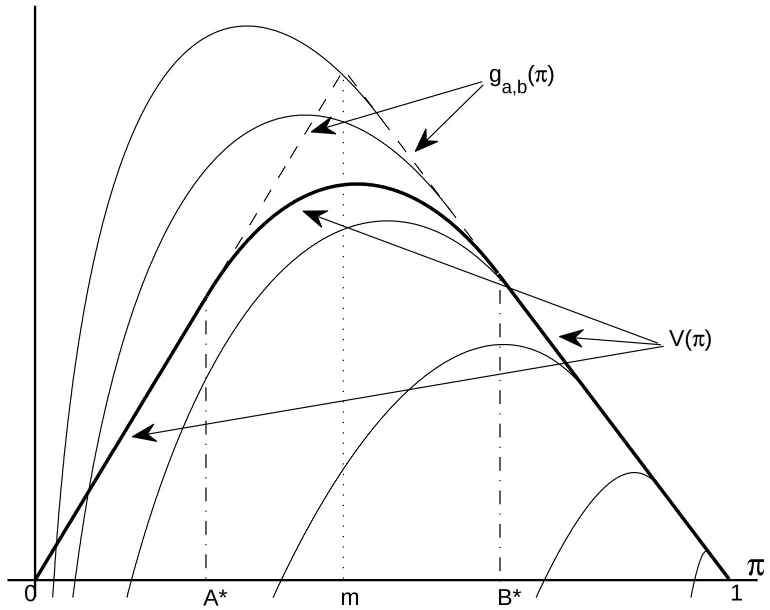

Figure 3.2: A computer drawing of the map $\pi \mapsto V(\pi)$ (bold curve), as expressed by (3.3.25), coinciding with the final payoff, given by (3.3.5). We set $a = 1000$, $b = 800$, $c = 1$, $\rho = 0.0933$. The optimal boundaries $A^\star$ and $B^\star$, where $\pi \mapsto V(\pi, B^\star)$ hits smoothly $\pi \mapsto g_{a,b}(\pi)$, are 0.2472.. and 0.6671.., respectively. The set $\{\pi \in [0,1] : V(\pi) = g_{a,b}(\pi)\} = [0, A^\star] \cup [B^\star, 1]$ is the stopping region, while the interval $(A^\star, B^\star)$ is the set of continued observation of the posterior probability process $(\pi_t)_{t\geq 0}$. The other curves represent the map $\pi \mapsto V(\pi, B)$, given by (3.3.18), for different values of $B = 0.5, 0.6, 0.7, 0.8, 0.9, 0.97$, showing the uniqueness of $A^\star$ and $B^\star$.

3.4 Conclusions

The disorder problem and the sequential testing for the drift of a time-homogeneous diffusion process satisfying (3.1.2) have been presented. We have seen that they can be solved reducing the initial problem into an optimal stopping problem for the posterior probability process and determining the solution of the associated free-boundary problem, for a second order differential operator.

After that we completed our work, we noticed that in Gapeev and Shiryaev [37, 38] a generalization of the problems of sequential detection and sequential testing for the drift of a large class of diffusion processes is provided. Our results can be obtained as special cases of their extension, which must be used when the evolution of the posterior probability processes (3.2.13) and (3.3.9) depends on the current state of the observation process: this would be the case, if we weakened the assumption (3.1.2) allowing the signal-to-noise ratio function to be dependent on the current state of the observations.

Chapter 4

On the Martingale and Free-Boundary Approaches in Sequential Detection Problems with Exponential Penalty for Delay

We study the connection between the martingale and free-boundary approaches in sequential detection problems for the drift of a diffusion process, with constant signal-to-noise ratio function, under the assumption of exponential penalty for the delay. Although the two methods appear to be very different at first sight, the fascinating result of this analysis is that they are intimately related. By means of the solution of a suitable free-boundary problem, we derive the decomposition of the reward function into the product between a gain function of the weighted likelihood ratio process and a positive martingale inside the continuation region.

4.1 Introduction

The Bayesian sequential detection (or disorder) problem can be informally stated as follows: we sequentially observe a stochastic process $X = (X_t)_{t\geq 0}$, which changes some of its statistical features at an unknown random time ϑ. So, before and after ϑ, the processes $X^1 = (X_t^1)_{t\geq 0}$ and $X^2 = (X_t^2)_{t\geq 0}$, respectively, take place. It is assumed that ϑ is concentrated on zero, with probability $\pi \in [0, 1)$, and is exponentially distributed, with probability $(1 - \pi)$. The goal is to construct the optimal alarm time $\tau^\star$, which is, in a certain sense, "as close as possible" to ϑ. It means that $\tau^\star$ minimizes, over all the stopping times τ of X, the expectation of a proper reward function, which is a trade-off between the measure of the frequency of a false alarm $(\tau < \vartheta)$ and the measure of the delay in detecting correctly the disorder $(\tau \geq \vartheta)$. The described optimal stopping problem can be approached through two different methods, which are now briefly summarized.

The *martingale* approach is based on the reduction of the reward function into the product between a positive martingale, having unitary value at time zero, and a gain function f of the underlying sufficient statistic (that is, the process which gives all the information to determine the stopping rule), with f presenting a unique minimizer $x^\star$. In this way, at each stopping time, the expected reward is the expectation of the product between the martingale and the function

f, the latter evaluated at the level achieved by the sufficient statistic; then, it is shown that if the time $\tau^\star$ at which the sufficient statistic reaches $x^\star$ has finite expectation, the Bayesian value function (the infimum, over all the stopping times, of the expected reward) will coincide with the value $f(x^\star)$ and $\tau^\star$ will be optimal. This technique was introduced by Beibel and Lerche [10] and recently extended by Christensen and Irle [21].

The *free-boundary* approach relies on the formulation of a suitable system (the free-boundary problem), consisting of a functional equation, like a differential equation, and a certain number of supplementary conditions (smooth fit, continuous fit, normal entrance, etc.); they allow to identify the boundaries, which separate the so called continuation and stopping regions, and to explicit the Bayesian value function. After that the free-boundary problem has been solved, a verification theorem shows, with the help of standard arguments, that the solution to the previous analytical problem coincides with that of the original optimal stopping problem.

In the literature, solutions to sequential detection problems for continuous time stochastic processes, with continuous sample paths, are derived exploiting either the free-boundary or the martingale approach. Shiryaev [72, chapter 4.4] (see also Shiryaev [74]) solved the disorder problem for the drift of a Wiener process with linear penalty for the delay, reducing the original optimal stopping problem for the posterior probability process into a free-boundary problem for an ordinary differential operator. Extensions of this result are given by Gapeev and Peskir [36], in the finite horizon formulation, by Feinberg and Shiryaev [30], in the generalized Bayesian formulation, and by Gapeev and Shiryaev [38], for the Bayesian disorder problem for a wide class of diffusion processes. On the other hand, Beibel [9] solved the Shiryaev's Wiener disorder problem, assuming that the delay for declaring correctly the disorder was exponentially penalized: unlike Shiryaev, Beibel's method was based on the martingale approach.

Maintaining the assumption of exponential penalty for the delay, the aim of the present chapter is to construct a bridge between the martingale and free-boundary approaches in sequential detection problems for the drift of diffusion processes, with constant signal-to-noise ratio function. This link will be evident when we show that the decomposition of the reward function, which the martingale approach of Beibel relies on, can be derived from the solution of the free-boundary problem. The original idea of such a decomposition is due to Gapeev and Lerche [34], who underline the connection between the two approaches in discounted optimal stopping problems and then apply their results to the rational valuation of some kinds of perpetual American options. Here, we extend their idea to disorder problems.

The chapter is organized as follows: in Section 4.2, a formal description of the sequential detection problem is provided and the main quantities, common to the two approaches, are derived; in Section 4.3, the solution of the free-boundary problem is obtained and it is shown to solve the original optimal stopping problem; in Section 4.4, the Beibel's martingale approach is briefly revisited; in Section 4.5, we show how the solution of the free-boundary problem can be exploited to derive the gain function and the positive martingale inside the continuation region; Section 4.6 concludes with a summary discussion.

4.2 Problem description

In this section, we formally introduce the sequential detection problem for the drift of a diffusion process, with constant signal-to-noise ratio function. The stochastic differential equation, satisfied by a generic function of the posterior probability and weighted-likelihood ratio processes (defined below), is recovered. It will play a prominent role in Sections 4.3 and 4.4.

Let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, P_\pi)$ be a filtered probability space, where:

$$P_\pi := \pi P^0 + (1-\pi)\int_0^\infty \lambda e^{-\lambda s} P^s\, ds, \tag{4.2.1}$$

with $\pi \in [0,1)$ and $\lambda > 0$ known and fixed. Let ϑ be an $\mathscr{F}_0$-measurable random variable, having a zero-modified exponential distribution, with parameter λ: $P_\pi(\vartheta = 0) = \pi$ and $P_\pi(\vartheta > t|\vartheta > 0) = e^{-\lambda t}$. It follows that P^s, $s \geq 0$, is a probability measure under which $\vartheta = s$ almost surely (a.s.). On the same probability space, a standard Brownian motion $W = (W_t)_{t\geq 0}$ is defined and is assumed to be independent of ϑ.

Let $X = (X_t)_{t\geq 0}$ be a time-homogeneous diffusion process, whose drift moves from $\mu_0(\cdot)$ to $\mu_1(\cdot)$ at the time ϑ. ϑ is known as disorder time or change-point and represents the moment at which the process changes its characteristics (for instance, the time at which a production system goes out of control). It results that X solves the following stochastic differential equation:

$$dX_t = \Big(\mu_0(X_t) + \mathbf{1}_{\{t\geq\vartheta\}}\big(\mu_1(X_t) - \mu_0(X_t)\big)\Big)dt + \sigma(X_t)dW_t, \quad X_0 = x_0. \tag{4.2.2}$$

We assume that $\mu_i(\cdot)$, $i = 0, 1$, and $\sigma(\cdot) > 0$ are continuously differentiable functions and satisfy the usual conditions (such as, the linear growth and the global Lipschitz), ensuring that equation (4.2.2) has a unique strong solution under $\vartheta = s$: hence, $P_\pi(\cdot|\vartheta = s) = P^s(\cdot)$ is the distribution law of a time-homogeneous diffusion process, whose drift changes from $\mu_0(\cdot)$ to $\mu_1(\cdot)$ at time $\vartheta = s$, $s \geq 0$. It will be assumed that the signal-to-noise ratio function

$$\rho^2(x) := \left(\frac{\mu_1(x) - \mu_0(x)}{\sigma(x)}\right)^2 = \rho^2 \in \mathbb{R} \tag{4.2.3}$$

is constant, independent of the current observations.

Denote by $\mathscr{F}_t^X = \sigma(X_s : 0 \leq s \leq t)$ the σ-algebra generated by X up to t and let τ be a stopping time of X, that is, an $\mathscr{F}_t^X$-measurable random variable, $t \geq 0$. Through the sequential observation of the process, we want to detect as accurately as possible the disorder time and the following two requirements must be considered: to avoid false alarms and to find out as soon as possible the change-point. In other words, the goal is to solve the *Bayesian disorder problem*:

$$V(\pi) := \inf_\tau \big[P_\pi(\tau < \vartheta) + cE_\pi[e^{\alpha(\tau-\vartheta)^+} - 1]\big], \quad c, \alpha > 0. \tag{4.2.4}$$

$P_\pi(\tau < \vartheta)$ is the probability of a false alarm and $cE_\pi[e^{\alpha(\tau-\vartheta)^+} - 1]$ is the expected exponential penalty, when the alarm has been sound at $\tau > \vartheta$; α can be interpreted as the instantaneous interest rate, compounding the losses due to the delay in detecting correctly the disorder and c is a constant, such that $1/(c\alpha)$ is the average cost to be suffered for each false alarm. The formulation (4.2.4) was initially studied for discrete time processes by Poor [69].

Let $(\pi_t)_{t\geq 0}$ be the posterior probability process, with $\pi_t := P_\pi(\vartheta \leq t|\mathscr{F}_t^X)$ and $\pi_0 = \pi$; let $(\psi_t)_{t\geq 0}$ and $(\phi_t)_{t\geq 0}$ be the likelihood ratio and weighted likelihood ratio processes, respectively, given by:

$$\psi_t := \frac{\pi_t}{1-\pi_t}, \quad \phi_t := \frac{E_\pi[e^{\alpha(t-\vartheta)^+}|\mathscr{F}_t^X]}{1-\pi_t} - 1 = \frac{E_\pi[\mathbf{1}_{\{\vartheta\leq t\}}e^{\alpha(t-\vartheta)^+}|\mathscr{F}_t^X]}{1-\pi_t}. \tag{4.2.5}$$

Denote by $P^s|\mathscr{F}_t^X$, $s, t \geq 0$, the restriction of P^s on $\mathscr{F}_t^X$ and define by $L_t := d(P^0|\mathscr{F}_t^X)/d(P^t|\mathscr{F}_t^X)$ the Radon-Nikodým derivative of $P^0|\mathscr{F}_t^X$ with respect to $P^t|\mathscr{F}_t^X$. According to Girsanov's theorem:

$$L_t = \exp\left\{\int_0^t \frac{\mu_1(X_s)-\mu_0(X_s)}{\sigma^2(X_s)}\,dX_s - \frac{1}{2}\int_0^t \frac{\mu_1^2(X_s)-\mu_0^2(X_s)}{\sigma^2(X_s)}\,ds\right\}. \tag{4.2.6}$$

Then:

$$\frac{d(P^s|\mathscr{F}_t^X)}{d(P^t|\mathscr{F}_t^X)} = \frac{L_t}{L_s}\mathbf{1}_{\{s<t\}} + \mathbf{1}_{\{s\geq t\}}. \tag{4.2.7}$$

The application of Bayes theorem and (4.2.6)-(4.2.7) give:

$$P_\pi(\vartheta \leq s|\mathscr{F}_t^X) = \frac{\pi L_t + (1-\pi)\left\{\int_0^{s\wedge t}\lambda e^{-\lambda u}\frac{L_t}{L_u}\,du - e^{-\lambda(s\vee t)} + e^{-\lambda t}\right\}}{\pi L_t + (1-\pi)\left\{\int_0^t \lambda e^{-\lambda u}\frac{L_t}{L_u}\,du + e^{-\lambda t}\right\}}, \tag{4.2.8}$$

$$P_\pi(\vartheta = 0|\mathscr{F}_t^X) = \frac{\pi L_t}{\pi L_t + (1-\pi)\left\{\int_0^t \lambda e^{-\lambda u}\frac{L_t}{L_u}\,du + e^{-\lambda t}\right\}}. \tag{4.2.9}$$

Differentiating with respect to s, with $0 < s \leq t$, we have

$$P_\pi(\vartheta \in ds|\mathscr{F}_t^X) = \frac{(1-\pi)\lambda e^{-\lambda s}\frac{L_t}{L_s}\,ds}{\pi L_t + (1-\pi)\left\{\int_0^t \lambda e^{-\lambda u}\frac{L_t}{L_u}\,du + e^{-\lambda t}\right\}}. \tag{4.2.10}$$

The explicit expressions of π_t and ϕ_t can now be obtained combining their definitions and (4.2.10); some algebra yields:

$$\pi_t = \frac{\pi L_t + (1-\pi)\int_0^t \lambda e^{-\lambda s}\frac{L_t}{L_s}\,ds}{\pi L_t + (1-\pi)\left\{\int_0^t \lambda e^{-\lambda u}\frac{L_t}{L_u}\,du + e^{-\lambda t}\right\}}, \tag{4.2.11}$$

$$\phi_t = e^{(\alpha+\lambda)t}L_t\Big(\frac{\pi}{1-\pi} + \int_0^t \frac{\lambda e^{-(\alpha+\lambda)s}}{L_s}\,ds\Big), \tag{4.2.12}$$

From (4.2.5), one can notice that the explicit expression of ψ_t can be straightforwardly obtained letting α go to zero in (4.2.12). For a further deepening on these formulae, see Shiryaev [72] and Bayraktar and Dayanik [7].

In order to recover the evolution of a generic function $v(\pi_t, \phi_t)$, we need to employ some techniques of stochastic calculus. Let

$$q(x) := \int_{x_0}^x \frac{\mu_1(\omega)-\mu_0(\omega)}{\sigma^2(\omega)}\,d\omega. \tag{4.2.13}$$

By virtue of Itô's formula applied to $q(X_t)$:

$$\int_0^t \frac{\mu_1(X_s)-\mu_0(X_s)}{\sigma^2(X_s)}\,dX_s = \int_{x_0}^{X_t} \frac{\mu_1(\omega)-\mu_0(\omega)}{\sigma^2(\omega)}\,d\omega \qquad (4.2.14)$$
$$-\frac{1}{2}\int_0^t \frac{\partial}{\partial x}\Big(\frac{\mu_1(x)-\mu_0(x)}{\sigma^2(x)}\Big)\Big|_{x=X_s}\sigma^2(X_s)\,ds.$$

Taking into account (4.2.6) and (4.2.14), Itô's formula can be applied to L_t:

$$dL_t = \frac{\mu_1(X_t)-\mu_0(X_t)}{\sigma^2(X_t)}L_t(dX_t-\mu_0(X_t)dt), \quad L_0 = 1. \qquad (4.2.15)$$

At this point, the evolution of the weighted likelihood ratio process is obtained by applying Itô's formula to (4.2.12); the evolution of the posterior probability process can be recovered first determining the evolution of ψ_t and then applying Itô's formula to $\pi_t = \psi_t/(1+\psi_t)$. Hence, we have

$$d\phi_t = \big(\lambda + (\alpha+\lambda)\phi_t + \rho^2\pi_t\phi_t\big)dt + \rho\phi_t d\widetilde{W}_t, \quad \phi_0 = \phi = \frac{\pi}{1-\pi}, \qquad (4.2.16)$$

$$d\pi_t = \lambda(1-\pi_t)dt + \rho\pi_t(1-\pi_t)d\widetilde{W}_t, \quad \pi_0 = \pi, \qquad (4.2.17)$$

where $\widetilde{W} = (\widetilde{W}_t)_{t\geq 0}$, with

$$\widetilde{W}_t := \int_0^t \frac{dX_s}{\sigma(X_s)} - \int_0^t \frac{(1-\pi_s)\mu_0(X_s)+\pi_s\mu_1(X_s)}{\sigma(X_s)}\,ds, \qquad (4.2.18)$$

is a standard Brownian motion under P_π (see Liptser and Shiryaev [54, Th. 7.12, p. 258]).

Let $C_S^{2,2}$, with $S := [0,1)\times[0,\infty)$, be the class of two-times continuously differentiable functions on S, let $v \in C_S^{2,2}$ and set $y_t := (\pi_t,\phi_t)$. Then, standard results based on the multidimensional version of Itô's formula imply that the evolution of $v(y_t)$ is given by:

$$dv(y_t) = (\mathbb{L}v)(y_t)dt + \rho\Big(\pi_t(1-\pi_t)\frac{\partial v}{\partial\pi}(y_t) + \phi_t\frac{\partial v}{\partial\phi}(y_t)\Big)d\widetilde{W}_t, \qquad (4.2.19)$$

where $\mathbb{L}$ is the infinitesimal generator of $(\pi_t,\phi_t)_{t\geq 0}$ and is given by

$$\mathbb{L} := \lambda(1-\pi)\frac{\partial}{\partial\pi} + \big(\lambda + (\lambda+\alpha)\phi + \rho^2\pi\phi\big)\frac{\partial}{\partial\phi} \qquad (4.2.20)$$
$$+\frac{1}{2}\rho^2\Big(\pi^2(1-\pi)^2\frac{\partial^2}{\partial\pi^2} + 2\pi(1-\pi)\phi\frac{\partial^2}{\partial\pi\partial\phi} + \phi^2\frac{\partial^2}{\partial\phi^2}\Big),$$

for all $(\pi,\phi)\in S$.

4.3 The free-boundary approach

In this section, the sequential detection problem is solved using the free-boundary approach. It is basically based on two steps: in the first one, a free-boundary problem is formulated and a solution is determined; in the second one, it is shown that the solution to the previous free-boundary problem solves the optimal stopping problem, too.

In order to construct the appropriate free-boundary problem, we need to rewrite the Bayesian value function (4.2.4) according to the Mayer-Lagrange formulation (see Chapter 2 or Peskir and

Shiryaev [67, p. 124]). Standard arguments (see Bayraktar and Dayanik [7]) show that (4.2.4) is equivalent to

$$V(\pi,\phi) = \inf_{\tau} E_{\pi,\phi}\left[1-\pi_\tau + \int_0^\tau (1-\pi_t)c\alpha\phi_t\,dt\right]. \tag{4.3.1}$$

where $P_{\pi,\phi}$ is the probability measure under which the process $(\pi_t,\phi_t)_{t\geq 0}$ starts at some point $(\pi,\phi)\in[0,1)\times[0,\infty)$. Following the reasoning in Gapeev and Shiryaev [38, section 3.1], a simple application of Itô's formula to $1-\pi_t$ yields

$$1-\pi_t = 1-\pi-\int_0^t \lambda(1-\pi_s)\,ds + N_t, \tag{4.3.2}$$

where $N=(N_t)_{t\geq 0}$, defined by

$$N_t := -\rho\int_0^t \pi_s(1-\pi_s)d\widetilde{W}_s, \tag{4.3.3}$$

is a continuous martingale under $P_{\pi,\phi}$, so that for any stopping time τ with finite expectation, according to the optional sampling theorem, $E_{\pi,\phi}[N_\tau]=0$. Hence, (4.3.1) and (4.3.2) imply that

$$V(\pi,\phi) = 1-\pi+\inf_{\tau} E_{\pi,\phi}\left[\int_0^\tau (1-\pi_t)(c\alpha\phi_t-\lambda)dt\right]. \tag{4.3.4}$$

It is evident that it is never optimal to stop (that is, declare the disorder) when $\phi_t<\lambda/(c\alpha)$, $t\geq 0$; then, all the pairs (π,ϕ), such that $\phi<\lambda/(c\alpha)$, belong to the continuation region

$$C := \{(\pi,\phi)\in[0,1)\times[0,\infty) : V(\pi,\phi)<1-\pi\}. \tag{4.3.5}$$

It follows that there exists a constant $A^\star\geq\lambda/(c\alpha)$, such that the continuation region assumes the form

$$C = \{(\pi,\phi)\in[0,1)\times[0,\infty) : \phi<A^\star\}, \tag{4.3.6}$$

implying that the stopping region is the closure of the set:

$$D := \{(\pi,\phi)\in[0,1)\times[0,\infty) : \phi>A^\star\}. \tag{4.3.7}$$

Hence,

$$\tau^\star := \inf\{t\geq 0 : \phi_t\geq A^\star\} \tag{4.3.8}$$

should be optimal in (4.3.1), with $(\phi_t)_{t\geq 0}$ that turns out to be the sufficient statistic of the problem (4.3.1), in the sense that it completely specifies the stopping rule.

By standard results on stochastic differential equations, the two-dimensional process (π_t,ϕ_t), solving the system (4.2.16) and (4.2.17), is a time-homogeneous strong Markov process. Thus, together with the general theory of optimal stopping problems (see Chapter 2 or, e.g., Peskir and Shiryaev [67, Chap. 3 and 4] or Shiryaev [72, Chap. 3]), we are naturally led to formulate the following free-boundary problem for the unknown function V and the unknown boundary

$A^\star$:

$$(\mathbb{L}V)(\pi,\phi) = -(1-\pi)c\alpha\phi, \quad (\pi,\phi)\in C, \tag{4.3.9}$$

$$V(\pi,\phi)|_{\phi=A^\star} = 1-\pi \quad \textit{(instantaneous stopping)}, \tag{4.3.10}$$

$$\left.\frac{\partial V(\pi,\phi)}{\partial\phi}\right|_{\phi=A^\star} = 0 \quad \textit{(smooth fit)}, \tag{4.3.11}$$

$$V(\pi,\phi) = 1-\pi, \quad (\pi,\phi)\in D, \tag{4.3.12}$$

$$V(\pi,\phi) < 1-\pi, \quad (\pi,\phi)\in C. \tag{4.3.13}$$

Justified by the form of the infinitesimal generator (4.2.20), we look for a solution to the above system of the following kind:

$$V(\pi,\phi) = (1-\pi)h(\phi), \tag{4.3.14}$$

with h being a twice continuously differentiable function. Since we have that $0 \le V(\pi,\phi) = (1-\pi)h(\phi) \le 1-\pi$, $h(\cdot)$ must be a bounded function, satisfying:

$$0 \le h(\phi) \le 1. \tag{4.3.15}$$

The expressions (4.2.20), (4.3.9) and (4.3.14) generate the following second order linear differential equation:

$$\frac{\rho^2}{2}\phi^2 h''(\phi) + \big(\lambda + (\lambda+\alpha)\phi\big)h'(\phi) - \lambda h(\phi) = -c\alpha\phi. \tag{4.3.16}$$

In order to solve (4.3.16), we proceed, as usual, finding two linearly independent solutions of the associated homogeneous equation and one particular solution of the complete equation.

Adopting the same notation used by Beibel [9, Sec. 2], set $\alpha' := 2\alpha/\rho^2$, $\lambda' := 2\lambda/\rho^2$ and

$$\gamma_1 := \frac{1}{2}(\lambda'+\alpha'-1) + \sqrt{\frac{1}{4}(\lambda'+\alpha'-1)^2 + \lambda'}, \tag{4.3.17}$$

$$\gamma_2 := 1 - \frac{1}{2}(\lambda'+\alpha'-1) + \sqrt{\frac{1}{4}(\lambda'+\alpha'-1)^2 + \lambda'}. \tag{4.3.18}$$

Then, the homogeneous equation associated to (4.3.16) can be equivalently written as:

$$\begin{aligned} &h'' + \left(\frac{2\gamma_1}{\phi} + (\gamma_1+\gamma_2)\frac{w'}{w} - w' - \frac{w''}{w'}\right)h' \\ &+\left(\left((\gamma_1+\gamma_2)\frac{w'}{w} - w' - \frac{w''}{w'}\right)\frac{\gamma_1}{\phi} + \frac{\gamma_1(\gamma_1-1)}{\phi^2} - \gamma_1\frac{w'^2}{w}\right)h = 0, \end{aligned} \tag{4.3.19}$$

where $w(\phi) = \lambda'/\phi$. (4.3.19) is a confluent hypergeometric differential equation having two linearly independent solutions, given by (see Abramowitz and Stegun [1, p. 505]):

$$g_1(\phi) = \left(\frac{\phi}{\lambda'}\right)^{-\gamma_1} M\left(\gamma_1, \gamma_1+\gamma_2, \frac{\lambda'}{\phi}\right), \quad g_2(\phi) = \left(\frac{\phi}{\lambda'}\right)^{-\gamma_1} U\left(\gamma_1, \gamma_1+\gamma_2, \frac{\lambda'}{\phi}\right),$$

where $M(a,b,z)$ and $U(a,b,z)$ are known as confluent hypergeometric functions of the first and second kind, respectively. Equivalently, they can be represented in integral form:

$$g_1(\phi) = \frac{\int_0^{\frac{\lambda'}{\phi}} e^u u^{\gamma_1-1}(\lambda'-\phi u)^{\gamma_2-1}\,du}{\lambda'^{\gamma_2-1}B(\gamma_1,\gamma_2)}, \tag{4.3.20}$$

$$g_2(\phi) = \frac{\int_0^\infty e^{-u} u^{\gamma_1 - 1} (\lambda' + \phi u)^{\gamma_2 - 1}\, du}{\lambda'^{\gamma_2 - 1} \Gamma(\gamma_1)}, \tag{4.3.21}$$

where $B(\cdot, \cdot)$ and $\Gamma(\cdot)$ are the beta and gamma functions.

A particular solution, $g_p(\phi)$, of the complete equation (4.3.16) can be searched among the first degree polynomials. One easily obtains $g_p(\phi) = -c(\phi + 1)$. The general solution of (4.3.16) is therefore:

$$h(\phi) = k_1 g_1(\phi) + k_2 g_2(\phi) - c(\phi + 1), \tag{4.3.22}$$

where k_1 and k_2 are constants to be determined. Notice that as $\phi \downarrow 0$, $g_1(\phi) \to +\infty$: because of (4.3.15), we must have $k_1 = 0$; setting $g \equiv g_2$ and $f \equiv k_2$ we have from (4.3.14):

$$V(\pi; \phi) = (1 - \pi)\big(f g(\phi) - c(\phi + 1)\big). \tag{4.3.23}$$

The constant f can be obtained through the instantaneous stopping condition (4.3.10):

$$f = \frac{1 + c(1 + A^\star)}{g(A^\star)}. \tag{4.3.24}$$

Let $x \mapsto f(x)$ be the map defined by

$$f(x) = \frac{c(x + 1) + 1}{g(x)} \tag{4.3.25}$$

and let

$$V(\pi, \phi; A) := (1 - \pi)\big(f(A) g(\phi) - c(\phi + 1)\big). \tag{4.3.26}$$

The application of the smooth fit condition (4.3.11) to (4.3.26) and the expression (4.3.25) imply that the optimal boundary $A^\star$ satisfies:

$$g'(A^\star) = c \frac{g(A^\star)}{1 + c(1 + A^\star)}. \tag{4.3.27}$$

The next proposition proves the uniqueness of the solution to the above equation.

Proposition 4.3.1 *There exists a unique $A^\star \in (0, \infty)$ satisfying* (4.3.27)*.*

Proof. Simple algebraic passages allow to rewrite (4.3.27) as:

$$\frac{g(z)}{g'(z)} - z - 1 = \frac{1}{c}. \tag{4.3.28}$$

Easily, one can see from (4.3.17) and (4.3.18) that $\gamma_1 > 0$, $\gamma_2 - 1 > 0$ and

$$\gamma_2 - 2 = -\frac{1}{2}(\lambda' + \alpha' + 1) + \sqrt{\frac{1}{4}(\lambda' + \alpha' + 1)^2 - \alpha'} < 0. \tag{4.3.29}$$

Thus,

$$g'(z) = \frac{(\gamma_2 - 1) \int_0^\infty e^{-u} u^{\gamma_1} (\lambda' + z u)^{\gamma_2 - 2}\, du}{(\lambda')^{\gamma_2 - 1} \Gamma(\gamma_1)} > 0, \tag{4.3.30}$$

$$g''(z) = \frac{(\gamma_2 - 1)(\gamma_2 - 2) \int_0^\infty e^{-u} u^{\gamma_1 + 1} (\lambda' + z u)^{\gamma_2 - 3}\, du}{(\lambda')^{\gamma_2 - 1} \Gamma(\gamma_1)} < 0, \tag{4.3.31}$$

for any $z > 0$. Moreover, $g(0) = g'(0) = 1$ and, as shown also in Beibel [9, Sec. 2],

$$g(z) = z^{\gamma_2 - 1} \frac{\Gamma(\gamma_1 + \gamma_2 - 1)}{\lambda'^{\gamma_2 - 1}\Gamma(\gamma_1)} (1 + o(1)), \tag{4.3.32}$$

$$g'(z) = (\gamma_2 - 1) z^{\gamma_2 - 2} \frac{\Gamma(\gamma_1 + \gamma_2 - 1)}{\lambda'^{\gamma_2 - 1}\Gamma(\gamma_1)} (1 + o(1)), \tag{4.3.33}$$

so that:

$$\frac{g(z)}{g'(z)} = \frac{z}{\gamma_2 - 1}(1 + o(1)), \tag{4.3.34}$$

as $z \to +\infty$. Denote by $l(z)$ the left-hand side of (4.3.28); then, $l(0) = 0$, $\lim_{z\to\infty} l(z) = +\infty$ and

$$l'(z) = \frac{-g(z)g''(z)}{g'^2(z)} > 0, \tag{4.3.35}$$

for any $z > 0$, from which the assertion. ∎

The first step of the free-boundary approach is concluded; the next one is to show that the solution to the free-boundary problem (4.3.9)-(4.3.13) is also a solution of the optimal stopping problem (4.3.1).

Before to prove the verification theorem, we need the following results.

Proposition 4.3.2 *Let $\tau_x := \inf\{t \geq 0 : \phi_t \geq x\}$ be the first exit time of the process $(\phi_t)_{t\geq 0}$ from the interval $[0, x]$, $x \geq 0$. Then, $E_{\pi,\phi}[\tau_x] < \infty$, for every $(\pi, \phi) \in [0, 1) \times [0, \infty)$.*

Proof. The proof is essentially based on the final part of the proof of Lemma 1 in Beibel [9]. If $x \leq \phi$, obviously $E_{\pi,\phi}[\tau_x] = 0$. So assume that $x > \phi$. From the definition of ϕ_t in (4.2.5) and the fact that $E_\pi[e^{\alpha(t-\vartheta)^+} | \mathscr{F}_t^X] \geq 1$, it results that $\tau_x \leq \inf\{t \geq 0 : 1/(1 - \pi_t) \geq 1 + x\}$. Then,

$$P_{\pi,\phi}(\tau_x > t) \leq P_{\pi,\phi}\big((1 + x)(1 - \pi_t) > 1\big) \leq (1 + x)E_{\pi,\phi}[1 - \pi_t] = (1 + x)(1 - \pi)e^{-\lambda t},$$

from which the assertion. ∎

Proposition 4.3.3 *The stopped process $(M_{t\wedge\tau^\star})_{t\geq 0}$, defined by*

$$M_t := \rho \int_0^t \left(\pi_s(1 - \pi_s)\frac{\partial V}{\partial \pi}(\pi_s, \phi_s; A^\star) + \phi_s \frac{\partial V}{\partial \phi}(\pi_s, \phi_s; A^\star) \right) d\widetilde{W}_s, \tag{4.3.36}$$

is a martingale under $P_{\pi,\phi}$, for all $(\pi, \phi) \in [0, 1) \times [0, \infty)$.

Proof. Denoted by $h(\phi) = f(A^\star)g(\phi) - c(\phi + 1)$, the expression (4.3.36) is equivalent to

$$M_t := \rho \int_0^t (1 - \pi_s) \big(\phi_s h'(\phi_s) - \pi_s h(\phi_s)\big) \, d\widetilde{W}_s. \tag{4.3.37}$$

Using the Burkholder-Davis-Gundy inequality (see Karatzas and Shreve [45, Th. 3.28, p. 166], with $m = 1$) and the fact that $\phi_t \leq A^\star$ on $0 \leq t \leq \tau^\star$, we have

$$E_{\pi,\phi}\left[\Big(\max_{0\leq s\leq\tau^\star} |M_s| \Big)^2 \right] \leq K \int_0^{\tau^\star} (1 - \pi_s)^2 \big(\phi_s h'(\phi_s) - \pi_s h(\phi_s)\big)^2 \, ds \leq H E_{\pi,\phi}[\tau^\star] < \infty,$$

where K and H are two constants and the last inequality is a consequence of Proposition 4.3.2. This completes the proof. ∎

Now, we can state the verification theorem.

Theorem 4.3.1 *The Bayesian risk function* (4.3.1) *is explicitly given by*

$$V(\pi,\phi) = \begin{cases} V(\pi,\phi;A^\star), & \text{if} \quad (\pi,\phi) \in [0,1) \times [0,A^\star), \\ 1-\pi, & \text{if} \quad (\pi,\phi) \in [0,1) \times [A^\star,\infty), \end{cases} \tag{4.3.38}$$

where $V(\pi,\phi;A^\star)$ *is given through* (4.3.26) *and* $A^\star$ *is the unique solution of* (4.3.27). *The stopping time* $\tau^\star$ *defined by* (4.3.8) *is optimal in* (4.3.1).

Proof. In order to show that the functions (4.3.1) and (4.3.38) are the same, denote the former by $V^\star(\pi;\phi)$.

By construction, $V(\pi,\phi)$ from (4.3.38) is two-times continuously differentiable on $[0,A^\star) \cup (A^\star\infty)$ and one-time continuously differentiable at $A^\star$ with respect to ϕ (because of (4.3.10) and (4.3.11)); moreover, the Lebesgue measure of those $t>0$ for which $\phi_t = A^\star$ is zero. Thus, Itô's formula applies in its standard form to $V(\pi,\phi)$ (see (4.2.19)):

$$V(\pi_t,\phi_t) = V(\pi,\phi) + \int_0^t (\mathbb{L}V)(\pi_s,\phi_s)\mathbf{1}_{\{\phi_s \neq A^\star\}}\, ds + \widetilde{M}_t, \tag{4.3.39}$$

where $\widetilde{M} = (\widetilde{M}_t)_{t\geq 0}$, defined by

$$\widetilde{M}_t := \rho \int_0^t \left(\pi_s(1-\pi_s)\frac{\partial V}{\partial \pi}(\pi_s,\phi_s) + \phi_s \frac{\partial V}{\partial \phi}(\pi_s,\phi_s) \right) d\widetilde{W}_s, \tag{4.3.40}$$

is a continuous local martingale under $P_{\pi,\phi}$. We observe that the expression into the brackets equals the bounded value $-\pi(1-\pi)$, if $(\pi,\phi) \in D$; this fact and the result of Proposition 4.3.3 imply that $\widetilde{M}$ is a martingale.

From (4.2.20), one can observe that $\mathbb{L}(1-\pi) = -\lambda(1-\pi)$. This consideration, the conditions (4.3.9) and (4.3.12) and the fact that $\phi > A^\star \geq \lambda/(c\alpha)$ on D imply

$$(\mathbb{L}V)(\pi,\phi) \geq -(1-\pi)c\alpha\phi, \tag{4.3.41}$$

for every $(\pi,\phi) \in [0,1) \times [0,\infty)$. The inequality (4.3.41), $V \leq 1-\pi$ (from (4.3.12) and (4.3.13)) and (4.3.39) show that

$$\begin{aligned} 1-\pi_t + \int_0^t (1-\pi_s)c\alpha\phi_s\, ds &\geq V(\pi_t,\phi_t) + \int_0^t (1-\pi_s)c\alpha\phi_s\, ds \\ &\geq V(\pi,\phi) + \widetilde{M}_t, \qquad t \geq 0. \end{aligned} \tag{4.3.42}$$

Hence, for any stopping time τ with finite expectation and the optional sampling theorem (so that, $E_{\pi,\phi}[\widetilde{M}_\tau] = 0$), one has

$$E_{\pi,\phi}\left[1-\pi_\tau + \int_0^\tau (1-\pi_t)c\alpha\phi_t\, dt\right] \geq V(\pi,\phi), \tag{4.3.43}$$

for all $(\pi,\phi) \in [0,1) \times [0,\infty)$. This shows that $V^\star \geq V$. Now, if we replace τ by $\tau^\star$, then, by (4.3.9),(4.3.10), (4.3.42) and the definition of $\tau^\star$, we find that

$$1-\pi_{\tau^\star} + \int_0^{\tau^\star} (1-\pi_t)c\alpha\phi_t\, dt = V(\pi,\phi) + \widetilde{M}_{\tau^\star}. \tag{4.3.44}$$

Taking the expectation on both sides of (4.3.44), since $\tau^\star$ is of finite expectation (so that the optional sampling theorem applies) and because of (4.3.43), it results $V^\star = V$. It also proves the optimality of $\tau^\star$. ∎

4.4 The martingale approach of Beibel

In this section, we review the martingale approach used by Beibel [9]. It relies on the decomposition of the reward function into the product between a positive martingale, with starting value equal to one, and a gain function f of the weighted likelihood ratio process $(\phi_t)_{t\geq 0}$, having a unique minimizer: it is optimal to stop as soon as ϕ_t reaches the minimizing argument of f.

Using the law of iterated expectation, it is easy to see that the Bayesian disorder problem (4.2.4) can be equivalently written as

$$V(\pi,\phi) = \inf_{\tau} E_{\pi,\phi}\left[R(\pi_\tau,\phi_\tau) - c\right], \tag{4.4.1}$$

where $R(\pi,\phi) := 1-\pi + c(1-\pi)(\phi+1)$ is the so called reward function. In order to be tight to Beibel [9], where it was assumed $\pi = \phi = 0$ (that is, the disorder time ϑ follows an exponential distribution of parameter $\lambda > 0$), we set $V := V(0,0)$ and $E := E_{0,0}$. Of course, the assumption of $0 \leq \pi < 1$ does not modify the line of arguments that we are going to analyze.

Collecting the term $1-\pi$ in $R(\pi,\phi)$, expression (4.4.1) becomes

$$V = \inf_{\tau} E\left[m_\tau f(\phi_\tau)\right] - c, \tag{4.4.2}$$

where $m = (m_t)_{t\geq 0}$ is defined by $m_t := (1-\pi_t)g(\phi_t)$, $f(x)$ is the gain function given by (4.3.25) and $g(\cdot)$ is a function to be found, such that m is a martingale, with $m_0 = 1$. The next theorem is the main result contained in Beibel [9].

Theorem 4.4.1 *Let m be a martingale, with $m_0 = 1$, $f(\cdot)$ be a function, with unique minimizer in $A^\star$, and let $\tau^\star := \inf\{t \geq 0 : \phi_t \geq A^\star\}$ be of finite expectation. Then, for every stopping time τ of finite expectation, $E[m_\tau f(\phi_\tau)] - c \geq f(A^\star) - c = V$ and $\tau^\star$ is optimal in* (4.4.2).

The proof is very simple: we just need to notice that for every stopping time τ of finite expectation, $E[m_\tau f(\phi_\tau)] \geq f(A^\star)E[m_\tau] = f(A^\star)$, because of the optional sampling theorem. The continuous sample path of $(\phi_t)_{t\geq 0}$, evident from (4.2.12) and (4.2.16), implies that $f(\phi_{\tau^\star}) = f(A^\star)$. Since $\tau^\star$ has finite expectation, according to Proposition 4.3.2, the result follows.

At this point, one needs to find a function $g(\cdot)$ such that $m = (m_t)_{t\geq 0}$ is a martingale, with $m_0 = 1$. In Section 4.2, the evolution of whatever "well behaved"' function $v(\cdot,\cdot)$ of $(\pi_t)_{t\geq 0}$ and $(\phi)_{t\geq 0}$ has been found and is given by (4.2.19). It is known that if $v(\cdot,\cdot)$ satisfies

$$\mathbb{L}v = 0, \tag{4.4.3}$$

then $(v(\pi_t,\phi_t))_{t\geq 0}$ is a continuous local martingale. $\mathbb{L}$ is the infinitesimal generator of $(\pi_t,\phi_t)_{t\geq 0}$ and is given by (4.2.20).

Replacing $v(\cdot,\cdot)$ by $(1-\pi)g(\phi)$ in (4.4.3), we get the second order differential equation, represented by the homogeneous equation associated to (4.3.16). Its two fundamental solutions are given (4.3.20) and (4.3.21), with the former being discarded, because of its lack of boundedness as $\phi \downarrow 0$. Therefore, set the function $g(\cdot)$, in $m = (m_t)_{t\geq 0}$, equal to (4.3.21). Observe that, since $\pi = 0$, $\phi = \pi/(1-\pi) = 0$ and $g(0) = 1$, then $m_0 = 1$ (in any case, if π were different from zero, in order to get $m_0 = 1$, we could modify in a proper manner $g(\cdot)$, because it is determined up to a multiplicative constant).

Coming back to equation (4.2.19), we see that $m_t = 1 + Q_t$, $t \geq 0$, where

$$Q_t := \rho \int_0^t (1 - \pi_s) \left(\phi_s g'(\phi_s) - \pi_s g(\phi_s) \right) \, d\widetilde{W}_s. \tag{4.4.4}$$

The same arguments used in Proposition 4.3.3 prove that $(Q_{t \wedge \tau^\star})_{t \geq 0}$ is a martingale, so that the same holds for $(m_{t \wedge \tau^\star})_{t \geq 0}$. Finally, the fact that $f(x) = (c(x+1)+1)/g(x)$ has a unique minimizer $A^\star > 0$ is proved in Lemma 2 of Beibel [9].

4.5 Connection of the two approaches

In this section, we see why the free-boundary and martingale approaches are strictly connected: especially, we show how the gain function and the positive martingale in the continuation region can be deduced from the solution of the free-boundary problem.

We can easily observe that the two approaches lead to the same solution: from (4.3.38) in Theorem 4.3.1, with $\pi = \phi = 0$, and Theorem 4.4.1, we see that $V(0,0) = V = f(A^\star) - c$, where $A^\star$ is determined as unique solution of the equation (4.3.27), which coincides with the equation $f'(x) = 0$. In other words, the solution $A^\star$ of (4.3.27) is a critical point of $f(x)$.

The most fascinating aspect is that the Beibel's decomposition of the reward function into the product between a martingale $m^\star = (m^\star_t)_{t \geq 0}$ (defined below), with $m^\star = m$ over the continuation region, and the gain function f can be obtained via the free-boundary method. This connection is highlighted in the following theorem.

Theorem 4.5.1 *The function $V(\pi, \phi; A^\star)$ defined in (4.3.26) admits the representation*

$$V(\pi, \phi; A^\star) = f(A^\star) E_{\pi,\phi}[m^\star_{\tau_\star}] - c, \quad (\pi, \phi) \in [0,1) \times [0, A^\star), \tag{4.5.1}$$

where $m^\star = (m^\star_t)_{t \geq 0}$, with $m^\star_t := (1 - \pi_{t \wedge \tau^\star}) g(\phi_{t \wedge \tau^\star})$, is a positive martingale and $A^\star$ is the unique global minimum of $f(x)$ on $(0, \infty)$.

Proof. Simple calculations or a direct comparison between (4.3.1) and (4.4.1) show that

$$\int_0^t (1 - \pi_s) c \alpha \phi_s \, ds = c(1 - \pi_t)(\phi_t + 1) - c. \tag{4.5.2}$$

Itô's formula applied to $V(\pi_{t \wedge \tau^\star}, \phi_{t \wedge \tau^\star}; A^\star)$ yields

$$\begin{aligned} V(\pi_{t \wedge \tau^\star}, &\phi_{t \wedge \tau^\star}; A^\star) \\ &= V(\pi, \phi; A^\star) + \int_0^{t \wedge \tau^\star} (\mathbb{L}V)(\pi_s, \phi_s; A^\star) \mathbf{1}_{\{\phi_s \neq A^\star\}} \, dt + M_{t \wedge \tau^\star}, \end{aligned} \tag{4.5.3}$$

where M_t is given by (4.3.36). According to (4.3.26),

$$V(\pi_{t \wedge \tau^\star}, \phi_{t \wedge \tau^\star}; A^\star) = (1 - \pi_{t \wedge \tau^\star}) \left(f(A^\star) g(\phi_{t \wedge \tau^\star}) - c(\phi_{t \wedge \tau^\star} + 1) \right). \tag{4.5.4}$$

Therefore, the definition of $\tau^\star$, (4.3.9) and (4.5.2)-(4.5.4) imply that

$$V(\pi,\phi;A^\star) + M_{t\wedge\tau^\star} = (1-\pi_{t\wedge\tau^\star})g(\phi_{t\wedge\tau^\star})f(A^\star) - c. \tag{4.5.5}$$

Since $(M_{t\wedge\tau^\star})_{t\geq 0}$ is a martingale, as shown in Proposition 4.3.3, and $\tau^\star$ has finite expected value, according to Proposition 4.3.2, applying the expectation to both sides of (4.5.5) and using the optional sampling theorem, we have

$$V(\pi,\phi;A^\star) = f(A^\star)E_\pi[(1-\pi_{t\wedge\tau^\star})g(\phi_{t\wedge\tau^\star})] - c. \tag{4.5.6}$$

This also proves that $m^\star = (m^\star_t)_{t\geq 0}$ is a martingale.

It remains to show that $A^\star$ is the unique global minimum on $(0,\infty)$ of $f(x)$. Suppose there exists a certain $A > 0$ such that $f(A) < f(A^\star)$: it would result that $V(\pi,\phi;A) < V(\pi,\phi;A^\star)$, which contradicts the optimality of $\tau_\star$. So, it must result $f(A) \geq f(A^\star)$, for any $A > 0$. Since $f'(x) = 0$ coincides with the equation (4.3.27) whose solution is unique, according to Proposition 4.3.1, the proof is complete. ∎

4.6 Conclusions

We have analyzed the connection between the martingale and free-boundary approaches in sequential detection problems for the drift of diffusion processes, which are characterized by a constant signal-to-noise ratio function. We concentrated on the case of exponential delay penalty, because as shown also in Beibel [9, Remark 2], it generalizes the linear delay penalty.

We have shown in Theorem 4.5.1 that the Beibel's decomposition of the reward function into the product between a positive martingale and the gain function of the sufficient statistic (the weighted likelihood ratio process) can be derived from the free-boundary method.

One can notice that the martingale approach relies on the continuous sample paths of the sufficient statistic, since the first time at which the weighted likelihood ratio process exceeds the optimal boundary coincides with the first time at which the weighted likelihood ratio process hits the optimal boundary. It means that the Bayesian value function can be represented according to Theorem 4.4.1. When one moves toward sequential detection problems involving stochastic processes with discontinuous sample paths (see Bayraktar et al. [6], Bayraktar and Dayanik [7], Bayraktar et al. [8], Dayanik [22], Gapeev [33] and Peskir and Shiryaev [66]), the sufficient statistic(s) may have jumps: in these cases, the representation of the Bayesian value function provided in Theorem 4.4.1 fails to hold and the resolution of the optimal stopping problem can be obtained via the free-boundary approach.

In summary, the martingale approach is easy to understand, but cannot be used when the sufficient statistic has discontinuous sample paths. On the other hand, the free-boundary approach is less intuitive, but allows us to specify the components of the martingale approach and can be also exploited when we observe processes with discontinuous trajectories.

Chapter 5

Sequential Testing Problems for Lévy Processes

We present the sequential testing of two simple hypotheses for a large class of Lévy processes. As usual in this framework, the initial optimal stopping problem is reduced to a free-boundary problem, solved through the principles of the smooth and/or continuous fit. The well known solutions of the Wiener and the Poisson sequential testing can be derived from our procedure. The exact solution for sequentially testing two simple hypotheses concerning the parameter p, $0 < p < 1$, of a negative binomial process is explicitly given.

5.1 Introduction

Let $X = (X_t)_{t \geq 0}$ be a Lévy process, having Lévy-Khintchine triplet $g = \{\gamma, \sigma^2,\ v\}$, where σ^2 and v are, respectively, the gaussian component and the Lévy measure. At time $t = 0$ we begin to observe X: we just know that its triplet g is either $g_0 = \{\gamma_0, \sigma^2, v_0\}$, with probability $1 - \pi$, or $g_1 = \{\gamma_1, \sigma^2, v_1\}$, with probability π. Hence, we want to decide optimally, that is, as soon as possible and minimizing a given risk function, if the process X is characterized by g_0 or g_1. In this chapter, we only refer to the Bayesian formulation of the problem, because the proof of the optimality of the Wald's sequential probability ratio test (SPRT) in the variational formulation (where no probabilistic assumption is formulated about the realization of g at time 0, see, e.g., Wald [79, 80] and Wald and Wolfowitz [81]) can be obtained exploiting the solution of the Bayesian problem and will be discussed in the next chapter.

Probems of sequential testing of two simple hypotheses for continuous-time processes were introduced by Dvoretzky et al. [28], who showed the optimality of the SPRT for the mean and the intensity of a Wiener and Poisson process, respectively. The Bayesian sequential testing of the drift of a Wiener process is due to Shiryaev [72, Sec. 4.2]: reasoning in terms of the a posteriori probability process, the initial optimal stopping problem was reduced to a free-boundary Stephan problem, for a second order differential operator, which was solved using the principle of smooth fit. Despite some attempts to solve the Bayesian sequential testing problem for the intensity of a Poisson process, see for example Romberg [70], the literature had to wait for almost 40 years, since the publications of Shiryaev in the 60's, in order to have a complete solution. Indeed, Peskir and Shiryaev [65] determine the solution to the initial optimal stopping problem for the posterior probability process as the solution of a free-boundary differential-difference Stephan problem, solved by the principles of smooth and continuous fit. The latter deserves a remarkable mention, since its discovery in Peskir and Shiryaev [65] as variational principle, on equal footing as the smooth fit, plays a key role when the observed process has discontinuous sample paths. Gapeev [32] started the study of sequential testing for compound Poisson processes, assuming the intensity of the underlying Poisson process equal to the mean of the mark distribution. An extension of this result was provided by Dayanik et al. [23] and Dayanik and Sezer [24], who presented an accurate numerical algorithm to determine the optimal boundaries, for the sequential testing of (multi-)hypotheses for a general compound Poisson process. Recently, Gapeev and Peskir [35] solved the Wiener sequential testing within the finite horizon formulation and Gapeev and Shiryaev [37] solved the sequential testing of two simple hypotheses about the drift of a wide class of diffusion processes.

The aim of this chapter is twofold: (1) to provide general results for the sequential testing of an extended class of Lévy processes, where the Wiener and the Poisson processes represent particular cases and (2) to provide the explicit solution to the sequential testing problem for the parameter p of a negative binomial process. This last problem was formulated, but not solved, in Dvoretzky et al. [28] and its solution seems not to appear in the literature.

The organization of this chapter is as follows. In Section 5.2, some basic notions on Lévy processes, which will be used in the rest of the chapter, are provided; Section 5.3 presents in a more formal way the problem stated in the introduction; in Sections 5.4 and 5.5, the general solution to the sequential testing problem for continuous and pure increasing jump Lévy processes is shown, by exploiting the principles of smooth and continuous fit at the optimal boundary points; using the results of Section 5.5, in Section 5.6 we provide the exact solution to the problem of sequentially testing two simple hypotheses about the parameter p of a negative binomial process, reducing the initial optimal stopping problem to a free-boundary Stephan problem, for an integro-differential operator. Section 5.7 concludes with a summary discussion.

5.2 Some elements about Lévy processes

Lévy processes represent one of the most important families of stochastic processes and, for such a reason, a huge literature about them can be found. After having remembered the definition of a Lévy process, some useful results, based on Sato [71], are stated: we will exploit them later for solving our problems.

Definition 5.2.1 $X = (X_t)_{t\geq 0}$ is a Lévy process if the following conditions are satisfied:

1. $X_0 = 0$ almost surely (a.s.);
2. for any $n \geq 1$ and $0 \leq t_0 < t_1 < \ldots < t_{n-1} < t_n$, the increments X_{t_0}, $X_{t_1} - X_{t_0}$, $\ldots$, $X_{t_n} - X_{t_{n-1}}$ are independent;
3. the increments are stationary, i.e., the distribution of $X_{t+h} - X_t$ only depends on h;
4. X is right-continuous with left limit.

Any Lévy process $X = (X_t)_{t\geq 0}$ is completely specified by the characteristic function $\hat{\mu}(z)$ of the (infinitely divisible) distribution P_{X_1}, as stated by the following theorem.

Theorem 5.2.1 (*Lévy-Khintchine representation*) *If X is a Lévy process, then*

$$\hat{\mu}(z) = \exp\left(i\gamma z - \frac{1}{2}\sigma^2 z^2 + \int_{\mathbb{R}} \left(e^{izx} - 1 - izx\mathbf{1}_D(x)\right) v(dx)\right), \quad z \in \mathbb{R}, \tag{5.2.1}$$

where $D = \{x : |x| \leq 1\}$, $\gamma \in \mathbb{R}$, $\sigma^2 \geq 0$ and v is a measure, satisfying

$$v(\{0\}) = 0 \quad \textit{and} \quad \int_{\mathbb{R}} (x^2 \wedge 1) v(dx) < \infty. \tag{5.2.2}$$

$g = \{\gamma, \sigma^2, v\}$ is called the *generating triplet* of X; σ^2 and v are called the *Gaussian component* and the *Lévy measure* of X, respectively. Notice that if

$$\int_D |x| v(dx) < \infty, \tag{5.2.3}$$

we can rewrite (5.2.1) as

$$\hat{\mu}(z) = \exp\left(i\widetilde{\gamma} z - \frac{1}{2}\sigma^2 z^2 + \int_{\mathbb{R}} \left(e^{izx} - 1\right) v(dx)\right), \quad z \in \mathbb{R}, \tag{5.2.4}$$

being

$$\widetilde{\gamma} = \gamma - \int_D x v(dx). \tag{5.2.5}$$

We define in this case $\widetilde{g} = \{\widetilde{\gamma}, \sigma^2, v\}$.

One of the main theorems concerning Lévy processes is the Lévy-Itô decomposition, which affirms that any Lévy process can be seen as the sum of three independent Lévy processes: a Wiener process with drift, a compound Poisson process and a square integrable pure jump martingale. When condition (5.2.3) holds, the following more restrictive theorem is valid.

Theorem 5.2.2 (*Lévy-Itô decomposition*) *Let X be a Lévy process having generating triplet $\widetilde{g} = \{\widetilde{\gamma}, \sigma^2, v\}$, satisfying (5.2.3). Then the process $X^J = (X_t^J)_{t\geq 0}$, defined by*

$$X_t^J = \sum_{s \leq t} (X_s - X_{s^-}), \tag{5.2.6}$$

has triplet $\widetilde{g}^J = \{0, 0, v\}$. The process $X^c = (X_t^c)_{t\geq 0}$, given by

$$X_t^c = X_t - X_t^J, \tag{5.2.7}$$

has triplet $g^c = \{\widetilde{\gamma}, \sigma^2, 0\}$. The two processes X^J and X^c are independent and are referred as the jump and continuous part of X, respectively.

Since we will deal with pure increasing jump Lévy processes with finite variation, the next two theorems are of considerable importance.

Theorem 5.2.3 *Let X be a Lévy process; it is increasing if and only if $\int_{-\infty}^0 v(dx) = 0$, $\sigma^2 = 0$, $\int_{(0,1]} x v(dx) < \infty$ and $\widetilde{\gamma} \geq 0$.*

Theorem 5.2.4 *If a Lévy process X satisfies the condition* (5.2.3) *and* $\sigma^2 = 0$, *then X has finite variation on* $(0, t]$, $\forall t \geq 0$.

The last result we need to face our problems is represented by the density transformation of Lévy processes.

Theorem 5.2.5 *Let X be a Lévy process, having triplet $g_0 = \{\gamma_0, \sigma^2, v_0\}$, under the probability measure P_0, and $g_1 = \{\gamma_1, \sigma^2, v_1\}$, under P_1. Let $(\mathscr{F}_t)_{t\geq 0}$ be a (increasing) sequence of sigma-algebras, which make X measurable, with respect to P_i, $i = 0, 1$. Denote by $P_i|\mathscr{F}_t$ the restriction of P_i to $\mathscr{F}_t$, $i = 0, 1$, and assume that $P_1|\mathscr{F}_t$ and $P_0|\mathscr{F}_t$ are mutually absolutely continuous $(P_1|\mathscr{F}_t \approx P_1|\mathscr{F}_t)$. Denote by*

$$\xi(x) = \frac{dv_1}{dv_0}(x), \tag{5.2.8}$$

$$\eta = \frac{1}{\sigma^2}\left(\gamma_1 - \gamma_0 - \int_{|x|\leq 1} x(v_1 - v_0)(dx)\right), \tag{5.2.9}$$

$$X_t^v = \lim_{\epsilon\downarrow 0}\left(\sum_{(s, x_s - x_{s-})\in H(t,\epsilon)} (X_s - X_{s^-}) - t\int_{\epsilon<|x|\leq 1} xv(dx)\right), \tag{5.2.10}$$

where dv_1/dv_0 is the Radon-Nikodym derivative of v_1 with respect to v_0 and $H(t, \epsilon) = \{(s, x) : s \leq t, |x| > \epsilon\}$. Then,

$$\frac{d(P_1|\mathscr{F}_t)}{d(P_0|\mathscr{F}_t)} = \exp\left\{\eta(X_t - X_t^{v_0}) - \frac{t}{2}\eta^2\sigma^2 - t\gamma\eta\right. \tag{5.2.11}$$

$$\left. + \lim_{\epsilon\downarrow 0}\left(\sum_{(s, x_s - x_{s-})\in H(t,\epsilon)} \log\left(\xi(X_s - X_{s^-})\right) - t\int_{|x|>\epsilon}(\xi(x) - 1)v_0(dx)\right)\right\}.$$

A deeper explanation and the proofs of the previous theorems can be found, for example, in Sato [71, Chap. 2, 4 and 6].

5.3 Formulation and preliminaries

At time $t = 0$, a trajectory of the Lévy process $X = (X_t)_{t\geq 0}$ is started to be observed; let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, P_\pi)$ be a filtered probability space, on which X is defined, where:

$$P_\pi = \pi P_1 + (1 - \pi)P_0, \tag{5.3.1}$$

with $\pi \in [0, 1]$. It is assumed that $P_1 \overset{\text{loc}}{\approx} P_2$, i.e., the restrictions of P_1 and P_2 to $\mathscr{F}_t$ are mutually absolutely continuous $(P_1|\mathscr{F}_t \approx P_2|\mathscr{F}_t)$, $\forall t \geq 0$. Let ϑ be an $\mathscr{F}_0$-measurable random variable, taking values 1 and 0, with probabilities $P_\pi(\vartheta = 1) = \pi$ and $P_\pi(\vartheta = 0) = 1 - \pi$; denote by $g_\vartheta = \{\gamma_\vartheta, \sigma^2, v_\vartheta\}$ the Lévy-Khintchine triplet of X. Then, X will be completely characterized by $g_1 = \{\gamma_1, \sigma^2, v_1\}$, with probability π, or by $g_0 = \{\gamma_0, \sigma^2, v_0\}$, with probability $1-\pi$. Equivalently, $P_\pi(X \in \cdot|\vartheta = i) = P_i(X \in \cdot)$, being $P_i(X \in \cdot)$ the distribution of a Lévy process, with triplet g_i,

$i = 0, 1$. By continuously updating the available information, through the sequential observation of the process, the aim is to test the two simple hypotheses

$$H_0 : \vartheta = 0 \qquad Vs \qquad H_1 : \vartheta = 1. \tag{5.3.2}$$

Let τ be a stopping time with respect to the filtration $(\mathscr{F}_t^X)_{t\geq 0}$, where $\mathscr{F}_t^X = \sigma\{X_s : 0 \leq s \leq t\}$ represents all the available information at time t; let d be an $\mathscr{F}_\tau^X$-measurable random variable, taking values 0, if H_0 is accepted, or 1, in the other case. This simply means that when we stop the process at time τ, if $d = 1$, we accept $\vartheta = 1$, else we reject it.

The pair (τ, d) is called sequential decision rule; notice that each decision rule entails a statistical loss, due to the cost of collecting the observations until the time τ and due to a wrong final choice, in deciding between H_0 and H_1. The former is in average $E(\tau)$ (without loss of generality, a sampling cost equal to 1 per unit of time has been set), while it is reasonable to assume that the latter has expectation $aP_\pi(d = 0, \vartheta = 1) + bP_\pi(d = 1, \vartheta = 0)$, with $a, b \geq 0$. Thus, the risk associated to (τ, d) is given by:

$$R(\tau, d) = E_\pi \left(\tau + a\mathbf{1}_{(d=0,\vartheta=1)} + b\mathbf{1}_{(d=1,\vartheta=0)}\right). \tag{5.3.3}$$

Among all the possible decision rules, we want to choose that minimizing (5.3.3): it is called the π-*Bayes decision rule*, $(\tau_\pi^\star, d_\pi^\star)$, and it is such that $V(\pi) = R(\tau_\pi^\star, d_\pi^\star)$, being

$$V(\pi) = \inf_{(\tau,d)} R(\tau, d). \tag{5.3.4}$$

One can easily verify that (5.3.4) is concave.

According to Shiryaev [72, Lemma1, pp. 166-167], expression (5.3.4) can be equivalently written as:

$$V(\pi) = \inf_\tau E_\pi \left(\tau + g_{a,b}(\pi_\tau)\right), \tag{5.3.5}$$

where $(\pi_t)_{t\geq 0}$, defined by $\pi_t = P_\pi(\vartheta = 1|\mathscr{F}_t^X)$, is the *posterior probability process* and $g_{a,b}(\pi) = a\pi \wedge b(1 - \pi)$, where $x \wedge y$ stands for $\min\{x, y\}$. Further, the optimal decision function $d_\pi^\star$ is given by:

$$d_\pi^\star = \begin{cases} 0 & \text{if} \quad \pi_{\tau_\pi^\star} < c \\ 1 & \text{if} \quad \pi_{\tau_\pi^\star} \geq c \end{cases}, \tag{5.3.6}$$

being $c = b/(a + b)$. The original problem (5.3.4) has been reduced to an optimal stopping problem for the process $(\pi_t)_{t\geq 0}$. Let $D = \{\pi \in [0, 1] : V(\pi) = g_{a,b}(\pi)\}$; according to the general theory of optimal stopping, (see, e.g., Peskir and Shiryaev [67, Chap. 1] or Shiryaev [72, Chap. 3]), the function $V(\pi)$ is pointwise dominated by $g_{a,b}(\pi)$ and $\tau_D = \inf\{t \geq 0 : \pi_t \in D\}$ is optimal in (5.3.5). These facts, together with the concavity of $V(\pi)$, imply the existence of two points A and B, $0 < A \leq c \leq B < 1$, such that $D = [0, A] \cup [B, 1]$.

Less formally, from expression (5.3.5), one may notice that the closer π_t to 0 or 1, the lower the chance that the loss will decrease, if the process is continued to be observed. This provides the intuition for guessing the existence of two points A and B, sucht that $\tau_{A,B} = \inf\{t \geq 0 : \pi_t \notin (A, B)\}$ is optimal in (5.3.5).

The likelihood ratio process $(\varphi_t)_{t\geq 0}$ will assume a great importance in our analysis; it is defined as the Radon-Nikodym derivative

$$\varphi_t = \frac{d\left(P_1|\mathscr{F}_t^X\right)}{d\left(P_0|\mathscr{F}_t^X\right)}. \tag{5.3.7}$$

The application of Bayes theorem allows us to write

$$\pi_t = \pi \frac{d\left(P_1|\mathscr{F}_t^X\right)}{d\left(P_\pi|\mathscr{F}_t^X\right)}, \tag{5.3.8}$$

being $P_\pi|\mathscr{F}_t^X = \pi P_1|\mathscr{F}_t^X + (1-\pi)P_0|\mathscr{F}_t^X$.
From (5.3.7) and (5.3.8) one gets

$$\pi_t = \left(\frac{\pi}{1-\pi}\varphi_t\right) \Big/ \left(1 + \frac{\pi}{1-\pi}\varphi_t\right). \tag{5.3.9}$$

The solution to the optimal stopping problem (5.3.5) requires that, as much as possible, the value function $V(\pi)$ be explicit and evaluating the unknown boundaries A and B: this can be reached by the formulation of a suitable free-boundary problem, involving the infinitesimal generator/operator of $(\pi_t)_{t\geq 0}$.

5.4 Solution of the problem for continuous Lévy processes

According to the Lévy-Itô decomposition of a Lévy process (see Theorem 5.2.2), it has almost surely continuous paths if and only if its Lévy measure is 0; so let X have the triplet $g_\vartheta = \{\gamma_\vartheta, \sigma^2, 0\}$. Notice that X is a Wiener process, having diffusion coefficient σ^2 and drift γ_ϑ. From (5.3.7) and Theorem 5.2.5, setting $\eta = (\gamma_1 - \gamma_0)/\sigma^2$, the likelihood ratio becomes:

$$\varphi_t = \exp\left(\eta X_t - \frac{t}{2}\eta^2\sigma^2 - t\gamma_0\eta\right) = \exp\left(\frac{\gamma_1-\gamma_0}{\sigma^2}\left(X_t - \frac{t}{2}(\gamma_1+\gamma_0)\right)\right). \tag{5.4.1}$$

From (5.4.1), (5.3.9) and the application of Itô's formula for continuous semimartingales, the evolution of the likelihood ratio and the a posteriori probability processes can be recovered:

$$d\varphi_t = -\frac{\gamma_0(\gamma_1-\gamma_0)}{\sigma^2}\varphi_t dt + \frac{\gamma_1-\gamma_0}{\sigma^2}\varphi_t dX_t, \tag{5.4.2}$$

$$d\pi_t = \frac{\gamma_1-\gamma_0}{\sigma^2}\pi_t(1-\pi_t)\Big(dX_t - \big(\gamma_0(1-\pi_t)+\gamma_1\pi_t\big)dt\Big). \tag{5.4.3}$$

Equation (5.4.3) can be used to determine the infinitesimal generator of the process $(\pi_t)_{t\geq 0}$. Let $f \in C^2[0,1]$; then by Itô's formula one obtains

$$\begin{aligned} f(\pi_t) &= f(\pi_0) + \int_0^t f'(\pi_s)\,d\pi_s + \frac{1}{2}\int_0^t f''(\pi_s)(d\pi_s)^2 \qquad (5.4.4) \\ &= f(\pi_0) + \frac{1}{2}\frac{(\gamma_1-\gamma_0)^2}{\sigma^2}\int_0^t f''(\pi_s)\pi_s^2(1-\pi_s)^2\,ds + \mathscr{M}_t, \end{aligned}$$

where $\mathscr{M} = (\mathscr{M}_t)_{t\geq 0}$, with $\mathscr{M}_t$ given by

$$\mathscr{M}_t = \frac{\gamma_1 - \gamma_0}{\sigma^2} \int_0^t f'(\pi_s)\pi_s(1-\pi_s)\Big(dX_s - \big(\gamma_0(1-\pi_s) + \gamma_1\pi_s\big)ds\Big), \tag{5.4.5}$$

is a local martingale with respect to $\left(\mathscr{F}_t^X\right)_{t\geq 0}$ and P_π, $\forall \pi \in [0,1]$. Hence, from (5.4.4), the infinitesimal generator of $(\pi_t)_{t\geq 0}$, acting on $f \in C^2[0,1]$, is

$$(\mathbb{L}f)(\pi) = \frac{1}{2}\frac{(\gamma_1 - \gamma_0)^2}{\sigma^2}\pi^2(1-\pi)^2 f''(\pi). \tag{5.4.6}$$

Independently of the realization of ϑ at time $t = 0$, X has null Lévy measure and is therefore almost surely continuous; then from equation (5.4.3), it is evident that $(\pi_t)_{t\geq 0}$ evolves continuously in time. This allows us to guess that the smooth fit condition holds at the boundary points A and B. Again from (5.4.3), one can notice that $(\pi_t)_{t\geq 0}$ is a time homogeneus (strong) Markov process. Thus, standard arguments based on the strong Markov property, (see Chapter 2 or, e.g., Peskir and Shiryaev [67, Chap. 3 and 4] or Shiryaev [72, Chap. 3]), lead to the formulation of the following free-boundary problem, for the unknown function V and the unknown points A and B:

$$\mathbb{L}V = -1 \qquad \text{for} \quad \pi \in (A, B), \tag{5.4.7}$$

$$V = g_{a,b} \qquad \text{for} \quad \pi \notin (A, B), \tag{5.4.8}$$

$$V < g_{a,b} \qquad \text{for} \quad \pi \in (A, B), \tag{5.4.9}$$

$$V(A) = aA, \tag{5.4.10}$$

$$V'(A) = a \qquad (\textit{smooth fit}), \tag{5.4.11}$$

$$V(B) = b(1-B), \tag{5.4.12}$$

$$V'(B) = -b \qquad (\textit{smooth fit}). \tag{5.4.13}$$

For a fixed $A < c$, through the conditions (5.4.7), (5.4.10) and (5.4.11) one gets that for $\pi \geq A$:

$$V(\pi, A) = aA + (\pi - A)\left(a - \frac{2\sigma^2}{(\gamma_1-\gamma_0)^2}\psi'(A)\right) + \frac{2\sigma^2}{(\gamma_1-\gamma_0)^2}\big(\psi(\pi) - \psi(A)\big), \tag{5.4.14}$$

and

$$V'(\pi, A) = \frac{2\sigma^2}{(\gamma_1-\gamma_0)^2}\big(\psi'(\pi) - \psi'(A)\big) + a, \tag{5.4.15}$$

where

$$\psi(\pi) = (1-2\pi)\log\left(\frac{\pi}{1-\pi}\right), \tag{5.4.16}$$

$$\psi'(\pi) = \frac{1}{\pi} - \frac{1}{1-\pi} + 2\log\left(\frac{1-\pi}{\pi}\right). \tag{5.4.17}$$

According to (5.4.12) and (5.4.13), the optimal boundaries $A^\star$ and $B^\star$ are uniquely determined as solution of the system of transcendental equations

$$V(B^\star, A^\star) = b(1-B^\star), \tag{5.4.18}$$

$$V'(B^\star, A^\star) = -b, \tag{5.4.19}$$

where $\pi \mapsto V(\pi, A)$ and $\pi \mapsto V'(\pi, A)$ are given by (5.4.14) and (5.4.15), respectively. After that $A^\star$ and $B^\star$ have been found, the solution $\pi \mapsto V(\pi, A^\star)$, for $\pi \in (A^\star, B^\star)$, is given through (5.4.14) (See Figure 5.1 below).

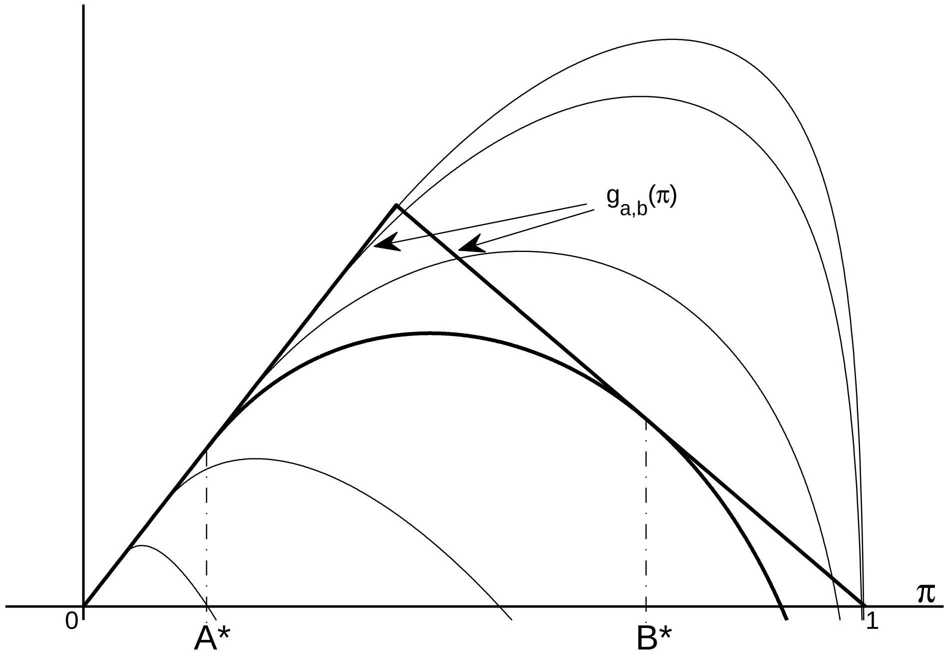

Figure 5.1: A computer drawing of the map $\pi \mapsto V(\pi, A^\star)$ (bold curve), for $\pi \geq A^\star$, with $a = 15$, $b = 10$, $\gamma_1 = -3$, $\gamma_0 = -2$ and $\sigma^2 = 1$. The unique pair of points satisfying (5.4.18) and (5.4.19) is $A^\star$ =0.1593.. and $B^\star$ =0.7206.. These represent the only points where $\pi \mapsto V(\pi, A^\star)$ hits smoothly $g_{a,b}(\pi)$. The other curves $\pi \mapsto V(\pi, A)$ are obtained setting A =0.05, 0.1, 0.2, 0.30, 0.35.

All the previous facts are summarized in the following theorem.

Theorem 5.4.1 *The π-Bayes decision rule $(\tau_\pi^\star, d_\pi^\star)$ for the problem* (5.3.4) *is explicitly given by*

$$\tau_\pi^\star = \inf\{t \geq 0 : \pi_t \notin (A^\star, B^\star)\}, \tag{5.4.20}$$

$$d_\pi^\star = \begin{cases} 0 & \textit{(accept } H_0\textit{)} \quad \textit{if} \quad \pi_{\tau_\pi^\star} \leq A^\star \\ 1 & \textit{(accept } H_1\textit{)} \quad \textit{if} \quad \pi_{\tau_\pi^\star} \geq B^\star \end{cases}, \tag{5.4.21}$$

where the optimal boundaries $A^\star$ and $B^\star$ are obtained as unique solution of the system of transcendental equations (5.4.18) *and* (5.4.19). *The explicit expression of the value function in* (5.3.5) *is given by*

$$V(\pi) = \begin{cases} V(\pi, A^\star) & \textit{for} \quad \pi \in (A^\star, B^\star) \\ g_{a,b}(\pi) & \textit{for} \quad \pi \in [0, A^\star] \cup [B^\star, 1] \end{cases}, \tag{5.4.22}$$

where $\pi \mapsto V(\pi, A)$ is given by (5.4.14).

Proof. One has to show that the system (5.4.18)-(5.4.19) has a unique solution and that the solution to the free-boundary problem (5.4.7)-(5.4.13) coincides with (5.3.5). One can recover a formal proof in Shiryaev [72, Sec. 4.2, pp. 184-185], (where $\gamma_1 = r$ and $\gamma_0 = 0$), or in Peskir and Shiryaev [67, pp. 290-292], (where $\gamma_1 = \mu$ and $\gamma_0 = 0$). ∎

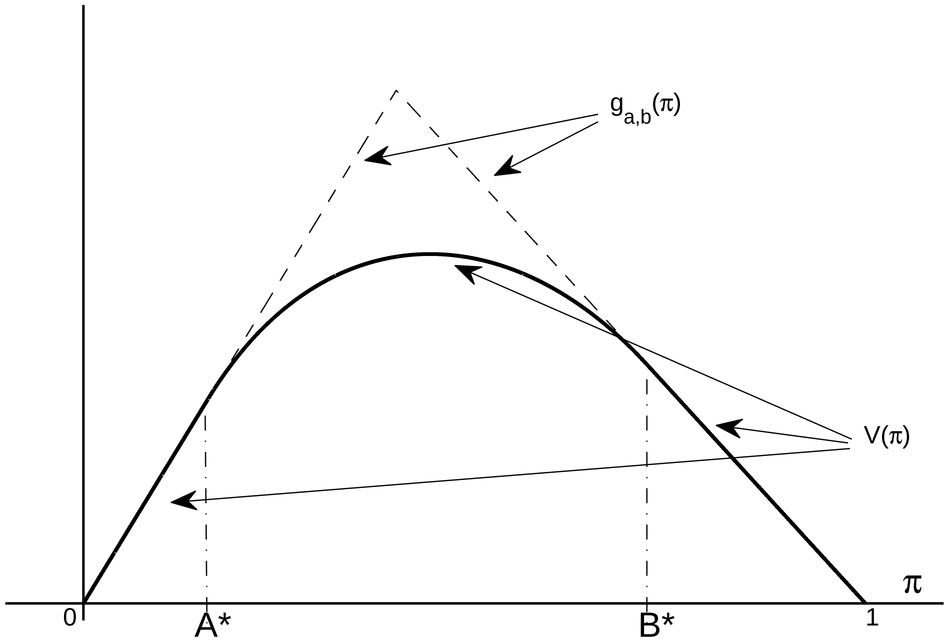

Figure 5.2: A computer drawing of the map $\pi \mapsto V(\pi)$, as expressed by (5.4.22), coinciding with the final payoff, given by (5.3.5). The same parameters of Figure 5.1 have been used. The set $D = \{\pi \in [0,1] : V(\pi) = g_{a,b}(\pi)\} = [0, A^\star] \cup [B^\star, 1]$ is the stopping region, while the interval $(A^\star, B^\star)$ is the set of continued observation of the posterior probability process $(\pi_t)_{t\geq 0}$.

5.5 Solution of the problem for pure increasing jump Lévy processes

Let X be a Lévy process with Lévy-Khintchine triplet $g_\vartheta = \{\gamma_\vartheta, 0, v_\vartheta\}$ and assume that the Lévy measure v_ϑ satisfies the condition

$$\int_{|x|\leq 1} |x|\, v_\vartheta(dx) < \infty. \tag{5.5.1}$$

Then, according to Theorem 5.2.4, X has finite variation on $(0, t)$, for any $t \in (0, \infty)$. If we define

$$\widetilde{\gamma}_\vartheta = \gamma_\vartheta - \int_{|x|\leq 1} x\, v_\vartheta(dx), \tag{5.5.2}$$

the process can be equivalently characterized by the triplet $\widetilde{g}_\vartheta = (\widetilde{\gamma}_\vartheta, 0, v_\vartheta)$, as we noticed in Section 5.2. Further, assume that

$$\int_{-\infty}^{0} v_\vartheta(dx) = 0, \tag{5.5.3}$$

$$\widetilde{\gamma}_\vartheta = 0. \tag{5.5.4}$$

Under these conditions, the process X is increasing and increases only by jumps (see Theorem 5.2.3).

5.5.1 Derivation of the infinitesimal operator

In order to sequentially test if either $\vartheta = 0$ or $\vartheta = 1$, we need to repeat the procedure followed in the previous section. According to Theorem 5.2.5, the likelihood ratio process (5.3.7) becomes:

$$\varphi_t = \exp\Big(\sum_{s\le t} \log\big(\xi(X_s - X_{s-})\big) - t\int \big(\xi(x) - 1\big)\, v_0(dx)\Big), \tag{5.5.5}$$

where $\xi(x)$ denotes the Radon-Nikodym derivative $\frac{dv_1}{dv_0}(x)$. The expressions (5.3.9) and (5.5.5), together with the application of Itô's formula for purely discontinuous semimartingales, allow to obtain the expressions for the evolution of $(\varphi_t)_{t\ge0}$ and $(\pi_t)_{t\ge0}$:

$$d\varphi_t = \varphi_{t-}\int \big(\xi(x) - 1\big)\big(\mu^X - v_0\big)\,(dx, dt), \tag{5.5.6}$$

$$d\pi_t = \int \frac{\pi_{t-}(1-\pi_{t-})\big(\xi(x) - 1\big)}{1 + \pi_{t-}\big(\xi(x) - 1\big)}\big(\mu^X - v^X\big)\,(dx, dt), \tag{5.5.7}$$

where μ^X, defined by:

$$\mu^X\big((0,t] \times A\big) = \sum_{s\le t} \mathbf{1}(\Delta X_s \in A), \quad \text{for} \quad A \in \mathscr{B}(\mathbb{R}^+\backslash\{0\}), \tag{5.5.8}$$

is the measure of jumps of the process X, and v^X, defined by:

$$v^X(dx) = \Big(1 + \pi_{t-}\big(\xi(x) - 1\big)\Big)v_0(dx) = (1-\pi_{t-})v_0(dx) + \pi_{t-}\xi(x)v_0(dx), \tag{5.5.9}$$

is a compensator of μ^X.

For determining the infinitesimal operator of $(\pi_t)_{t\ge0}$, let $f \in C^1[0,1]$; by the application of Itô's formula one gets:

$$\begin{aligned}
f(\pi_t) =& f(\pi_0) + \int_0^t f'(\pi_{s-})\,d\pi_s + \sum_{0\le s\le t}\big(\Delta f(\pi_s) - f'(\pi_{s-})\Delta\pi_s\big) \\
=& f(\pi_0) - \int_0^t\int_0^\infty f'(\pi_{s-})\Big(\pi_{s-}(1-\pi_{s-})\big(\xi(x) - 1\big)\Big)\,v_0(dx)\,ds \\
&+ \int_0^t\int_0^1 \big(f(\pi_{s-} + y) - f(\pi_{s-})\big)\mu^\pi(dy, ds)
\end{aligned}$$

$$
\begin{aligned}
&= f(\pi_0) - \int_0^t \int_0^\infty f'(\pi_{s^-}) \Big(\pi_{s-} (1-\pi_{s^-}) \big(\xi(x) - 1 \big) \Big)\, v_0(dx)\, ds \\
&\quad + \int_0^t \int_0^1 \big(f(\pi_{s^-} + y) - f(\pi_{s^-}) \big) v^\pi(dy, ds) \\
&\quad + \int_0^t \int_0^1 \big(f(\pi_{s^-} + y) - f(\pi_{s^-}) \big) (\mu^\pi - v^\pi)(dy, ds),
\end{aligned} \tag{5.5.10}
$$

being μ^π and v^π the measure of jumps and the corresponding compensator of the process $(\pi_t)_{t \geq 0}$. The above expression can be rewritten in a more compact form as:

$$
f(\pi_t) = f(\pi_0) + \int_0^t \big(\mathbb{L} f \big)(\pi_{s^-})\, ds + \mathscr{M}_t, \tag{5.5.11}
$$

where $\mathscr{M} = (\mathscr{M}_t)_{t \geq 0}$, with

$$
\mathscr{M}_t = \int_0^t \int_0^1 \big(f(\pi_{s^-} + y) - f(\pi_{s^-}) \big) (\mu^\pi - v^\pi)(dy, ds), \tag{5.5.12}
$$

is a local martingale, with respect to $(\mathscr{F}_t^X)_{t \geq 0}$ and P_π, $\forall \pi \in [0,1]$. The process $(\pi_t)_{t \geq 0}$ jumps whenever X jumps; further one can easily verify that

$$
\pi + \Delta \pi = \frac{\pi \xi(x)}{1 + \pi \big(\xi(x) - 1 \big)}. \tag{5.5.13}
$$

Thus, using (5.5.9), (5.5.10) and (5.5.13), the expression of the infinitesimal operator $\mathbb{L}$ can be recovered:

$$
\begin{aligned}
\big(\mathbb{L} f \big)(\pi) &= - f'(\pi) \pi (1-\pi) \int_0^\infty \big(\xi(x) - 1 \big) v_0(dx) \\
&+ \int_0^\infty \left(f\left(\frac{\pi \xi(x)}{1 + \pi \big(\xi(x) - 1 \big)} \right) - f(\pi) \right) \Big((1-\pi) v_0(dx) + \pi \xi(x) v_0(dx) \Big).
\end{aligned} \tag{5.5.14}
$$

5.5.2 The free-boundary problem and solution of the optimal stopping problem

Without loss of generality, assume that

$$
\xi(x) > 1. \tag{5.5.15}
$$

Then, from (5.5.7) and (5.5.9), it is evident that $(\pi_t)_{t \geq 0}$ evolves continuously towards 0 and jumps towards 1 at the times of the jumps of the process X. This fact provides an intuitive support for guessing that the smooth fit holds at A, but not at B, where the continuous fit just holds.

This consideration and the strong Markov property of the process $(\pi_t)_{t \geq 0}$, clear from (5.5.7) and (5.5.9), together with the general theory of optimal stopping (see Chapter 2 or, e.g., Peskir and Shiryaev [67, Chap. 3 and 4] or Shiryaev [72, Chap. 3]), lead to the formulation of the following free-boundary problem for the unknown function V and the unknwon points A and B:

$$\mathbb{L}V = -1 \qquad \text{for} \quad \pi \in (A, B), \tag{5.5.16}$$

$$V = g_{a,b} \qquad \text{for} \quad \pi \notin (A, B), \tag{5.5.17}$$

$$V < g_{a,b} \qquad \text{for} \quad \pi \in (A, B), \tag{5.5.18}$$

$$V(A^+) = aA \qquad (\textit{continuous fit}) \tag{5.5.19}$$

$$V'(A) = a \qquad (\textit{smooth fit}), \tag{5.5.20}$$

$$V(B^-) = b(1 - B) \qquad (\textit{continuous fit}). \tag{5.5.21}$$

If a solution to this free-boundary problem can be determined, the next theorem holds.

Theorem 5.5.1 *For a fixed $B > c$, let $\pi \mapsto V(\pi, B)$ be the map solving* (5.5.16) *and* (5.5.21)*, for $\pi \le B$, and assume that the function $V(\pi)$, solving the free-boundary problem* (5.5.16)-(5.5.21)*, is such that $\big(\mathbb{L}V\big)(\pi) \ge -1$, $\forall \pi \in [0, 1]$.*

(I) If

$$\lim_{B \downarrow c} V'(B^-, B) < a, \tag{5.5.22}$$

the π-Bayes decision rule $(\tau^\star_\pi, d^\star_\pi)$ for the problem (5.3.5) *is explicitly given by:*

$$\tau^\star_\pi = \inf\{t \ge 0 : \pi_t \notin (A^\star, B^\star)\}, \tag{5.5.23}$$

$$d^\star_\pi = \begin{cases} 0 & (\textit{accept } H_0) \quad \textit{if} \quad \pi_{\tau^\star_\pi} \le A^\star \\ 1 & (\textit{accept } H_1) \quad \textit{if} \quad \pi_{\tau^\star_\pi} \ge B^\star \end{cases}, \tag{5.5.24}$$

where $A^\star$ and $B^\star$, with $0 < A^\star < c < B^\star$, are obtained as unique solution of the system of transcendental equations:

$$V(A^\star, B^\star) = aA^\star, \tag{5.5.25}$$

$$V'(A^\star, B^\star) = a. \tag{5.5.26}$$

The explicit expression of the value function in (5.3.5) *is given by*

$$V(\pi) = \begin{cases} V(\pi, B^\star) & \textit{for} \quad \pi \in (A^\star, B^\star) \\ g_{a,b}(\pi) & \textit{for} \quad \pi \in [0, A^\star] \cup [B^\star, 1] \end{cases}; \tag{5.5.27}$$

(II) if (5.5.22) *does not hold, the value function $V(\pi)$ from* (5.3.5) *equals $g_{a,b}(\pi)$, $\forall \pi \in [0, 1]$, so that the π-Bayes decision rule $(\tau^\star_\pi, d^\star_\pi)$ becomes trivial: $\tau^\star_\pi = 0$ and $d^\star_\pi = 0$, i.e., accept H_0, if $\pi < c$, or $d^\star_\pi = 1$, i.e., accept H_1, if $\pi > c$. If $\pi = c$, both the decisions are equally good.*

Proof. (I) The condition (5.5.22) is necessary and sufficient, for having a unique solution of the system (5.5.25) and (5.5.26). Indeed, from a geometric view point, it means that for $B > c$, close enough to c, the map $\pi \mapsto V(\pi, B)$ intersects $\pi \mapsto a\pi$, at some $\pi < B$. Further, the two curves $\pi \mapsto V(\pi, B')$ and $\pi \mapsto V(\pi, B'')$, with $B' < B''$, do not intersect on $(0, B']$ (see Peskir and Shiryaev [65, Remark 2.2, p. 850]). Thus, moving B on the interval $(c, 1)$, it is straightforward to see that there exists a unique pair of points $A^\star$ and $B^\star$, satisfying (5.5.25) and (5.5.26).

Denote by $V^\star$ the solution to the free-boundary problem (5.5.16)-(5.5.21). We have to show that $V^\star$ coincides with the payoff V, given by (5.3.5). By construction, $V^\star$ is C^1 on $[0,1]\backslash\{B^\star\}$, but C^0 at $B^\star$. Then, since the time spent by the process $(\pi_t)_{t\geq 0}$ at $B^\star$ is of Lebesgue measure zero, Itô's formula can be applied to $V^\star(\pi_t)$ and from (5.5.11) one obtains:

$$V^\star(\pi_t) = V^\star(\pi_0) + \int_0^t \big(\mathbb{L}V^\star\big)(\pi_{s^-})\,ds + \mathscr{M}_t, \tag{5.5.28}$$

being $\pi_0 = \pi$. Since $\mathscr{M} = (\mathscr{M}_t)_{t\geq 0}$ is a martingale, with respect to $(\mathscr{F}_t^X)_{t\geq 0}$ and P_π, $\forall\pi\in[0,1]$, for the optional sampling theorem $E_\pi(\mathscr{M}_\tau) = 0$, when the stopping time τ of X satisfies $E_\pi(\tau) < \infty$. Taking the expectation on both sides of (5.5.28) and under the assumption $\big(\mathbb{L}V^\star\big)(\pi) \geq -1$, $\forall\pi\in[0,1]$, obviously satisfied by (5.5.16) on $(A^\star, B^\star)$, one has:

$$\begin{aligned} E_\pi\big(V^\star(\pi_\tau)\big) &= V^\star(\pi_0) + E_\pi\left(\int_0^\tau \big(\mathbb{L}V^\star\big)(\pi_{s^-})\,ds\right) \\ &\geq V^\star(\pi_0) - E_\pi(\tau). \end{aligned} \tag{5.5.29}$$

From (5.5.29), (5.5.17) and (5.5.18), it results that

$$V^\star(\pi_0) \leq E_\pi\big(V^\star(\pi_\tau) + \tau\big) \leq E_\pi\big(g_{a,b}(\pi_\tau) + \tau\big), \tag{5.5.30}$$

for all stopping times τ with finite expectation, which implies that $V^\star(\pi) \leq V(\pi)$, $\forall\pi\in[0,1]$.

Because of the consistency of $(\pi_t)_{t\geq 0}$, (i.e., this process will converge sooner or later to 0 or 1, depending on, respectively, the true triplet g_0 or g_1 characterizing X), $E_\pi\big(\tau_\pi^\star\big) < \infty$. Thus, reapplying the expectation on both sides of (5.5.28) and for (5.5.16) and (5.5.23), we obtain:

$$E_\pi\big(V^\star(\pi_{\tau_\pi^\star})\big) = V^\star(\pi_0) - E_\pi(\tau_\pi^\star). \tag{5.5.31}$$

It is evident from (5.5.17) and (5.5.23) that $V^\star(\pi_{\tau_\pi^\star}) = g_{a,b}(\pi_{\tau_\pi^\star})$; then, from the expression (5.5.31), we have:

$$V^\star(\pi_0) = E_\pi\big(g_{a,b}(\pi_{\tau_\pi^\star}) + \tau_\pi^\star\big). \tag{5.5.32}$$

Therefore, (5.5.30) and (5.5.32) imply that $V^\star(\pi) = V(\pi)$, $\forall\pi\in[0,1]$, and $\tau_\pi^\star$ is optimal in (5.3.5). According to Shiryaev (1978, Theorem 17, p.161), the smooth fit condition (5.5.20) must hold at $A^\star$.

(II) It is obvious that if the expression (5.5.22) is violated, then there are no points where, for $B > c$, $\pi \mapsto V(\pi, B)$ intersects $\pi \mapsto a\pi$ and, consequently, where $\pi \mapsto V(\pi, B)$ hits smoothly $\pi \mapsto a\pi$: hence, the set of continued observation $(A^\star, B^\star)$ vanishes and the π-Bayes decision rule becomes trivial. More arguments can be found in Peskir and Shiryaev [65, pp. 849-850]. ∎

5.5.3 Examples

We are going to present two examples of pure increasing processes, already examined in the literature, concerning the sequential testing of the parameters characterizing the involved distributions. Especially, the results in the previous sections can be exploited for constructing the free-boundary problem (5.5.16)-(5.5.21), when X is a Poisson process, with intensity λ_ϑ, and when X is a compound Poisson process, whose intensity $1/\lambda_\vartheta$ equals the mean of the exponential distribution of its jumps.

1) Poisson process: let X be a Lévy process, having triplet $\widetilde{g}_\vartheta = \{0, 0, \lambda_\vartheta \delta_1\}$, where $\lambda_1 > \lambda_0$ and δ_1 is the measure putting unit mass on 1. Notice that $\xi(x) = \lambda_1/\lambda_0$ and the length of the jumps is always 1. From the expressions (5.5.5)-(5.5.9) and (5.5.14), one obtains:

$$\varphi_t = \exp\left(X_t \log \frac{\lambda_1}{\lambda_0} - t(\lambda_1 - \lambda_0) \right), \tag{5.5.33}$$

$$d\varphi_t = \varphi_{t^-} \int \left(\frac{\lambda_1}{\lambda_0} - 1 \right) \left(\mu^X - \lambda_0 \delta_1(x) \right)(dx, dt) = \varphi_{t^-} \left(\frac{\lambda_1}{\lambda_0} - 1 \right) (dX_t - \lambda_0\, dt), \tag{5.5.34}$$

$$\begin{aligned} d\pi_t &= \int \frac{\pi_{t^-}(1-\pi_{t^-}) \left(\frac{\lambda_1}{\lambda_0} - 1 \right)}{1 + \pi_{t^-} \left(\frac{\lambda_1}{\lambda_0} - 1 \right)} \left(\mu^X - v^X \right)(dx, dt) \\ &= \frac{\pi_{t^-}(1-\pi_{t^-})(\lambda_1 - \lambda_0)}{\lambda_1 \pi_{t^-} + \lambda_0 (1 - \pi_{t^-})} \Big(dX_t - \big(\lambda_1 \pi_{t^-} + \lambda_0 (1-\pi_{t^-}) \big) dt \Big), \end{aligned} \tag{5.5.35}$$

$$\begin{aligned} \big(\mathbb{L} f\big)(\pi) =& - f'(\pi)\pi(1-\pi) \int_0^\infty \left(\frac{\lambda_1}{\lambda_0} - 1 \right) \lambda_0 \delta_1(dx) \\ &+ \int_0^\infty \left(f\left(\frac{\pi \frac{\lambda_1}{\lambda_0}}{1 + \pi\big(\frac{\lambda_1}{\lambda_0} - 1\big)} \right) - f(\pi) \right) \Big((1-\pi)\lambda_0 \delta_1(dx) + \pi \lambda_1 \delta_1(dx) \Big), \end{aligned} \tag{5.5.36}$$

from which:

$$\begin{aligned} \big(\mathbb{L} f\big)(\pi) =& - f'(\pi)\pi(1-\pi)(\lambda_1 - \lambda_0) \\ &+ \big(\lambda_1 \pi + \lambda_0 (1-\pi) \big) \left(f\left(\frac{\lambda_1 \pi}{\lambda_1 \pi + \lambda_0 (1-\pi)} \right) - f(\pi) \right). \end{aligned} \tag{5.5.37}$$

The solution V of the system (5.5.16)-(5.5.21), involving the differential-difference operator (5.5.37), is shown in Peskir and Shiryaev [65, pp. 844-845], with the condition (5.5.22) explicitly given by $\lambda_1 - \lambda_0 > 1/a + 1/b$. According to Theorem 5.5.1, one just needs to prove that $\mathbb{L}V \geq -1$, (see Peskir and Shiryaev [65, pp. 848-849]).

2) Compound Poisson process with exponential jumps: let X be a Lévy process, with $\widetilde{g}_\vartheta = \{0, 0, v_\vartheta(dx) = \mathbf{1}_{\{x>0\}} e^{-\lambda_\vartheta x} dx\}$, where $\lambda_0 > \lambda_1$. Now, $\xi(x) = e^{(\lambda_0 - \lambda_1)x}$. Applying (5.5.5)-(5.5.9) and (5.5.14), we have:

$$\varphi_t = \exp\left(X_t(\lambda_0 - \lambda_1) - t \left(\frac{\lambda_0 - \lambda_1}{\lambda_0 \lambda_1} \right) \right), \tag{5.5.38}$$

$$d\varphi_t = \varphi_{t^-} \int_0^\infty \left(e^{(\lambda_0 - \lambda_1)x} - 1 \right) \left(\mu^X - e^{-\lambda_0 x} \right)(dx, dt), \tag{5.5.39}$$

$$d\pi_t = \int_0^\infty \frac{\pi_{t^-}(1-\pi_{t^-}) \left(e^{-\lambda_1 x} - e^{-\lambda_0 x} \right)}{e^{-\lambda_1 x} \pi_{t^-} + e^{-\lambda_0 x}(1-\pi_{t^-})} \Big(\mu^X - \big(e^{-\lambda_1 x} \pi_{t^-} + e^{-\lambda_0 x} (1 - \pi_{t^-}) \big) \Big) (dx, dt), \tag{5.5.40}$$

$$
\begin{aligned}
\big(\mathbb{L}f\big)(\pi) = & -f'(\pi)\pi(1-\pi)\frac{\lambda_0-\lambda_1}{\lambda_0\lambda_1} \\
& + \int_0^\infty \left(f\left(\frac{\pi e^{-\lambda_1 x}}{\pi e^{-\lambda_1 x} + (1-\pi)e^{-\lambda_0 x}}\right) - f(\pi)\right)\Big(\pi e^{-\lambda_1 x} + (1-\pi)e^{-\lambda_0 x}\Big)dx. \qquad (5.5.41)
\end{aligned}
$$

The solution V of the system (5.5.16)-(5.5.21), involving the integro-differential operator (5.5.41), is shown in Gapeev [32], with (5.5.22) given by $1/\lambda_1 - 1/\lambda_0 > 1/a + 1/b$. Again, from Theorem 5.5.1, one needs to show that $\mathbb{L}V \geq -1$.

Remark 5.5.1 There could be situations where we need to deal with a Lévy process X, presenting both the continuous and the jump parts. Thus, assume that X has Lévy-Khintchine triplet $g_\vartheta = \{\gamma_\vartheta, \sigma^2, v_\vartheta\}$, with v_ϑ satisfying the conditions (5.5.1), (5.5.3) and (5.5.15). Denote by $\gamma_\vartheta^J = \int_{|x|\leq 1} x\, v_\vartheta(dx)$. According to the Lévy-Itô decomposition (see Theorem 5.2.2) the process $X^J = (X_t^J)_{t\geq 0}$, defined by $X_t^J = \sum_{s\leq t}(X_s - X_{s^-})$, has triplet $g_\vartheta^J = \{\gamma_\vartheta^J, 0, v_\vartheta\}$, or equivalently $\widetilde{g}_\vartheta^J = \{0, 0, v_\vartheta\}$, while the process $X^c = (X_t^c)_{t\geq 0}$, given by $X_t^c = \big(X_t - X_t^J\big)$, has triplet $g_\vartheta^c = \{\widetilde{\gamma}_\vartheta, \sigma^2, 0\}$, where $\widetilde{\gamma}_\vartheta = \gamma_\vartheta - \gamma_\vartheta^J$ represents the drift of X^c; further, the two processes X^J and X^c are independent. To sequentially test if either $\vartheta = 0$ or $\vartheta = 1$, the results of Sections 5.4 and 5.5 can be applied to X^c and X^J, respectively. Especially, the following sequential decision rule appears to be reasonable: let

$$\tau_\pi^J = \inf\{t \geq 0 : \pi_t^J \notin (A^{J,\star}, B^{J,\star})\}, \qquad (5.5.42)$$

$$\tau_\pi^c = \inf\{t \geq 0 : \pi_t^c \notin (A^{c,\star}, B^{c,\star})\}, \qquad (5.5.43)$$

where $\pi^J = \big(\pi_t^J\big)_{t\geq 0}$, $\pi^c = (\pi_t^c)_{t\geq 0}$, $A^{J,\star}$, $B^{J,\star}$, $A^{c,\star}$ and $B^{c,\star}$ have obvious meaning. Thus, we can set:

$$\tau_\pi^\star = \tau_\pi^J \wedge \tau_\pi^c, \qquad (5.5.44)$$

$$i = \begin{cases} J & \text{if} \quad \tau_\pi^\star = \tau_\pi^J \\ c & \text{if} \quad \tau_\pi^\star = \tau_\pi^c \end{cases}, \qquad (5.5.45)$$

$$d_\pi^\star = \begin{cases} 0 & (\text{accept } H_0) \quad \text{if} \quad \pi^i_{\tau_\pi^\star} \leq A^{i,\star} \\ 1 & (\text{accept } H_1) \quad \text{if} \quad \pi^i_{\tau_\pi^\star} \geq B^{i,\star} \end{cases}. \qquad (5.5.46)$$

A more rigorous procedure will be illustrated in Chapter 8.

5.6 Solution of the problem for a negative binomial process

5.6.1 Description of the negative binomial process

The negative binomial process (n.b.p.) with parameter p, $0 < p < 1$, is a pure increasing Lévy process, whose increment $X_{t+h} - X_t$ has a negative binomial distribution, with parameters h and p. Its Lévy-Khintchine triplet is $\widetilde{g} = \{0, 0, v(\{k\})\}$, where $v(\{k\}) = (1-p)^k/k$, $k = 1, 2, \ldots$. This is a process which admits compound Poisson representation, where the intensity of the underlying Poisson process is $\lambda = -\log p$ and the mark distribution is $\rho(\{k\}) = -(1-p)^k/(k \log p)$, $k = 1, 2, \ldots$, i.e., the jumps have a logarithmic distribution, with parameter p.

Despite in Dayanik and Sezer [24] an accurate numerical procedure for solving the sequential testing problem for any compound Poisson process is provided, we give the explicit solution for the negative binomial one. Indeed, in Dvoretzky et al. [28, p. 264], the authors write that "a complication [with respect to the sequential testing for a Poisson process] is caused by the fact that the probability that the chance variable will exceed one in a small time interval is of the same order of magnitude as the probability that the chance variable will be one". Looking at the Lévy-Khintchine triplet, this means that, unlike the Poisson process, whose Lévy measure is completely concentrated on one, the Lévy measure of a n.b.p. concentrates on the positive integers.

The n.b.p. is used in many fields, (like physics, geoscience, birth and death processes, accident statistics, internet traffic) to model phenomena occurring in clusters. A review of the literature, the distributional properties and the applications of such a process can be found in Kozubowski and Podgórski [46]. Moreover, the "clumping" feature of several issues related to agricultural studies, where the negative binomial distribution holds an important role for modeling data (see, e.g., Anscombe [3], Mukhopadhyay [59], Mukhopadhyay and de Silva [60], Mulekar et al. [61], Plant and Wilson [68], Young [85]), enlarges the number of possible frameworks, where the n.b.p. can be successfully applied.

5.6.2 Infinitesimal operator

Let $X = (X_t)_{t \geq 0}$ be a n.b.p. with parameter p_ϑ, $0 < p_\vartheta < 1$, having, thus, Lévy measure $v_\vartheta(x) = (1 - p_\vartheta)^x / x$, $x = 1, 2, \ldots$. As discussed in Section 5.3, ϑ is a random variable, taking values either 0, with probability $1-\pi$, or 1, with probability π; through the sequential observation of the process X, we want to test the null hyppothesis $\vartheta = 0$, against the alternative one $\vartheta = 1$, minimizing (5.3.3). Without loss of generality, it is assumed that $p_0 > p_1$. Notice that

$$\xi(x) = \frac{dv_1}{dv_0}(x) = \left(\frac{q_1}{q_0}\right)^x, \quad x = 1, 2, \ldots, \tag{5.6.1}$$

where $q_i = 1 - p_i$, $i = 0, 1$. From (5.5.5), we have:

$$\begin{aligned} \varphi_t &= \exp\left(\sum_{s \leq t} (X_s - X_{s^-}) \log\left(\frac{q_1}{q_0}\right) - t \sum_{x=1}^{\infty} \left(\left(\frac{q_1}{q_0}\right)^x - 1\right) \frac{q_0^x}{x}\right) \\ &= \exp\left(X_t \log\left(\frac{q_1}{q_0}\right) + t \log\left(\frac{p_1}{p_0}\right)\right). \end{aligned} \tag{5.6.2}$$

The evolution of $(\varphi_t)_{t \geq 0}$ and $(\pi_t)_{t \geq 0}$ can be easily recovered from the equations (5.5.6) and (5.5.7); what is relevant for our analysis is the expression of the infinitesimal operator of $(\pi_t)_{t \geq 0}$.

From (5.5.14) and some simple algebraic passages, we obtain:

$$\begin{aligned}\big(\mathbb{L}f\big)(\pi) =& f'(\pi)\pi(1-\pi)\log\left(\frac{p_1}{p_0}\right) + f(\pi)\big((1-\pi)\log p_0 + \pi\log p_1\big) \\ &+ \sum_{x=1}^{\infty} f\left(\frac{\pi q_1^x}{\pi q_1^x + (1-\pi)q_0^x}\right)\frac{(\pi q_1^x + (1-\pi)q_0^x)}{x}.\end{aligned} \tag{5.6.3}$$

The explicitation of the value function (5.3.5) and the computation of the optimal boundaries $A^\star$ and $B^\star$ require to solve the Stephan problem (5.5.16)-(5.5.21): (5.5.16) becomes an integro-differential equation, defined by the infinitesimal operator (5.6.3).

5.6.3 Solution of the free-boundary problem

The way we use to solve the free-boundary problem (5.5.16)-(5.5.21), together with (5.6.3), is along lines similar to the reasoning followed in Peskir and Shiryaev [65, pp. 842-846]; but, as we have already preannounced, here the problem is more complicated, since the Lévy measure of X has as domain the set of the positive integer numbers.

For a fixed $B > c$, consider the equation (5.5.16) and (5.6.3) on the interval $(0, B]$. Define the "step" function:

$$S(\pi, x) = \frac{\pi q_1^x}{\pi q_1^x + (1-\pi)q_0^x}, \quad x = 1, 2, \dots, \tag{5.6.4}$$

for $\pi \le B$. One can check that this function satisfies the following properties:

$$S(\pi, 1) > \pi; \tag{5.6.5}$$

$$\frac{\partial S(\pi, x)}{\partial \pi} > 0; \tag{5.6.6}$$

$$S(\pi, x+1) > S(\pi, x); \tag{5.6.7}$$

$$\lim_{x\to\infty} S(\pi, x) = 1; \tag{5.6.8}$$

$$S(\pi, x) = S\big(S(\pi, x-1), 1\big). \tag{5.6.9}$$

Determine the sequence of points $\dots < B_n < B_{n-1} < \dots < B_1 < B_0 = B$, such that $S(B_n, 1) = B_{n-1}$, for $n \ge 1$. Easily, one has:

$$B_n = \frac{q_0^n B}{q_0^n B + q_1^n(1-B)}, \quad n = 0, 1, \dots \tag{5.6.10}$$

Denote $I_n = (B_n, B_{n-1}]$, for $n \ge 1$, and define the "distance" function:

$$d(\pi, B) = 1 + \left[\log\left(\frac{B}{1-B}\frac{1-\pi}{\pi}\right) \Big/ \log\left(\frac{q_1}{q_0}\right)\right], \tag{5.6.11}$$

for $\pi \le B$, where $[y]$ is the integer part of y. Notice that d is such that

$$\pi \in I_n \Longleftrightarrow d(\pi, B) = n, \quad 0 < \pi \le B. \tag{5.6.12}$$

Consider the equation (5.5.16) and (5.6.3) on the interval $I_1 = (B_1, B]$; because of the properties (5.6.6), (5.6.7), (5.6.8) and by construction of B_1, $S(\pi, x)$ ranges in the interval

$(B,1]$. Then, for (5.5.17), we can set $V\big(S(\pi,x)\big) = b\big(1-S(\pi,x)\big)$, for $\pi \in I_1$ and $x \geq 1$: thus, we have a first order linear differential equation and by using the continuity condition (5.5.21) at B, a unique solution $\pi \mapsto V(\pi,B)$ on I_1 can be recovered. Especially, for $\pi \in I_1$, we have:

$$V(\pi,B) = \frac{\pi^{\gamma_0}}{(1-\pi)^{\gamma_1}} C_1 - \pi \underbrace{\left(\frac{k_0-k_1}{k_0k_1} + b\right)}_{=H_1} + \underbrace{\left(b - \frac{1}{k_0}\right)}_{=G_1}, \tag{5.6.13}$$

where we set

$$k_i = \log p_i \quad \text{and} \quad \gamma_i = \frac{k_i}{k_0-k_1}, \quad i=0,1, \tag{5.6.14}$$

with C_1 given by:

$$C_1 = \frac{(1-B)^{\gamma_1}}{B^{\gamma_0}} \left(B\frac{k_0-k_1}{k_0k_1} + \frac{1}{k_0}\right). \tag{5.6.15}$$

Consider then the equation (5.5.16) and (5.6.3) on the interval $I_2 = (B_2, B_1]$; because of the properties (5.6.6), (5.6.7), (5.6.8), (5.6.9) and by construction of B_2 and B_1, $S(\pi,1)$ ranges in I_1, while $S(\pi,x)$, for $x \geq 2$, ranges in $(B,1]$. Hence, setting $V\big(S(\pi,1)\big)$ equal to the solution found on I_1, evaluated in $S(\pi,1)$, $V\big(S(\pi,x)\big) = b\big(1-S(\pi,x)\big)$, for $x \geq 2$ and imposing a continuity condition on $I_2 \cup I_1$ at B_1, we obtain a unique solution $\pi \mapsto V(\pi,B)$ on I_2:

$$\begin{aligned} V(\pi,B) =& \frac{\pi^{\gamma_0}}{(1-\pi)^{\gamma_1}} \left(C_2 + \frac{C_1}{k_0-k1} \frac{q_1^{\gamma_0}}{q_0^{\gamma_1}} \log\left(\frac{\pi}{1-\pi}\right)\right) \\ & - \pi \underbrace{\left(\frac{k_0-k_1}{k_0k_1} + b + \frac{q_0}{k_0^2} - \frac{q_1}{k_1^2}\right)}_{=H_2} + \underbrace{\left(b - \frac{1}{k_0} + \frac{q_0}{k_0^2}\right)}_{=G_2}, \end{aligned} \tag{5.6.16}$$

where

$$C_2 = C_1 \left(1 - \frac{1}{k_0-k_1}\frac{q_1^{\gamma_0}}{q_0^{\gamma_1}} \log\left(\frac{\pi}{1-\pi}\right)\right) + \frac{(1-B_1)^{\gamma_1}}{B_1^{\gamma_0}} \Big(B_1(H_2-H_1) - (G_2-G_1)\Big). \tag{5.6.17}$$

Suppose now to be on the generic interval $I_n = (B_n, B_{n-1}]$, on which we consider the equation (5.5.16) and (5.6.3); because of the properties (5.6.6), (5.6.7), (5.6.8), (5.6.9) and by construction of $B_n, B_{n-1}, \ldots, B_1$, $S(\pi,1)$ ranges in I_{n-1}, $S(\pi,2)$ ranges in I_{n-2}, …, $S(\pi,n-1)$ ranges in I_1 and $S(\pi,x)$ ranges in $(B,1]$, for $x \geq n$. So setting $V\big(S(\pi,1)\big)$ equal to the solution found on I_{n-1}, evaluated in $S(\pi,1)$, $V\big(S(\pi,2)\big)$ equal to the solution found on I_{n-2}, evaluated in $S(\pi,2)$, …, $V\big(S(\pi,n-1)\big)$ equal to the solution found on I_1, evaluated in $S(\pi,n-1)$, and $V\big(S(\pi,x)\big) = b\big(1-S(\pi,x)\big)$, for $x \geq n$, and imposing a continuity condition on $\cup_{i=1}^n I_i$ at B_{n-1}, we have a unique solution $\pi \mapsto V(\pi,B)$ on I_n:

$$V(\pi,B) = \frac{\pi^{\gamma_0}}{(1-\pi)^{\gamma_1}} \phi_n(\pi) - \pi H_n + G_n, \tag{5.6.18}$$

where:

$$\phi_n(\pi) = C_n + \sum_{j=1}^{n-1} \beta(j) \int \frac{\phi_{n-j}\big(S(\pi,j)\big)}{\pi(1-\pi)} \, d\pi, \tag{5.6.19}$$

$$\beta(n) = \left(\frac{q_1^{\gamma_0}}{q_0^{\gamma_1}}\right)^n \frac{1}{n(k_0-k_1)}, \tag{5.6.20}$$

with $\{C_n\}$, $\{H_n\}$, $\{G_n\}$, $n \geq 1$, being constants satisfying the following recurrence relationships (C_1 is given by (5.6.15)):

$$\begin{aligned} C_{n+1} =& \phi_n(B_n) - \sum_{j=1}^{n} \beta(j) \int \frac{\phi_{n+1-j}\big(S(\pi,j)\big)}{\pi(1-\pi)}\, d\pi \bigg|_{\pi=B_n} \\ &+ \frac{(1-B_n)^{\gamma_1}}{B_n^{\gamma_0}} \Big(B_n(H_{n+1}-H_n) - (G_{n+1}-G_n) \Big), \end{aligned} \tag{5.6.21}$$

$$\begin{aligned} H_n =& -\frac{1}{k_1} \left(\sum_{j=1}^{n-1} \frac{q_1^j}{j} (H_{n-j} - G_{n-j}) \right) \\ &- \frac{1}{k_0} \left(\left(\sum_{j=1}^{n-1} \frac{q_0^j}{j} G_{n-j} \right) - b\left(k_0 + \sum_{j=1}^{n-1} \frac{q_0^j}{j} \right) \right) + \frac{k_0 - k_1}{k_0 k_1}, \end{aligned} \tag{5.6.22}$$

$$G_n = -\frac{1}{k_0} \left(\left(\sum_{j=1}^{n-1} \frac{q_0^j}{j} G_{n-j} \right) - b\left(k_0 + \sum_{j=1}^{n-1} \frac{q_0^j}{j} \right) + 1 \right). \tag{5.6.23}$$

By adopting the distance function (5.6.11), the general solution of the functional equation (5.5.16) and (5.6.3), satisfying the continuity condition (5.5.21) at B, can be expressed for each $\pi \leq B$:

$$V(\pi, B) = \frac{\pi^{\gamma_0}}{(1-\pi)^{\gamma_1}} \phi_{d(\pi,B)}(\pi) - \pi H_{d(\pi,B)} + G_{d(\pi,B)}, \tag{5.6.24}$$

with $\phi_{d(\pi,B)}(\pi)$, $C_{d(\pi,B)}$, $H_{d(\pi,B)}$ and $G_{d(\pi,B)}$ which follows in an obvious manner from (5.6.19), (5.6.21)-(5.6.23). Observe that, by construction, the map $\pi \mapsto V(\pi, b)$ is C^1 on $(0, B)$, but C^0 at B.

The smooth fit condition (5.5.20) requires to evaluate the first derivative of $\pi \mapsto V(\pi, B)$; notice that when we compute it, $d(\pi, B)$ can be seen as independent of π:

$$\begin{aligned} V'(\pi, B) =& \frac{\pi^{\gamma_0 - 1}}{(1-\pi)^{\gamma_1+1}} \Bigg((\gamma_0 - \pi) C_{d(\pi,B)} + \sum_{j=1}^{d(\pi,B)-1} \beta(j) \Bigg((\gamma_0 - \pi) \\ &\times \int \frac{\phi_{d(\pi,B)-j}\big(S(\pi,j)\big)}{\pi(1-\pi)}\, d\pi + \phi_{d(\pi,B)-j}\big(S(\pi,j)\big) \Bigg) \Bigg) - H_{d(\pi,B)}, \end{aligned} \tag{5.6.25}$$

for $0 < \pi \leq B$.

It is verified that $\pi \mapsto V(\pi, B)$ is concave on $(0, B)$, $\lim_{\pi \to 0} V(\pi, B) = -\infty$, $\forall B \in [c, 1]$; further, from (5.6.13), we have that $V(\pi, B) < 0$, as $B \uparrow 1$, $\forall \pi \in (0, 1)$, (see Figure 5.3). As stated in Theorem 5.5.1, the system (5.5.25) and (5.5.26) has a unique solution if and only if the condition (5.5.22) is satisfied. Observe that when $B \downarrow c$, B_1 is strictly less than c, which means that (5.5.22) can be directly checked using (5.6.13): easy calculations show that (5.5.22) holds if and only if

$$\log \left(\frac{p_0}{p_1} \right) > \frac{1}{a} + \frac{1}{b}. \tag{5.6.26}$$

In this case, the optimal boundary points $A^\star$ and $B^\star$ are obtained as solution of the system of transcendental equations $V(A^\star, B^\star) = aA^\star$ and $V'(A^\star, B^\star) = a$, where the maps $\pi \mapsto V(\pi, B)$

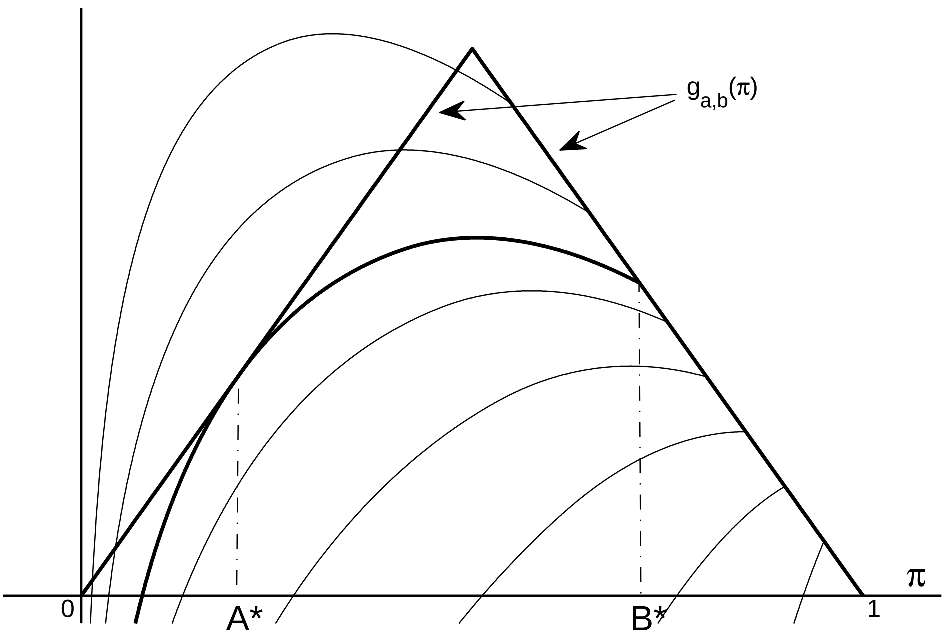

Figure 5.3: A computer drawing of the map $\pi \mapsto V(\pi, B^\star)$ (bold curve), for $\pi \leq B^\star$, with $a = b = 8$, $p_0 = 0.8$ and $p_1 = 0.3$. The unique pair of points satisfying (5.5.25) and (5.5.26) is $A^\star$ =0.2004.. and $B^\star$ =0.7142.. These represent the only points where $\pi \mapsto V(\pi, B^\star)$ hits smoothly $a\pi$. Observe that, unlike Figure 5.1, at $B^\star$ the smooth fit does not hold. The other curves $\pi \mapsto V(\pi, B)$ are obtained setting B =0.55, 0.65, 0.75, 0.8,. . ., 0.95.

and $\pi \mapsto V'(\pi, B)$ are given by (5.6.24) and (5.6.25). Once $A^\star$ and $B^\star$ have been determined, the solution $\pi \mapsto V(\pi, B^\star)$, for $\pi \in (A^\star, B^\star)$, is given through (5.6.24).

The next theorem states all the previous facts.

Theorem 5.6.1 *The π-Bayes decison rule $(\tau^\star_\pi, d^\star_\pi)$ for the sequential testing of two simple hypotheses about the parameter p of a n.b.p., with $p_0 > p_1$:*

(I) is explicitly given by (5.5.23) *and* (5.5.24)*, if the condition* (5.6.26) *holds; the optimal boundaries $A^\star$ and $B^\star$ are obtained as unique solution of the system of transcendental equations* (5.5.25) *and* (5.5.26)*, where $\pi \mapsto V(\pi, B)$ and $\pi \mapsto V'(\pi, B)$ are provided by* (5.6.24) *and* (5.6.25)*. The value function $V(\pi)$ of* (5.3.5) *coincides, on $(A^\star, B^\star)$, with $\pi \mapsto V(\pi, B^\star)$ (given by means of equation* (5.6.24)*) and, on $[0, 1]\backslash(A^\star, B^\star)$, with $\pi \mapsto g_{a,b}(\pi)$;*

(II) becomes trivial, if the condition (5.6.26) *fails to hold (see point (II) of Theorem 5.5.1).*

Proof. Denote by $V^\star(\pi)$ the solution to the free-boundary (5.5.16)-(5.5.21) and (5.6.3); according to Theorem 5.5.1 we only need to show that $(\mathbb{L}V^\star)(\pi) \geq -1$, $\forall \pi \in [0, 1]$. This condition is obviously satisfied by construction on the interval $(A^\star, B^\star)$. For $\pi \in (B^\star, 1)$, $(\mathbb{L}V^\star)(\pi) = 0$, since $\mathbb{L}f = 0$ if $f(\pi) = b(1 - \pi)$; further, because of the smooth and the continuous fit, $(\mathbb{L}V^\star)(A^\star) = -1$. It remains to prove that $(\mathbb{L}V^\star)(\pi) \geq -1$, for $\pi \in (0, A^\star)$.

Consider on the interval $\left(S^{-1}(A^\star,1),A^\star\right]$ the equation $(\mathbb{L}V)(\pi)=-1$, upon imposing $V(\pi)=V(\pi,B^\star)$, for $A^\star<\pi<B^\star$, and $V(\pi)=b(1-\pi)$, for $\pi\geq B^\star$. Solve the specified equation, with the initial condition $V(A^\star)=V(A^\star,B^\star)+c$, $c\geq 0$. From (5.6.18), we get the unique solution $\pi\mapsto V_c(\pi)$, with $V_c(\pi)=V(\pi,B^\star)+\frac{\pi^{\gamma_0}}{(1-\pi)^{\gamma_1}}\frac{(1-A^\star)^{\gamma_1}}{A^{\star\gamma_0}}c$, for each fixed c. Notice that the curves $\pi\mapsto V_c(\pi)$ do not intersect on $\left(S^{-1}(A^\star,1),A^\star\right]$ for different c's. One can observe that $\exists\, c_0>0$, such that, $\forall c\in[0,c_0)$ the maps $\pi\mapsto V_c(\pi)$ and $\pi\mapsto a\pi$ intersect on $\left(S^{-1}(A^\star,1),A^\star\right]$, while for $c\geq c_0$, they do not intersect on such an interval. For $c\in[0,c_0)$, denote by π_c this intersection point (obviously, $\pi_0=A^\star$) and notice that as $c\uparrow c_0$, π_c decreases continuously toward $S^{-1}(A^\star,1)$. Thus, since $V_c'(\pi_c)>a=V'^\star(\pi_c)$, $V_c(\pi_c)=a\pi_c=V^\star(\pi_c)$ and $V_c\left(S(\pi_c,x)\right)=V^\star\left(S(\pi_c,x)\right)$, for $x\geq 1$, from the expresssion of the infinitesimal operator (5.6.3), we have $(\mathbb{L}V^\star)(\pi_c)\geq(\mathbb{L}V_c)(\pi_c)=-1$.

In analogous way, the previuous reasoning can be extendend to the interval $\left(S^{-1}(A^\star,n),\right.$ $\left.S^{-1}(A^\star,n-1)\right]$, $n\geq 1$, with $S^{-1}(A^\star,0)=A^\star$. Consider on such an interval the equation $(\mathbb{L}V)(\pi)=-1$, upon imposing $V(\pi)=V(\pi,B^\star)$, for $S^{-1}(A^\star,n-1)<\pi<B^\star$, and $V(\pi)=b(1-\pi)$, for $\pi\geq B^\star$; solve the specified equation, with the initial condition $V\left(S^{-1}(A^\star,n-1)\right)=V\left(S^{-1}(A^\star,n-1),B^\star\right)+k_n+c$, where $k_n\geq 0$ is given by $V\left(S^{-1}(A^\star,n-1),B^\star\right)+k_n=aS^{-1}(A^\star,n-1)$ (observe that $k_1=0$) and $c\geq 0$. Applying (5.6.18), we can obtain the unique solution $\pi\mapsto V_c(\pi)$, with $V_c(\pi)=V(\pi,B^\star)+\frac{\pi^{\gamma_0}}{(1-\pi)^{\gamma_1}}\times\frac{(1-S^{-1}(A^\star,n-1))^{\gamma_1}}{(S^{-1}(A^\star,n-1))^{\gamma_0}}(k_n+c)$, for each fixed c. Notice that the curves $\pi\mapsto V_c(\pi)$ do not intersect on $\left(S^{-1}(A^\star,n),S^{-1}(A^\star,n-1)\right]$ for different c's. One can observe that $\exists\, c_0>0$, such that, $\forall c\in[0,c_0)$ the maps $\pi\mapsto V_c(\pi)$ and $\pi\mapsto a\pi$ intersect on $\left(S^{-1}(A^\star,n),S^{-1}(A^\star,n-1)\right]$, while for $c\geq c_0$, they do not intersect on such an interval. For $c\in[0,c_0)$, denote by π_c this intersection point (obviously, $\pi_0=S^{-1}(A^\star,n-1)$) and notice that as $c\uparrow c_0$, π_c decreases continuously toward $S^{-1}(A^\star,n)$. Thus, since $V_c'(\pi_c)>a=V'^\star(\pi_c)$, $V_c(\pi_c)=a\pi_c=V^\star(\pi_c)$ and $V_c\left(S(\pi_c,x)\right)\leq V^\star\left(S(\pi_c,x)\right)$, for $x\geq 1$, from the expresssion of the infinitesimal operator (5.6.3), we have $(\mathbb{L}V^\star)(\pi_c)\geq(\mathbb{L}V_c)(\pi_c)=-1$. ∎

Remark 5.6.1 The π-Bayes decision rule (5.5.23) and (5.5.24) can be rewritten in this case in the following way:

$$\tau_\pi^\star=\inf\{t\geq 0:\ Z_t\notin(\widetilde{A}^\star,\widetilde{B}^\star)\},\tag{5.6.27}$$

$$d_\pi^\star=\begin{cases}0 & \text{(accept } H_0\text{)} \quad \text{if} \quad Z_{\tau_\pi^\star}\leq\widetilde{A}^\star\\ 1 & \text{(accept } H_1\text{)} \quad \text{if} \quad Z_{\tau_\pi^\star}\geq\widetilde{B}^\star\end{cases},\tag{5.6.28}$$

where:

$$Z_t=X_t+\mu t,\tag{5.6.29}$$

$$\mu=\log\left(\frac{p_1}{p_0}\right)\Big/\log\left(\frac{1-p_1}{1-p_0}\right),\tag{5.6.30}$$

$$\widetilde{A}^\star=\log\left(\frac{A^\star}{1-A^\star}\frac{1-\pi}{\pi}\right)\Big/\log\left(\frac{1-p_1}{1-p_0}\right),\tag{5.6.31}$$

$$\widetilde{B}^\star=\log\left(\frac{B^\star}{1-B^\star}\frac{1-\pi}{\pi}\right)\Big/\log\left(\frac{1-p_1}{1-p_0}\right).\tag{5.6.32}$$

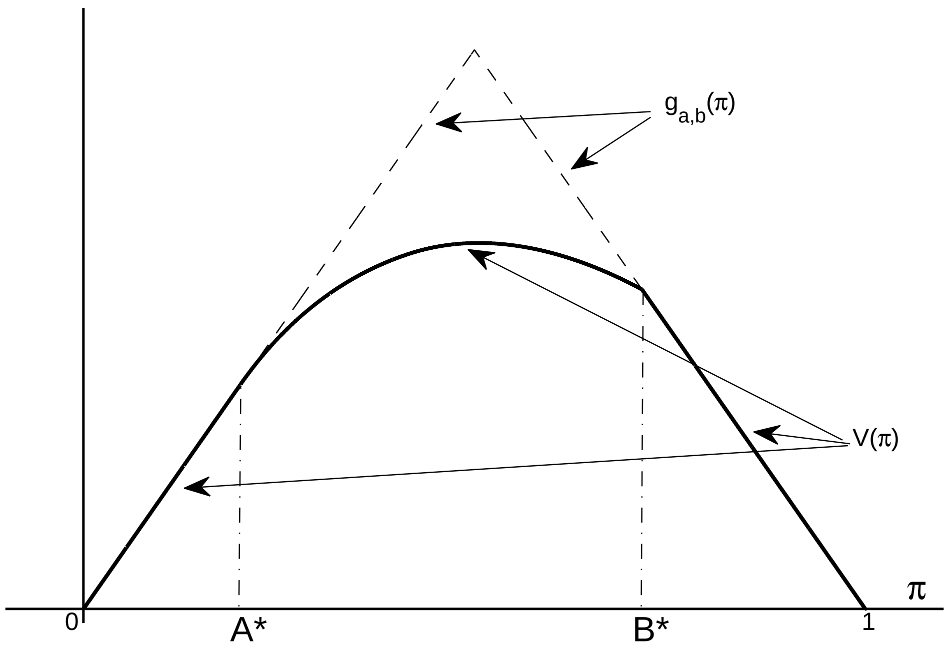

Figure 5.4: A computer drawing of the map $\pi \mapsto V(\pi)$, coinciding with the final payoff, given by (5.3.5). The same parameters of Figure 5.3 have been used. The set $D = \{\pi \in [0,1] : V(\pi) = g_{a,b}(\pi)\} = [0, A^\star] \cup [B^\star, 1]$ is the stopping region, on which $V(\pi) = g_{a,b}(\pi)$, while the interval $(A^\star, B^\star)$ is the set of continued observation of the posterior probability process $(\pi_t)_{t\geq 0}$, where $V(\pi) = V(\pi, B^\star)$ (given by (5.6.24)).

In other words, we can apply the following equivalent optimal sequential procedure: observe the process (X_t) and evaluate the correspondent stochastic quantity (Z_t); as soon as (Z_t) enters either $(-\infty, \widetilde{A}^\star]$ or $[\widetilde{B}^\star, \infty)$ stop the observation; in the first case we accept $H_0 : \vartheta = 0$ (i.e., $p = p_0$), while in the second one we accept $H_1 : \vartheta = 1$ (i.e., $p = p_1$). Obviously, the condition (5.6.26) must be satisfied: the two optimal boundaries $\widetilde{A}^\star$ and $\widetilde{B}^\star$ are determined through (5.6.31) and (5.6.32), after that $A^\star$ and $B^\star$ have been obtained, as solution of (5.5.25) and (5.5.26).

5.7 Conclusions

We have presented the sequential testing of two simple hypotheses, concerning the Lévy-Khintchine triplet of a Lévy process. Especially, we concentrated on continuous paths and pure jump increasing Lévy processes, with the latter having finite variation. We have shown that the Shiryaev [72] and Peskir and Shiryaev [65] sequential testing for the mean and the intensity of a Wiener and Poisson process, respectively, fall into the more general setting of sequential testing for Lévy processes.

Our key contribution has been to provide the exact solution to the Bayesian sequential testing of a negative binomial process: indeed, after a deep investigation, we had not found trace of its solution in the related literature, although this problem (in the variational formulation) was posed by Dvoretzky et al. [28], about sixty years ago.

Another interesting problem is the sequential testing for a gamma process: it is a Lévy process, but, unlike the Poisson and the negative binomial ones, has infinite Lévy measure on the real positive half-line. This problem will be analyzed it in Chapter 7.

Chapter 6

On the Wald's Sequential Probability Ratio Test for Lévy Processes

The Wald's sequential probability ratio test (SPRT) of two simple hypotheses regarding the Lévy-Khintchine triplet of a wide family of Lévy processes is analyzed: we concentrate on continuous paths and pure increasing jump Lévy processes. Appealing to the theory of Markov processes, we employ a general method for determining the stopping boundaries and the expected length of the SPRT, for a given admissible pair (α, β) of error probabilities. The well known results of the Wiener and Poisson sequential testing can be derived accordingly. The explicit solution for the SPRT of two simple hypotheses about the parameter $p \in (0, 1)$ of a Lévy negative binomial process is shown.

6.1 Introduction

Lots of physical, social, economic, financial and biologic random phenomena are often modeled by Lévy processes: they are stochastic processes starting from 0, with independent and stationary increments and right-continuous with left limit trajectories. The characteristic function $\hat{\mu}_t(z)$ of a Lévy process $X = (X_t)_{t \geq 0}$ can be expressed at any time $t \geq 0$ in terms of the so called Lévy-Khintchine triplet $g = \{\gamma, \sigma^2, v\}$:

$$\hat{\mu}_t(z) = \exp\left(it\gamma z - \frac{1}{2}t\sigma^2 z^2 + t\int_{\mathbb{R}} \left(e^{izx} - 1 - izx\mathbf{1}_{\{|x|\leq 1\}}(x)\right) v(dx)\right), \tag{6.1.1}$$

for any $z \in \mathbb{R}$, where $\gamma \in \mathbb{R}$, $\sigma^2 \geq 0$ and $v(\cdot)$ is the so called Lévy measure on $\mathbb{R}$, satisfying $v(\{0\}) = 0$ and $\int (x^2 \wedge 1)\, v(dx) < \infty$.

It is assumed that at time $t = 0$, we begin the observation of a Lévy process $X = (X_t)_{t\geq 0}$ with unknown triplet g; nevertheless, our knowledge and experience allow us to discriminate between two possible hypotheses: $g = g_0$ and $g = g_1$, where $g_i = \{\gamma_i, \sigma^2, v_i\}$, $i = 0, 1$. The goal is to sequentially test which of the two hypotheses is true: while in Chapter 5 (or Buonaguidi and Muliere [16]) the Bayesian version of the problem has been studied, the present chapter aims to solve its classical counterpart, known as *fixed error probability* or *variational* formulation. This can be explained as follows: once the first and second type error probabilities have been fixed, we want to find a stopping rule, which allows us to "optimally" interrupt the observation of X, in order to reach a decision about g. The stopping rule is defined to be optimal if it has the smallest expected observation time (under both the hypotheses) in the class of those stopping rules, whose error probabilities do not exceed the given ones.

This problem was designed by Wald [79, 80], who introduced the notion of sequential probability ratio test (SPRT) for discrete time processes. Later, Dvoretzky et al. [28] started the analysis of the sequential testing for continuous time stochastic processes: they determined the stopping boundaries and the expected observation time of the SPRT for the drift and the intensity of a Wiener and Poisson process, respectively. A formal proof of the optimal character of the SPRT for the drift of a Wiener process was obtained by Shiryaev [72, Sec. 4.2]; Irle and Schmitz [43] gave a rigorous proof of the optimality of the SPRT, when the observed log-likelihood ratio process is continuous in time and has independent and stationary increments; Bhat [14] derived the optimality properties of the SPRT for certain parameters of some diffusion-type and jump processes. Peskir and Shiryaev [65, Sec. 3] characterized the set of all the admissible pairs of first and second type error probabilities, outside which the SPRT for the intensity of a Poisson process does not exist. The analysis was successively extended to compound Poisson processes with exponential jumps by Gapeev [32].

The Bayesian formulation, which relies on the existence of a priori distribution about the propriety of the two hypotheses to be tested and on the adoption of a stopping rule minimizing the Bayesian risk (that is, the expected loss due to the sampling cost and a final incorrect choice), is strictly connected with the fixed error probability formulation: indeed, the former was exploited by Wald and Wolfowitz [81] and Peskir and Shiryaev [65, Sec. 3] as technical device for achieving the proof of the optimality of the latter, in the case of i.i.d. random variables and a Poisson process, respectively. Other fundamental works within the Bayesian framework are due to Shiryaev [72, Sec. 4.2], Gapeev and Peskir [35], Gapeev and Shiryaev [37] and Shiryaev and Zhitlukhin [75], for establishing the correct drift of a Brownian motion or a more general diffusion process, and to Gapeev [32], Dayanik and Sezer [24], Dayanik et al. [23] and Dayanik and Sezer [25], for identifying the parameters of compound Poisson or jump-diffusion processes.

As we said, in this chapter we deal with the fixed error probability formulation only. In particular: (1) we extend the well known results about the sequential testing for the Wiener and Poisson process to continuous paths and pure increasing jump Lévy processes. For the former, we derive the results shown in Shiryaev [72] with slightly different techniques, as well as the moment generating function of the optimal time we need to reach a decision; for the latter, we generalize the results of Dvoretzky et al. [28] and Peskir and Shiryaev [65] and it will be noted that the optimal stopping rule, given by the SPRT, exists only when the pair of first and second type error probabilities belongs to the "admissible" region; (2) we use the previous results for deriving the exact expression of the SPRT for the parameter $p \in (0, 1)$ of a negative binomial process. This problem was formulated in Dvoretzky et al. [28], but the lack of its solution in the related literature, as well as the possibility to use such a process for modeling a huge number of phenomena, make our task very challenging from a theoretical and applied view point.

The chapter is organized as follows. In Section 6.2, we describe more formally the problem stated in the Introduction. The stopping boundaries, the set of the admissible error probabilities and the expected observation time of the SPRT are obtained in Sections 6.3 and 6.4, for continuous paths and pure increasing jump Lévy processes, respectively; in Section 6.5, the explicit solution of the SPRT for the parameter p of a negative binomial process is derived. Section 6.6 concludes the chapter with a summary discussion.

All the theorems on Lévy processes we will make use are based on Sato [71]; they have been recalled in the previous chapter.

6.2 Continuous time sequential probability ratio test

Let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, P_i)$, $i = 0, 1$, be a filtered probability space, hosting a Lévy process $X = (X_t)_{t\geq 0}$, having Lévy-Khintchine triplet $g_0 = \{\gamma_0, \sigma^2, v_0\}$, under P_0, and $g_1 = \{\gamma_1, \sigma^2, v_1\}$, under P_1, with $P_1|\mathscr{F}_t$ and $P_0|\mathscr{F}_t$ (where $P_i|\mathscr{F}_t$ is the restriction of P_i to $\mathscr{F}_t$, $i = 0, 1$), which are assumed to be mutually absolutely continuous, for any $t \geq 0$. At time $t = 0$, the observation of a path of X is begun, in order to sequentially test the following two simple hypotheses:

$$H_0 : g = g_0 \quad Vs \quad H_1 : g = g_1. \tag{6.2.1}$$

By the pair (τ, d), we denote a sequential decision rule: τ is a stopping time of X, that is, an $\mathscr{F}_t^X$-measurable random variable, $t \geq 0$, where $\mathscr{F}_t^X = \sigma\{X_s : 0 \leq s \leq t\}$ is the natural filtration generated by X up to t; d is the decision function associated to τ, that is, an $\mathscr{F}_\tau^X$-measurable random variable, taking on value i, if H_i must be accepted, $i = 0, 1$. Let $\beta(d) = P_0(d = 1)$ and $\alpha(d) = P_1(d = 0)$ be the probabilities of error of the first and second kind, respectively, associated to (τ, d). Given two positive real numbers α and β, with $\alpha + \beta < 1$, we desire to find the optimal decision rule $(\hat{\tau}, \hat{d})$ in the class $\Delta(\alpha, \beta) = \{(\tau, d) : \alpha(d) \leq \alpha,\ \beta(d) \leq \beta\}$. Here, optimality means that $(\hat{\tau}, \hat{d})$ simultaneously minimizes the expected length of the observation time, both under P_0 and P_1:

$$E_0[\hat{\tau}] \leq E_0[\tau] \quad \text{and} \quad E_1[\hat{\tau}] \leq E_1[\tau], \quad \forall (\tau, d) \in \Delta(\alpha, \beta). \tag{6.2.2}$$

Once $(\hat{\tau}, \hat{d})$ has been found, the fixed error probability (or variational) sequential testing problem is solved. From the definition of the SPRT for discrete time processes, see Wald [79, 80], it is not hard to guess that the optimal sequential decision rule has the following structure:

$$\hat{\tau} = \inf\{t \geq 0 : \log(\varphi_t) \notin (\hat{A}, \hat{B})\}, \tag{6.2.3}$$

$$\hat{d} = \begin{cases} 0 \quad (\text{accept } H_0), & \text{if} \quad \log(\varphi_{\hat{\tau}}) \leq \hat{A}, \\ 1 \quad (\text{accept } H_1), & \text{if} \quad \log(\varphi_{\hat{\tau}}) \geq \hat{B}, \end{cases} \tag{6.2.4}$$

where $\varphi = (\varphi_t)_{t\geq 0}$, defined by

$$\varphi_t = \frac{d(P_1|\mathscr{F}_t^X)}{d(P_0|\mathscr{F}_t^X)}, \tag{6.2.5}$$

is the likelihood ratio process and $\hat{A}$ and $\hat{B}$ are two constants, chosen so that $\hat{A} < 0 < \hat{B}$ and satisfying

$$P_1(\hat{d} = 0) = \alpha, \tag{6.2.6}$$

$$P_0(\hat{d} = 1) = \beta. \tag{6.2.7}$$

To stay close to the special kinds of Lévy processes which have already been analyzed in this context (Wiener process with drift, Poisson process and compound Poisson process with exponential jumps), we study the case when X is either a *continuous paths* or a *pure increasing jump* Lévy process. It will be seen in Sections 6.3 and 6.4 that in these situations the likelihood ratio (6.2.5) takes the form

$$\varphi_t = \exp\big((X_t - \mu t)k\big), \tag{6.2.8}$$

where μ, $k \in \mathbb{R}$, when X has continuous paths, and μ, $k \in \mathbb{R}^+$, when X is purely increasing by jumps. Then, it is easy to see that (6.2.3) and (6.2.4) can be equivalently expressed as:

$$\hat{\tau} = \inf\{t \geq 0 : Z_t = A \text{ or } Z_t \geq B\}, \tag{6.2.9}$$

$$\hat{d} = \begin{cases} 0 \quad (\text{accept } H_0), & \text{if} \quad Z_{\hat{\tau}} = A, \\ 1 \quad (\text{accept } H_1), & \text{if} \quad Z_{\hat{\tau}} \geq B, \end{cases} \tag{6.2.10}$$

where the process $Z = (Z_t)_{t \geq 0}$ is defined by

$$Z_t = X_t - \mu t, \tag{6.2.11}$$

and $A(= \hat{A}/k) < 0 < B(= \hat{B}/k)$ are two constants, still solving (6.2.6) and (6.2.7). Because of the class of processes we consider and the expressions (6.2.8) and (6.2.11), we may notice in (6.2.9) and (6.2.10) that $\{Z_{\hat{\tau}} \leq A\} = \{Z_{\hat{\tau}} = A\}$, while $\{Z_{\hat{\tau}} \geq B\} = \{Z_{\hat{\tau}} = B\}$ only when X is a continuous paths Lévy process. Indeed, the process Z always moves downwards continuously: this implies that the first time it is below the threshold A coincides with the first time at which it hits A; the same reasoning holds for B when X has continuous sample paths, but, of course, it does not hold when X, and so Z, increase by jumps.

The next theorem states the optimality of the decision rule $(\hat{\tau}, \hat{d})$.

Theorem 6.2.1 *Let X be a continuous paths or a pure increasing jump Lévy process, with Lévy-Khintchine triplet g_i, under P_i, $i = 0, 1$. In the sequential testing of the two simple hypotheses* (6.2.1)*, for a fixed pair (α, β) of error probabilities, with $\alpha + \beta < 1$, the decision rule $(\hat{\tau}, \hat{d}) \in \Delta(\alpha, \beta)$, expressed by* (6.2.9) *and* (6.2.10) *and characterized by* (6.2.6) *and* (6.2.7)*, is optimal for the problem* (6.2.2) *and is unique P_i-a.s., $i = 0, 1$.*

Proof. The claims follow immediately from the fact that the process Z has independent and stationary increments and from Irle and Schmitz [43, Th. 4.2]. An alternative proof can be derived by combining the results of the previous chapter, Theorems 5.4.1 and 5.5.1, and the three-steps procedure described in Peskir and Shiryaev [65, pp. 852-854], in order to pass from the Bayesian to the fixed error probability solution. ∎

In the next sections we will see how the constants A and B and the expressions of $E_i[\hat{\tau}]$, $i = 0, 1$, characterizing the decision rule (6.2.9) and (6.2.10), can be determined, when X is a continuous paths or a pure increasing jump Lévy process. We recall that if X is a Lévy process under a probability measure P, then X is a (strong) Markov process with starting point 0 and transition probability function $P(X_t \in B | X_s = x) = P_{X_{t-s}}(B - x)$, for any $s \geq 0$, $t > s$ and $B \in \mathscr{B}(\mathbb{R})$ (see Sato [71, Th. 10.5, p. 57]). From this result and (6.2.11), it immediately follows that Z is a Markov process under P_i, $i = 0, 1$. This consideration is of paramount importance, since it allows us to borrow relevant techniques from the Markov process theory and to use them for solving our problems.

6.3 SPRT: continuous path Lévy processes

Let X be a Lévy process with Lévy-Khintchine triplet $g_i = \{\gamma_i, \sigma^2, 0\}$, under P_i, $i = 0, 1$. It is easily deduced that X has P_i-a.s. continuous sample paths, that is, X is a Wiener process with diffusion coefficient σ^2 and drift γ_i under P_i: $X_t = \gamma_i t + \sigma W_t$, where $W = (W_t)_{t \geq 0}$ is a standard Wiener process. According to the density transformation for Lévy processes (see Sato [71, Th. 33.2, p. 219]) the likelihood ratio (6.2.5) takes the form

$$\varphi_t = \exp\left(\frac{\gamma_1 - \gamma_0}{\sigma^2}\left(X_t - \frac{t}{2}(\gamma_1 + \gamma_0)\right)\right), \tag{6.3.1}$$

so that (6.2.8) holds with $\mu = (\gamma_1 + \gamma_0)/2$ and $k = (\gamma_1 - \gamma_0)/\sigma^2$; the process $Z = (Z_t)_{t \geq 0}$ follows in an obvious manner from (6.2.11). The procedure for determining the stopping thresholds A and B and the average observation time is now analyzed with additional arguments with respect to Shiryaev [72, Sec. 4.2].

6.3.1 Optimal boundaries

It is straightforward to see that the following relationships hold:

$$P_1(Z_{\hat{\tau}} = A) = P_1(\varphi_{\hat{\tau}} = e^{Ak}) = e^{Ak} P_0(Z_{\hat{\tau}} = A) = e^{Ak}(1 - P_0(Z_{\hat{\tau}} = B)), \tag{6.3.2}$$

$$P_0(Z_{\hat{\tau}} \geq B) = P_0(\varphi_{\hat{\tau}} = e^{Bk}) = e^{-Bk} P_1(Z_{\hat{\tau}} = B) = e^{-Bk}(1 - P_1(Z_{\hat{\tau}} = A)), \tag{6.3.3}$$

where the second equalities in (6.3.2) and (6.3.3) follow from (6.2.5) and (6.3.1), while the last equality in (6.3.2) and the first equality in (6.3.3) are due to the fact that Z can cross B only continuously. According to (6.2.6), (6.2.7) and using (6.3.2) and (6.3.3), we get the following expressions for a fixed pair $(\alpha, \beta) \in (0, 1) \times (0, 1)$, with $\alpha + \beta < 1$:

$$A = \log\left(\frac{\alpha}{1 - \beta}\right) \Big/ k, \quad B = \log\left(\frac{1 - \alpha}{\beta}\right) \Big/ k. \tag{6.3.4}$$

An alternative way to derive (6.3.4) is offered by the theory of Markov processes. Let P_i^z be the probability measure under which X has triplet $g_i = \{\gamma_i, \sigma^2, 0\}$, $i = 0, 1$, and Z starts at $z \in \mathbb{R}$.

A simple application of Itô's formula implies that the infinitesimal generator $\mathbb{L}_i$ of Z, under P_i^z, acts on a function $w \in C^2(\mathbb{R})$ like

$$(\mathbb{L}_i w)(z) = (-1)^{1+i}\frac{\gamma_1 - \gamma_0}{2}\, w'(z) + \frac{\sigma^2}{2}\, w''(z), \quad i = 0, 1. \tag{6.3.5}$$

The functions $f(z) = P_1^z(Z_{\hat{\tau}} = A)$ and $h(z) = P_0^z(Z_{\hat{\tau}} = B)$ must satisfy the following systems (see, e.g., Peskir and Shiryaev [67, p. 130]):

$$\begin{cases} (\mathbb{L}_1 f)(z) = 0, & \text{if } z \in (A, B), \\ f(A) = 1, \\ f(B) = 0, \end{cases} \qquad \begin{cases} (\mathbb{L}_0 h)(z) = 0, & \text{if } z \in (A, B), \\ h(A) = 0, \\ h(B) = 1, \end{cases} \tag{6.3.6}$$

whose solutions are

$$f(z) = \frac{e^{kA}\left(e^{k(B-z)} - 1\right)}{e^{kB} - e^{kA}}, \quad h(z) = \frac{e^{kz} - e^{kA}}{e^{kB} - e^{kA}}. \tag{6.3.7}$$

Since the process Z starts at $z = 0$, from (6.2.6), (6.2.7) and (6.3.7) we obtain

$$\alpha = \frac{e^{kA}\left(e^{kB} - 1\right)}{e^{kB} - e^{kA}}, \quad \beta = \frac{1 - e^{kA}}{e^{kB} - e^{kA}}, \tag{6.3.8}$$

so that, for a fixed pair $(\alpha, \beta) \in (0, 1) \times (0, 1)$, with $\alpha + \beta < 1$, the expressions (6.3.4) are easily recovered.

Denoted by $\mathscr{A}$ the set of admissible error probabilities (α, β), for which the solution (A, B) to (6.2.6) and (6.2.7) exists, we see that $\mathscr{A} = \{(\alpha, \beta) : \alpha \in (0, 1),\, 0 < \beta < 1 - \alpha\}$. Indeed, we observe that the right-hand side of the second expression in (6.3.8) is a continuous and strictly decreasing function of B, so that taking the limit as $B \downarrow 0$, we obtain $\beta < 1$. This condition, together with the constraint $\alpha + \beta < 1$, implies that $0 < \beta < 1 - \alpha$, for a fixed $\alpha \in (0, 1)$.

6.3.2 Expected observation times

Let $M_i(z, u) = E_i^z[e^{u\hat{\tau}}]$ be the moment generating function of $\hat{\tau}$ under P_i^z, $i = 0, 1$. Then, according to the Markov process theory (see, e.g., Peskir and Shiryaev [67, p. 131]), $M_i(z, u)$ solves the system:

$$\begin{cases} (\mathbb{L}_i M_i)(z, u) = -u M_i(z, u), & \text{if } z \in (A, B), \\ M_i(z, u) = 1, & \text{if } z = A \text{ or } z = B. \end{cases} \tag{6.3.9}$$

Hence, (6.3.5) and the first row of (6.3.9) lead to

$$\frac{\partial^2}{\partial z^2} M_i(z, u) + (-1)^{1+i} k \frac{\partial}{\partial z} M_i(z, u) + \frac{2u}{\sigma^2} M_i(z, u) = 0, \quad i = 0, 1, \tag{6.3.10}$$

whose solution, taking into account the boundary conditions in (6.3.9), is

$$M_i(z, u) = \frac{\left(e^{p_i(u)B} - e^{p_i(u)A}\right)e^{q_i(u)z} + \left(e^{q_i(u)A} - e^{q_i(u)B}\right)e^{p_i(u)z}}{e^{q_i(u)A + p_i(u)B} - e^{p_i(u)A + q_i(u)B}}, \tag{6.3.11}$$

where $p_i(u) = \big((-1)^i k - \sqrt{k^2 - 8u/\sigma^2}\,\big)/2$, $q_i(u) = \big((-1)^i k + \sqrt{k^2 - 8u/\sigma^2}\,\big)/2$, $i = 0, 1$. Then,

$$E_0[\hat{\tau}] = \frac{\partial}{\partial u} M_0(z,u)\Big|_{u=0,z=0} = \frac{2}{\gamma_1 - \gamma_0}\left(\frac{(e^{kB}-1)(B-A)}{e^{kB}-e^{kA}} - B\right), \tag{6.3.12}$$

$$E_1[\hat{\tau}] = \frac{\partial}{\partial u} M_1(z,u)\Big|_{u=0,z=0} = \frac{2}{\gamma_1 - \gamma_0}\left(\frac{(e^{kB}-e^{k(A+B)})(B-A)}{e^{kB}-e^{kA}} + A\right). \tag{6.3.13}$$

Remark 6.3.1 Denoted by $g_i(z) = E_i^z[\hat{\tau}]$, $i = 0, 1$, so that $g_i(z) = \partial/(\partial u) M_i(z,u)|_{u=0}$, we see from (6.3.10) that $g_i(z)$ solves

$$g_i''(z) + (-1)^{1+i} k g_i'(z) + \frac{2}{\sigma^2} = 0, \quad i = 0, 1. \tag{6.3.14}$$

This is the equation solved in Shiryaev [72, Sec. 4.2, p. 187], that, together with the natural boundary conditions $g_i(A) = g_i(B) = 0$ and the fact that Z starts at $z = 0$, allows us to recover (6.3.12) and (6.3.13).

Remark 6.3.2 It is easily seen that the process Z admits the following Itô representation under P_i^z, $i = 0, 1$:

$$dZ_t = (-1)^{1+i}\frac{\gamma_1 - \gamma_0}{2}\, dt + \sigma\, dW_t, \quad Z_0 = z. \tag{6.3.15}$$

Its scale function $S_i(z)$, speed measure $m_i(dz)$ and Green function $G_i^{A,B}(z,y)$ on $[A,B]$ are therefore given by

$$S_i(z) = \int^z \exp\left(-\int^y (-1)^{1+i}\frac{\gamma_1-\gamma_0}{\sigma^2}\, dx\right) dy = (-1)^i \frac{e^{(-1)^i kz}}{k}, \quad z \in \mathbb{R}, \tag{6.3.16}$$

$$m_i(dz) = \frac{2}{S_i'(z)\sigma^2}\, dz = \frac{2}{e^{(-1)^i kz}\sigma^2}\, dz, \tag{6.3.17}$$

$$G_i^{A,B}(z,y) = \begin{cases} \frac{(-1)^i k^{-1}\left(e^{(-1)^i kB} - e^{(-1)^i kz}\right)\left(e^{(-1)^i ky} - e^{(-1)^i kA}\right)}{e^{(-1)^i kB} - e^{(-1)^i kA}}, & A \le y \le z, \\ \frac{(-1)^i k^{-1}\left(e^{(-1)^i kB} - e^{(-1)^i ky}\right)\left(e^{(-1)^i kz} - e^{(-1)^i kA}\right)}{e^{(-1)^i kB} - e^{(-1)^i kA}}, & z \le y \le B, \end{cases} \tag{6.3.18}$$

being the numerators of the first and second case of (6.3.18) equal to $\big(S_i(B) - S_i(z)\big)\big(S_i(y) - S_i(A)\big)$ and $\big(S_i(B) - S_i(y)\big)\big(S_i(z) - S_i(A)\big)$, respectively, and the common denominator equal to $S_i(B) - S_i(A)$. By standard results about one-dimensional diffusions (see, e.g., Peskir and Shiryaev [67, pp. 200-201]), we have

$$E_i^z[\hat{\tau}] = \int_A^B G_i^{A,B}(z,y)\, m_i(dy), \quad i = 0, 1. \tag{6.3.19}$$

Simple but tedious calculations and the fact that $z = 0$ lead to the expressions (6.3.12) and (6.3.13).

6.4 SPRT: pure increasing jump Lévy processes

Let $X = (X_t)_{t\geq 0}$ be a Lévy process with characteristic function under P_j, $j = 0, 1$, given by

$$\hat{\mu}_{j,t}(z) = \exp\left(t\int_0^\infty \left(e^{izx} - 1\right) v_j(dx)\right), \quad z \in \mathbb{R}, \quad t \geq 0. \tag{6.4.1}$$

A direct comparison between (6.1.1) and (6.4.1) shows that $\int_{-\infty}^0 v_i(dx) = 0$, $\int_0^1 xv_i(dx) = \gamma_i$, $i = 0, 1$. The following properties of X are therefore readily inferred by means of standard arguments (see Sato [71, Chap. 2 and 4]): X has Lévy-khintchine triplet $\widetilde{g}_i = \{\widetilde{\gamma}_i, \sigma^2, v_i\} = \{0, 0, v_i\}$, where $\widetilde{\gamma}_i = \gamma_i - \int_0^1 xv_i(dx)$, $i = 0, 1$; X has finite variation on $(0, t]$, $\forall t \geq 0$; X is non-decreasing and increases by jumps only. Thus, X is a pure increasing jump process.

The well known density transformation for Lévy processes (see Sato [71, Th. 33.2, p. 219]) leads to the following representation of the likelihood ratio (6.2.5):

$$\varphi_t = \exp\Big(\sum_{s\leq t} \log\left(\xi(X_s - X_{s^-})\right) - t\int \left(\xi(x) - 1\right) v_0(dx)\Big), \tag{6.4.2}$$

where $\xi(x) = dv_1(x)/dv_0(x)$ is the Radon-Nikodym derivative between the Lévy measures v_1 and v_0, that X has under P_1 and P_0, respectively. It is assumed that

$$\xi(x) > 1, \quad \forall x \in S, \tag{6.4.3}$$

$$P_{i,X_1} \in NEF, \quad i = 0, 1, \tag{6.4.4}$$

where $S \subseteq (0, \infty)$ is the common support of v_0 and v_1, P_{i,X_1} is the probability distribution of X_1 under P_i and NEF stands for natural exponential family. Assumption (6.4.3) does not entail any loss of generality; (6.4.4) is equivalent to the condition imposed by Dvoretzky et al. [28], for which the conditional distribution of X_s, $0 \leq s \leq t$, given X_t, is independent of $i = 0, 1$, for any $t \geq 0$, that is, X_t is a sufficient statistic for the process X. Under these assumptions, it is easily seen that (6.4.2) becomes

$$\begin{aligned}\varphi_t &= \exp\Big(h\sum_{s\leq t}(X_s - X_{s-}) - t\int \left(\xi(x) - 1\right) v_0(dx)\Big) \\ &= \exp\Big(hX_t - t\int \left(\xi(x) - 1\right) v_0(dx)\Big).\end{aligned} \tag{6.4.5}$$

Therefore, the representation (6.2.8) holds with $k = h$ and $\mu = k^{-1}\int \big(\xi(x) - 1\big) v_0(dx)$, with k, $\mu \in \mathbb{R}^+$.

6.4.1 Optimal boundaries and admissible error probabilities

For some $A < 0 < B$, consider the stopping time $\hat{\tau}$, provided by (6.2.9). Because of our assumptions, the process Z, defined by (6.2.11), creeps (linearly) downwards. Hence, A can be easily recovered by the following relationship:

$$P_1(Z_{\hat{\tau}} = A) = P_1(\varphi_{\hat{\tau}} = \exp(Ah)) = e^{Ak}P_0(Z_{\hat{\tau}} = A) = e^{Ak}(1 - P_0(Z_{\hat{\tau}} \geq B)), \tag{6.4.6}$$

where the second equality follows from (6.2.5) and (6.4.5). From (6.2.6) and (6.2.7) and the above expression, we have

$$A = \log\left(\frac{\alpha}{1-\beta}\right) \Big/ k. \tag{6.4.7}$$

The determination of the point B is more complex, since Z can pass over it only by jumps. It requires to derive the infinitesimal generator of the process Z.

Proposition 6.4.1 *Let P_i^z, $z \in \mathbb{R}$, $i = 0, 1$, be the probability measure under which X is a Lévy process with generating triplet $\widetilde{g}_i = \{0, 0, v_i\}$ and the process Z starts at z. Then, the infinitesimal generator $\mathbb{L}_i$ of Z acts like*

$$(\mathbb{L}_i f)(z) = -\mu f'(z) + \int_0^\infty \big(f(z+y) - f(z)\big) v_i(dy), \quad i = 0, 1. \tag{6.4.8}$$

Proof. Let $f \in C^1(\mathbb{R})$; by the application of Itô's formula for non-continuous semimartingales, we have:

$$\begin{aligned}
&f(Z_t) - f(z) \\
&\quad = \int_0^t f'(Z_{s^-})\, dX_s - \mu \int_0^t f'(Z_{s^-})\, ds + \sum_{s \le t} \Big(\Delta f(Z_s) - f'(Z_{s^-})\Delta Z_s\Big) \\
&\quad = -\mu \int_0^t f'(Z_{s^-})\, ds + \int_0^t \int_0^\infty \big(f(Z_{s^-} + y) - f(Z_{s^-})\big)\, \rho_i(dy, ds) \\
&\quad = -\mu \int_0^t f'(Z_{s^-})\, ds + \int_0^t \int_0^\infty \big(f(Z_{s^-} + y) - f(Z_{s^-})\big)\, (\rho_i - v_i)(dy, ds) \\
&\qquad + \int_0^t \int_0^\infty \big(f(Z_{s^-} + y) - f(Z_{s^-})\big)\, v_i(dy)\, ds,
\end{aligned} \tag{6.4.9}$$

where ρ_i is the measure of jumps of X, and so of Z, too, and the Lévy measure v_i is its compensator. The above expression can be written in compact form as

$$f(Z_t) = f(z) + \int_0^t (\mathbb{L}_i f)(Z_{s^-}) ds + \mathscr{M}_t, \tag{6.4.10}$$

where $\mathscr{M} = (\mathscr{M}_{t \ge 0})$, defined by

$$\mathscr{M}_t = \int_0^t \int_0^\infty \big(f(Z_{s^-} + y) - f(Z_{s^-})\big)\, (\rho_i - v_i)(dy, ds), \tag{6.4.11}$$

is a local martingale, with respect to $(\mathscr{F}_t^X)_{t \ge 0}$ and P_i^z, $i = 0, 1$. Hence, $(\mathbb{L}_i f)(z)$ takes the form (6.4.8). ∎

Denote by $f(z) = P_0^z(Z_{\hat{\tau}} \ge B)$; from general Markov process theory (see, e.g., Peskir and Shiryaev [67, p. 130]), $z \mapsto f(z)$ solves the following system:

$$(\mathbb{L}_0 f)(z) = 0, \quad \text{if } z \in (A, B) \setminus \mathscr{D}, \tag{6.4.12}$$

$$f(A) = 0, \tag{6.4.13}$$

$$f(z) = 1, \quad \text{if } z \ge B, \tag{6.4.14}$$

where $\mathscr{D}$ is the set of points in (A,B) where $z \mapsto f(z)$ is not eventually differentiable. According to (6.4.14) and $\lim_{z\to B^-} f(z) < 1$ (since Z creeps downwards and jumps upwards when X jumps), we observe that f is not continuous at B; further, this discontinuity together with (6.4.8) and (6.4.12) allow us to notice that $\mathscr{D} = \{x > 0 : v_0(\{x\}) > 0\}$: in other words, f is not differentiable at the points from which the process Z can jump to B with positive probability.

Since f, solving (6.4.12)-(6.4.14), depends on $A = A(\alpha,\beta)$ and B and Z starts at 0, we denote the value $f(0)$ by $g(A,B)$. Then, from (6.2.7) and (6.2.10) it is clear that

$$\beta = g(A(\alpha,\beta),B). \tag{6.4.15}$$

As $g(A(\alpha,\beta),B)$ is a continuous and decreasing function of B, a necessary and sufficient condition can be given on β to determine the unknown B in (6.2.10), after having fixed $A = A(\alpha,\beta)$ from (6.4.7):

$$\beta < \lim_{B\downarrow 0} g(A(\alpha,\beta),B). \tag{6.4.16}$$

Notice from (6.4.7) that the above right-hand side is a decreasing function of β: hence, there exists a unique $\beta^\star(\alpha)$, which makes the inequality (6.4.16) into an equality. Therefore, defining

$$G(\alpha) = \lim_{B\downarrow 0} g(A(\alpha,\beta^\star(\alpha)),B), \tag{6.4.17}$$

the set $\mathscr{A}$ of admissible error probabilities, for which a solution (A,B) to (6.2.6) and (6.2.7) exists, is

$$\mathscr{A} = \big\{(\alpha,\beta) : \alpha \in (0,1),\, 0 < \beta < G(\alpha)\big\}. \tag{6.4.18}$$

For a fixed $(\alpha,\beta) \in \mathscr{A}$, A and B are then determined through (6.4.7) and (6.4.15).

6.4.2 Expected observation times

Once $(\alpha,\beta) \in \mathscr{A}$ is chosen and the correspondent boundaries A and B are obtained, the expressions of $E_i[\hat{\tau}]$, $i = 0,1$, can be derived. Denote by $M_i(z,u) = E_i^z[e^{u\hat{\tau}}]$ the moment generating function of $\hat{\tau}$, under P_i^z, $i = 0,1$. General Markov process theory (see, e.g., Peskir and Shiryaev [67, p. 131]) leads to the formulation of the following system for the unknown map $z \mapsto M_i(z,u)$:

$$(\mathbb{L}_i M_i)(z,u) = -uM_i(z,u), \quad \text{if } z \in (A,B)\setminus\mathscr{D}, \tag{6.4.19}$$

$$M_i(A,u) = 1, \tag{6.4.20}$$

$$M_i(z,u) = 1, \quad \text{if } z \geq B. \tag{6.4.21}$$

The functional equation (6.4.19), defined through (6.4.8), and the boundary conditions (6.4.20) and (6.4.21) imply that $z \mapsto M_i(z,u)$ has a discontinuity at B and a discontinuous derivative on $\mathscr{D}$. Of course, once $z \mapsto M_i(z,u)$ has been explicitly recovered, $E_i[\hat{\tau}]$ is given by $\partial/(\partial u) M_i(z,u)|_{u=0,z=0}$, $i = 0,1$.

6.4.3 Examples

We now illustrate how the above results can be applied to recover the explicit expressions of the SPRT for two pure increasing jump processes already examined in the literature: the Poisson process and the compound Poisson process with exponential jumps.

Example 6.4.1 Let X be a Lévy process with generating triplet $\widetilde{g}_i = \{0, 0, \lambda_i\delta_1\}$, where δ_1 denotes the probability measure concentrated at 1: then, X is a Poisson process with intensity $\lambda_i > 0$, under P_i, $i = 0, 1$. By assuming that $\lambda_1 > \lambda_0$, we notice that $\xi(x) = \lambda_1/\lambda_0 > 1$, for $x = 1$. From (6.4.2), one has

$$\varphi_t = \exp\left(X_t \log\frac{\lambda_1}{\lambda_0} - (\lambda_1 - \lambda_0)t\right), \tag{6.4.22}$$

so that (6.2.8) holds with $k = \log(\lambda_1/\lambda_0)$ and $\mu = (\lambda_1 - \lambda_0)/\log(\lambda_1/\lambda_0)$. According to (6.4.8) and (6.4.12), we immediately see that the map $z \mapsto f(z) = P_0^z(Z_{\hat{\tau}} \geq B)$ solves on the interval (A, B) the difference-differential equation

$$-\mu f'(z) + \lambda_0\big(f(z+1) - f(z)\big) = 0, \tag{6.4.23}$$

provided in Dvoretzky et al. [28, eq. 4.13]. Through (6.4.23) and the boundary conditions (6.4.13) and (6.4.14), one can see that $f(z)$ is not continuous at B and $\mathscr{D} = \{B - 1\}$, if $A < B - 1$. The expression of $f(z)$ is given in Dvoretzky et al. [28, 4.15] and in Peskir and Shiryaev [65, eq. 3.21]. According to (6.4.15)-(6.4.18) one can determine the set $\mathscr{A}$, whose expression is given in Peskir and Shiryaev [65, eq. 3.24 and 3.26]. For a fixed $(\alpha, \beta) \in \mathscr{A}$, from (6.4.7) $A(\alpha, \beta) = \log\big(\alpha/(1-\beta)\big)/\log(\lambda_1/\lambda_0)$; B is then determined as solution of (6.4.15). According to (6.4.8) and (6.4.19), the function $z \mapsto M_i(z, u)$ must solve

$$-\mu\frac{\partial}{\partial z}M_i(z,u) + (u - \lambda_i)M_i(z,u) + \lambda_i M_i(z+1,u) = 0, \quad i = 0, 1. \tag{6.4.24}$$

This equation appears in Dvoretzky et al. [28, eq. 4.20]. Using (6.4.24) and (6.4.20)-(6.4.21), $M_i(z, u)$ can be derived: it is given in Dvoretzky et al. [28, eq. 4.21].

Example 6.4.2 Let X be a Lévy process having generating triplet $\widetilde{g}_i = \{0, 0, e^{-\lambda_i x}\mathbf{1}_{\{x>0\}}dx\}$, under P_i, $i = 0, 1$: X is therefore a compound Poisson process, whose intensity $1/\lambda_i$ is equal to the mean of the exponential distribution of the height of its jumps. Assume that $\lambda_0 > \lambda_1$: then, $\xi(x) = e^{(\lambda_0 - \lambda_1)x} > 1$, for any $x \in (0, \infty)$. From (6.4.2), we have

$$\varphi_t = \exp\left(X_t(\lambda_0 - \lambda_1) - \frac{\lambda_0 - \lambda_1}{\lambda_0\lambda_1}t\right). \tag{6.4.25}$$

According to (6.2.8), $k = \lambda_0 - \lambda_1$ and $\mu = 1/(\lambda_0\lambda_1)$. From (6.4.8) and (6.4.12), it results that $z \mapsto f(z) = P_0^z(Z_{\hat{\tau}} \geq B)$ solves on (A, B) the integro-differential equation

$$-\frac{f'(z)}{\lambda_0\lambda_1} - \frac{f(z)}{\lambda_0} + \int_0^\infty f(z+y)e^{-\lambda_0 y}\,dy = 0. \tag{6.4.26}$$

It is easy to see that (6.4.26), the boundary condition (6.4.14) and the change of variable $x = z+y$ lead to

$$-\frac{e^{-\lambda_0 z}}{\lambda_0\lambda_1}f'(z) - \frac{e^{-\lambda_0 z}}{\lambda_0}f(z) + \int_z^B f(x)e^{-\lambda_0 x}\,dx + \frac{e^{-\lambda_0 B}}{\lambda_0} = 0, \tag{6.4.27}$$

that, differentiated with respect to z, becomes the second order differential equation

$$f''(z) - (\lambda_0 - \lambda_1)f'(z) = 0. \tag{6.4.28}$$

The boundary condition (6.4.13) and $-e^{-\lambda_0 B}f'(B) - \lambda_1 e^{-\lambda_0 B}f(B) + \lambda_1 e^{-\lambda_0 B} = 0$ (the latter obtained replacing z by B in (6.4.27)) imply that the solution of (6.4.28) is

$$f(z) = \frac{\lambda_1\left(e^{(\lambda_0-\lambda_1)z} - e^{(\lambda_0-\lambda_1)A}\right)}{\lambda_0 e^{(\lambda_0-\lambda_1)B} - \lambda_1 e^{(\lambda_0-\lambda_1)A}}, \tag{6.4.29}$$

from which we observe that $f(z)$ is not continuous at B and $\mathscr{D} = \{\emptyset\}$. According to (6.4.7), $A(\alpha,\beta) = \log\big(\alpha/(1-\beta)\big)/(\lambda_0 - \lambda_1)$; (6.4.16)-(6.4.18) allow us to recover $\mathscr{A}$, with $G(\alpha) = \lambda_1(1-\alpha)/\lambda_0$. For $(\alpha,\beta) \in \mathscr{A}$, (6.4.15) leads to $B = \log\big(\lambda_1(1-\alpha)/(\lambda_0\beta)\big)/(\lambda_0 - \lambda_1)$. Using (6.4.8) and (6.4.19), one sees that $z \mapsto M_i(z,u)$ solves

$$-\frac{1}{\lambda_0\lambda_1}\frac{\partial}{\partial z}M_i(z,u) + (u - \lambda_i^{-1})M_i(z,u) + \int_0^\infty M_i(z+y,u)e^{-\lambda_i y}\,dy = 0, \quad i = 0,1. \tag{6.4.30}$$

From the above equation and the boundary conditions (6.4.20) and (6.4.21), one can recover the explicit expression of $z \mapsto M_i(z,u)$ in a similar way as (6.4.29) was derived. Finally, $E_i[\hat\tau] = \frac{\partial}{\partial u}M_i(z,u)|_{u=0,\,z=0}$, $i = 0,1$. All the previous results are presented in Gapeev [32].

Remark 6.4.1 So far, we have analyzed the SPRT for either continuous paths or pure increasing jump Lévy processes. Let now X be a Lévy process with Lévy-Khintchine triplet $g_i = \{\gamma_i, \sigma^2, v_i\}$, under P_i, $i = 0,1$. Assume that v_i satisfies $\int_{-\infty}^0 v_i(dx) = 0$ and $\int_0^1 x v_i(dx) < \infty$: according to the Lévy-Itô decomposition (see Theorem 5.2.2 or Sato [71, Th. 19.3, p. 121]), $X = X^J + X^c$, where $X^J = (X_t^J)_{t\ge0}$ and $X^c = (X_t^c)_{t\ge0}$ are two independent Lévy processes, defined by $X_t^J = \sum_{s\le t}(X_s - X_{s-})$ and $X_t^c = X_t - X_t^J$, with triplets $\widetilde{g}_i^J = \{0,0,v_i\}$ and $g_i^c = \{\widetilde{\gamma}_i, \sigma^2, 0\}$, respectively, under P_i. Thus, the results of Sections 6.3 and 6.4 can be applied. In particular, the following sequential procedure seems to be reasonable for the problem (6.2.1) and (6.2.2): let $(\alpha,\beta) \in \mathscr{A}^J$ and determine the stopping boundaries $A^c < 0 < B^c$ and $A^J < 0 < B^J$ for the processes $Z^c = (Z_t^c)_{t\ge0}$ and $Z^J = (Z_t^J)_{t\ge0}$, respectively; the meaning of all these quantities follows in an obvious manner from the previous decomposition and Sections 6.2-6.4. Let

$$\hat\tau^c = \inf\{t \ge 0 : Z_t^c = A^c \text{ or } Z_t^c = B^c\}, \tag{6.4.31}$$

$$\hat\tau^J = \inf\{t \ge 0 : Z_t^J = A^J \text{ or } Z_t^J \ge B^J\}. \tag{6.4.32}$$

Therefore, we can set

$$\hat{\tau} = \hat{\tau}^c \wedge \hat{\tau}^J, \tag{6.4.33}$$

$$l = \begin{cases} c, & \text{if } \hat{\tau} = \hat{\tau}^c, \\ J, & \text{if } \hat{\tau} = \hat{\tau}^J \end{cases} \tag{6.4.34}$$

$$\hat{d} = \begin{cases} 0 \quad (\text{accept } H_0), & \text{if} \quad Z^l_{\hat{\tau}} = A^l, \\ 1 \quad (\text{accept } H_1), & \text{if} \quad Z^l_{\hat{\tau}} \geq B^l. \end{cases} \tag{6.4.35}$$

6.5 SPRT for a negative binomial process

We say that $X = (X_t)_{t \geq 0}$ is a negative binomial process (n.b.p.) of parameter $p \in (0, 1)$, if it has independent and stationary increments and they have negative binomial distribution:

$$P(X_t - X_s = x) = \frac{\Gamma(x + t - s)}{\Gamma(x + 1)\Gamma(t - s)} p^{t-s}(1 - p)^x, \quad x = 0, 1, 2, \ldots, \quad t > s. \tag{6.5.1}$$

The n.b.p. is a pure increasing jump Lévy process, having Lévy-Khintchine triplet $g = \{1 - p, 0, v(\{x\})\}$, or, equivalently, $\widetilde{g} = \{0, 0, v(\{x\})\}$, where $v(\{x\}) = (1-p)^x/x$, $x = 1, 2, \ldots$. X can also be characterized as a compound Poisson process, where the waiting times between jumps are i.i.d. exponential random variables, with mean $1/\log(1/p)$, and the height of the jumps follows the discrete logarithmic distribution $\eta(\{x\}) = -(1 - p)^x/(x \log p)$, $x = 1, 2, \ldots$

Because of its nature of counting process, the n.b.p. represents a valid alternative to the Poisson process, especially when data are overdispersed and can be used to model phenomena occurring in clusters; these reasons, together with the strict connection with the negative binomial distribution, make the n.b.p. a flexible and applicable stochastic process in an extremely large number of fields: some of them are distribution theory (Anscombe [3], Barndorff [5], Vaillant [77]), agriculture and pest management (Mukhopadhyay [59], Mukhopadhyay and de Silva [60], Plant and Wilson [68], Young [85]), cosmology (Carruthers and Minh [18]) and entomology (Mulekar et al. [61], Nedelman [62], Wilson and Room [83]). For an application to hydrology, a review on the distributional properties and a complete list of references about the n.b.p., see Kozubowski and Podgórski [46].

On the basis of the results of Section 6.4, we derive the explicit expression of the SPRT for two simple hypotheses about the parameter p of a n.b.p.: to the best of our knowledge, the exact solution is not provided in the related literature. Indeed, Dvoretzky et al. [28, p. 264] stated that, unlike the Poisson process, "a complication [for the SPRT of a n.b.p.] is caused by the fact that the probability that the chance variable will exceed one in a small time interval is of the same order of magnitude as the probability that the chance variable will be one". In other words, since the n.b.p. has jumps whose length can take any positive integer number, the derivation of the associated SPRT is tougher than that of a Poisson process, which has jumps of unitary length.

The Bayesian solution of the problem has been given in Chapter 5 (or see Buonaguidi and Muliere [16, Sec. 6]).

6.5.1 Optimal boundaries and set of admissible error probabilities

Let X be a n.b.p. of parameter $p_i \in (0,1)$, under P_i, $i=0,1$; X has therefore Lévy-Khintchine triplet $\widetilde{g}_i = \{0,0,v_i\}$, where $v_i(\{x\}) = (1-p_i)^x/x$, $x = 1,2,\ldots$. From the sequential observation of a trajectory of X, we want to test the two hypotheses $H_0 : p = p_0$ and $H_1 : p = p_1$ by a stopping rule, that, for a given pair of error probabilities (α,β), solves the problem (6.2.2).

Without loss of generality, assume that $p_0 > p_1$; then, setting $q_i = 1-p_i$, $i=0,1$, we have

$$\xi(x) = \frac{dv_1}{dv_0}(x) = \left(\frac{q_1}{q_0}\right)^x > 1, \quad x = 1,2,\ldots \tag{6.5.2}$$

Using (6.4.2) and the well known identity $\sum_{x=1}^{\infty}(1-y)^x/x = -\log y$, $y \in (0,1)$, the likelihood ratio becomes

$$\varphi_t = \exp\left(X_t \log\left(\frac{q_1}{q_0}\right) - t\log\left(\frac{p_0}{p_1}\right)\right). \tag{6.5.3}$$

We observe that (6.2.8) is valid with $k = \log(q_1/q_0)$ and $\mu = \log(p_0/p_1)/\log(q_1/q_0)$. Consider now the process $Z = (Z_t)_{t\geq 0}$ in (6.2.11); from (6.4.7), the lower boundary A for the optimal stopping rule (6.2.9) and (6.2.10) is

$$A = \log\left(\frac{\alpha}{1-\beta}\right) \Big/ \log\left(\frac{q_1}{q_0}\right). \tag{6.5.4}$$

In order to determine the upper boundary B, we need the expression of the infinitesimal generator $\mathbb{L}$ of Z. Let P_i^z be the probability measure under which the process starts at z and X is a negative binomial process with parameter p_i, $i=0,1$. Then, according to Lemma 4.1, we have that $\mathbb{L}_i$ acts on $z \mapsto w(z) \in C^1(\mathbb{R})$ like

$$(\mathbb{L}_i w)(z) = -\mu w'(z) + \log p_i\, w(z) + \sum_{y=1}^{\infty} w(z+y)\frac{(1-p_i)^y}{y}, \quad i = 0,1. \tag{6.5.5}$$

As in Subsection 6.4.1, define $f(z) = P_0^z(Z_{\hat{\tau}} \geq B)$; then, the map $z \mapsto f(z)$ must solve the system (6.4.12)-(6.4.14), with $\mathscr{D} = \{B-1,\ldots,B-n^\star\}$, where $n^\star = \max\{n \in \mathbb{N} : B-n > A\}$. The functional equation (6.4.12), defined through the integro-differential operator (6.5.5), appears immediately more complicated than the correspondent equation (6.4.23) for the Poissonian case: this is due to the fact that while the Lévy measure of a Poisson process puts all its mass on the singleton $\{1\}$, the Lévy measure of a n.b.p. concentrates on the positive integer numbers (see Figure 6.1 above).

Denote $I_n = [B-n, B-n+1)$, for $n \geq 1$, and $\delta(z,B) = \lceil B-z \rceil$, where $\lceil x \rceil$ stays for the smallest integer greater than or equal to x. We observe that

$$\delta(z,B) = n \quad \Longleftrightarrow \quad z \in I_n. \tag{6.5.6}$$

Consider the equation (6.4.12) and (6.5.5) on the interval $I_1 = [B-1,B)$: it results that $(z+y) \geq B$ for $y \geq 1$, so that, because of (6.4.14), we can set $f(z+y) = 1$, for $z \in I_1$ and $y \geq 1$.

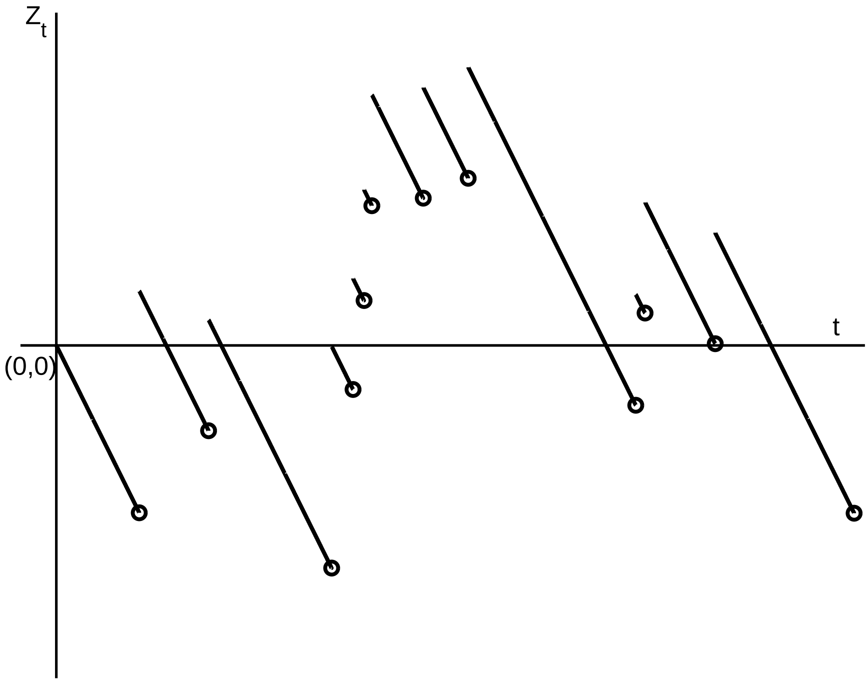

Figure 6.1: A simulated path of the process $Z = (Z_t)_{t\geq 0}$, as defined in (6.2.11). X is a n.b.p. with parameter $p_0 = 0.6$, under P_0, and $p_1 = 0.5$, under P_1. It is assumed that the true hypothesis is $p = p_0$. For every pair $A < 0 < B$, it is evident that the stopping time $\hat{\tau}$ from (6.2.9) coincides with the first time t at which $Z_t = A$ or $Z_t \geq B$. The magnitude of the jumps of Z, and so of X, too, can be any positive integer number, as we observe from the figure, unlike the unitary height of the jumps of a Poisson process.

Then, we have a first order linear differential equation, for which a unique solution $z \mapsto f(z)$ on I_1 is easily found:

$$f(z) = 1 + Ce^{\frac{\log p_0}{\mu}z}, \quad z \in I_1, \tag{6.5.7}$$

where C is a constant to be determined. Consider now the equation (6.4.12) and (6.5.5) on the interval $I_2 = [B-2, B-1)$: of course, $z+1$ ranges in the interval I_1, while $z+y \geq B$ for $y \geq 2$. Hence, setting $f(z+1)$ equal to the solution determined on I_1, evaluated at $z+1$, $f(z+y) = 1$, for $y \geq 2$ because of (6.4.14), and imposing a continuity condition on $I_2 \cup I_1$ at $B-1$, we obtain the unique solution

$$f(z) = 1 + Ce^{\frac{\log p_0}{\mu}z}\left(1 - \frac{q_0}{\mu}e^{\frac{\log p_0}{\mu}}(B-1-z)\right), \quad z \in I_2. \tag{6.5.8}$$

Move now on the generic interval $I_n = [B-n, B-n+1)$ and consider the equation (6.4.12) and (6.5.5): since $z+1$ ranges in I_{n-1}, $z+2$ ranges in I_{n-2}, $\ldots$, $z+n-1$ ranges in I_1 and $z+y \geq B$ for $y \geq n$, setting $f(z+1)$ equal to the solution found on I_{n-1}, evaluated at $z+1$, $f(z+2)$ equal to the solution found on I_{n-2}, evaluated at $z+2$, $\ldots$, $f(z+n-1)$ equal to the solution found on I_1, evaluated at $z+n-1$, $f(z+y) = 1$, for $y \geq n$ according to (6.4.14), and imposing a continuity condition on $\cup_{i=1}^{n} I_i$ at $B-n+1$, we obtain a unique solution $z \mapsto f(z)$ on I_n. Using (6.5.6), it takes the following expression:

$$f(z) = 1 + Ce^{\frac{\log p_0}{\mu} z}\big(F_1(z,B) - F_2(z,B) + F_3(z,B)\big), \tag{6.5.9}$$

where F_1, F_2 and F_3 are functions given by

$$F_1(z,B) = \sum_{j=0}^{\delta(z,B)-1} \frac{(-1)^j}{j!}\left(\frac{q_0}{\mu} e^{\frac{\log p_0}{\mu}}(B-j-z)\right)^j, \tag{6.5.10}$$

$$F_2(z,B) = \sum_{j=2}^{\delta(z,B)-1} \frac{q_0^j}{j\mu} e^{j\frac{\log p_0}{\mu}}(B-j-z), \tag{6.5.11}$$

$$\begin{aligned} F_3(z,B) = \sum_{j=1}^{\delta(z,B)-2} \frac{q_0^j}{j\mu} e^{j\frac{\log p_0}{\mu}} \int \Big[&\mathbf{1}_{\{j\geq 2\}}\big(F_1(z+j,B)-1\big) \\ &- F_2(z+j,B) + F_3(z+j,B)\Big]dz. \end{aligned} \tag{6.5.12}$$

The integral sign in (6.5.12) is meant as antiderivative with null constant of integration. The constant C is easily determined through (6.4.13) and (6.5.9); then

$$f(z) = 1 - e^{\frac{\log p_0}{\mu}(z-A)} \frac{F_1(z,B) - F_2(z,B) + F_3(z,B)}{F_1(A,B) - F_2(A,B) + F_3(A,B)}, \quad A \leq z < B. \tag{6.5.13}$$

Notice that $z \mapsto f(z)$ is not continuous at B and is of class C^1 on (A,B), but C^0 at $\{B-1,\dots,B-n^\star\}$ (see Figure 6.2).

From (6.2.7) and (6.2.10) and using (6.5.13), we have

$$\beta = 1 - e^{-A(\alpha,\beta)\frac{\log p_0}{\mu}} \frac{F_1(0,B) - F_2(0,B) + F_3(0,B)}{F_1(A(\alpha,\beta),B) - F_2(A(\alpha,\beta),B) + F_3(A(\alpha,\beta),B)}, \tag{6.5.14}$$

where $A = A(\alpha,\beta)$ is given by (6.5.4) and the above right-hand side coincides with $g(A(\alpha,\beta),B)$ in (6.4.15). Since it is verified that the latter is continuous in B and decreases to 0 as $B \uparrow \infty$, the condition (6.4.16) reads as (notice that $\lim_{B\downarrow 0}\big(F_1(0,B) - F_2(0,B) + F_3(0,B)\big) = F_1(0,0^+) = 1$):

$$\beta < 1 - \frac{e^{-A(\alpha,\beta)\frac{\log p_0}{\mu}}}{F_1(A(\alpha,\beta),0) - F_2(A(\alpha,\beta),0) + F_3(A(\alpha,\beta),0)}. \tag{6.5.15}$$

We observe that the right-hand side of (6.5.15) is a decreasing function of β: therefore, there exists a unique $\beta^\star(\alpha) > 0$ at which the equality in (6.5.15) is attained. Thus, defining

$$G(\alpha) = 1 - \frac{e^{-A(\alpha,\beta^\star(\alpha))\frac{\log p_0}{\mu}}}{F_1(A(\alpha,\beta^\star(\alpha)),0) - F_2(A(\alpha,\beta^\star(\alpha)),0) + F_3(A(\alpha,\beta^\star(\alpha)),0)}, \tag{6.5.16}$$

the set of the admissible error probabilities $\mathscr{A}$ takes the form (6.4.18) (see Figure 6.3). Then, for a fixed pair $(\alpha,\beta) \in \mathscr{A}$, once A is determined by (6.5.4), B is obtained as unique solution of (6.5.14).

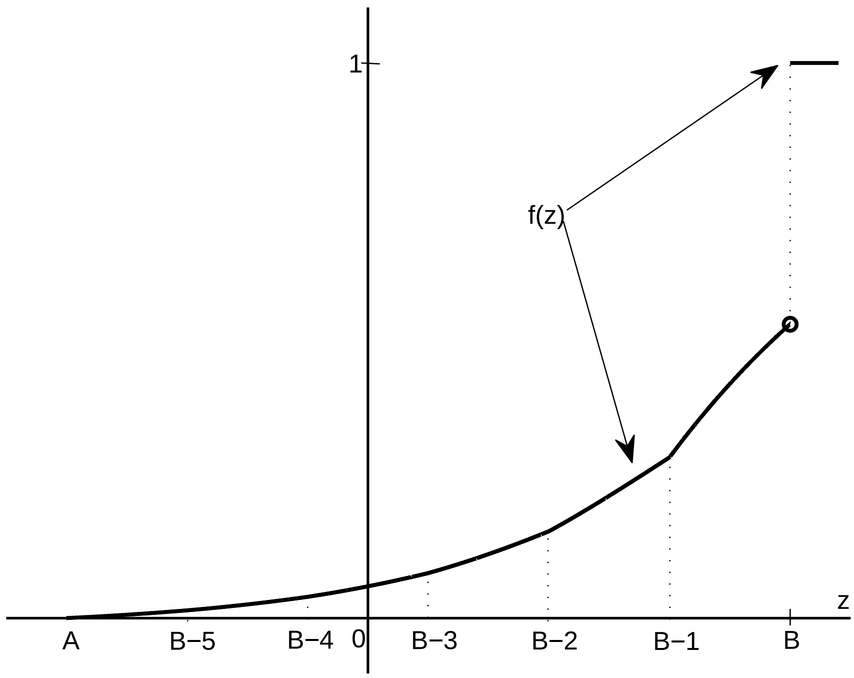

Figure 6.2: A computer drawing of the map $z \mapsto f(z) = P_0^z(Z_{\hat{\tau}} \geq B)$ as expressed by (6.5.13), which is the unique solution of (6.4.12)-(6.4.14) and (6.5.5). We set $p_0 = 0.6$, $p_1 = 0.3$, $A = -2.5$ and $B = 3.5$. One should observe that the map is discontinuous at B and not differentiable at $B-1 = 2.5$, $B-2 = 1.5$, $B-3 = 0.5$, $B-4 = -0.5$, $B-5 = -1.5$. Indeed, denoted by $f_l'(z)$ and $f_r'(z)$ the left and right derivative of $f(z)$ at z, we have $f_l'(B-1) = 0.1411..$ and $f_r'(B-1) = 0.2930..$, $f_l'(B-2) = 0.0881..$ and $f_r'(B-2) = 0.1185..$, $f_l'(B-3) = 0.0525..$ and $f_r'(B-3) = 0.0606..$, $f_l'(B-4) = 0.0307..$ and $f_r'(B-4) = 0.0331..$, $f_l'(B-5) = 0.0178..$ and $f_r'(B-5) = 0.0186..$

6.5.2 Expected observation times

As in Subsection 6.4.2, let $M_i(z,u)$ be the moment generating function $E_i^z[e^{u\hat{\tau}}]$ of $\hat{\tau}$ from (6.2.9), under P_i^z, $i = 0,1$. The map $z \mapsto M_i(z,u)$ must solve on the set $(A,B) \setminus \mathscr{D}$ the functional equation (6.4.19), defined through the integro-differential operator (6.5.5):

$$-\mu\frac{\partial}{\partial z}M_i(z,u) + (u + \log p_i)M_i(z,u) + \sum_{y=1}^{\infty} M_i(z+y,u)\frac{q_i^y}{y} = 0, \quad i = 0,1. \tag{6.5.17}$$

The condition (6.4.21) and calculations similar to those used for deriving (6.5.9)-(6.5.12) lead to

$$\begin{aligned} M_i(z,u) =& C_{\delta(z,B)}(u) + K(u)e^{\frac{u+\log p_i}{\mu}z}\big(F_1(z,u,B) - F_2(z,u,B) + F_3(z,u,B)\big) \\ &+ e^{-\frac{u+\log p_i}{\mu}(B-\delta(z,B)+1-z)}L_{\delta(z,B)}(z,u,B), \end{aligned} \tag{6.5.18}$$

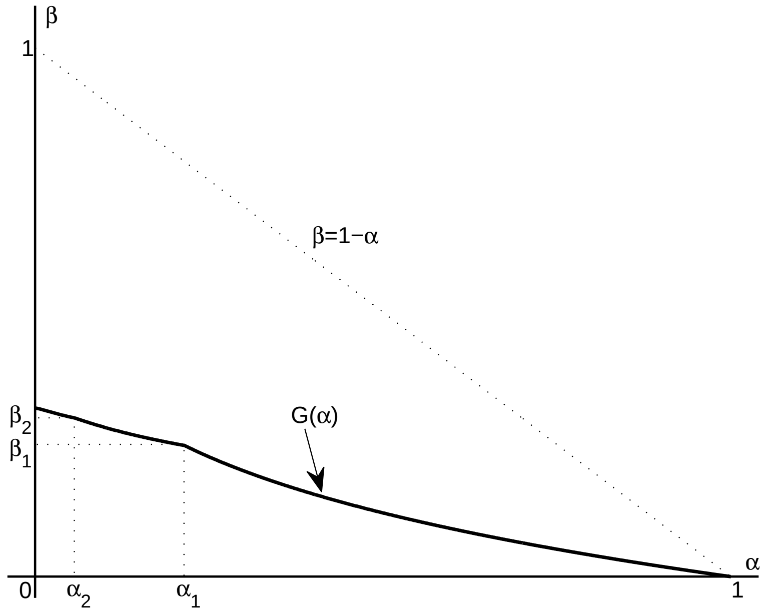

Figure 6.3: A computer drawing of the map $\alpha \mapsto G(\alpha)$, given by (6.5.16). We set $p_0 = 0.8$ and $p_1 = 0.3$. The area under the bold curve is the set of admissible probabilities $\mathscr{A}$, outside which a solution A and B to (6.2.6) and (6.2.7) does not exist. The function $G(\alpha)$ has infinitely many humps: they correspond to the pairs (α, β), at which $B = 0^+$ and $A = B - n$, $n \geq 1$, and it is due to the lack of differentiability of the map (6.5.13) at $B - n$, $n \geq 1$. From the picture the most evident ones are for $\alpha_1 = 0.2150..$ and $\beta_1 = 0.2479..$ (where $B = 0^+$ and $A = B - 1$) and $\alpha_2 = 0.0570..$ and $\beta_2 = 0.3001..$ (where $B = 0^+$ and $A = B - 2$). We observe that $G(0^+) < 1$: it is consistent with the fact that the process Z is a supermartingale under P_0. Indeed, for $t > s$, easy calculations show that $E_0[Z_t|\mathscr{F}_s^X] = Z_s + (t-s)m(p_0, p_1)$, with $m(p_0,p_1) = (q_0/p_0 - \log(p_0/p_1)/\log(q_1/q_0)) < 0$, on $\{(p_0, p_1) : p_i \in (0,1), i = 0, 1, \, p_0 > p_1\}$.

where $C_{\delta(z,B)}$, F_1, F_2, F_3 and $L_{\delta(z,B)}$ are given by

$$C_{\delta(z,B)}(u) = \frac{1}{u + \log p_i}\left(\log p_i + \sum_{j=1}^{\delta(z,B)-1} \frac{q_i^j}{j}\big(1 - C_{\delta(z,B)-j}(u)\big)\right), \tag{6.5.19}$$

$$F_1(z,u,B) = \sum_{j=0}^{\delta(z,B)-1} \frac{(-1)^j}{j!}\left(\frac{q_i}{\mu} e^{\frac{u+\log p_i}{\mu}}(B - j - z)\right)^j, \tag{6.5.20}$$

$$F_2(z,u,B) = \sum_{j=2}^{\delta(z,B)-1} \frac{q_i^j}{j\mu} e^{j\frac{u+\log p_i}{\mu}}(B - j - z), \tag{6.5.21}$$

$$\begin{aligned} F_3(z,u,B) = \sum_{j=1}^{\delta(z,B)-2} \Bigg(\frac{q_i^j}{j\mu} e^{j\frac{u+\log p_i}{\mu}} \int \Big[& \mathbf{1}_{\{j \geq 2\}}\big(F_1(z+j,u,B) - 1\big) \\ & - F_2(z+j,u,B) + F_3(z+j,u,B)\Big] dz\Bigg), \end{aligned} \tag{6.5.22}$$

$$L_{\delta(z,B)}(z,u,B) = \mathbf{1}_{\{\delta(z,B)\geq 2\}}\Big(e^{-\frac{u+\log p_i}{\mu}} L_{\delta(z,B)-1}(B-\delta(z,B)+1,u,B) + C_{\delta(z,B)-1}(u)$$
$$- C_{\delta(z,B)}(u)\Big) + \sum_{j=1}^{\delta(z,B)-2} \frac{q_i^j}{j\mu} \int_{B-\delta(z,B)+1}^{z} L_{\delta(z,B)-j}(s+j,u,B)ds, \quad (6.5.23)$$

with $K(u)$ in (6.5.18) determined so that (6.4.20) is satisfied.

Then, since Z starts at 0, $E_i[\hat{\tau}] = \partial/(\partial u)M_i(z,u)|_{u=0,z=0}$, $i = 0, 1$.

Remark 6.5.1 If we define $g_i(z) = \partial/(\partial u)M_i(z,u)|_{u=0}$, $i = 0, 1$, we observe from (6.5.17) that the map $z \mapsto g_i(z)$ solves on $(A,B) \setminus \mathscr{D}$

$$-\mu g_i'(z) + 1 + \log p_i\, g_i(z) + \sum_{y=1}^{\infty} g_i(z+y)\frac{q_i^y}{y} = 0, \quad i = 0, 1, \quad (6.5.24)$$

with the natural boundary conditions $g_i(A) = 0$ and $g_i(z) = 0$, for $z \geq B$. Its solution is

$$g_i(z) = Q_{\delta(z,B)} + Ke^{\frac{\log p_i}{\mu}z}\big(F_1(z,B) - F_2(z,B) + F_3(z,B)\big)$$
$$+ e^{-\frac{\log p_i}{\mu}(B-\delta(z,B)+1-z)} L_{\delta(z,B)}(z,B), \quad (6.5.25)$$

where F_1, F_2 and F_3 coincide with (6.5.10)-(6.5.12) (p_0 and q_0 are replaced by p_i and q_i) and

$$Q_{\delta(z,B)} = -\frac{1}{\log p_i}\left(1 + \sum_{j=1}^{\delta(z,B)-1} \frac{q_i^j}{j} Q_{\delta(z,B)-j}\right), \quad (6.5.26)$$

$$L_{\delta(z,B)}(z,B) = \mathbf{1}_{\{\delta(z,B)\geq 2\}}\Big(e^{-\frac{\log p_i}{\mu}} L_{\delta(z,B)-1}(B-\delta(z,B)+1,B) + Q_{\delta(z,B)-1}$$
$$- Q_{\delta(z,B)}\Big) + \sum_{j=1}^{\delta(z,B)-2} \frac{q_i^j}{j\mu} \int_{B-\delta(z,B)+1}^{z} L_{\delta(z,B)-j}(s+j,B)ds, \quad (6.5.27)$$

$$K = -e^{-\frac{\log p_i}{\mu}A}\, \frac{Q_{\delta(A,B)} + e^{-\frac{\log p_i}{\mu}(B-\delta(A,B)+1-A)} L_{\delta(A,B)}(A,B)}{F_1(A,B) - F_2(A,B) + F_3(A,B)}. \quad (6.5.28)$$

Therefore, $E_i[\hat{\tau}] = g_i(0)$, $i = 0, 1$.

6.6 Conclusions

The Wald's SPRT for the Lévy-Khintchine triplet of an important class of Lévy processes has been analyzed. After having recalled the structure of the optimal stopping rule, that is, the stopping rule that, for a fixed pair of first and second type error probabilities, presents the smallest expected observation time among the decision rules having at most the same error probabilities, we concentrated on the SPRT for continuous paths and pure increasing jump Lévy processes.

We recovered the explicit expression of the stopping boundaries and the expected observation time for the SPRT about the drift of a continuous paths Lévy process (Brownian motion). Even though these results were already shown by Shiryaev [72, Sec. 4.2], we proposed different methods to derive them and, in addition to the latter work, we provided the expression of the moment generating function of the optimal stopping time.

Then, the analysis moved toward pure increasing jump Lévy processes: we generalized the results contained in Dvoretzky et al. [28] and Peskir and Shiryaev [65, Sec. 3]. In particular, through the derivation of the infinitesimal generator of the process Z and the powerful theory of Markov processes, we saw how to characterize the set of admissible error probabilities and the stopping boundaries for the SPRT and its expected observation time. Our main contribution has been the exploitation of these results in order to explicitly obtain the SPRT of a negative binomial process: this problem seems to be faced here for the first time, although was formulated at the beginning of the second half of the past century by Dvoretzky et al. [28].

Chapter 7

A Collocation Method for the Sequential Testing of a Gamma Process

We study the Bayesian problem of sequential testing of two simple hypotheses about the parameter $\alpha > 0$ of a Lévy gamma process. The initial optimal stopping problem is reduced to a free-boundary problem, where at the unknown boundary points, separating the stopping and continuation set, the principles of the smooth and/or continuous fit hold and the unknown value function satisfies on the continuation set a linear integro-differential equation. Due to the form of the Lévy measure of a gamma process, determining the solution of this equation and the boundaries is not an easy task. Hence, instead of solving the problem analytically, we use a collocation technique: the value function is replaced by a truncated series of polynomials with unknown coefficients, that, together with the boundary points, are determined by forcing the series to satisfy the boundary conditions and, at fixed points, the integro-differential equation. The proposed numerical technique is finally employed in well understood problems to assess its efficiency.

7.1 Introduction

Establishing the correct distributional properties of a sequentially observed stochastic process is of fundamental importance in many practical problems, as well as a challenging task from a theoretical view point. In this chapter it is assumed that at time $t = 0$ we begin to follow the evolution of a Lévy gamma process $X = (X_t)_{t\geq 0}$ with parameter $\alpha > 0$: its sequential testing consists of picking a stopping time τ of X and a decision function d, expressing which of the two simple hypotheses initially formulated about α might be accepted at time τ, so that a risk value function is minimized. The problem is analyzed within the Bayesian framework, where a priori distribution on the correctness of the hypotheses is given and the goal is the minimization of the sum between the expected cost of the observation process and the expected loss one suffers if a final wrong decision is made.

Problems of sequential testing for continuous time processes have been widely studied in the literature and can be distinguished in two areas depending on the sample paths of the observed process: the first area contains the works of Shiryaev [72, sec. 4.2], Gapeev and Peskir [35], Gapeev and Shiryaev [37] and Shiryaev and Zhitlukhin [75], where solutions to the Bayesian sequential testing for the drift of a Wiener process or a more general diffusion process are provided; the second area includes the works of Peskir and Shiryaev [65], Gapeev [32], Dayanik and Sezer [24, 25], Dayanik et al. [23] and Ludkovski and Sezer [55], who study problems of sequential testing for jumping processes of compound Poisson type. Hence, while in the first area the analyzed processes have continuous patterns, in the second one the observed processes show the feature to jump a finite number of times on any finite time interval.

The novelty of this chapter is the analysis of the Bayesian sequential testing for a gamma process, which is a purely jump process with infinitely many positive jumps on any finite time interval. The value function and the optimal stopping boundaries of the initial optimal stopping problem for the posterior probability process are shown to be the solution of a free-boundary problem: the value function satisfies at the stopping boundaries the principles of the smooth and/or continuous fit and solves on the continuation set a linear integro-differential equation. Unfortunately a complication arises: determining an explicit solution of the free-boundary problem appears to be extremely complex. This requires devising a suitable numerical technique.

The successive approximation scheme adopted in Dayanik and Sezer [24] for the sequential testing of a compound Poisson process cannot be applied, since the gamma process is not a compound Poisson process. A collocation approach is thus developed. It relies on replacing the value function in the free-boundary problem with a truncated series of polynomials with unknown coefficients (in particular, the Chebyshev polynomials are used) and forcing it to solve the boundary conditions and, at a fixed number of points, the integro-differential equation. The number of points is chosen so that, taking into account the boundary conditions, the number of equations coincides with the number of the unknown variables, that is, the coefficients of the series and the stopping boundaries. This approach is a modification of the well known collocation method, widespread in mathematical physics and engineering for solving boundary value problems (that is, the boundary points are known in advance), and is another interesting feature of our work. Its efficiency is illustrated in problems where exact solutions are available.

The chapter is organized as follows: in Section 7.2 the basic concepts on the collocation method for a linear Volterra integro-differential equation are illustrated; in Section 7.3, after having recalled the main properties of a gamma process and having defined more formally the problem initially introduced, the original optimal stopping problem for the posterior probability process is reduced to a free-boundary problem; in Section 7.4, we show how its numerical solution can be accurately derived by a collocation approach; in Section 7.5, we compare exact and collocation solutions of well understood sequential testing problems. Section 7.6 concludes with a summary discussion.

The sequential testing for a gamma process was already thought by Dvoretzky et al. [28, p. 255], but, to the best of our knowledge, a solution has never been provided. This study is a natural continuation of the arguments contained in Chapter 5 (see also Buonaguidi and Muliere [16]) and is motivated by the extensive use of the gamma process in risk theory (Dufresne et al. [26]), degradation and failure models (Lawless and Crowder [51], Park and Padgett [63]), maintenance and reliability (Van Noortwijk [78]).

7.2 Preliminaries on the collocation method

As it will be seen later, the sequential testing for a gamma process requires solving a free-boundary problem, characterized by a linear integro-differential equation and some boundary conditions, where, as the expression "free-boundary" suggests, the boundaries are unknown. In this section, we provide the basic elements on the collocation method for a boundary value problem (that is, the boundaries are known in advance), involving a linear Volterra integro-differential equation. We will extend these arguments in Section 7.4 for solving our problem.

7.2.1 Collocation method for a linear Volterra integro-differential equation

Let $\mathbb{T}$ be a linear Volterra integro-differential operator acting on a function f belonging to its domain of definition as

$$(\mathbb{T}f)(x) = f'(x) - g(x) - h(x)f(x) - \int_A^x k(x,z)f(z)\,dz, \tag{7.2.1}$$

for $x \in I = [A, B] \subset \mathbb{R}$ where $g(x)$, $h(x)$ and $k(x,z)$, $x \in I$ and $A \leq z \leq x$, are known functions. Consider now the functional equation

$$(\mathbb{T}f)(x) = 0, \tag{7.2.2}$$

along with the boundary condition

$$f(A) = p, \tag{7.2.3}$$

where p is a fixed number. It is assumed that the boundary value problem (7.2.2)-(7.2.3) admits a unique solution f on I that we want to determine. Often this task cannot be analytically accomplished, so that we need numerical techniques allowing us to approximate f as accurately as desired: one of them is the so called collocation method (see, for example, Kress [47, Sec. 12.4]).

Let us briefly explain its main idea. Let $\Phi = \{\phi_i\}_{i\geq 0}$ be a known basis for f and denote by f_n an approximation of f obtained as linear combination of the first $n+1$ basis functions:

$$f(x) \approx f_n(x) = \sum_{i=0}^{n} w_i \phi_i(x), \quad x \in I, \tag{7.2.4}$$

so that

$$f'(x) \approx f_n'(x) = \sum_{i=0}^{n} w_i \phi_i'(x), \quad x \in I. \tag{7.2.5}$$

Choosing n points, known as collocation nodes, $x_i \in I$, $i = 1, .., n$, the problem (7.2.2)-(7.2.3) boils down to computing the coefficients w_i by solving the following system of $n+1$ linear equations:

$$(\mathbb{T}f_n)(x_i) = 0, \quad i = 1, .., n, \tag{7.2.6}$$

$$f_n(A) = p. \tag{7.2.7}$$

Two problems naturally arise: the choice of an appropriate basis for f and of the truncation limit n.

7.2.2 The Chebyshev polynomials

In addition to the uniqueness of f for the problem (7.2.2)-(7.2.3), assume that f is continuous on I. Then, according to the Waierstrass approximation theorem, f can be uniformly approximated on I by polynomials. One could be tempted to use as Φ the family $\{x^i\}_{i\geq 0}$: its drawback is the lack of the orthogonality property.

Let us recall that a family of functions $\{\psi_i\}_{i\geq 0}$ is said to be orthogonal on I with respect to the weighting function $\eta(x)$ if

$$\int_I \psi_i(x)\psi_j(x)\eta(x)\,dx = \begin{cases} 0, & i \neq j \\ \lambda_j, & i = j \end{cases}. \tag{7.2.8}$$

The idea is that the information set of an element of a family of orthogonal functions does not overlap with the one expressed by another member of the family. Therefore, if we choose as basis for f a family of orthogonal polynomials, the performances in the numerical approximation of f are improved, due to a better identification of the coefficients w_i in (7.2.4).

A well known family of orthogonal polynomials is the family of Chebyshev polynomials: their detailed description can be found in Hamming [42, Sec. 2.28 and 2.29] and Lanczos [50, Chap. 7]; here, we illustrate their main properties, which explain why they represent one of the most important family of polynomials (and, maybe, the most important one) in approximation theory.

The Chebyshev polynomials $\{T_i\}_{i\geq 0}$ are defined by

$$T_n(x) = \cos[n(\arccos(x))], \quad n \geq 0, \quad x \in [-1, 1]. \tag{7.2.9}$$

The trigonometric identity

$$\cos(n+1)\theta + \cos(n-1)\theta = 2\cos\theta\cos n\theta \tag{7.2.10}$$

and the substitution $\theta = \arccos(x)$ in (7.2.10) lead to the recurrence relationship

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x), \quad n \geq 2. \tag{7.2.11}$$

Since $T_0(x) = 1$ and $T_1(x) = x$, $x \in [-1, 1]$, from (7.2.11) it is easily seen that $\{T_i\}_{i\geq 0}$ is a family of polynomials. It presents some remarkable features: 1) Chebyshev polynomials are orthogonal on $[-1, 1]$ with respect to the weighting function $\eta(x) = (1 - x^2)^{-1/2}$:

$$\int_{-1}^{1} \frac{T_m(x)T_n(x)}{\sqrt{1-x^2}}\,dx = \int_0^{\pi} \cos m\theta \cos n\theta\,d\theta = \begin{cases} 0, & m \neq n \\ \frac{\pi}{2}, & m = n \neq 0 \\ \pi, & m = n = 0 \end{cases}; \tag{7.2.12}$$

2) the zeros of the n-th degree polynomial T_n are given by

$$x_j = \cos\left(\left(j - \frac{1}{2}\right)\frac{\pi}{n}\right), \quad j = 1, .., n; \tag{7.2.13}$$

3) for $n \geq 0$, derivatives are easy to compute; for instance:

$$T'_n(x) = \frac{n\sin[n\arccos(x)]}{\sin[\arccos(x)]}, \quad T''_n(x) = \frac{nx\sin[n\arccos(x)]}{\big(\sin[\arccos(x)]\big)^3} - \frac{n^2 T_n(x)}{\big(\sin[\arccos(x)]\big)^2}; \tag{7.2.14}$$

4) the shifted Chebyshev polynomials on the interval $I = [A, B]$, $\{T_i^I\}_{i\geq 0}$, along with their first and second derivatives, $\{T_i^{'I}\}_{i\geq 0}$ and $\{T_i^{''I}\}_{i\geq 0}$, are given, for $n \geq 0$ and $x \in I$, by

$$T_n^I(x) = T_n\left(2\frac{x-A}{B-A} - 1\right), \tag{7.2.15}$$

$$T_n^{'I}(x) = \frac{2}{B-A}T'_n\left(2\frac{x-A}{B-A} - 1\right), \quad T_n^{''I}(x) = \frac{4}{(B-A)^2}T''_n\left(2\frac{x-A}{B-A} - 1\right); \tag{7.2.16}$$

5) Chebyshev expansions are usually one of the most rapidly convergent expansions for functions.

Properties 1-5 appropriately justify the use of Chebyshev polynomials as basis for f; in particular, according to the fifth property, which does not hold only in isolated cases, “low degree” polynomials often lead to satisfactory approximations; in turn, this reflects in a saving of time during numerical computations.

7.2.3 Accuracy of the solution

Once a basis for the function f in the problem (7.2.2)-(7.2.3) has been chosen, we should determine the length n of the expansion in (7.2.4).

The truncated series (7.2.4), whose coefficients are obtained as solution of (7.2.6)-(7.2.7), approximately solves (7.2.2), in the sense that if we replace (7.2.4) and (7.2.5) in (7.2.2), then $(\mathbb{T}f_n)(x) \approx 0$, $x \in I$. This suggests we could increase n until

$$\sup_{x\in I} |(\mathbb{T}f_n)(x)| < \epsilon \tag{7.2.17}$$

for a fixed $\epsilon > 0$. Of course, since it is not practically possible to evaluate $(\mathbb{T}f_n)(x)$ for any $x \in I$, we can consider a set of equally spaced nodes in I (not the collocation nodes, where $(\mathbb{T}f_n)(x)$ is almost exactly zero) to assess the quality of the computed solution.

Alternatively, defining

$$\delta_n = \sup_{x\in I} |f_n(x) - f_{n-1}(x)|, \quad n \geq 1, \tag{7.2.18}$$

we might increase n until $\delta_n < \delta$, for a specified $\delta > 0$.

Remark 7.2.1 When f is approximated by $f_n = \sum_{i=0}^n w_i T_i^I$, it is well known that the distance $\sup_{x\in I} |f(x) - f_n(x)|$ is minimized if the collocation nodes are the zeros of T_n^I given by

$$x_j^I = \frac{(B-A)(x_j+1)}{2} + A, \quad j = 1, .., n, \tag{7.2.19}$$

where x_j is given in (7.2.13). We notice that the zeros of T_n^I can be used as collocation nodes only if I is known: this does not occur in free-boundary problems, where A and B must be determined. We will face this problem in Section 7.4.

7.3 Sequential testing of a gamma process

In this section, we begin the analysis of the sequential testing for a gamma process. The interest in this problem was raised at the end of Chapter 5 (see also Buonaguidi and Muliere [16, p. 69]).

A gamma process $X = (X_t)_{t\geq 0}$ of parameter $\alpha > 0$ is a Lévy process with Lévy-Khintchine representation

$$E\left[e^{izX_t}\right] = \exp\left\{t\int_0^\infty (e^{izx}-1)\frac{e^{-\alpha x}}{x}\,dx\right\} = \left(\frac{\alpha}{\alpha - iz}\right)^t, \quad z \in \mathbb{R}, \tag{7.3.1}$$

where $v(dx) = x^{-1}e^{-\alpha x}\mathbf{1}_{(0,\infty)}(dx)$ is the so called Lévy measure. Using standard arguments based on Sato [71, Chap. 4], the following properties are easily inferred from (7.3.1): 1) X is a purely jump process; 2) X is not a compound Poisson process and for any finite time interval the jumping times of X are countable and dense in $[0,\infty)$; 3) X is a subordinator, that is $t \mapsto X_t$ is increasing (in particular this map is strictly increasing and not continuous anywhere a.s.); 4) X has sample paths of finite variation; 5) X_t, $t \geq 0$, has gamma distribution, whose density is given by

$$f_t(x;\alpha) = \frac{\alpha^t}{\Gamma(t)}x^{t-1}e^{-\alpha x}\mathbf{1}_{(0,\infty)}(x). \tag{7.3.2}$$

The second property is a direct consequence of $v(\mathbb{R}) = \infty$, while the fourth one arises from $\int_{-1}^1 |x|\,v(dx) < \infty$. For a deeper investigation on the properties of the gamma process we refer to Kyprianou [48, Sec. 1.2 and 2.6], James et al. [44] and Yor [84]. Figure 7.1 below shows two simulated paths of a gamma process.

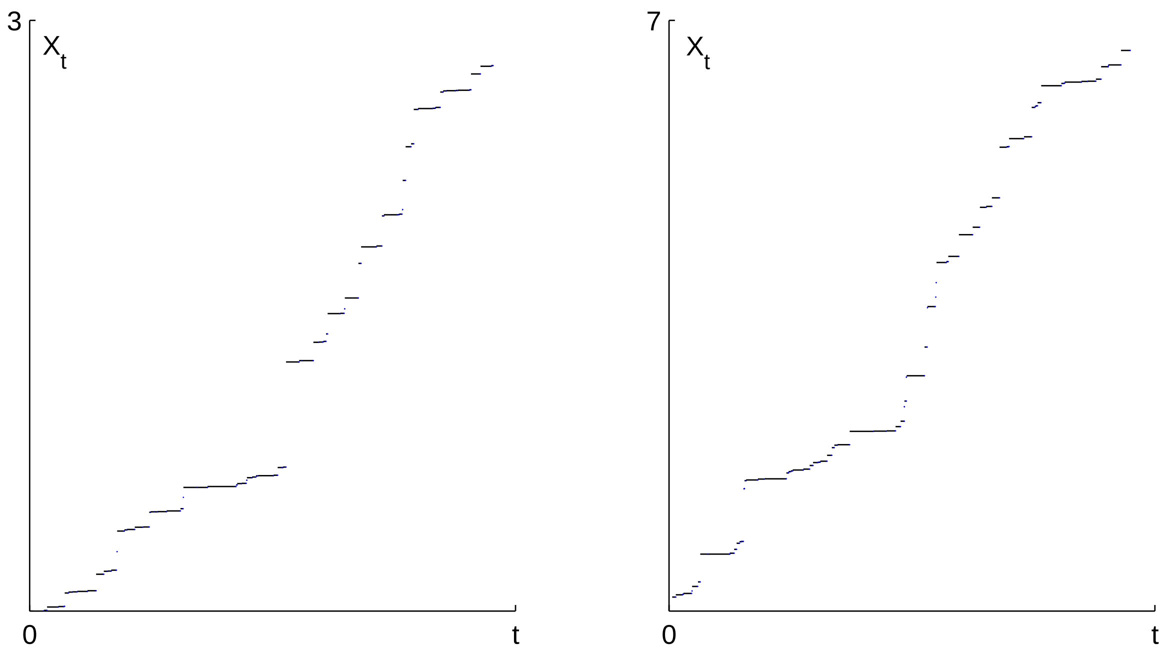

Figure 7.1: Two simulated paths of a gamma process $X = (X_t)_{t\geq 0}$. We set $\alpha = 5$ and $\alpha = 3$ for the left and right drawing, respectively.

7.3.1 Formulation of the problem

On the filtered statistical space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \{P_1, P_0\})$ the stochastic process $X = (X_t)_{t\geq 0}$ is defined and is assumed to be a gamma process of parameter $\alpha_i > 0$ under P_i, $i = 0, 1$. Let α be an $\mathscr{F}_0$-measurable random variable independent of X; under the probability measure P_π, defined by

$$P_\pi = \pi P_1 + (1-\pi)P_0, \quad \pi \in [0,1], \tag{7.3.3}$$

α takes value α_1, with probability π, and α_0, with probability $1-\pi$, where π is given. In order to test the two simple hypotheses

$$H_0 : \alpha = \alpha_0 \quad Vs \quad H_1 : \alpha = \alpha_1, \quad \alpha_0 > \alpha_1, \tag{7.3.4}$$

we are allowed to sequentially observe X. Let $\mathscr{F}_t^X = \sigma\{X_s : 0 \leq s \leq t\}$ and denote by (τ, d) a sequential decision rule, where τ is a stopping time of X, that is an $\mathscr{F}_t^X$-measurable random variable, $t \geq 0$, and d, called decision function, is an $\mathscr{F}_\tau^X$-measurable random variable, which, at the time τ, takes value i, if H_i, $i = 0, 1$, must be accepted.

The Bayesian problem of sequentially testing (7.3.4) requires computing

$$V(\pi) = \inf_{(\tau,d)} E_\pi \left[\tau + a\mathbf{1}_{(d=0,\alpha=\alpha_1)} + b\mathbf{1}_{(d=1,\alpha=\alpha_0)}\right], \quad a, b > 0, \tag{7.3.5}$$

and determining the π-Bayes decision rule $(\tau_\pi^\star, d_\pi^\star)$ at which the infimum in the above expression is attained. By means of standard arguments based on Shiryaev [72, Lemma 1, pp. 166-167], one can show that (7.3.5) is equivalent to the following optimal stopping problem

$$V(\pi) = \inf_\tau E_\pi \left[\tau + g_{a,b}(\pi_\tau)\right], \tag{7.3.6}$$

where $(\pi_t)_{t\geq 0}$, with $\pi_t = P_\pi(\alpha = \alpha_1 | \mathscr{F}_t^X)$, is the posterior probability process, $g_{a,b}(\pi) = a\pi \wedge b(1-\pi)$ and the π-Bayes decision rule is $d_\pi^\star = 1$, if $\pi_{\tau_\pi^\star} \geq c$, and $d_\pi^\star = 0$, if $\pi_{\tau_\pi^\star} < c$, being $c = b/(a+b)$.

Denoted by $D = \{\pi \in [0,1] : V(\pi) = g_{a,b}(\pi)\}$, the structure of the value function (7.3.6) and the general theory of optimal stopping (see, e.g., Peskir and Shiryaev [67, Chap. 1] or Shiryaev [72, Chap. 3]) imply that $\tau_\pi^\star = \inf\{t \geq 0 : \pi_t \in D, \pi_0 = \pi\}$ and that there exist two points A and B, with $0 < A \leq c \leq B < 1$, such that $D = [0, A] \cup [B, 1]$. D is called stopping set, while its complement (A, B) is the continuation set.

Let $(\varphi_t)_{t\geq 0}$ be the likelihood ratio process, defined by $\varphi_t = d\left(P_1|\mathscr{F}_t^X\right) / d\left(P_0|\mathscr{F}_t^X\right)$; according to Sato [71, Th. 33.2, p. 219],

$$\begin{aligned}\varphi_t &= \exp\left\{(\alpha_0 - \alpha_1)X_t - t\int_0^\infty \left(e^{(\alpha_0-\alpha_1)x} - 1\right)\frac{e^{-\alpha_0 x}}{x}\,dx\right\} \\ &= \exp\left\{(\alpha_0 - \alpha_1)X_t - \log\left(\frac{\alpha_0}{\alpha_1}\right)t\right\},\end{aligned} \tag{7.3.7}$$

where we used under the appropriate assumptions the well-known Frullani's formula

$$\int_0^\infty \frac{f(px) - f(qx)}{x}\,dx = [f(0) - f(\infty)]\log\left(\frac{q}{p}\right). \tag{7.3.8}$$

For further reference set

$$Y_t = (\alpha_0 - \alpha_1)X_t - \log\left(\frac{\alpha_0}{\alpha_1}\right)t. \tag{7.3.9}$$

A simple application of Bayes theorem shows that

$$\pi_t = \frac{\pi e^{Y_t}}{1 + \pi(e^{Y_t} - 1)}. \tag{7.3.10}$$

Let $\mu^X\big((0,t] \times H\big) = \sum_{s \le t} \mathbf{1}(\Delta X_s \in H)$, $H \in \mathscr{B}(\mathbb{R}^+ \backslash \{0\})$, be the measure of jumps of the process X; then, the expressions (7.3.7) and (7.3.10), together with a straightforward application of Itô's formula for purely jump Lévy processes, lead to the following stochastic differential equations:

$$\begin{aligned} d\varphi_t = &- \log\left(\frac{\alpha_0}{\alpha_1}\right)\varphi_{t-}\,dt \\ &+ \varphi_{t-}\int_0^\infty \left(e^{(\alpha_0-\alpha_1)x} - 1\right)\mu^X(dx, dt),\ \varphi_0 = 1, \end{aligned} \tag{7.3.11}$$

$$\begin{aligned} d\pi_t = &- \log\left(\frac{\alpha_0}{\alpha_1}\right)\pi_{t-}(1-\pi_{t-})dt \\ &+ \int_0^\infty \frac{\pi_{t-}(1-\pi_{t-})\left(e^{(\alpha_0-\alpha_1)x} - 1\right)}{1 + \pi_{t-}\left(e^{(\alpha_0-\alpha_1)x} - 1\right)}\mu^X(dx, dt),\ \pi_0 = \pi. \end{aligned} \tag{7.3.12}$$

7.3.2 Reduction of the optimal stopping problem to a free-boundary problem

We reduce the optimal stopping problem (7.3.6) to a free-boundary problem for the value function $V(\pi)$ and the boundaries A and B defining the stopping region D. To accomplish this task we need to determine the infinitesimal operator of $(\pi_t)_{t \ge 0}$ and show some properties of the function $V(\pi)$.

Proposition 7.3.1 *Let $f \in C^1[0,1]$; then*

$$f(\pi_t) = f(\pi) + \int_0^t (\mathbb{L}f)(\pi_{s-}) + \mathscr{M}_t, \tag{7.3.13}$$

where $\mathbb{L}$ is the infinitesimal operator of $(\pi_t)_{t\ge 0}$ defined by

$$\begin{aligned} (\mathbb{L}f)(\pi) = &- \log\left(\frac{\alpha_0}{\alpha_1}\right) f'(\pi)\pi(1-\pi) \\ &+ \int_0^\infty \left[f\left(\frac{\pi e^{-\alpha_1 x}}{(1-\pi)e^{-\alpha_0 x} + \pi e^{-\alpha_1 x}}\right) - f(\pi)\right]\frac{(1-\pi)e^{-\alpha_0 x} + \pi e^{-\alpha_1 x}}{x}\,dx \end{aligned} \tag{7.3.14}$$

and $\mathscr{M} = (\mathscr{M}_t)_{t \ge 0}$, given by

$$\begin{aligned} \mathscr{M}_t = \int_0^t \int_0^\infty &\left[f\left(\frac{\pi_{s-}e^{-\alpha_1 x}}{(1-\pi_{s-})e^{-\alpha_0 x} + \pi_{s-}e^{-\alpha_1 x}}\right) - f(\pi_{s-})\right] \\ &\times \left(\mu^X(dx, ds) - \frac{(1-\pi_{s-})e^{-\alpha_0 x} + \pi_{s-}e^{-\alpha_1 x}}{x}\,dx\,ds\right), \end{aligned} \tag{7.3.15}$$

is a local martingale with respect to $(\mathscr{F}_t^X)_{t\ge 0}$ and P_π, $\forall \pi \in [0,1]$.

Proof. The expressions (7.3.13)-(7.3.15) can be obtained by applying the results of Section 5.5 (or Buonaguidi and Muliere [16, Sec. 5.2]) or can be derived by using Ito's formula and (7.3.12):

$$\begin{aligned}
f(\pi_t) =& f(\pi) + \int_0^t f'(\pi_{s-})\, d\pi_s + \sum_{0\le s\le t} \left(\Delta f(\pi_s) - f'(\pi_{s-})\Delta\pi_s\right)\\
=& f(\pi) - \int_0^t \log\left(\frac{\alpha_0}{\alpha_1}\right) f'(\pi_{s^-})\pi_{s-}(1-\pi_{s^-})\, ds\\
&+ \int_0^t\int_0^1 \left[f(\pi_{s^-}+z) - f(\pi_{s^-})\right]\mu^\pi(dz,ds)\\
=& f(\pi) - \int_0^t \log\left(\frac{\alpha_0}{\alpha_1}\right) f'(\pi_{s^-})\pi_{s-}(1-\pi_{s^-})\, ds\\
&+ \int_0^t\int_0^1 \left[f(\pi_{s^-}+z) - f(\pi_{s^-})\right]v^\pi(dz)\, ds\\
&+ \int_0^t\int_0^1 \left[f(\pi_{s^-}+z) - f(\pi_{s^-})\right]\left(\mu^\pi(dz,ds) - v^\pi(dz)\, ds\right), \qquad (7.3.16)
\end{aligned}$$

where μ^π and v^π are the jumping measure and the associated compensator of $(\pi_t)_{t\ge0}$. From (7.3.12) one may notice that the magnitude of its jumps is

$$\Delta\pi_t = \frac{\pi_{t^-}(1-\pi_{t^-})\left(e^{(\alpha_0-\alpha_1)x}-1\right)}{1+\pi_{t^-}\left(e^{(\alpha_0-\alpha_1)x}-1\right)}, \qquad (7.3.17)$$

so that

$$\pi_{t^-} + \Delta\pi_t = \frac{\pi_{t^-}e^{-\alpha_1 x}}{(1-\pi_{t^-})e^{-\alpha_0 x} + \pi_{t^-}e^{-\alpha_1 x}}. \qquad (7.3.18)$$

Hence, the replacement in (7.3.16) of $(\pi_{s^-}+z)$ with (7.3.18) and the integration over $(0,\infty)$ with respect to μ^X and its compensator $(1-\pi)v_0 + \pi v_1$, being $v_i(dx) = x^{-1}e^{-\alpha_i x}\mathbf{1}_{(0,\infty)}(dx)$, $i = 0, 1$, complete the proof. ∎

Now, let us show that the value function $V(\pi)$ is continuous.

Proposition 7.3.2 *The map $\pi \mapsto V(\pi)$ in (7.3.6) is concave and thus continuous on $[0,1]$.*

Proof. Let π_1, $\pi_2 \in [0,1]$ and $\lambda \in [0,1]$. From (7.3.3), it is easy to notice that $P_{\lambda\pi_1+(1-\lambda)\pi_2} = \lambda P_{\pi_1} + (1-\lambda)P_{\pi_2}$. Hence,

$$\begin{aligned}
V\big(\lambda\pi_1 + (1-\lambda)\pi_2\big) &= \inf_\tau E_{\lambda\pi_1+(1-\lambda)\pi_2}\left[\tau + g_{a,b}(\pi_\tau)\right]\\
&= \inf_\tau \Big\{\lambda E_{\pi_1}\left[\tau + g_{a,b}(\pi_\tau)\right] + (1-\lambda)E_{\pi_2}\left[\tau + g_{a,b}(\pi_\tau)\right]\Big\}\\
&\ge \lambda \inf_\tau E_{\pi_1}\left[\tau + g_{a,b}(\pi_\tau)\right] + (1-\lambda)\inf_\tau E_{\pi_2}\left[\tau + g_{a,b}(\pi_\tau)\right]\\
&= \lambda V(\pi_1) + (1-\lambda)V(\pi_2). \qquad (7.3.19)
\end{aligned}$$

∎

The next proposition proves that $V(\pi)$ is smooth from the right at A, whenever the interval $(A,B) \neq \emptyset$.

Proposition 7.3.3 *If the optimal stopping boundary A is strictly less than $c = b/(a+b)$, then $V(\pi)$ from (7.3.6) is differentiable from the right at A and we have*

$$V'_+(A) = a. \tag{7.3.20}$$

Proof. Since on (A,B) we have $V(\pi) < g_{a,b}(\pi)$, for any $\epsilon > 0$ such that $A+\epsilon < c$, it results

$$\frac{V(A+\epsilon)-V(A)}{\epsilon} \le \frac{a(A+\epsilon)-aA}{\epsilon} = a, \tag{7.3.21}$$

so that $V'_+(A) \le a$, where the right-hand derivative exists because of the concavity of $\pi \mapsto V(\pi)$.

In order to show that the reverse inequality holds, fix $\epsilon > 0$ so that $A+\epsilon < c$ and consider the stopping time $\tau^\star_{A+\epsilon}$, that, according to the arguments of Subsection 6.3.1, is optimal for $V(A+\epsilon)$. We recall that $\tau^\star_{\pi+\epsilon}$ is the first exit time from (A,B) of the process $(\pi_t)_{t\ge 0}$, starting at $\pi_0 = \pi+\epsilon$. Then, from (7.3.3) and similarly to Gapeev and Peskir [35], we have

$$\begin{aligned} V(A+\epsilon) - V(A) & \\ &\ge E_{A+\epsilon}\left[\tau^\star_{A+\epsilon} + g_{a,b}(\pi_{\tau^\star_{A+\epsilon}})\right] - E_A\left[\tau^\star_{A+\epsilon} + g_{a,b}(\pi_{\tau^\star_{A+\epsilon}})\right] \\ &= \sum_{i=0}^{1} E_i\left[S_i(A+\epsilon) - S_i(A)\right], \end{aligned} \tag{7.3.22}$$

where

$$S_i(\pi) = \frac{1+(-1)^i(1-2\pi)}{2}\left(\tau^\star_{A+\epsilon} + a\frac{\pi e^{Y_{\tau^\star_{A+\epsilon}}}}{1+\pi(e^{Y_{\tau^\star_{A+\epsilon}}}-1)} \wedge b\frac{1-\pi}{1+\pi(e^{Y_{\tau^\star_{A+\epsilon}}}-1)}\right). \tag{7.3.23}$$

Then, according to the mean value theorem, there exist $\xi_i \in (A, A+\epsilon)$, $i = 0,1$, such that

$$\sum_{i=0}^{1} E_i\left[S_i(A+\epsilon) - S_i(A)\right] = \epsilon\sum_{i=0}^{1} E_i\left[S'_i(\xi_i)\right], \tag{7.3.24}$$

being

$$\begin{aligned} S'_i(\pi) = (-1)^{i-1}&\left(\tau^\star_{A+\epsilon} + a\frac{\pi e^{Y_{\tau^\star_{A+\epsilon}}}}{1+\pi(e^{Y_{\tau^\star_{A+\epsilon}}}-1)} \wedge b\frac{1-\pi}{1+\pi(e^{Y_{\tau^\star_{A+\epsilon}}}-1)}\right) \\ &+ \frac{1+(-1)^i(1-2\pi)}{2}\left(a\mathbf{1}_{\left\{\pi_{\tau^\star_{A+e}}<c\right\}} - b\mathbf{1}_{\left\{\pi_{\tau^\star_{A+e}}>c\right\}}\right)\frac{e^{Y_{\tau^\star_{A+\epsilon}}}}{\left[1+\pi(e^{Y_{\tau^\star_{A+\epsilon}}}-1)\right]^2}. \end{aligned} \tag{7.3.25}$$

From the definition of $\tau^\star_{\pi+\epsilon}$ and simple calculations, one has

$$\begin{aligned} \tau^\star_{A+\epsilon} &= \inf\{t \ge 0 : \pi_t \notin (A,B),\ \pi_0 = A+\epsilon\} \\ &\le \inf\left\{t \ge 0 : Y_t \le \log\left(\frac{A}{1-A}\frac{1-(A+\epsilon)}{A+\epsilon}\right)\right\} =: \gamma_\epsilon. \end{aligned} \tag{7.3.26}$$

According to Sato [71, Th 43.20, p. 323],

$$P_i\left[\lim_{t\downarrow 0} t^{-1}Y_t = -\log\left(\frac{\alpha_0}{\alpha_1}\right)\right] = 1, \quad i = 0,1, \tag{7.3.27}$$

meaning that the starting point 0 of $Y = (Y_t)_{t \geq 0}$ is regular for $(-\infty, 0)$ (that is, with probability 1, Y, starting at 0, enters $(-\infty, 0)$ immediately). From (7.3.26) and (7.3.27), it results $\gamma_\epsilon \downarrow 0$ P_i-a.s. as $\epsilon \downarrow 0$, $i = 0, 1$. Therefore, $\tau^\star_{A+\epsilon} \downarrow 0$ and $Y_{\tau^\star_{A+\epsilon}} \to 0$ as $\epsilon \downarrow 0$ P_i-a.s., $i = 0, 1$. Hence, from (7.3.25)

$$S_i'(\xi_i) \to (-1)^{i-1} aA + \frac{1 + (-1)^i (1 - 2A)}{2} a, \quad P_i\text{-a.s., } i = 0, 1, \text{ as } \epsilon \downarrow 0. \tag{7.3.28}$$

Since $S_i'(\xi_i) + (-1)^i \tau^\star_{A+\epsilon}$ is obviously bounded, for $i = 0, 1$, from (7.3.22), (7.3.24), (7.3.28), $E_i[\tau^\star_{A+\epsilon}] \to 0$ as $\epsilon \downarrow 0$, $i = 0, 1$, and the bounded convergence theorem we have

$$V_+'(A) = \lim_{\epsilon \downarrow 0} \frac{V(A+\epsilon) - V(A)}{\epsilon} \geq \lim_{\epsilon \downarrow 0} \sum_{i=0}^{1} E_i \left[S_i'(\xi_i) \right] = a, \tag{7.3.29}$$

which, combined with (7.3.21), completes the proof. ∎

Proposition 7.3.2 and 7.3.3 formally justify the so called principles of the smooth and continuous fit, stating that the value function $V(\pi)$ must be smooth at A and just continuous at B. The discovery of the continuous fit condition as variational principle alike the smooth fit is due Peskir and Shiryaev [65]: it can be explained by noticing that the process $(\pi_t)_{t \geq 0}$, defined through (7.3.10) and (7.3.12), creeps downwards and jumps upwards, so that the boundary A is continuously crossed, while B, at which the smooth fit breaks down, is passed by jumps only (see Figure 7.2).

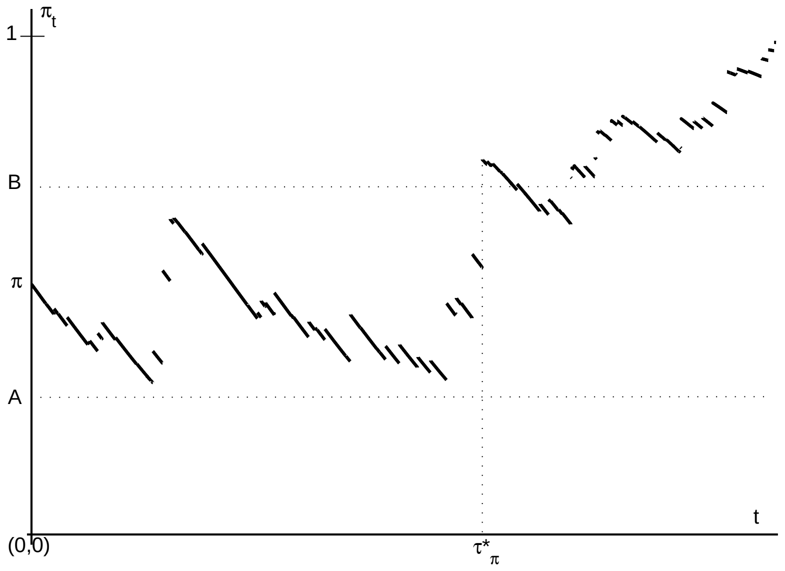

Figure 7.2: A simulated path of the posterior probability process $(\pi_t)_{t \geq 0}$ as defined by (7.3.10) and (7.3.12), with $\alpha_0 = 5$ and $\alpha_1 = 3$. It is assumed that the true hypothesis is $\alpha = \alpha_1$.

The above facts, the strong Markov property of $(\pi_t)_{t \geq 0}$, evident from (7.3.12), and the general theory of optimal stopping (see Chapter 2 or, e.g., Peskir and Shiryaev [67, Chap. 3 and

4] and Shiryaev [72, Chap. 3]) naturally lead to the formulation of the following free-boundary problem, for the unknown function V and the unknown boundaries A and B:

$$\mathbb{L}V = -1 \qquad \text{for} \quad \pi \in (A, B), \tag{7.3.30}$$

$$V = g_{a,b} \qquad \text{for} \quad \pi \notin (A, B), \tag{7.3.31}$$

$$V < g_{a,b} \qquad \text{for} \quad \pi \in (A, B), \tag{7.3.32}$$

$$V(A_+) = aA \qquad (\textit{continuous fit}), \tag{7.3.33}$$

$$V'(A) = a \qquad (\textit{smooth fit}), \tag{7.3.34}$$

$$V(B_-) = b(1 - B) \qquad (\textit{continuous fit}) \tag{7.3.35}$$

7.3.3 Existence and uniqueness of the solution

We are going to prove that if a solution to the free-boundary problem (7.3.30)-(7.3.35) exists, then it is unique.

For a fixed $B > c$, consider on the interval $I_B = (0, B]$ the integro-differential equation defined by (7.3.14) and (7.3.30). Denote by $V(\pi; B)$, $\pi \in I_B$, its solution. Notice that the function

$$S(\pi, x) = \frac{\pi e^{-\alpha_1 x}}{(1-\pi)e^{-\alpha_0 x} + \pi e^{-\alpha_1 x}}, \quad \pi \in I_B, \quad x \geq 0, \tag{7.3.36}$$

appearing in (7.3.14), is increasing in x, $\lim_{x\to\infty} S(\pi, x) = 1$ and, according to (7.3.31) and (7.3.35), leads us to set $V(S(\pi, x); B) = b(1 - S(\pi, x))$, whenever $\pi \in I_B$ and $x \geq x^\star(\pi; B) := \log\left(\frac{1-\pi}{\pi}\frac{B}{1-B}\right)\Big/(\alpha_0 - \alpha_1)$. Hence, $V(\pi; B)$ satisfies

$$(\mathbb{L}_B V)(\pi; B) = 0, \quad \pi \in I_B, \tag{7.3.37}$$

$$V(B; B) = b(1 - B), \tag{7.3.38}$$

where $\mathbb{L}_B$ is the operator defined by

$$\begin{aligned}(\mathbb{L}_B f)(\pi) &= -\log\left(\frac{\alpha_0}{\alpha_1}\right) f'(\pi)\pi(1-\pi) \\ &+ b(1-\pi)\int_{x^\star(\pi;B)}^{\infty} \frac{e^{-\alpha_0 x}}{x}\,dx - f(\pi)\int_{x^\star(\pi;B)}^{\infty} \frac{(1-\pi)e^{-\alpha_0 x} + \pi e^{-\alpha_1 x}}{x}\,dx \\ &+ \int_0^{x^\star(\pi;B)} \big[f(S(\pi,x)) - f(\pi)\big]\frac{(1-\pi)e^{-\alpha_0 x} + \pi e^{-\alpha_1 x}}{x}\,dx + 1, \quad \pi \in I_B.\end{aligned} \tag{7.3.39}$$

Proposition 7.3.4 *For any fixed $B > c$, (7.3.37)-(7.3.38) has a unique continuously differentiable solution $V(\pi; B)$, $\pi \in I_B$.*

Proof. Define $f(y) = V(\pi; B)$, with $\pi = e^y/(1 + e^y)$; it is not difficult to show that f solves

$$\begin{aligned}f'(y) = &-\frac{1}{\lambda} - \frac{b}{\lambda}\frac{e^{\gamma y}}{1+e^y}\int_{B^o}^{\infty}\frac{e^{-\gamma z}}{z-y}\,dz + f(y)\frac{e^{\gamma y}}{(1+e^y)\lambda}\int_{B^o}^{\infty}\frac{(1+e^z)e^{-\gamma z}}{z-y}\,dz \\ &- \frac{e^{\gamma y}}{(1+e^y)\lambda}\int_y^{B^o}[f(z) - f(y)]\frac{(1+e^z)e^{-\gamma z}}{z-y}\,dz, \quad y^\star \leq y < B^o,\end{aligned} \tag{7.3.40}$$

$$f(B^o) = \frac{b}{1+e^{B^o}}, \tag{7.3.41}$$

where $y^\star$ is any arbitrary finite number smaller than B^o, $B^o = \log(B/(1-B))$, $\gamma = \alpha_0/(\alpha_0-\alpha_1)$ and $\lambda = \log(\alpha_1/\alpha_0)$. The representation (7.3.40)-(7.3.41) is equivalent to (7.3.37)-(7.3.38), but has the advantage of directly appearing as a linear Volterra integro-differential equation of the second kind (meaning that one limit of integration is variable and the unknown function f also occurs outside the integral). We observe that (7.3.40) seems to be outside the scope of any existing theory on integro-differential equations, because one has to consider the difference $f(z) - f(y)$ in the last integral (and not just $f(z)$ like in the canonical representation (7.2.1)), in order to make it finite. This is caused by the lack of integrability of the map $z \mapsto (1 + e^z)e^{-\gamma z}/(z-y)$ on (y, B^o), which, in turn, is a consequence of the Lévy measure of a gamma process. Then, we proceed as follows: first we analyze "regular versions" of (7.3.40)-(7.3.41), for which the existence and uniqueness of solutions can be proved by resorting to standard theory; then, we verify that the limit of these solutions is indeed a solution of (7.3.40)-(7.3.41).

Let $0 < \epsilon \le 1$ and denote by $f_\epsilon(y)$ the function solving the following "regular" problem:

$$f'_\epsilon(y) = g(y) + h_\epsilon(y) f_\epsilon(y) + \int_y^{B^o} k_\epsilon(y,z) f_\epsilon(z)\, dz, \quad y^\star \le y < B^o, \tag{7.3.42}$$

$$f_\epsilon(B^o) = \frac{b}{1+e^{B^o}}, \tag{7.3.43}$$

where

$$g(y) = -\frac{1}{\lambda} - \frac{b}{\lambda}\frac{e^{\gamma y}}{1+e^y}\int_{B^o}^{\infty} \frac{e^{-\gamma z}}{z-y}\, dz, \tag{7.3.44}$$

$$h_\epsilon(y) = \frac{e^{\gamma y}}{(1+e^y)\lambda}\left[\int_y^{B^o} \frac{(1+e^z)e^{-\gamma z}}{(z-y)^{1-\epsilon}}\, dz + \int_{B^o}^{\infty} \frac{(1+e^z)e^{-\gamma z}}{z-y}\, dz\right], \tag{7.3.45}$$

$$k_\epsilon(y,z) = -\frac{e^{\gamma y}}{(1+e^y)\lambda}\frac{(1+e^z)e^{-\gamma z}}{(z-y)^{1-\epsilon}}. \tag{7.3.46}$$

Expressing (7.3.42)-(7.3.43) as a system of integral equations

$$w_\epsilon(y) = g(y) + h_\epsilon(y) f_\epsilon(y) + \int_y^{B^o} k_\epsilon(y,z) f_\epsilon(z)\, dz, \tag{7.3.47}$$

$$f_\epsilon(y) = \frac{b}{1+e^{B^o}} - \int_y^{B^o} w_\epsilon(z)\, dz, \tag{7.3.48}$$

or, more compactly,

$$F_\epsilon(y) = G_\epsilon(y) + \int_y^{B^o} K_\epsilon(y,z) F_\epsilon(z)\, dz, \tag{7.3.49}$$

where

$$F_\epsilon(y) = \begin{bmatrix} w_\epsilon(y) \\ f_\epsilon(y) \end{bmatrix}, \quad G_\epsilon(y) = \begin{bmatrix} g(y) + h_\epsilon(y) b/\big(1+e^{B^o}\big) \\ b/\big(1+e^{B^o}\big) \end{bmatrix},$$

$$K_\epsilon(y,z) = \begin{bmatrix} -h_\epsilon(y) & k_\epsilon(y,z) \\ -1 & 0 \end{bmatrix},$$

and using the matrix norm $||K_\epsilon(y,z)|| = \max\{h_\epsilon(y) + |k_\epsilon(y,z)|, 1\}$, the following facts are easily verified: i) $G_\epsilon(y)$ is a continuous function of y, in the sense that its components are all continuous; ii) for every continuous vector function s and all $y \le n_1 \le n_2 \le B^o$, $\int_{n_1}^{n_2} K_\epsilon(y,z)s(z)\,dz$ is a continuous function of y; iii) every component of $K_\epsilon(y,z)$ is absolutely integrable with respect to z, for $y^\star \le y < B^o$; iv) $\exists\, y^\star = y_0 < y_1 < \ldots < y_n = B^o$ such that, for all $i = 0, \ldots, n-1$, $\int_{y_i}^{\min\{y, y_{i+1}\}} ||K_\epsilon(y,z)||\,dz \le p < 1$, where p is independent of y and i; v) for $y^\star \le y \le B^o$, $\lim_{\delta \downarrow 0} \int_{y-\delta}^{y} ||K_\epsilon(y-\delta, z)||\,dz = 0$. Then, according to Linz [53, Th. 3.2, p. 32], we can conclude that for any $0 < \epsilon \le 1$, there exists only one continuous solution $F_\epsilon(y)$ to (7.3.49), that is, the integro-differential equation (7.3.42)-(7.3.43) has a unique continuously differentiable solution f_ϵ.

A direct analysis based on the existence and uniqueness of f_ϵ, $0 < \epsilon \le 1$, shows that $\{f_\epsilon\}$ and $\{f'_\epsilon\}$ are Cauchy sequences and therefore are uniform convergent on $[y^\star, B^o]$. Then

$$f(y) := \lim_{\epsilon \downarrow 0} f_\epsilon(y), \quad f'(y) := \lim_{\epsilon \downarrow 0} f'_\epsilon(y), \quad y^\star \le y \le B^o, \tag{7.3.50}$$

exist and we have that f is continuously differentiable with derivative f'. Further, since

$$\lim_{\epsilon \downarrow 0} \frac{f_\epsilon(z) - f_\epsilon(y)}{(z-y)^{1-\epsilon}} = \frac{f(z) - f(y)}{(z-y)} \quad \text{and} \quad \left| \frac{f_\epsilon(z) - f_\epsilon(y)}{(z-y)^{1-\epsilon}} \right| \le C_y \tag{7.3.51}$$

for any $z \in [y, B^o]$ and $0 < \epsilon \le 1$, where C_y is a constant depending on y, from the bounded convergence theorem we get

$$\begin{aligned} f'(y) = \lim_{\epsilon \downarrow 0} f'_\epsilon(y) &= -\frac{1}{\lambda} - \frac{b}{\lambda}\frac{e^{\gamma y}}{1+e^y} \int_{B^o}^\infty \frac{e^{-\gamma z}}{z-y}\,dz + \lim_{\epsilon \downarrow 0} f_\epsilon(y) \frac{e^{\gamma y}}{(1+e^y)\lambda} \int_{B^o}^\infty \frac{(1+e^z)e^{-\gamma z}}{z-y}\,dz \\ &\quad - \frac{e^{\gamma y}}{(1+e^y)\lambda} \lim_{\epsilon \downarrow 0} \int_y^{B^o} [f_\epsilon(z) - f_\epsilon(y)] \frac{(1+e^z)e^{-\gamma z}}{(z-y)^{1-\epsilon}}\,dz \\ &= -\frac{1}{\lambda} - \frac{b}{\lambda}\frac{e^{\gamma y}}{1+e^y} \int_{B^o}^\infty \frac{e^{-\gamma z}}{z-y}\,dz + f(y) \frac{e^{\gamma y}}{(1+e^y)\lambda} \int_{B^o}^\infty \frac{(1+e^z)e^{-\gamma z}}{z-y}\,dz \\ &\quad - \frac{e^{\gamma y}}{(1+e^y)\lambda} \int_y^{B^o} [f(z) - f(y)] \frac{(1+e^z)e^{-\gamma z}}{z-y}\,dz, \quad y^\star \le y < B^o. \end{aligned} \tag{7.3.52}$$

Hence, f from (7.3.50) is a continuously differentiable solution of (7.3.40)-(7.3.41), that is, (7.3.37)-(7.3.38) admits a continuously differentiable solution $V(\pi; B)$, $\pi \in I_B$.

The arguments contained in Remark 7.3.1 below finally show that $V(\pi; B)$ is unique. ∎

The map $\pi \mapsto V(\pi; B)$, $\pi \in I_B$, hits the map $\pi \mapsto b(1-\pi)$ at B, because of the continuous fit principle (7.3.35) and (7.3.38). The condition ensuring the existence of a special (unique) pair of $A^\star$ and $B^\star$, at which the map $\pi \mapsto V(\pi; B^\star)$ smoothly hits $\pi \mapsto a\pi$ and hits $\pi \mapsto b(1-\pi)$, respectively, is given in the next proposition.

Proposition 7.3.5 *There exist a unique function V and a unique pair of points $A^\star$ and $B^\star$, which solve the free-boundary problem* (7.3.30)-(7.3.35)*, defined through the integro-differential operator* (7.3.14)*, if and only if*

$$\lim_{B \downarrow c} V'(B^-; B) < a. \tag{7.3.53}$$

In this case we have

$$V(\pi) = \begin{cases} V(\pi; B^\star) & \textit{for} \quad \pi \in (A^\star, B^\star) \\ g_{a,b}(\pi) & \textit{for} \quad \pi \in [0, A^\star] \cup [B^\star, 1] \end{cases}, \tag{7.3.54}$$

where the map $\pi \mapsto V(\pi; B)$, $\pi \in I_B$, is the unique continuously differentiable solution of (7.3.37)-(7.3.38) *and $A^\star$ and $B^\star$ uniquely solve*

$$V(A^\star; B^\star) = aA^\star, \quad V'(A^\star; B^\star) = a. \tag{7.3.55}$$

Proof. The existence and uniqueness of the map $\pi \mapsto V(\pi; B)$, $\pi \in I_B$, $c < B < 1$, has been previously verified. The necessity and sufficiency of (7.3.53) for having a unique pair $A^\star$ and $B^\star$ solving (7.3.55), and therefore a unique solution of the free-boundary problem (7.3.30)-(7.3.35), arise from the following reasoning.

A direct verification based on the arguments of Section 7.4 (or the more formal proof given by Peskir and Shiryaev [65, Remark 2.2, p. 850]) shows that the maps $\pi \mapsto V(\pi; B')$ and $\pi \mapsto V(\pi; B'')$, $B' < B''$, do not intersect on the interval $(0, B']$ (see Figure 7.3). Condition (7.3.53) guarantees that for $B > c$, close enough to c, $\pi \mapsto V(\pi; B)$ crosses $\pi \mapsto a\pi$ at some $\pi < c$. Then moving B from c to 1, it is easily seen that there exists a unique pair $A^\star$ and $B^\star$ satisfying (7.3.55). In other words, there exists a unique pair $A^\star$ and $B^\star$ at which V, provided by (7.3.54), is consistent with (7.3.33)-(7.3.35). ∎

7.3.4 Optimality of the solution

The next theorem connects the free-boundary problem (7.3.30)-(7.3.35) with the optimal stopping problem (7.3.6).

Theorem 7.3.1 *The π-Bayes decision rule $(\tau^\star_\pi, d^\star_\pi)$ for the sequential testing of the two simple hypotheses* (7.3.4) *concerning the parameter α of a gamma process:*

I) if (7.3.53) *and $\partial(\mathbb{L}V)(\pi)/\partial\pi \le 0$, $\pi \in [0, A^\star)$, hold, is given by $\tau^\star_\pi = \inf\{t \ge 0 : \pi_t \notin (A^\star, B^\star)\}$, $d^\star_\pi = 0$ (accept $H_0 : \alpha = \alpha_0$), if $\pi_{\tau^\star_\pi} \le A^\star$, and $d^\star_\pi = 1$ (accept $H_1 : \alpha = \alpha_1$), if $\pi_{\tau^\star_\pi} \ge B^\star$. The stopping boundaries $0 < A^\star < c < B^\star < 1$ and the value function V in* (7.3.6) *are given by means of* (7.3.54) *and* (7.3.55)*;*

II) if (7.3.53) *does not hold, becomes trivial: $\tau^\star_\pi = 0$, $d^\star_\pi = 0$, if $\pi < c$, and $d^\star_\pi = 1$, if $\pi \ge c$. The value function $V(\pi)$ is therefore equal to $g_{a,b}(\pi)$, for $\pi \in [0, 1]$.*

Proof. The second statement is obvious and more arguments can be found in Peskir and Shiryaev [65, pp. 849-850]. According to Theorem 5.5.1 or Buonaguidi and Muliere [16, Th 5.1, p. 58], for proving the first part of the theorem we only need to check that $(\mathbb{L}V)(\pi) \ge -1$, for $\pi \in [0, 1]$, where $\mathbb{L}$ is given in (7.3.14). By construction, this condition is satisfied on the interval $(A^\star, B^\star)$. For $\pi \in (B^\star, 1]$, on which $V(\pi) = b(1 - \pi)$, a simple application of the Frullani's integral (7.3.8) shows that $(\mathbb{L}V)(\pi) = 0$. When $\pi = A^\star$, the smooth and continuous fit conditions (7.3.33) and

(7.3.34) imply $(\mathbb{L}V)(A^\star) = -1$. Finally, one can easily show that $(\mathbb{L}V)(A^\star_-) = -1$ that, along with $\partial(\mathbb{L}V)(\pi)/\partial\pi \le 0$ for $\pi \in [0, A^\star)$, completes the proof. ∎

Remark 7.3.1 The following probabilistic argument can be used to prove that for any $B > c$ the map $\pi \mapsto V(\pi; B)$, $\pi \in I_B$, solving (7.3.37)-(7.3.38), is unique. Let $g(\pi) = (m\pi + q) \wedge b(1-\pi)$, where $\pi \mapsto m\pi + q$ is the line hitting smoothly $\pi \mapsto V(\pi, B)$ at some $Z < B$. Consider now the optimal stopping problem (7.3.6) with $g(\pi)$ in place of $g_{a,b}(\pi)$ and denote by $V(\pi)$ the correspondent value function. Define $V^\star(\pi) = V(\pi; B)$, for $\pi \in (Z, B)$, being $V(\pi; B)$ a solution to (7.3.37)-(7.3.38), and $V^\star(\pi) = g(\pi)$, for $\pi \in [0, Z] \cup [B, 1]$. Then, the same arguments of Theorem 7.3.1 implies that $V(\pi) = V^\star(\pi)$, for $\pi \in [0, 1]$. Since Z is arbitrary, the claim is verified.

Remark 7.3.2 The π-Bayes decision rule given in point (I) of the Theorem 7.3.1 can also be expressed as

$$\tau^\star_\pi = \inf\{t \ge 0 : \; Z_t \notin (\widetilde{A}^\star, \widetilde{B}^\star)\}, \tag{7.3.56}$$

$$d^\star_\pi = \begin{cases} 0 \quad (\text{accept } H_0) \quad \text{if} \quad Z_{\tau^\star_\pi} \le \widetilde{A}^\star \\ 1 \quad (\text{accept } H_1) \quad \text{if} \quad Z_{\tau^\star_\pi} \ge \widetilde{B}^\star \end{cases}, \tag{7.3.57}$$

where:

$$Z_t = X_t - \mu t, \tag{7.3.58}$$

$$\mu = \log\left(\frac{\alpha_0}{\alpha_1}\right) \Big/ (\alpha_0 - \alpha_1), \tag{7.3.59}$$

$$\widetilde{A}^\star = \log\left(\frac{1-\pi}{\pi}\frac{A^\star}{1-A^\star}\right) \Big/ (\alpha_0 - \alpha_1), \tag{7.3.60}$$

$$\widetilde{B}^\star = \log\left(\frac{1-\pi}{\pi}\frac{B^\star}{1-B^\star}\right) \Big/ (\alpha_0 - \alpha_1). \tag{7.3.61}$$

Thus, the following equivalent sequential procedure can be adopted: observe $X = (X_t)_{t\ge0}$ and evaluate the process $Z = (Z_t)_{t\ge0}$, as defined in (7.3.58). As soon as Z enters $(-\infty, \widetilde{A}^\star]$ or $[\widetilde{B}^\star, \infty)$ stop the observation: accept $H_0 : \alpha = \alpha_0$ in the first case and $H_1 : \alpha = \alpha_1$ in the second one. We notice that since $\widetilde{A}^\star$ and $\widetilde{B}^\star$ in (7.3.60) and (7.3.61) can be determined once $A^\star$ and $B^\star$ have been computed as solution of (7.3.55), (7.3.53) must hold.

7.4 A collocation method for the free-boundary problem

As seen in Subsection 7.3.3, the solution to the free-boundary problem (7.3.30)-(7.3.35) requires evaluating the map $\pi \mapsto V(\pi; B)$, solving (7.3.37)-(7.3.38), and, if the condition (7.3.53) is satisfied, determining $A^\star$ and $B^\star$ at which $\pi \mapsto V(\pi; B^\star)$ smoothly hits $\pi \mapsto a\pi$ and crosses $\pi \mapsto b(1-\pi)$, respectively.

The problem is that explicitly finding $V(\pi; B)$ is not an easy task. This is due to presence of the integration variable x at the denominator of the fraction in the last integral of (7.3.39), which makes the integro-differential equation (7.3.37) extremely difficult to solve: one can observe that the source of this complication is the Lévy measure of a gamma process.

In this section we approach the free-boundary problem (7.3.30)-(7.3.35) numerically. In particular, we propose a modified version of the collocation method presented in Section 7.2, which allows us to get very accurate solutions.

7.4.1 Identifying the continuation set

Denote by $\{T_i^\star\}_{i\geq 0}$ the family of shifted Chebyshev polynomials on the interval $I = [0, 1]$, that is, $T_i^\star = T_i^I$, being T_i^I defined by (7.2.15), $i \geq 0$. For a fixed $B > c$ and a sufficiently large $n \geq 0$, consider an approximation $V_n(\pi; B)$ of $V(\pi; B)$ given by

$$V(\pi; B) \approx V_n(\pi; B) = \sum_{i=0}^{n} w_i(B) T_i^\star(\pi). \tag{7.4.1}$$

As seen in Section 7.2 and according to (7.3.37)-(7.3.38), the $n + 1$ coefficients $w_i(B)$ can be determined as solution of the following linear system of $n + 1$ equations

$$(\mathbb{L}_B V_n)(\pi_i; B) = 0, \quad i = 1, .., n, \tag{7.4.2}$$

$$V_n(B; B) = b(1 - B), \tag{7.4.3}$$

where $\mathbb{L}_B$ is defined in (7.3.39) and $\{\pi_1, .., \pi_n\}$ are n collocation nodes in $I_B = (0, B]$. As n increases, the uniform convergence of $V_n(\pi; B)$ to $V(\pi; B)$ on any compact interval is ensured by the Waierstrass approximation theorem and the continuity of $V(\pi; B)$, as stated in Proposition 7.3.4; the latter also guarantees that the coefficients $w_i(B)$, solution to (7.4.2)-(7.4.3), are well identified, due to the uniqueness of $V(\pi; B)$.

Solving the problem (7.4.2)-(7.4.3) for several values of B allows us to check if (7.3.53) is satisfied and, in this case, to have a plausible idea on the continuation set $(A^\star, B^\star)$. Let us explain this claim by means of two examples.

In the first one, we set $a = b = 0.5$ (hence $c = 0.5$), $\alpha_0 = 5$, $\alpha_1 = 1$ and we fix $n = 8$ in (7.4.1); Figure 7.3-a below shows that even for values of B very close to c (we used $B = 0.51, 0.55, 0.59$), the maps $\pi \mapsto V_n(\pi; B)$, $\pi \in I_B = (0, B]$, obtained as solutions of (7.4.2)-(7.4.3) (we used as collocation nodes a set of n equally spaced nodes in $[0.1, B]$), never intersect the map $\pi \mapsto a\pi$. It means that (7.3.53) fails to hold: the free-boundary problem does not admit a solution and the solution of the optimal stopping problem (7.3.6) becomes trivial (see point (II) of Theorem 7.3.1).

In the second example, we set $a = b = 5$ (hence $c = 0.5$), $\alpha_0 = 5$, $\alpha_1 = 1$ and $n = 8$ in (7.4.1); then, the system (7.4.2)-(7.4.3) has been solved for $B = 0.55, 0.58, 0.61, 0.64, 0.67, 0.70$ (again, a set of n equally spaced collocation nodes in $[0.1, B]$ has been used). The correspondent maps $\pi \mapsto V_n(\pi; B)$ are shown in Figure 7.3-b below: one can observe that (7.3.53) is satisfied, since there exist values of $B > c$ for which $\pi \mapsto V_n(\pi; B)$ intersects $\pi \mapsto a\pi$; thus, moving B on

$(c,1)$ from the left to the right, one can notice the existence of a unique pair of points $A^\star$ and $B^\star$ at which the continuous and smooth fit conditions (7.3.33)-(7.3.35) hold. We observe that $A^\star \in (0.2, 0.3)$ and $B^\star \in (0.64, 0.67)$ (of course, we can make these intervals more precise by solving (7.4.2)-(7.4.3) for values of $B \in (0.64, 0.67)$).

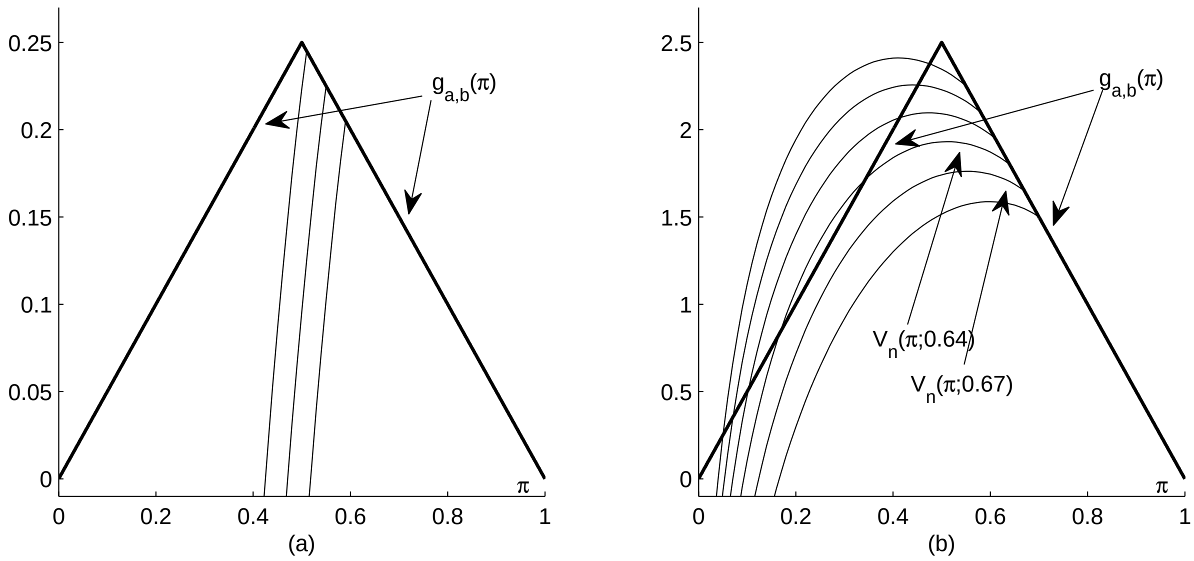

Figure 7.3: Two computer drawings of the maps $\pi \mapsto V_n(\pi; B)$ solving (7.4.2)-(7.4.3), with $n = 8$. On the left, a situation where the maps $\pi \mapsto V_n(\pi; B)$ never cross $\pi \mapsto a\pi$, even when $B \downarrow c$: the free-boundary problem (7.3.30)-(7.3.35) does not have a solution, so that the optimal stopping problem (7.3.6) becomes trivial; on the right, the more interesting situation where the condition (7.3.53) holds: it is evident that there exist $A^\star \in (0.2, 0.3)$ and $B^\star \in (0.64, 0.67)$ such that $\pi \mapsto V_n(\pi; B^\star)$ hits smoothly $\pi \mapsto a\pi$ at $A^\star$.

7.4.2 Extension of the collocation method

Once we have checked the free-boundary problem (7.3.30)-(7.3.35) admits a solution (that is, (7.3.53) is satisfied), we have to compute the optimal boundary points $A^\star$, $B^\star$ and the map $\pi \mapsto V(\pi; B^\star)$, $\pi \in (A^\star, B^\star)$. This task requires an extension of the collocation method presented in Section 7.2 and adopted in the previous subsection, because the interval $(A^\star, B^\star)$ on which $V(\pi; B^\star)$ is defined is unknown, as well as $V(\pi; B^\star)$ itself.

For a sufficiently large $n \geq 0$, let $V_n(\pi; B^\star)$ be an approximation of $V(\pi; B^\star)$, expressed as linear combination of the first $n+1$ shifted Chebyshev polynomials on $[0, 1]$:

$$V(\pi; B^\star) \approx V_n(\pi; B^\star) = \sum_{i=0}^{n} w_i(B^\star) T_i^\star(\pi), \quad \pi \in I_{B^\star} = (0, B^\star]. \tag{7.4.4}$$

Solving the free-boundary problem (7.3.30)-(7.3.35) reduces therefore to determining the $n+1$ coefficients $w_i(B^\star)$ and the two points $A^\star$ and $B^\star$. Since the map $\pi \mapsto V(\pi; B^\star)$ solves on $I_{B^\star}$

the integro-differential equation (7.3.37)-(7.3.38) and satisfies (7.3.55), our problem boils down to solving the following system of $n+3$ non-linear equations:

$$(\mathbb{L}_{B^\star} V_n)(\pi_i; B^\star) = 0, \quad i = 1, .., n, \tag{7.4.5}$$
$$V_n(A^\star; B^\star) = aA^\star, \tag{7.4.6}$$
$$V_n'(A^\star; B^\star) = a, \tag{7.4.7}$$
$$V_n(B^\star; B^\star) = b(1 - B^\star), \tag{7.4.8}$$

where $\mathbb{L}_{B^\star}$ is defined by (7.3.39) and the n collocation nodes $\{\pi_1, .., \pi_n\}$ are chosen so that they are less than $B^\star$. We observe that even though $B^\star$ is not known, the procedure developed in Subsection 7.4.1 for identifying the continuation set allows us to reasonably establish an open neighbourhood of $B^\star$, say (k_1, k_2). Then, we can fix $\pi_i \le k_1$, $i = 1, .., n$. The system (7.4.5)-(7.4.8) can be handled by means of standard numerical techniques: the $n+1$ coefficients $w_i(B^\star)$ and $A_n^\star$ and $B_n^\star$, approximating the true values $A^\star$ and $B^\star$, are well identified and rapidly computed, as consequence of the uniqueness argument of Proposition 7.3.5.

Once the solution to (7.4.5)-(7.4.8) has been determined , according to Theorem 7.3.1 point (I), the following approximated π-Bayes decision rule can be used to test the two simple hypotheses (7.3.4) for a gamma process of parameter α:

$$\tau_{n,\pi}^\star = \inf\{t \ge 0 : \pi_t \notin (A_n^\star, B_n^\star)\}, \tag{7.4.9}$$
$$d_{n,\pi}^\star = \begin{cases} 0 \quad (\text{accept } H_0) \quad \text{if} \quad \pi_{\tau_{n,\pi}^\star} \le A_n^\star \\ 1 \quad (\text{accept } H_1) \quad \text{if} \quad \pi_{\tau_{n,\pi}^\star} \ge B_n^\star \end{cases}. \tag{7.4.10}$$

The value function $V(\pi)$ from (7.3.6) and (7.3.54) can be approximated by

$$V_n(\pi) = \begin{cases} V_n(\pi; B_n^\star) & \text{for} \quad \pi \in (A_n^\star, B_n^\star) \\ g_{a,b}(\pi) & \text{for} \quad \pi \in [0, A_n^\star] \cup [B_n^\star, 1] \end{cases}. \tag{7.4.11}$$

Similarly to Subsection 7.2.3, we can assess the quality of the approximation in two ways: the first one relies on the fact that $V_n(\pi; B_n^\star)$ must satisfy $(\mathbb{L}_{B_n^\star} V_n)(\pi; B_n^\star) \approx 0$, for any $\pi \in [A_n^\star, B_n^\star)$. Then, we can increase n until

$$M_n = \sup_{\pi \in [A_n^\star, B_n^\star)} |(\mathbb{L}_{B_n^\star} V_n)(\pi; B_n^\star)| < \epsilon, \quad \epsilon > 0. \tag{7.4.12}$$

The second one is based on the convergence of $\{V_n\}$: defining by

$$\rho_n = \sup_{\pi \in (0,1)} \left| \frac{V_n(\pi) - V_{n-1}(\pi)}{V_{n-1}(\pi)} \right|, \quad n \ge 1, \tag{7.4.13}$$

the maximum relative distance between V_n and V_{n-1}, we can increase n until $\rho_n < \delta$, $\delta > 0$.

To illustrate the above procedure, let us continue the analysis of the second example described in the previous subsection, where we checked that the free-boundary problem (7.3.30)-(7.3.35) admits a unique solution when $a = b = 5$, $\alpha_0 = 5$ and $\alpha_1 = 1$; further, we observed that $A^\star \in (0.2, 0.3)$ and $B^\star \in (0.64, 0.67)$. For different values of n in (7.4.4) and n equally spaced

Table 7.1

n	I	$A_n^\star$-$B_n^\star$	M_n	ρ_n
4	$[0.1, 0.64]$	0.2577-0.6457	0.1251	-
6	$[0.1, 0.64]$	0.2525-0.6503	0.0207	0.0157
8	$[0.1, 0.64]$	0.2541-0.6510	0.0143	0.0013
40	$[0.01, 0.64]$	0.2541-0.6511	0.0060	1.8×10^{-4}

collocation nodes in the interval I, the table 7.1 shows the values of $A_n^\star$, $B_n^\star$, obtained as solution of (7.4.5)-(7.4.8), M_n and ρ_n (the latter expresses the maximum relative distance between the value functions V_n associated to two consecutive n of Table 7.1).

From Table 7.1 we notice that the value function V_n and the boundaries $A_n^\star$ and $B_n^\star$ are almost the same when $n = 8$ and $n = 40$: this is due to the rapid convergence of the series of Chebyshev polynomials. Figure 7.4-a below shows the maps $\pi \mapsto V_n(\pi; B_n^\star)$ and $\pi \mapsto V_n(\pi)$ when $n = 8$; Figure 7.4-b shows that $(\mathbb{L}V)(\pi) \approx (\mathbb{L}V_n)(\pi)$ is decreasing on $(0, A_n^\star)$: then, Theorem 7.3.1 point (I) applies.

7.5 Use of the collocation method in well known problems

In this section, we apply the collocation approach illustrated in Section 7.4 to four problems of sequential testing, for which explicit solutions are available in the literature. In particular, we consider the sequential testing of two simple hypotheses for a Wiener process with drift (Shiryaev [72, Sec. 4.2]), a Poisson process (Peskir and Shiryaev [65]), a compound Poisson process with exponential jumps (Gapeev [32]) and a negative binomial process (Chapter 5 or Buonaguidi and Muliere [16]).

Analogously to Subsection 7.3.1, let P_0 and P_1 be the probability measures under which the two hypotheses H_0 and H_1 we want to test are true with probability one, respectively, P_π be the probability measure defined in (7.3.3) and $\pi_t = P_\pi(H_1 \text{ is true } | \mathscr{F}_t^X)$, $t \geq 0$, be the posterior probability process. The goal is to solve the optimal stopping problem (7.3.6): we compute the value functions and the optimal boundary points by means of our method and we compare them with the exact ones. This allows us to assess the efficiency of the proposed numerical technique.

7.5.1 Sequential testing of a Wiener process

Let $X = (X_t)_{t \geq 0}$ be a Wiener process with drift γ, that is, $X_t = \gamma t + \sigma W_t$, where $\sigma > 0$ and $W = (W_t)_{t \geq 0}$ is a standard Wiener process. The two hypotheses to sequentially test are

$$H_0 : \gamma = \gamma_0 \quad Vs \quad H_1 : \gamma = \gamma_1. \tag{7.5.1}$$

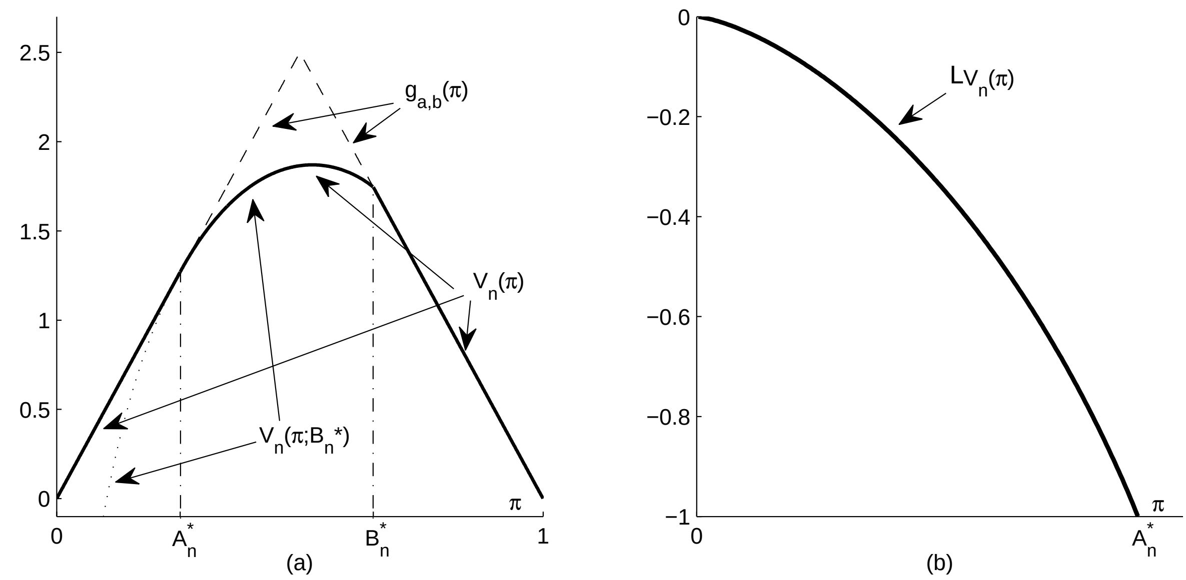

Figure 7.4: (a) A computer drawing of the map $\pi \mapsto V_n(\pi)$ (bold curve), as defined in (7.4.11), with $a = b = 5$, $\alpha_0 = 5$, $\alpha_1 = 1$ and $n = 8$ in (7.4.4). The set $D = [0, A_n^\star] \cup [B_n^\star, 1]$ is the stopping region, where $V_n = g_{a,b}$, while $(A_n^\star, B_n^\star) = (0.2541.., 0.6510..)$ is the continuation set, on which $V_n(\pi) = V_n(\pi; B_n^\star)$. We notice that $V_n(\pi)$ is differentiable at $A_n^\star$, while is just continuous at $B_n^\star$, in accordance with the principle of continuous and smooth fit (7.3.33)-(7.3.35). (b) A computer drawing of the map $\pi \mapsto (\mathbb{L}V_n)(\pi)$, $\pi \in [0, A_n^\star)$, with $\mathbb{L}$ given by (7.3.14). The same parameters of Figure 7.4-a have been used. We notice that $\pi \mapsto (\mathbb{L}V_n)(\pi)$ is strictly decreasing on $[0, A_n^\star)$: according to Theorem 7.3.1 point (I), the solution of the free-boundary problem (7.3.30)-(7.3.35) coincides with that of the optimal stopping problem (7.3.6).

It is well known that π_t is given by (7.3.10), with Y_t replaced by

$$Y_t^\gamma = \frac{\gamma_1 - \gamma_0}{\sigma^2}\left(X_t - \frac{t}{2}(\gamma_1 + \gamma_0)\right), \tag{7.5.2}$$

and that the infinitesimal generator $\mathbb{L}^\gamma$ of $(\pi_t)_{t\geq 0}$ is

$$(\mathbb{L}^\gamma f)(\pi) = \frac{1}{2}\frac{(\gamma_1 - \gamma_0)^2}{\sigma^2}\pi^2(1-\pi)^2 f''(\pi). \tag{7.5.3}$$

One can show that the unknown value function V from (7.3.6) and the unknown boundaries A and B satisfy the free-boundary problem (7.3.30)-(7.3.35) (with $\mathbb{L}^\gamma$ in place of $\mathbb{L}$), as well as the smooth fit condition at B

$$V'(B) = -b. \tag{7.5.4}$$

For a fixed $B > c$, let $V(\pi; B)$, $\pi \in (0, B]$, be the function solving (7.3.30), (7.3.35), (7.5.3) and (7.5.4) (see Shiryaev [72, eq. 4.70] or Section 5.4). V is thus expressed by (7.3.54) and the optimal stopping boundaries $A^\star$ and $B^\star$ are the unique solution of (7.3.55).

If we approximate $V(\pi; B^\star)$ by $V_n(\pi; B^\star)$ as in (7.4.4), the problem reduces to determining the $n+1$ coefficients of $V_n(\pi; B^\star)$, $A^\star$ and $B^\star$, that is, the following system of $n+3$ non-linear equations must be solved:

$$
\begin{aligned}
&(\mathbb{L}^\gamma V_n)(\pi_i; B^\star) = -1, \quad i = 1, .., n-1, && (7.5.5)\\
&V_n(A^\star; B^\star) = aA^\star, && (7.5.6)\\
&V_n'(A^\star; B^\star) = a, && (7.5.7)\\
&V_n(B^\star; B^\star) = b(1 - B^\star), && (7.5.8)\\
&V_n'(\pi; B^\star) = -b. && (7.5.9)
\end{aligned}
$$

We notice that the expressions (7.5.3) and (7.5.5) require evaluating the second derivative of the shifted Chebyshev polynomials, which is given by (7.2.14) and (7.2.16); moreover, the absence of jumps in the paths of X implies that the operator (7.5.3) does not involve integrals of the function which is applied to and this allows us to fix the $n-1$ collocation nodes π_i in the entire interval $[0, 1]$. Finally, once (7.5.5)-(7.5.9) has been solved, the approximated value function $V_n(\pi)$ is given by (7.4.11).

For a numerical application, we retrieve the example analyzed in Chapter 5, Figures 5.1 and 5.2, where setting $a = 15$, $b = 10$, $\sigma^2 = 1$, $\gamma_0 = -2$ and $\gamma_1 = -3$, the exact values $A^\star = 0.1593..$ and $B^\star = 0.7206..$ were obtained. The collocation approach (7.5.5)-(7.5.9) with $n = 8$ and $n-1$ equally spaced collocation nodes in $[0.1, 0.8]$ leads to very satisfactory results: $A^\star \approx A_n^\star = 0.1606..$ and $B^\star \approx B_n^\star = 0.7206..$; further, denoted by

$$||V, V_n|| = \sup_{\pi \in (0,1)} \left| \frac{V_n(\pi) - V(\pi)}{V(\pi)} \right| \tag{7.5.10}$$

the maximum relative distance between the exact value function V and its approximation V_n, we get $||V, V_n|| = 9.08 \times 10^{-4}$.

7.5.2 Sequential testing of a Poisson process

Let $X = (X_t)_{t \geq 0}$ be a sequentially observed Poisson process with intensity $\lambda > 0$; the aim is to test

$$H_0 : \lambda = \lambda_0 \quad Vs \quad H_1 : \lambda = \lambda_1, \quad \lambda_1 > \lambda_0. \tag{7.5.11}$$

The posterior probability π_t takes the expression (7.3.10), with Y_t substituted by

$$Y_t^\lambda = \log\left(\frac{\lambda_1}{\lambda_0}\right) X_t - t(\lambda_1 - \lambda_0); \tag{7.5.12}$$

the infinitesimal generator of $(\pi_t)_{t \geq 0}$ is

$$
\begin{aligned}
(\mathbb{L}^\lambda f)(\pi) = &-(\lambda_1 - \lambda_0) f'(\pi)\pi(1-\pi)\\
&+ \big(\lambda_1\pi + \lambda_0(1-\pi)\big)\left[f\left(\frac{\lambda_1 \pi}{\lambda_1\pi + \lambda_0(1-\pi)} \right) - f(\pi) \right].
\end{aligned} \tag{7.5.13}
$$

The optimal stopping problem (7.3.6) can be reduced to the free-boundary problem (7.3.30)-(7.3.35) (with $\mathbb{L}^\lambda$ in place of $\mathbb{L}$): its analytical solution was derived by Peskir and Shiryaev [65].

Let us describe how the proposed collocation approach can be applied. Let $\pi \mapsto V(\pi; B)$, $\pi \in I_B = (0, B]$ and $B > c$, be the map solving the difference-differential equation defined by (7.3.30), (7.3.31), (7.3.35) and (7.5.13); if we define the "step" and "distance" functions

$$S(\pi) = \frac{\lambda_1 \pi}{\lambda_1 \pi + \lambda_0 (1-\pi)}, \quad \pi \in I_B, \tag{7.5.14}$$

$$d^\lambda(\pi, B) = 1 + \left\lfloor \log\left(\frac{B}{1-B}\frac{1-\pi}{\pi}\right) \Big/ \log\left(\frac{\lambda_1}{\lambda_0}\right) \right\rfloor, \quad \pi \in I_B, \tag{7.5.15}$$

where $\lfloor x \rfloor$ is the integer part of x, it is not difficult to see that (7.3.30), (7.3.31), (7.3.35) and (7.5.13) imply that $V(\pi; B)$ solves (7.3.37)-(7.3.38), with $\mathbb{L}_B$ replaced by

$$\begin{aligned}(\mathbb{L}_B^\lambda f)(\pi) = &- (\lambda_1 - \lambda_0) f'(\pi)\pi(1-\pi) + \big(\lambda_1 \pi + \lambda_0(1-\pi)\big) \\ &\times \left\{ \left[b(1 - S(\pi)) \mathbf{1}_{\{d^\lambda(\pi,B)=1\}} + f(S(\pi)) \mathbf{1}_{\{d^\lambda(\pi,B)>1\}} \right] - f(\pi) \right\} + 1. \end{aligned} \tag{7.5.16}$$

As in Subsection 7.4.1, approximating $V(\pi; B)$ by $V_n(\pi; B)$ from (7.4.1) and solving the system (7.4.2)-(7.4.3) for the operator (7.5.16) and different values of $B > c$ allow us to check if the necessary and sufficient condition (7.3.53) for the existence of a solution to the free-boundary problem (7.3.30)-(7.3.35) is satisfied and to individuate reasonable neighbourhoods of $A^\star$ and $B^\star$. Once this operation has been accomplished, the next step is to approximate $V(\pi; B^\star)$ by $V_n(\pi; B^\star)$ from (7.4.4) and solve (7.4.5)-(7.4.8) for the operator (7.5.16): in this way, the approximating boundaries $A_n^\star$ and $B_n^\star$ and the coefficients involved in the expression of $V_n(\pi; B^\star)$ can be computed. The approximating value function V_n takes the expression (7.4.11).

Let us consider the numerical example analyzed by Peskir and Shiryaev [65, Fig. 2 and 3], where $a = b = 2$, $\lambda_0 = 1$, and $\lambda_1 = 5$. The exact values of the optimal boundaries are $A^\star = 0.2253..$ and $B^\star = 0.7050...$ The first part of the above procedure leads us to establish $A^\star \in (0.2, 0.3)$ and $B^\star \in (0.68, 0.72)$ (we fixed $n = 8$ and solved (7.4.2)-(7.4.3) and (7.5.16) for $B = 0.65, 0.68, 0.72$ and n equally spaced collocation nodes in $[0.1, B]$). Then, we solved (7.4.5)-(7.4.8) and (7.5.16) for $n = 8$ and n equally spaced collocation nodes in $[0.1, 0.68]$. We obtained the following very good approximations: $A^\star \approx A_n^\star = 0.2245..$, $B^\star \approx B_n^\star = 0.7048..$ and $||V, V_n|| = 2.59 \times 10^{-3}$.

7.5.3 Sequential testing of a compound Poisson process with exponential jumps

Let $X = (X_t)_{t \geq 0}$ be a compound Poisson process, whose intensity is $1/\eta$, $\eta > 0$, and the distribution of its jumps is negative exponential of parameter $\eta > 0$. We want to test

$$H_0 : \eta = \eta_0 \quad Vs \quad H_1 : \eta = \eta_1, \quad \eta_0 > \eta_1. \tag{7.5.17}$$

It is straightforward to show that π_t is given by (7.3.10), where Y_t is replaced by

$$Y_t^\eta = (\eta_0 - \eta_1) X_t - t\left(\frac{\eta_0 - \eta_1}{\eta_0 \eta_1}\right), \tag{7.5.18}$$

and the infinitesimal generator of $(\pi_t)_{t\geq 0}$ is

$$(\mathbb{L}^\eta f)(\pi) = -f'(\pi)\pi(1-\pi)\frac{\eta_0-\eta_1}{\eta_0\eta_1} - f(\pi)\left(\frac{\pi}{\eta_1}+\frac{1-\pi}{\eta_0}\right) + \int_0^\infty f\left(\frac{\pi e^{-\eta_1 x}}{\pi e^{-\eta_1 x}+(1-\pi)e^{-\eta_0 x}}\right)\left(\pi e^{-\eta_1 x}+(1-\pi)e^{-\eta_0 x}\right)dx. \tag{7.5.19}$$

The optimal stopping problem (7.3.6) can be reduced to the free-boundary problem (7.3.30)-(7.3.35) (with $\mathbb{L}$ replaced by $\mathbb{L}^\eta$): its solution was obtained by Gapeev [32].

We see that $V(\pi;B)$, $\pi\in I_B$, solution of (7.3.30), (7.3.31), (7.3.35) and (7.5.19), satisfies (7.3.37)-(7.3.38), where (7.3.37) is defined through the operator

$$\begin{aligned}(\mathbb{L}_B^\eta f)(\pi) &= -\frac{\eta_0-\eta_1}{\eta_0\eta_1}f'(\pi)\pi(1-\pi)\\ &\quad - f(\pi)\left(\frac{\pi}{\eta_1}+\frac{1-\pi}{\eta_0}\right)+\frac{b(1-\pi)}{\eta_0}\left(\frac{1-\pi}{\pi}\frac{B}{1-B}\right)^{-\frac{\eta_0}{\eta_0-\eta_1}}\\ &\quad + \int_0^{d^\eta(\pi,B)} f\left(S^\eta(\pi,x)\right)\left(\pi e^{-\eta_1 x}+(1-\pi)e^{-\eta_0 x}\right)dx+1,\end{aligned} \tag{7.5.20}$$

being $S^\eta(\pi,x)$ and $d^\eta(\pi,B)$ defined by

$$S^\eta(\pi,x)=\frac{\pi e^{-\eta_1 x}}{(1-\pi)e^{-\eta_0 x}+\pi e^{-\eta_1 x}},\quad \pi\in I_B,\quad x\geq 0, \tag{7.5.21}$$

$$d^\eta(\pi;B)=\log\left(\frac{1-\pi}{\pi}\frac{B}{1-B}\right)\Big/(\eta_0-\eta_1),\quad \pi\in I_B. \tag{7.5.22}$$

The same arguments of Section 7.4 and Subsection 7.5.2 can be used to derive approximations of the value function V and the boundaries $A^\star$ and $B^\star$.

For a numerical example, we set $a=b=1$, $\eta_0=0.5$ and $\eta_1=0.1$. The exact boundaries are $A^\star=0.1632..$ and $B^\star=0.7455...$ The solutions of (7.4.2)-(7.4.3) and (7.5.20) for $B=0.68, 0.72, 0.76$, $n=8$ and n equally spaced collocation nodes in $[0.1,B]$ allow us to identify $A^\star\in(0.1,0.2)$ and $B^\star\in(0.72,0.76)$. Very good approximations are then obtained as solution of (7.4.5)-(7.4.8) and (7.5.20), for $n=8$ and n equally spaced collocation nodes in $[0.1,0.72]$: $A^\star\approx A_n^\star=0.1639..$, $B^\star\approx B_n^\star=0.7456..$ and $||V,V_n||=4.59\times 10^{-4}$.

7.5.4 Sequential testing of a negative binomial process

Let $X=(X_t)_{t\geq 0}$ be a negative binomial process of parameter $0<p<1$, that is, X has independent and stationary increments and the probability that $X_t=x$ is

$$\frac{\Gamma(x+t)}{\Gamma(x+1)\Gamma(t)}p^t(1-p)^x,\quad x=0,1,2,... \tag{7.5.23}$$

The posterior probability π_t for the sequential testing of the two simple hypotheses

$$H_0: p=p_0\quad Vs\quad H_1: p=p_1,\quad p_0>p_1, \tag{7.5.24}$$

is provided by (7.3.10), with Y_t replaced by

$$Y_t^p=\log\left(\frac{q_1}{q_0}\right)X_t - t\log\left(\frac{p_0}{p_1}\right), \tag{7.5.25}$$

where $q_i = 1 - p_i$, $i = 0, 1$. The infinitesimal operator of $(\pi_t)_{t\geq 0}$ takes the form

$$\begin{aligned}\big(\mathbb{L}^p f\big)(\pi) = &\log\left(\frac{p_1}{p_0}\right) f'(\pi)\pi(1-\pi) + f(\pi)\big((1-\pi)\log p_0 + \pi \log p_1\big) \\ &+ \sum_{x=1}^{\infty} f\left(\frac{\pi q_1^x}{\pi q_1^x + (1-\pi) q_0^x}\right) \frac{\pi q_1^x + (1-\pi) q_0^x}{x}. \end{aligned} \tag{7.5.26}$$

Also in this case, the optimal stopping problem (7.3.6) can be reduced to the free-boundary problem (7.3.30)-(7.3.35) (with $\mathbb{L}^p$ in place of $\mathbb{L}$): its explicit solution was derived in Chapter 5 (or Buonaguidi and Muliere [16, Sec. 6]).

Denoted by $V(\pi; B)$, $\pi \in I_B$, the map solving (7.3.30), (7.3.31), (7.3.35) and (7.5.26), and defined the “step” and “distance” functions

$$S^p(\pi, x) = \frac{\pi q_1^x}{\pi q_1^x + (1-\pi) q_0^x}, \quad \pi \in I_B, \quad x = 1, 2, .., \tag{7.5.27}$$

$$d^p(\pi, B) = 1 + \left\lfloor \log\left(\frac{B}{1-B}\frac{1-\pi}{\pi}\right) \Big/ \log\left(\frac{q_1}{q_0}\right)\right\rfloor, \quad \pi \in I_B, \tag{7.5.28}$$

it is not difficult to verify that $V(\pi; B)$ must solve (7.3.37)-(7.3.38) for the operator

$$\begin{aligned}(\mathbb{L}_B^p f)(\pi) = &\log\left(\frac{p_1}{p_0}\right) f'(\pi)\pi(1-\pi) + f(\pi)\big((1-\pi)\log p_0 + \pi \log p_1\big) \\ &+ \sum_{x=1}^{d^p(\pi,B)-1} \left(f\left(S^p(\pi; x)\right) \frac{\pi q_1^x + (1-\pi) q_0^x}{x}\right) \\ &- b(1-\pi)\left(\log p_0 + \sum_{x=1}^{d^p(\pi,B)-1} \frac{q_0^x}{x}\right) + 1. \end{aligned} \tag{7.5.29}$$

Repeating now step by step the procedure of Section 7.4 and Subsection 7.5.2, approximations of V, $A^\star$ and $B^\star$ can be easily computed

For a numerical illustration, let us exploit the example in Chapter 5, Figures 5.3 and 5.4, where $a = b = 8$, $p_0 = 0.8$ and $p_1 = 0.3$. The exact stopping boundaries are $A^\star = 0.2004..$ and $B^\star = 0.7142...$ Fixing $B = 0.64, 0.68, 0.72$, $n = 8$ and n equally spaced nodes in $[0.1, B]$, solving (7.4.2)-(7.4.3) and (7.5.29) shows that $A^\star \in (0.2, 0.3)$ and $B^\star \in (0.68, 0.72)$. The solution of the system (7.4.5)-(7.4.8) and (7.5.29), for $n = 8$ and n equally spaced collocation nodes in $[0.1, 0.68]$, leads to satisfactory results: $A^\star \approx A_n^\star = 0.2004..$, $B^\star \approx B_n^\star = 0.7174..$ and $||V, V_n|| = 4.92 \times 10^{-3}$.

7.6 Conclusions

We considered the sequential testing of two simple hypotheses for a Lévy gamma process. Our study represented an attempt to extend the existing literature on sequential testing to processes with infinite jump activity on finite time intervals.

Initially, we approached the problem from a probabilistic-analytic view point: we shown some properties of the value function, like the smoothness and/or continuity at the stopping boundaries, and we constructed the free-boundary problem that the value function and the boundaries must satisfy. Then, we verified that if the free-boundary problem admits a solution, it is unique and coincides with that of the original optimal stopping problem.

Since deriving an explicit solution of the free-boundary problem was very hard, we proposed a numerical collocation approach. The value function was approximated by a linear combination of Chebyshev polynomials: we shown that its coefficients and the two stopping boundaries can be determined as solution of a system of non-linear equations, obtained by forcing the linear combination to solve a complex integro-differential equation, at fixed and properly chosen collocation nodes, and the boundary conditions, which are in accordance with the smooth and continuous fit principles. The performances of our approximation method were finally evaluated in explicitly solved sequential testing problems, where we obtained very good approximations of the exact solutions.

We remark that the presented collocation approach can be adapted to other optimal stopping problems (like sequential detection and optimal prediction problems), whose solutions are difficult to determine.

Chapter 8

On the Sequential Testing for Lévy Processes with Diffusion and Jump Components

We study the Bayesian problem of sequential testing of two simple hypotheses about the Lévy-Khintchine triplet of a Lévy process, having diffusion component, represented by a Brownian motion with drift, and jump component of finite variation. The method of proof consists of reducing the original optimal stopping problem to a free-boundary Stephan problem. We show it is characterized by a second order integro-differential equation, that the unknown value function solves on the continuation region, and by the smooth fit principle, which holds at the unknown boundary points. Since determining an explicit solution of the free-boundary problem is extremely tough, we approach the latter by a collocation method: the value function is replaced by a linear combination of polynomials with unknown coefficients, that, along with the boundary points, are determined by allowing the linear combination to satisfy the boundary conditions and, at fixed points, the integro-differential equation. Our technique is illustrated by several examples.

8.1 Introduction

A Lévy process $X = (X_t)_{t \geq 0}$ is a continuous time real-valued stochastic process starting at 0, with independent and stationary increments and right continuous with left limit trajectories. Denoted by $\hat{\mu}_t(z) = E[e^{izX_t}]$, $z \in \mathbb{R}$, the characteristic function of X_t, $t \geq 0$, it is well known that $\hat{\mu}_t(z) = e^{t\phi(z)}$, where $\phi(\cdot)$ is the so called characteristic exponent, which takes the following expression:

$$\phi(z) = i\gamma z - \frac{1}{2}\sigma^2 z^2 + \int_{\mathbb{R}} \left(e^{izx} - 1 - izx\mathbf{1}_{\{|x| \leq 1\}}(x)\right) v(dx). \tag{8.1.1}$$

It is immediate to notice that $\phi(\cdot)$ depends on the triplet $g = \{\gamma, \sigma^2, v(\cdot)\}$, known as Lévy-Khintchine or generating triplet, where $\gamma \in \mathbb{R}$, $\sigma^2 \geq 0$ is the diffusion coefficient and $v(\cdot)$ is the Lévy measure on $\mathbb{R}$, which is responsible of the jump structure of X.

Under the assumption that X presents diffusion component and jump component of finite variation, the following problem is faced: at time $t = 0$ a Lévy process X is begun to be observed; the goal is to sequentially test the two simple hypotheses initially formulated on the

Lévy-Khintchine triplet of X. This problem is analyzed according to the Bayesian formulation, in the sense that a prior Bernoulli distribution on the two hypotheses is given and the aim is to determine a rule that allows us to optimally interrupt the observation of X and make the appropriate choice. Optimality means that the best trade-off between the expected cost of the observation process and the expected loss due to a final incorrect decision is returned.

The problem described above naturally appears in many applied disciplines. In finance, the price of a share can be modeled by an exponentiated Lévy process, where the continuous/diffusion component expresses the appreciation rate and the volatility of the share, while its bounces, due to sudden shocks in the market, are incorporated in the jump component (see, e.g., Mordecki [58]). In signal detection theory, the intensity of a certain type of wave can be described by a Lévy process with diffusion and jump components, with the latter expressing instantaneous drops in the intensity of the signal, due to the appearance of obstacles. In maintenance and reliability, the degradation process of a machinery is suitable for being expressed by a Lévy process, whose jump component models the natural degradation, while the diffusion component includes the effects of some repairs, leading to a non-monotone senescence process of the machine (see, e.g., Paroissin and Rabehasaina [64]). In all these situations, discriminating between two hypotheses about the distributional properties of the sequentially observed Lévy process is evidently an important issue.

The sequential testing for continuous time stochastic processes has been extensively studied in the literature: Shiryaev [72, sec. 4.2], Gapeev and Peskir [35], Gapeev and Shiryaev [37] and Shiryaev and Zhitlukhin [75] dealt with the Bayesian sequential testing for the drift of a Wiener or a more general diffusion process; Peskir and Shiryaev [65], Gapeev [32], Dayanik and Sezer [24], Dayanik et al. [23] and Ludkovski and Sezer [55] analyzed the Bayesian sequential testing for compound Poisson processes; in Chapter 7 we considered the sequential testing of a gamma process. In these works, the diffusion and jump components were separately tested.

The purpose of this chapter is to study the sequential testing for a Lévy process, which exhibits diffusion and jump components. A similar problem was analyzed by Dayanik and Sezer [25], for the sequential testing of independent Brownian motions and compound Poisson processes. Our study differs from theirs for at least three reasons: the jump component we consider, the way we approach the problem and the numerical scheme we use for computing the solution. Indeed, we concentrate on Lévy processes whose jump component has finite variation on any finite time interval: this class of processes is larger than that of the aforementioned work and allows us to consider in the jump part other processes, like the gamma process, which are important in applications (see, e.g., Park and Padgett [63]). This implies that we cannot use the successive approximation scheme and the related numerical algorithm provided in Dayanik and Sezer [25]. We show that the value function and the stopping boundaries of the initial optimal stopping problem can be obtained as solution of a free-boundary Stephan problem, characterized by the principle of the smooth fit and by a second order integro-differential operator.

The high complexity of the free-boundary problem requires employing a numerical scheme, in order to determine its solution. A collocation method is therefore devised: we substitute the value function in the free-boundary problem with a linear combination of Chebyshev polynomials, whose coefficients are unknown. These and the stopping boundaries can be computed as solution of a system of equations, obtained by forcing the linear combination to satisfy the boundary conditions, arising from the smooth fit principle, and the integro-differential equation, at a fixed number of points. This numerical scheme, which extends the more classical collocation method for boundary value problems, is easy to implement and is illustrated through several examples.

The chapter is organized as follows: in Section 8.2, we recall some elements on Lévy processes and we define more formally the problem initially introduced; in Section 8.3, we reduce the initial optimal stopping problem to a free-boundary Stephan problem, by determining the infinitesimal generator of the posterior probability process and proving the smoothness of the value function at the stopping boundaries; in Section 8.4, we briefly recall the collocation method and we analyze how it can be exploited for solving the free-boundary problem: we use as illustrative example the case in which the jump component of the observed Lévy process is represented by a gamma process. In Section 8.5, we provide some results for the application of our method to other interesting examples of Lévy processes with diffusion and jump components. Section 8.6 concludes with a summary discussion.

8.2 Preliminaries

In this section, we recall some important results on Lévy processes, which will greatly help us later for solving our problem. Then, we formally introduce the Bayesian sequential testing for the Lévy-Khintchine triplet of a Lévy process.

8.2.1 Decomposition of a Lévy process

The distributional properties of a Lévy process $X = (X_t)_{t\geq 0}$ are completley contained in the characteristic exponent (8.1.1), or, equivalently, in the Lévy-Khintchine triplet $g = \{\gamma, \sigma^2, v(\cdot)\}$, where $v(\cdot)$ satisfies $v(\{0\}) = 0$ and $\int_{\mathbb{R}}(x^2 \wedge 1)\, v(dx) < \infty$. In the rest of the chapter, it will be assumed that

$$\int_{|x|\leq 1} |x|\, v(dx) < \infty. \tag{8.2.1}$$

Under (8.2.1), we notice that $\phi(\cdot)$ from (8.1.1) is equivalent to $\widetilde{\phi}(\cdot)$, defined by

$$\widetilde{\phi}(z) = i\widetilde{\gamma}z - \frac{1}{2}\sigma^2 z^2 + \int_{\mathbb{R}} \left(e^{izx} - 1\right)\, v(dx), \quad z \in \mathbb{R}, \tag{8.2.2}$$

where

$$\widetilde{\gamma} = \gamma - \int_{|x|\leq 1} x\, v(dx). \tag{8.2.3}$$

We denote by $\widetilde{g} = \{\widetilde{\gamma}, \sigma^2, v(\cdot)\}$ the Lévy-Khintchine triplet associated to the representation (8.2.2) and (8.2.3). The constant $\widetilde{\gamma}$ is also known as drift of X.

A fundamental result in the theory of Lévy processes and at the basis of the analysis that will be later developed is the Lévy-Ito decomposition theorem, which affirms that any Lévy process can be split into the sum of two independent Lévy processes: a Wiener process with drift and a compensated sum of independent jumps. When (8.2.1) holds, a simplified version of the theorem can be formulated.

Theorem 8.2.1 *Let X be a Lévy process, whose Lévy measure $v(\cdot)$ satisifes* (8.2.1)*. Let $X^j = (X_t^j)_{t\geq 0}$ and $X^c = (X_t^c)_{t\geq 0}$ be the processes defined by*

$$X_t^j = \sum_{s\leq t}(X_s - X_{s^-}), \quad X_t^c = X_t - X_t^j. \tag{8.2.4}$$

Then, X^j and X^c are Lévy processes, whose characteristic exponents $\widetilde{\phi}^j(\cdot)$ and $\phi^c(\cdot)$ are given by

$$\widetilde{\phi}^j(z) = \int_{\mathbb{R}} \left(e^{izx} - 1\right) v(dx), \quad \phi^c(z) = i\widetilde{\gamma}z - \frac{1}{2}\sigma^2 z^2, \quad z \in \mathbb{R} \tag{8.2.5}$$

The map $t \mapsto X_t^c$ is continuous and the two processes X^j and X^c are independent.

Proof. See Sato [71, Th. 19.3, p. 121]. ∎

From (8.2.5), we see that the generating triplet of X^j and X^c are $\widetilde{g}^j = \{0, 0, v(\cdot)\}$ and $g^c = \{\widetilde{\gamma}, \sigma^2, 0\}$, respectively. According to well-known arguments based on Sato [71, Chap. 4], the following properties are easily inferred: 1) X^j has trajectories of finite variation on $(0, t]$, $\forall t \geq 0$; 2) if $v(\mathbb{R}) < \infty$, X^j has a finite number of jumps on any finite time interval and the time between jumps have exponential distribution with mean $1/v(\mathbb{R})$, while if $v(\mathbb{R}) = \infty$, the jumping times of X^j are countable and dense in $[0, \infty)$; 3) X^j is a purely jump process, in the sense that it evolves by jumps only; 4) X^c is a Wiener process with drift $\widetilde{\gamma}$ and diffusion coefficient σ^2, that is, $X_t^c = \widetilde{\gamma} + \sigma W_t$, where $W = (W_t)_{t\geq 0}$ is a standard Wiener process.

The first property directly comes from condition (8.2.1). The second and third properties imply that if $v(\mathbb{R}) < \infty$, X^j is a compound Poisson process and its trajectories are piecewise constant a.s.; instead, if $v(\mathbb{R}) = \infty$, then X^j has infinite jump activity on any finite time interval and, therefore, its trajectories are discontinuous anywhere a.s.

We refer to X^j and X^c as the *jump* and *diffusion* (or continuous) components of X. Figure 8.1 below shows two simulated paths of Lévy processes with diffusion and jump components: in the left drawing, the jump component is a compound Poisson process, that is, $v(\mathbb{R}) < \infty$, while in the right drawing, although the associated Lévy measure still satisifies (8.2.1), the jump component is not of compound Poisson type, that is, $v(\mathbb{R}) = \infty$.

8.2.2 Formulation of the problem

Let $X = (X_t)_{t\geq 0}$ be a Lévy process defined on the filtered statistical space $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \{P_0, P_1\})$, where P_i is the probability measure under which X has Lévy-Khintchine triplet $g_i = \{\gamma_i, \sigma^2, v_i(\cdot)\}$, $i = 0, 1$. It is assumed that $\sigma^2 > 0$, v_i satisfies (8.2.1) and is not identically zero, $i = 0, 1$, and $v_0 \approx v_1$, that is, v_0 and v_1 are mutually absolutely continuous. Let ϑ be an $\mathscr{F}_0$-measurable random variable independent of X and denote by

$$P_\pi = \pi P_1 + (1 - \pi)P_0, \quad \pi \in [0, 1], \tag{8.2.6}$$

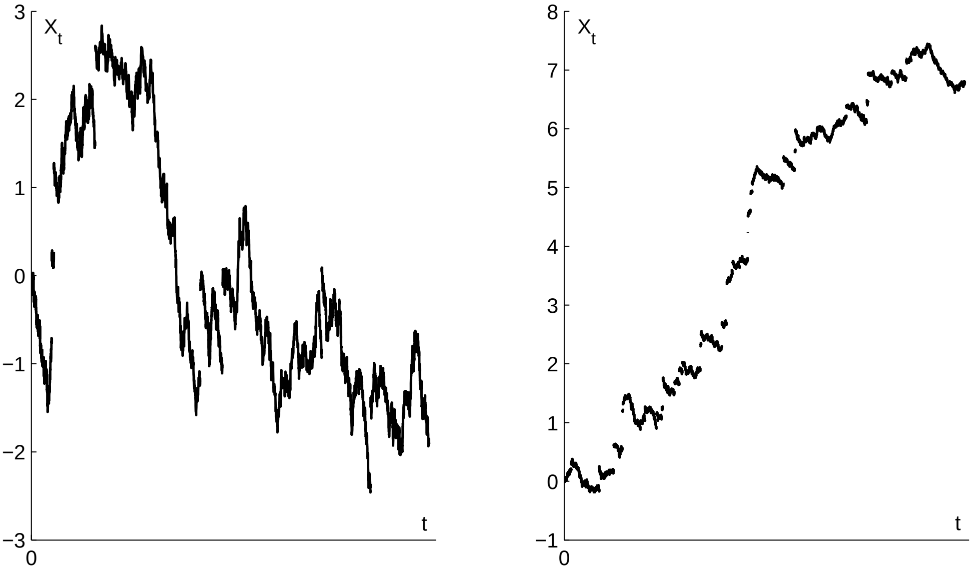

Figure 8.1: Two simulated paths of a Lévy process with diffusion and jump components. In the left drawing, the jump component is represented by a negative binomial process, whose Lévy measure is $v(\{x\}) = (1-p)^x/x$, with $x = 1, 2, \ldots$ and $p \in (0,1)$. Then, $v(\mathbb{R}) = \sum_{x=1}^{\infty}(1-p)^x/x = -\log p < \infty$. In the right drawing, the jump component is expressed by a gamma process, whose Lévy measure is $v(x) = x^{-1}e^{-\alpha x}$, with $x > 0$ and $\alpha > 0$; hence, $v(\mathbb{R}) = \int_0^\infty x^{-1}e^{-\alpha x}\,dx = \infty$.

the probability measure, under which ϑ takes on values 0 and 1, with probability $1-\pi$ and π, respectively, where π is known and fixed. Then, under P_π, X is a Lévy process with triplet $g_\vartheta = \{\gamma_\vartheta, \sigma^2, v_\vartheta(\cdot)\}$. By continuously observing X, we want to test the two simple hypotheses

$$H_0 : \vartheta = 0 \quad \text{and} \quad H_1 : \vartheta = 1. \tag{8.2.7}$$

Let $\mathscr{F}_t^X = \sigma\{X_s : 0 \le s \le t\}$ be the sigma-algebra generated by X up to t and define by (τ, d) a sequential decision rule, where τ is a stopping time of X, that is an $\mathscr{F}_t^X$-measurable random variable, $t \ge 0$, and d, known as decision function, is an $\mathscr{F}_\tau^X$-measurable random variable: once the process is stopped at time τ, d takes value i if the hypothesis H_i, $i = 0, 1$, must be accepted. The Bayesian sequential testing of (8.2.7) consists of expliciting as much as possible the value function

$$V(\pi) = \inf_{(\tau,d)} E_\pi\left[\tau + a\mathbf{1}_{(d=0,\vartheta=1)} + b\mathbf{1}_{(d=1,\vartheta=0)}\right], \quad a, b > 0, \tag{8.2.8}$$

and establishing the optimal π-Bayes decision rule $(\tau_\pi^\star, d_\pi^\star)$, at which the infimum in (8.2.8) is reached.

Let $(\pi_t)_{t\ge0}$ be the posterior probability process, defined by $\pi_t = P_\pi(\vartheta = 1|\mathscr{F}_t^X)$, and $g_{a,b}(x) = ax \wedge b(1-x)$. It is well known (see Shiryaev [72, Lemma 1, pp. 166-167]) that (8.2.8) is equivalent to the optimal stopping problem

$$V(\pi) = \inf_\tau E_\pi\left[\tau + g_{a,b}(\pi_\tau)\right], \tag{8.2.9}$$

where the optimal decision function is given by $d^\star_\pi = 1$, if $\pi_{\tau^\star_\pi} > c$, and $d^\star_\pi = 0$, if $\pi_{\tau^\star_\pi} \le c$, where $c = b/(a+b)$.

Denote by D and C the stopping and continuation regions, that is, $D = \{\pi \in [0,1] : V(\pi) = g_{a,b}(\pi)\}$ and $C = \{\pi \in [0,1] : V(\pi) < g_{a,b}(\pi)\}$; then, standard arguments based on the theory of optimal stopping (see, e.g., Peskir and Shiryaev [67, ch. 1] or Shiryaev [72, ch. 3]) and the structure of (8.2.9) imply that $\tau^\star_\pi = \inf\{t \ge 0 : \pi_t \notin C, \pi_0 = \pi\}$ and that there exist two points A and B, with $0 < A \le c \le B < 1$, so that $D = [0, A] \cup [B, 1]$ and $C = (A, B)$. Later, it will be clear that C is never empty, that is $0 < A < c < B < 1$, unlike what we analyzed in Section 5.5 and Chapter 7, where the continuation region could vanish.

Denoted by $P_i|\mathscr{F}^X_t$ the restriction of P_i to $\mathscr{F}^X_t$, $i = 0, 1$; let $\varphi_t = d(P_1|\mathscr{F}^X_t)/\ d(P_0|\mathscr{F}^X_t)$; then, from Sato [71, Th. 33.2, p. 219], we have

$$\begin{aligned}\varphi_t = \exp\Bigg\{ &\frac{\widetilde{\gamma}_1 - \widetilde{\gamma}_0}{\sigma^2}\left(X^c_t - t\frac{\widetilde{\gamma}_1 + \widetilde{\gamma}_0}{2}\right)\\ &+ \sum_{s\le t} \log\big(\xi(X_s - X_{s-})\big) - t\int_{\mathbb{R}} \big(\xi(x) - 1\big)\, v_0(dx)\Bigg\},\end{aligned} \tag{8.2.10}$$

where $\widetilde{\gamma}_i$ is given by (8.2.3), $i = 0, 1$, X^c is the continuous part of X, as defined in (8.2.4), and $\xi(x) = \frac{dv_1}{dv_0}(x)$ is meant as Radon-Nikodym derivative. $(\varphi_t)_{t\ge 0}$ is called likelihood ratio process. The expression of π_t directly follows from a simple application of Bayes theorem:

$$\pi_t = \frac{\pi\varphi_t}{1 + \pi(\varphi_t - 1)}. \tag{8.2.11}$$

Let $\mu^X\big((0,t]\times H\big) = \sum_{s\le t} \mathbf{1}(\Delta X_s \in H)$, $H \in \mathscr{B}(\mathbb{R}\backslash\{0\})$, be the measure of jumps of the process X and $v^X(dx, dt) = \big((1 - \pi_{t-})v_0(dx) + \pi_{t-}v_1(dx)\big)dt$ be its compensator; then, the expressions (8.2.10) and (8.2.11) and a straightforward application of Itô's formula lead to the following stochastic differential equations:

$$\begin{aligned}d\varphi_t = &-\frac{(\widetilde{\gamma}_1 - \widetilde{\gamma}_0)\widetilde{\gamma}_0}{\sigma^2}\,\varphi_{t-}dt + \frac{\widetilde{\gamma}_1 - \widetilde{\gamma}_0}{\sigma^2}\,\varphi_{t-}dX^c_t\\ &+ \varphi_{t-}\int_{\mathbb{R}}\Big(\xi(x) - 1\Big)(\mu^X - v_0)(dx, dt), \quad \varphi_0 = 1,\end{aligned} \tag{8.2.12}$$

$$\begin{aligned}d\pi_t = &\frac{\widetilde{\gamma}_1 - \widetilde{\gamma}_0}{\sigma^2}\pi_{t-}(1 - \pi_{t-})d\widetilde{X}^c_t\\ &+ \int_{\mathbb{R}} \frac{\pi_{t-}(1 - \pi_{t-})\big(\xi(x) - 1\big)}{1 + \pi_{t-}\big(\xi(x) - 1\big)}(\mu^X - v^X)(dx, dt), \quad \pi_0 = \pi,\end{aligned} \tag{8.2.13}$$

where $\widetilde{X}^c = (\widetilde{X}^c_t)_{t\ge 0}$, defined by

$$\widetilde{X}^c_t = X^c_t - \left(\widetilde{\gamma}_0\int_0^t (1 - \pi_{s-})ds + \widetilde{\gamma}_1\int_0^t \pi_{s-}ds\right), \tag{8.2.14}$$

is a Wiener process with diffusion coefficient σ^2 under P_π.

8.3 The free-boundary approach

In this section, we reduce the optimal stopping problem (8.2.9) to a free-boundary problem for the value function $V(\pi)$ and the boundaries A and B, defining the stopping and continuation region. Then, we will prove that under some conditions the solution of the free-boundary problem coincides with that of the optimal stopping problem (8.2.9).

8.3.1 Reduction of the optimal stopping problem to a free-boundary problem

In order to construct the appropriate free-boundary problem, we need to determine the infinitesimal generator of the posterior probability process $(\pi_t)_{t\geq 0}$.

Proposition 8.3.1 *Let $f \in C^2[0,1]$; then,*

$$f(\pi_t) = f(\pi) + \int_0^t \big((\mathbb{L}^c + \mathbb{L}^j)f\big)(\pi_{s-})\, ds + \mathscr{M}_t, \tag{8.3.1}$$

where $\mathbb{L}^c + \mathbb{L}^j$ is the infinitesimal operator of $(\pi_t)_{t\geq 0}$ defined by

$$(\mathbb{L}^c f)(\pi) = \frac{1}{2}\frac{(\widetilde{\gamma}_1 - \widetilde{\gamma}_0)^2}{\sigma^2}\, \pi^2(1-\pi)^2 f''(\pi), \tag{8.3.2}$$

$$\begin{aligned}(\mathbb{L}^j f)(\pi) = &- \pi(1-\pi)f'(\pi)\int_{\mathbb{R}} \big(\xi(x)-1\big)v_0(dx)\\ &+ \int_{\mathbb{R}} \left(f\left(\frac{\pi\xi(x)}{1+\pi\big(\xi(x)-1\big)}\right) - f(\pi)\right)\Big((1-\pi)v_0(dx) + \pi v_1(dx)\Big),\end{aligned} \tag{8.3.3}$$

and $\mathscr{M} = (\mathscr{M}_t)_{t\geq 0}$, with $\mathscr{M}_t$ given by

$$\begin{aligned}\mathscr{M}_t = &\frac{\widetilde{\gamma}_1 - \widetilde{\gamma}_0}{\sigma^2}\int_0^t f'(\pi_{s-})\pi_{s-}(1-\pi_{s-})d\widetilde{X}^c_s\\ &+ \int_0^t\int_0^1 \big(f(\pi_{s-}+y) - f(\pi_{s-})\big)(\mu^\pi - v^\pi)(dy, ds),\end{aligned} \tag{8.3.4}$$

is a local martingale with respect to $(\mathscr{F}_t^X)_{t\geq 0}$ and P_π. The quantities μ^π and v^π in (8.3.4) *are the measure of jumps of $(\pi_t)_{t\geq 0}$ and its compensator, respectively.*

Proof. Let $(\pi_t^c)_{t\geq 0}$ be the continuous part of $(\pi_t)_{t\geq 0}$; its evolution is directly obtained through (8.2.13):

$$d\pi_t^c = \frac{\widetilde{\gamma}_1 - \widetilde{\gamma}_0}{\sigma^2}\pi_{t-}(1-\pi_{t-})d\widetilde{X}^c_t - \pi_{t-}(1-\pi_{t-})\left(\int_{\mathbb{R}} \big(\xi(x)-1\big)v_0(dx)\right) dt. \tag{8.3.5}$$

Using Ito's formula for non-continuous semimartingales and (8.3.5), one has

$$
\begin{aligned}
f(\pi_t) =& f(\pi) + \int_0^t f'(\pi_{s^-}) d\pi_s^c + \frac{1}{2}\int_0^t f''(\pi_{s^-})(d\pi_s^c)^2 + \sum_{s\le t}\left(f(\pi_s) - f(\pi_{s^-})\right)\\
=& f(\pi) + \int_0^t \pi_{s^-}(1-\pi_{s^-})f'(\pi_{s^-})\left(\frac{\widetilde{\gamma}_1-\widetilde{\gamma}_0}{\sigma^2} d\widetilde{X}_s^c - \int_{\mathbb{R}}\big(\xi(x)-1\big)v_0(dx)\,ds\right)\\
&+ \frac{1}{2}\frac{(\widetilde{\gamma}_1-\widetilde{\gamma}_0)^2}{\sigma^2}\int_0^t \pi_{s^-}^2(1-\pi_{s^-})^2 f''(\pi_{s^-})ds + \int_0^t\int_0^1 \big(f(\pi_{s^-}+y) - f(\pi_{s^-})\big)\mu^\pi(dy,ds)\\
=& f(\pi) + \int_0^t \pi_{s^-}(1-\pi_{s^-})f'(\pi_{s^-})\left(\frac{\widetilde{\gamma}_1-\widetilde{\gamma}_0}{\sigma^2} d\widetilde{X}_s^c - \int_{\mathbb{R}}\big(\xi(x)-1\big)v_0(dx)\,ds\right)\\
&+ \frac{1}{2}\frac{(\widetilde{\gamma}_1-\widetilde{\gamma}_0)^2}{\sigma^2}\int_0^t \pi_{s^-}^2(1-\pi_{s^-})^2 f''(\pi_{s^-})ds + \int_0^t\int_0^1 \big(f(\pi_{s^-}+y) - f(\pi_{s^-})\big)v^\pi(dy,ds)\\
&+ \int_0^t\int_0^1 \big(f(\pi_{s^-}+y) - f(\pi_{s^-})\big)\big(\mu^\pi - v^\pi\big)(dy,ds). \qquad (8.3.6)
\end{aligned}
$$

From (8.2.13), we observe that

$$
\pi_{t^-} + \Delta\pi_t = \pi_{t^-} + \frac{\pi_{t^-}(1-\pi_{t^-})\big(\xi(x)-1\big)}{1+\pi_{t^-}\big(\xi(x)-1\big)} = \frac{\pi_{t^-}\xi(x)}{1+\pi_{t^-}\big(\xi(x)-1\big)}, \qquad (8.3.7)
$$

so that

$$
\begin{aligned}
&\int_0^t\int_0^1 \big(f(\pi_{s^-}+y) - f(\pi_{s^-})\big)v^\pi(dy,ds)\\
&\qquad = \int_0^t\int_{\mathbb{R}}\left(f\left(\frac{\pi_{s^-}\xi(x)}{1+\pi_{s^-}\big(\xi(x)-1\big)}\right) - f(\pi_{s^-})\right)v^X(dx,ds), \qquad (8.3.8)
\end{aligned}
$$

being $v^X(dx,ds) = \big((1-\pi_{s^-})v_0(dx)+\pi_{s^-}v_1(dx)\big)ds$. The expressions (8.3.6) and (8.3.8) complete the proof. ∎

The value function $V(\pi)$ has some interesting analytical features. By means of (8.2.6) and (8.2.9), it is easy to prove the map $\pi \mapsto V(\pi)$ is concave and therefore continuous on $[0,1]$ (see Proposition 7.3.2). The next proposition proves that $V(\pi)$ is differentiable at the boundary points A and B.

Proposition 8.3.2 *Let $0 < A < c < B < 1$; then, $V(\pi)$ from (8.2.9) is differentiable from the right and left at A and B, respectively, and we have*

$$
V'(A_+) = a, \quad V'(B_-) = -b. \qquad (8.3.9)
$$

Proof. Since $V(\pi) < g_{a,b}(\pi)$ for $\pi \in (A,B)$, for any $\epsilon > 0$ such that $c < B-\epsilon < B$, we have

$$
\frac{V(B)-V(B-\epsilon)}{\epsilon} \ge \frac{b(1-B) - b(1-B+\epsilon)}{\epsilon} = -b, \qquad (8.3.10)
$$

so that $V'(B_-) \ge -b$, where the left-hand derivative exists because of the concavity argument of $\pi \mapsto V(\pi)$.

Let us show that the reverse inequality holds. For any $\epsilon > 0$ such that $c < B-\epsilon < B$, consider the stopping time $\tau^\star_{B-\epsilon}$, that, as seen in Subsection 8.2.2, is optimal for $V(B-\epsilon)$. We recall that $\tau^\star_{\pi-\epsilon}$ is the first time that $(\pi_t)_{t\geq 0}$ leaves (A, B), with $\pi_0 = \pi - \epsilon$. Then, (8.2.6) and arguments similar to those used by Gapeev and Peskir [35] and in Proposition 7.3.3 imply

$$\begin{aligned} &V(B) - V(B-\epsilon) \\ &\quad \leq E_B\left[\tau^\star_{B-\epsilon} + g_{a,b}(\pi_{\tau^\star_{B-\epsilon}})\right] - E_{B-\epsilon}\left[\tau^\star_{B-\epsilon} + g_{a,b}(\pi_{\tau^\star_{B-\epsilon}})\right] \\ &\quad = \sum_{i=0}^{1} E_i\left[S_i(B) - S_i(B-\epsilon)\right], \end{aligned} \tag{8.3.11}$$

where

$$S_i(\pi) = \frac{1+(-1)^i(1-2\pi)}{2}\left(\tau^\star_{B-\epsilon} + a\frac{\pi\varphi_{\tau^\star_{B-\epsilon}}}{1+\pi(\varphi_{\tau^\star_{B-\epsilon}}-1)} \wedge b\frac{1-\pi}{1+\pi(\varphi_{\tau^\star_{B-\epsilon}}-1)}\right). \tag{8.3.12}$$

According to the mean value theorem, there exist $\omega_i \in (B-\epsilon, B)$, $i = 0, 1$, such that

$$\sum_{i=0}^{1} E_i\left[S_i(B) - S_i(B-\epsilon)\right] = \epsilon \sum_{i=0}^{1} E_i\left[S_i'(\omega_i)\right], \tag{8.3.13}$$

where

$$\begin{aligned} S_i'(\pi) &= (-1)^{i-1}\left(\tau^\star_{B-\epsilon} + a\frac{\pi\varphi_{\tau^\star_{B-\epsilon}}}{1+\pi(\varphi_{\tau^\star_{B-\epsilon}}-1)} \wedge b\frac{1-\pi}{1+\pi(\varphi_{\tau^\star_{B-\epsilon}}-1)}\right) \\ &+ \frac{1+(-1)^i(1-2\pi)}{2}\left(a\mathbf{1}_{\left\{\pi_{\tau^\star_{B-\epsilon}}<c\right\}} - b\mathbf{1}_{\left\{\pi_{\tau^\star_{B-\epsilon}}>c\right\}}\right)\frac{\varphi_{\tau^\star_{B-\epsilon}}}{\left[1+\pi(\varphi_{\tau^\star_{B-\epsilon}}-1)\right]^2}. \end{aligned} \tag{8.3.14}$$

From the definition of $\tau^\star_{\pi-\epsilon}$ and (8.2.11), one has

$$\begin{aligned} \tau^\star_{B-\epsilon} &= \inf\{t \geq 0 : \pi_t \notin (A, B),\ \pi_0 = B-\epsilon\} \\ &\leq \inf\left\{t \geq 0 : \log(\varphi_t) \geq \log\left(\frac{B}{1-B}\frac{1-(B-\epsilon)}{B-\epsilon}\right)\right\} =: \eta_\epsilon. \end{aligned} \tag{8.3.15}$$

According to Sato [71, Th. 43.21, case 6, p. 324] and (8.2.10), it results that the starting point 0 of $(\log(\varphi_t))_{t\geq 0}$ is regular for $(0, \infty)$ (that is, with probability 1, the log-likelihood ratio process, starting at 0, immediately enters $(0, \infty)$). This implies that $\eta_\epsilon \downarrow 0$ P_i-a.s. as $\epsilon \downarrow 0$, $i = 0, 1$. Then, from (8.3.15), $\tau^\star_{B-\epsilon} \downarrow 0$ and $\varphi_{\tau^\star_{B-\epsilon}} \to 1$ as $\epsilon \downarrow 0$ P_i-a.s., $i = 0, 1$. Accordingly, from (8.3.14)

$$S_i'(\omega_i) \to (-1)^{i-1}b(1-B) - \frac{1+(-1)^i(1-2B)}{2}b, \quad P_i\text{-a.s.},\ i = 0, 1, \text{ as } \epsilon \downarrow 0. \tag{8.3.16}$$

Since $S_i'(\omega_i) + (-1)^i\tau^\star_{B-\epsilon}$ is bounded, for $i = 0, 1$, from (8.3.11), (8.3.13), (8.3.16), the fact that $E_i[\tau^\star_{B-\epsilon}] \to 0$ as $\epsilon \downarrow 0$, $i = 0, 1$, and the bounded convergence theorem, we have

$$V'(B_-) = \lim_{\epsilon\downarrow 0}\frac{V(B) - V(B-\epsilon)}{\epsilon} \leq \lim_{\epsilon\downarrow 0}\sum_{i=0}^{1} E_i\left[S_i'(\omega_i)\right] = -b, \tag{8.3.17}$$

which, combined with (8.3.10), leads to the desired results.

The right differentiability of $V(\pi)$ at A is proved analogously. ∎

Proposition 8.3.2 formally proves the so called *smooth fit principle*, stating that the two optimal boundary points must be chosen so that the value function is smooth at those points. This result can be intuitively explained through the analysis of sample paths of the posterior probability process: from (8.2.11) and (8.2.13), we observe that even though it jumps when the underlying Lévy process jumps, the presence of the diffusion component (Brownian motion with drift) allows $(\pi_t)_{t\geq 0}$ to continuously cross A and B (see Figure 8.2 below). In other words, if $(\pi_t)_{t\geq 0}$ starts at A or B, it immediately enters the stopping region D. We refer to Alili and Kyprianou [2] for a further analysis on the smooth fit principle.

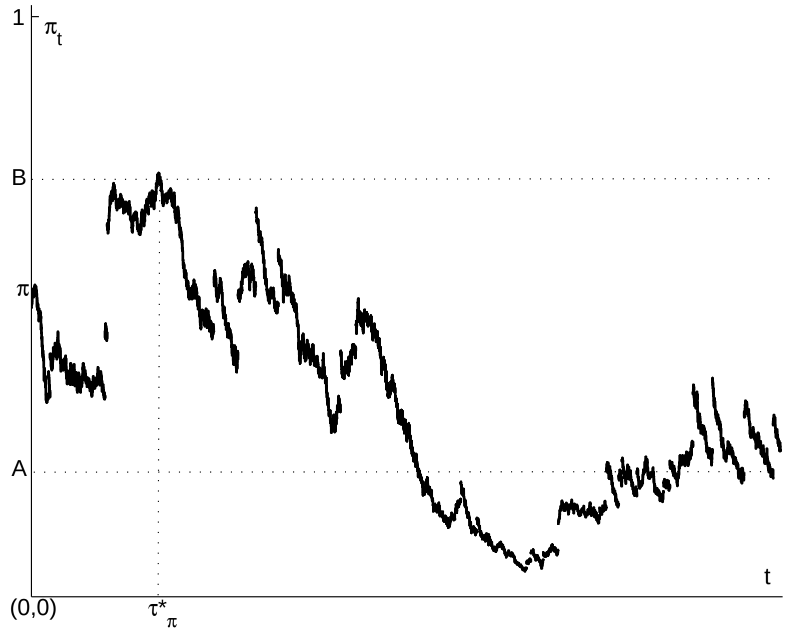

Figure 8.2: A simulated path of the posterior probability process $(\pi_t)_{t\geq 0}$, as defined by (8.2.11) and (8.2.13). The underlying Lévy process has as jump component a gamma process, that is, X has generating triplet $g_\vartheta = \{\gamma_\vartheta, \sigma^2, v_\vartheta(dx) = x^{-1}e^{-\alpha_\vartheta x}\mathbf{1}_{(0,\infty)}(dx)\}$, with $\alpha_\vartheta > 0$. It is assumed that $\alpha_0 > \alpha_1$ and that the true hypothesis is $H_0 : \vartheta = 0$. The drawing looks very similar even when the jump component is expressed by a compound Poisson process.

From (8.2.13), one may notice that $(\pi_t)_{t\geq 0}$ is a strong Markov process. This observation, the previous results and the general theory of optimal stopping (see Chapter 2 or, e.g., Peskir and Shiryaev [67, Chap. 3 and 4] and Shiryaev [72, Chap. 3]) entail the formulation of the following free-boundary Stephan problem, for the unknown function V and the unknown boundaries A

and B:

$$(\mathbb{L}^c + \mathbb{L}^j)V = -1 \qquad \text{for} \quad \pi \in (A, B), \tag{8.3.18}$$

$$V = g_{a,b} \qquad \text{for} \quad \pi \notin (A, B), \tag{8.3.19}$$

$$V < g_{a,b} \qquad \text{for} \quad \pi \in (A, B), \tag{8.3.20}$$

$$V(A_+) = aA \qquad (\textit{continuous fit}), \tag{8.3.21}$$

$$V'(A) = a \qquad (\textit{smooth fit}), \tag{8.3.22}$$

$$V(B_-) = b(1 - B) \qquad (\textit{continuous fit}), \tag{8.3.23}$$

$$V'(B) = -b \qquad (\textit{smooth fit}). \tag{8.3.24}$$

We observe that the presence of the second order integro-differential operator $\mathbb{L}^c + \mathbb{L}^j$ in (8.3.18) makes the above free-boundary problem very difficult to solve. A numerical approach will be discussed in Section 8.4.

8.3.2 Optimality of the free-boundary problem solution

The next theorem ensures under some mild assumptions that the solution of the optimal stopping problem (8.2.9) can be obtained by solving the free-boundary problem (8.3.18)-(8.3.24).

Theorem 8.3.1 *For a given $B > c$, let $\pi \mapsto V(\pi; B)$ be the map solving* (8.3.18)-(8.3.19) *and* (8.3.23)-(8.3.24) *on $I_B = (0, B]$. Further, assume that the map $\pi \mapsto V(\pi)$ solving the free-boundary problem* (8.3.18)-(8.3.24) *is such that $(\mathbb{L}^j V)(\pi) \geq -1$, $\pi \in [0, A] \cup [B, 1]$. Then, the π-Bayes decision rule $(\tau^\star_\pi, d^\star_\pi)$ for the optimal stopping problem* (8.2.9) *is given by:*

$$\tau^\star_\pi = \inf\{t \geq 0 : \pi_t \notin (A^\star, B^\star)\}, \tag{8.3.25}$$

$$d^\star_\pi = \begin{cases} 0 & (\textit{Accept } H_0), \quad \textit{if } \pi_{\tau^\star_\pi} \leq A^\star, \\ 1 & (\textit{Accept } H_1), \quad \textit{if } \pi_{\tau^\star_\pi} \geq B^\star, \end{cases} \tag{8.3.26}$$

where $A^\star$ and $B^\star$, with $0 < A^\star < c < B^\star < 1$, are obtained as unique solution of the following transcendental equations:

$$V(A^\star; B^\star) = aA^\star, \quad V'(A^\star; B^\star) = a. \tag{8.3.27}$$

The value function (8.2.9) *is given by*

$$V(\pi) = \begin{cases} V(\pi; B^\star), & \pi \in (A^\star, B^\star), \\ g_{a,b}(\pi), & \pi \in [0, A^\star] \cup [B^\star, 1]. \end{cases} \tag{8.3.28}$$

Proof. We begin by proving the uniqueness of $A^\star$ and $B^\star$ solving (8.3.27). Since the map $\pi \mapsto V(\pi; B)$ satisifies (8.3.24), $\lim_{B \downarrow c} V'(B_-, B) = -b < a$. It means that there exist values of $B \in (c, 1)$, such that the maps $\pi \mapsto V(\pi; B)$ intersect the map $\pi \mapsto a\pi$ at some point in $(0, c)$. Further, Peskir and Shiryaev [65, Remark 2.2, p. 850] or direct verification arguments (see Figure 8.3) show that for $B' < B''$, the maps $\pi \mapsto V(\pi; B')$ and $\pi \mapsto V(\pi; B'')$ do not intersect on $(0, B']$. Then, moving B on $(c, 1)$ from the left to the right, one can observe that there exists a unique pair of points $A^\star$ and $B^\star$ at which $\pi \mapsto V(\pi; B^\star)$ smoothly hits $\pi \mapsto a\pi$ and, by construction, $\pi \mapsto b(1-\pi)$, respectively.

Now, let us show that the functions expressed in (8.2.9) and (8.3.28) are the same. To this aim, denote the latter by $V^\star$. By construction, $V^\star(\pi)$ is C^2 on $[0,1] \setminus \{A^\star, B^\star\}$ and C^1 at $A^\star$ and $B^\star$. Since the time spent by $(\pi_t)_{t\geq 0}$ at $A^\star$ and $B^\star$ is of Lebesgue measure zero, Itô's formula (8.3.1) can be applied to $V^\star(\pi_t)$:

$$V^\star(\pi_t) = V^\star(\pi) + \int_0^t \big((\mathbb{L}^c + \mathbb{L}^j)V^\star\big)(\pi_{s^-})ds + \mathscr{M}_t^\star, \tag{8.3.29}$$

where $\mathscr{M}_t^\star$ takes the expression (8.3.4), with f replaced by $V^\star$. Since $V^\star$ and $V^{\star\prime}$ are obviously bounded, $\mathscr{M}^\star = (\mathscr{M}_t^\star)_{t\geq 0}$ is a martingale with respect to $(\mathscr{F}_t^X)_{t\geq 0}$ and P_π. Hence, for any $\mathscr{F}_t^X$-measurable stopping time τ, $t \geq 0$, with $E_\pi[\tau] < \infty$, the optional sampling theorem implies $E_\pi[\mathscr{M}_\tau^\star] = 0$. Further, (8.3.18), the assumption $(\mathbb{L}^j V^\star)(\pi) \geq -1$, for $\pi \in [0, A] \cup [B, 1]$, and the fact that $(\mathbb{L}^c V^\star)(\pi) = 0$ on the same set guarantee that $\big((\mathbb{L}^c + \mathbb{L}^j)V^\star\big)(\pi) \geq -1$, $\forall \pi \in [0,1]$, with the exception of $A^\star$ and $B^\star$, where $V^\star$ is not C^2. These considerations and (8.3.29) imply that

$$E_\pi[V^\star(\pi_\tau)] = V^\star(\pi) + E_\pi\left[\int_0^\tau \big((\mathbb{L}^c + \mathbb{L}^j)V^\star\big)(\pi_{s^-})ds\right] \geq V^\star(\pi) - E_\pi[\tau]. \tag{8.3.30}$$

From (8.3.30), (8.3.19) and (8.3.20), one has

$$V^\star(\pi) \leq E_\pi[V^\star(\pi_\tau) + \tau] \leq E_\pi[\tau + g_{a,b}(\pi_\tau)], \tag{8.3.31}$$

for all stopping times of X with finite expectation. This shows that $V^\star$ is a lower bound for V.

Due to the consistency of $(\pi_t)_{t\geq 0}$, that is, $\lim_{t\to\infty} \pi_t = i$, if H_i is true, $i = 0, 1$, $E_\pi[\tau_\pi^\star] < \infty$. Therefore, from (8.3.30), (8.3.18) and (8.3.25) it results that

$$E_\pi[V^\star(\pi_{\tau_\pi^\star})] = V^\star(\pi) - E_\pi[\tau_\pi^\star]. \tag{8.3.32}$$

According to (8.3.19) and (8.3.25), $V^\star(\pi_{\tau_\pi^\star}) = g_{a,b}(\pi_{\tau_\pi^\star})$, so that $V^\star(\pi) = E_\pi[\tau_\pi^\star + g_{a,b}(\pi_{\tau_\pi^\star})]$. This equality and (8.3.31) prove that $V(\pi) = V^\star(\pi)$, $\forall \pi \in [0,1]$, and $\tau_\pi^\star$ is optimal in (8.2.9). ∎

Remark 8.3.1 The above results concern all the situations where both the diffusion and jump components of the sequentially observed Lévy process X have different generating triplets under the two hypotheses (8.2.7). According to Theorem 8.2.1, $X = X^c + X^j$ and the Lévy-Khintchine triplets of X^c and X^j are $g_i^c = \{\widetilde{\gamma}_i, \sigma^2, 0\}$ and $\widetilde{g}_i^j = \{0, 0, v_i(\cdot)\}$, respectively, under H_i, $i = 0, 1$. Hence, we treated the case where $\widetilde{\gamma}_0 \neq \widetilde{\gamma}_1$ and $v_0 \neq v_1$. Indeed, if $\widetilde{\gamma}_0 \neq \widetilde{\gamma}_1$ but $v_0 = v_1$, the problem (8.2.7)-(8.2.9) reduces to the well-known sequential testing for the drift of a Wiener process, studied by Shiryaev [72, sec. 4.2] (see also Section 5.4 or Buonaguidi and Muliere [16, Sec. 4]), for which one needs to observe X^c, only. On the other hand, if $v_0 \neq v_1$ but $\widetilde{\gamma}_0 = \widetilde{\gamma}_1$, we are in the framework analyzed in Section 5.5 (or in Buonaguidi and Muliere [16, sec. 5]), so that the observation of X^j only turns out to be relevant for the solution of the problem (8.2.7)-(8.2.9).

8.4 Numerical approach to the free-boundary problem

Theorem 8.3.1 states that the solution of the optimal stopping problem (8.2.9) can be derived by solving the free-boundary problem (8.3.18)-(8.3.24). This task turns out to be extremely difficult, because it requires to solve the second order integro-differential equation (8.3.18), defined through the operators (8.3.2) and (8.3.3). Hence, instead of approaching the free-boundary problem analytically, we compute its solution numerically, using a numerical scheme based on the collocation method.

In this section, first we briefly recall the basic elements of the collocation method for boundary value problems, then we show how to properly extend it to our free-boundary problem, when the observed Lévy process is given by the sum of a Wiener and a gamma process. We selected this process as illustrative example, since the gamma process is not a compound Poisson process and therefore falls outside the theory developed by Dayanik and Sezer [25].

8.4.1 Collocation method for a boundary value problem

A boundary value problem is specified by a functional equation and, eventually, a certain number of boundary conditions, used for determining the unknown constants that can arise from the solution of the equation. We underline that the boundary points for this kind of problems are known, unlike a free-boundary problem. For instance, given the operator $\mathbb{K}$ defined by

$$(\mathbb{K}f)(x) = f''(x) - g(x) - h_1(x)f'(x) - h_2(x)f(x) - \int_A^x k(x,z)f(z)\,dz, \tag{8.4.1}$$

$x \in I = [A, B]$, where $g(x)$, $h_1(x)$, $h_2(x)$ and $k(x,z)$ are known functions and A and B are known, consider the following boundary value problem for the unknown function f:

$$(\mathbb{K}f)(x) = 0 \tag{8.4.2}$$

$$f(B) = \alpha, \quad f'(B) = \beta. \tag{8.4.3}$$

Suppose that a solution of (8.4.2) and (8.4.3) cannot be determined, but it is unique and continuous on I; in order to obtain an approximation of f, the collocation method can be applied (see, e.g., Kress [47, sec. 12.4]). The key element of this approach is represented by the Waierstrass approximation theorem, which states that any continuous function on a closed interval can be uniformly approximated by a linear combination of polynomials. It means that, denoted by $\Psi = \{\psi_i\}_{i\geq 0}$ a known family of polynomials, we can express f as

$$f(x) = \sum_{i=0}^{\infty} w_i\psi_i(x), \quad x \in I, \tag{8.4.4}$$

being $\{w_i\}_{i\geq 0}$ the unknown coefficients of the above series: hence, the sequence $\{f_n\}_{n\geq 0}$, given by

$$f_n(x) = \sum_{i=0}^{n} w_i\psi_i(x), \tag{8.4.5}$$

uniformly converges to f, as $n \to \infty$. So, for a sufficiently large and fixed n, a good approximation of f can be obtained by determining the $n+1$ coefficients $\{w_0, \dots, w_n\}$ in (8.4.5). Let $\{x_1, \dots, x_{n-1}\}$ be $n-1$ points in I, also known as *collocation nodes*; then, the coefficients $\{w_0, \dots, w_n\}$ can be computed by forcing f_n to satisfy (8.4.2) at x_j, $j = 1, \dots, n-1$, and (8.4.3):

$$(\mathbb{K} f_n)(x_j) = 0 \quad j = 1, \dots, n-1, \tag{8.4.6}$$

$$f_n(B) = \alpha, \quad f_n'(B) = \beta. \tag{8.4.7}$$

In this way, a linear system of $n+1$ equations for $n+1$ unknown variables is obtained.

It is clear that before solving the system (8.4.6) and (8.4.7) we shall fix the family of polynomials Ψ. A customary choice in approximation theory is the family of Chebyshev polynomials, denoted here by $T = \{T_i\}_{i \geq 0}$ and defined by

$$T_n(x) = \cos[n(\arccos(x))], \quad n \geq 0, \quad x \in [-1, 1]. \tag{8.4.8}$$

If the function f to be found has as domain the interval $[0,1]$, it can be approximated by the shifted Chebyshev polynomials on the interval $[0,1]$, denoted by $T^\star = \{T_i^\star\}_{i \geq 0}$ and defined by

$$T_n^\star(x) = T_n(2x - 1), \quad x \in [0, 1]. \tag{8.4.9}$$

We refer to Section 7.2 for the illustration of the main properties of Chebyshev polynomials and the choice of the truncation limit n, as well as the main references about this topic. Here, we recall that the choice of the collocation nodes in a boundary value problem is simple: the zeros of T_n (or $T_n^\star$, depending of the interval of interest). Indeed, it is well known that in this case the distance $\sup_{x \in I} |f(x) - f_n(x)|$ is minimized. Unfortunately in a free-boundary problem the interval where the functional equation must be solved is not known a priori, so that the zeros of T_n cannot be used.

8.4.2 Collocation method and free-boundary problem: the Wiener-gamma process

In order to discuss how the collocation method can be efficiently extended to our problem, we consider as an example the case where the jump component is a gamma process.

Let X be a Lévy process with generating triplet $g_\vartheta = \{\gamma_\vartheta, \sigma^2, v_\vartheta(dx) = x^{-1} e^{-\alpha_\vartheta x} \mathbf{1}_{(0,\infty)}(dx)\}$ under P_π, with $\alpha_\vartheta > 0$. According to (8.2.3) and Theorem 8.2.1, $X = X^c + X^j$, where X^c and X^j have triplets $g_\vartheta^c = \{\widetilde{\gamma}_\vartheta, \sigma^2, 0\}$ and $\widetilde{g}_\vartheta^j = \{0, 0, v_\vartheta(dx)\}$, respectively, where, under the hypothesis $\vartheta = i$, $i = 0, 1$,

$$\widetilde{\gamma}_i = \gamma_i - (1 - e^{-\alpha_i})/\alpha_i. \tag{8.4.10}$$

One may recognize X^j as a gamma process with parameter α_ϑ (see Kyprianou [48, Sec. 1.2 and 2.6]). Simple calculations show that (8.2.10), (8.2.12) and (8.2.13) become

$$\varphi_t = \exp\left\{ \frac{\widetilde{\gamma}_1 - \widetilde{\gamma}_0}{\sigma^2} \left(X_t^c - t \frac{\widetilde{\gamma}_1 + \widetilde{\gamma}_0}{2} \right) + (\alpha_0 - \alpha_1) X_t^j - \log\left(\frac{\alpha_0}{\alpha_1}\right) t \right\}, \tag{8.4.11}$$

$$d\varphi_t = -\frac{(\widetilde{\gamma}_1 - \widetilde{\gamma}_0)\widetilde{\gamma}_0}{\sigma^2}\,\varphi_{t^-}dt + \frac{\widetilde{\gamma}_1 - \widetilde{\gamma}_0}{\sigma^2}\,\varphi_{t^-}dX_t^c$$
$$- \log\left(\frac{\alpha_0}{\alpha_1}\right)\varphi_{t^-}dt + \varphi_{t^-}\int_0^\infty \left(e^{(\alpha_0-\alpha_1)x} - 1\right)\mu^X(dx,dt),\ \varphi_0 = 1, \tag{8.4.12}$$

$$d\pi_t = \frac{\widetilde{\gamma}_1 - \widetilde{\gamma}_0}{\sigma^2}\pi_{t^-}(1-\pi_{t^-})d\widetilde{X}_t^c - \log\left(\frac{\alpha_0}{\alpha_1}\right)\pi_{t^-}(1-\pi_{t^-})dt$$
$$+ \int_0^\infty \frac{\pi_{t^-}(1-\pi_{t^-})\left(e^{(\alpha_0-\alpha_1)x} - 1\right)}{1 + \pi_{t^-}\left(e^{(\alpha_0-\alpha_1)x} - 1\right)}\,\mu^X(dx,dt),\ \pi_0 = \pi. \tag{8.4.13}$$

As seen in Section 8.3, the optimal stopping problem (8.2.9) can be reduced to the free-boundary problem (8.3.18)-(8.3.24), where $\mathbb{L}^c + \mathbb{L}^j$, the infinitesimal generator of $(\pi_t)_{t\geq 0}$, is defined through (8.3.2), (8.3.3) and (8.4.10) and $\mathbb{L}^j$ takes the following expression:

$$(\mathbb{L}^j f)(\pi) = -\log\left(\frac{\alpha_0}{\alpha_1}\right)f'(\pi)\pi(1-\pi)$$
$$+ \int_0^\infty \left[f\left(\frac{\pi e^{-\alpha_1 x}}{(1-\pi)e^{-\alpha_0 x} + \pi e^{-\alpha_1 x}}\right) - f(\pi)\right]\frac{(1-\pi)e^{-\alpha_0 x} + \pi e^{-\alpha_1 x}}{x}\,dx. \tag{8.4.14}$$

Without loss of generality it is assumed that $\alpha_0 > \alpha_1$. For a fixed $B > c$, let $\pi \mapsto V(\pi; B)$ be the map solving (8.3.18), (8.3.19), (8.3.23) and (8.3.24) on the interval $I_B = (0, B]$. Define

$$S(\pi, x) = \frac{\pi e^{-\alpha_1 x}}{(1-\pi)e^{-\alpha_0 x} + \pi e^{-\alpha_1 x}}, \quad \pi \in I_B, \quad x \geq 0, \tag{8.4.15}$$

and observe that $S(\pi, x)$ is increasing in x, $\lim_{x\to\infty} S(\pi, x) = 1$ and $S(\pi, x) \geq B$ only if $x \geq d(\pi; B)$, where

$$d(\pi; B) = \log\left(\frac{1-\pi}{\pi}\frac{B}{1-B}\right)\Big/(\alpha_0 - \alpha_1), \quad \pi \in I_B. \tag{8.4.16}$$

Therefore, $V(\pi; B)$ satisfies on I_B the following boundary value problem:

$$(\mathbb{L}_B V)(\pi; B) = -1, \quad \pi \in I_B \tag{8.4.17}$$
$$V(B; B) = b(1-B), \quad V'(B_-; B) = -b, \tag{8.4.18}$$

where $\mathbb{L}_B$ is the operator defined by

$$(\mathbb{L}_B f)(\pi) = (\mathbb{L}^c f)(\pi) - \log\left(\frac{\alpha_0}{\alpha_1}\right)f'(\pi)\pi(1-\pi)$$
$$+ b(1-\pi)\int_{d(\pi;B)}^\infty \frac{e^{-\alpha_0 x}}{x}\,dx - f(\pi)\int_{d(\pi;B)}^\infty \frac{(1-\pi)e^{-\alpha_0 x} + \pi e^{-\alpha_1 x}}{x}\,dx$$
$$+ \int_0^{d(\pi;B)} \left[f(S(\pi,x)) - f(\pi)\right]\frac{(1-\pi)e^{-\alpha_0 x} + \pi e^{-\alpha_1 x}}{x}\,dx, \quad \pi \in I_B. \tag{8.4.19}$$

Proposition 8.4.1 *Given $B > c$,* (8.4.17)-(8.4.18)*, defined through* (8.4.19)*, has a unique twice continuously differentiable solution $V(\pi; B)$, $\pi \in I_B$.*

Proof. Define $f(y) = V(\pi; B)$, with $\pi = e^y/(1+e^y)$; denote by $B^o = \log(B/(1-B))$, $k = (\widetilde{\gamma}_1 - \widetilde{\gamma}_0)^2/(2\sigma^2)$ and $\rho = \alpha_0/(\alpha_0 - \alpha_1)$. Then, easy calculation shows that (8.4.17)-(8.4.18) is equivalent to

$$
\begin{aligned}
f''(y) = & -\frac{1}{k} - \frac{be^{\rho y}}{k(1+e^y)} \int_{B^o}^{\infty} \frac{e^{-\rho z}}{z-y}\, dz \\
& + f(y) \frac{e^{\rho y}}{k(1+e^y)} \int_{B^o}^{\infty} \frac{e^{-\rho z}(1+e^z)}{z-y}\, dz + f'(y) \left(\frac{1-e^y}{1+e^y} + \frac{\log(\alpha_0/\alpha_1)}{k} \right) \\
& - \frac{e^{\rho y}}{k(1+e^y)} \int_y^{B^o} [f(z) - f(y)] \frac{(1+e^z)e^{-\rho z}}{z-y}\, dz,
\end{aligned} \tag{8.4.20}
$$

$$
f(B^o) = \frac{b}{1+e^{B^o}}, \quad f'(B^o_-) = -\frac{be^{B^o}}{(1+e^{B^o})^2}, \tag{8.4.21}
$$

with $y \in [y^\star, B^o]$, where $y^\star$ is an arbitrary finite number smaller than B^o. The functional equation defined through (8.4.17) and (8.4.19) has been turned into the linear Volterra integro-differential equation (8.4.20): this equation is said to be of the second kind, since the unknown function f is also outside the integral. Unfortunately, we cannot directly apply the standard results on integro-differential equations for proving the existence and uniqueness of f, since the map $z \mapsto (1+e^z)e^{-\rho z}/(z-y)$ is not integrable on $[y, B^o]$: this means that the last integral in (8.4.20) cannot be split and therefore we cannot obtain the canonical representation (8.4.1) and (8.4.2). Hence, first we consider "regular versions" of (8.4.20), that is, equations which admit twice continuously differentiable solutions, then we verify that the limit of this solutions is a solution of (8.4.20)-(8.4.21).

For a given $0 < \epsilon \leq 1$, let $f_\epsilon(y)$ be the solution of the following "regular" problem:

$$
f''_\epsilon(y) = g(y) + h_1(y) f'_\epsilon(y) + h_{\epsilon,2}(y) f_\epsilon(y) + \int_y^{B^o} k_\epsilon(y,z) f_\epsilon(z)\, dz \tag{8.4.22}
$$

$$
f_\epsilon(B^o) = \frac{b}{1+e^{B^o}}, \quad f'_\epsilon(B^o_-) = -\frac{be^{B^o}}{(1+e^{B^o})^2}, \tag{8.4.23}
$$

where

$$
g(y) = -\frac{1}{k} - \frac{be^{\rho y}}{k(1+e^y)} \int_{B^o}^{\infty} \frac{e^{-\rho z}}{z-y}\, dz, \tag{8.4.24}
$$

$$
h_1(y) = \frac{1-e^y}{1+e^y} + \frac{\log(\alpha_0/\alpha_1)}{k}, \tag{8.4.25}
$$

$$
h_{\epsilon,2}(y) = \frac{e^{\rho y}}{k(1+e^y)} \left[\int_y^{B^o} \frac{(1+e^z)e^{-\rho z}}{(z-y)^{1-\epsilon}}\, dz + \int_{B^o}^{\infty} \frac{e^{-\rho z}(1+e^z)}{z-y}\, dz \right], \tag{8.4.26}
$$

$$
k_\epsilon(y,z) = -\frac{e^{\rho y}}{k(1+e^y)} \frac{(1+e^z)e^{-\rho z}}{(z-y)^{1-\epsilon}}. \tag{8.4.27}
$$

According to Linz [53, Th. 3.2 p. 32], it is well known that for a fixed $0 < \epsilon \leq 1$, (8.4.22)-(8.4.23) has a unique twice continuously differentiable solution f_ϵ.

A direct analysis based on the existence and uniqueness of f_ϵ shows that $\{f_\epsilon\}$, $\{f'_\epsilon\}$ and $\{f''_\epsilon\}$, $0 < \epsilon \leq 1$ are Cauchy sequences as $\epsilon \downarrow 0$ and therefore are uniform convergent on any compact interval $[y^\star, B^o]$. Therefore,

$$
f(y) := \lim_{\epsilon \downarrow 0} f_\epsilon(y), \quad f'(y) := \lim_{\epsilon \downarrow 0} f'_\epsilon(y), \quad f''(y) := \lim_{\epsilon \downarrow 0} f''_\epsilon(y), \quad y^\star \leq y \leq B^o, \tag{8.4.28}
$$

exist and f is twice continuously differentiable with first and second derivative f' and f''. Then, from (8.4.22), (8.4.23) and (8.4.28) we see that

$$\begin{aligned} f''(y) = \lim_{\epsilon\downarrow 0} f''_\epsilon(y) = g(y) + \lim_{\epsilon\downarrow 0} f_\epsilon(y)\frac{e^{\rho y}}{k(1+e^y)}\int_{B^o}^{\infty}\frac{e^{-\rho z}(1+e^z)}{z-y}\,dz \\ + h_1(y)\lim_{\epsilon\downarrow 0} f'_\epsilon(y) - \frac{e^{\rho y}}{k(1+e^y)}\lim_{\epsilon\downarrow 0}\int_y^{B^o}[f_\epsilon(z)-f_\epsilon(y)]\frac{(1+e^z)e^{-\rho z}}{(z-y)^{1-\epsilon}}\,dz \\ = g(y) + f(y)\frac{e^{\rho y}}{k(1+e^y)}\int_{B^o}^{\infty}\frac{e^{-\rho z}(1+e^z)}{z-y}\,dz + h_1(y)f'(y) \\ - \frac{e^{\rho y}}{k(1+e^y)}\lim_{\epsilon\downarrow 0}\int_y^{B^o}[f_\epsilon(z)-f_\epsilon(y)]\frac{(1+e^z)e^{-\rho z}}{(z-y)^{1-\epsilon}}\,dz. \end{aligned} \tag{8.4.29}$$

Further, since

$$\lim_{\epsilon\downarrow 0}\frac{f_\epsilon(z)-f_\epsilon(y)}{(z-y)^{1-\epsilon}} = \frac{f(z)-f(y)}{(z-y)} \quad \text{and} \quad \left|\frac{f_\epsilon(z)-f_\epsilon(y)}{(z-y)^{1-\epsilon}}\right| \le C_y \tag{8.4.30}$$

for any $z \in [y, B^o]$ and $0 < \epsilon \le 1$, where C_y is a constant depending on y, the application of the bounded convergence theorem implies that

$$\lim_{\epsilon\downarrow 0}\int_y^{B^o}[f_\epsilon(z)-f_\epsilon(y)]\frac{(1+e^z)e^{-\rho z}}{(z-y)^{1-\epsilon}}\,dz = \int_y^{B^o}[f(z)-f(y)]\frac{(1+e^z)e^{-\rho z}}{(z-y)}\,dz. \tag{8.4.31}$$

The expressions (8.4.29) and (8.4.31) show that f, as defined in (8.4.28), is a twice continuously differentiable solution of (8.4.20)-(8.4.21). Then, (8.4.17)-(8.4.18), given by means of (8.4.19), has a twice continuously differentiable solution $V(\pi; B)$, $\pi \in I_B$.

The uniqueness of $V(\pi; B)$ is proved analogously as in Remark 7.3.1. ∎

Proposition 8.4.1 allows us to apply the collocation method recalled in Subsection 8.4.1 to the boundary value problem (8.4.17) and (8.4.18). For a given $B > c$ and a sufficiently large $n \ge 0$, denote by

$$V_n(\pi; B) = \sum_{i=0}^{n} w_i(B)T_i^\star(\pi) \tag{8.4.32}$$

an approximation of $V(\pi; B)$ on I_B, where $\{T_i^\star\}_{i\ge 0}$ are the shifted Chebyshev polynomials on $[0,1]$, defined in (8.4.9), and $\{w_0, \dots, w_n\}$ are the unknown coefficients of the linear combination, whose dependence on B is highlighted in (8.4.32). Since $V(\pi; B)$ satisfies (8.4.17)-(8.4.18), once $n-1$ collocation nodes $\{\pi_1, \dots, \pi_{n-1}\}$ in I_B have been chosen, the $n+1$ coefficients can be computed by solving the following system of $n+1$ linear equations:

$$(\mathbb{L}_B V_n)(\pi_i; B) = -1, \quad i = 1, \dots, n-1, \tag{8.4.33}$$

$$V_n(B; B) = b(1-B), \quad V'_n(B_-; B) = -b. \tag{8.4.34}$$

The continuity of $V(\pi; B)$ and the Waierstrass approximation theorem imply the uniform convergence of $V_n(\pi; B)$ to $V(\pi; B)$ on any compact interval as n increases; the uniqueness of $V(\pi; B)$ guarantees that the system (8.4.33)-(8.4.34) is not ill conditioned.

We may observe that solving the system for different values of $B > c$ allows us to get plausible neighborhoods of the two optimal stopping boundaries $A^\star$ and $B^\star$. This statement is explained by means of the following numerical example. Set $a = b = 5$, $\sigma^2 = 1$, $\widetilde{\gamma}_0 = 2$, $\alpha_0 = 5$, $\widetilde{\gamma}_1 = 3$, $\alpha_1 = 1$ and fix $n = 12$ in (8.4.32). Solve the system (8.4.33)-(8.4.34) for $B = 0.67$, 0.72, 0.77, 0.82, 0.87, by choosing $n - 1 = 11$ collocation nodes in I_B (we used $n - 1 = 11$ equally spaced points in $[0.1, B]$). The maps $\pi \mapsto V_n(\pi; B)$ are shown in Figure 8.3: it is evident that moving B from c to 1, there exists a unique pair of points $A^\star$ and $B^\star$ at which the smooth and continuous fit conditions (8.3.21)- (8.3.24) are fulfilled. In our example, we notice that $A^\star \in (0.15, 0.25)$ and $B^\star \in (0.77, 0.82)$ (a higher precision on these intervals can be reached by fixing other values of B in $(0.77, 0.82)$ and solving again (8.4.33)-(8.4.34)).

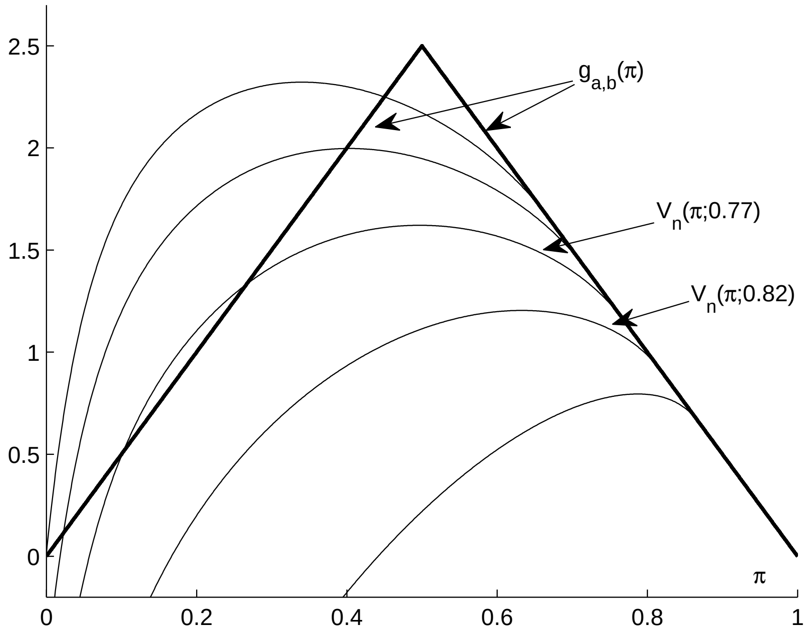

Figure 8.3: A computer drawing of the maps $\pi \mapsto V_n(\pi; B)$ for the sequential testing of a Lévy process composed of a Brownian motion with drift and a gamma process, with $a = b = 5$, $\sigma^2 = 1$, $\widetilde{\gamma}_0 = 2$, $\alpha_0 = 5$, $\widetilde{\gamma}_1 = 3$, $\alpha_1 = 1$. We set $B = 0.67, 0.72, 0.77, 0.82, 0.87$ and $n = 12$ in (8.4.32). One may observe that the smooth fit condition (8.4.34), arising from (8.4.20), implies that $\pi \mapsto V_n(\pi; B)$ has left derivative at B equal to $-b$. The drawing of $V_n(\pi; B)$ for a general Lévy process with diffusion and jump components looks very much the same.

Now, let us explain how to properly extend the collocation method to the free-boundary problem (8.3.18)-(8.3.24), in order to compute accurate approximations of the map $\pi \mapsto V(\pi; B^\star)$ and of the optimal stopping boundaries $A^\star$ and $B^\star$. For a fixed and sufficiently large $n \geq 0$, let

$$V_n(\pi; B^\star) = \sum_{i=0}^{n} w_i(B^\star) T_i^\star(\pi) \tag{8.4.35}$$

be an approximation of $V(\pi; B^\star)$, for $\pi \in I_{B^\star} = (0, B^\star]$. Since the map $\pi \mapsto V(\pi; B^\star)$ solves (8.4.17)-(8.4.18) and satisfies (8.3.27), the free-boundary problem (8.3.18)-(8.3.24) reduces to solving the following system of $n + 3$ non-linear equations, for the $n + 3$ unknown variables $A^\star$,

$B^\star$ and $w_0(B^\star), \ldots, w_n(B^\star)$:

$$(\mathbb{L}_{B^\star} V_n)(\pi_i; B^\star) = -1, \quad i = 1, .., n-1, \tag{8.4.36}$$
$$V_n(A^\star; B^\star) = aA^\star, \tag{8.4.37}$$
$$V_n'(A^\star; B^\star) = a, \tag{8.4.38}$$
$$V_n(B^\star; B^\star) = b(1 - B^\star), \tag{8.4.39}$$
$$V_n'(B^\star_-; B^\star) = -b. \tag{8.4.40}$$

The operator $\mathbb{L}_{B^\star}$ appearing in (8.4.36) is given in (8.4.19) and $\{\pi_1, \ldots, \pi_{n-1}\}$ are $n-1$ collocation nodes in $I_{B^\star}$. $B^\star$ is not known a priori so that the choice of the collocation nodes could be problematic; indeed, this difficulty is overcome by looking at the solutions of (8.4.33)-(8.4.34) for different values of B. As seen before, this leads us to establish a neighborhood (k_1, k_2) of $B^\star$, so that we can fix $\pi_1 < \ldots < \pi_{n-1} \leq k_1$. The approximation of $A^\star$ and $B^\star$ are denoted in the sequel by $A^\star_n$ and $B^\star_n$: they are obtained, along with $w_0(B^\star_n), \ldots, w_n(B^\star_n)$, as solution of (8.4.36)-(8.4.40). The uniqueness of $A^\star$ and $B^\star$, as well as $V(\pi; B^\star)$, makes the above system rapidly solvable and its solution well identifiable, even when n is large.

From Theorem 8.3.1, we can now naturally define the following approximations of the π-Bayes decision rule $(\tau^\star_\pi, d^\star_\pi)$ and of the value function $V(\pi)$ for a fixed value $n \geq 0$ in (8.4.35):

$$\tau^\star_{n,\pi} = \inf\{t \geq 0 : \pi_t \notin (A^\star_n, B^\star_n)\}, \tag{8.4.41}$$

$$d^\star_{n,\pi} = \begin{cases} 0 \quad \text{(Accept } H_0), & \text{if } \pi_{\tau^\star_{n,\pi}} \leq A^\star_n, \\ 1 \quad \text{(Accept } H_1), & \text{if } \pi_{\tau^\star_{n,\pi}} \geq B^\star_n, \end{cases} \tag{8.4.42}$$

$$V_n(\pi) = \begin{cases} V_n(\pi; B^\star_n) & \text{for} \quad \pi \in (A^\star_n, B^\star_n) \\ g_{a,b}(\pi) & \text{for} \quad \pi \in [0, A^\star_n] \cup [B^\star_n, 1] \end{cases}. \tag{8.4.43}$$

The choice of n can be made on the basis of the two criteria discussed at the end of Section 7.2. Since $V_n(\pi)$ approximately solves $(\mathbb{L}_{B^\star_n} V_n)(\pi) + 1 \approx 0$, $\pi \in (A^\star_n, B^\star_n)$, we can define

$$M_n = \sup_{\pi \in (A^\star_n, B^\star_n)} |(\mathbb{L}_{B^\star_n} V_n)(\pi) + 1| \tag{8.4.44}$$

and increase n until $M_n < \epsilon$, for a given $\epsilon > 0$. The other criterion is based on the evaluation of the sequence $\{\rho_n\}_{n \geq 1}$, given by

$$\rho_n = \sup_{\pi \in [0,1]} |V_n(\pi) - V_{n-1}(\pi)|, \tag{8.4.45}$$

so that we can increase n until $\rho_n < \epsilon$, $\epsilon > 0$.

We can now conclude the discussion of the numerical example previously begun, where we observed that $A^\star \in (0.15, 0.25)$ and $B^\star \in (0.77, 0.82)$. For different values of n and $n-1$ equally spaced collocation nodes in the interval I, the next table shows $A^\star_n$ and $B^\star_n$, obtained as solution of the system (8.4.36)-(8.4.40), M_n and ρ (the latter, similarly to (8.4.45), denotes the maximum distance between the V_ns of two consecutive rows of Table 8.1).

Table 8.1

n	I	$A_n^\star$-$B_n^\star$	M_n	ρ
6	$[0.1, 0.77]$	0.1834-0.7675	0.1872	-
8	$[0.1, 0.77]$	0.1786-0.7763	0.0905	0.0202
10	$[0.1, 0.77]$	0.1782-0.7775	0.0623	0.0030
12	$[0.1, 0.77]$	0.1783-0.7777	0.0377	0.0006
20	$[0.1, 0.77]$	0.1783-0.7777	0.0103	0.0002

From Table 8.1 we observe that the boundaries $A_n^\star, B_n^\star$ remain stable up to the second decimal digit from $n = 8$; for $n = 12$ and $n = 20$ they are the same up to the fourth decimal digit and the distance between the associated value functions V_ns is just 0.0002. Figure 8.4 shows the map $\pi \mapsto V_n(\pi)$, as defined by (8.4.43), with $n = 12$. We recall that for the optimality of the free-boundary problem solution, Theorem 3.1 requires $(\mathbb{L}^j V)(\pi) \geq -1$, $\pi \in [0, A] \cup [B, 1]$: for the case of a gamma process, this condition was verified in Section 7.4.2.

8.5 Examples

In the previous section, the application and extension of the collocation method to solving the free-boundary problem (8.3.18)-(8.3.24) have been illustrated by assuming that the jump component of X was a gamma process. In this section, we concentrate on some other examples of sequential testing for Lévy processes with diffusion and jump components. In particular, we consider the cases where the latter is expressed by a Poisson process, a compound Poisson process with exponential jumps and a negative binomial process.

8.5.1 The Wiener-Poisson process

Let X be a Lévy process with generating triplet $g_\vartheta = \{\gamma_\vartheta, \sigma^2, \lambda_\vartheta \delta_1\}$, $\lambda_\vartheta > 0$, where δ_1 is the measure putting unit mass on 1. From the results of subsection (8.2.1) it is easily seen that X^c, the continuous component of X, has triplet $g_\vartheta^c = \{\widetilde{\gamma}, \sigma^2, 0\}$, with $\widetilde{\gamma}_\vartheta = \gamma_\vartheta - \lambda_\vartheta$, while the jump component X^j is a Poisson process of intensity λ_ϑ and has therefore triplet $\widetilde{g}_\vartheta^j = \{0, 0, \lambda_\vartheta \delta_1\}$. From (8.2.10), we see that

$$\varphi_t = \exp\left\{ \frac{\widetilde{\gamma}_1 - \widetilde{\gamma}_0}{\sigma^2} \left(X_t^c - t\frac{\widetilde{\gamma}_1 + \widetilde{\gamma}_0}{2} \right) + \log\left(\frac{\lambda_1}{\lambda_0}\right) X_t^j - t(\lambda_1 - \lambda_0) \right\}. \tag{8.5.1}$$

The infinitesimal generator of $(\pi_t)_{t\geq 0}$ is given by (8.3.2) and (8.3.3), where

$$\begin{aligned}(\mathbb{L}^j f)(\pi) = & - \pi(1-\pi) f'(\pi)(\lambda_1 - \lambda_0) \\ & + (\lambda_1 \pi + \lambda_0(1-\pi)) \left[f\left(\frac{\lambda_1 \pi}{\lambda_1 \pi + \lambda_0(1-\pi)} \right) - f(\pi) \right].\end{aligned} \tag{8.5.2}$$

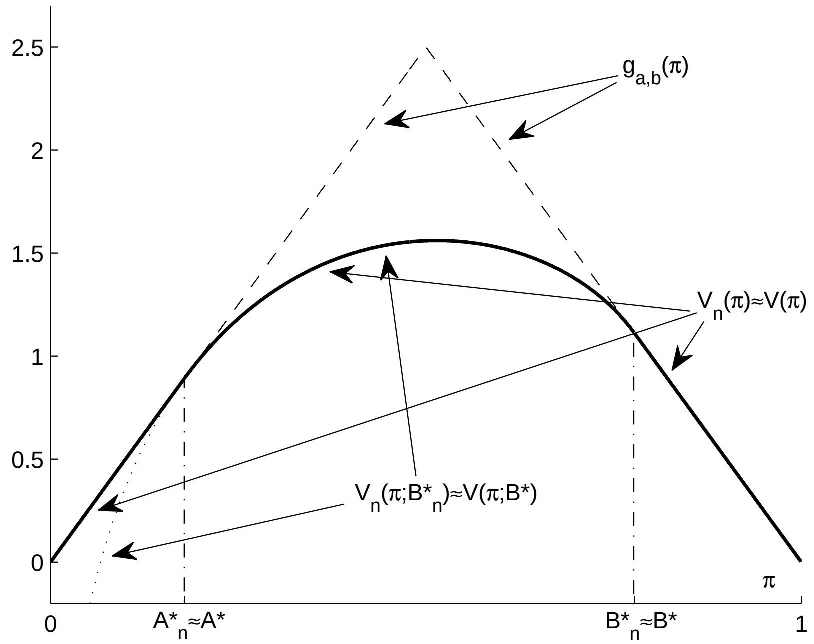

Figure 8.4: A computer drawing of the map $\pi \mapsto V_n(\pi)$ (bold curve), as defined by (8.4.43), with $n = 12$, for the sequential testing of a Lév process, composed of a Brownian motion and a gamma process. The same parameters of Figure 8.3 have been used. The interval $C = (A^\star_n, B^\star_n) = (0.1783, 0.7777)$ is the numerical continuation region: as soon as the posterior probability process $(\pi_t)_{t\geq 0}$ exits C, it is optimal to stop the observation. We notice that at the two boundaries $A^\star_n \approx A^\star$ and $B^\star_n \approx B^\star$ the smooth fit principle holds, according to (8.3.22) and (8.3.24).

Assuming without loss of generality that $\lambda_0 < \lambda_1$, we observe that $S(\pi)$, defined by

$$S(\pi) = \frac{\lambda_1 \pi}{\lambda_1 \pi + \lambda_0(1-\pi)}, \tag{8.5.3}$$

is increasing. Following the reasoning of Peskir and Shiryaev [65, p. 844], fix $B > c$ and define the intervals $I_n = (B_n, B_{n-1}]$, $n \geq 1$, where $\cdots < B_2 < B_1 < B_0 =: B$ are obtained so that $S(B_n) = B_{n-1}$. Then,

$$B_n = \frac{\lambda_0^n B}{\lambda_0^n B + \lambda_1^n (1-B)}, \quad n = 0, 1, \ldots \tag{8.5.4}$$

and one can notice that $\pi \in I_n$ only if $d(\pi; B) = n$, being

$$d(\pi;B) = 1 + \left\lfloor \log\left(\frac{B}{1-B}\frac{1-\pi}{\pi}\right) \Big/ \log\left(\frac{\lambda_1}{\lambda_0}\right)\right\rfloor, \quad \pi \in I_B = (0, B], \tag{8.5.5}$$

where $\lfloor x \rfloor$ is the smallest integer greater than or equal to x. Let $\pi \mapsto V(\pi; B)$ be the map solving (8.3.18), (8.3.19), (8.3.23) and (8.3.24), for $\pi \in I_B = (0, B]$, that is, let $V(\pi; B)$ satisfy the boundary value problem (8.4.17)-(8.4.18), for the operator $\mathbb{L}_B$ defined by

$$\begin{aligned}(\mathbb{L}_B f)(\pi) =& (\mathbb{L}^c f)(\pi) - (\lambda_1 - \lambda_0) f'(\pi)\pi(1-\pi) + \big(\lambda_1\pi + \lambda_0(1-\pi)\big) \\ &\times \left\{ \Big[b(1 - S(\pi))\mathbf{1}_{\{d(\pi,B)=1\}} + f(S(\pi))\mathbf{1}_{\{d(\pi,B)>1\}} \Big] - f(\pi) \right\},\end{aligned} \tag{8.5.6}$$

with $\mathbb{L}^c$ given by (8.3.2). The next proposition proves the uniqueness of $V(\pi; B)$.

Proposition 8.5.1 *For a fixed $B > c$, the boundary value problem* (8.4.17)-(8.4.18)*, defined through* (8.5.6)*, has a unique continuously differentiable solution $V(\pi; B)$, $\pi \in I_B$.*

Proof. Let $\pi \in I_1$, that is $d(\pi; B) = 1$. From (8.4.17) and (8.5.6) one has

$$V''(\pi; B) - \frac{(\lambda_1 - \lambda_0)}{k\pi(1-\pi)} V'(\pi; B) - \frac{\lambda_1 \pi + \lambda_0(1-\pi)}{k\pi^2(1-\pi)^2} V(\pi; B) + \frac{b\lambda_0(1-\pi)+1}{k\pi^2(1-\pi)^2} = 0, \qquad (8.5.7)$$

with $k = (\widetilde{\gamma}_1 - \widetilde{\gamma}_0)^2/(2\sigma^2)$. According to the existence and uniqueness theorem for second-order linear differential equations, we observe that (8.5.7), along with the initial conditions (8.4.18), has a unique solution $\pi \mapsto V(\pi; B)$ on I_1. Move further and consider $\pi \in I_2$, that is, $d(\pi; B) = 2$. Then, (8.4.17), (8.5.6) and the use of the solution found on I_1 define a second-order linear differential equation, that, by imposing a continuity and differentiability condition at B_1, has a unique solution $\pi \mapsto V(\pi; B)$ over I_2. Move now on the generic interval I_n, that is, $d(\pi; B) = n$; (8.4.17), (8.5.6) and the solution found on I_{n-1} bear a second-order linear differential equation, which has a unique solution $\pi \mapsto V(\pi; B)$ on I_n, once a continuity and differentiability condition at B_{n-1} are given. This completes the proof. ∎

Proposition 8.5.1 and the Waierstrass approximation theorem ensure that $V_n(\pi; B)$, defined in (8.4.32) and solving (8.4.33)-(8.4.34) for the operator (8.5.6), converges uniformly to $V(\pi; B)$ on any compact interval, as $n \to \infty$. Repeating the same procedure of Subsection 8.4.2, we can identify precise neighborhoods of the optimal boundaries $A^\star$ and $B^\star$ satisfying (8.3.27), by solving (8.4.33)-(8.4.34) for different values of B. The function $V(\pi; B^\star)$ and $A^\star$ and $B^\star$ are then approximated by $V_n(\pi; B_n^\star)$ (given in (8.4.35)), $A_n^\star$ and $B_n^\star$, which are numericaly computed by solving the system of non-linear equations (8.4.36)-(8.4.40), defined through (8.5.6). The value function $V_n(\pi) \approx V(\pi)$ is finally given by (8.4.43).

We observe that the condition $(\mathbb{L}^j V(\pi)) \geq -1$, $\pi \in [0, A] \cup [B, 1]$, used in Theorem 8.3.1 for proving the optimality of the free-boundary problem solution, was shown by Peskir and Shiryaev [65, pp. 848-849].

8.5.2 The Wiener-compound Poisson with exponential jumps process

Let X be a Lévy process with triplet $g_\vartheta = \{\gamma_\vartheta, \sigma^2, v_\vartheta(dx) = e^{-\lambda_\vartheta x}\mathbf{1}_{(0,\infty)}(dx)\}$, $\lambda_\vartheta > 0$, that is, X can be decomposed into the sum of the two independent processes X^c, with $g_\vartheta^c = \{\widetilde{\gamma}_\vartheta, \sigma^2, 0\}$ and

$$\widetilde{\gamma}_\vartheta = \gamma_\vartheta + \frac{1}{\lambda_\vartheta} e^{-\lambda_\vartheta} + \frac{1}{\lambda_\vartheta^2}\left(e^{-\lambda_\vartheta} - 1\right), \qquad (8.5.8)$$

and X^j, identified by $\widetilde{g}_\vartheta^j = \{0, 0, v_\vartheta(dx) = e^{-\lambda_\vartheta x}\mathbf{1}_{(0,\infty)}(dx)\}$. X^j is therefore a compound Poisson process, whose intensity is $1/\lambda_\vartheta$ and whose jumps have exponential distribution of parameter λ_ϑ. The likelihood ratio (8.2.10) takes the form

$$\varphi_t = \exp\left\{\frac{\widetilde{\gamma}_1 - \widetilde{\gamma}_0}{\sigma^2}\left(X_t^c - t\frac{\widetilde{\gamma}_1 + \widetilde{\gamma}_0}{2}\right) + (\lambda_0 - \lambda_1)X_t^j - t\left(\frac{\lambda_0 - \lambda_1}{\lambda_0 \lambda_1}\right)\right\} \qquad (8.5.9)$$

and the infinitesimal generator of $(\pi_t)_{t\geq 0}$ is provided by (8.3.2) and (8.3.3), with

$$\begin{aligned}(\mathbb{L}^j f)(\pi) =& - f'(\pi)\pi(1-\pi)\frac{\lambda_0-\lambda_1}{\lambda_0\lambda_1} - f(\pi)\left(\frac{\pi}{\lambda_1}+\frac{1-\pi}{\lambda_0}\right)\\ &+ \int_0^\infty f\left(\frac{\pi e^{-\lambda_1 x}}{\pi e^{-\lambda_1 x}+(1-\pi)e^{-\lambda_0 x}}\right)\left(\pi e^{-\lambda_1 x}+(1-\pi)e^{-\lambda_0 x}\right)dx. \qquad (8.5.10)\end{aligned}$$

Under the assumption $\lambda_0 > \lambda_1$, fix $B > c$ and define the quantities $S_\lambda(\pi, x)$ and $d_\lambda(\pi; B)$ as in (8.4.15) and (8.4.16), with λ_0 and λ_1 in place of α_0 and α_1. Denote by $\pi \mapsto V(\pi; B)$ the map solving (8.3.18), (8.3.19), (8.3.23) (8.3.24) on $I_B = (0, B]$; then, $V(\pi; B)$ satisfies (8.4.17)-(8.4.18), defined through the integro-differential opeartor

$$\begin{aligned}(\mathbb{L}_B f)(\pi) =& (\mathbb{L}^c f)(\pi) - \frac{\lambda_0-\lambda_1}{\lambda_0\lambda_1} f'(\pi)\pi(1-\pi)\\ &- f(\pi)\left(\frac{\pi}{\lambda_1}+\frac{1-\pi}{\lambda_0}\right) + \frac{b(1-\pi)}{\lambda_0}\left(\frac{1-\pi}{\pi}\frac{B}{1-B}\right)^{-\frac{\lambda_0}{\lambda_0-\lambda_1}}\\ &+ \int_0^{d_\lambda(\pi;B)} f\big(S_\lambda(\pi,x)\big)\left(\pi e^{-\lambda_1 x}+(1-\pi)e^{-\lambda_0 x}\right)dx, \qquad (8.5.11)\end{aligned}$$

where $\mathbb{L}^c$ is provided by (8.3.2). The uniqueness of $V(\pi; B)$ is stated in the next proposition.

Proposition 8.5.2 *For a fixed $B > c$, the boundary value problem* (8.4.17)-(8.4.18)*, defined through* (8.5.11)*, has a unique twice continuously differentiable solution $V(\pi; B)$, $\pi \in I_B$.*

Proof. Let $f(y) = V(\pi; B)$, with $\pi = e^y/(1+e^y)$; denote by $B^o = \log(B/(1-B))$, $k = (\widetilde{\gamma}_1 - \widetilde{\gamma}_0)^2/(2\sigma^2)$ and $\rho = \lambda_0/(\lambda_0 - \lambda_1)$. Then, (8.4.17)-(8.4.18) and (8.5.11) can equivalently be written as

$$f''(y) = g(y) + h_1(y)f'(y) + h_2(y)f(y) + \int_y^{B^o} k(y,z)f(z)\,dz, \qquad (8.5.12)$$

$$f(B^o) = \frac{b}{1+e^{B^o}}, \quad f'(B^o_-) = -\frac{be^{B^o}}{(1+e^{B^o})^2}. \qquad (8.5.13)$$

where

$$g(y) = -\frac{1}{k}\left(1+\frac{b}{(1+e^y)\lambda_0}\left(\frac{B}{1-B}e^{-y}\right)^{-\rho}\right), \qquad (8.5.14)$$

$$h_1(y) = \frac{1-e^y}{1+e^y} + \frac{\lambda_0-\lambda_1}{k\lambda_0\lambda_1}, \qquad (8.5.15)$$

$$h_2(y) = \frac{1}{k(1+e^y)}\left(\frac{1}{\lambda_0}+\frac{e^y}{\lambda_1}\right), \qquad (8.5.16)$$

$$k(y,z) = -\frac{e^{\rho y}}{k(1+e^y)(\lambda_0-\lambda_1)}\frac{1+e^z}{e^{\rho z}}. \qquad (8.5.17)$$

We turned (8.4.17) and (8.5.11) into a linear Volterra integro-differential equation, that, along with the two initial conditions (8.5.13) and according to Linz [53, exercise 3.19, p. 50], has a unique twice continuously differentiable solution $f(y)$, $y \leq B^o$. Because of the equivalence between $f(y)$ and $V(\pi; B)$, we obtain the desired result. ∎

This proposition justifies the application of the procedure described in Subsection 8.4.2 and at the end of Subsection 8.5.1 to numerically computing the solution of the free-boundary problem (8.3.18)-(8.3.24). The condition $(\mathbb{L}^j V)(\pi) \geq -1$, $\pi \in [0, A] \cup [B, 1]$, arises from the results derived by Gapeev [32].

8.5.3 The Wiener-negative binomial process

Let X be a Lévy process, characterized by the triplet $g_\vartheta = \{\gamma_\vartheta, \sigma^2, v_\vartheta(x) = (1 - p_\vartheta)^x / x\}$, where $p_\vartheta \in (0, 1)$ and $x = 1, 2, \ldots$ Then X can be recognized as the sum of the independent processes X^c, with $g^c_\vartheta = \{\widetilde{\gamma}_\vartheta, \sigma^2, 0\}$, being, from (8.2.3), $\widetilde{\gamma}_\vartheta = \gamma_\vartheta - (1 - p_\vartheta)$, and X^j, with $\widetilde{g}_\vartheta = \{0, 0, v_\vartheta(x) = (1 - p_\vartheta)^x / x\}$, $x = 1, 2, \ldots$ X^j is also known as negative binomial process: it is a compound Poisson process, whose intensity is $-\log p_\vartheta$ and whose jumps have logarithmic distribution $\phi(\{x\})$, $x = 1, 2, \ldots$, of parameter p_ϑ, given by $\phi(\{x\}) = -(1 - p_\vartheta)^x / (x \log p_\vartheta)$. A detailed description of this process can be found in Kozubowski and Podgórski [46].

According to (8.2.10), the likelihood ratio becomes

$$\varphi_t = \exp\left\{ \frac{\widetilde{\gamma}_1 - \widetilde{\gamma}_0}{\sigma^2} \left(X^c_t - t \frac{\widetilde{\gamma}_1 + \widetilde{\gamma}_0}{2} \right) + X^j_t \log\left(\frac{q_1}{q_0}\right) - t \log\left(\frac{p_0}{p_1}\right) \right\}, \tag{8.5.18}$$

where $q_i = 1 - p_i$, $i = 0, 1$. The infinitesimal generator of $(\pi_t)_{t \geq 0}$ is expressed by (8.3.2) and (8.3.3), with $\mathbb{L}^j$ taking the form

$$\begin{aligned}\big(\mathbb{L}^j f\big)(\pi) =& f'(\pi)\pi(1-\pi)\log\left(\frac{p_1}{p_0}\right) + f(\pi)\big((1-\pi)\log p_0 + \pi \log p_1\big) \\ &+ \sum_{x=1}^{\infty} f\left(\frac{\pi q_1^x}{\pi q_1^x + (1-\pi) q_0^x}\right) \frac{(\pi q_1^x + (1-\pi) q_0^x)}{x}.\end{aligned} \tag{8.5.19}$$

Assume that $p_0 > p_1$, fix $B > c$ and define the function

$$S_p(\pi, x) = \frac{\pi q_1^x}{\pi q_1^x + (1-\pi) q_0^x}, \quad \pi \in I_B = (0, B], \quad x = 1, 2, \ldots \tag{8.5.20}$$

Using the same arguments as in Subsection 5.6.3 (or Buonaguidi and Muliere [16, Subsec. 6.3]), determine the sequence of points $\cdots < B_2 < B_1 < B_0 =: B$, so that $S_p(B_{n-1}, 1) = B_n$, $n \geq 1$. Then, we have

$$B_n = \frac{q_0^n B}{q_0^n B + q_1^n (1 - B)}, \quad n = 0, 1, \ldots \tag{8.5.21}$$

Let $I_n = (B_n, B_{n-1}]$, $n \geq 1$, and define $d_p(\pi; B)$, $\pi \in I_B$, as in (8.5.5), with λ_1 and λ_0 replaced by q_1 and q_0, respectively. One may observe that $\pi \in I_n$ only if $d_p(\pi; B) = n$.

Denote by $\pi \mapsto V(\pi; B)$ the map solving (8.3.18), (8.3.19), (8.3.23) and (8.3.24) on I_B, that is, from the above construction, $\pi \mapsto V(\pi; B)$ solves the free-boundary problem (8.4.17)-(8.4.18) for the operator $\mathbb{L}_B$, given by

$$(\mathbb{L}_B f)(\pi) = (\mathbb{L}^c f)(\pi) - \log\left(\frac{p_0}{p_1}\right) f'(\pi)\pi(1-\pi) + f(\pi)\big((1-\pi)\log p_0 + \pi \log p_1\big) + \sum_{x=1}^{d_p(\pi;B)-1} \left(f\left(S_p(\pi;x)\right) \frac{\pi q_1^x + (1-\pi) q_0^x}{x} \right) - b(1-\pi)\left(\log p_0 + \sum_{x=1}^{d_p(\pi;B)-1} \frac{q_0^x}{x}\right), \tag{8.5.22}$$

with $\mathbb{L}^c$ expressed by (8.3.2). The next proposition shows the uniqueness of $\pi \mapsto V(\pi; B)$.

Proposition 8.5.3 *For a given $B > c$, the boundary value problem* (8.4.17)-(8.4.18)*, defined by means of* (8.5.22)*, has a unique continuously differentiable solution $V(\pi; B)$, $\pi \in I_B$.*

Proof. Define $k = (\widetilde{\gamma}_1 - \widetilde{\gamma}_0)^2/(2\sigma^2)$ and consider $\pi \in I_1$, that is, $d_p(\pi; B) = 1$. Then (8.4.17) and (8.5.22) lead to

$$V''(\pi;B) - \frac{\log(p_0/p_1)}{k\pi(1-\pi)} V'(\pi;B) + \frac{(1-\pi)\log p_0 + \pi \log p_1}{k\pi^2(1-\pi)^2} V(\pi;B) + \frac{1 - b\log p_0(1-\pi)}{k\pi^2(1-\pi)^2} = 0. \tag{8.5.23}$$

The initial conditions (8.4.18) and the existence and uniqueness theorem for second-order linear differential equations guarantee that (8.5.23) has a unique solution $\pi \mapsto V(\pi; B)$ on I_1. Let now $\pi \in I_2$, that is, $d_p(\pi; B) = 2$: (8.4.17) and (8.5.22), upon using the solution found on I_1, define a second-order linear differential equation that, by imposing a continuity and differentiability condition at B_1, has a unique solution $\pi \mapsto V(\pi; B)$ on I_2. In general, if we choose $\pi \in I_n$, that is, $d_p(\pi; B) = n$, we observe that (8.4.17), (8.5.22) and the use of the solutions found on $I_{n-1}, \ldots, I_1$ generate a second-order linear differential equation that, along with a continuity and differentiability condition at B_{n-1}, has a unique solution $\pi \mapsto V(\pi; B)$ over I_n, $n \geq 1$. Therefore, $\pi \mapsto V(\pi; B)$, $\pi \in I_B$, is unique. ∎

This result allows us to apply the collocation method and the scheme described in Subsection 8.4.1 and at the end of Subsection 8.5.1, in order to obtain a numerical solution of the free-boundary problem (8.3.18)-(8.3.24). The condition $(\mathbb{L}^j V)(\pi) \geq -1$, $\pi \in [0, A] \cup [B, 1]$, was proved in Subsection 5.6.4 (or Buonaguidi and Muliere [16, Subsection 6.4]).

Remark 8.5.1 Let $V_c(\pi)$ and $V_j(\pi)$ be the value functions defined as in (8.2.8) or (8.2.9), associated to the problem of sequentially testing the two simple hypotheses (8.2.7) for the Lévy-Khintchine triplet of X^c and X^j, respectively. As we can see from Figure 8.5 below, the value function $V(\pi)$ (bold solide line), relative to the sequential testing of the triplet of $X = X^c + X^j$, is always dominated by $V_c(\pi)$ (bold dotted line) and $V_j(\pi)$ (bold dashed line), that is, $V(\pi) \leq V_c(\pi)$ and $V(\pi) \leq V_j(\pi)$, $\pi \in [0, 1]$. This fact can be intuitively explained: the observation of the two components X^c and X^j allows us to have a bigger amount of information about the true realization of ϑ and this reduces the total risk of the decisional process (for a

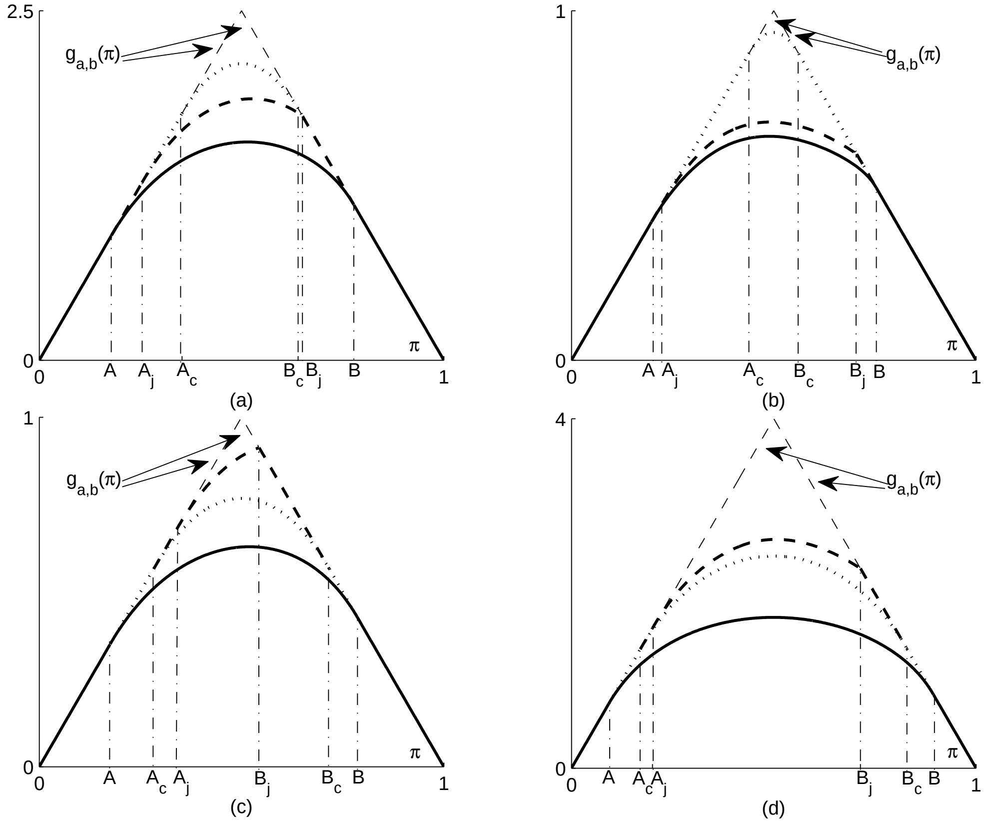

Figure 8.5: Computer drawings of the functions $V(\pi)$ (bold solid line), $V_c(\pi)$ (bold dotted line) and $V_j(\pi)$ (bold dashed line). The optimal boundaries A and B, A_c and B_c, A_j and B_j associated to $V(\pi)$, $V_c(\pi)$ and $V_j(\pi)$, respectively, are shown. a) X^j is a gamma process; we set $a = b = 5$, $\widetilde{\gamma}_0 = 2$, $\widetilde{\gamma}_1 = 3$, $\sigma^2 = 1$, $\alpha_0 = 5$ and $\alpha_1 = 1$. We have $(A, B) = (0.1783, 0.7777)$, $(A_j, B_j) = (0.2541, 0.6510)$ and $(A_c, B_c) = (0.3529, 0.6471)$. b) X^j is a Poisson process; we set $a = b = 2$, $\widetilde{\gamma}_0 = 2$, $\widetilde{\gamma}_1 = 3$, $\sigma^2 = 1$, $\lambda_0 = 1$ and $\lambda_1 = 5$. It results $(A, B) = (0.1981, 0.7532)$, $(A_j, B_j) = (0.2253, 0.7050)$ and $(A_c, B_c) = (0.4381, 0.5619)$. c) X^j is a compund Poisson process with exponential jumps; we set $a = b = 2$, $\widetilde{\gamma}_0 = 2$, $\widetilde{\gamma}_1 = 4$, $\sigma^2 = 1$, $\lambda_0 = 3$ and $\lambda_1 = 0.5$. We obtain $(A, B) = (0.1757, 0.7858)$, $(A_c, B_c) = (0.2829, 0.7171)$ and $(A_j, B_j) = (0.3396, 0.5435)$. d) X^j is a negative binomial process; we set $a = b = 8$, $\widetilde{\gamma}_0 = 2$, $\widetilde{\gamma}_1 = 3.4$, $\sigma^2 = 1$, $p_0 = 0.8$ and $p_1 = 0.3$. We have $(A, B) = (0.0922, 0.8949)$, $(A_c, B_c) = (0.1702, 0.8298)$ and $(A_j, B_j) = (0.2004, 0.7142)$.

related discussion, see also Dayanik and Sezer [25, Sec. 6]).
Another interesting fact concerns the analytic properties of $V(\pi)$, $V_c(\pi)$ and $V_j(\pi)$. Indeed, one can notice that the maps $\pi \mapsto V(\pi)$ and $\pi \mapsto V_c(\pi)$ are C^1 at the boundaries of their continuation regions (A, B) and (A_c, B_c), respectively, while $\pi \mapsto V_j(\pi)$ is C^1 at the lower boundary A_j, but just C^0 at B_j. This different behaviour is due to the presence of the Brownian motion component in the first two cases, implying that the associated posterior probability processes immediately enter the stopping regions, if they start at one of the two boundaries. Instead in our above

examples, it is easy to see that the posterior probability process of X^j moves in the following way: it creeps downwards and jumps upwards, so that if it starts at A_j, it immediately enters the stopping region, but if it starts at B_j, it initially remains in the continuation set. Then, $\pi \mapsto V_j(\pi)$ is continuous but not differentiable at B_j.

8.6 Conclusions

The Bayesian formulation of the problem of sequential testing two simple hypotheses about the Lévy-Khintchine triplet of a Lévy process, showing diffusion component and jump component of finite variation, has been analyzed.

First, we derived the infinitesimal generator of the posterior probability process and we proved the differentiability of the value function at the boundaries of the continuation region. Hence, the initial optimal stopping problem was reduced to a free-boundary Stephan problem and we verified that its solution coincides with that of the optimal stopping problem.

Second, we proposed a numerical method for solving the free-boundary problem, since we observed that determining its solution is a hard task, because of the second order integro-differential equation to be solved on the continuation set. Our scheme was basically developed in three steps: verification of the uniqueness and continuity of the solution of the integro-differential equation; identification of neighborhoods of the optimal boundary points by means of the use of the collocation method in a boundary value problem; computation of the approximating value function and of the optimal boundaries, through the solution of a system of non-linear equations. Our procedure was discussed in details for four types of Lévy processes with diffusion and jump components.

We observe that in the “variational” or ”fixed error probability” formulation, the sequential testing of two simple hypotheses about the distributional features of a Lévy process with diffusion and jump components remains an open problem. Indeed, even though some results were obtained by Dvoretzky et al. [28], Peskir and Shiryaev [65, sec. 3], Gapeev [32] and in Chapter 6 (or see Buonaguidi and Muliere [17]), the diffusion and jump components of the observed Lévy process have never been jointly analyzed in that context.

Chapter 9

Conclusions

In this thesis, we studied the problems of sequential detection and sequential testing of two simple hypotheses for continuous time strong Markov processes. Their solutions were obtained by appropriately reducing the original optimal stopping problem to a free-boundary problem. Now we summarize the results presented in Chapters 3-8 and we discuss some possible lines of future research.

In Chapter 3, we dealt with the problems of sequential detection and sequential testing for the drift of a time-homogeneous diffusion process. We considered the special case of constant signal-to-noise ratio function. We have shown that the infinitesimal generators of the posterior probability processes related to the appearance of the disorder time and to the propriety of the alternative hypothesis coincide with the ones obtained by Shiryaev [72, Sec. 4.2]. Then, we apply his technique and the smooth fit principle in order to construct and solve the associated free-boundary problems.

In Chapter 4, we analyzed the deep connection between the martingale and free-boundary approaches in the context of sequential detection with exponential penalty for the delay. First we solved the problem of Beibel [9], by exploiting the free-boundary approach technique and then, in Theorem 4.5.1, we showed that the latter naturally entails the Beibel's decomposition of the reward function into the product between a gain function of the sufficient statistic of the problem and a martingale over the continuation region. We believe that this intimate relationship between the two approaches could be further investigated in the case where the assumption of constant signal-to-noise ratio function is weakened.

In Chapter 5, we studied the optimal stopping problem associated to the Bayesian formulation of sequential testing of two simple hypotheses for the Lévy-Khintchine triplet of a Lévy process. Our main contribution has been to extend the results of Peskir and Shiryaev [65, Sec. 2] to the class of pure increasing jump Lévy processes. In particular we made use of the continuous fit principle introduced by Peskir and Shiryaev [65], in order to characterize the optimal stopping boundaries; in Theorem 5.5.1, we showed the conditions under which the solution of the free-boundary problem coincides with that of the optimal stopping problem. Then, we provided the explicit solution of the Bayesian sequential testing for a negative binomial process, by solving the associated free-boundary problem for an integro-differential operator.

In Chapter 6, we analyzed the fixed error probability or variational problem of sequential testing for Lévy processes. Making use of the powerful theory of Markov processes, we employed a rather general method for the determination of the stopping boundaries, the expected length and the set of admissible error probabilities of the associated sequential probability ratio test. The proposed technique was applied to the sequential testing for the drift of a Wiener process, for which, in addition to the results obtained by Shiryaev [72, sec. 4.2], we derived the explicit expression of the moment generating function of the optimal stopping time. Then, we extended the results of Dvoretzky et al. [28] and Peskir and Shiryaev [65, Sec. 3] to the class of pure increasing jump Lévy processes and we determined the explicit expression of the sequential probability ratio test for a negative binomial process.

In Chapter 7, we solved a problem that was opened at the end of Chapter 5 (see also Buonaguidi and Muliere [16, p. 69]), that is, the Bayesian sequential testing for a gamma process. The initial optimal stopping problem was turned into a free-boundary problem for an integro-differential operator. Due to the sample paths of a gamma process, we saw that the smooth fit principle breaks down at one boundary point, which is uniquely characterized by the continuous fit. Although the explicit solution of the integro-differential equation satisfied by the value function on the continuation set is hard to obtain, we proved the conditions under which the free-boundary problem has a unique solution. In this case, under some general assumptions, the solution of the free-boundary problem equals the one of the optimal stopping problem. In the second part of the chapter, we proposed a numerical scheme, based on the collocation method, for obtaining an approximated solution of the free-boundary problem with any degree of accuracy and we verified its very good performances in problems where explicit solutions are available.

In Chapter 8, we completed the analysis begun in Chapter 5: we concentrated on the sequential testing for Lévy processes presenting both diffusion and jump components. The free-boundary problem associated to the initial optimal stopping problem is now characterized by a second order integro-differential operator and by the smooth fit principle, which holds on both the two boundary points of the continuation region. These facts are due to the presence of the Wiener process, which entails the appearance of the second derivative in the infinitesimal generator and makes the boundary points regular for the stopping region. In this situation, the free-boundary problem always admits a solution, that, as shown in Theorem 8.3.1, coincides, under general conditions, with that of the original optimal stopping problem. The huge complexity of the free-boundary problem led us to apply, by making the appropriate modifications, the numerical scheme proposed in the previous chapter, in order to compute numerically the free-boundary problem solution. We illustrated the method through its application to several types of Lévy processes with diffusion and jump components.

We underline that the techniques used for the sequential testing of Lévy processes in Chapters 5, 7 and 8 can be exploited in order to derive interesting results about the problem of sequential detection for the Lévy-Khintchine triplet of the same class of Lévy processes we treated in this thesis.

Further, the numerical scheme devised in Chapters 7 and 8 is flexible enough for being adapted to other problems of optimal stopping, whose solution through the free-boundary formulation cannot be explicitly recovered.

In the context of sequential analysis, future lines of research could concern:

- the sequential testing, in the fixed error probability formulation, of a gamma process and a Lévy process with diffusion and jump components;
- sequential testing problems of multiple simple hypotheses for the Lévy-Khintchine triplet of a Lévy process;
- problems of sequential testing and sequential detection for the class of α-stable Lévy processes;
- sequential testing and detection problems for multivariate Lévy processes.

We conclude by briefly discussing a couple of statistical problems that could arise from the reading of this thesis and that could represent a hint for future studies. One of the main assumptions our analysis relies on is the knowledge of the distributional properties that the observed ongoing process has under the two hypotheses to be tested (sequential testing) or before and after the occurrence of the disorder time (sequential detection).

In practical situations this assumption does not always hold, because of, for example, the lack of a sufficient amount of information. In this case, the issue to be addressed is about the "goodness of fit" of the proposed procedures: what if the assumptions on the process are wrong? Are our stopping rules close to be optimal? How do they perform for low-intensity processes?

The second issue concerns all the situations where, in addition to the time, some covariates are available. In this framework, it appears to be rather natural to link the mean parameter of the ongoing process (like the drift of a Brownian motion or the intensity of a Poisson process) to a function of a linear combination of the covariates and to sequentially estimate the associated coefficients. An improvement of the sequential procedures should therefore be expected: the presence of covariates should lead to a reduction of the uncertainty and, thus, to anticipate the correct stoppage of the process. We believe that the problem of devising optimal stopping rules when some covariates are given deserves a deep investigation.

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