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Abstract

The thesis is comprised of three loosely related essays. All essays consider the interplay between broader monetary policy, and a subset of the fixed income term structure. The conditioning effect of institutional demand on monetary policy actions is considered in the second and third essays.

The first essay is an empirical study of the gradual Fed monetary policy reaction function where the policy instrument is the shortest available instrument on the interest rate term structure, the Federal Funds Rate. The study suggests that the inertia in the Fed reaction function is due to intentional higher-order smoothing. The persistency in the data cannot be explained by variables omitted from simple Taylor rules such as time-varying equilibrium inflation and real rate targets. Furthermore, term-structure evidence indicates that since the mid-1990's the partial adjustment mechanism is anticipated by market participants.

In the second essay I construct an equilibrium model of the interest rate term structure. I consider the impact of a deep interest rate derivative market as well as institutional demand features on the cost of debt and broader monetary policy. The key result is that frictions such as bond-swap basis, limited access, and maturity specific risk are critical for outright monetary transactions to work, else any term specific bond purchases or sales can be undone by arbitrageurs with access to the derivative market.

The third essay describes an equilibrium model of the term structure of credit spreads with term-specific credit financing demand. An analytical solution for equilibrium yields is derived for the economy with long-term investors. It illustrates how an institutional framework imposing a strict capital regime and focussing on mark-to-market valuation can contribute to the lack of long term credit provision and credit market volatility. Conversely, incentivising investors to optimize over a longer time-horizon can lower the equilibrium cost of long-term financing and increase its equilibrium supply.

Chapter 1

US short rates: A macro-finance framework to address the persistency puzzle

1.1 Introduction

The question of whether the Federal Reserve adopts a policy of interest rate gradualism has not been answered conclusively. The persistency in the Federal Funds Rate (FFR) time series is well-documented (Sack (1998); Clarida et al. (1998); Lowe and Ellis (1997)). Some economists (Clarida et al. (2000); Levin et al. (1999); Sack (2000); Coibion and Gorodnichenko (2011)) interpret this persistence as evidence that the Federal Reserve implements a partial adjustment mechanism of which the simplest form is $i_t = (1-\rho)i_t^* + \rho i_{t-1}$ where i_t is the actual policy rate, i_t^* is the target policy rate and ρ is the degree of policy inertia. Others (Rudebusch (2002); Poole (2003)) attribute the persistence to a series of serially correlated shocks not captured by ordinary Taylor rules. Shocks of the form $u_t = \rho u_{t-1} + \epsilon_t$, where ϵ_t are $i.i.d$, endogenous to the policymaker's target rate i_t^* but exogenous to the econometrician's policy function can lead the econometrician to conclude falsely that the central bank implements an inertial[1] policy. I construct a framework in which the two opposing theories can be tested. The framework augments real-time data models of the FFR in the macro literature with innovations in the short-rate finance lit-

[1] inertial, gradual and smooth are used interchangeably throughout the paper.

erature. Tests using this macro-finance approach show that the Fed did adopt a gradual policy in the post-1980 period and provide strong evidence for time varying equilibrium inflation and real rates. I also revisit the term structure evidence against smoothing and find that since the mid-1990's the market has priced in a gradual Fed reaction function. However neither the macro-finance factors nor the partial adjustment mechanism are able to account fully for the significant heteroskedasticity of short rates first identified in Chan et al. (1992).

The theoretical rationale for an inertial monetary policy function has been discussed in the literature (Clarida et al. (2000); Woodford (1999); Goodfriend (1991)). A central bank can influence policy both through its current actions as well as altering private sector expectations of its future actions. Woodford (1999) shows that when the universe of possible central bank actions is expanded to include a commitment to future actions, the value of the social welfare function corresponding to the optimal policy is improved relative to the optimal discretionary policy. The improvement is achieved not because a smoothing motive appears explicitly in the social welfare function (it does not[2]), but because of the expectations channel of monetary policy. Central banks that commit credibly to a path of future policy actions influence market expectations of future short term rates. The change in expectations of future short term rates in turn affects aggregate demand and output beyond the spot effect. Thus a similar outcome to a fully discretionary policy can be achieved under credible commitment, but with significantly lower short term interest rate volatility. Rudebusch (2006, 2002) acknowledges the validity of the expectations channel in principle but questions its empirical relevance. He argues that the joint likelihood of the Fed having a fully credible policy and market participants being sufficiently forward looking is low historically. The debate is ongoing, an actuality acknowledged by Bernanke in a Federal Reserve speech (Bernanke (2004)).

[2]Variance of the *changes* in the level of interest rates does not appear in the social loss function of Woodford (1999), although it does appear in Rudebusch and Svensson (1999). The loss function in Woodford (1999) is given by: $W = E_0(\sum_{t=0}^{\infty} \beta^t L_t)$, where $L_t = \pi_t^2 + \lambda_x(x_t - x^*)^2 + \lambda_r(r_t - r^*)^2$, for equilibrium levels x^* of the output gap and r^* of the nominal interest rate. π_t, x_t and r_t represent the time t value of inflation, output gap and the nominal interest rate respectively. λ_x and λ_r are penalty scalars applied to the variation in levels of the output gap and nominal interest rates respectively, and β is the intertemporal discount factor.

Whether or not the Fed did indeed smooth interest rates is therefore purely an empirical question to which simple nested versions of the Taylor rules have been unable to respond conclusively.[3] The question is relevant both for policymakers and market agents. Policy inertia affects policymakers' ability to properly estimate their model parameters. The conditional response of an economy to an exogenuous shock will be affected by whether the central bank is smoothing interest rate changes over time. Equally, market agents will be eager to understand Fed policy in response to shocks so as to better forecast the effect of monetary policy on their market variables of interest.[4]

The macroeconomic and finance literature approach the empirical modelling of the short term interest rate in remarkably different ways. The difference in approach can be attributed to differing objectives. The macroeconomic literature relates changes in the FFR with contemporaneous macroeconomic factors. The emphasis is on describing a linear relationship between the policy rate and structural variables, rather than a characterisation of the full (conditional or unconditional) distribution of changes in interest rates. Instead the finance literature focusses on models that price consistently securities whose payoff depends on the evolution of the short rate.[5] This requires analytically tractable models that are sufficiently flexible to capture the full forward distribution of the short rates. Despite the different approaches, there is no theoretical reason why cyclical macroeconomic models cannot be improved through stylized findings from the finance literature. In particular Taylor models can be improved by including proxies for time-varying equilibrium rates of inflation and real rates, as well as a characterisation of the second moment of interest rate changes.

[3]Griliches (1967) discusses the identification problem in distinguishing between serial correlation and a partial adjustment mechanism in a time series. Figure 1.4 shows that two simple extensions to the Taylor rule provide a reasonable ex-post fit.

[4]The current economic regime of exceptionally low nominal interest rates does not negate its relevance. The zero-rate boundary has forced the Fed to implement forward looking policies, whether or not economic agents find them credible. The purpose of the forward looking policies is to stimulate aggregate demand such that the FFR is adjusted upward, not least because this will distance the policy rate from its lower bound. The question of this paper affects the interpretation of the Fed's stance towards how this adjustment would be managed.

[5]Here I intend short rate to mean true "short rate", the rate at which funds deposited with the Fed for the shortest possible period (overnight) are remunerated. In reality short rate models are calibrated typically to the 1 month tbill rate or 1 year treasury yield.

Cieslak and Povala (2011) decompose the term structure of interest rates into three frequencies: a persistent “generational frequency”, a “business cycle” frequency related to transitory monetary policy intervention, and a term premium. In a structural model of the short rate only the first two are relevant. Traditional macroeconomic model factors capture only the business cycle frequency, whereas the focus in the finance literature is on the generational frequency and is often expressed as a time-varying mean rate. Fama (2006), Cieslak and Povala (2011) and Kozicki and Tinsley (2001) amongst others relate the generational frequency to time-varying long term inflation expectations, although the exact functional form of their proxy for inflation expectations differ. Others proxy the time varying mean with factors imputed from the the term structure of interest rates. These factors can be easily observable (the long yield in the model of Schaefer and Schwartz (1984)) or latent (the time varying mean in Balduzzi et al. (1998) or Hull and White (1994))[6].The recent work of Favero et al. (2012) has also related a component of the “generational frequency” to changes in the equilibrium real rate caused by demographic shifts.

It is not clear whether the observed inertia in the time series can be explained by these ’generational frequency’ factors excluded from simple augmented Taylor rules, consistent with the omitted persistent variable argument of Rudebusch (2006), or whether the inertia found in the time series is as a result of an explicit smoothing motive on the part of the Fed. The proposed blended macro-finance model combines a description of the generational frequency together with a description of the business cycle frequency. It allows us to test whether the generational frequency factors excluded from augmented Taylor rules account for the persistency in the time series. Variables describing the business cycle monetary shocks are based on real-time Fed expectations rather than ex-post revisions, which deviate significantly from their real time counterpart and are found to affect significantly estimations of the Fed reaction function (Orphanides (2001, 2003)). The tests show that these factors influence significantly the Fed reaction function in the post 1980 period, but

[6]In fact the spot rate evolution in any model nested by the general framework of Heath et al. (1992) can be expressed as a linear transformation of forward rates.

cannot account for the serial correlation in the time series. The results strongly suggest a Fed smoothing motive when the macro-economic factors are adjusted for real time Fed expectations. They also indicate a higher order Fed smoothing motive as opposed to the often used first order smoothing function.

Another unresolved issue is the significant heteroskedascity in the interest rate time series observed by Chan et al. (1992) and Brenner et al. (1996). It is not obvious whether the heteroskedasticity can be attributed to omitted macro-economic variables in the reduced form finance models, or whether it is caused by some other unspecified mechanism. One possible explanation for the observed heteroskedascity is a higher order smoothing function of the Fed not captured by Markovian short-rate models. Alternatively a 1-lag smoothing function may be sufficient to model the Fed inertial mechanism, but the implicit 1-lag smoothing function used in Markovian short-rate models misspecifies the true Fed reaction function due to the monthly frequency of the data usually employed. I use the blended macro-finance model to test for heteroskedascity in the quarterly FFR time series and find that the heteroskedascity remains even when one accounts for both the generational frequency and macro business cycle factors, as well as a higher order smoothing mechanism. The result suggests that the Fed deviates more (less) from traditional rules when interest rates are high (low), which corresponds to periods of high (low) inflation. It indicates either that policymakers assume the elasticity of the economy to deviations from the augmented Taylor rule to be higher (lower) in a state that corresponds to low (high) interest rates, or simply that other variables not included in the model are assigned a greater weight when interest rates are high.

The rest of the paper proceeds as follows: Section 2) provides a more detailed overview of the different approaches to modelling the short rate in the macroeconomic and finance literature; I consider the main developments in each literature and study their empirical properties in the post-1980 period; in Section 3) I formulate a generalized model and set out the econometric framework for testing; Section 4) describes the data used in the empirical investigation; in Section 5) I review the results of the estimation; in Section 6) I address directly some of the evidence in the literature against smoothing and Section 7) concludes.

1.2 Review of the short rate literature

1.2.1 Macroeconomic models of the short rate

The Taylor rule formulated in Taylor (1993) is widely considered the baseline macroeconomic model of the Fed reaction function. It can be expressed as:

$$i_t = i^* + \bar{\phi}_\pi(\pi_t - \pi^*) + \phi_q(q_t - q^*) \tag{1.1}$$

where i_t is the nominal short term interest rate, and π_t and q_t are the current level of inflation and real output. i^*, π^* and q^* are the baseline nominal interest rate, the target inflation rate and the level of potential real output respectively. $\bar{\phi}_\pi$ and ϕ_q are response parameters to inflation and real output. In his original formulation Taylor set the baseline interest rate i^* equal to the sum of the equilibrium real rate r^* and current inflation π_t. The rule could be recast as:

$$i_t = c + \phi_\pi \pi_t + \phi_x x_t \tag{1.2}$$

where x_t is the real output gap, $\phi_\pi = 1 + \bar{\phi}_\pi$, $\phi_x = \phi_q$, and $c = r^* - \bar{\phi}_\pi \pi^*$ is a scalar.

The simple rule provides an unsatisfactory fit to the FFR over the period 1981-2007 using quarterly data. [7] Estimating the rule using ordinary OLS gives:

$$i_t = \underset{(0.00)}{0.02} + \underset{(0.16)}{1.59}\pi_t - \underset{(0.11)}{0.07}x_t + k_t \tag{1.3}$$

with parameter standard errors in parentheses and k_t is the time t error. The R^2 value is 52%. The inflation coefficient estimate is not far off the initial parameter values $\phi_\pi = 1.5$ suggested by Taylor, but the coefficient on the real output gap is way off. A comparison of the estimated rule and the suggested rule are given in figure 1.1. Also shown in figure 1.1 are the errors k_t, which are heteroskedastic and serially correlated. The distribution

[7] GDP deflator and Real GDP data is from the Department of Commerce: Bureau of Economic Analysis, real potential GDP is obtained from the Congressional Budget Office and the FFR effective rate is from the Federal Reserve website.

of the error terms together with the relatively low R^2 value points to a mispecified model.

Numerous authors have sought to explain the discreprancies between the actual policy rate and the predicted values under the estimated rule. The extensions of the simple model can be classified under three broad categories: the inclusion of additional contemporaneous macro-economic variables, adjustments for real-time data used by the Fed, and inclusion of lagged variables corresponding to a Fed smoothing motive. I provide a brief overview of each extension below.

Common macro-economic variables added to the simple Taylor rule include real GDP growth, exchange rates and other asset prices. Ireland (2004) includes real GDP growth because it is directly observable as opposed to the real output gap, a function of the latent real potential GDP variable. However, including real GDP growth in equation (1.3) does not materially improve the fit of the model (R^2 value is 53%):

$$i_t = \underset{(0.01)}{0.01} + \underset{(0.16)}{1.61}\pi_t - \underset{(0.11)}{0.08}.x_t + \underset{(0.36)}{0.33}dy_t + k_t \tag{1.4}$$

where dy_t denotes real GDP growth. The effect of exchange rates on the Central Bank reaction function has been considered by amongst others Obstfeld and Rogoff (1995) and Chadha et al. (2004). Obstfeld and Rogoff (1995) conclude that exchange rates should not substitute real economic variables in the monetary policy function, and Chadha et al. (2004) find only small effects of changes in exchange rates on the monetary policy reaction function of the Fed. The role of asset prices is emphasised in Cecchetti (2000) and Goodhart and Hofmann (2000). Goodhart and Hofmann (2000) argue that asset prices embed information about future demand, which if not included in the Central Bank reaction function leads to a sub-optimal outcome. The argument is rejected on theoretical grounds by Mishkin (2001) and Bernanke and Gertler (2001). Both acknowledge that asset prices are an important element of the monetary transmission mechanism, but emphasise an explicit asset price target is suboptimal. Chadha et al. (2004) also finds weak evidence of the inclusion of asset price levels in the Fed reaction function.

The second extension category adjusts for the informational problems associated with forecasts of the Taylor rule based on ex-post data. Data is subject to extensive revisions after the end of the time period to which the data relates. Orphanides (2001) and Orphanides (2003) demonstrate that policy reaction functions based on ex-post data can be spurious [8]. Consider a real-time estimate of inflation and the output gap subject to error given by $\pi_t = \pi_{t|t} + \varepsilon_t^{\pi}$ and $x_t = x_{t|t} + \varepsilon_t^{x}$ where $\pi_{t|t}$ and $x_{t|t}$ denote the time t estimate of inflation and output gap based on information at time t. π_t and x_t denote the final ex-post revision of inflation and output gap respectively. The 'true' monetary policy rule will be a function of the estimates of the macro-variables $\pi_{t|t}$ and $x_{t|t}$ rather than their final values. Since the errors ε_t^{π} and ε_t^{x} are correlated with $\pi_{t|t}$ and $x_{t|t}$ an empirical estimation based on explanatory variables π_t and x_t can lead to biased estimates of ϕ_π and ϕ_x in (1.3). In principle the GMM framework is sufficiently flexible to deal with the "errors-in-variables" problem, but Consolo and Favero (2009) warn against the problems associated with weak instruments. This warning together with the startling revisions of the estimates suggest real-time estimates of the data should be used in an empirical investigation of the policy rule.

Figure 1.2 illustrates the difference between the final revisions of inflation and the output gap, and the real time estimates of these variables. Real time estimates of inflation and output growth are from the Greenbook Forecasts provided by the FOMC, based on the meeting date closest to the middle of each quarter. The real-time dataset for the real output gap that extends to 1987 is augmented with estimates of the real output gap constructed using the Hodrick-Prescott filter. The real-time estimates of real output used in the Hodrick-Prescott filter are from the Croushore and Stark (2001) dataset.[9] The

[8]Umino (2014) has highlighted the uncertainty generated from data revisions in the Japanese context.
[9]Available from the Federal Reserve bank of Philadelphia website: http://www.phil.frb.org/research-

estimates of the real output gap are broadly consistent with the results in Orphanides and Van Norden (2002)[10], who emphasise the notable variation in the estimates of the real-time output gap relative to their final revisions. Particularly noticeable is the variation during the period 1981 to 1985 where the real time estimates of the output gap seemed to be overestimated significantly, sometimes by as much as 5 percentage points.

Reestimating the coefficients of the basic Taylor rule model using real time estimates improves the fit significantly:

$$i_t = \underset{(0.00)}{0.01} + \underset{(0.08)}{1.64}\pi_{t|t} + \underset{(0.08)}{0.53}.x_{t|t} + k_t \tag{1.5}$$

The R^2 of the above estimation is 79%. Note that the coefficients ϕ_π=1.64, $\phi_x = 0.5$ and c = 0.01 are close to Taylor's original suggestions. The improvement in the R^2 value is consistent with Orphanides (2003) finding that historical policy analysis is severely altered by the inclusion of real-time data. The improvement can also be seen graphically in Figure 1.3, which displays the time series of the policy rate implied by the suggested and estimated Taylor rules using real-time data.

The third extension to the simple Taylor rule relates to the core question of this paper. The error terms in the Taylor model are serially correlated (The Durbin-Watson statistic for the estimate in (1.3) is 0.5). Economists have either accounted for this through an inertial mechanism or by assuming persistent exogenous shocks. Augmenting (1.3) with a partial adjustment mechanism of the form $i_t = (1-\rho)i_t^* + \rho i_{t-1}$ gives the following estimates:

$$i_t = \underset{(0.01)}{0.00} + \underset{(0.07)}{0.12}\pi_t + \underset{(0.03)}{0.10}x_t + \underset{(0.03)}{0.92}i_{t-1} + k_t \tag{1.6}$$

The model provides an excellent ex-post fit for interest rates: $R^2 = 95\%$. The inflation co-

and-data/real-time-center/real-time-data/

[10]In fact Orphanides and Van Norden (2002) observe that there are two sources of variation: the real-time availability of real output data and the statistical method used to estimate real potential output. Appendix 1.8.1 outlines the Hodrick-Prescott filter, one such method.

efficient is close to 0.12 , the value suggested by Taylor adjusted by the inertial coefficient. However, the coefficient on the real output gap remains far off the adjusted coefficient suggested by Taylor (0.04). Here we consider the often used 1 lag adjustment but some authors (Clarida et al. (2000); Coibion and Gorodnichenko (2008)) have considered multiple lags.

Estimates of (1.3) augmented with persistent errors of the form $u_t = \rho u_{t-1} + \epsilon_t$ are:[11]

$$i_t = \underset{(0.01)}{0.00} - \underset{(0.07)}{0.10}\pi_t + \underset{(0.10)}{0.43}x_t + u_t \tag{1.7}$$

$$u_t = \underset{(0.01)}{0.94}u_{t-1} + \epsilon_t \tag{1.8}$$

The large persistent coefficient of 0.94 is consistent with Coibion and Gorodnichenko (2008) and is similar to the partial adjusmtent estimate of 0.92 given in (1.6). In fact figure 1.4 shows that the two sets of estimates track each other closely. The small coefficient for inflation is also consistent with Coibion and Gorodnichenko (2008), and either implies that the Taylor Principle was not satisfied during the period, or more likely the model above is misspecified.

1.2.2 Properties of the short rate - observations from the finance literature

1.2.2.1 Short rate heteroskedascity

The significant heteroskedascity in the short rate time series is observed by Chan et al. (1992), who compare the empirical properties of the 1 factor class of short rate models over the period 1964 - 1989 in a GMM framework. They test models with a discrete-time econometric specification:

$$i_{t+1} - i_t = \alpha + \beta i_t + \epsilon_{t+1} \tag{1.9}$$

[11] Estimates obtained using the Cochrane-Orcutt method.

$$E[\epsilon_{t+1}] = 0, \;\; E[\epsilon_{t+1}^2] = \sigma^2 i_t^{2\gamma} \tag{1.10}$$

The set of of moment conditions are given in Appendix 1.8.2. The generalized form nests the Merton, Vasicek, CIR SR, Dothan, Geometric Brownian Motion (GBM), Brennan Schwartz (BS), CIR VR and CEV models. The main finding is that model performance is most sensitive to the heteroskedastic term γ and the most successful models in this class were those with high values of γ. I run the same empirical tests over the period 1981-2007, and find results broadly consistent with Chan et al. (1992). The results in Table 1.2 show the nested models with $\gamma < 1$ are rejected at a 95% confidence interval, whereas models with $\gamma \geq 1$ are not.

Two aspects of the data used in the empirical investigation of Chan et al. (1992) require further explanation. First, the data used is at a monthly frequency. This is typical of empirical finance models of the short rate and contrasts with the quarterly Federal Funds data used in macroeconomic analysis of the short rate. Quarterly data is used commonly in macro models as the frequency of Federal Reserve board meetings is bi-quarterly. Using data at a higher frequency in a structural model of the policy rate would be disingenous. Second, the short rate is based on the (annualized) yield of a traded instrument - 1 month Treasury Bills - rather than a policy rate. Figure 1.5 compares the 1 month Treasury Bill rate (at a quarterly frequency) with three commonly used measures of the FFR: the FFR target rate, FFR effective rate and the estimate of the FFR effective rate given in the FOMC greenbook at the middle of each quarter[12]. The three measures of the FFR are almost indistinguishable at a quarterly frequency, whereas the differences between these and the 1 month Treasury bill rate are slightly greater. However table 1.1, which gives the summary statistics of the time series in figure 1.5, shows that the distributional attributes of the 1 month Treasury bill rate are quite similar to the three FFR measures. Differences in estimation results using quarterly FFR data and monthly 1 month T-Bill data should be attributed mostly to the data frequency.

[12]The target FFR rate is the monetary policymakers' desired level for the overnight rate at which Federal Reserve depositary institutions lend their balances to other depositary institutions. The FFR effective rate is a weighted average of the *actual* rates at which the lending takes place. In the US the target rate is 'effected' typically in the repo market although recently there have been periods where the link between the repo rate and the FFR target rate has not been stable, see for example Bech et al. (2012).

I rerun the GMM estimation using quarterly FFR data. The results are less compelling than the estimation using monthly one month T-Bill data. None of the models with $\gamma < 1$ are rejected at a 95% level of confidence. Yet all of the models are rejected at an 80% confidence level, which suggests that square interest rate changes are related to the levels of interest rates. However, from the estimates it is not clear if this observed relationship is due to the omission of structural macroeconomic variables in the reduced form model, or whether the Fed deviates more (less) from the policy rule when interest rates are high (low).

1.2.2.2 Stochastic mean rates in reduced form models

Reduced form 2-factor models were introduced to account for the observation that short-term interest rates reverted to a mean rate, which was itself variable. The time varying mean is also presented as a possible explanation for the significant heteroskedascity observed by Chan et al. (1992). The 2-factor models include the G2++ model of Brigo and Mercurio (2007), the model of Hull and White (1994), the Vasicek model with stochastic central tendency described in Balduzzi et al. (2000) (BDFS), and the model of Schaefer and Schwartz (1984) (SSW). Here I focus on the BDFS and SSW models, which cover the cases where the long term mean is stationary (BDFS) and non-stationary (SSW).

A discrete form econometric specification of a simplified version of the BDFS model is given by:

$$i_{t+1} - i_t = \kappa(\mu_t - i_t) + \epsilon_{r,t+1} \tag{1.11}$$

$$\mu_{t+1} - \mu_t = a + b\mu_t + \epsilon_{\mu,t+1} \tag{1.12}$$

$$E[\epsilon_{r,t}] = 0,\ E[\epsilon_{r,t}^2] = \eta_r^2,\ E[\epsilon_{\mu,t}] = 0,\ E[\epsilon_{\mu,t}^2] = \eta_\mu^2,\ E[\epsilon_{r,t}\epsilon_{\mu,t}] = \rho\eta_r\eta_\mu \tag{1.13}$$

where μ_t is the time varying mean. Although the mean is time-varying i_t is stationary as the mean μ_t reverts to a constant rate $\frac{a}{-b}$.

In the SSW model the short rate is defined as the sum of the long rate (l_t) and the short rate spread (s_t), where l_t is subject to permanent shocks and s_t is assumed orthogonal to

l_t. The discrete form specification of the SSW model is given by:

$$i_t = l_t + s_t \tag{1.14}$$

$$s_{t+1} - s_t = m(\theta - s_t) + \epsilon_{s,t+1} \tag{1.15}$$

$$l_{t+1} - l_t = \varpi + \epsilon_{l,t+1} \tag{1.16}$$

$$E[\epsilon_{s,t}] = 0,\ E[\epsilon_{s,t}^2] = \eta_s^2,\ E[\epsilon_{l,t}] = 0,\ E[\epsilon_{l,t}^2] = \eta_l^2,\ E[\epsilon_{s,t}\epsilon_{l,t}] = 0 \tag{1.17}$$

The identifying moment conditions used to estimate the BDFS and SSW models are given in Appendix 1.8.2.

The most important empirical feature of the SSW and BDFS models for the purposes of constructing a structural model of the short rate is the distributional property of the long term mean rate. Summary distributional statistics for the BDFS and the SSW model are provided in table 1.4 and the time series is plotted in figure 1.6. The standard deviation of the long term mean rates of both the BDFS and SSW models are similar in levels (0.0245 and 0.0279 respectively) and differences (0.008 and 0.006). Figure 1.5 shows that the long term BDFS and SSW rates track each other closely, representing a similar source of underlying variation. In the next section I compare the properties of the BDFS and SSW long rates with the structural long rate proxies.

The estimation of both the BDFS model and the SSW model embed information from the term structure. In the case of the SSW model the required information is explicit, the consol yield or otherwise the yield of the longest available bond. Instead the BDFS model imposes a restriction on the relative prices of bonds of different maturity, from which a proxy for the time-varying mean rate is imputed[13]. The use of term structure information in the estimation implies the time-varying mean rate aggregates the generational, business cycle and risk premium frequencies identified in Cieslak and Povala (2011). But a real world description of the short rate should exclude the risk premium frequency. Moreover the business cycle frequency is captured by the macroeconomic models described in

[13]See Appendix 1.8.2 for details.

section 1.2.1. The structural proxies discussed in the next section attempt to model the generational frequency, without contamination from the other two frequencies.

1.2.2.3 Structural proxies for the persistent component of short rates

Cieslak and Povala (2011) and Fama (2006) both relate the persistent component of short rates to time-varying inflation expectations. Time varying inflation expectations can be linked to a modified Taylor rule of the form:

$$i_t = r^* + \pi_t^* + \bar{\phi}_\pi(\pi_t - \pi_t^*) + \phi_x x_t \tag{1.18}$$

where the equilibrium inflation rate π_t^* is now time-varying. The Cieslak and Povala (2011) and Fama (2006) proxies for inflation expectations are both backward-looking. Fama (2006) uses the average of the past five year short rate as a proxy, which given quarterly data can be expressed as:

$$\gamma_t^{FAMA} = \frac{1}{\tau_F}\sum_{k=1}^{\tau_F} r_{t-k} \tag{1.19}$$

where $\tau_F = 20$. Cieslak and Povala (2011) appeal to the learning literature and use a proxy of the form:

$$\gamma_t^{CP} = \left(\sum_{k=0}^{\tau_C-1} v^k\right)^{-1} \sum_{k=0}^{\tau_C-1} v^k \pi_{t-k} \tag{1.20}$$

where the weighting factor is set at[14] $v = 0.9609$ based on inflation surveys and the truncation point is set at 10 years corresponding to $\tau_C = 40$.

Figure 1.6 plots the estimate of the BDFS time-varying mean, the SSW long rate proxy, γ_t^{FAMA} and γ_t^{CP}, and table 1.4 provides some descriptive statistics of these time series. The results reveal a contrast in the volatility of the estimates. The volatility of first differences of the BDFS and SSW terms are 0.8% and 0.6% respectively, against 0.1% for both the FAMA and CP proxy. The autocorrelation (first four lags) show that the FAMA and CP proxies are more persistent than the BDFS and SSW terms. These discrepan-

[14] The actual value used in Cieslak and Povala (2011) is 0.9868 is based on monthly data. We convert this to $0.9609 = 0.9868^3$ for quarterly data.

cies provide some empirical support for the assertion that the BDFS and SSW proxies are contaminated by the risk premium and business cycle variation, and γ_t^{CP} or γ_t^{FAMA} are better able to distill the changes in inflation expectations occurring at a generational frequency from the data.

Recently, Favero et al. (2012) relate the persistent component of interest rate changes to changes in the age composition of the population. Intuitively, population demographics affect the natural real rate of an economy by inter alia changing the proportion of employed people in the economy and influencing the aggregate marginal propensity to save. There is a large literature on life-cycle modelling that analyses these and other effects (e.g. Merton (1969)). The effect of demographics on monetary policy can be linked to a modified Taylor rule of the form:

$$i_t = r_t^* + \pi_t + \bar{\phi}_\pi(\pi_t - \pi^*) + \phi_q(q_t - q_t^*) \tag{1.21}$$

where the equilibrium real rate r_t^* is now time-varying. As a proxy for population demographics I use the inverse of the MY variable cited in Favero et al. (2012) and originally proposed in Geanakoplos et al. (2004). The proxy, denoted YM, is the ratio of young persons (aged 20-29) to middle aged persons (aged 40-49) in the US. The autocorrelation statistics for the YM variable in the last row of table 1.4 show that the demographic variable has a very high degree of persistency both in levels and first differences, greater than the other four time-varying mean proxies. This statistic serves as a caution that the quarter of a century of data used in the empirical section may not be sufficient to capture a meaningful relationship between YM and the FFR.

This review of the finance short-rate literature has highlighted a number of points worthy of consideration when constructing the econometric framework. To summarise, there is strong evidence that the short rate reverts to a time-varying mean rate. The time-varying mean could represent either a time-varying inflation expectation or a time-varying equilibrium rate in the economy. Structural time-varying inflation proxies such as the FAMA and CP term seem to be superior to proxies imputed from the interest rate term structure such as in the BDFS or SSW model. The structural proxies are not contaminated by the 'risk premium' frequency as described in Cieslak and Povala (2011). An obvious proxy for the time-varying natural real rate is a demographic variable such as the ratio of young to middle aged persons YM. Finally, heteroskedasticity of the short rate is a feature of reduced-form models, although it is not clear whether this feature is attributable to omitted structural variables.

1.3 Econometric approach

Motivated by the considerations in section 1.2, I consider a general model that includes real time estimates of the Taylor rule variables as well as variables that operate at the genational frequency, which affect the equilibrium real and inflation rates. The model also nests both a partial adjustment and persistent shock mechanism, and includes heteroskedastic errors. The model can also be seen as an augmented ARMA(M, N) model with heteroskedastic errors, where the model has been augmented by macro business cycle factors and generational factors. Formally, I estimate the parameters of nested versions of the following generalised model:

$$i_t = c + \phi_\pi \pi_{t|t} + \phi_x x_{t|t} + \phi_\gamma \gamma_t + k_t \tag{1.22}$$

$$k_t = \sum_{j=1}^{M} \delta_j i_{t-j} + u_t \tag{1.23}$$

$$u_t = \sum_{l=1}^{N} \rho_l u_{t-l} + \epsilon_t \tag{1.24}$$

$$E[\epsilon_{t+1}] = 0, \;\; E[\epsilon_{t+1}^2] = \sigma^2 i_t^{2\vartheta} \tag{1.25}$$

where i_t is the quarterly value of the FFR and c is a scalar. $\pi_{t|t}$ and $x_{t|t}$ are the estimates of the time t inflation rate and output gap based on information at time t, with scalar coefficients $\phi_\pi$and ϕ_x. γ_t is the time t Q-dimensional vector of variables intended to capture time varying equilibrium real and inflation rates, with ϕ_γ a vector of scalar coefficients. δ_j, $j \in \{1, \ldots, M\}$ is the coefficient of the jth lagged interest rate and ρ_l, $l \in \{1, \ldots, N\}$ the coefficient of the lth lagged error term, u_{t-l}. ϵ_t, the component of the error term independent of lagged shocks has unknown distribution with the first two moments given by (1.25), where $\vartheta \geq 0$ is a scalar.

The partial adjustment mechanism is described by (1.23) and the persistent shock mechanism by (1.24). Heteroskedastic errors are captured by ϑ. If $\vartheta \neq 0$ ϵ_t is heteroskedastic and is conditioned by i_t.

The approach is to test (1.22) - (1.25) using the Generalized Method of Moments (GMM) as per Hansen (1982). The motivation for using the GMM framework is similar to Chan et al. (1992). GMM is an appropriate choice as it does not require the errors ϵ_t to be homoskedastic and the conditional distribution of the errors may not be normal. Define θ as the vector of parameters with elements c, ϕ_π, ϕ_x, ϑ and the elements of the vectors ϕ_γ, δ and ρ, where $\delta = \begin{pmatrix} \delta_1 & \dots & \delta_M \end{pmatrix}$and $\rho = \begin{pmatrix} \rho_1 & \dots & \rho_M \end{pmatrix}$. The vector of moments is given by:

$$f_t(\theta) = Vec\begin{pmatrix} f_{t,1}(\theta) & f_{t,2}(\theta) & f_{t,3}(\theta) \end{pmatrix} \tag{1.26}$$

where

$$f_{t,1}(\theta) = u_t Z_t$$

$$f_{t,2}(\theta) = \epsilon_t \begin{pmatrix} u_{t-1} & \dots & u_{t-N} \end{pmatrix}^T$$

$$f_{t,3}(\theta) = (\epsilon_t^2 - \sigma^2 i_t^{2\vartheta}) \begin{pmatrix} 1 & i_t \end{pmatrix}^T$$

$$Z_t = \begin{pmatrix} 1 & \pi_{t|t} & x_{t|t} & \gamma_{t,1} & \dots & \gamma_{t,Q} & i_{t-N-1} & \dots & i_{t-N-M} \end{pmatrix}^T$$

The GMM condition is given by[15]:

$$E[f_t(\theta)] = 0 \tag{1.27}$$

[15]Description of the GMM procedure below follows the description in Chan et al. (1992)

where $E[f_t(\theta)]$ replaces the sample counterpart $g_K(\theta) = \frac{1}{K}\sum_{t=1}^{K} f_t(\theta)$. We estimate the parameter vector θ by minimizing:

$$J_K(\theta) = g_K^T(\theta) W_K(\theta) g_K(\theta) \tag{1.28}$$

where $W_K(\theta)$ is a positive-definite symettric weighting matrix. In the general unrestricted model the parameters are exactly identified so the optimum θ sets $J_K(\theta) = 0$. In nested versions of the model the weighting matrix is relevant. The optimum choice for $W_K(\theta)$ is $W_K(\theta) = S(\theta)^{-1}$ where $S(\theta) = E[f_t(\theta) f_t^T(\theta)]$. The estimated covariance matrix for θ is then given by:

$$\frac{1}{K}(\nabla^T(\hat{\theta})\hat{W}_K(\theta)\nabla(\hat{\theta}))^{-1} \tag{1.29}$$

where $\hat{W}_K(\theta) = \hat{S}(\theta)^{-1}$ and $\nabla(\hat{\theta})$ is the Jacobian matrix of $g_K(\theta)$ with respect to θ based on estimated values of θ. The matrix in (1.29) is used to test the significance of the individual parameters in the GMM estimation.

Differences in (1.28) evaluated at the parameter estimates under the unrestricted and restricted hypothesis can be used to assess the hypothesised restriction. The likelihood-ratio test for parameter restrictions of the form $H_0 : \theta_{i_1} = 0, \theta_{i_2} = 0 \ldots \theta_{i_q} = 0$ uses the fact that:

$$L = (J_K(\bar{\theta}) - J_K(\hat{\theta})) \tag{1.30}$$

is asymptotically distributed χ_q^2, where $\hat{\theta}$ is the estimated value of θ without the restrictions under H_0 and $\bar{\theta}$ is the estimated value of θ with the parameter restrictions. The weighting matrix used is based on the estimates of the unrestricted model. It is easy to see that if the number of orthogonality conditions in the unrestricted model equals the number of estimated parameters then $J_K(\hat{\theta}) = 0$ and $L = J_K(\bar{\theta})$, which is asymptotically distributed χ_q^2.

Note the instrument variables in $f_{t,1}(\theta)$ include i_{t-k}, $N+1 \leq k \leq N+M$. Using the lagged instrumental variables $i_{t-1} \ldots i_{t-N}$ as instruments would result in biased estimation, since these variables are not orthogonal to the error terms $u_{t-1}, \ldots, u_{t-N}$ (Hayashi

(2000)). To see this consider the restriction relating to i_{t-k}, $k \leq N$:

$$E_t[i_{t-k}u_t] = E_t[i_{t-k}(\sum_{l=1}^{N} \rho_l u_{t-l} + \epsilon_t)] = E_t[u_{t-k}(\sum_{l=1}^{N} \rho_l u_{t-l} + \epsilon_t)] = \rho_l E_t[u_{t-l}^2] \neq 0$$

In the application of the model we set $N = M = 3$. This allows us to test higher order partial adjustment mechanisms than the typical 1-lag smoothing function, whilst ensuring that the number of testable nested models does not explode. For the generational frequency factor we first set $\gamma_t = \gamma_t^{CP}$ where $\gamma_t^{CP}$is the Cieslak-Povala term defined earlier in expression (1.20). So γ_t is a scalar and $Q = 1$. It may seem contradictory to use a term that contains a weighted average of past actual inflation values (γ_t^{CP}) together with the real time estimate of inflation $\pi_{t|t}$. The justification for including the real-time estimate of inflation is clear. The Central Bank estimates inflation in real time to determine whether the increase in the price of goods has deviated from its long term equilibrium rate. The long term equilibrium rate however is affected by market participants' expectation of long term inflation, an expectation that changes slowly over time. Appealing to the learning literature, market participants' expectation of long-term future inflation is formed based on a weighted average of the past *experienced* increases in the prices of goods. Therefore it is appropriate to use actual price increases in the formulation of the proxy for long term inflation expectations.

In subsequent tests I also consider alternative specifications of γ_t. The first alternative specification sets $\gamma_t = l_t$, where l_t is the yield on a long term bond. Although l_t is a market implied variable and therefore will be affected by time-varying risk premia or hedging demands of institutional investors, it will be useful to check that the results relative to the partial adjustment mechanism are robust to alternative specifications of γ_t. The second alternative sets $\gamma_t = \begin{pmatrix} \gamma_t^{CP} & YM_t \end{pmatrix}$, where YM_t is the change in the proportion of young persons to middle-aged persons, defined in section 1.2.2.3. Favero et al. (2012) find the inverse of YM_t to have substantive predictive power for changes in the term structure of interest rates. The second alternative specification will reveal whether the conclusions are sensitive to the inclusion of YM_t.

The nested models are also compared using four alternative performance metrics. The first metric (R_1^2) is the ordinary R^2 coefficient of determination statistic used in OLS. The second metric (R_2^2) is the proportion of total variation in squared interest rates explained by the model. The third metric (R_3^2) and fourth metrics (R_4^2) correspond to the alternative measures of model performance in Chan et al. (1992). R_3^2 is the proportion of total variation in interest rate *changes* explained by the model, and R_4^2 is calculated as the proportion of total variation in square interest rate *changes* explained by the model. The third and fourth measures allow us to compare how well the models explain the first and second moments of the distribution of interest rate changes. Specifically these measures provide a direct evaluation of the explanatory power of the generational and business cycle frequency factors, not included in the class of models in Chan et al. (1992).

1.4 Data

The FFR data is obtained from the Board of Governors of the Federal Reserve System H.15 series. I use data covering the period 1981-2007 at quarterly intervals, corresponding to the date of the FOMC meeting closest to the middle of each quarter. The period 1981-2007 excludes the pre-Volcker years, when it is generally accepted that monetary policy was conducted differently, see e.g Clarida et al. (2000). A full set of real-time inflation estimates also exists for this period, obtained from the Fed's real-time Greenbook dataset[16]. For the bulk of this period (Q4:1987 onwards) there also available real-time estimates of the output gap used by the staff of the Fed to forecast the variables included in the Greenbook. I extend this dataset to 1981 by using real time estimates of real output from the dataset of Croushore and Stark (2001) and estimate the real potential output trend with the Hodrick-Prescott filter (see Appendix 1.8.1). To calculate γ_t^{CP} I use quarterly values of the core inflation measure (less energy and food) used by Cieslak and Povala (2011). The 10 year constant maturity yield, also available in the Federal Reserve System H.15 series, provides a proxy for the long rate l_t. Finally the inverse of the MY variable cited in Favero et al. (2012) and originally proposed in Geanakoplos et al. (2004) is used as a proxy for demographic changes.

[16]Available from the Federal Reserve Bank of Philadelphia: http://www.phil.frb.org/research-and-data/real-time-center/greenbook-data/

1.5 Results

Tables 1.5 and 1.6 reports the parameter estimates, minimized GMM values (J-statistic and χ^2 probability), and values of the four performance metrics R_k^2, $k = 1, 2, 3, 4$ for the unrestricted model and a number of nested alternatives. The nested alternatives include models with heteroskedastic errors (table 1.5) and models with the restriction $\vartheta = 0$ (table 1.6).

There are a few noteworthy features in these tables. Firstly, the models with heteroskedastic errors all have higher p-values than the models with homoskedastic errors. For a few of the models this difference is large, which suggests that the Fed does deviate from the augmented Taylor rule more when interest rates are higher. Further support is provided by the R_4^2 measure, which is consistently higher for the models with heteroskedastic errors. Secondly, the models that include the generational frequency factor (γ_t^{CP} in this case) perform better based on the minimized GMM criterion than the models without the generational frequency factor. The discrepancy between the two sets of models is most evident in table 1.5 where the best-fitting models amongst 1a -10a have significantly larger p-values than the best-fitting models amongst models 11a - 20a. The result is consistent with Cieslak and Povala (2011), which finds γ_t^{CP} to be an important determinant of the Fed reaction function over and above the responses to transient inflation deviations. It is useful to note that the result is not weakened with the inclusion of the business cycle variables nor the partial adjustment and persistent shock mechanisms included in the model.

Most startling is the difference between the minimized GMM criterion of the models that include a partial adjustment mechanism and those that don't. Focussing on table 1.5, the minimized GMM criterion of models 2a, 4a, 7a, 10a, 12a, 14a, 17a and 20a (all with the restriction $\delta_1 = \delta_2 = \delta_3 = 0$) are improved significantly with the addition of a partial adjustment mechanism. The significant improvement in fit when the restrictions are lifted provides compelling evidence that the Fed did smooth the FFR over the period 1981-2007. The generational frequency factor γ_t^{CP}, whilst significant, cannot account for all the persistence in the time series. Another interesting feature of the results is that the models with a higher order smoothing function outperform the models with only a first order smoothing function. For example in table 1.5, models 3a and 6a outperform model 9a, and models 13a and 16a outperform model 19a.

Model 6a performs well under the GMM criterion (p-value is 0.7) and has a reasonably small number of parameters. The model is described by the expressions:

$$i_t = c + \phi_\pi \pi_{t|t} + \phi_x x_{t|t} + \phi_\gamma \gamma_t + \delta_1 i_{t-1} + \delta_2 i_{t-2} + \epsilon_t \tag{1.31}$$

$$E[\epsilon_{t+1}] = 0, \;\; E[\epsilon^2_{t+1}] = \sigma^2 i_t^{2\vartheta} \tag{1.32}$$

and hereon is denoted the "preferred model." Considering its low number of parameters, the preferred model is able to capture a surprising amount of the variation in the *changes* and *squared changes* in the FFR over time. This is evidenced both statistically by the reasonably high R^2_3 and R^2_4 values of 0.47 and 0.51, and graphically. Figure 1.7 shows a time series of the changes in the FFR together with the changes in the fitted values of the preferred model as well as the deviation of the fitted value from the FFR of the previous period. It is clear from the figure that changes in fitted values from the preferred model track the FFR closely. Figure 1.8 plots a measure of realised volatility, defined as absolute changes in the short rate, against the conditional volatility implied by the preferred model. The preferred model correctly identifies the increase in volatility and the subsequent drop in volatility in the 1990-1995 period, as well as the increase in volatility around 2002. The preferred model also captures the increase in volatility in 2005. But the model also overstates the volatility in the period 1985-1990. Overall the preferred model provides a reasonable description of realised volatility over the period.

Table 1.7 reports a series of Newey-West pairwise tests with the preferred model either as a restricted nested model or an alternative unrestricted model. The results provide clear evidence for a higher order partial adjustment mechanism. The restricted models with no partial adjustment mechanism and with a 1 lag partial adjustment mechanism are both rejected at a 99.5% confidence interval in favour of the preferred model. The pairwise tests also reject the restriction of no generational frequency term as well as rejecting the alternative of homoskedastic errors. In the last two rows of Table 1.7 the preferred model appears as the restricted model. The two rows relate to the tests of the restriction of the persistent shock terms and the third order partial adjustment lag. In both cases these restrictions cannot be rejected at a 90% confidence interval. Whilst this does not rule out the presence of persistent shocks there is not compelling statistical evidence to support the hypothesis.

The results of the GMM tests with the alternative specifications of the generational frequency vector γ_t do not differ substantively from the main results described above[17]. The preferred model described by expressions (1.31) to (1.32) with $\gamma_t = l_t$, where l_t is the long yield, has a low J statistic (p-value 0.93). The results of the analogous Newey-West tests are similar and are shown in table 1.8. The restricted models with 1-lag or no partial adjustment mechanism are rejected in favour of the preferred model. The models with no generational frequency factor or homoskedastic errors are also rejected in the pairwise tests. As in the main results in table 1.7 in the two tests where the preferred model appears as the restricted model, the restrictions cannot be rejected at a 90% confidence interval.

The second alternative specification of the preferred model where the generational factor is augmented by a demographic variable, $\gamma_t = \begin{pmatrix} \gamma_t^{CP} & YM_t \end{pmatrix}$ also has a low J statistic (p-value 0.96). The results of the Newey-West tests are given in table 1.9. Again the results are similar to the main results in 1.7. In the two tests where the preferred model appears as a restricted model, none of the restrictions are rejected. As with the first alternative specification the restricted models with 1-lag or no partial adjustment mechanism are rejected in favour of the preferred model. The model with homoskedastic errors is rejected in favour of the preferred model and so is the model that sets $\phi_{\gamma,1} = 0$, where $\phi_{\gamma,i}$ is the cofficient of the ith generational frequency factor. The test of the model restriction $\phi_{\gamma,2} = 0$ is not rejected, implying that the restriction on the demographic variable is not

[17]The full GMM results are omitted but available on request. Only the pairwise Newey-West tests are reported.

statistically significant.

A plausible explanation is that the effect of changing demographics is not captured by the period of a quarter of a century (approximately) over which we have real time data. Long term expectations embedded in the long end of the interest rate curve may provide a better means to extract the effect of demographics on the short rate, albeit at the cost of contamination by the 'risk premium' frequency.

1.6 Addressing the evidence against gradualism

Rudebusch (2002) and Rudebusch and Wu (2008) argue that the persistency in the FFR represents inadequacies in the augmented Taylor rule specification of the FFR. In their view the Fed responds to other variables, which are persistent and lead to observed serial correlation in the FFR time series. Both Rudebusch (2002) and Rudebusch and Wu (2008) appeal to the term structure of interest rates for evidence. In Rudebusch (2002) the term structure is invoked explicitly as he finds that forward interest rates further than one quarter ahead implied by Eurodollar futures do not predict a significant proportion of the variation in the short interest rate. In Rudebusch and Wu (2008) the invocation of the term structure is implicit as they set up a macro-finance model specifying the one-period short rate to be a linear function of a state vector comprised of latent (interpreted as level and slope) and macro-economic variables (output and inflation). The state vector is modelled as a Gaussian VAR(1) process and the price of risk is a linear function of the state vector. Under these conditions analytical expressions for the price of term τ zero-coupon bonds exist, and the dynamics of the state vector allows for both a partial adjustment and persistent shock mechanism. The model, estimated by maximum likelihood using US Treasury zero-coupon yields with maturities up until five years, dismisses the inertial policy interpretation.

An initial rebuttal to these arguments is provided by the results in the previous section. The two obvious candidates for omitted persistent variables in ordinary Taylor rules are time varying inflation expectations and change in population demographics. Both variables are included in the tests in section 3.3 and do not alter the conclusion that the Fed does implement a gradual policy. The estimation in section 3.3 focusses on changes in the short rate and does not attempt to link these changes with term structure variables. As such it cannot address the poor FFR predictability of the Eurodollar futures market.

I investigate the issue of poor predictability of interest rate changes by considering the forward interest rates implied by the 3 month, 6 month and 1 year treasuries. Treasuries are used rather than Eurodollar futures as these are not affected by changes in the actual or perceived risk embedded in the Libor rate, and therefore provide a better proxy of the expectation of the true short rate[18]. First consider the relation:

$$i_{t+1} - i_t = c + \varphi \left(E_t \left(i_{t+1} \right) - i_t \right) + k_t \tag{1.33}$$

where $E_t(i_{t+1})$ is the expectation of the one-step-ahead forecast of interest rates implied by the treasury market through:

$$y_{t,t+2} \approx 0.5 \left(y_{t,t+1} + E_t \left(i_{t+1} \right) \right) \tag{1.34}$$

where $y_{t,t+2}$ is the time t yield on a 2 quarter (6m) treasury, and $y_{t,t+1}$ is the time t yield on a 1 quarter (3m) treasury. I estimate (1.33)over the period 1981-2007 as[19]:

$$i_{t+1} - i_t = -\underset{0.07}{0.11} + \underset{0.07}{0.39} \left(E_t \left(i_{t+1} \right) - i_t \right) + k_t \tag{1.35}$$

with an $R^2 = 0.23$. The R^2 value is higher than reported in Rudebusch (2002) for the period 1988-2000, but is not overwhelming. More interesting is that the results of the estimates change dramatically when considering the period 1997-2007:

$$i_{t+1} - i_t = -\underset{0.03}{0.08} + \underset{0.05}{0.77} \left(E_t \left(i_{t+1} \right) - i_t \right) + k_t \tag{1.36}$$

[18] An even better proxy would be the rates on overnight indexed swaps, but data is not available over the period as the market was not liquid.

[19] The start of the quarter for the estimation corresponds to the FOMC meeting date closest to the middle of each quarter.

The $R^2 = 0.85$, which suggests that the 1 to 2 quarter forward rates predict a substantial amount of the 1 step ahead changes in interest rates. This result can also be seen visually in figure 1.9, which shows that from the mid-1990's $E_t\left(i_{t+1,t+2}\right) - i_t$ tracks $i_{t+1} - i_t$ closely. Perhaps the best hypothesis for the improved fit is that the Volcker and Greenspan eras had established a precedent of interest rate smoothing, which was eventually understood and then anticipated by market participants. This interpretation is consistent with the theoretical rationale for smoothing put forward by Woodford (1999), where a central bank should include a gradual mechanism to exploit optimally the expectations channel. The optimality is predicated on the credibility of the central bank and its ability to deliver on past promises. The estimation reveals that after ten years of the Greenspan reign the expectations channel could be exploited.

Another plausible reason for the poor fit observed by Rudebusch (2002) in the period 1988-2000, is that market participants anticipated a smoothed approach to monetary policy but the exact timing of the incremental interest rate changes was not understood. I consider the predictive power of the 1 to 4 quarter forward rates implied by treasury yields:

$$\frac{1}{L-1}\sum_{l=1}^{L-1} i_{t+l} - i_t = c + \varphi E_t\left(\frac{1}{L-1}\sum_{l=1}^{L-1} i_{t+l} - i_t\right) + k_t \tag{1.37}$$

where

$$E_t\left(\frac{1}{L}\sum_{l=1}^{L} i_{t+l}\right) \approx \frac{L}{L-1}\left(y_{t,t+4} - \frac{1}{L} y_{t,t+1}\right) \tag{1.38}$$

and I set $L = 4$. The result of the estimation over the entire period 1981-2007 is:

$$\frac{1}{3}\sum_{l=1}^{3} i_{t+l} - i_t = -\underset{(0.05)}{0.18} + \underset{(0.05)}{0.37} E_t\left(\frac{1}{3}\sum_{l=1}^{3} i_{t+l} - i_t\right) + k_t \tag{1.39}$$

with an R^2 value of 36%. This is larger than the 23% over the same period for the 1-step ahead forecast. Estimation results over the period 1997-2007 are not reported but like the one-step-ahead forecasts the R^2 values are significantly higher.

The rejection of the inertial policy in the macro-financial model requires further analysis of the dynamics of the short rate and the latent variables in the model. The dynamic equations are:

$$i_t = c + L_t^m + S_t^m \tag{1.40}$$

$$L_t^m = \rho_L L_{t-1}^m + (1 - \rho_L)\,\pi_t + \varepsilon_{L,t} \tag{1.41}$$

$$S_t^m = \rho_S S_{t-1}^m + (1 - \rho_S)\,(g_x x_t + g_\pi\,(\pi_t - L_t^m)) + u_{S,t} \tag{1.42}$$

$$u_{S,t} = \rho_u u_{S,t-1} + \varepsilon_{S,t} \tag{1.43}$$

where L_t^m and S_t^m are latent factors x_t and π_t are measures of the output gap and inflation and $c, \rho_L$and ρ_S are scalars. The dynamics of inflation and the output gap are based on a standard New Keynesian specification and are omitted here.

The latent variables L_t^m and S_t^m closely approximate the level and slope factors from a PCA analysis of the yields of bonds of maturity up to 5 years, with correlation 0.98 and 0.97 respectively. Rudebusch and Wu (2008) argue that if $\rho_S \neq 0$ the central bank reaction function is inert and if $\rho_u \neq 0$ shocks are persistent. But Cieslak and Povala (2011) find the level factor in a PCA analysis to relate primarily to the long-term inflation target and to transitory short-rate expectations, and the slope factor to relate primarily to short rate expectations and risk premia. If the level and slope factor already include transitory short rate expectations, and market participants expect the Fed to smooth interest rates, then the inclusion of L_t^m and S_t^m already controls for smoothing. Thus $\rho_S = 0$ can be consistent with an inertial Fed policy. Moreover, the time varying risk premia embedded in S_t^m contaminates the analysis and makes the interpretation of expression 1.42 difficult.

1.7 Concluding remarks

In this paper I have defined an econometric framework to test variants of the Fed policy reaction function in the post-Volcker era. The framework is informed by features that distinguish both the macro and finance short rate literature. The framework nests a partial adjustment and a persistent shock mechanism, and uses real-time rather than ex-post data. Appealing to observations in the finance literature, structural proxies for time-varying inflation expectations and equilibrium real rate are included as well as a heteroskedastic term to model the conditional volatility of interest rate changes. The structural proxies are not contaminated by the 'risk premium' frequency, unlike the long term mean variables in reduced form two factor models.

I find that the time-varying inflation expectations are an important attribute of the Fed reaction function. The tests also suggest that the Fed does implement a partial adjustment mechanism, a result robust to the inclusion of persistent generational frequency factors. Moreover, the results point to a Fed partial adjustment mechanism of order greater than 1. The heteroskedastic term in the model is able to capture adequately the realised volatility of interest rate changes. There is no strong support for the persistent exogenous shock story, although it cannot be statistically ruled out. The impact of the demographic factor is weak, possibly owing to only a quarter-century of data over which there exists real time data.

I also address the concerns in Rudebusch (2002), Rudebusch (2006) and Rudebusch and Wu (2008) by showing first that the partial adjustment result is robust to the most obvious persistent factors that affect the equilibrium inflation and real rate of the economy. I show that the prices of financial instruments whose value derives from future realisations of the short rate, imply that market did actually anticipate a gradual Fed approach in the period 1997 - 2007. This result is consistent with a market learning story and improved credibility of the Fed. Finally, I show that the rejection of the inertial policy in the nested macro-financial model of Rudebusch and Wu (2008) can be explained by the inclusion of macro-finance factors that already control for smoothing.

These conclusions point to a few obvious areas of future research. Firstly, a similar analysis could be performed for other countries where central bank real-time data exists. Secondly, a study of the relative importance of the time-varying inflation rate, the time-varying real rate, and smoothing motive in explaining central bank deviations from simple Taylor rules over different historical periods would be useful. Thirdly, it would be interesting to understand what cost there is (if any) to agents who are not aware of the central bank smoothing motive, and if the implied cross-subsidy is beneficial for society.

1.8 Appendix

1.8.1 The Hodrick-Prescott Filter

I use the Hodrick-Prescott filter (the 'H-P filter') to estimate the value of potential GDP for the period 1981 (Q1) - 1987 (Q4), where real-time real output gap data is not available from the FOMC Greenbook. The H-P filter is one of many possible choices of which the Beveridge-Nelson, Watson, Clark and Harvey-Jaeger filters are examples. The relative properties of these filters are discussed in Orphanides and Van Norden (2002). The purpose of the filter is to extract the trend component from a time series. Formally, consider a series x_t that can be decomposed as $z_t = \zeta_t + \eta_t$ where ζ_t is a time-varying trend component and $\eta_t$is a cyclical component. The HP filter solves for ζ where:

$$\zeta = \underset{\zeta}{argmin} \sum_{t=1}^{K} (z_t - \zeta_t)^2 + \lambda \sum_{t=1}^{K-2} ((\zeta_{t+2} - \zeta_{t+1}) - (\zeta_{t+1} - \zeta_t))^2$$

and $\zeta = (\zeta_1 \zeta_2 \ldots \zeta_K)$. The HP filter penalises the sum of square second differences of the trend component by an amount λ. For quarterly data Hodrick and Prescott (1997) suggest setting $\lambda = 1600$. The filter can be implemented by solving the linear equation:

$$(I + \lambda H^T H)\zeta^T = \mathbf{z^T}$$

where $\mathbf{z} = (z_1 z_2 \dots z_T)$, I is the $T \times T$ identity matrix, and H is the $(T-2) \times T$ second order finite difference matrix:

$$H = \begin{pmatrix} 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \end{pmatrix}$$

Substituting the real output series into the vector $\mathbf{z}$, the estimate of the real output gap at time t, x_t, is given by: $x_t = \frac{z_t - \zeta_t}{\zeta_t}$.

I apply the HP filter to real output data obtained from Croushore and Stark (2001) dataset over the period 1966 to 1987. The period 1966 - 1981 is used as a burn-in period. Only the estimated values in the period 1981-1987 are used in the econometric analysis. The result is given in figure 1.10.

1.8.2 Identifying moment conditions of reduced form models

The generalized class of models in Chan et al. (1992) given in (1.9) - (1.10) can be estimated by GMM with the following identifying moment conditions:

$$E \begin{bmatrix} \epsilon_{t+1} \\ \epsilon_{t+1} i_t \\ \epsilon_{t+1}^2 - \sigma^2 i_t^{2\gamma} \\ \left(\epsilon_{t+1}^2 - \sigma^2 i_t^{2\gamma}\right) i_t \end{bmatrix} = 0 \tag{1.44}$$

The estimation is performed both using monthly one month t-bill data from the CRSP database and quarterly FFR effective rates from the FFR effective rates from the Board of Governors of the Federal Reserve System H.15 series. Balduzzi et al. (2000) show that the BDFS model implies a proxy process for the time-varying mean μ_t, of discretized form:

$$\mu_t = a_0 + a_1 \left[\left(\frac{1 - e^{-\kappa \tau_1}}{\kappa} \right) log(P_1) - \left(\frac{1 - e^{-\kappa \tau_2}}{\kappa} \right) log(P_2) \right] \tag{1.45}$$

where P_1 is the price of a bullet bond maturing at τ_1 and P_2 is the price of a bullet bond maturing at τ_2. a_0 and a_1 are parameters estimated from the relation:

$$i_{t+1} - i_t = \kappa(\mu_t - r_t) + \epsilon_{r,t+1} \tag{1.46}$$

using identifying moment conditions of the form:

$$E[\epsilon_{r,t+1} Z_t] = 0 \tag{1.47}$$

$$E[\left(\epsilon^2_{r,t+1} - \eta^2_r\right) Z_t] = 0 \tag{1.48}$$

where Z_t is a vector of instruments. The estimation of the time varying mean rate in Figure 1.6 is performed using quarterly FFR effective data for r_t and 1 and 2 year zero-coupon bond prices from the CRSP Fama-Bliss file for P_1 and P_2. I use the instrument vector $Z_t = \begin{bmatrix} 1 & i_t \end{bmatrix}$ for the estimation. The proxy for the long rate in the SSW model is the 10 year constant maturity yield.

1.9 Figures and Tables

Figure 1.1: Estimated and suggested policy rate under simple Taylor rules
The estimated Taylor model rule (dashed) with revised data is:$i_t = 0.02 + 1.59\pi_t - 0.07x_t + k_t$ and the suggested Taylor model (dotted) is: $i_t = 0.01 + 1.5\pi_t + 0.5x_t + k_t$, where i_t, π_t, x_t and k_t are the FFR, inflation, real output gap, and error values at time t. Values for actual inflation are derived from the implicit GDP deflator, and the real output gap is given by: $\frac{Real\,GDP - Real\,Potential\,GDP}{Real\,Potential\,GDP}$.

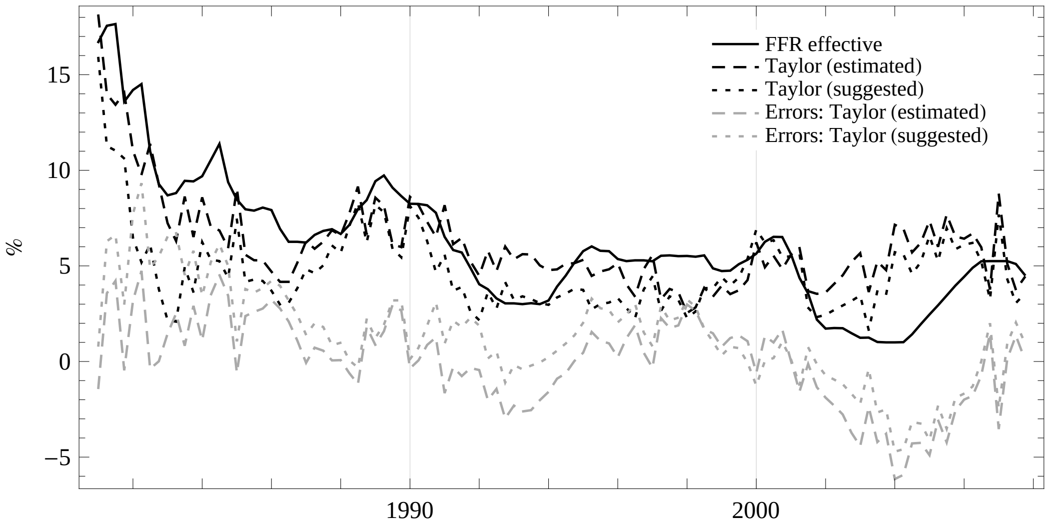

Figure 1.2: Final revisions against real time estimates
Values for final revisions of inflation (black, full) and final real output gap (black, dashed) are the same as used in figure 1. Real time estimates of inflation and output growth are from the FOMC Greenbook Forecasts, based on the meeting date closest to the middle of each quarter. Data from 1981Q1 - 1987Q4 is estimated using the Hodrick-Prescott filter. (See appendix section 1.8.1).

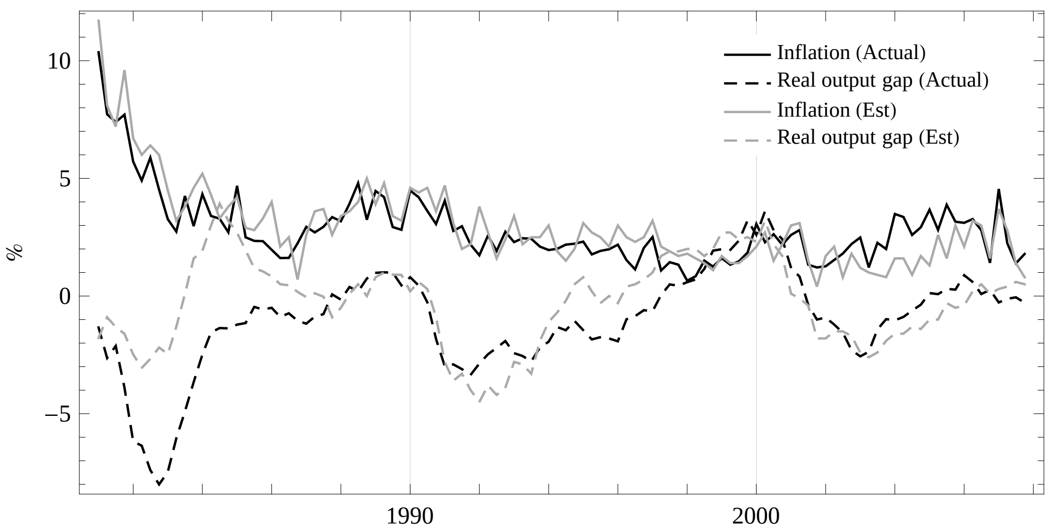

Figure 1.3: Real time estimates of the Taylor rule
Real time estimates of the Taylor rule model is: $i_t = 0.01 + 1.64\pi_{t|t} + 0.53.x_{t|t} + k_t$, where i_t, $\pi_{t|t}$ and $x_{t|t}$ are the time t values of the FFR, real time estimates of inflation and the real output gap and $k_1, k_2, k_3 \ldots$ are $i.i.d\,(0, \sigma^2)$. The suggested Taylor model is: $i_t = 0.01 + 1.5\pi_{t|t} + 0.5\pi_{t|t} + k_t$. Plots of the estimated values of k_t under both models are in grey. Dataset is the same as figure 1.2.

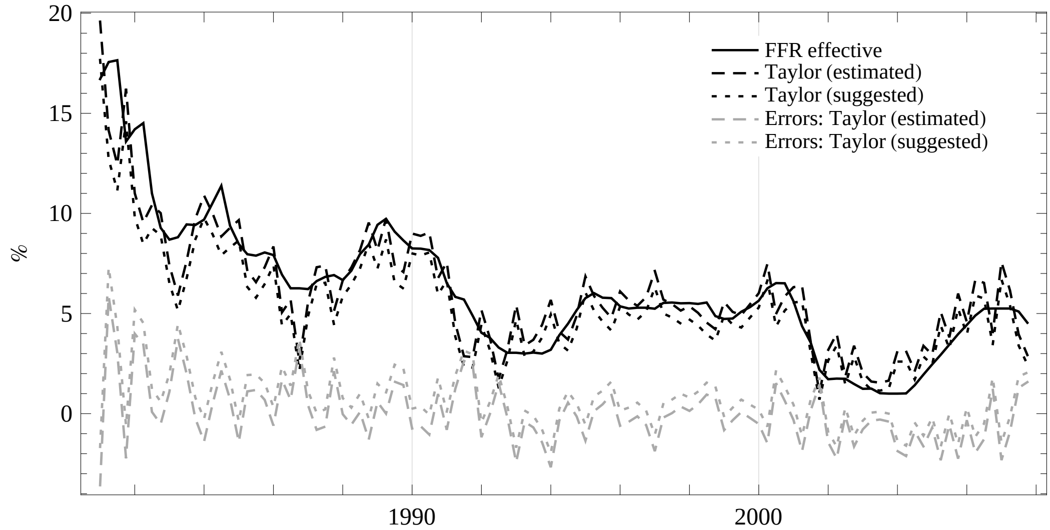

Figure 1.4: FFR against predicted rates under two simple augmented Taylor rules
The basic Taylor rule model is: $i_t = c + \phi_\pi \pi_t + \phi_x x_t + k_t$, where i_t, $\pi_t$and x_t are the time t values of the FFR, inflation and the real output gap and $k_1, k_2, k_3 \ldots$ are $i.i.d\,(0, \sigma^2)$. Under the Taylor rule with partial adjustment (black, dashed) $k_t = \delta_1 i_{t-1} + \epsilon_t$, and under the persistent shocks model (black, dotted) $k_t = \rho_1 k_{t-1} + \epsilon_t$ where $\epsilon_1, \epsilon_2, \epsilon_3 \ldots$ are $i.i.d\,(0, \sigma^2)$. Plots of the estimated values of $\epsilon_t$under both models are in grey. Actual inflation, and real output gap data is the same as figure 1.1.

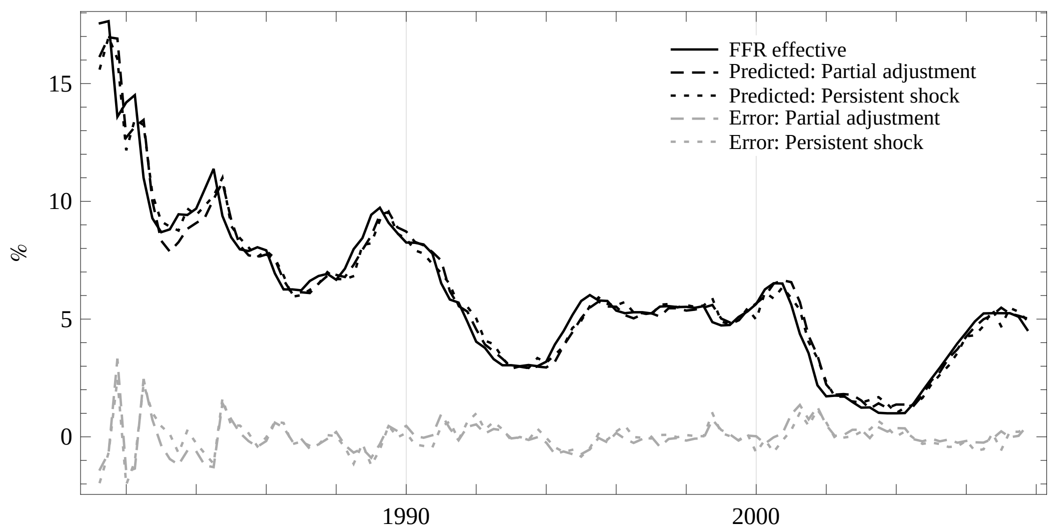

Figure 1.5: Comparison of different measures of the short rate
Figure below compares three measures of the FFR with the 1 month Treasury Bill rate (annualized). The three measures of the FFR are the FFR target rate, FFR effective rate and the estimate of the FFR effective rate given in the FOMC Greenbook forecasts. Data for the FFR target rate is from Bloomberg, FFR effective rate is from Board of Governors of the Federal Reserve System H.15 series and 1 month treasury bill rates are from CRSP. Data from the FOMC meeting date closest to the middle of each quarter is used.

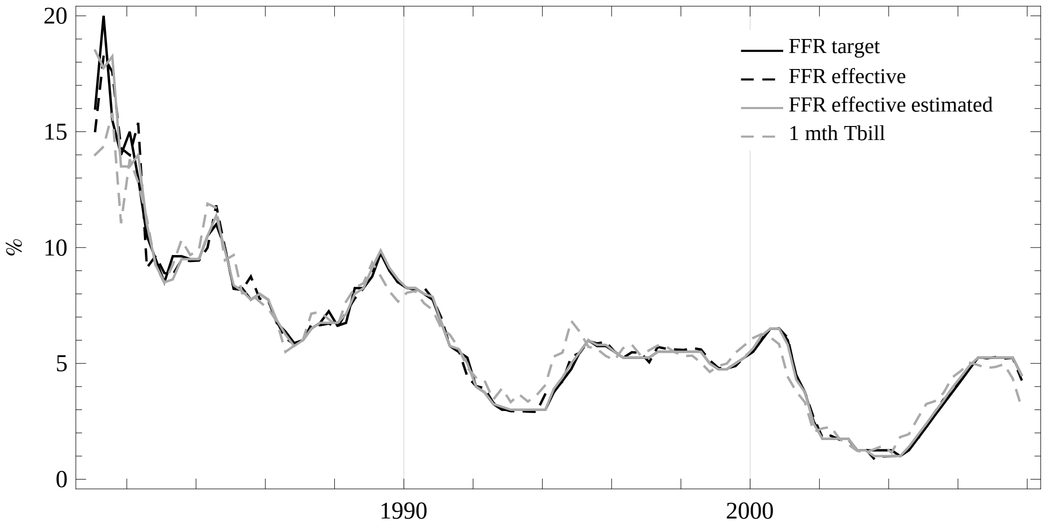

Figure 1.6: Comparison of proxies for the time-varying mean rate
The graph below compares the estimated time-varying long rates in the model of Balduzzi et al. (2000) (BDFS) and Schaefer and Schwartz (1984) (SSW), the time-varying inflation estimates of Fama (2006) (FAMA) and Cieslak and Povala (2011) (CP), as well as the ratio of young to middle-aged (YM) demographic factor over the period 1981-2007 based on quarterly intervals.

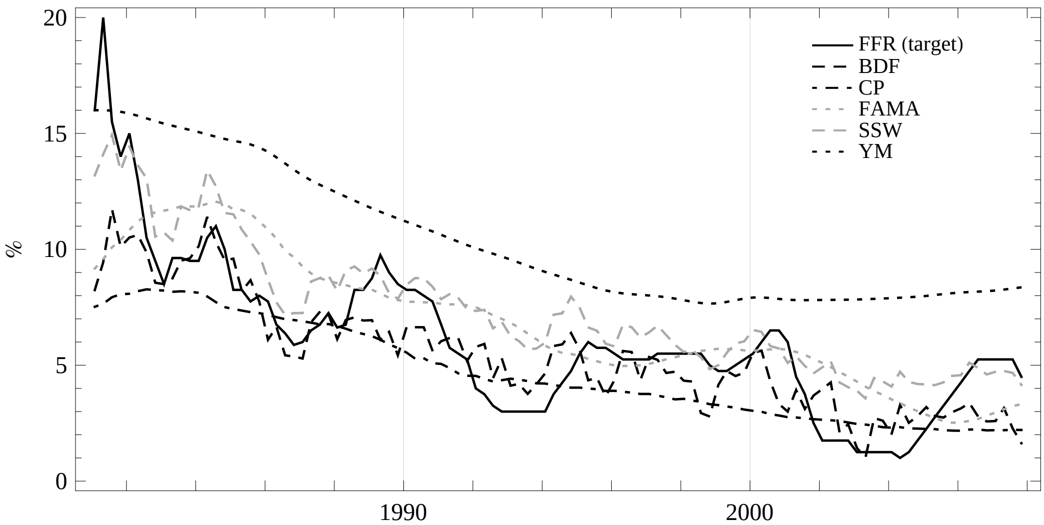

Figure 1.7: Performance of the preferred model (I)
Plot of changes in the FFR (full) against changes implied by the preferred model (dashed). The dashed grey line gives the deviation of the fitted value from the FFR of the previous period. The preferred model is given by: $i_t = 0.25 + 0.14\pi_{t|t} + 0.09x_{t|t} + 0.13\gamma_t + 1.13i_{t-1} - 0.34i_{t-2} + \epsilon_t$ where $E[\epsilon_{t+1}] = 0$ and $E[\epsilon_{t+1}^2] = 0.07^2 i_t^{2\times0.97}$

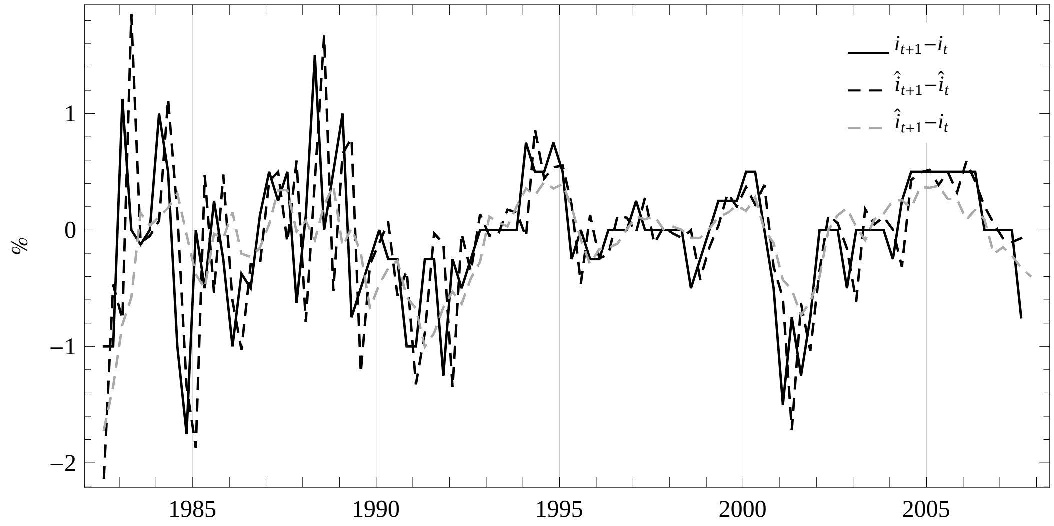

Figure 1.8: Performance of the preferred model (II)
Plot of absolute changes in the FFR against conditional volatility implied by preferred model. The change in the Federal Funds Rate is depicted by the full black line. The dotted black line depicts the conditional volatility implied by the preferred model. The preferred model is the same as in figure 1.7.

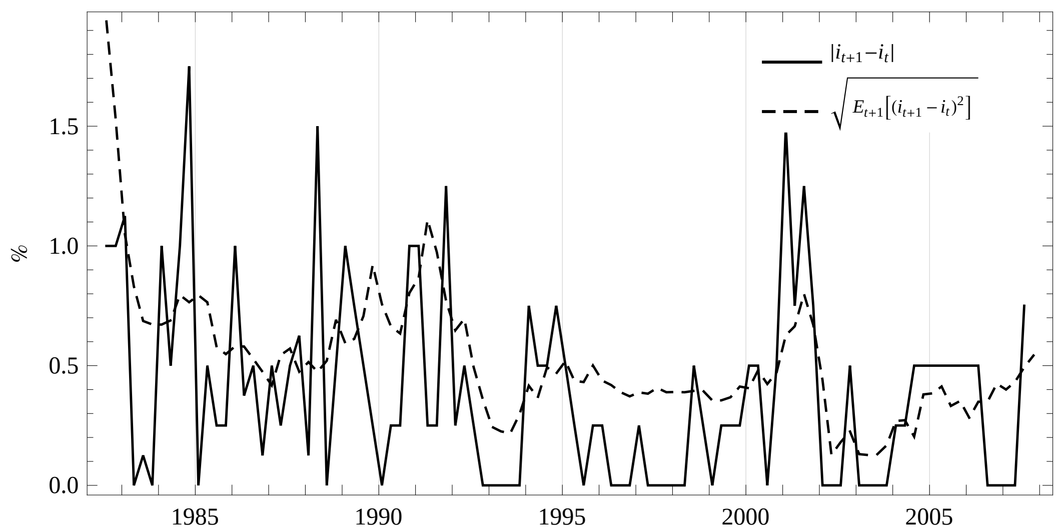

Figure 1.9: Plot of changes in the FFR against one-step-ahead expected change implied from treasuries

The dashed line is the one-step-ahead expected change in the short rate $E_t(i_{t+1}) - i_t$ implied from the treasury market, derived through the relation $y_{t,t+2} \approx 0.5(y_{t,t+1} + E_t(i_{t+1}))$ where $y_{t,t+2}$ is the yield on 6-month treasuries and $y_{t,t+1}$ is the yield on 3-month treasuries.

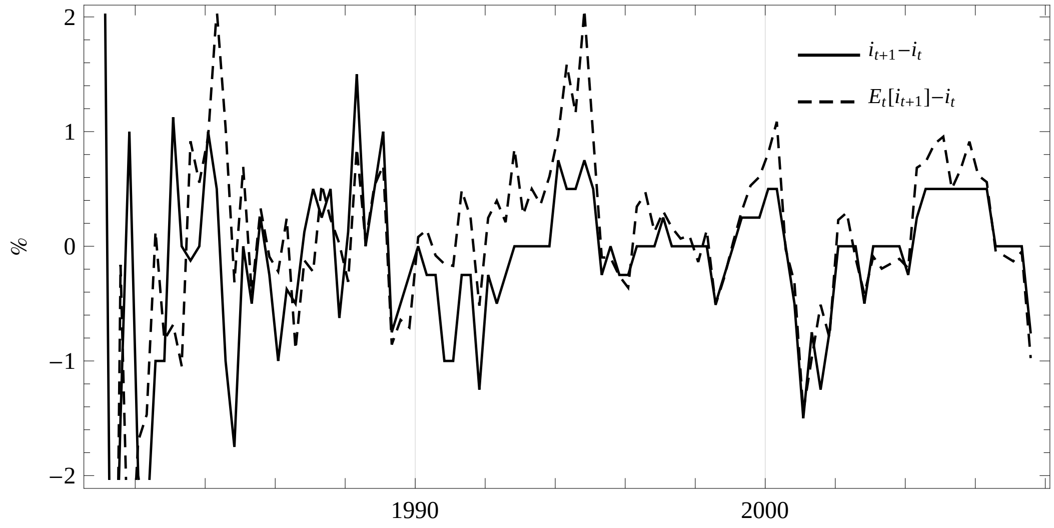

Figure 1.10: Hodrick-Prescott filter estimates of the real output gap

HP filter applied to real output data obtained from Croushore and Stark (2001) dataset over the period 1966 to 1987. The period 1966 - 1981 is used as a burn-in period. Only the estimated values in the period 1981-1987 are used in the econometric analysis. The HP filter solves for γ where $\gamma = \underset{\gamma}{argmin} \sum_{t=1}^{K} (z_t - \gamma_t)^2 + \lambda \sum_{t=1}^{K-2} ((\gamma_{t+2} - \gamma_{t+1}) - (\gamma_{t+1} - \gamma_t))^2$

Table 1.1: Descriptive statistics of different measures of the short rate
The table below shows descriptive statistics of the FFR target, FFR effective, estimated FFR effective and 1 month Treasury Bill rate (annualized). The four time series are identical to those described in figure 1.5. Descriptive statistics of the differences of each measure are given in the rows below the respective measure.

	Mean	Std Dev	Skewness	Kurtosis	Min	Max	ρ_1	ρ_2	ρ_3	ρ_4
Target	0.0609	0.0339	1.233	5.5366	0.01	0.2	0.967	0.9395	0.9037	0.8468
	-0.0011	0.0087	-0.5692	12.5161	-0.045	0.04	0.0529	-0.0378	0.3348	0.0445
Effective	0.0611	0.0339	1.1718	5.1196	0.0088	0.1826	0.9598	0.9243	0.8915	0.8294
	-0.001	0.0095	-2.502	19.6515	-0.0625	0.0323	-0.0128	-0.058	0.2638	0.0937
Estimate	0.0612	0.0345	1.2846	5.6068	0.01	0.185	0.9759	0.9391	0.8962	0.8413
	-0.0013	0.0077	-2.6971	15.353	-0.0475	0.01	0.2475	0.1012	0.3201	0.1026
1 mth Tbill	0.061	0.0303	0.8234	3.7802	0.0115	0.1581	0.9583	0.9269	0.8901	0.8481
	-0.001	0.0087	-1.2198	10.7769	-0.0476	0.0276	-0.1333	0.083	0.0621	0.0733

Table 1.2: Empirical comparison of 1-factor finance models in the period 1981-2007 (monthly)
I rerun the GMM estimation of the generalized model $dr_t = (\alpha + \beta r_t)dt + \sigma r_t^{\gamma} dZ_t$ as per Chan et al. (1992) over the period 1981-2007 using the 1 month t-bill as a proxy for the short rate and monthly data frequency. Moment conditions are given in appendix 1.8.2. T-statistics are in parentheses and R_1^2 and R_2^2 are the proportion of actual yield changes and squared yield changes explained by the predictive values for each model respectively. χ^2statistics and P-values of tests of the restricted models are also reported. 1 month T-Bill (annualized) yield data is from the CRSP database.

	α	β	σ^2	γ	χ^2	P-val	d.f	R_1^2	R_2^2
Unrestricted	0.0168	-0.3889	0.1509	1.0833				0.03	0.18
	(1.74)	(-1.78)	(1.07)	(6.4)					
Merton	0.0002	0	0.0002	0	13.3965	0.0012	2	0	-0.03
	(0.02)		(0.25)						
Vasicek	0.0195	-0.4379	0.0002	0	11.8327	0.0006	1	0.03	-0.03
	(2.03)	(-2.01)	(0.26)						
CIR SR	0.0203	-0.4595	0.0048	0.5	6.4709	0.011	1	0.03	0.07
	(2.11)	(-2.11)	(0.48)						
Dothan	0	0	0.093	1.	3.2503	0.3546	3	0	0.17
			(1.02)						
GBM	0	-0.0241	0.092	1.	3.3974	0.1829	2	0	0.17
		(-0.11)	(1.01)						
B-S	0.018	-0.4164	0.0951	1.	0.2328	0.6294	1	0.03	0.17
	(1.88)	(-1.91)	(0.99)						
CIR VR	0	0	0.1021	1.5	7.2252	0.0651	3	0	-0.05
			(1.18)						
CEV	0	-0.0019	0.0159	1.131	2.8653	0.0905	1	0	-0.06
		(-0.1)	(1.14)	(7.05)					

Table 1.3: Empirical comparison of 1-factor finance models in the period 1981-2007 (quarterly)

I rerun the estimation described in Table 1.2 using the target FFR as a proxy for the short rate and at a quarterly data frequency.

	α	β	σ^2	γ	χ^2	P-val	d.f	R_1^2	R_2^2
Unrestricted	0.0133	-0.2875	0.9819	1.6263				0.08	0.7
	(1.1)	(-1.22)	(0.57)	(4.57)					
Merton	-0.0021	0	0.0001	0	3.4824	0.1753	2	0	-0.03
	(-0.16)		(0.08)						
Vasicek	0.0178	-0.3867	0.0001	0	2.8786	0.0898	1	0.07	-0.03
	(1.51)	(-1.68)	(0.07)						
CIR SR	0.0186	-0.4022	0.0024	0.5	2.4102	0.1205	1	0.06	0.07
	(1.58)	(-1.76)	(0.11)						
Dothan	0	0	0.0396	1.	3.7558	0.2891	3	-0.02	0.29
			(0.25)						
GBM	0	-0.0495	0.0394	1.	3.0849	0.2139	2	0.02	0.29
		(-0.19)	(0.25)						
B-S	0.0203	-0.4347	0.0385	1.	1.5753	0.2094	1	0.06	0.28
	(1.74)	(-1.91)	(0.22)						
CIR VR	0	0	0.1345	1.5	2.7656	0.4292	3	-0.02	0.16
			(0.66)						
CEV	0	-0.0086	0.5741	1.7945	1.0258	0.3111	1	-0.01	0.28
		(-0.13)	(1.02)	(9.32)					

Table 1.4: Descriptive statistics of time-varying mean rate proxies

The table below displays descriptive statistics of the estimated time-varying long rates in the model of Balduzzi et al. (2000) (BDFS) and Schaefer and Schwartz (1984) (SSW), the time-varying inflation estimates of Fama (2006) (FAMA) and Cieslak and Povala (2011) (CP), and the young to middle-aged (YM) demographic factor.

	Mean	Std Dev	Skewness	Kurtosis	Min	Max	ρ_1	ρ_2	ρ_3	ρ_4
r_t	0.0609	0.0339	1.233	5.5366	0.01	0.2	0.967	0.9395	0.9037	0.8468
	-0.0011	0.0087	-0.5692	12.5161	-0.045	0.04	0.0529	-0.0378	0.3348	0.0445
BDFS	0.0547	0.0245	0.5989	2.6746	0.0099	0.1174	0.9435	0.9021	0.8762	0.839
	-0.0006	0.0082	0.0311	2.8808	-0.0214	0.0229	-0.1412	-0.0718	0.0212	-0.079
CP	0.0468	0.0207	0.4264	1.7479	0.0217	0.0827	0.9993	0.9978	0.996	0.994
	-0.0005	0.0008	0.2955	6.0763	-0.0026	0.0028	0.568	0.4525	0.3865	0.4185
FAMA	0.0685	0.0282	0.3778	2.0869	0.0252	0.1205	0.9982	0.9932	0.9858	0.9763
	-0.0005	0.0017	0.5496	4.384	-0.0053	0.0052	0.9071	0.8345	0.756	0.6385
SSW	0.0737	0.0279	0.9267	3.0586	0.0357	0.1494	0.9763	0.9524	0.9261	0.8984
	-0.0008	0.006	-0.1099	5.235	-0.0251	0.0157	-0.0009	0.061	-0.046	-0.0569
YM	0.1037	0.029	0.7621	2.0222	0.0766	0.16	0.9997	0.999	0.9977	0.9961
	-0.0007	0.0008	-0.1748	2.257	-0.0027	0.0007	0.9796	0.9269	0.8592	0.7963

Table 1.5: GMM estimates of alternative macro-finance models of the Federal Funds rate

The test statistic J is distributed χ^2 with degree of freedom equal to the number of parameter restrictions (d.f). $R^2_{2k-1}$is the proportion of total variation in interest rate levels ($k = 1$) and interest rate changes ($k = 2$) explained by the model. R^2_{2k} is the proportion of total variation in square interest rate levels ($k = 1$) and square interest rate changes ($k = 2$) explained by the model. The system of equations tested are (1.22) - (1.25) in section 1.3.

Model	c	ϕ_γ	ϕ_π	ϕ_x	δ_1	δ_2	δ_3	ρ_1	ρ_2	ρ_3	σ	ϑ	J	pval	d.f	R^2_1	R^2_2	R^2_3	R^2_4
1a.	0.09	0.09	0.03	0.06	1.59	-0.73	0.03	-0.37	-0.28	-0.08	0.09	0.91	0	-	-	0.97	0.94	0.44	0.59
2a.	0.23	0.57	1.06	0.59	0	0	0	0	0	0	0.5	0.19	24.67	0	6.	0.85	0.84	-1.54	-7.08
3a.	0.26	0.14	0.1	0.13	1.14	-0.36	0.01	0	0	0	0.08	0.96	2.18	0.54	3.	0.97	0.95	0.48	0.52
4a.	0.21	0.64	0.95	0.57	0	0	0	0.27	-0.1	0.07	0.69	0.09	27.08	0	3.	0.88	0.87	-1.02	-4.51
5a.	0.16	0.12	0.07	0.1	1.36	-0.52	0	-0.22	-0.18	0	0.09	0.89	0.2	0.91	2.	0.97	0.95	0.47	0.58
6a.	0.25	0.14	0.09	0.13	1.13	-0.34	0	0	0	0	0.07	0.97	2.17	0.7	4.	0.97	0.95	0.47	0.51
7a.	0.31	0.63	0.93	0.57	0	0	0	0.26	-0.03	0	0.7	0.06	28.21	0	4.	0.88	0.87	-1.01	-4.17
8a.	0.44	0.17	0.22	0.23	0.68	0	0	0.28	0	0	0.09	0.83	8.04	0.09	4.	0.97	0.94	0.44	0.42
9a.	0.43	0.16	0.25	0.25	0.67	0	0	0	0	0	0.1	0.79	16.09	0.01	5.	0.96	0.94	0.38	0.42
10a.	0.33	0.64	0.91	0.57	0	0	0	0.28	0	0	0.69	0.07	28.57	0	5.	0.89	0.87	-0.96	-3.82
11a.	-0.08	0	-0.09	-0.06	2.23	-1.27	0.11	-0.45	-0.36	-0.17	0.16	0.74	0.48	0.78	2.	0.94	0.9	0.01	0.34
12a.	1.25	0	1.63	0.65	0	0	0	0	0	0	0.66	0.16	32.01	0	7.	0.73	0.74	-3.58	-22.65
13a.	0.36	0	0.09	0.08	1.36	-0.54	0.06	0	0	0	0.07	0.96	8.71	0.07	4.	0.96	0.94	0.37	0.48
14a.	1.96	0	1.41	0.62	0	0	0	0.3	-0.2	0.13	1.05	0.03	30.28	0	4.	0.79	0.78	-2.67	-17.11
15a.	0.21	0	0.02	0.02	1.76	-0.81	0	-0.28	-0.27	0	0.1	0.84	3.29	0.35	3.	0.96	0.93	0.24	0.44
16a.	0.36	0	0.08	0.07	1.38	-0.49	0	0	0	0	0.07	0.99	8.8	0.12	5.	0.96	0.93	0.34	0.42
17a.	2.01	0	1.41	0.61	0	0	0	0.25	-0.16	0	1.23	0.06	30.48	0	5.	0.76	0.77	-3.03	-21.55
18a.	0.57	0	0.2	0.17	0.79	0	0	0.22	0	0	0.07	0.93	15.59	0.01	5.	0.96	0.94	0.38	0.37
19a.	0.53	0	0.26	0.2	0.77	0	0	0	0	0	0.07	0.92	19.7	0	6.	0.96	0.93	0.32	0.32
20a.	1.58	0	1.51	0.64	0	0	0	0.15	0	0	0.85	0.02	32.65	0	6.	0.76	0.76	-3.05	-17.38

Table 1.6: GMM estimates of alternative macro-finance models of the Federal Funds rate (1981-2007)

Parameters are estimated for all the models in table 1.5 with the additional restriction that the errors are homoskedastic ($\vartheta = 0$).

Model	c	ϕ_γ	ϕ_π	ϕ_x	δ_1	δ_2	δ_3	ρ_1	ρ_2	ρ_3	σ	ϑ	J	pval	d.f	R_1^2	R_2^2	R_3^2	R_4^2
1b.	0.19	0.07	0.22	0.13	1.75	-1.42	0.47	-0.75	-0.41	-0.01	0.35	0	5.3	0.07	2.	0.97	0.95	0.46	0.43
2b.	0.27	0.56	1.06	0.59	0	0	0	0	0	0	0.69	0	25.23	0	7.	0.85	0.84	-1.53	-6.98
3b.	0.36	0.14	0.15	0.17	1.02	-0.28	0	0	0	0	0.36	0	18.43	0	4.	0.97	0.95	0.48	0.46
4b.	0.28	0.63	0.95	0.57	0	0	0	0.28	-0.11	0.07	0.8	0	27.58	0	4.	0.88	0.87	-1.	-4.4
5b.	0.17	0.1	0.12	0.1	1.39	-0.57	0	-0.39	-0.25	0	0.33	0	12.02	0.01	3.	0.97	0.95	0.47	0.51
6b.	0.36	0.13	0.13	0.16	1.03	-0.27	0	0	0	0	0.36	0	18.06	0	5.	0.97	0.95	0.47	0.45
7b.	0.33	0.63	0.92	0.57	0	0	0	0.26	-0.04	0	0.78	0	28.52	0	5.	0.88	0.87	-1.	-4.22
8b.	0.5	0.18	0.25	0.26	0.64	0	0	0.21	0	0	0.37	0	14.02	0.02	5.	0.97	0.95	0.42	0.37
9b.	0.46	0.16	0.27	0.26	0.65	0	0	0	0	0	0.36	0	19.26	0	6.	0.96	0.94	0.37	0.35
10b.	0.35	0.64	0.91	0.57	0	0	0	0.27	0	0	0.77	0	28.83	0	6.	0.89	0.87	-0.96	-3.88
11b.	0.2	0	0.25	0.11	1.96	-1.77	0.65	-0.84	-0.54	-0.14	0.38	0	6.25	0.04	2.	0.96	0.95	0.37	0.43
12b.	1.27	0	1.64	0.65	0	0	0	0	0	0	0.85	0	32.78	0	8.	0.73	0.74	-3.58	-22.27
13b.	0.43	0	0.14	0.11	1.25	-0.47	0.07	0	0	0	0.35	0	22.25	0	5.	0.96	0.94	0.4	0.45
14b.	1.93	0	1.41	0.63	0	0	0	0.3	-0.19	0.13	1.	0	30.43	0	5.	0.79	0.78	-2.65	-17.32
15b.	0.19	0	0.1	0.05	1.6	-0.69	0	-0.36	-0.29	0	0.35	0	13.79	0.01	4.	0.96	0.93	0.34	0.45
16b.	0.42	0	0.12	0.1	1.25	-0.39	0	0	0	0	0.34	0	20.42	0	6.	0.96	0.94	0.38	0.39
17b.	1.36	0	1.61	0.64	0	0	0	0.19	-0.16	0	0.86	0	30.11	0	6.	0.75	0.76	-3.22	-20.62
18b.	0.65	0	0.24	0.2	0.76	0	0	0.19	0	0	0.33	0	19.24	0	6.	0.96	0.94	0.36	0.31
19b.	0.56	0	0.28	0.2	0.76	0	0	0	0	0	0.33	0	21.38	0	7.	0.96	0.93	0.31	0.25
20b.	2.37	0	1.23	0.65	0	0	0	0.39	0	0	0.99	0	38.64	0	7.	0.81	0.81	-2.23	-11.89

Table 1.7: Pairwise comparisons of alternative FFR model specifications

The models defined by the expressions (1.22) - (1.25) and the restrictions below are compared using pairwise Newey-West tests. The reference point for the tests is the model with restrictions $\delta_3 = \rho_1 = \rho_2 = \rho_3 = 0$, which is used either as the unrestricted alternative or as the restricted model.

Unrestricted model (initial restrictions)	Restricted model (additional restrictions)	d.f	$J \sim \chi^2_{d.f}$	$p - value$
$\delta_3 = \rho_1 = \rho_2 = \rho_3 = 0$	$\delta_2 = 0$	1	19.09	0
$\delta_3 = \rho_1 = \rho_2 = \rho_3 = 0$	$\delta_1 = \delta_2 = 0$	2	741.42	0
$\delta_3 = \rho_1 = \rho_2 = \rho_3 = 0$.	$\phi_\gamma = 0$	1	7.92	0.0049
$\delta_3 = \rho_1 = \rho_2 = \rho_3 = 0$	$\vartheta = 0$	1	23.95	0
$\delta_3 = \rho_3 = 0$	$\rho_1 = \rho_2 = 0$	2	1.27	0.5296
$\rho_1 = \rho_2 = \rho_3 = 0$	$\delta_3 = 0$	1	0.03	0.8553

Table 1.8: Pairwise comparisons of alternative FFR model specification with long yield as generational factor

The models defined by the expressions (1.22) - (1.25) and the restrictions below are compared using pairwise Newey-West tests. $\gamma_t = l_t$ where l_t is the ten year yield on US treasuries. The reference point for the tests is the model with restrictions $\delta_3 = \rho_1 = \rho_2 = \rho_3 = 0$, which is used either as the unrestricted alternative or as the restricted model.

Unrestricted model (initial restrictions)	Restricted model (additional restrictions)	d.f	$J \sim \chi^2_{d.f}$	$p - value$
$\delta_3 = \rho_1 = \rho_2 = \rho_3 = 0$	$\delta_2 = 0$	1	20.974	0
$\delta_3 = \rho_1 = \rho_2 = \rho_3 = 0$	$\delta_1 = \delta_2 = 0$	2	944.809	0
$\delta_3 = \rho_1 = \rho_2 = \rho_3 = 0$.	$\phi_\gamma = 0$	1	11.4054	0.0007
$\delta_3 = \rho_1 = \rho_2 = \rho_3 = 0$	$\vartheta = 0$	1	29.7368	0
$\delta_3 = \rho_3 = 0$	$\rho_1 = \rho_2 = 0$	2	0.5329	0.7661
$\rho_1 = \rho_2 = \rho_3 = 0$	$\delta_3 = 0$	1	0.0247	0.8750

Table 1.9: Pairwise comparisons of second alternative FFR model specification with demographic variable

The models defined by the expressions (1.22) - (1.25) and the restrictions below are compared using pairwise Newey-West tests. $\gamma_t = \left(\gamma_t^{CP} \quad YM_t \right)$ where YM_t is a demographic variable defining the ratio of young to middle aged. The reference point for the tests is the model with restrictions $\delta_3 = \rho_1 = \rho_2 = \rho_3 = 0$, which is used either as the unrestricted alternative or as the restricted model.

Unrestricted model (initial restrictions)	Restricted model (additional restrictions)	d.f	$J \sim \chi^2_{d.f}$	$p - value$
$\delta_3 = \rho_1 = \rho_2 = \rho_3 = 0$	$\phi_{\gamma,2} = 0$	1	0.02	0.8917
$\delta_3 = \rho_1 = \rho_2 = \rho_3 = 0$	$\delta_2 = 0$	1	18.69	0
$\delta_3 = \rho_1 = \rho_2 = \rho_3 = 0$	$\delta_1 = \delta_2 = 0$	2	724.15	0
$\delta_3 = \rho_1 = \rho_2 = \rho_3 = 0$.	$\phi_{\gamma,1} = 0$	1	3.18	0.0744
$\delta_3 = \rho_1 = \rho_2 = \rho_3 = 0$	$\vartheta = 0$	1	25.48	0
$\delta_3 = \rho_3 = 0$	$\rho_1 = \rho_2 = 0$	2	1.23	0.5397
$\rho_1 = \rho_2 = \rho_3 = 0$	$\delta_3 = 0$	1	0.04	0.8458

Chapter 2

An equilibrium term structure model of interest rates with limited access, local demand and a swap market

2.1 Background

Recent large central bank Outright Monetary Transaction (OMT) programmes have brought the theory of preferred habitat investment in the government bond market to the forefront of public policy.[1] Indeed without a preferred habitat theory these programmes are difficult to justify. Yet there are a number of basic questions relating to the theory that are unanswered. How do you rationalize maturity-specific bond demand in a general equilibrium framework? How do changes in the sources of local demand affect equilibrium yields? How does the presence of an interest rate derivative market affect equilibium bond yields and does it impede maturity-specific policymaker intervention? I address these questions by constructing an equilibrium model, which includes a devel-

[1] A detailed description of OMT programmes is discussed in Cø euré (2013). For an empirical discussion of the recent asset purchase programs see D'Amico et al. (2012), D'Amico and King (2013), Krishnamurthy and Vissing-Jorgensen (2011), Meaning and Zhu (2011), Swanson et al. (2011).

oped swap market where a subset of agents can hedge their interest rate risk, subject to bond-swap basis risk. The model also endogenises local demand factors that provide an explicit motive for (quasi) term-specific bond demand.

The interest rate swap market is one of the most liquid OTC derivative markets. In 2011, notional outstanding for US interest rate swaps was around $160 trillion [2], far exceeding the $15 trillion[3] of US government debt outstanding in the same year. Interest rate swaps switch fixed (floating) interest rate exposure into floating (fixed) exposure, allowing bond holders to eliminate a large proportion of a bond's price volatility. Such a sizeable market in both absolute and relative terms is relevant in an equilibrium analysis of the interest rate term structure. Its *absolute* size provides agents with a deep market for hedging interest rate exposure, and ensures the swap curve is less susceptible to time varying local demand. In the stripped down model of Vayanos and Vila (2009)[4] risk averse arbitrageurs integrate the bond market, but are limited in removing distortions in the term structure created by maturity clienteles. The limitation is brought about by the price volatility risk, induced by short rate volatility, to which the arbitrageur would be exposed. Therefore, in their model the interplay between short rate induced price volatility, risk aversion and maturity specific demand is crucial. Instead in the model described here agents with access to the swap market integrate the bond market. If a particular bond is considered cheap due to lack of maturity specific bond demand, these agents can purchase the bonds and hedge out the interest rate risk in the swap market. This trade earns the investor the bond 'z-spread', reflecting the difference between the bond yield and the swap spread.

Like in the Vayanos and Vila (2009) model agents with access to the swap market are limited somewhat in their ability to remove maturity clientele driven distortions, but the

[2]Data from Bank of International Settlement statistics and available on the website: http://www.bis.org/statistics/otcder/dt21a21b.pdf

[3]Data from Treasury Direct: http://www.treasurydirect.gov/

[4]Subsequent works include Greenwood and Vayanos (2010) and Greenwood and Vayanos (2014).

rationale differs. In this model the residual bond-swap basis risk creates the limitation rather than the short rate induced price volatility risk. The basis risk is smaller than the naked short rate risk and has different characteristics. This is seen clearly in Figure 2.1, which shows a time series of bond yields and bond-swap basis for the 1 and 5 year maturity points. Figure 2.2 shows the same is true for the 'presumed' bond reference rate, the FFR, and the basis between the swap reference rate (Libor) and the FFR.

The large *relative* size of the swap market also contributes to its relevance. A monetary policy programme that targets specific curve points through changes in bond supply will not have a first order effect on the swap market. To the extent that agents have access to the swap market and bond-swap basis volatility is low, and are therefore able to perform 'limited arbitrage', the effect of an OMT programme will be reduced.

In this paper I also consider the source of the maturity-specific demand. Maturity-specific demand is likely to arise for one of 3 reasons. Firstly, an agent may have liabilities on their balance sheet with a specific maturity, and valued with reference to the Treasury curve. This is the case for example with pension funds and insurers. The existence of these liabilities incentivises these agents to invest in government securities of similar duration. The agents may decide to deviate from the liability position in order to improve portfolio return, but the agent is exposed to liability mismatch risk. Ultimately the agent positions himself at a point where the tradeoff between portfolio return and liability mismatch risk is optimal .[5]

Secondly, an agent may only have access to a certain segment of the term structure. This is the case for example with households who have access to shorter maturity bonds through fixed deposits and money market funds offered by retail banks, but are unable to

[5]This is not too far from reality. Asset managers who offer LDI solutions perform this role. They are provided with a risk budget and a set of liabilities to match on behalf of the institution with fixed liabilities, and their role is to maximise return subject to these constraints.

access longer term bonds unless they are financially sophisticated. Limited access creates maturity-specific demand as funds that would have been assigned to other segments if a full portfolio optimization were possible, are assigned to less optimal segments. Finally, maturity-specific demand can arise if bonds with certain maturity characteristics confer benefits to the investor over and above the yield produced. This may be the case for example with shorter term Treasuries that are more likely to be eligible for collateral posting purposes or loan security purposes than longer term Treasuries, or may be subject to less of a haircut than longer term Treasuries. The additional liquidity option of the shorter term Treasuries has an economic value that is reflected in the price.

In this version of the model I explicitly endogenise the first two points and appeal informally to the third. I demonstrate that the interplay between bond-swap basis volatility, local clientele demand through limited access and predetermined liabilities, and the extent to which agents have access to the swap market plays an important role in the term structure equilibrium. The swap curve provides a natural reference curve to understand these effects as it cannot be subject to explicit curve point intervention in the same way as the bond term structure. Swap supply is by definition zero as short and long positions cancel. Moreover, there is no segmentation due to limited access in the swap market as agents with access to the swap curve have access to the entirety of the term structure.

The model also provides a definitive motive for the existence of maturity clienteles. In the Vayanos and Vila (2009) model maturity clienteles do not substitute between bonds of different maturities. This is the result of a portfolio optimization for T year lived agents with min-max preferences where the optimum portfolio is an allocation to only the T year bond and the private technology, *or equivalent*. Since the general model has a finite N factors in a market with infinite instruments, an equivalent portfolio can in theory be constructed in infinite ways. This fact is difficult to reconcile with the original linear demand function for a specific bond maturity.

The paradox is addressed partially in this paper by introducing a term-specific idiosyncratic factor. Ceteris parabis, institutions prefer to purchase a T year bond in order to hedge a T year liability. Deviating from the T year bond institutions trade off an increase in idosyncratic risk (and possibly outright short rate risk) for an increase in portfolio return. The approach is consistent with Cont (2005) who observes that traders and portfolio managers will prefer to hedge with an instrument of the same maturity and concludes that these practices imply the existence of a risk that is specific to the maturity of the claim. Whilst there is strong empirical evidence to support this representation[6], a precise explanation remains elusive. Heuristically, the idiosyncratic factor can be related to institutional flow, transaction costs, market participant inertia, out of model risks and the expectation of the future haircuts required for posting as a repoable asset, but the true mechanism remains a black box[7].

The results of the model have implications for monetary policy and the cost of government debt, especially at longer maturities. Increased agent access to the swap market and lower bond-swap reference rate basis volatility can reduce the cost of debt especially if the swap market is 'two-sided'. The effect is larger if the bond reference rate volatility is high. However, if access is high and basis volatility is low, explicit curve point interventions will have a limited effect. Agents with access to the swap market will undo the intervention in the bond market, hedge their position in the swap market and run a low basis risk. I find the share of overall equilibrium bond demand amongst investors with access to the swap market is large and there is an implicit subsidy from the agents without access to the swap market to agents with access to the swap market. This subsidy will be proportionally higher if only a small fraction of agents have access to the swap market.

[6]See for example Bouchaud et al. (1999) and Bouchaud et al. (1998).

[7]Although the source of the maturity-specific risk is opaque, what is most important is that market participants assume such a friction exists, regardless of whether they can pinpoint its cause.

The results of the model also show that the impact of changes in agent liabilities are similar to changes in bond supply with the sign reversed, provided the liability is valued off the Treasury curve. An agent liability is simply a short position in the government bond of equal maturity. Significant amounts of bond purchases by a central bank at maturities where there are equally large amounts of agent liabilities is akin to a large negative supply of bonds at those maturities. I also find term-specific risk to be a subtle but important feature of the model. For small levels of term-specific risk bonds are quasi-substitutable. They are simply scaled up versions of each other, where the scaling factor is the short-rate sensitivity of the bond. Any curve point intervention can then be seen as simply an addition or deduction to the overall supply of short-rate risk in the market. For maturity-specific interventions such as a twist operation to work as intended, agent assumed maturity-specific risk should be large.

The rest of the paper is organized as follows: Section 2.2 describes the model setup including the market instruments, risky processes, and conjectured bond and swap pricing functions. I set out the first order conditions with respect to bond and swap demand for each agent, and solve for the equilibrium bond demand for each agent and the equilibrium bond price. Section 2.3 describes the main results of the model and Section 2.4 provides some brief empirical results related to the qualitative results of the model. Finally, Section 2.5 concludes and describes some possible extensions.

2.2 The model

The model below is developed in a continuous bond supply setting, and builds on the work of Vayanos and Vila (2009).

2.2.1 Market instruments and processes

Time is continuous in the model. At time t there are zero-coupon (ZC) bonds supplied in the maturity interval $(0, \Lambda_2]$, where at any time t a bond with maturity τ pays 1 dollar at time $T = t + \tau$. The supply of the τ -maturity bond is given by k_τ . $P^B_{t,\tau} = P^B(t,T)$ is the time t zero-coupon bond price paying \$1 at time T, and is determined endogenously in the model[8]. The short rate r_t is given exogenously and is assumed to follow a 1-factor Vasicek process :

$$dr_t = \alpha_r\left(\mu_r - r_t\right)dt + \sigma_r dB^r_t \tag{2.1}$$

r_t can be interpreted practically as the (annualized) remuneration on overnight funds deposited with the central bank, broadly equivalent to the Federal Funds Rate (FFR). I choose the simplest possible specification for the short rate. The intention is to highlight the effect of the swap curve, institutional demand and other frictions on the interest rate term structure, rather than seek an accurate expression of the policy reaction function.

At each time t there is an infinitely deep market for zero-coupon interest rate swaps [9] in the maturity interval $(0, \Lambda_2]$. $P^S_{t,\tau} = P^S(t,T)$ is the time t price of the fixed leg of the ZC swap[10], which pays \$1 at time $T = t + \tau$. The floating leg of the zero-coupon swaps references a swap short rate $m_t \equiv r_t + s_t$, which is a spread above the risk free short rate.[11] The spread s_t is given exogenously and evolves as a 1-factor Vasicek process.

$$ds_t = \alpha_s\left(\mu_s - s_t\right)dt + \sigma_s dB^s_t \tag{2.2}$$

[8]The bond price is a function of time t and final maturity date T. It is denoted as $P^B_{t,\tau}$, $\tau \equiv T - t$, for consistency with the notation in Vayanos and Vila (2009)

[9]In reality zero-coupon interest rate swaps are not liquid. Interest rate swaps are typically traded in par format, but this should not detract from the model in the same way that hypothetical ZC bonds do not.

[10]Including the notional exchange

[11]m_t is assumed to evolve continuously. In reality the swap reference rate is subject to periodic (but not continuous) resets.

where B_t^s evolves independently to B_t^r[12]. s_t may be interpreted practically as the spread between the (normalized) 6 month Libor rate and the FFR. Figure 2.2 displays a historical time series of this spread for the US market. I ignore the counterparty risk embedded in the swap, an accepted practice in the recent literature on bond-swap basis, see for example Fujii and Takahashi (2009).[13]

Appealing to the local deformation literature (see e.g. Kennedy (1994), Bouchaud et al. (1999), Goldstein (2000), Cont (2005)) each swap and bond is subject to an exogenous maturity-specific factor, $\zeta_{t,\tau}^S = \zeta^S(t,T)$, and $\zeta_{t,\tau}^B = \zeta^B(t,T)$. The local factor renders the swap unreplicable by any other combination of securities and eliminates the possibility of simple arbitrage arguments. $\zeta_{t,\tau}^S$and $\zeta_{t,\tau}^B$ are stationary Gaussian random fields with increments along the time dimension distributed as:

$$d\zeta_{t,\tau}^i \equiv \zeta_{t+dt,\tau-dt}^i - \zeta_{t,\tau}^i \sim N\left(0, \sigma_{\zeta^i}^2 dt\right) \quad i \in \{B, S\} \tag{2.3}$$

and correlation across the maturity dimension given by:

$$Cor(d\zeta_{t,\tau_1}^i, d\zeta_{t,\tau_2}^j) = \begin{cases} \delta^{|\tau_2-\tau_1|} & i = j \\ 0 & i \neq j \end{cases} \tag{2.4}$$

where $0 < \delta \leq 1$. A random field with these properties can be generated using the simple approach described in Goldstein (2000). A brief review of the approach is given in the appendix 2.6.1. The correlation structure of local shocks given above is a decreasing function of the distance between maturities $|\tau_2 - \tau_1|$, and the speed of decrease as a function of $|\tau_2 - \tau_1|$ is controlled by the parameter δ. In order for the interpretation of a local shock to hold I assume δ is close to zero.[14]

[12]This assumption can be relaxed but it complicates the subsequent expressions without substantially adding to the results.

[13]Counterparty risk on interest rate swaps is reduced significantly through frequent collateral posting arrangements. Newer studies such as Feldhütter and Lando (2008) instead focus on the risks in the swap reference rate spread and the liquidity option provided by treasuries.

[14]In fact in the discrete bond setup where the maturity gap Δt between bonds/swaps is not too small,

Finally, define $P^{BS}_{t,\tau,u}$ as a package of a ZC bond with maturity τ and an equal maturity fixed-for-floating ZC swap struck at u with notional $P^{B}_{t,\tau}$.[15]Without loss of generality assume the agents enter in zero market-value swaps when choosing their optimum portfolio.[16] Hereon I drop the subscript u referring to the strike when it is clear u is chosen to set the market-value of the swap to zero.

We can now use our definition of the bond-swap package, together with the outright bond, to replicate a swap position. The swap position is equivalent to a long position in a bond and a short position in the bond-swap package. Thus it is sufficient for the agent first order conditions derived in the following sections to refer only to the distributions of bond price returns and bond-swap package returns. Defining the instruments in this way aids intuition and facilitates the interpretation of the results.

2.2.2 Agents

There are 3 representative agents in this economy. Agent 1 is broadly intended to represent households, Agent 2 represents typical institutions and Agent 3 represents sophisticated institutions. At each time t Agent 1 has access to bonds in the non-empty maturity interval $(0, \Lambda_1]$. Agent 2 has access to the full bond term structure $(0, \Lambda_2]$, $0 < \Lambda_1 \leq \Lambda_2$, and Agent 3 has access to the full bond term structure $(0, \Lambda_2]$ as well as access to swaps in the maturity interval $(0, \Lambda_2]$. In addition all agents have access to the cash rate r_t . Total funds available for investment in a bond portfolio are split amongst agents as: $\theta_1 : \theta_2 : \theta_3$, where $\theta_3 = 1 - \theta_1 - \theta_2$. Agent i has fixed nominal liabilities valued on the treasury curve

setting $\delta = 0$ would be a reasonable assumption. However in the continuous case the assumption, apart from being unrealistic, would render the maturity-specific risk inconsequential as it is diversified out by integrating over any non-empty interval.

[15]As per market convention, the notional of the ZC swap refers to the value of the fixed leg, including the notional exchange, at inception. A τ maturity ZC fixed-for-floating swap with notional N and fixed rate u pays $N\left(e^{u\tau} - 1\right)$ and receives $N\left(e^{\int_0^\tau r_t dt} - 1\right)$ at time τ.

[16]A position in an 'off-market' swap can be replicated by a position in a zero-market swap and an amount (possibly negative, if the swap is 'out-the-money') invested at the cash rate.

payable at τ, with market value given by $l_\tau^{(i)}$, expressed as a proportion of θ_i . Household treasury-based liabilities are set equal to zero for all maturities, $l_\tau^{(1)} \equiv 0$. Similar to the arbitrageurs in Vayanos and Vila (2009) Agent i is endowed with preferences over the bond portfolio that trades off mean and variance of dollar returns. Let $x_{t,\tau}^{(i)}$ be the proportion of the bond portfolio allocated to the maturity τ bond by Agent i , and $y_{t,\tau}^{(3)}$ be the proportion of the third Agent's portfolio allocated to a bond-swap package. $d\bar{W}_t^{(i)}$ denotes the change in bond portfolio wealth, $W_t^{(i)}$, of Agent i net of liabilities. Then the budget constraint for agents 1 and 2 are given by:

$$d\bar{W}_t^{(i)} = \left[W_t^{(i)} - \int_0^{\Lambda_i} x_{t,\tau}^{(i)} d\tau\right] r_t dt + \int_0^{\Lambda_i} \left(x_{t,\tau}^{(i)} - l_\tau^{(i)}\right) \frac{dP_{t,\tau}^B}{P_{t,\tau}^B} d\tau,\ i \in \{1,2\} \tag{2.5}$$

and the budget constraint for agent 3 is:

$$d\bar{W}_t^{(3)} = \left[W_t^{(3)} - \int_0^{\Lambda_2} \left(x_{t,\tau}^{(3)} + y_{t,\tau}^{(3)}\right) d\tau\right] r_t dt + \int_0^{\Lambda_2} \left(x_{t,\tau}^{(3)} - l_\tau^{(3)}\right) \frac{dP_{t,\tau}^B}{P_{t,\tau}^B} d\tau + \int_0^{\Lambda_2} y_{t,\tau}^{(3)} \frac{dP_{t,\tau}^{BS}}{P_{t,\tau}^{BS}} d\tau \tag{2.6}$$

As in Vayanos and Vila (2009) agents are endowed with mean-variance preferences over changes in net wealth, where one possible interpretation of these preferences is the existence of an overlapping generation of agents that live for an infinitesimal time interval. In addition agents have linear and separable preferences over the term specific properties embedded in the bonds and swaps, which could relate to transactability, flow, out of model risks and repoability.

The optimization problem of agent i, is:

$$\max_{\left\{x_{t,\tau}^{(i)}\right\}_{\tau\in(0,\Lambda_{min(i,2)})}} E_t\left[dW_t^{(i)}\right] - \frac{\gamma}{2} V_t\left[dW_t^{(i)}\right] + \Phi_t^{(i)} dt \tag{2.7}$$

where γ is a measure of risk aversion and $V[\cdot]$ is the variance function. $\Phi_t^{(i)}$ is a linear function of time t maturity-specific bond and swap factors only:

$$\Phi_t^{(i)} = \begin{cases} -\int_0^{\Lambda_i} x_{t,\tau}^{(i)} \zeta_{t,\tau}^B d\tau & i \in \{1,2\} \\ -\int_0^{\Lambda_3} (x_{t,\tau}^{(i)} + y_{t,\tau}^{(i)}) \zeta_{t,\tau}^B d\tau + \int_0^{\Lambda_i} y_{t,\tau}^{(i)} \zeta_{t,\tau}^S d\tau & i = 3 \end{cases} \tag{2.8}$$

2.2.3 Pricing functions

2.2.3.1 Swap market pricing function

Swap market prices are given exogenously in the model and are affine in r_t and s_t. Each swap is also subject to the maturity specific factor $\zeta_{t,\tau}^S$, scaled by term to maturity. The price function of the fixed leg of the swap is given by:

$$P_{t,\tau}^S = e^{-r_t D(\tau) - s_t E(\tau) - F(\tau) - \tau \zeta_{t,\tau}^S} \tag{2.9}$$

The local demand factor can be viewed as an exogenous market friction anticipated by swap market participants. The friction results in a residual risk for a portfolio of swaps whose short rate sensitivity is completely hedged, unless of course the sum of short and long positions in each swap is identically equal to zero.

Change in the price of the fixed leg of the swap, is given by[17]:

$$dP_{t,\tau}^S / P_{t,\tau}^S \ = -dr_t D(\tau) - ds_t E(\tau) + r_t D'(\tau) dt + s_t E'(\tau) dt \ + F'(\tau)\, dt - \tau d\zeta_{t,\tau}^S + \zeta_{t,\tau}^S dt + \Pi^S(\tau) dt$$

(2.10)

where

$$\Pi^S(\tau) = \frac{1}{2} V\left[dP_{t,\tau}^S / P_{t,\tau}^S\right] = \frac{1}{2}\left(D(\tau)^2 + E(\tau)^2 + \tau^2 \sigma_{\zeta^S}^2\right)$$

[17] In an extension of this model one may want to consider defining $\zeta_{t,\tau}^S$ as a mean reverting process with mean 0.

is a convexity term. Pricing in the swap market is set exogenously in the model through the following relation:

$$E_t\left[dP^S_{t,\tau}/P^S_{t,\tau}\right] \equiv \left(r_t + s_t + \zeta^S_{t,\tau} + D(\tau)\lambda_r\sigma_r + E(\tau)\lambda_s\sigma_s\right) dt \tag{2.11}$$

where $(r_t + s_t)\, dt$ is the increase in the floating leg of the swap position. The net expected return of a floating for fixed swap is then:

$$E_t\left[dP^S_{t,\tau}/P^S_{t,\tau}\right] - (r_t + s_t)\, dt = \left(\zeta^S_{t,\tau} + D(\tau)\lambda_r\sigma_r + E(\tau)\lambda_s\sigma_s\right) dt \tag{2.12}$$

λ_r is interpreted broadly as the price of one unit of short interest rate risk and λ_s is the price of one unit of spread risk.[18] Alternatively, λ_r and λ_s indicate the aggregate position of institutional swap market participants with respect to interest rate and swap spread risk. If $\lambda_r = \lambda_s = 0$ then the swap market is 'two-sided' , and is only subject to temporary maturity-specific idiosyncratic shocks that lead to a non-zero expected return from a swap position .[19]

By matching terms, we derive the following expressions for $D(\tau)$,$E(\tau)$, and $F(\tau)$:

$$D(\tau) = \frac{1 - e^{-\alpha_r\tau}}{\alpha_r} \tag{2.13}$$

$$E(\tau) = \frac{1 - e^{-\alpha_s\tau}}{\alpha_s} \tag{2.14}$$

$$F(\tau) = \sum_{i\in\{r,s\}} \frac{\left(3 + e^{-2\tau\alpha_i} - 4e^{-\tau\alpha_i} - 2\tau\alpha_i\right)\sigma_i^2}{4\alpha_i^3} - \frac{1}{6}\tau^3\sigma^2_{\zeta^S} + \Upsilon(\tau) \tag{2.15}$$

$$\Upsilon(\tau) = \sum_{i\in\{r,s\}} \frac{\left(-1 + e^{-\tau\alpha_i}\right)\lambda_i\sigma_i}{\alpha_i^2} + \frac{\tau\lambda_i\sigma_i}{\alpha_i} \tag{2.16}$$

[18] Note the (infinite) maturity specific demand factors in the model means λ_r and λ_s does not follow from the absence of arbitrage, unlike in traditional finite factor asset pricing models.

[19] If $\lambda_r, \lambda_s < 0$ then heuristically there are more i) unfunded fixed institutional liabilities with projected floating rate inflows than ii) unfunded institutional floating rate liabilities with projected future fixed rate inflows than projected fixed asset cashflows. An example of i) would be corporations with fixed debt servicing payments but with flexible service contracts or product prices. An example of ii) would be banks that issue fixed rate mortgages and loans funded with bank deposits.

where $\Upsilon(\tau)$ is the adjustment factor that accounts for a (possibly) lop-sided swap market. Note that $\Upsilon(\tau) = 0$ when $\lambda_r = \lambda_s = 0$.

2.2.3.2 Bond market pricing function

In contrast to the swap pricing above, bonds are priced endogenously in the model. Agents conjecture that bond prices are of the form:

$$P_{t,\tau}^{B} = e^{-r_t A(\tau) - C(\tau) - \Xi(\tau)\zeta_{t,\tau}^{B}} \tag{2.17}$$

$A(\tau)$,$C(\tau)$ and $\Xi(\tau)$ are determined endogenously through the agent first order conditions and the market clearing condition. The time t instantaneous return of a bond maturing at time τ is:

$$dP_{t,\tau}^{B}/P_{t,\tau}^{B} = -A(\tau)dr_t + r_t A'(\tau)dt + C'(\tau)\,dt + \Xi'(\tau)\zeta_{t,\tau}^{B}dt - \Xi(\tau)d\zeta_{t,\tau}^{B} + \Pi^{B}(\tau) \tag{2.18}$$

and expectation and volatility given by:

$$E_t\left[dP_{t,\tau}^{B}/P_{t,\tau}^{B}\right] = -\alpha_r\left(\mu_r - r_t\right)A(\tau)dt + r_t A'(\tau)dt + C'(\tau)\,dt + \Xi'(\tau)\zeta_{t,\tau}^{B}dt + \Pi^{B}(\tau)dt \tag{2.19}$$

$$V\left[dP_{t,\tau}^{B}/P_{t,\tau}^{B}\right] = A(\tau)^2\sigma_r^2 dt + \Xi(\tau)^2\sigma_{\zeta^B}^2 dt$$

where $\Pi^{B}(\tau)$ is the convexity term:

$$\Pi^{B}(\tau)dt = \frac{1}{2}V\left[dP_{t,\tau}^{B}/P_{t,\tau}^{B}\right] \tag{2.20}$$

The expectation and volatility of return on the bond swap package is:

$$E_t\left[dP_{t,\tau}^{BS}/P_{t,\tau}^{BS}\right] = E_t\left[dP_{t,\tau}^{B}/P_{t,\tau}^{B}\right] - \zeta_{t,\tau}^{S} - D(\tau)\lambda_r\sigma_r - E(\tau)\lambda_s\sigma_s \tag{2.21}$$

$$V\left[dP_{t,\tau}^{BS}/P_{t,\tau}^{BS}\right] = (D(\tau) - A(\tau))^2\sigma_r^2 + E^2(\tau)\sigma_s^2 + \Xi(\tau)^2\sigma_{\zeta^B}^2 + \tau^2\sigma_{\zeta^S}^2 \tag{2.22}$$

The three terms on the right of expression (2.21) arise from the net expected return of a fixed for floating swap, which is simply minus the expected return of a floating for fixed swap given in (2.12).

2.2.4 Agent first order conditions

The first order and market clearing conditions for the three agents are given below and allow us to solve for $A(\tau)$, $C(\tau)$ and $\Xi(\tau)$.

***Agent 1*:**

Recall the optimization problem of Agent 1 given by (2.7) and the budget constraint in (2.5). Appendix 2.6.2 shows that $\Xi(\tau) = \tau$ is a necessary condition for equilibrium to hold. I substitute this condition directly into the F.O.C's to simplify the algebra somewhat. For all $0 < \tau \leq \Lambda_1$, the first order conditions are[20]:

$$\phi(\tau)r_t + \bar{\phi}(\tau) = \gamma\sigma_r^2 \int_0^{\Lambda_1} \left(x_{t,s}^{(1)} - l_s^{(1)}\right) A(s)dsA(\tau) + \sigma_{\zeta^B}^2 \tau\gamma \int_0^{\Lambda_1} \left(x_{t,s}^{(1)} - l_s^{(1)}\right) \delta^{|\tau-s|} sds \tag{2.23}$$

where:

$$\phi(\tau) = A(\tau)\alpha^r + A'(\tau) - 1 \tag{2.24}$$

$$\bar{\phi}(\tau) = -A(\tau)\alpha^r \mu_r + C'(\tau) + \Pi^B(\tau) \tag{2.25}$$

Setting $z_{t,\tau}^{(i)} \equiv x_{t,\tau}^{(i)} - l_\tau^{(i)}$ the expression simplifies to:

$$\phi(\tau)r_t + \bar{\phi}(\tau) = \gamma\sigma_r^2 \int_0^{\Lambda_1} z_{t,s}^{(1)} A(s)dsA(\tau) + \sigma_{\zeta^B}^2 \tau\gamma \int_0^{\Lambda_1} z_{t,\tau}^{(1)} \delta^{|\tau-s|} sds \tag{2.26}$$

[20]Ignoring maturity specific terms that add to zero when $\Xi(\tau) = \tau$.

Agent 2:

The conditions for agent 2 are similar bar the upper limit of the integral. For all $0 < \tau \leq \Lambda_2$ we have:

$$\phi(\tau)r_t + \bar{\phi}(\tau) = \gamma\sigma_r^2 \int_0^{\Lambda_2} z_{t,s}^{(2)} A(s) ds A(\tau) + \sigma_{\zeta^B}^2 \tau\gamma \int_0^{\Lambda_2} z_{t,\tau}^{(2)} \delta^{|\tau - s|} sds \tag{2.27}$$

Agent 3:

The optimization problem and budget constraint for Agent 3 are given in (2.7) and (2.6) respectively. For all $0 < \tau \leq \Lambda_2$ the first order condition with respect to $z_{t,\tau}^{(3)}$ is:

$$\phi(\tau)r_t + \bar{\phi}(\tau) = \gamma(\sigma_r^2 \mathfrak{I}_1 + \sigma_{\zeta^B}^2 \mathfrak{I}_2) \tag{2.28}$$

where $\mathfrak{I}_1$ and $\mathfrak{I}_2$ are defined as:

$$\begin{aligned} \mathfrak{I}_1 &= \int_0^{\Lambda_2} \left(\left(z_{t,s}^{(3)} + y_{t,s}^{(3)} \right) A(s) - D(s) y_{t,s}^{(3)} \right) ds A(\tau) \\ \mathfrak{I}_2 &= \int_0^{\Lambda_2} \left(z_{t,s}^{(3)} + \ y_{t,s}^{(3)} \right) \delta^{|\tau - s|} sds \end{aligned}$$

For all $0 < \tau \leq \Lambda_2$ the first order condition with respect to $y_{t,\tau}^{(3)}$ is given by:

$$\phi(\tau)r_t + \bar{\phi}(\tau) - D(\tau)\lambda_r\sigma_r - E(\tau)\lambda_s\sigma_s = \gamma(\sigma_{\mathfrak{r}}^2 \mathfrak{I}_3 + \sigma_{\zeta^B}^2 \mathfrak{I}_4) \tag{2.29}$$

where $\mathfrak{I}_3$ and $\mathfrak{I}_4$ are defined as:

$$\begin{aligned} \mathfrak{I}_3 &= \int_0^{\Lambda_2} \left(\left(z_{t,s}^{(3)} + y_{t,s}^{(3)} \right) A(s) - D(s) y_{t,s}^{(3)} \right) ds (A(\tau) - D(\tau)) + \frac{\sigma_s^2}{\sigma_r^2} \int_0^{\Lambda_2} y_{t,s}^{(3)} E(s) ds E(\tau) \\ \mathfrak{I}_4 &= \tau \int_0^{\Lambda_2} \left(z_{t,\tau}^{(3)} + \ y_{t,\tau}^{(3)} \right) \delta^{|\tau - s|} sds + \frac{\sigma_{\zeta^S}^2}{\sigma_{\zeta^B}^2} \tau \int_0^{\Lambda_2} y_{t,\tau}^{(3)} \delta^{|\tau - s|} sds \end{aligned}$$

Finally, the market clearing condition is:

$$\theta_1 x_{t,\tau}^{(1)} 1_{\tau < \Lambda_1} + \theta_2 x_{t,\tau}^{(2)} + \theta_3 \left(x_{t,\tau}^{(3)} + y_{t,\tau}^{(3)} \right) = k_\tau \tag{2.30}$$

(2.30) can be rewritten as:

$$\theta_1(z_{t,\tau}^{(1)} + l_\tau^{(1)}) + \theta_2(z_{t,\tau}^{(2)} + l_\tau^{(2)}) + \theta_3\left(z_{t,\tau}^{(3)} + l_\tau^{(3)} + y_{t,\tau}^{(3)}\right) = k_\tau \tag{2.31}$$

2.2.5 Equilibrium prices and demand

The first order conditions (2.26) - (2.29) have the form of Fredholm integral equations of the first type. The reader will note the similarity with a Volterra integral equation, with the exception of the constant limits of integration. Fredholm integral equations often arise in the theory of signal processing and have been studied comprehensively (see for example Pogorzelski (1966) or Polyanin and Manzhirov (2012)). The factor multiplying the demand variable within the integral is termed the kernel of the integral equation. In (2.26) and (2.27) the kernel, $K(\tau, s)$ is:

$$K(\tau, s) = \gamma\sigma_r^2 A(s)A(\tau) + \sigma_{\zeta^B}^2 \tau\gamma\delta^{|\tau-s|}s \tag{2.32}$$

The kernel in (2.32), and the kernels of the agent 3 F.O.C's are non-degenerate, which implies there does not exist a function $p(s)$ such that:

$$\int_0^{\Lambda_i} K(\tau, s)p(s)ds = 0 \tag{2.33}$$

The result can be traced back to the non-replicability of bonds of a specific maturity. It is the infinite dimensional analog of a sufficient condition for matrix invertibility in finite dimensions. The kernels in (2.26) - (2.29) are non-degenerate because there are no redundant bonds in the asset universe.

For analytical tractability I reduce the conditions (2.26) - (2.29) to a set of Fredholm integral equations of the second kind, with degenerate kernels.

Proposition 1. *For small δ, (2.26), (2.27), (2.28) and (2.29) can be approximated as:*

$$\phi(\tau)r_t + \bar{\phi}(\tau) = \gamma\sigma_r^2 \int_0^{\Lambda_1} z_{t,s}^{(1)} A(s)dsA(\tau) + \sigma_{\zeta^B}^2 \tau^2 \gamma z_{t,\tau}^{(1)} \kappa \tag{2.34}$$

$$\phi(\tau)r_t + \bar{\phi}(\tau) = \gamma\sigma_r^2 \int_0^{\Lambda_2} z_{t,s}^{(2)} A(s)dsA(\tau) + \sigma_{\zeta^B}^2 \tau^2 \gamma z_{t,\tau}^{(2)} \kappa \tag{2.35}$$

$$\phi(\tau)r_t + \bar{\phi}(\tau) = \gamma(\sigma_r^2 \mathfrak{I}_1 + \sigma_{\zeta^B}^2 \tilde{\mathfrak{I}}_2) \tag{2.36}$$

$$\phi(\tau)r_t + \bar{\phi}(\tau) - D(\tau)\lambda_r\sigma_r - E(\tau)\lambda_s\sigma_s = \gamma(\sigma_{\mathfrak{r}}^2 \mathfrak{I}_3 + \sigma_{\zeta^B}^2 \tilde{\mathfrak{I}}_4) \tag{2.37}$$

where $\tilde{\mathfrak{I}}_2$, and $\tilde{\mathfrak{I}}_4$ are defined as:

$$\tilde{\mathfrak{I}}_2 = \left(z_{t,\tau}^{(3)} + \ y_{t,\tau}^{(3)}\right)\tau^2\kappa$$
$$\tilde{\mathfrak{I}}_4 = \tau^2 \left(z_{t,\tau}^{(3)} + \ y_{t,\tau}^{(3)}\right)\kappa + \frac{\sigma_{\zeta^S}^2}{\sigma_{\zeta^B}^2}\tau^2 y_{t,\tau}^{(3)}\kappa$$

Proof. See appendix.

□

Note that in (2.34)-(2.37) the demand variables appear both within the integral and as a separate term. Closed form solutions for integral equations with kernels specified in the first order conditions exist and are used to solve the system of equations (2.34) - (2.37) together with the market clearing condition (2.30). To simplify, I drop the scaling factor κ in the rest of the text as $\sigma_{\zeta^B}^2$ and $\sigma_{\zeta^S}^2$ can be adjusted according. Alternatively, the results can be interpreted as corresponding to the case where $\kappa = 1$ so $log\delta = -2$.

The next result states that in equilibrium the bond and swap sensitivity to the short rate must be equal.

Proposition 2. *The system (2.30) - (2.37) can only be satisfied if $A(\tau) = D(\tau)$*

Proof. See appendix. □

The result in Proposition 2 is not surprising. In the model of Vayanos and Vila (2009) maturity clienteles are assumed to have a demand function linear in yield. An increase in the short rate increases aggregate maturity clientele demand for bonds. For markets to clear, arbitrageurs must increase their short position in bonds through the carry trade. In order to entice arbitrageurs to hold the short position in bonds, the expected return on a bond as a function of the short rate ($\phi(\tau)$ in this model) should be negative. But in this model agents are not assumed to have linear demand in yield. The heterogeneity of the agents arises from limited access and predetermined liabilities. Since the term structure of liabilities of agents is assumed to remain constant over time, $A(\tau) = D(\tau)$ must hold else all agents would demand more bonds as a function of r_t when $\phi(\tau) > 0$ and vice versa, which is not compatible with the market clearing condition.

Whilst the result in Proposition 2 may be problematic in terms of explaining the carry trade and the bond risk premia as a function of r_t, it makes the model more tractable and focusses attention on the impact of the swap curve on changes in bond demand and supply.[21] Moreover, it is reasonable to assume that bond and swap sensitivity to changes in the short rate are equal. In fact the empirical study in section 2.4.3 suggests this might be the case. Additionally, practitioners are likely to assume it is true, regardless of whether or not it actually is, and this is reflected in their individual optimization problems.

The next proposition gives the Agent 1 and 2 net bond demand function. For the next

[21] Similar results to Vayanos and Vila (2009) can be obtained by assuming in this model that the market value of bond supply and liabilities is a decreasing linear function of r_t. For example setting the bond supply at time t equal to $k_{t,\tau} = k^0_\tau + k^1_\tau r_t$ where $k^1_\tau < 0$. As the short rate rises demand must fall for market to clear. Ceteris parabis, demand will fall if $\phi(\tau) < 1$. Since $A(\tau)$ is of the form $\frac{1-e^{-\alpha_A \tau}}{\alpha_A}$ it follows that $\alpha_A > \alpha_r$ and the rate of mean reversion implied by the price of bonds is greater than the real world mean reversion rate.

propositions, it will be convenient to define the following notation:

$$\vartheta_i = \left(\sigma^2_{\zeta^B} + \sigma^2_r \int_0^{\Lambda_i} \frac{D(s)^2}{s^2}\, ds\right)^{-1}, \quad i \in \{1,2\}$$

$$\vartheta_3 = \vartheta_2$$

$$I_{M,N} = I_{N,M} = \int_0^{\Lambda_2} \frac{M(s)N(s)}{s^2}\, ds$$

Proposition 3. *Under mild regularity conditions*[22] *the Agent 1 and 2 demand for the term τ bond, net of term τ liabilities, is given by:*

$$z^{(i)}_{t,\tau} = b_i \frac{\bar{\phi}(\tau)}{\tau^2} + c_i(\tau) \int_0^{\Lambda_i} \frac{D(s)\bar{\phi}(s)}{s^2}\, ds, \quad i \in \{1,2\} \tag{2.38}$$

where:

$$b_i = \frac{1}{\gamma \sigma^2_{\zeta^B}}$$

$$c_i(\tau) = \frac{-\sigma^2_r \vartheta_i D(\tau)}{\sigma^2_{\zeta^B} \gamma \tau^2}$$

Proof. See appendix

□

The demand function in (2.38) can be broadly interpreted as the sum of a linearly increasing function of expected bond return and a (negative) term reflecting the marginal additional short rate risk of an additional unit of the bond to the investors' portfolio. Both terms are normalized by local bond volatility. The latter term is a function of the portfolio of bond returns, which affects the demand at each maturity and hence the marginal increase in short-rate risk of an additional unit of the bond. The term is decreasing in both short rate volatility and the sensitivity of the bond to short rate risk. This result is

[22] See appendix for more details

intuitive and not novel. Investors require a larger return in an economy where short rate volatility is high and on bonds with longer duration. Also note that the linear term is the same for Agent 1 and 2. The demand functions differ with respect to the 'marginal risk' term as the Agent 2 marginal risk is evaluated relative to the investors portfolio, which has a wider universe than for Agent 1.

We can use the result in proposition 3 to express the Agent 3 bond demand in terms of bond-swap package demand and Agent 2 bond demand. Rearrangement of the Agent 3 first order condition (2.36) gives[23]:

$$z^{(3)}_{t,\tau} = z^{(2)}_{t,\tau} - y^{(3)}_{t,\tau} + \sigma_r^2 \vartheta_3 \frac{D(\tau)}{\tau^2} \int_0^{\Lambda_2} D(s) y^{(3)}_{t,s}\, ds, \tag{2.39}$$

If bond-swap package demand is positive at each maturity then the total Agent 3 bond demand for maturity τ, $z^{(3)}_{t,\tau} + y^{(3)}_{t,\tau}$, is greater than the Agent 2 bond demand $z^{(2)}_{t,\tau}$. In fact a weaker condition is necessary for $z^{(3)}_{t,\tau} + y^{(3)}_{t,\tau} > z^{(2)}_{t,\tau}$ to hold. Provided the short rate duration weighted sum of bond-swap package demand from Agent 3 is positive, the Agent 3 total bond demand will be greater than the Agent 2 bond demand. Broadly speaking, in an economy where Agents are enticed to create the bond-swap package, for example if bond yields are sufficiently above the corresponding fixed swap yield and bond swap basis is low, overall bond demand from Agents with access to the swap market will be greater than from similar Agents without access to the swap market.

The next two propositions give the bond and bond-swap package demand in equilibrium as a function of $\bar{\phi}(\tau)$.

Proposition 4. *Under mild regularity conditions Agent 3 demand for the term τ bond net of term τ liabilities, $z^{(3)}_{t,\tau}$, has the form:*

[23]To see this compare (2.36) with (2.35) and substitute $\bar{\phi}(\tau)$ with $\bar{\phi}(\tau) - \gamma\sigma^2_{\zeta^B} y^{(3)}_{t,\tau}$ in the Agent 2 bond demand function (2.38).

$$z_{t,\tau}^{(3)} = a_3(\tau) + b_3 \frac{\bar{\phi}(\tau)}{\tau^2} + c_3(\tau) \int_0^{\Lambda_2} \frac{D(s)\bar{\phi}(s)}{s^2} \, ds \tag{2.40}$$

where:

$$b_3 = \frac{1}{\gamma \sigma_{\zeta B}^2}$$

$$c_3(\tau) = -\frac{\sigma_r^2 \left(D(\tau) \left(\sigma_{\zeta B}^2 + \sigma_{\zeta S}^2 \right) \left(I_{E,E}\sigma_s^2 + \sigma_{\zeta S}^2 \right) - I_{D,E}\sigma_s^2 \sigma_{\zeta B}^2 E(\tau) \right)}{\left(Q_1 + Q_2 \right) \sigma_{\zeta B}^2 \tau^2}$$

$$Q_1 = \gamma \sigma_r^2 \left(\sigma_s^2 \sigma_{\zeta B}^2 \left(I_{D,D} I_{E,E} - I_{D,E}^2 \right) + I_{D,D}\sigma_{\zeta S}^2 \left(I_{E,E}\sigma_s^2 + \sigma_{\zeta B}^2 \right) + I_{D,D}\sigma_{\zeta S}^4 \right)$$

$$Q_2 = \sigma_{\zeta B}^2 \sigma_{\zeta S}^2 \left(I_{E,E}\sigma_s^2 + \sigma_{\zeta S}^2 \right)$$

and $a_3(\tau)$ is given in appendix 4, and is independent of $\bar{\phi}(u),\ u \in (0, \Lambda_2]$.

Proof. See appendix.

□

Proposition 5. *Under mild regularity conditions Agent 3 demand for the term* τ *bond-swap package has the form:*

$$y_{t,\tau} = a_4(\tau) + c_4(\tau) \int_0^{\Lambda_2} \frac{D(s)\bar{\phi}(s)}{s^2} \, ds \tag{2.41}$$

where:

$$a_4(\tau) = \frac{-D(\tau)\gamma\sigma_r^2 \int_0^{\Lambda_2} a_3(s)D(s)ds - \gamma\sigma_{\zeta B}^2 \tau^2 a_3(\tau)}{\gamma\sigma_{\zeta B}^2 \tau^2}$$

$$c_4(\tau) = \frac{\left(-\gamma\sigma_r^2 b_3 D(\tau) - D(\tau)\gamma\sigma_r^2 \int_0^{\Lambda_2} c_3(s)D(s)ds - \gamma\sigma_{\zeta B}^2 \tau^2 c_3(\tau) \right)}{\gamma\sigma_{\zeta B}^2 \tau^2}$$

Proof. See Appendix.

□

The expressions for Agent 3 bond and bond-swap package demand in (2.40) and (2.41) are not as easily interpretable as the expressions for Agent 1 and Agent 2 bond demand.

(2.40) is the sum of a linear term in bond return, a term related to the marginal additional short rate risk of an additional bond unit and a term unrelated to $\bar{\phi}(\tau)$, with all terms normalized by local volatility. The coefficient on bond return for Agent 3 is identical to that of Agent 2, i.e.$b_3 = b_2$. $a_3(\tau)$, the Agent 3 demand term independent of $\bar{\phi}(\tau)$, depends on the short rate and spread risk premium in the swap market, λ_r and λ_s. In the case where the swap market is perfectly 'two-sided' and $\lambda_r = \lambda_s = 0$, $a_3(\tau)$ collapses to zero. In this case the only difference between Agent 2 demand and Agent 3 bond demand is the marginal short-rate risk adjustment term.

Similar to Agent 2, the marginal short-rate risk adjustment term depends on the convolution of the short rate sensitivity and the normalized return of bonds. However, the coefficient $c_3(\tau)$ is more complicated than $c_2(\tau)$ and is a function of bond-swap basis, bond and swap local volatility, and short rate and swap spread sensitivity of τ maturity swaps. It is interesting to compare the outright and total bond demand in the limiting case when bond-swap basis volatility equals zero, $\sigma_s = 0$, and the swap market is perfectly 'two-sided'. Since $b_3 = b_2$ we need only compare $c_3(\tau)$ and $c_3(\tau) + c_4(\tau)$ respectively with $c_2(\tau)$. In the limiting case we have:

$$\begin{aligned} c_3(\tau) &= \frac{-\sigma_r^2 \bar{\vartheta}_3 D(\tau)}{\sigma^2_{\zeta^B} \gamma \tau^2} \\ &\leq c_2(\tau) \end{aligned} \tag{2.42}$$

where $\bar{\vartheta}_3 = \left(\sigma^2_{\zeta^B} \frac{\sigma^2_{\zeta^S}}{\sigma^2_{\zeta^B} + \sigma^2_{\zeta^S}} + \sigma_r^2 I_{DD} \right)^{-1}$ and the inequality in (2.42) follows from $\frac{\sigma^2_{\zeta^S}}{\sigma^2_{\zeta^B} + \sigma^2_{\zeta^S}} \leq 1$. So outright bond demand for Agent 3 is less than or equal to Agent 2 bond demand in the limiting case. Equality holds when $\sigma_{\zeta^S} \to \infty$ so that Agent 3 is precluded from hedging in the swap market and it is as if Agent 3 has no access. Total bond demand in the limiting case is given by:

$$\begin{aligned} c_3(\tau) + c_4(\tau) &= \frac{-\sigma_r^2 \vartheta_3 D(\tau)}{\sigma_{\zeta^B}^2 \gamma \tau^2} \left(1 - \sigma_r^2 I_{DD} \bar{\vartheta}_3 \frac{\sigma_{\zeta^B}^2}{\sigma_{\zeta^B}^2 + \sigma_{\zeta^S}^2} \right) \\ &\geq c_2(\tau) \end{aligned} \tag{2.43}$$

where the inequality follows from $\frac{\sigma_{\zeta^B}^2}{\sigma_{\zeta^B}^2 + \sigma_{\zeta^S}^2} \geq 0$. So total bond demand for the agent who can construct the bond-swap package is greater than or equal to the total bond demand for the agent without access to the bond-swap package. Equality again holds when $\sigma_{\zeta^S} \to \infty$. (2.43) also shows the disparity in total bond demand between the agent with access and the agent without access is increasing in short rate volatility σ_r, and decreasing in σ_{ζ^S} and the mean reversion coefficient α_r. These effects are intuitive. An increase in short rate volatility means the bond-swap package is less risky relative to the outright bond, and leads to greater demand for the bond in package form. The sensitivity of the bond to short rate risk is decreasing in α_r, which reduces the relative riskiness of the outright bond to the bond-swap package. Increasing σ_{ζ^S} leads to increased friction between the bond and swap market, and increases the risk of the bond-swap package whilst leaving the riskiness of the outright bond unchanged.

We can combine (2.38), (2.40) and (2.41) together with the market clearing condition (2.31) to obtain the following condition:

$$\bar{a}(\tau) + \bar{b}\frac{\bar{\phi}(\tau)}{\tau^2} + \bar{c}(\tau) \int_0^{\Lambda_2} \frac{D(s)\bar{\phi}(s)}{s^2} ds + \bar{d}(\tau) \int_0^{\Lambda_2} \frac{D(s)\bar{\phi}(s)(\mathrm{s} > \Lambda_1)}{s^2} ds = \bar{k}(\tau) \tag{2.44}$$

where $\bar{a}(\tau)$, $\bar{b}(\tau)$, $\bar{c}(\tau)$, $\bar{d}(\tau)$ and $\bar{k}(\tau)$ are given by the following linear equation:

$$\begin{pmatrix} \bar{a}(\tau) \\ \bar{b}(\tau) \\ \bar{c}(\tau) \\ \bar{d}(\tau) \\ \bar{k}(\tau) \end{pmatrix} = \begin{pmatrix} 0 & 0 & a_3(\tau) + a_4(\tau) & 0 \\ b_1 \mathbf{1}(\tau < \Lambda_1) & b_2 & b_3 & 0 \\ c_1(\tau)\mathbf{1}(\tau < \Lambda_1) & c_2(\tau) & c_3(\tau) + c_4(\tau) & 0 \\ -c_1(\tau)\mathbf{1}(\tau < \Lambda_1) & 0 & 0 & 0 \\ 0 & -l_\tau^{(2)} & -l_\tau^{(3)} & k_\tau \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ 1 \end{pmatrix}$$

To arrive at (2.44) liabilities are assumed to be valued off the treasury curve. This is akin to Agent i starting with a short position in $l_\tau^{(i)}$ bonds. Agents find their optimum bond position without considering their liabilities, and then add $l_\tau^{(i)}$ of the τ maturity bond to cover the short position. The solution to (2.44) is given in the following proposition.

Proposition 6. *The solution to the integral equation (2.44) is:*

$$\frac{\bar{\phi}(\tau)}{\tau^2} = f(\tau) + \lambda(A_1 g_1(\tau) + A_2 g_2(\tau)) \tag{2.45}$$

where:

$$\lambda = -1$$

$$f(\tau) = \frac{\bar{k}(\tau) - \bar{a}(\tau)}{\bar{b}(\tau)}$$

$$g_1(\tau) = \frac{\bar{c}(\tau)}{\bar{b}(\tau)},\ g_2(\tau) = \frac{\bar{d}(\tau)}{\bar{b}(\tau)},$$

$$h_1(\tau) = D(\tau)\ h_2(\tau) = D(\tau)\mathbf{1}(\tau > \Lambda_1)$$

and the values for A_i, $i=$ 1,2, are given by the solution to the linear equation:

$$(\mathbf{I} - \lambda\mathbf{S}) \cdot \mathbf{A} = \mathbf{f}$$

$$\mathbf{A} = \begin{pmatrix} A_1 & A_2 \end{pmatrix}^T,$$

$$\mathbf{S} = (s_{ij})\,, \quad s_{ij} = \int_0^{\Lambda_2} h_i(s) g_j(s) ds, \qquad i,j = 1,2$$

$$\mathbf{f} = \begin{pmatrix} f_1 & f_2 \end{pmatrix}^T, \quad f_i = \int_0^{\Lambda_2} h_i(s) f(s) ds$$

The values for A_i, $i=$ 1,2 can be computed using Cramer's rule:

$$A_1 = \frac{f_1 - \lambda(f_1 s_{22} - f_2 s_{12})}{(s_{11}s_{22} - s_{12}s_{21})\lambda^2 - (s_{11} + s_{22})\lambda + 1}$$

$$A_2 = \frac{f_2 - \lambda(f_2 s_{11} - f_1 s_{21})}{(s_{11}s_{22} - s_{12}s_{21})\lambda^2 - (s_{11} + s_{22})\lambda + 1}$$

Proof. See appendix.

□

Note if there is no Agent 1 in the market the result in Proposition 6 collapses to:

$$\frac{\bar{\phi}(\tau)}{\tau^2} = \frac{\bar{k}(\tau) - \bar{a}(\tau)}{\bar{b}(\tau)} - A_1 \frac{\bar{c}(\tau)}{\bar{b}(\tau)}, \tag{2.46}$$

where

$$A_1 = \left(1 + \int_0^{\Lambda_2} \frac{\bar{c}(s)}{\bar{b}(s)} D(s) ds\right)^{-1} \int_0^{\Lambda_2} \left(\frac{\bar{k}(s) - \bar{a}(s)}{\bar{b}(s)}\right) D(s) ds$$

We can solve for $C(\tau)$ by combining (2.45) together with the relation:

$$\bar{\phi}(\tau) = -A(\tau)\alpha^r \mu_r + C'(\tau) + \Pi^B(\tau)$$

and the boundary condition $C(0) = 0$.

2.3 Results

I consider some of the more obvious implications of the model. I perturb the main parameters and note the effect on the term structure, and individual agent demand and portfolio returns.

2.3.1 Agent access to the swap market and the cost of debt

If a larger proportion of Agents have access to the swap market, there are more market integrators who can purchase bonds and hedge (subject to bond-swap basis) in the swap market. A greater proportion of agents are therefore able to construct a lower volatility package with the bonds. This raises the demand across maturities for bonds and reduces the cost of debt (see panel A in 2.7). The effect is most marked at longer maturities, where the discreprancy in price volatility between bond-swap packages and pure bond packages is greater. The increasing discreprancy in price volatility by maturity leads to a stronger increase in demand for the longer maturities when access to the swap market is augmented. The marginal effect of increased swap market access on yields is non-linear and largest in absolute terms when access to the swap market is low. This is analogous to the marginal effect of an increment to the arbitrageur budget (however defined) on bond yields being greatest when the budget is low.

The effect of access to the swap market on the equilibrium bond demand function of an individual agent at longer maturities is seen in figure 2.9. The quantity of 5 year bonds demanded by Agent 3 is greater than Agent 2 regardless of the level of access (Panel A). The divergence between Agent 3 bond demand and Agent 2 demand is greatest when the proportion of agents with access is low. In this case the yield discrepancy between the bond and swap market is larger and agents who are able to straddle both markets purchase more bonds and hedge their risk with swaps. Panel B shows that the share of Agent 3 bond demand as a proportion of total bond demand increases with access. For large levels of access agents with access to the swap market own almost the entirety of 5 year bonds issued.

2.3.2 Volatility of the swap spread and the cost of debt

A volatile swap reference short rate relative to the 'true' short rate deters investors with access to the swap market from constructing the bond-swap package. A more volatile reference rate reduces aggregate demand for bonds, and hence increases yield, across all maturities. Panel B in figure 2.7 illustrates the effect of bond-swap basis volatility on the term structure of yields and shows that the sensitivity of yields is increasing in maturity. Similar reasoning to the effect of increasing access applies. At longer maturities the discrepancy between bond price volatility and package volatility is the greatest. Any perturbation in the swap reference spread volatility is amplified by maturity so that the change in this discrepancy is largest at longer maturities. This has practical relevance with Libor-based swaps being by far the most liquid reference rate for interest rate swaps. Libor - FFR spread is volatile (see 2.3), discouraging investors from purchasing the bond-swap package, increasing yields at longer maturities.

2.3.3 Bond supply and the cost of debt

The effect of changes in bond supply on the term structure of interest rates in the model is dependent on the bond/swap reference rate basis risk and the degree of access to the swap market. In the Vayanos and Vila (2009) model arbitrageurs with access to the full term structure act on the distortion created by changes in bond supply in a limited manner. In the one-factor version of their model, agents are constrained by aversion to short rate risk. Instead in this model agents with access to the swap curve act on the distortion in the deep swap market subject to bond-swap reference rate basis (and a small amount of idiosyncratic risk), which is significantly smaller than interest rate risk. The effect of a change in supply can be seen in figure 2.8. Panel A and B show the effect of a constant increase in bond supply (from 0.1 to 0.15) across the term structure for different levels of access and bond-swap basis risk respectively. The effect of the increase in supply is greatest at the longest maturities, and when access is low and basis is high. When access is low, there are less market agents who can absorb the increase in supply and hedge their risk in the swap market. This forces the agents with no access to the swap market to absorb some of the extra supply. These agents require compensation for the large naked short-rate risk they run, since they are risk averse. Bond price sensitivity to short rate risk is increasing in term, so the change in yield due to changes in bond supply is also increasing in term. For large bond-swap basis agents with access to the swap market find absorbing the extra bond supply more risky as the hedge is more risky and therefore requires greater compensation.

Bond supply driven monetary policy is therefore effective when access to the swap market is low but has limitations in an environment where a large proportion of agents have access to a deep derivative market and/or the basis risk between the physical and derivative market is small. The result conditions the view that bond-supply is an important driver of bond-yields (see for example Greenwood and Vayanos (2010)). Moreover, the results show that in models such as Cochrane (2014) where there is a direct mapping between bond supply and interest rates, the introduction of a swap curve can affect significantly the correspondence between the two. In the limiting case where the size of the swap market is infinite no link between bond supply and interest rates would exist.

2.3.4 Liability shocks and the cost of debt

A liability on an agent balance sheet valued off the treasury curve can simply be seen as a short position in a bond of equivalent maturity. Agents optimise with respect to demand net of liabilities, and then simply increase their bond holding to neutralise the existing short positions on their balance sheet. Analytically, the Agent's first order conditions can be expressed in terms of net demand, $z_{t,\tau}^{(i)} \equiv x_{t,\tau}^{(i)} - l_\tau^{(i)}$, and the functional form of $z_{t,\tau}^{(i)}$ in equilibrium does not depend on $l_\tau^{(i)}$. However, the liabilities do affect the market clearing condition as the condition is expressed in terms of actual (rather than net) bond demand. A one unit increase in liability of a particular maturity for a particular agent will lead to a one unit increase in agent bond demand, ceteris parabis. Thus the affect of an increase in agent liabilities is similar to a reduction in bond supply at the same maturity point of the liability. To see this mathematically, define $l(\tau) = \sum_{i=1}^{3} l_\tau^{(i)}\theta_i$ and $k(\tau) = k_\tau$. Then the following set of equalities hold:

$$\frac{d\bar{\phi}(\tau)}{dk(\tau)} = \frac{d\bar{\phi}(\tau)}{d\bar{k}(\tau)}\frac{d\bar{k}(\tau)}{dk(\tau)} = \frac{-d\bar{\phi}(\tau)}{d\bar{k}(\tau)}\frac{d\bar{k}(\tau)}{dl(\tau)} = \frac{-d\bar{\phi}(\tau)}{dl(\tau)} \tag{2.47}$$

An additional implication of the model is if liability valuation were to be moved onto a swap curve, the net effect would be similar to an 'increase' in bond supply. Agents with swap curve based liabilities begin with a short position in a fixed rate swap. In this model, the short position can be hedged in the infinitely deep swap market without affecting swap market pricing. Agents would be incentivised to move out of bonds and into similar maturity swaps, thereby increasing the yield on bonds. Clearly the assumption that swap-market pricing would remain unmoved is an oversimplication but the change in liability valuation would have a disproportionate affect on the bond market due to the depth in the swap market and the fact that all else equal agents will hedge with the liability valuation instrument to avoid basis risk. Once valued on the swap curve, liability shocks will not have further effects on yield.

2.3.5 Agent access to the swap market and portfolio returns

Agent access to the swap market also has implications for agent welfare. We can see the effect of access on portfolio returns in figure 2.10. Panel A shows the expected returns for agents with access to the swap market increases considerably as access reduces. As access reduces there are fewer agents that can take advantage of the dislocation between the bond and swap market. Those of the remaining fewer agents with access profit in expectations from the higher relative bond returns and increase the average return of their portfolio. Similarly, the portfolio returns of the agents without access increases, as there are less agents to reduce bond returns by purchasing the bond-swap package, which increases the equilibrium price of bonds. Since average portfolio returns increase when agent access reduces there is a cross-subsidisation from the average tax payer to bond holders as the government pays more interest on its debt. If we assume the average tax-payer is the aggregate agent, then we can see from the increasing disparity between Agent 2 and Agent 3 expected returns that there is a net expected cross-subsidisation from agents without access (less sophisticated) to agents with access (more sophisticated). I construct a statistic Θ to capture the expected transfer of wealth from less to more sophisticated agents:

$$\Theta = \frac{\theta_3}{\theta_2}(ER_3 - ER_2) \tag{2.48}$$

where $\theta_2 = 1 - \theta_3$ since $\theta_1 = 0$ and ER_i is the expected portfolio return for Agent i.

Panel B of Figure 2.10 plots Θ for varying values of access to the swap market (θ_3). When swap market access is zero there is clearly no cross-subsidisation. With a small amount of access there is a large cross subsidisation between a large proportion of agents to a small proportion of agents. As the amount of access increases expected return for the agents with access falls considerably, but the cost of the smaller discrepancy in return is borne by a small proportion of agents. But the discrepancy falls faster than the reduction in 'unsophisticated' agents so the cross-subsidisation statistic falls.

2.3.6 Impact of curve point intervention on the cost of debt

The previous results assumed a constant bond supply or liability shock across the term structure. One may wonder what happens if supply is changed only at specific points. What would happen for example if the central bank performed an operation 'twist', whereby bonds were sold (purchased) at the short end and purchased (sold) at the long end? Agents in this model consider bonds to be quasi-substitutable in the sense that they are assumed to be sensitive primarily to one risk factor (changes in short rates), whilst retaining a small idiosyncratic risk component. Ignoring the idiosyncratic risk component, an increase in τ year bond supply can be seen simply as an increase in the supply of $D(\tau)$ units of interest rate risk. Thus an increase of $D(t_2)$ units of supply of a t_1 maturity bond is seen by the agents as quasi-equivalent to an increase in $D(t_1)$ units of a t_2 maturity bond. Since bonds are assumed to be quasi-substitutable, the effect of increasing (decreasing) supply at a subset of points on the curve is to decrease (increase) yields across the entirety of the term structure.

Consider the effect of two 'twist' operations in figure 2.11, where bonds are sold at shorter maturities and purchased at longer maturities. In all cases access to the swap market is assumed to be low. In the first 'twist' operation change in supply increases linearly as a function of term but overall change in bond supply is zero. This operation increases yields at all maturities, since it increases the overall supply of units of short rate risk. Bonds at longer maturities have a greater sensitivity to short rate risk than shorter bonds. In the second 'twist' operation the total change in $D(\tau)$ weighted bond supply is unchanged. More bonds are sold at shorter maturities than are purchased at longer maturities. In the second operation the term structure is left almost unchanged.

The results of curve point intervention reported here are broadly consistent with the 1-factor model in Vayanos and Vila (2009). In the 1-factor Vayanos and Vila (2009) model bonds can be considered completely substitutable; a bond of one maturity is simply a scaled version of a bond of a different maturity where the scaling factor is the bond sensitivity to the short rate. Thus an increase in bonds of a specific maturity can be easily interpreted as an increase in supply of units of the short rate risk factor. However the introduction of a deep swap market with extensive agent access can temper these results. Any increase or decrease in supply in the short-rate risk factor is offset by the availability of an assumed infinitely deep interest rate swap market where short rate risk can be hedged. Since a curve 'twist' operation can be considered as a change in units of supply of short rate risk (excluding idiosyncratic risk), the results of the previous section can be applied. A constant supply shock where there is extensive access to the swap market or low bond-swap basis has little effect on the term structure. Similarly a twist operation that changes the net supply of short rate risk also has little effect. Unpublished results where I have rerun the same twist operations in figure 2.11 for the case where there is extensive access to the swap market confirm this.

2.3.7 Impact of short rate volatility

Panel A in Figure 2.13 shows the effect of an increase in short-rate volatility, σ_r, on yields of different maturity. The results are consistent with intuition. As rate volatility increases the yield on the bonds increase. Panel B shows the effect of an increase in 1% in short rate volatility on the 5 year bond yield, for varying levels of access and basis. Consistent with intuition, the effect is greatest for low access and high basis, and smallest for high access and low basis. When basis is high or access is low there are less bonds purchased in package format, so the market is more sensitive to an increase in outright interest rate volatility. The equilibrium yield adjusts so that the expected return for the majority holding the bond in outright format compensates them for the additional interest rate risk borne. Interesting is the relative impact of access and bond-swap basis on changes in yields. The second derivative of long yields with respect to short rate volatility and access exceeds the second derivative with respect to short rate volatility and basis. This suggests that in the absence of long-dated liabilities, which provide a natural hedge for the long-dated bonds, demand for the long-dated bonds is dominated by those who construct the bond swap package. The package is not sensitive to the volatility of the short-rate, but volatility of bond-swap basis.

2.3.8 Effect of swap market risk risk premia

Swap market pricing is given exogenously in the model and acts as a natural reference curve. Assuming there is non-zero access to the swap market, changes in risk premia in the swap market will be reflected in bond yields though the trades of agents with access to both markets. Swap market risk premia also affects the compositional mix of bond demand between the bond-swap package and outright bond demand. Panel A of figure 2.15 shows the derivative of bond demand with respect to the swap spread risk premium, λ_s. Total bond demand falls, more bonds are demanded in an outright format and there is a reduction in the demand for the bond-swap package. The rationale for this is clear. An increase in λ_s reduces the expected return of the bond-swap package, as the bond-swap package investor is short the fixed leg of the swap. This deters agents with access from constructing the bond-swap package and substituting this risk for outright bond risk. Since the risk of the outright bond is larger, less bonds are purchased in outright format than sold through disinvestment in the bond-swap package, thereby reducing total bond demand. Similar results hold for the derivative with respect to the short rate risk premium (Panel B of figure 2.15), λ_r, although the effect on demand in absolute terms is greater. In the example given the short rate mean reversion coefficient, $\alpha_r = 0.1$, is smaller than the swap spread mean reversion coefficient, $\alpha_s = 0.4$. It follows that the swap sensitivity to short rate risk is greater than to swap spread risk for a particular maturity point, $D(\tau) > E(\tau)$. Moreover the volatility of the short rate, $\sigma_r = 0.017$, is assumed to be larger than the volatility of the swap reference spread, $\sigma_s = 0.002$. Since λ_r and λ_s represent the effect on net return per unit of short rate risk and swap spread risk respectively, the absolute effect of a specified increase in λ_r will be greater than the effect of the same increase in λ_s.

2.3.9 Effect of term-specific risk

Term-specific risk is a subtle but crucial component of the model. Although small, it creates a necessary friction to prevent the disparity between the pricing of risk in the bond market and swap market from being arbitraged by the agents with access to both markets. Consider the limiting case where $\sigma_{\zeta \mathrm{S}} = \sigma_{\zeta \mathrm{B}} = 0$. In this case there is only one risk factor in the bond market, the short rate r_t. This risk factor is a subset of the two risk factors in the swap market, r_t and s_t. Since there are swaps available at more than one maturity point, the swap market is complete and the price of r_t and s_t risk is implied by the exogenously given prices of the instruments[24]. The price of the fixed leg of the zero-coupon swap is:

[24] Here I mean specifically the price of the fixed leg cashflows of the swap. Market convention is to quote the fixed rate payable under the swap from which the price of the fixed leg cashflows can be deduced.

$$P^S_{t,\tau} = e^{-r_t D(\tau) - s_t E(\tau) - F(\tau)} \tag{2.49}$$

and the price of the bond is:

$$P^B_{t,\tau} = e^{-r_t D(\tau) - s_t E(\tau) - F(\tau)} \mid_{s_t=0,\sigma_s=0} \tag{2.50}$$

To see why (2.49) implies (2.50) recall that λ_r is defined as the market price of short rate risk in the swap market and suppose:

$$\frac{E_t(dP^B_{t,\tau}/P^B_{t,\tau}) - r_t}{\sigma_r D(\tau)} < \lambda_r \tag{2.51}$$

Agents with access to the swap market can arbitrage the market by creating a risk free bond swap package. Agents short the bond, invest the cash in the rolling short rate, and replicate a swap that pays the 'true' risk free floating rate (without the reference rate spread) and receives the risk free fixed rate. If the inequality in (2.51) is reversed agents will execute the opposite trade by purchasing the bond, borrowing cash and replicating a swap that pays a fixed rate and receives the risk free short rate.

A τ- maturity swap referencing the true risk free rate, r_t, can be constructed with any two swaps of maturity τ_1 and τ_2, $\tau_1 \neq \tau_2$. The weightings w_1 and w_2 that create a swap sensitive only to fluctuations in the true risk free rate can be found by solving the linear equations:

$$\begin{bmatrix} D(\tau_1) & D(\tau_2) \\ E(\tau_1) & E(\tau_2) \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} D(\tau) \\ 0 \end{bmatrix}$$

In the example above there is no definitive motive for there to be 'maturity clienteles'. Agents with future liabilities discounted by the treasury bond curve can hedge their risk with infinite combinations of different bonds. If an agent has a 1$ notional τ - maturity liability it can be hedged perfectly either by 1$ notional of the bond with price $P^B_{t,\tau}$ or any other combination of bonds that replicate it. Each τ- maturity bond is perfectly substitutable with $\bar{w}$ units of the $\bar{\tau}$ maturity bond and a residual cash amount, where:

$$\bar{w} = \frac{D(\tau)}{D(\bar{\tau})} \tag{2.52}$$

Moreover, we can see clearly from this example that the bond term structure access constraint $x^{(1)}_{t,\tau} = 0$, $\Lambda_1 < \tau \leq \Lambda_2$ imposed on Agent 1 is inconsequential if $\sigma_{\zeta \mathrm{S}} = \sigma_{\zeta \mathrm{B}} = 0$. Any non-negative allocation to a τ- maturity bond, $\Lambda_1 < \tau \leq \Lambda_2$, (long-maturities) in a full unconstrained optimization can be replicated by bonds in the maturity interval $0 < \tau \leq \Lambda_1$ (short maturities). The increase in demand for the short-maturity bonds and decrease in long-maturity bonds by Agent 1 can be offset perfectly by Agents 2 and 3 who exploit the perfect substitutability of bonds by reducing their exposure to (or taking short positions in) the short-maturity bonds and demanding more of the long-maturity bonds. In this case Agents 2 and 3, who have access to the full term-structure of bonds, are able to integrate perfectly the short- and long-maturity bond market.

A similar logic can be applied to the Vayanos and Vila (2009) model. Maturity clienteles are motivated in their model by a series of overlapping investor generations living for a time interval T, who consume at the end of their life. Investors have access to the full portfolio of bonds and a private technology. Under the preference assumptions in their model the optimum portfolio at time t is *equivalent* to a portfolio consisting of $\hat{y}_{t,\tau}$ units of the bond maturing at time T and zero units of all other bonds, where $\hat{y}_{t,\tau}$is defined in Vayanos and Vila (2009). In fact since there is only a finite number N factors and an infinite number of instruments, the optimum portfolio can be constructed in an infinite number of ways. This is difficult to reconcile with the linear demand for a specific maturity assumed at the outset.

The idiosyncratic risk in both the bond and swap markets are therefore necessary frictions to ensure that there are indeed quasi-'maturity clienteles', who ex-ante have a preference for investing at a specific maturity. Firstly, a τ- maturity liability can only be perfectly hedged by an equivalent notional amount of the τ-maturity bond. Secondly, it becomes more costly in utility terms to replicate the short rate sensitivity of the time τ- liability with bonds of a different maturity as the replicating bonds are subject to different idiosyncratic risk factors. Consider the net volatility of the position of an agent with 1$ liability in τ time and a $\bar{w}$ unit bond position in the $\bar{\tau}$ bond where $\bar{w}$ has the form in 2.52:

$$V[\frac{\bar{w}dP^B_{t,\bar{\tau}}}{P^B_{t,\bar{\tau}}} - \frac{dP^B_{t,\tau}}{P^B_{t,\tau}}] = \sigma^2_{\zeta\mathrm{B}}(\tau^2 - 2\bar{w}\delta^{|\bar{\tau}-\tau|}\bar{\tau}\tau + \bar{w}^2\bar{\tau}^2) \tag{2.53}$$

Panel A in figure 2.14 plots the function given in (2.53) for variable values of τ and $\bar{\tau}$. The figure shows that the residual risk of hedging a long-dated liability with short-dated bonds is high, whereas the residual risk of hedging a liability with a similar maturity bond is low. Initially it may be surprising that the residual risk of hedging a short-dated liability with a long-dated bond is also low. This is because the duration of the short-dated liability is low, so a relatively small percentage of long-dated bonds are required to hedge this risk and the absolute effect of idiosyncratic yield volatility on the liability valuation is small.

Panel B in figure 2.14 shows the impact of idiosyncratic risk on the efficacy of term-specific bond-purchase programmes. The z-axis gives the difference in the change in yield for a 0.1 supply shock between the large idiosyncratic risk case ($\sigma_{\zeta^B} = \sigma_{\zeta^S} = 1\%$) and the small idiosyncratic risk case ($\sigma_{\zeta^B} = \sigma_{\zeta^S} = 0.2\%$). The difference between the two increases with the size of the basis, and decreases with access other than for very small values of access. As access reduces, agents with access to the swap market demand an increasing amount of bonds in equilibrium and their sensitivity to idiosyncratic risk increases. Yet for very low levels of access the proportion of bonds owned by the arbitrageurs is small, and the fact that the bond-swap package is more difficult to create due to idiosyncratic risk becomes less relevant in the determination of equilibrium bond yields.

2.3.10 Effect of restricting full bond term-structure access

Agent 1 does not have access to the full bond term-structure. This agent could represent households who have access to money market funds that have strict maturity limits on their investments. In figure 2.16 I study the effect on equilibrium yields if the proportion of agents who have restricted bond market access increases. The full line in Panel A gives the equilibrium yields when there is no restricted bond market access ($\theta_1 = 0, \theta_2 = 1$). Instead the dashed line corresponds to the case where thirty percent of investable funds pertain to agents who cannot access bonds beyond the two year maturity ($\theta_1 = 0.3$, $\theta_2 = 0.7$, $\Lambda_1 = 2$). In order to keep the effects of the derivative market separate, no agents have access to the swap market in this example ($\theta_3 = 0$). We can see from the figure that the shorter-dated yields are tighter and the longer-dated yields wider in the case where some agents have restricted access. Agents with restricted access are forced to assign a greater proportion of funds to shorter dated maturities than they would if a full optimization where possible. Conversely the demand of the restricted agents for longer-dated bonds falls to zero by definition. The effect is similar to a central bank twist operation that sells longer-dated bonds and purchases shorter-dated bonds. The operation is partly offset by agents with full access who short (or purchase less of) the shorter-dated bonds and invest more in the longer-dated bonds. However these agents are not able to fully reverse the effects of the restricted access as each bond is subject to idiosyncratic risk. Shorting the short-dated bonds and increasing exposure to longer-dated bonds such that the marginal increase in short rate exposure from the short positions is neutralized leaves the agent with access with a residual risk, which needs to be compensated.

The disparity highlighted in Panel A seems to reduce when the rate of short rate reversion increases. In panel B the rate of reversion is increased from $\alpha_r = 0.1$ in Panel A to $\alpha_r = 0.4$. A possible explanation for the reduction in the disparity is that the increased concavity of $A(\tau)$ renders longer-dated bonds more similar to shorter-dated ones. Therefore the residual risk of creating a marginal short-rate neutral position by shorting the short-dated bonds to purchase the longer-dated bonds is less. This is over and above the effect of the lower overall supply of short-rate risk reducing equilibrium yields.

2.4 Some brief empirical investigations

2.4.1 Bond term structure and swap reference rate basis volatility

I begin by testing the theoretical prediction that the bond term structure should steepen upward with increasing swap reference rate basis volatility, ceteris parabis. In the model described in this paper the volatility of s_t, σ_s , is an exogenously given constant. In the results section I analysed the effect on the bond term structure of perturbing σ_s . Although σ_s is forward looking, I use the past n working day Libor - FFR basis volatility as a proxy for σ_s :

$$\hat{\sigma}_{s,t} = \sqrt{\frac{1}{n-1}\sum_{k=0}^{n-1}\left(\Delta \mathrm{s}_{(t-k)}\ \right)^2}$$

where s_t is the difference between the 6 month USD libor rate and the Federal Funds target rate, and$\Delta \mathrm{s}_t = s_t - s_{t-1}$ [25] Time has been rebased to working days for the above for notational convenience. Figure 2.3 plots the volatility indicator $\hat{\sigma}_{s,t}$, FFR and USD 6 month Libor rate in the period 1990 - 2013 for $n = 252$.[26]

I consider the effect of swap reference rate basis volatility on bond term structure steepness, by regressing the difference between the τ year and 1 year bond yield on $\hat{\sigma}_{s,t}$ for $\tau \in \{1, 2, 3, 4, 5, 7, 10\}$:

$$-\frac{p^B_{t,\tau}}{\tau} + p^B_{t,1} = \alpha^\tau + \beta^\tau \hat{\sigma}_{s,t} + \epsilon^\tau_t$$

[25] Changing the proxy to the sample standard deviation $\sqrt{\frac{1}{n-1}\sum_{k=0}^{n-1}\left(\Delta \mathrm{s}_{(t-k)} - \overline{\Delta \mathrm{s}_t}\right)^2}$ where $\overline{\Delta \mathrm{s}_t} = \frac{1}{n}\sum_{k=0}^{n-1}\Delta \mathrm{s}_{t-k}$ has little effect on the results.

[26] The results are robust to other time lags.

The estimated coefficients for β^τ are given in the first column of estimates in table 2.1. Estimates up to 5 year maturity are significantly different from zero, and the signs of the estimates are consistent with predictions of the model ie larger bond-swap basis volatility is related to steeper bond term structure. Heuristic support for the regression results can be obtained by examining the time series of 5 year and 10 year excess bond yields (above the 1 year yield) alongside the time series of $\hat{\sigma}_{s,t}$, given in figure 2.5.

A legitimate empirical concern is that $\hat{\sigma}_{s,t}$ is simply a proxy for s_t , which in turn is a proxy for general credit conditions and/or aggregate risk aversion. This would affect agent appetite to consume pure interest rate or basis risk, and would lead to steeper curves. However, the plot of $\hat{\sigma}_{s,t}$ against s_t in figure 2.4 suggests there is significant independent variation in $\hat{\sigma}_{s,t}$. For example between 2001 and 2004 the volatility of Libor-FFR basis increased significantly, whilst the basis fell from 80 bps to -50 bps. Nevertheless, I attempt to control for the effect of s_t as an explanatory variable. The inclusion of s_t has little effect and the estimates (second column in table 2.1) are broadly unchanged. For robustness I also include the estimates with $n = 60$ working day lags (columns 3 and 4). The sign of the estimates are the same but the confidence intervals are sufficiently large that the null hypothesis of β^τ =0 cannot be rejected across maturities.

Another interesting feature of the estimates is the initial increase in the coefficients along the term structure and then decrease beyond the five year term. The increasing sensitivity is predicted by the theoretical model. At the short end of the curve the majority of demand is from agents without recourse to the swap market. This is either because it is the only part of the curve to which they have access (Agent 1), or because the unavailability of a swap market reduces the attractiveness of the longer term bonds (Agent 2). For these agents, changes in the volatility of the reference rate has little effect on their optimisation problem, other than through the eventual change in equilibrium price. At longer maturities the proportion of demand pertaining to agents with access to the swap market increases. Since the viability of the bond-swap package is sensitive to σ_s , longer maturities should be more sensitive to σ_s . What is not explained by the model is why the estimated coefficients decrease beyond the 5 year point. One possible explanation is that investors at the very long maturities are 'inert' and do not reoptimize frequently. Also, the larger confidence intervals at the longer maturities suggest that other factors such as liability shocks may overwhelm the effect of $\hat{\sigma}_{s,t}$. Finally, yields are by definition normalized by the term τ. It is plausible that the sensitivity of prices to $\hat{\sigma}_{s,t}$ is increasing in term, but the rate at which the sensitivity increases is decreasing, leading to the yield sensitivity decreasing beyond a certain point.

2.4.2 Bond swap basis sensitivity to swap reference rate basis volatility

We now turn our attention to the effect of the reference rate volatility on bond swap basis. Figure 10 includes a time series of the difference between the 5 year bond yield and the 5 year swap yield. As expected, the difference between bond and swap yield decreases (becomes more negative) when $\hat{\sigma}_{s,t}$ increases.$\hat{\sigma}_{s,t}$ is positively correlated with s_t and the swap yield is an increasing function (at least in our model) of s_t . Empirically it is difficult to distinguish between the effect of s_t and$\hat{\sigma}_{s,t}$ on bond swap basis, as they almost always positively comove in depressed markets and it is in depressed markets where their influence on bond-swap basis should be greatest. As a first step we adjust the difference between the 5 year bond and swap yield by the difference between 6m Libor and the FFR (the "adjusted B-S basis").

This adjustment (see Figure 2.6) has a significant effect. The adjusted bond-swap basis spikes both during the mid 1990's when there was a prolonged period of FFR hikes, and in the period around the 2008 financial crisis. In both cases the Libor-FFR basis was particularly volatile. These observations are consistent with the theory discussed in section 2.3.2. However the behaviour of the adjusted basis around the dot-com bubble in early 2000's is puzzling. The adjusted basis falls sharply with little increase in the level of Libor - FFR basis volatility. I leave this topic for further investigation.

2.4.3 Testing the assumption of equal bond and swap sensitivity to the short rate

To test the assumption of equal bond and swap sensitivity to the short rate, we recall the following expressions:

$$P^B_{t,\tau} = e^{-r_t A(\tau) - C(\tau) - \Xi(\tau)\zeta^B_{t,\tau}}$$

where $P^B_{t,\tau}$ and $P^S_{t,\tau}$ are the τ year zero coupon bond and swap prices at time t . Since the functions $A(\cdot), C(\cdot), \Xi(.), D(\cdot), E(\cdot)$ and $F(\cdot)$ are all independent of r_t we can write the following expressions:

$$p^B_{t,\tau} = \alpha^B - r_t A(\tau) + \epsilon^B_t$$

$$p^S_{t,\tau} = \alpha^S - r_t D(\tau) + \epsilon^S_t$$

where $p^B_{t,\tau}$ and $p^S_{t,\tau}$ are the natural logarithms of $P^B_{t,\tau}$ and $P^S_{t,\tau}$ respectively. ϵ^B_t and ϵ^S_t are exogenous with respect to r_t ie $E(\ \epsilon^B_t\ |r_u\ ) = 0$ and $E(\ \epsilon^S_t\ |r_u\ ) = 0\ \forall t, u \in \mathbb{R}$. We estimate $A(\tau)$ and $D(\tau)$ by projecting $p^B_{t,\tau}$ and $p^S_{t,\tau}$ onto the Federal funds rate using quarterly data from January 1990 to June 2013. The estimates $\hat{A}(\tau)$ and $\hat{D}(\tau)$, normalized by τ , for $\tau \in \{1, 2, 3, 4, 5, 7, 10\}$, are given in table A2. Estimates $\hat{A}(\tau)$ and $\hat{D}(\tau)$ are similar across the term structure. For maturities $\tau \in \{2, 3, 4\}$ the null hypothesis that $A(\tau) = D(\tau)$ cannot be rejected at the 99% confidence interval. For robustness, the regression was also estimated in first differences and did not reject the hypothesis $A(\tau) = D(\tau)$ at any maturity $\tau \in \{1, 2, 3, 4, 5, 7, 10\}$. Thus the assumption $A(\tau) = D(\tau)$ seems to be justified by the empirical results.

Although out of the scope of this paper, there is an outstanding question as to why the estimates of $\hat{A}(\tau)$ and $\hat{D}(\tau)$ based on a level regression differ significantly from the estimates from the first differences regression. Possible explanations may be the discrete nature of Federal Fund Rate changes due to only two FOMC meetings taking place per quarter or the market anticipating a Fed policy of smoothed interest rate changes.

2.5 Concluding remarks

I have presented a model that takes a step towards understanding the effect of term-specific demand and monetary policy on the term structure of interest rates in an economy where there is a large interest-rate swap market, and some agents are constrained in their access to the bond market, the swap market or both. An arbitrageur in this model with access to both the bond and swap market is faced with term-specific risk and bond-swap basis risk. These act as important frictions that prevent the arbitrageur with access to both markets from arbitraging (i) the differences in prices between markets or (ii) term structure dislocation due to changes in liability structure or bond supply. I find in the absence of these frictions curve point intervention by the central bank through OMT's can be reversed completely by the arbitrageur. In this case the swap market acts as a pure reference curve and the central bank would either need to affect agents' expectation of the forward evolution of the short rate (termed "forward guidance") in order to reduce longer-term yields, or affect structural attributes of the participants that take positions in the market on an unfunded basis. Conversely, I find in the presence of the frictions described in the model OMT's have a role to play. The effectiveness of such a programme increases with the proportion of funds available to agents without swap market access, bond-swap reference rate basis volatility, and the size of term-specific risk. The programme effectiveness also varies with agent liabilities and increases with the volatility of the policy rate. In all cases there is a cross-subsidisation from agents with less access to those with more access.

I caution against concluding that a central bank looking to impact the yield curve through OMT's would be better served maintaining or exacerbating the frictions in the model. When frictions are large OMT's may be more effective, but the overall cost of debt may initally be larger. This is likely to be the case if the swap market is 'balanced'. Reducing the frictions can lead to a lower cost of debt, and may reduce the need to intervene in the physical bond market in the first place.

Finally, I tested the model prediction that the term-structure of bonds steepens and adjusted bond-swap basis increases with reference rate volatility. The results were generally consistent with the theory, although a causal relation cannot be deduced as the reference rate may act simply as a proxy for other causal variables not included in the analysis.

There are a number of avenues for future research. Firstly, a rich set of implications in the case where the Central Bank varies bond supply as a function of the short rate has not been explored. Secondly, the mathematical description of both the short rate and reference rate spread processes could be improved. The reference rate spread process is likely to be more volatile when its value increases, so a process with stochastic volatility may be preferred. The idiosyncratic risk process in the model could be changed, and an improved understanding of the underlying reasons for idiosyncratic risk should be sought both through theoretical and empirical work. Finally, placing a large interest rate derivative market in a well-defined macro-economic model and specifying an endogenous motive to transact in the derivative market may be difficult but would be very worthwhile.

2.6 Appendix

2.6.1 Construction of the local demand factor

In order to generate the random field $\zeta^S_{t,\tau} = \zeta^S(t,T)$, $\tau = T - t$, with variance and correlation structure given by (2.3) - (2.4) an entire function $d\zeta^S_{t,\tau}$ over T should be generated for each time increment. Goldstein (2000) describes a simple way to do this using the Ornstein-Uhlenbeck process, which I describe here with minor modifications. $d\zeta^S(t,t)$ is generated from a normal distribution:

$$d\zeta^S(t,t) \sim N\left(0, \sigma^2_{\zeta^S} dt\right)$$

The rest of the field $d\zeta^S(t,\cdot)$ is then generated by:

$$d\zeta^S(t,T) = d\zeta^S(t,t)e^{-\varpi(T-t)} + \sqrt{2\varpi\sigma^2_{\zeta^S}} \int_t^T e^{-\varpi(T-u)} dw_t(u)$$

where:

$$E\left[dw_t(u)\right] = 0$$

$$cov\left[dw_t(u_1), dw_t(u_2)\right] = \begin{cases} dt du_1 & u_1 = u_2 \\ 0 & u_1 \neq u_2 \end{cases}$$

we then have

$$\begin{aligned} var\left[d\zeta^S(t,T)\right] &= e^{-2\varpi(T-t)}\sigma^2_{\zeta^S} dt + \left(1 - e^{-2\varpi(T-t)}\right)\sigma^2_{\zeta^S} dt \\ &= \sigma^2_{\zeta^S} dt \end{aligned}$$

and

$$\begin{aligned} Cov\left[d\zeta^S(t,T_1), d\zeta^S(t,T_2)\right] &= e^{-\varpi(T_1+T_2-2t)}\sigma^2_{\zeta^S} dt + e^{-\varpi(T_1+T_2-2t)}\left(e^{2\varpi(T_1 \wedge T_2 - t)} - 1\right)\sigma^2_{\zeta^S} dt \\ &= e^{-\varpi|T_1-T_2|}\sigma^2_{\zeta^S} dt \end{aligned}$$

and the correlation is:

$$Corr\left[d\zeta^S(t,T_1), d\zeta^S(t,T_2)\right] = e^{-\varpi|T_1-T_2|}$$

or in the notation consistent with Vayanos and Vila (2009):

$$Corr\left[\zeta^S_{t,\tau_1}, \zeta^S_{t,\tau_2}\right] = e^{-\varpi|\tau_1-\tau_2|}$$

where $\tau_1 = T_1 - t$ and $\tau_2 = T_2 - t$.

2.6.2 Agent preferences over term-specific factors and consistency of conjectured bond price form

Recall agents have linear preferences over the term-specific bond and swap factors, with $\Phi_t^{(i)}$, $i = \{1,2\}$ given by:

$$\Phi_t^{(i)} = -\int_0^{\Lambda_i} x^{(i)}_{t,\tau}\zeta^B_{t,\tau} d\tau \tag{2.54}$$

and the preference function for Agent 3 is:

$$\Phi_t^{(3)} = -\int_0^{\Lambda_i} (x^{(i)}_{t,\tau} + y^{(i)}_{t,\tau})\zeta^B_{t,\tau} d\tau + \int_0^{\Lambda_i} y^{(i)}_{t,\tau}\zeta^S_{t,\tau} d\tau \tag{2.55}$$

If the price of the bond is:

$$P^B_{t,\tau} = e^{-r_t A(\tau) - C(\tau) - \Xi(\tau)\zeta^B_{t,\tau}} \tag{2.56}$$

then $\Xi(\tau) = \tau$ is the only form of $\Xi(\tau)$ for which the first order conditions of all agents would hold for all values of $\zeta^B_{t,\tau}$. To see this first note that $V_t\left[d\bar{W_t}^{(i)}\right]$ for $i \in \{1,2,3\}$ in expression (2.7) does not depend on $\zeta^B_{t,\tau}$ and $\zeta^S_{t,\tau}$ (only on $\sigma_{\zeta^S}$and σ_{ζ^B}). Then for Agent $i \in \{1,2\}$ the optimization problem in (2.7) can be rewritten as:

$$\underset{\left\{x^{(i)}_{t,\tau}\right\}_{\tau\in(0,\Lambda_i)}}{max} \left(\int_0^{\Lambda_i} x^{(i)}_{t,\tau}(\Xi'(\tau)-1)\zeta^B_{t,\tau} d\tau\right) dt - E_t\left[\left(\left(\int_0^{\Lambda_i} x^{(i)}_{t,\tau}\Xi(\tau) d\tau\right) d\zeta^B_{t,\tau}\right] + o_{i,1}(.) \tag{2.57}$$

and for Agent 3:

$$\max_{\left\{x_{t,\tau}^{(3)}\right\}_{\tau\in(0,\Lambda_2)}} \left(\int_0^{\Lambda_i}(x_{t,\tau}^{(3)}+y_{t,\tau}^{(3)})(\Xi'(\tau)-1)\zeta_{t,\tau}^{B}d\tau\right)dt-E_t\left[-\left(\int_0^{\Lambda_i}(x_{t,\tau}^{(3)}+y_{t,\tau}^{(3)})\Xi(\tau)d\tau\right)d\zeta_{t,\tau}^{B}\right]+o_{3,1}(.) \tag{2.58}$$

where $o_{i,1}(.)$ for all i does not depend on $\zeta_{t,\tau}^{B}$and $\zeta_{t,\tau}^{S}$. The first order condition with respect to $x_{t,\tau}^{(i)}$for each Agent i is:

$$\Xi'(\tau)\zeta_{t,\tau}^{B}-\zeta_{t,\tau}^{B}+o_{i,2}(.)=0$$

where again $o_{i,2}(.)$ does not depend on $\zeta_{t,\tau}^{B}$and $\zeta_{t,\tau}^{S}$. Since these conditions must hold for all values of $\zeta_{t,\tau}^{B}$ it follows that $\Xi'(\tau)=1$ and $\Xi(\tau)=\tau$. The first order conditions with respect to $y_{t,\tau}^{(3)}$ are consistent with this solution.

2.6.3 Proof of Proposition 1

I show that the first order condition (2.26) can be approximated by (2.34) for small δ. The approximation of the other first order conditions follows similar techniques. (2.26) can be rewritten as:

$$\frac{\phi(\tau)r_t+\bar{\phi}(\tau)}{\tau}=\gamma\sigma_r^2\int_0^{\Lambda_1}z_{t,s}^{(1)}A(s)ds\frac{A(\tau)}{\tau}+\sigma_{\zeta^B}^2\gamma\Omega(\tau) \tag{2.59}$$

where $\Omega(\tau)$ is defined as:

$$\Omega(\tau)=\int_0^{\tau}z_{t,s}^{(1)}\delta^{\tau-s}sds+\int_{\tau}^{\Lambda_1}z_{t,s}^{(1)}\delta^{s-\tau}sds$$

Setting $\mathcal{E}_1$ equal to the right hand side of (2.59) differentiated twice with respect to τ, and noticing that:

$$\Omega''(\tau)=(log\delta)^2\Omega(\tau)+2z_{t,\tau}^{(1)}\tau log\delta$$

gives:

$$\mathcal{E}_1 = \gamma\sigma_r^2 \int_0^{\Lambda_1} z_{t,s}^{(1)} A(s)ds \left(\frac{A''(\tau)}{\tau} - 2\frac{A'(\tau)}{\tau^2} + 2\frac{A(\tau)}{\tau^3} \right) + \sigma_{\zeta^B}^2 \gamma (log\delta)^2 \Omega + 2\sigma_{\zeta^B}^2 \gamma (log\delta) \tau z_{t,\tau}^{(1)} \tag{2.60}$$

Differentiating the left hand side of (2.59) twice with respect to τ, and denoting the result as $\mathcal{E}_2$ yields:

$$\mathcal{E}_2 = \frac{\phi''(\tau)r_t + \bar{\phi}''(\tau)}{\tau} - 2\frac{\phi'(\tau)r_t + \bar{\phi}'(\tau)}{\tau^2} + 2\frac{\phi(\tau)r_t + \bar{\phi}(\tau)}{\tau^3} \tag{2.61}$$

Solving for $\Omega(\tau)$ in (2.59), substituting the result in (2.60), and equating with (2.61) gives:

$$\phi(\tau)r_t + \bar{\phi}(\tau) = \gamma\sigma_r^2 \int_0^{\Lambda_1} z_{t,s}^{(1)} A(s)dsA(\tau) + 2\left(\sigma_{\zeta^B}^2 \gamma \tau^2 z_{t,\tau}^{(1)}\right) \upsilon + o(\upsilon^2)$$

where $\upsilon = -\frac{1}{log\delta}$. For small values of δ we can ignore the higher order terms in υ, $o(\upsilon^2)$. Setting $\kappa = 2\upsilon$ we obtain (2.34).

2.6.4 Proof of proposition 2

I assume $\theta_1 = 0$, so there are only Agents 2 and 3 in the market . The proof for $\theta_1 \neq 0$ is similar but the formulae are more complicated. Substituting the market clearing condition (2.30) into the first Agent 3 first order condition (2.36) and matching terms in r_t gives:

$$\phi(\tau) = -\gamma\sigma_r^2 \int_0^{\Lambda_2} \frac{\theta_2}{\theta_3} \hat{z}_{t,s}^{(2)} A(s)dsA(\tau) - \gamma\sigma_r^2 \int_0^{\Lambda_2} \hat{y}_{t,s}^{(3)} D(s)dsA(\tau) - \gamma\sigma_{\zeta^B}^2 \frac{\theta_2}{\theta_3} \hat{z}_{t,\tau}^{(2)} \tau^2 \tag{2.62}$$

where $\hat{z}_{t,s}^{(i)}$ and $\hat{y}_{t,\tau}^{(3)}$ are the coefficients of the r_t term in the agents bond and bond-swap package demand function respectively:

$$z_{t,\tau}^{(i)} = \bar{z}_{t,\tau}^{(i)} + \hat{z}_{t,\tau}^{(i)} r_t$$

$$y_{t,\tau}^{(i)} = \bar{y}_{t,\tau}^{(3)} + \hat{y}_{t,\tau}^{(3)} r_t$$

The Agent 2 first order condition (2.35) together with (2.62) gives:

$$(1+\frac{\theta_2}{\theta_3})\phi(\tau) = -\gamma\sigma_r^2 \int_0^{\Lambda_2} \hat{y}_{t,s}^{(3)} D(s) ds A(\tau) \tag{2.63}$$

Recall $\phi(\tau) = A(\tau)\alpha^r + A'(\tau) - 1$ so (2.63) is a first order partial differential equation in $A(\tau)$ with solution:

$$A(\tau) = \frac{1-e^{-\alpha_A \tau}}{\alpha_A} \tag{2.64}$$

$$\alpha_A = \alpha_r + \frac{\gamma\sigma_r^2 \int_0^{\Lambda_2} \hat{y}_{t,s}^{(3)} D(s) ds}{(1+\frac{\theta_2}{\theta_3})} \tag{2.65}$$

The terms in r_t of (2.35) form a Fredholm linear integral equation of the second type with solution:

$$\hat{z}_{t,\tau}^{(2)} = \frac{\phi(\tau)}{\sigma_{\zeta^B}^2 \tau^2 \gamma} - \frac{\sigma_r^2}{\sigma_{\zeta^B}^2 \gamma} \left(\sigma_{\zeta^B}^2 + \sigma_r^2 \int_0^{\Lambda_2} \frac{\mathrm{A}(s)^2}{s^2} ds \right)^{-1} \int_0^{\Lambda_2} \frac{\phi(s)A(s)}{s^2} ds \frac{A(\tau)}{\tau^2} \tag{2.66}$$

Substituting for $\phi(\tau) = (\alpha_r - \alpha_A)A(\tau)$ gives:

$$\hat{z}_{t,\tau}^{(2)} = \frac{(\alpha_r - \alpha_A)A(\tau)}{\sigma_{\zeta^B}^2 \tau^2 \gamma} - \frac{\sigma_r^2(\alpha_r - \alpha_A)}{\sigma_{\zeta^B}^2 \gamma} \left(\sigma_{\zeta^B}^2 + \sigma_r^2 \int_0^{\Lambda_2} \frac{\mathrm{A}(s)^2}{s^2} ds \right)^{-1} \int_0^{\Lambda_2} \frac{A(s)^2}{s^2} ds \frac{A(\tau)}{\tau^2}$$
(2.67)

Taking the difference of the two Agent 3 first order conditions (2.37) and (2.36), matching terms in r_t and substituting for $\hat{z}_{t,\tau}^{(2)}$ gives:

$$\frac{\sigma_s^2 \int_0^T E(s)\hat{y}_{t,s}^{(3)} ds E(\tau)}{\sigma_{\zeta^S}^2 \tau^2} + \hat{y}_{t,\tau}^{(3)} = \frac{\varrho(\alpha_A - \alpha_r)D(\tau)}{\tau^2} \tag{2.68}$$

where

$$\varrho = -\frac{1}{\sigma_{\zeta^S}^2 \gamma} \left(1 + \frac{\theta_2}{\theta_3} \left(1 + \frac{\sigma_r^2}{\sigma_{\zeta^B}^2} \int_0^{\Lambda_2} \frac{A(s)^2}{s^2} ds \right)^{-1} \right)$$

(2.68) is another Fredholm linear integral equation with solution:

$$\hat{y}_{t,s}^{(3)} = (\alpha_A - \alpha_r)\varrho \left(\frac{D(\tau)}{\tau^2} - \varsigma \frac{E(\tau)}{\tau^2} \right) \tag{2.69}$$

$$\varsigma = \frac{\sigma_s^2}{\sigma_{\zeta^S}^2}\left(1+\frac{\sigma_s^2}{\sigma_{\zeta^s}^2}\int_0^{\Lambda_2}\frac{E(s)^2}{s^2}ds\right)^{-1}\int_0^{\Lambda_2}\frac{E(s)D(s)}{s^2}ds$$

Substituting (2.69) into (2.65) gives the following equality:

$$(\alpha_A-\alpha_r)\left(1+\frac{\theta_2}{\theta_3}\right)=(\alpha_A-\alpha_r)\,\varrho\gamma\sigma_r^2\left(\int_0^{\Lambda_2}\frac{D(s)^2}{s^2}ds-\varsigma\int_0^{\Lambda_2}\frac{D(s)E(s)}{s^2}ds\right) \tag{2.70}$$

(2.70) is satisfied if $\alpha_A=\alpha_r$ or

$$\left(1+\frac{\theta_2}{\theta_3}\right)=\varrho\gamma\sigma_r^2\left(\int_0^{\Lambda_2}\frac{D(s)^2}{s^2}ds-\varsigma\int_0^{\Lambda_2}\frac{D(s)E(s)}{s^2}ds\right) \tag{2.71}$$

(2.71) can be rewritten as:

$$-\frac{\sigma_{\zeta^B}^2(\theta_2+\theta_3)\left(\sigma_r^2 I_{D,D}\left(\sigma_s^2 I_{E,E}+\sigma_{\zeta^S}^2\right)-\sigma_r^2\sigma_s^2 I_{D,E}^2+\sigma_s^2\sigma_{\zeta^S}^2 I_{E,E}+\sigma_{\zeta^S}^4\right)}{\sigma_r^2\left(\theta_3\sigma_r^2 I_{D,D}\left(\sigma_s^2 I_{E,E}+\sigma_{\zeta^S}^2\right)-\theta_3\sigma_r^2\sigma_s^2 I_{D,E}^2+\sigma_{\zeta^S}^2(\theta_2+\theta_3)\left(\sigma_s^2 I_{E,E}+\sigma_{\zeta^S}^2\right)\right)}=\int_0^{\Lambda_2}\frac{A(s)^2}{s^2}ds \tag{2.72}$$

where $I_{J,K}=\int_0^{\Lambda_2}\frac{J(s)K(s)}{s^2}ds$

By the Cauchy-Schwarz inequality $I_{D,E}^2\le I_{D,D}I_{E,E}$, so the left hand side of (2.72) is less than or equal to zero. The equality then can only hold if $A(s)=0$ for all s, which is absurd. $\alpha_A=\alpha_r$ remains the only solution to (2.70).

2.6.5 Proof of Proposition 3

Proof. Substituting $D(\tau)$ for $A(\tau)$ in the Agent i, $i\in\{1,2\}$, first order conditions (2.34)-(2.35) gives:

$$\frac{\bar{\phi}(\tau)}{\sigma_{\zeta^B}^2\tau^2\gamma}=\frac{\sigma_r^2}{\sigma_{\zeta^B}^2\tau^2}\int_0^{\Lambda_i}z_{t,s}^{(i)}D(s)dsD(\tau)+z_{t,\tau}^{(i)} \tag{2.73}$$

Suppressing the Agent indicator and time subscript, (2.73) can be rewritten as:

$$z_\tau - \lambda \int_0^{\Lambda_i} z_s g(\tau)h(s)ds = f(\tau) \tag{2.74}$$

where $\lambda = \frac{-\sigma_r^2}{\sigma_{\zeta B}^2}$, $\quad g(\tau) = \frac{D(\tau)}{\tau^2}$, $\quad f(\tau) = \frac{\bar{\phi}(\tau)}{\sigma_{\zeta B}^2 \tau^2 \gamma}$, $\quad h(s) = D(s)$.

(2.74) is a Fredholm linear integral equation of the second type with a degenerate kernel. Provided $\lambda \neq \int_0^{\Lambda_i} \mathrm{g}(s)h(s)ds$ the solution (see Polyanin and Manzhirov (2012) or Mikhlin (1960)) to (2.74) is:

$$z_\tau = f(\tau) + \lambda k g(\tau) \tag{2.75}$$

where:

$$k = \left(1 - \lambda \int_0^{\Lambda_i} g(s)h(s)ds\right)^{-1} \int_0^{\Lambda_i} f(s)h(s)ds$$

Substituting for the values of λ, $g(\tau)$, $f(\tau)$ and $h(s)$ in 2.75 gives 2.38. In the case $\lambda = \int_0^{\Lambda_i} \mathrm{g}(s)h(s)ds$, if $\int_0^{\Lambda_i} f(s)h(s)ds = 0$ then the solution is:

$$z_\tau = \frac{\bar{\phi}(\tau)}{\sigma_{\zeta B}^2 \tau^2 \gamma} + \chi \frac{D(\tau)}{\tau^2}$$

where χ is an arbitrary constant. If $\int_0^{\Lambda_i} f(s)h(s)ds \neq 0$ there is no solution. Polyanin and Manzhirov (2012) has more details.

□

2.6.6 Proof of proposition 4

Proof. Use the Agent 3 first order condition (2.36) to express bond swap package demand $y_{t,\tau}^{(3)}$ in terms of Agent 3 pure bond demand $z_{t,\tau}^{(3)}$. Again supressing the agent indicator and time subscript we obtain:

$$y_\tau = \frac{\bar{\phi}(\tau) - \gamma\sigma_r^2 \int_0^{\Lambda_2} z_s D(s)dsD(\tau) - z_\tau \tau^2 \gamma \sigma_{\zeta B}^2}{\sigma_{\zeta B}^2 \gamma \tau^2} \tag{2.76}$$

Substitution into the second Agent 3 first order condition (2.37) gives:

$$z_\tau - \lambda \int_0^{\Lambda_2} z_s g_1(\tau) h_1(s) ds - \lambda \int_0^{\Lambda_2} z_s g_2(\tau) h_2(s) ds = f(\tau)$$

where

$$\lambda = \frac{-1}{\sigma^2_{\zeta \mathrm{S}}}$$

$$f(\tau) = \frac{1}{\gamma \sigma^2_{\zeta \mathrm{B}}} \frac{\bar{\phi}(\tau)}{\tau^2} + \frac{\Pi^S(\tau)}{\gamma \sigma^2_{\zeta \mathrm{S}} \tau^2} + \frac{\sigma_s^2}{\gamma \sigma^2_{\zeta \mathrm{S}} \sigma^2_{\zeta \mathrm{B}}} \int_0^{\Lambda_2} \frac{E(s)\bar{\phi}(\tau)}{s^2} ds \frac{E(\tau)}{\tau^2}$$

$$g_1(\tau) = \frac{\sigma_r^2 \sigma_s^2}{\tau^2 \sigma^2_{\zeta \mathrm{B}}} E(\tau) \int_0^{\Lambda_2} \frac{E(s)D(s)}{s^2} ds + \frac{(\sigma^2_{\zeta \mathrm{B}} + \sigma^2_{\zeta \mathrm{S}})\sigma_r^2 D(\tau)}{\sigma^2_{\zeta \mathrm{B}} \tau^2}$$

$$g_2(\tau) = \frac{\sigma_s^2}{\tau^2} E(\tau)$$

$$h_1(s) = D(s)$$

$$h_2(s) = E(s)$$

$$\Pi^S(\tau) = D(\tau)\lambda_r \sigma_r + E(\tau)\lambda_s \sigma_s$$

This is a Fredholm linear integral equation of the second kind, with degenerate kernel $K(\tau, s) = g_1(\tau)h_1(s) + g_2(\tau)h_2(s)$. Provided λ is not equal to one of the characteristic values:

$$\lambda_{1,2} = \frac{s_{11} + s_{22} \pm \sqrt{(s_{11} - s_{22})^2 + 4 s_{12} s_{21}}}{2(s_{11}s_{22} - s_{12}s_{21})}$$

where $s_{ij} = \int_0^{\Lambda_2} h_i(s) g_j(s) ds$ for $i, j = 1, 2$ a closed form solution exists (see for example Polyanin and Manzhirov (2012)). The solution is of the form:

$$z_\tau = f(\tau) + \lambda(A_1 g_1(\tau) + A_2 g_2(\tau)) \tag{2.77}$$

where:

$$A_1 = \frac{f_1 - \lambda(f_1 s_{22} - f_2 s_{12})}{(s_{11}s_{22} - s_{12}s_{21})\lambda^2 - (s_{11} + s_{22})\lambda + 1}$$

$$A_2 = \frac{f_2 - \lambda(f_2 s_{11} - f_1 s_{21})}{(s_{11}s_{22} - s_{12}s_{21})\lambda^2 - (s_{11} + s_{22})\lambda + 1}$$

$$f_1 = \int_0^{\Lambda_2} f(s)h_1(s)ds$$

$$f_2 = \int_0^{\Lambda_2} f(s)h_2(s)ds$$

Substituting the values for λ, $f(\tau)$, $g_1(\tau)$, $g_2(\tau)$, $h_1(s)$, and $h_2(s)$ into 2.77 and some rearrangement yields 2.40 with:

$$b_3 = \frac{1}{\gamma\sigma_{\zeta\mathrm{B}}^2}$$

$$c_3(\tau) \quad = \quad -\frac{\sigma_r^2 \left(D(\tau)\left(\sigma_{\zeta\mathrm{B}}^2 + \sigma_{\zeta\mathrm{S}}^2\right)\left(I_{E,E}\sigma_s^2 + \sigma_{\zeta\mathrm{S}}^2\right) - I_{D,E}\sigma_s^2\sigma_{\zeta\mathrm{B}}^2 E(\tau)\right)}{(Q_1+Q_2)\sigma_{\zeta\mathrm{B}}^2\tau^2}$$

$$a_3(\tau) = \frac{\Pi^S(\tau)}{\gamma\tau^2\sigma_{\zeta\mathrm{S}}^2} + a_{3,D}(\tau)\frac{D(\tau)}{\tau^2} + a_{3,E}(\tau)\frac{E(\tau)}{\tau^2}$$

where $a_{3,D}(\tau)$ and $a_{3,E}(\tau)$ are defined by:

$$a_{3,D}(\tau) = \frac{-\sigma_r^2\left(\sigma_{\zeta\mathrm{B}}^2 + \sigma_{\zeta\mathrm{S}}^2\right)\left((I_{E,E}I_{\Pi,D} - I_{D,E}I_{\Pi,E})\sigma_s^2 + I_{\Pi,D}\sigma_{\zeta\mathrm{S}}^2\right)}{(Q_1+Q_2)\sigma_{\zeta\mathrm{S}}^2}$$

$$a_{3,E}(\tau) = \frac{\sigma_s^2 E(\tau)\left((I_{D,E}I_{\Pi,D} - I_{D,D}I_{\Pi,E})\sigma_r^2\sigma_{\zeta\mathrm{B}}^2 - I_{\Pi,E}\left(I_{D,D}\sigma_r^2 + \sigma_{\zeta\mathrm{B}}^2\right)\sigma_{\zeta\mathrm{S}}^2\right)}{(Q_1+Q_2)\sigma_{\zeta\mathrm{S}}^2}$$

and Q is given by:

$$Q_1 = \gamma\sigma_r^2\left(\sigma_s^2\sigma_{\zeta^B}^2\left(I_{D,D}I_{E,E} - I_{D,E}^2\right) + I_{D,D}\sigma_{\zeta^S}^2\left(I_{E,E}\sigma_s^2 + \sigma_{\zeta^B}^2\right) + I_{D,D}\sigma_{\zeta^S}^4\right)$$

$$Q_2 = \sigma_{\zeta^B}^2\sigma_{\zeta^S}^2\left(I_{E,E}\sigma_s^2 + \sigma_{\zeta^S}^2\right)$$

For the cases where λ is equal to one (or both) characteristic value(s) see Polyanin and Manzhirov (2012).

□

2.6.7 Proof of proposition 5

Proof. Substitute 2.40, the expression for $z_{t,\tau}^{(3)}$in proposition 3, into 2.76. □

2.6.8 Proof of proposition 6

Proof. The general market clearing condition (2.44) in proposition 5 defines a Fredholm linear integral equation in $\frac{\bar{\phi}(\tau)}{\tau^2}$ with kernel $\sum_{i=1}^{2} g_i(\tau)h_i(s)$ where $g_i(\tau)$ and $h_i(s)$ for $i=$ 1,2 are as defined in the proposition. The solution to an integral equation of this form was given in proposition 4 by (2.77). Substitution of $\lambda,\ f(\tau),\ g_i(\tau),$ and $h_i(s)$ for $i = 1, 2$ gives (2.45).

□

2.7 Figures and tables

Figure 2.1: Bond-swap basis

The figure shows a time series of the 1- and 5-year zero-coupon Treasury yields and the 1 year and 5 year treasury swap basis in the period 1990 - 2013. The n-year basis is defined as the n- year zero-coupon Treasury yield less the n- year zero-coupon swap rate.

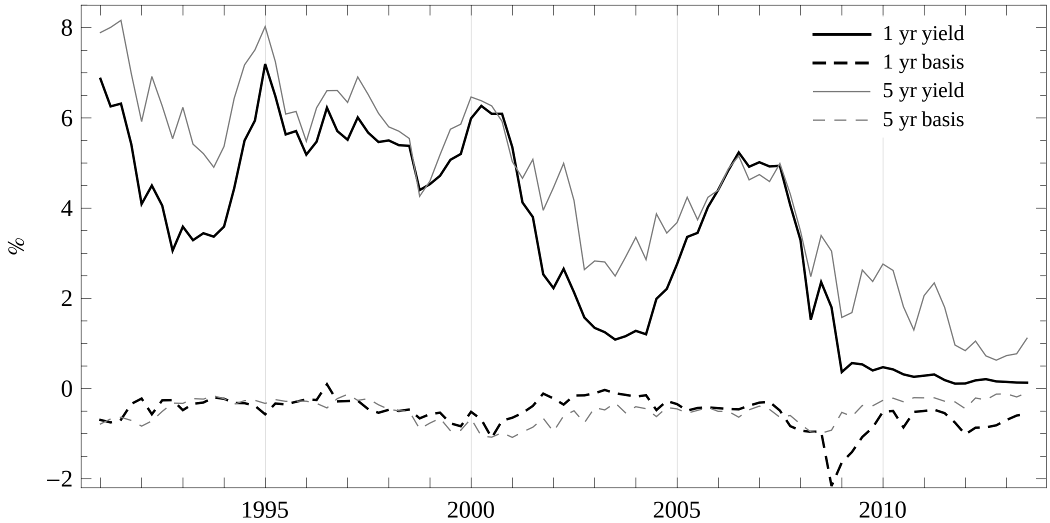

Figure 2.2: Swap reference rate - FFR spread

The figure shows a time series of the Federal Funds Rate (FFR), USD 6 month LIBOR rate and the USD 6 month LIBOR rate - FFR basis in the period 1990 - 2013.

Figure 2.3: Volatility proxy $\hat{\sigma}_{s,t}$ against 6m Libor and the FFR
The figure below gives a time series of the volatility proxy for the USD 6 month LIBOR rate - FFR basis, $\hat{\sigma}_{s,t}$, together with the FFR and USD 6 month LIBOR rate in the period 1990 - 2013 for n = 252.

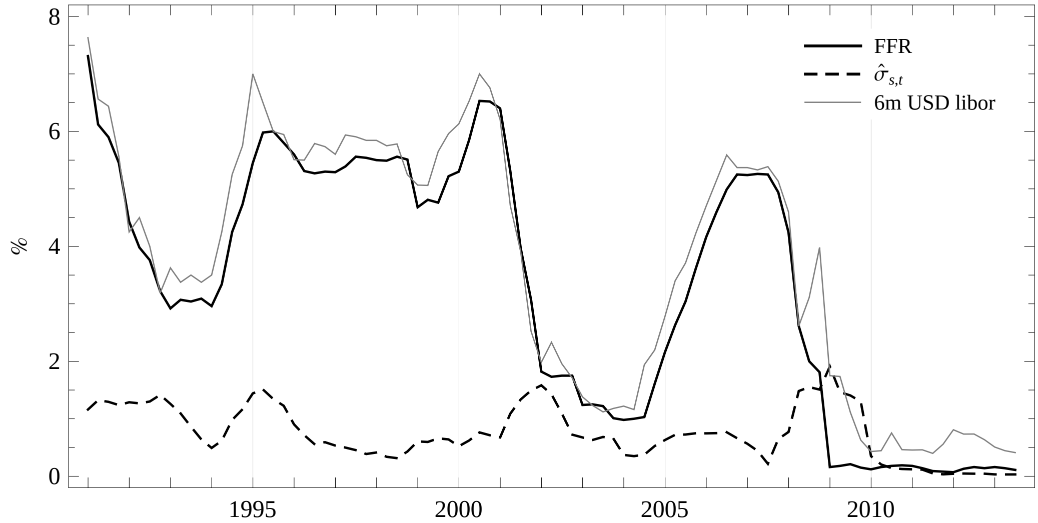

Figure 2.4: Volatility proxy $\hat{\sigma}_{s,t}$ against 6m Libor - FFR basis
The figure below plots a time series of the volatility proxy for the USD 6 month LIBOR rate - FFR basis, $\hat{\sigma}_{s,t}$, against the USD 6 month LIBOR rate - FFR basis over the period 1990 - 2013.

Figure 2.5: 5 and 10 year excess bond yields, and the volatility indicator
This figure shows the 5 and 10 year excess bond yields over the 1 year treasury yield, with the proxy for bond-swap basis volatility, $\hat{\sigma}_{s,t}$, over the period 1990 - 2013. The τ year excess bond yield is given by $-\frac{p^B_{t,\tau}}{\tau} + p^B_{t,1}$ where $p^B_{t,\tau}$ is the price of a τ-year bond.

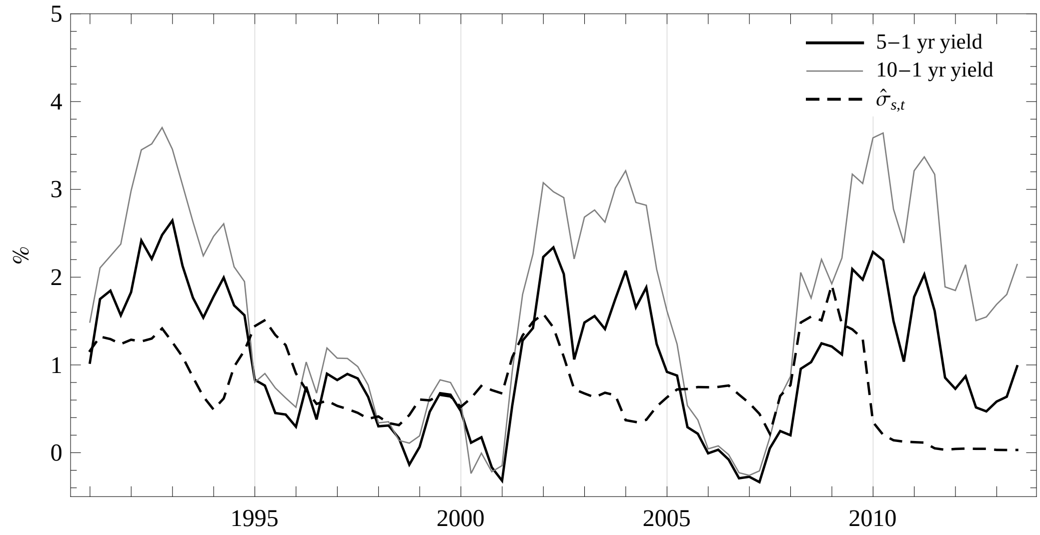

Figure 2.6: 5 year bond swap basis and 5 year adjusted bond swap basis, and the volatility indicator
This figure shows the 5 year bond-swap basis adjusted by the difference between the 6 month Libor rate and the FFR, the unadjusted 5 year bond-swap basis and the basis volatility indicator $\hat{\sigma}_{s,t}$. The τ- year adjusted bond-swap basis is given by $-\frac{p^B_{t,\tau}}{\tau} + \frac{p^S_{t,\tau}}{\tau} + (s_t - r_t)$, where $-\frac{p^B_{t,\tau}}{\tau}$ is the yield on a τ- year zero coupon bond, $-\frac{p^S_{t,\tau}}{\tau}$ is the τ year zero coupon swap rate, s_t is the 6 month Libor rate and $r_t$is the FFR.

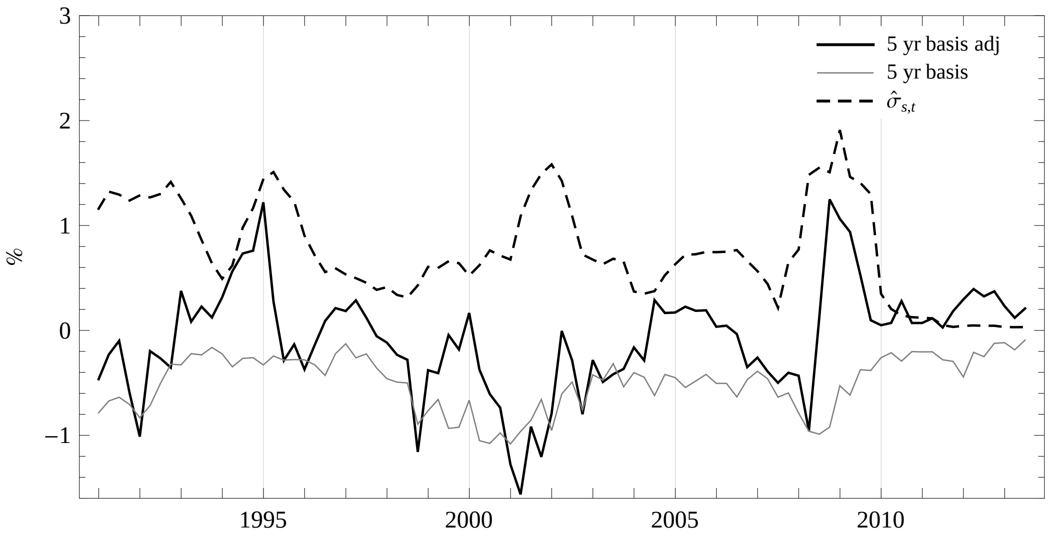

Figure 2.7: Effect of increased swap market access and bond-swap basis volatility Panel A shows the effect of varying access to the swap market (θ_3) on the yield curve in the maturity interval $[0, 10]$. Panel B shows the effect of varying bond-swap basis volatility on the yield curve. Parameters are given in table 2.3.

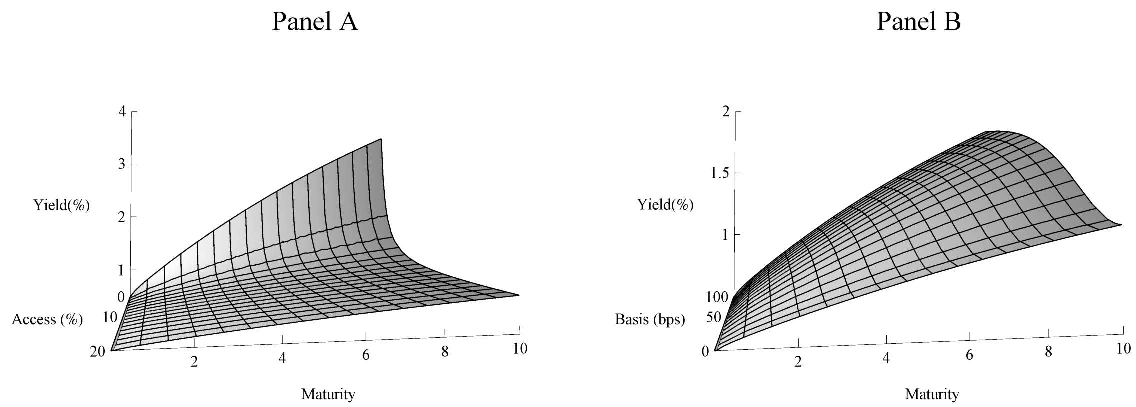

Figure 2.8: Effect of changes in bond supply varying swap access and bond-swap basis volatility
Panel A shows the change in equilibrium yield for a 0.05 unit increase (baseline $k(\tau) = 0.1$) in bond supply across the term structure for varying levels of swap market access (θ_3). Panel B shows the change in equilibrium yield for the same shock for varying bond-swap basis volatility levels. Parameters are given in table 2.3.

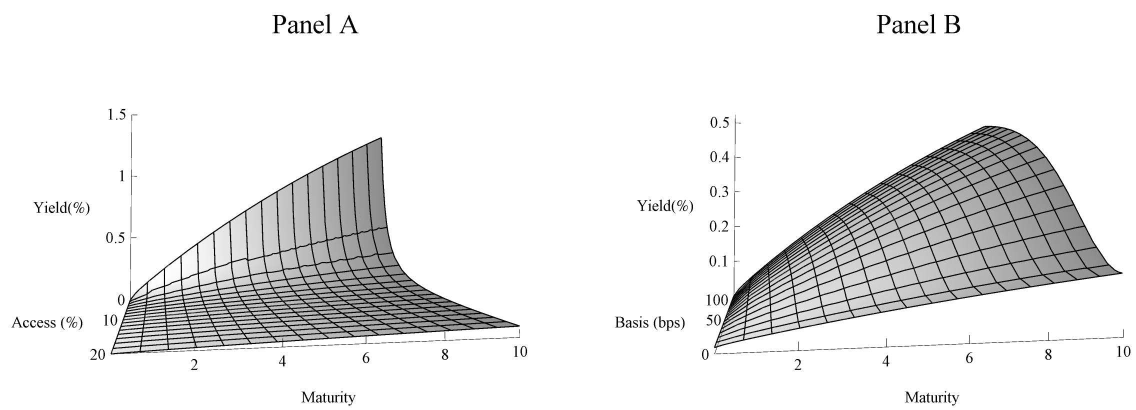

Figure 2.9: Agent access and bond demand
Panel A shows the Agent 2 and Agent 3 equilibrium 5 year bond demand for varying levels of access. Panel B shows the Agent 2 and Agent 3 share of total equilibrium bond demand for different levels of access. Parameters used are given in table 2.3. Agent 2 bond demand in Panel A is scaled up by a factor of 10.

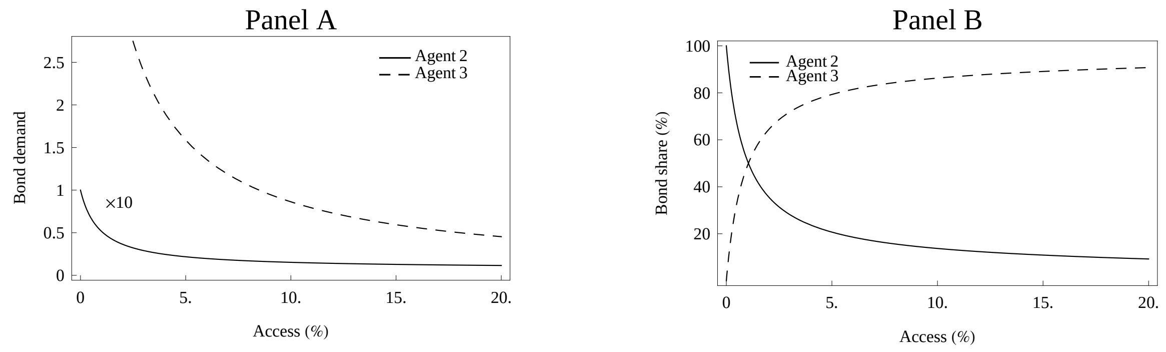

Figure 2.10: Expected portfolio returns by Agent access
Panel A shows the expected portfolio returns of each Agent for varying levels of access to the swap market. Panel B plots the cross-subsidisation statistic between agents without access to the swap market to agents with access. The statistic is defined in equation (2.48) and the expected returns in Θ are expressed in %. Parameters are given in table 2.3. Agent 3 expected portfolio returns in Panel A are scaled down by a factor of 10.

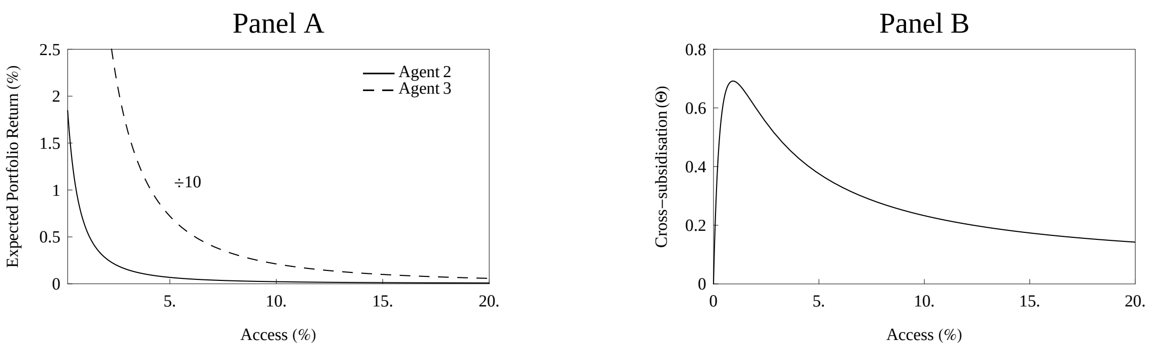

Figure 2.11: Effect of bond supply 'twists' on the term structure
Panel A shows the functional form of two different bond-supply shocks. The first shock ("Twist 1") is linearly decreasing in maturity, and the unweighted integral of the function is equal to zero. The second shock ("Twist 2") is a non-linear decreasing function of maturity. Under Twist 2 the overall sensitivity to the short rate of the outstanding bonds is unchanged as the $D(\tau)$ weighted integral of the function is equal to zero. Panel B shows the effect of Twist 1 and Twist 2 on the yield curve when bonds are quasi-substitutable subject to a scaling factor. Parameters are given in table 2.3.

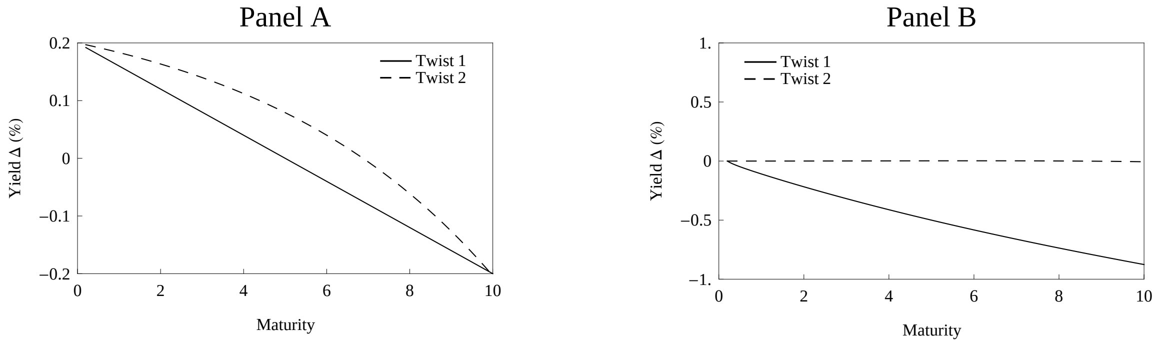

Figure 2.12: Effect of bond supply 'twists' on the term structure.
Panel A shows the functional form of two non-linear bond-supply shocks, "Twist 3" and "Twist 4" respectively. Panel B shows the effect of these shocks when there is low access to the swap market, high bond-swap reference rate basis and large bond and swap idiosyncratic risk. Parameters are given in table 2.3.

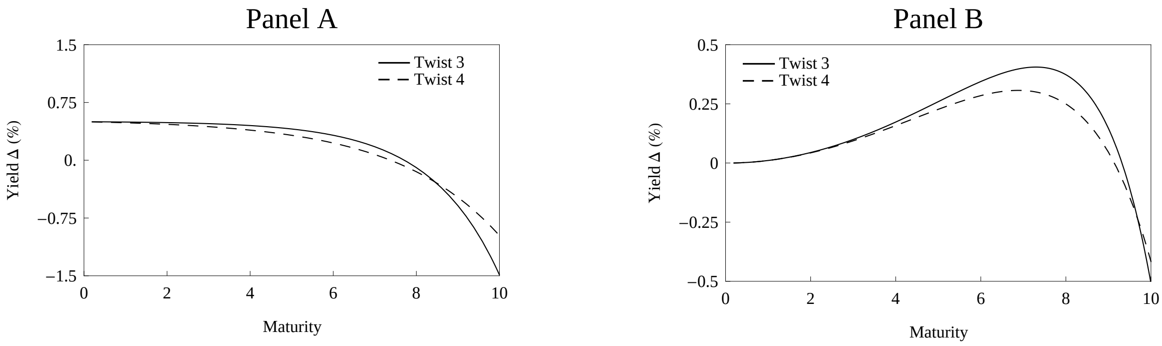

Figure 2.13: Effect of short rate volatility for varying access and bond-swap basis Panel A in the figure below shows the change in the yield curve for varying levels of short rate risk σ_r. Panel B shows the change in 5 year bond yield corresponding to a 100bps increase in short rate volatility (baseline 100bps) for different levels of access and bond swap basis. Parameters are given in table 2.3.

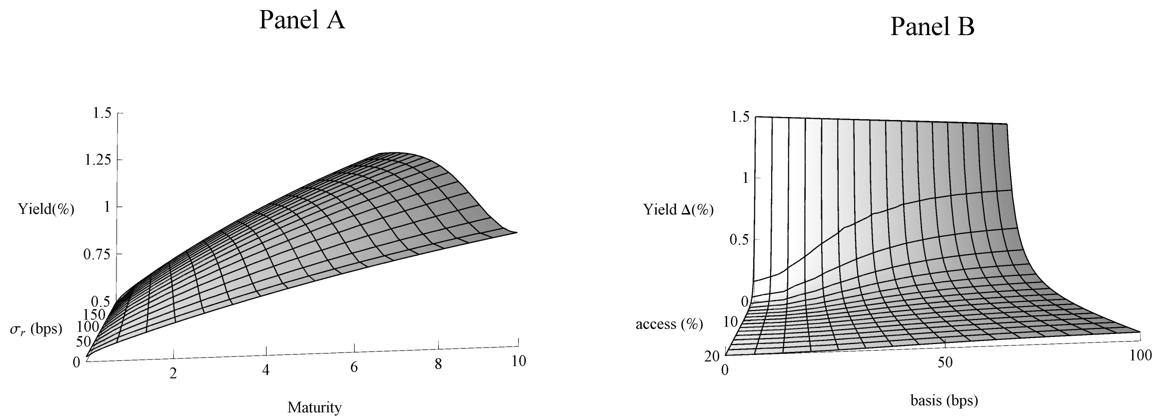

Figure 2.14: Residual risk of hedging a liability with a bond of different maturity Panel A below shows the residual risk of hedging the short rate risk of a τ- year maturity liability, with a $\bar{\tau}$-year maturity bond. The plot gives the volatility of a combined short one unit τ- maturity bond position and a $\bar{w} = \frac{D(\tau)}{D(\bar{\tau})}$ position in the $\bar{\tau}$-year maturity bond with residual risk given in (2.53). Panel B shows the difference in the change in yield between a high-idiosyncratic risk economy ($\sigma_{\zeta^B} = \sigma_{\zeta^S} = 1\%$) and a low idiosyncratic risk economy ($\sigma_{\zeta^B} = \sigma_{\zeta^S} = 0.2\%$), for a 0.1 unit increase in bond supply across the term-structure (baseline supply = 0). Parameters are given in table 2.3.

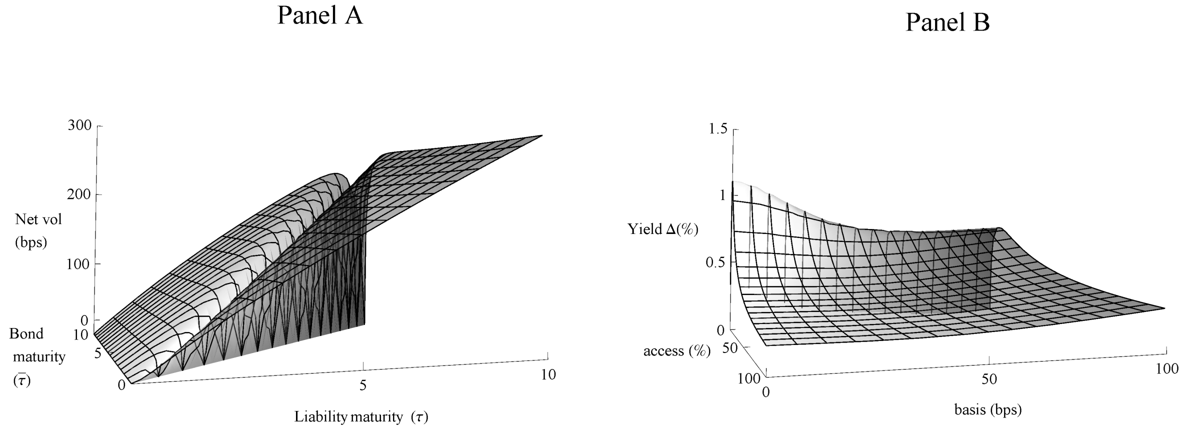

Figure 2.15: Effect of swap market short rate and swap spread risk premia
Panel A in the figure below shows the derivative of bond demand with respect to the spread risk premium in the swap market (λ_s) for agents with access. Panel B shows the derivative of bond demand with respect to swap market short rate risk premium (λ_r). The dashed line represents marginal demand for bonds in outright format, the dotdashed line represents the demand for bonds in package format and the full line represents total demand for bonds. λ_s and λ_r are assumed to be expressed in basis points (bps). Parameters are given in table 2.3.

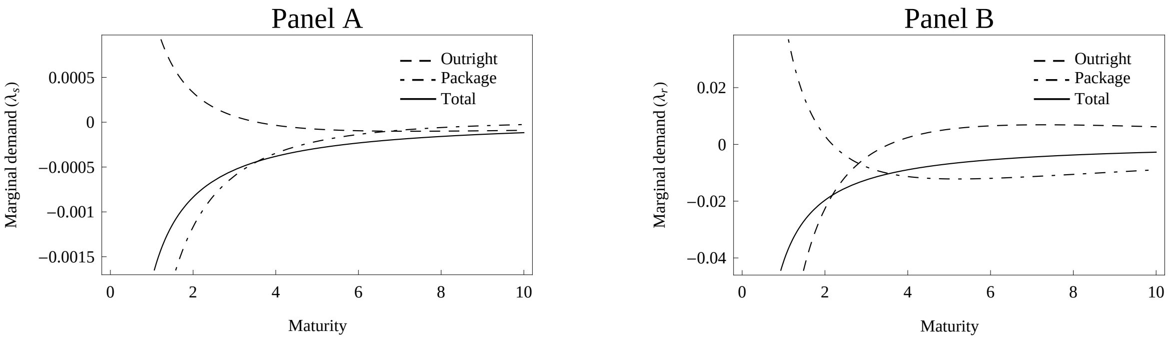

Figure 2.16: Effect of limited access to the bond term-structure
Panel A below shows the effect of increasing the proportion of agents with restricted access to the bond-term structure. The full line in Panel A shows the equilibrium term-structure when all agents have access to the full bond term-structure($\theta_1 = 0,\ \theta_2 = 1$), and the dashed line corresponds to the case when thirty percent of agents are restricted to maturities less than 2. ($\Lambda_1 = 2,\ \theta_1 = 0.3,\ \theta_2 = 0.7$). Panel B is the same plot with the short-rate reversion increased from $\alpha_r = 0.1$ toα_r=0.4.

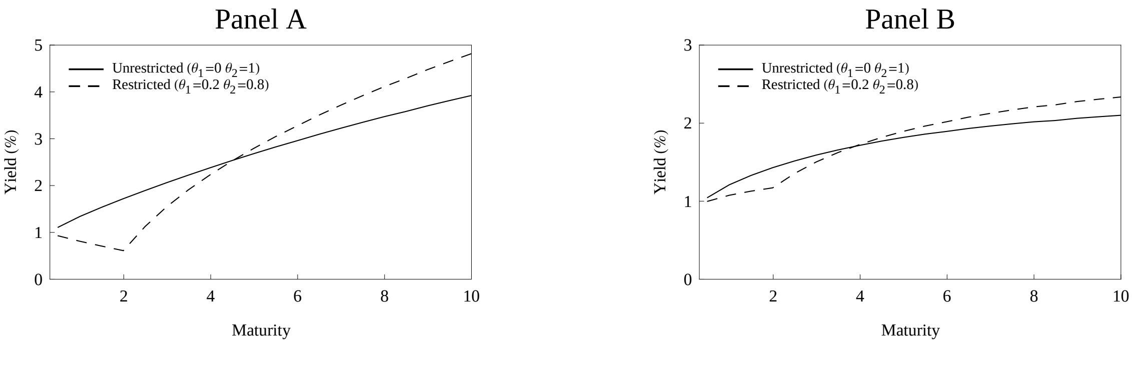

Table 2.1: Basis volatility and bond term-structure steepness

The table below shows the estimated β^τ from regressions: $-\frac{P^B_{t,\tau}}{\tau} + P^B_{t,1} = \alpha^\tau + \beta^\tau \hat{\sigma}_{s,t} + \epsilon^\tau_t$ (Model 1) and $-\frac{P^B_{t,\tau}}{\tau} + P^B_{t,1} = \alpha^\tau + \beta^\tau \hat{\sigma}_{s,t} + \gamma^\tau s_t + \epsilon^\tau_t$ (Model 2) for a 252 day and 60 day lag period.

τ	$\hat{\beta}^\tau$- Model 1(252d lag)	$\hat{\beta}^\tau$- Model 2 (252d lag)	$\hat{\beta}^\tau$- Model 1 (60d lag)	$\hat{\beta}^\tau$- Model 2 \n(60d lag)
2	0.25	0.24	0.15	0.14
	$(0.08, 0.38)$	$(0.07, 0.41)$	$(0.00, 0.26)$	$(-0.01, 0.28)$
3	0.37	0.35	0.18	0.17
	$(0.12, 0.56)$	$(0.09, 0.61)$	$(-0.04, 0.35)$	$(-0.05, 0.39)$
4	0.43	0.41	0.18	0.16
	$(0.09, 0.69)$	$(0.06, 0.77)$	$(-0.12, 0.40)$	$(-0.14, 0.46)$
5	0.48	0.47	0.17	0.15
	$(0.04, 0.82)$	$(0.02, 0.92)$	$(-0.21, 0.46)$	$(-0.23, 0.54)$
7	0.46	0.45	0.10	0.08
	$(-0.10, 0.89)$	$(-0.13, 1.03)$	$(-0.38, 0.46)$	$(-0.40, 0.57)$
10	0.36	0.35	0.01	-0.00
	$(-0.32, 0.88)$	$(-0.35, 1.05)$	$(-0.56, 0.44)$	$(-0.58, 0.58)$

‘

Table 2.2: Bond and swap sensitivity to short term policy rate changes

Regression estimates $\hat{A}(\tau)$ and $\hat{D}(\tau)$. (99% Confidence intervals in parentheses)

τ	$\hat{A}(\tau)/\tau$	$\hat{D}(\tau)/\tau$	$(\hat{A}(\tau)$-$\hat{D}(\tau))/\tau$
1	0.91	0.88	0.03
	$(0.84, 0.97)$	$(0.80, 0.95)$	$(-0.01, 0.07)$
2	0.90	0.92	-0.02
	$(0.82, 0.98)$	$(0.84, 1.00)$	$(-0.04, 0.01)$
3	0.86	0.88	-0.01
	$(0.77, 0.95)$	$(0.78, 0.97)$	$(-0.04, 0.01)$
4	0.81	0.84	-0.02
	$(0.72, 0.91)$	$(0.73, 0.94)$	$(-0.05, 0.00)$
5	0.77	0.80	-0.04
	$(0.66, 0.87)$	$(0.69, 0.91)$	$(-0.06, -0.01)$
7	0.69	0.74	-0.05
	$(0.58, 0.80)$	$(0.63, 0.85)$	$(-0.08, -0.02)$
10	0.60	0.68	-0.08
	$(0.48, 0.72)$	$(0.56, 0.80)$	$(-0.10, -0.05)$

Table 2.3: Parameters used in the generation of figures

Below I document the values of the parameters used to produce some of the figures in the paper. "-" indicates that the parameter is not appicable and "~" indicates that the parameter is varied within the figure.

Figure	Λ_1	Λ_2	α_r	α_s	μ_r	σ_r	σ_s	σ_{ζ^S}	σ_{ζ^B}	γ	λ_1	λ_2	$l_1(\tau)$	$l_2(\tau)$	$l_3(\tau)$	$k(\tau)$	θ_1	θ_2	θ_3
2.7(A)	–	10	0.1	0.4	0.015	0.017	0.002	0.002	0.002	7	0	0	0	0	0	0.1	0	~	~
2.7(B)	–	10	0.1	0.4	0.015	0.017	~	0.002	0.002	7	0	0	0	0	0	0.1	0	0.95	0.05
2.8(A)	–	10	0.1	0.4	0.015	0.017	0.002	0.002	0.002	7	0	0	0	0	0	~	0	~	~
2.8(B)	–	10	0.1	0.4	0.015	0.017	~	0.002	0.002	7	0	0	0	0	0	~	0	0.95	0.05
2.9(A)	–	10	0.1	0.4	0.015	0.017	0.002	0.002	0.002	7	0	0	0	0	0	0.1	0	~	~
2.9(B)	–	10	0.1	0.4	0.015	0.017	0.002	0.002	0.002	7	0	0	0	0	0	0.1	0	~	~
2.10(A)	–	10	0.1	0.4	0.015	0.017	0.002	0.002	0.002	7	0	0	0	0	0	0.1	0	~	~
2.10(B)	–	10	0.1	0.4	0.015	0.017	0.002	0.002	0.002	7	0	0	0	0	0	0.1	0	~	~
2.11(B)	–	10	0.1	0.4	0.015	0.017	0.007	0.002	0.002	7	0	0	0	0	0	~	0	0.98	0.02
2.12(B)	–	10	0.1	0.4	0.015	0.017	0.007	0.01	0.01	7	0	0	0	0	0	~	0	0.98	0.02
2.13(A)	–	10	0.1	0.4	0.015	~	0.002	0.002	0.002	7	0	0	0	0	0	0.1	0	0.95	0.05
2.13(B)	–	10	0.1	0.4	0.015	~	~	0.002	0.002	7	0	0	0	0	0	0.1	0	~	~
2.14(A)	–	–	0.1	–	–	–	–	0.002	0.002	–	–	–	–	–	–	–	–	–	–
2.14(B)	–	10	0.1	0.4	0.015	0.017	~	~	~	7	0	0	0	0	0	~	0	~	~
2.15(A)	–	10	0.1	0.4	–	0.017	0.002	0.001	0.001	5	~	~	–	–	–	–	–	–	–
2.16(A)	2	10	0.1	–	0.015	0.017	–	–	0.002	7	0	0	0	0	0	0.1	~	~	0
2.16(B)	2	10	0.4	–	0.015	0.017	–	–	0.002	7	0	0	0	0	0	0.1	~	~	0

The functional form of the twists in figure 2.11 and figure 2.12 is: $\triangle k(\tau) = a + b\tau + ce^{d\tau}$. For (i) Figure 2.11 twist 1, (ii) Figure 2.11 twist 2, (iii)2.12 twist 1 and (iv) 2.12 twist 2 the following parameters were used: (i) $a = 0.2$, $b = -0.04$, $c = 0$, $d = 0$; (ii) $a = 0.2902$, $b = 0$, $c = -0.0902$, $d = 0.17$; (iii) $a = 0.5049$, $b = 0$, $c = -0.0049$, $d = 0.6$ and (iv) $a = 0.5276$, $b = 0$, $c = -0.0276$, $d = 0.4$. Baseline value is $k(\tau) = 0$.

Chapter 3

Pro-cyclical regulation and the term structure of credit spreads

3.1 Introduction

The credit spread term structure displays interesting time dynamics around the 2008 financial crisis. Figures 3.1 and 3.2 show that at the end of 2008, option adjusted spreads (OAS) increased sharply across all maturities and rating classes. At the same time, the term structure of credit inverted, sloping significantly downward as the market priced in a significant probability of short-term default. Figure 3.3 shows that at the end of 2008 the OAS on 10-15 year bonds were on average more than 150bps lower than the OAS on 1 to 3 year bonds. Yet after the US government passed the Troubled Asset Relief Program ("TARP"), which bailed out systemic US institutions the spread between short term bonds and long term bonds narrowed, then inverted again in 2010 and crept up to unprecedently high levels of around 100bps. From 2012 to 2014 this spread has hovered around 100bps or more strikingly, around 60% of the current OAS of 10 - 15 year bonds (150bps). Moreover figure 3.4 shows the absolute difference between AAA spreads and spreads of lower rated bonds is greater now than the levels immediately preceding the 2008 crisis.[1]

The increase in the cost of long term financing, and the volatility of the credit mar-

[1] Although these spread differences to AAA have fallen significantly recently.

kets are both issues that concern policymakers. It is widely accepted that the provision of credit financing at all maturities is important for a functioning society, indeed the credit channel is an important channel through which monetary policy can effect a change in real output (Bernanke and Gertler (1995); Mishkin (2007)). Recently, the societal benefits of *long-term* capital provision have been highlighted by policymakers (see eg. World Economic Forum (2011) report). In fact long-term capital is particularly relevant when an economy faces a structural downturn and supply-side policy intervention, which is often of a long-term nature, is required to boost aggregate output. The World Economic Forum (2011) estimates that global infrastructure needs are now as high as $3 trillion per annum, a figure unable to be met with public finances. Yet it is precisely during a downturn when the lack of long-term financing is most acute for all but the safest institutions. In the absence of long-term financing institutions are forced to use short-term funding, if they can afford to fund themselves at all. The institutions consume the risk of not being able to roll their debt and defacto enter into the maturity transformation business.

The volatility of the credit markets well beyond that explainable by the financial accelerator defined in Bernanke et al. (1994) is a consequence of the broader issue of procyclicality in the financial system. Policymakers (see eg. Borio et al. (2001) and Papaioannou et al. (2013)) have attributed some of the pro-cyclical behaviour to short-termism amongst selected institutional agents. The short-termist behaviour can in turn be attributed, inter alia, to reporting, regulatory and shareholder pressures. In particular an increasing shift to mark-to-market valuation principles has contributed to pro-cyclical behaviour amongst banks, insurers and pension funds. Given the size of assets under management (around $80 trillion)[2] it has led to a strong call by some (eg. Barton and Wiseman (2014)) for short-termism to end.

The suggestion seems intuitive especially in markets where there are natural providers of long term funding (eg insurers with large annuity books and pension funds), and natural consumers of long term funding (eg. infrastructure projects). Obligating one side

[2]Figure in Papaioannou et al. (2013) and includes pension funds, life insurers, US endowments, mutual funds, Sovereign Wealth Funds and Foreign reserves.

of the funding trade to risk manage to a time horizon different from their natural time horizon seems counter-intuitive, and leads to a wedge in the market for credit.

I construct a simple model of the credit market to illustrate this idea. In the model there are corporate maturity clienteles that demand financing for projects of specific maturities, and investors that trade the risk and return of the bonds at different maturities. The model has similar features to the Vayanos and Vila (2009) model but is applied in a different context (credit rather than government securities), and agents have different time horizions.[3]

I consider two extreme versions of the model. In one version agents are subject to short-term mark-to-market valuations. Agents trade off risk and return at an infinitesimal time horizon. In the second version of the model agents trade off risk and return over a much longer time horizon. In this version equilibrium credit provision is greater, the average maturity of corporate debt is larger, and the average cost of credit for firms is reduced. Agent behaviour is also less pro-cyclical in this economy. The sensitivity of credit spreads to a change in underlying 'real world' probability of default is lower, and the same is true for a risk aversion shock.

It is understood that both versions of the model described are extreme representations of reality. In fact institutions are more likely to lie on a spectrum with the two poles being the two versions of the model. Intra-day hedge fund traders will lie very close to the pole described by the first version of the model, and some Scandinavian sovereign wealth funds would lie close to the pole described by the second version. Calibrating where each institutional type in each jurisdiction fits on the spectrum is beyond the scope of the paper. Tower and Impavido (2009) is a useful reference that compares different valuation metrics

[3]Note the assumption of maturity specific demand (without a term-specific factor) is more acceptable here than in the original Vayanos and Vila (2009) context. In the government security context, maturity specific demand from institutional investors is difficult to reconcile with an N factor model. In theory any bond can be replicated in infinite ways and so there is no reason (given the assumptions in the model) that such maturity specific demand should exist. In this model the maturity specific demand is for credit funding. It is highly unlikely that corporates would be able to hedge their maturity specific funding requirements with funding of different maturities, by for example gearing up on exposure to short term funding to hedge the long-term funding risk, or shorting exposure to the debt of a number of other corporates. Non-financial corporates are not in the maturity transformation business and are faced with a number of frictions that make this implausible.

for pension schemes across selected OECD countries. What is clear is that a debate exists around what pole is most appropriate (see eg. Allen and Carletti (2008)). The purpose of this paper is to explain via a simple model why a long-term valuation metric is best both in terms of provision of long-term capital and market stability.

The rest of this short paper proceeds as follows: Section 2) sets out a description of the model. In Section 3) I discuss some of the important results from the model. Section 4) concludes and provides some avenues for future research.

3.2 The model

Time is continuous in the model. At time t there are zero-coupon (ZC) risky corporate bonds supplied in the maturity interval $(0, \Lambda]$, where a bond with maturity τ pays 1 dollar at time $T = t + \tau$. $P_{t,\tau} = P(t, T)$. is the time t zero-coupon bond price paying \$1 at time T, and is determined endogenously in the model[4]. The dollar amount of zero coupon risky corporate bonds issued in equilibrium, $k(\tau)d\tau$, is also determined endogenously.

The corporate bonds at each maturity are issued by an infinite number of corporations subject to the same level of systemic default risk. Any corporation-specific risk is assumed to be entirely hedgeable by investors. The systemic hazard rate $\tilde{\lambda}_t$ is given exogenously and is assumed to follow a simple Ornstein-Uhlenbeck process:

$$d\tilde{\lambda}_t = \alpha \left(\mu - \tilde{\lambda}_t\right) dt + \sigma dB_t \tag{3.1}$$

Prices of the corporate bonds in equilibrium are an affine function of the systemic hazard rate $\tilde{\lambda}_t$:

$$P_{t,\tau} = e^{-\tilde{\lambda}_t A(\tau) - C(\tau)}$$

where $A(\tau)$ and $C(\tau)$ are quantities determined in equilibrium.

[4]The bond price is a function of time t and final maturity date$T$. It is denoted as $P_{t,\tau}$, $\tau \equiv T - t$, for consistency with the notation in Vayanos and Vila (2009)

Agents are endowed with preferences over the bond portfolio that trade off mean and variance of wealth. In the two versions of the model the tradeoff occurs at an infinitesimal (Agent 1) and Λ (Agent 2) time horizon respectively. Agent 2 is a buy and hold investor, so once invested he cannot change the composition of his portfolio. In order to focus on credit specific dynamics, we abstract completely from issues relating to the risk free rates on government bonds and assume the existence of an infinitely deep government bond term-structure where bonds are issued and redeemed at par. Thus Agents have a constant time-preference of wealth, and are indifferent between a dollar earned today or in the future[5].

The optimization problem of agent 1, is:

$$\max_{\{x_{t,\tau}\}_{\tau\in(0,\Lambda)}} E_t\left[dW_t\right] - \frac{\gamma}{2}V_t\left[dW_t\right] \tag{3.2}$$

where $x_{t,\tau}$ is the dollar amount of the portfolio assigned to the τ maturity bond, γ is the risk aversion coefficient, and $E_t\left[\right]$and $V_t\left[\right]$ are the expectation and variance functions respectively. The instantaneous change in wealth is given by:

$$dW_t = \int_0^\Lambda x_{t,\tau}\frac{dP_{t,\tau}}{P_{t,\tau}}d\tau \tag{3.3}$$

where it is understood that $dP_{t,\tau}$ is the price increment with respect to time t. The optimization problem of agent 2, is:

$$\max_{\{x_{t,\tau}\}_{\tau\in(0,\Lambda)}} E_t\left[W_{t+\Lambda} - W_t\right] - \frac{\gamma}{2}V_t\left[W_{t+\Lambda} - W_t\right] \tag{3.4}$$

The expression for the long term change in wealth can be rewritten as:

$$W_{t+\Lambda} - W_t = \int_0^\Lambda x_{t,\tau}L_{t,\tau}d\tau \tag{3.5}$$

where $L_{t,\tau}$ is defined as the return of an investment in τ maturity bonds at time t over the Λ time horizon. Demand for credit financing from corporates at maturity τ, $k_{t,\tau}$ is a

[5]These features can be added to the model but have been omitted to keep the formulae as simple as possible

decreasing function of the yield of the bond:

$$k_{t,\tau} = \rho\tau(R_{t,\tau} - \beta)d\tau \tag{3.6}$$

where $R_{t,\tau}$ is the yield on the τ maturity bond at time t and $\rho < 0$ for all τ. Note ρ has the opposite sign to the demand for investment in the model Vayanos and Vila (2009). The infinitesimal return of the τ maturity bonds are similar to Vayanos and Vila (2009) and is given by:

$$\frac{dP_{t,\tau}}{P_{t,\tau}} = A'(\tau)\tilde{\lambda}_t dt + C'(\tau)dt - A(\tau)\alpha\left(\mu - \tilde{\lambda}_t\right)dt + \frac{1}{2}A(\tau)^2\sigma^2 dt - A(\tau)\sigma_r dB_t - \tilde{\lambda}_t dt \tag{3.7}$$

The return of an investment in τ maturity bonds at time t over the Λ time horizon, $L_{t,\tau}$ to first order is given by:

$$\begin{aligned} L_{t,\tau} &\backsimeq A(\tau)\tilde{\lambda}_t + C(\tau) - \int_t^{t+\tau} \tilde{\lambda}_s ds \\ &= \int_0^\tau \left(\left(A(0) + A'(s)\right)\tilde{\lambda}_t + \left(C(0) + C'(s)\right) - \tilde{\lambda}_{t+s} \right) ds \end{aligned} \tag{3.8}$$

Substituting (3.7) into (3.3) and deriving (3.2) with respect to $x_{t,\tau}$ yields the short-term investor first order conditions:

$$A'(\tau)\tilde{\lambda}_t + C'(\tau) - A(\tau)\alpha\left(\mu - \tilde{\lambda}_t\right) + \frac{1}{2}A(\tau)^2\sigma^2 - \tilde{\lambda}_t = \gamma\sigma^2 A(\tau)\int_0^\Lambda x_{t,s}A(s)ds \tag{3.9}$$

(3.9) is identical to the first-order condition in Vayanos and Vila (2009), except that the short term rate r_t is replaced by the hazard rate $\tilde{\lambda}_t$. In order to arrive at the first order conditions for the long-tem investor we first substitute the approximate expression (3.8) for $L_{t,\tau}$in (3.5):

$$W_{t+\Lambda} - W_t = \int_0^\Lambda x_{t,\tau} \int_0^\tau \left(\left(A(0) + A'(s)\right)\tilde{\lambda}_t + \left(C(0) + C'(s)\right) - \tilde{\lambda}_{t+s} \right) ds d\tau \tag{3.10}$$

By changing the order of integration (3.10) can be rewritten as:

$$W_{t+\Lambda} - W_t = \int_0^\Lambda y_{t,s} \left(\left(A(0) + A'(s)\right)\tilde{\lambda}_t + \left(C(0) + C'(s)\right) - \tilde{\lambda}_{t+s} \right) ds \tag{3.11}$$

where $y_{t,s} = \int_s^\Lambda x_{t,\tau} d\tau$ is the cumulative sum of bond demand from s to the last point on the term structure Λ. Using the expectation and covariance function for an Ornstein-Uhlenbeck process we note that:

$$E_t\left[\int_0^\Lambda y_{t,s}\tilde{\lambda}_{t+s}ds\right] = \int_0^\Lambda y_{t,s}\left(\tilde{\lambda}_t e^{-\alpha s} + \mu(1-e^{-\alpha s})\right)ds \tag{3.12}$$

and

$$V_t\left[\int_0^\Lambda y_{t,s}\tilde{\lambda}_{t+s}ds\right] = \int_0^\Lambda \int_0^\Lambda y_{t,s}y_{t.u}\frac{\sigma^2}{2\alpha}\left(e^{-\alpha|u-s|} - e^{-\alpha(u+s)}\right)dsdu \tag{3.13}$$

(3.4), (3.5), (3.11), (3.12) and (3.13) and the conditions $A(0) = C(0)=0$ together give the following first order conditions with respect to $y_{t,\tau}$ for the agent with a long-term investment horizon:

$$(A'(\tau) - e^{-\alpha\tau})\tilde{\lambda}_t + C'(\tau) - \alpha\mu D(\tau) = \frac{\gamma\sigma^2}{2\alpha}\int_0^\Lambda y_{t,s}\left(e^{-\alpha|\tau-s|} - e^{-\alpha(\tau+s)}\right)ds \tag{3.14}$$

where I have defined $D(\tau) = \frac{1-e^{-\alpha\tau}}{\alpha}$. If we denote the left hand side of (3.14) as $f(\tau)$ and differentiate twice with respect to τ we obtain:

$$f^{(2)}(\tau) = \frac{\alpha\gamma\sigma^2}{2}\left(\int_0^\Lambda y_{t,s}\left(e^{-\alpha|\tau-s|} - e^{-\alpha(\tau+s)}\right)ds\right) - 2\gamma\sigma^2 y_{t,\tau} \tag{3.15}$$

Substituting the right hand side of (3.14) into (3.15) and rearranging yields:

$$y_{t,\tau} = \left(\frac{f^{(2)}(\tau) - \alpha^2 f(\tau)}{-\sigma^2\gamma}\right) \tag{3.16}$$

For markets to clear $k_{t,\tau} = x_{t,\tau}$ must hold. The market clearing condition together with (3.16) and the definition of $y_{t,\tau}$ gives:

$$\left(\frac{f^{(2)}(\tau) - \alpha^2 f(\tau)}{-\sigma^2\gamma}\right) = \int_\tau^\Lambda k_{t,s}ds \tag{3.17}$$

Substituting the demand function (3.6) into (3.17) and matching terms in $\tilde{\lambda}_t$ gives:

$$\frac{A^{(3)}(\tau) - \alpha^2 A'(\tau)}{-\sigma^2\gamma} = \int_\tau^\Lambda \rho A(s)ds \tag{3.18}$$

$$\frac{C^{(3)}(\tau) - \alpha^2 C'(\tau)}{-\sigma^2\gamma} = \int_\tau^\Lambda \rho(C(s) - \beta)ds \tag{3.19}$$

Both (3.18) and (3.19) when differentiated again are fourth order ordinary differential equations:

$$\frac{A^{(4)}(\tau) - \alpha^2 A^{(2)}(\tau)}{-\sigma^2\gamma} = -\rho A(\tau) \tag{3.20}$$

$$\frac{C^{(4)}(\tau) - \alpha^2 C^{(2)}(\tau)}{-\sigma^2\gamma} = -\rho(C(\tau) - \beta\tau) \tag{3.21}$$

Fourth order partial differential equations of the type in (3.20) and (3.21) have closed form general analytical solutions of the form given in appendix 3.5.1. The boundary conditions for equation (3.20) are given by:

$$A(0) = 0 \tag{3.22}$$

$$A'(0) = 1 \tag{3.23}$$

$$A^{(2)}(\Lambda) = -\alpha A'(\Lambda) \tag{3.24}$$

$$A^{(3)}(\Lambda) = \alpha^2 A'(\Lambda) \tag{3.25}$$

(3.22) and (3.23) are obtained by letting $\tau \to 0$ in the F.O.C (3.14). (3.25) is obtained by setting $\tau = \Lambda$ in (3.18). Finally (3.24) is found by setting $\tau = \Lambda$ in (3.14) and integrating by parts. Similarly, the boundary conditions for equation (3.21) are given by:

$$C(0) = 0 \tag{3.26}$$

$$C'(0) = 0 \tag{3.27}$$

$$C^{(2)}(\Lambda) = -\alpha(C'(\Lambda) - \mu) \tag{3.28}$$

$$C^{(3)}(\Lambda) = \alpha^2(C'(\Lambda) - \mu) \tag{3.29}$$

The general solutions (3.30) and (3.31) together with the boundary conditions (3.22)-(3.29) define the solutions to $A(\tau)$ and $C(\tau)$ in equilibrium when agents are long-term investors. The equilibrium solutions when agents are short-term investors are similar to those in Vayanos and Vila (2009) and are given in the appendix for convenience.

3.3 Results

3.3.1 Credit term structure sensitivity to changes in the spot hazard rate is lower in the economy with long-term investors

When the spot hazard rate increases, demand for credit financing falls as the overall yield increases. In order to reduce the investment demand so that markets can clear, it must be that the expected return for investors with long positions is negative relative to the position prior to the increase in yield. Conversely, when the spot hazard rate decreases demand for credit financing increases as the overall yield falls. For markets to clear, the expected return for investors must increase relative to the original position. In order to generate these dynamics the implied mean reversion in equilibrium must be greater than the mean reversion if prices were to accord with the expectations hypothesis. When the hazard rate is low the high level of reversion to a larger hazard rate implied in the pricing results in, ceteris parabis, a lower relative price and higher expected return. On the other hand when the hazard rate is high the greater reversion to a lower hazard rate implies that prices are relatively higher and one expects a lower relative return.

The result for short-term investors is comparable to the findings in Vayanos and Vila (2009). In their model the maturity clienteles are on the investor side. As yields rise maturity clienteles demand more bonds in equilibrium. Since markets must clear arbitrageurs must be enticed to demand less bonds in equilibrium, and similarly the mechanism through which this occurs is through a faster level of risk-neutral mean reversion. Instead in the model presented here the maturity clienteles are on the bond issuer side. Higher yields lead to less issuance so the short-term investor, similar to the arbitrageur, demands less in equilibrium.

Figure 3.5 plots the functional form of $A(\tau)$ in both the case of a short-term investor market and a long-term investor market, as well as when the expectations hypothesis holds. The figure indicates the mean reversion in the long-term investor economy is faster than the mean reversion in the short-term investor economy, and confirms that both are faster than the level of mean reversion in an economy where the expectations hypothesis holds (denoted 'real world'). Long-term investors are more inert as they are sensitive to the cumulative (weighted) sum of hazard rates over the time horizon. Since they are more inert, long-term investors require a greater incentive to move from their existing positions when the hazard rates increase. Panel A in figure 3.5 corresponds to the case when credit demand sensitivity to yields is lower ($\rho = -1$), and panel B illustrates the case when credit demand sensitivity to yields is higher ($\rho = -3$).

3.3.2 Disparity in credit term structure sensitivity is increasing in credit demand sensitivity, spot hazard rate volatility and risk-aversion

One can see clearly from the results in figure 3.5 that the disparity between the form of $A(\tau)$ increases with the level of credit demand sensitivity ($-\rho$). The result is intuitive. If credit demand on the issuer side is more sensitive to changes in yields, the greater 'inertia' of the long-term investor will be amplified and so will the disparity in values of $A(\tau)$. One implication of the result in 3.3.1 is that the volatility of the short term price return on a bond of a particular maturity is larger in an economy with short-term investors than long-term investors[6]. We can see this by observing that the volatility of returns of a bond of maturity τ is $\sigma^2 A(\tau)^2$. It follows from the relationship between $-\rho$ and $A(\tau)$ that the discrepancy in equilibrium volatility between a short-term investor market and a long-term investor market will increase in credit demand sensitivity.

Differences between long-term investor equilibrium values of $A(\tau)$ and the values of $A(\tau)$ under the expectations hypothesis also increase with risk aversion (γ) and hazard rate volatility (σ). We can see the joint effects of (i) risk aversion and credit demand sensitivity, and (ii) risk aversion and hazard rate volatility in panel A and B of figure 3.6 respectively. The dependent variable of the contour plots is the difference at the 10 year

[6]It may seem counterintuitive to refer to short-term price volatility in an economy with long-term investors. Recall our interpretation of the long-term investment economy is an economy consisting of overlapping generations of long-term investors. As each new generation of long-term investors optimizes over a long-term investment horizon prices will change.

maturity point. As risk aversion increases the agent must be compensated with a greater expected return for a given reduction in yield in order to entice him to deviate from the existing position and soak up the extra demand for credit financing. Similar reasoning holds for the opposite case. Hazard rate volatility has the same directional effect as risk aversion. Since agents are risk averse, an increase in volatility of the bond position will require a greater change in expected returns for agents to increase or reduce their holdings so that markets will clear, for a given yield induced change in demand for credit financing.

3.3.3 The cost of long term credit provision is less in the economy with long-term investors

In the economy with long-term investors the cost of long-term financing is generally lower than in the economy with short-term investors. The result is made clear in figure 3.7, which plots the term-structure of yield in (i) the economy with long-term investors, (ii) the economy with short-term investors, and (iii) an economy where the expectations hypothesis holds. Panels A and B correspond to the cases when absolute credit financing demand is larger ($\beta = 0.3$) and smaller ($\beta = 0.15$) respectively. The results suggest the difference in the cost of long-term financing between the short-term investor and long-term investor market is greater when the absolute demand for financing is greater.

Panel A of figure 3.8 shows how the disparity between bond yields in the two economies varies jointly with credit demand, β, and the spot hazard rate, $\tilde{\lambda}_t$. Although the disparity does increase slightly with the spot hazard rate, the figure shows clearly that the influence of the parameterβ is far greater. The figure also suggests the effect of the parameter β is independent of $\tilde{\lambda}_t$. The hazard rate affects the yield disparity through $A(\tau)$, which is lower in the long-term investor economy than the short-term investor economy. The effect of a disparity in $A(\tau)$ on yields is greater when the hazard rate is large, and this is why we see the hazard rate is positively related with bond yield differences between the two economies. Instead the influence of β is not reflected at all in $A(\tau)$, rather its effect is wholly reflected in the parameter $C(\tau)$.

The effect of credit volatility on the difference in yield between the two economies is non-linear. Panel B of figure 3.8 shows that the difference in yields increases with credit volatility, σ, and then decreases. It seems the difference between the long-term and short-term investor's metric is accentuated with larger credit volatility until a point after which the negative effect (on the difference between the two metrics) of a decrease in equilibrium demand outweighs the positive effect of the increase in credit volatility.

These subtle effects should not obscure the main message from this subsection. The long-term investor equilibrium yield is lower than the short-term investor yield and results indicate this difference increases with the absolute demand for credit financing β. The policy implication of this result is that countries in which where there is a large demand for long-term financing can benefit most from creating incentives for investors to optimize over a multi-period horizon. Assuming the long-term projects creating the demand for financing are implicitly or explicitly financed by the government or a parastatal institution, the burden of not facilitating a long-term investment philosophy will fall ultimately on the tax-payer.

3.3.4 The amount of bond financing is greater in the economy with long-term investors

We have established the long-term equilibrium yields of the bonds are normally lower in the economy with long-term investors. Since the demand for credit financing is assumed to be decreasing in yield it follows that the long-term equilibrium supply of credit financing is usually greater in the economy with long-term investors. Panel A of figure 3.9 shows the difference in equilibrium bond supply with two different values of credit volatility, $\sigma = 0.01$ and $\sigma = 0.02$. For the less volatile case the equilibrium bond supply in the long-term investor economy is greater or approximately equal to the bond supply in the short-term investor economy at all maturity points. The discrepancy in bond supply seems to increase with maturity at the longer maturities. In the more volatile case the equilibrium bond supply at short-maturities is actually lower for long-term investors but increases significantly (relative to short-term investors) at longer maturities. Panel B of

3.9 shows the difference in bond supply corresponding to the high volatility case in Panel A, with two different values of risk aversion ($\gamma = 5$,$\gamma = 8$). For the higher risk aversion case with long-term investors, it seems less (relative) financing is provided in equilibrium at shorter maturities in order to provide the greater financing at longer maturities.

Although we observe a difference in equilibrium bond supply between the two markets, we might have expected a greater disparity. Greater disparities are generated if we assume that the agents calibrate their models differently. For example, a short-term investor will calibrate to past credit-spread volatility as he is primarily exposed to the changes in spreads of the bonds. Instead the long-term investor would calibrate to realised default volatilities. But actual defaults have historically been found to account for only a small fraction of credit spreads (see for example Huang and Huang (2012)). One obvious explanation for this puzzle offered by Chen (2010) is that the marginal utilities of investors is greater during recessions. In our model this would imply a time-varying risk aversion parameter γ, a feature from which we have abstracted. The short-term investor would account for agents increase in risk aversion by increasing the parameter σ. In Panel A of figure 3.10 we can see the effect on equilibrium bond supply if the short-term agent increases σ in his first order conditions from 0.01 to 0.02, whilst the long-term investor sets σ at 0.01. The difference in equilibrium bond supply in this case is around 30%.

We can also capture the effect on the term-structure of the recent regulatory push for financial institutions to set aside a larger amount of capital against the short-term mark-to-market losses on their credit portfolio. Panel B of figure 3.10 shows the disparity in equilibrium bond supply when the short-term agent has a higher coefficient of risk aversion ($\gamma_s = 8$) than the long-term agent ($\gamma_l = 5$), and compares this to the case when the coefficient of risk aversion for both agents is equal ($\gamma_s = \gamma_l = 5$). We can see clearly the disparity in bond-supply is far greater at all maturities when the short-term investor is more risk averse, rising to 20% at the 10 year point. The results indicate that regulation forcing financial institutions to assign a significant amount of capital against short-term price volatility can contribute to a problem of credit supply at longer maturities.

3.4 Concluding remarks

I derive an analytical solution to a model similar to the preferred-habitat model of Vayanos and Vila (2009), but applied to credit financing and with long-term investors. The long-term investors are exposed to the risk of realised defaults over the longer time horizon, rather than short-term mark to market changes in the prices of bonds. I compare the equilibrium term structure of credit in an economy with the long-term investors to results if all investors have a short-term investment horizon. I find the sensitivity to the hazard rate is lower in the economy with long-term investors, and the difference to the sensitivity of implied by the expectations hypothesis is increasing in risk aversion, hazard rate volatility and credit demand sensitivity to yields. I find that yields in the economy with long-term investors are generally lower, and the difference is positively affected by the absolute demand for credit financing. I also find that the hazard rate volatility increases the difference in yields up to a point. Since demand for financing is negatively related to yields I find equilililibrium supply of long-maturity bonds to be greater in the economy with long-term investors. If short-term investors calibrate to a higher level of credit volatility to reflect the difference between credit spread volatility and actual default volatility, the disparity in equilibrium supply can be large. Moreover the disparity is accentuated if the short-term investor is more risk averse than the long-term investor. The results indicate that the regulatory push to assign significant capital buffers against short-term mark-to-market changes in credit spreads may have a detrimental effect on long-term credit provision.

There are three obvious areas where the paper can be extended. Firstly, time-varying risk aversion can be endogenised in the model, which would likely lead to greater disparity in equilibrium bond supply when both investor types use similar credit volatility and risk aversion values in their first order conditions. Time-varying risk aversion would affect the short-term investor through the risk-aversion parameter in the first-order condition as well as increase the price volatility of the corporate bond. Instead the long-term investor would only be afflicted by the first effect because the distribution of actual defaults over time is not influenced by the change in investor risk-aversion. Secondly, the results of the extended model with a more general credit demand function needs to be explored to understand how spikes in credit demand at individual maturities propagate through the credit curve in both economies. Thirdly, the theoretical conclusions of the model should be taken to the data and compared across rating structures and jurisdictions. In particular it would be worthwhile to pinpoint jurisdictions where long-term investment is incentivised and understand how it has influenced the evolution of long-term credit spreads.

3.5 Appendix

3.5.1 General solutions to $A(\tau)$ and $C(\tau)$ for long-term investors

The general solution to the fourth order ordinary differential equation:

$$\frac{A^{(4)}(\tau)-\alpha^2 A^{(2)}(\tau)}{-\sigma^2\gamma}=-\rho A(\tau)$$

has an analytical solution, see for example Zaitsev and Polyanin (2012). The solution is given by:

$$A(\tau)=c_1 e^{\frac{\tau\sqrt{\alpha^2-\sqrt{\alpha^4-4\nu}}}{\sqrt{2}}}+c_2 e^{-\frac{\tau\sqrt{\alpha^2-\sqrt{\alpha^4-4\nu}}}{\sqrt{2}}}+c_3 e^{\tau\sqrt{\frac{1}{2}\sqrt{\alpha^4-4\nu}+\frac{\alpha^2}{2}}}+c_4 e^{\tau\left(-\sqrt{\frac{1}{2}\sqrt{\alpha^4-4\nu}+\frac{\alpha^2}{2}}\right)} \quad (3.30)$$

where $\nu=-\sigma^2\rho\gamma$ and the coefficientsc_i $i\in\{1,2,3,4\}$ are determined by the boundary conditions. The general solution to the fourth order ordinary differential equation:

$$\frac{C^{(4)}(\tau)-\alpha^2 C^{(2)}(\tau)}{-\sigma^2\gamma}=-\rho(C(\tau)-\beta)$$

is given by:

$$C(\tau)=d_1 e^{\frac{\tau\sqrt{\alpha^2-\sqrt{\alpha^4-4\nu}}}{\sqrt{2}}}+d_2 e^{-\frac{\tau\sqrt{\alpha^2-\sqrt{\alpha^4-4\nu}}}{\sqrt{2}}}+d_3 e^{\frac{\tau\sqrt{\sqrt{\alpha^4-4\nu}+\alpha^2}}{\sqrt{2}}}+d_4 e^{-\frac{\tau\sqrt{\sqrt{\alpha^4-4\nu}+\alpha^2}}{\sqrt{2}}}+\Phi(\tau) \quad (3.31)$$

where:

$$\Phi(\tau)=\frac{\alpha^3(\mu-1)e^{-\alpha\tau}}{\nu}+\beta\tau$$

and the coefficients and d_i $i\in\{1,2,3,4\}$ are again determined by the boundary conditions. Although some of the exponential terms in (3.30) and (3.31) may be complex-valued, the complex terms fall away when multiplied by the coefficients c_i and d_i provided the variables in the boundary conditions are real-valued. The solution can then be computed simply using a symbolic software package such as Mathematica.

3.5.2 General solutions to $A(\tau)$ and $C(\tau)$ for short term investors

Here I report for convenience the equilibrium values of $A(\tau)$ and $C(\tau)$ when agent investment horizon is infinitesimal. The expressions are similar to those found in the 1-factor model of Vayanos and Vila (2009) , although the context is different. In equilibrium the following hold:

$$A(\tau) = \frac{1 - e^{-\hat{\alpha}\tau}}{\hat{\alpha}}$$
$$C(\tau) = \hat{\alpha}\hat{\kappa} \int_0^{\tau} \mathrm{A}(u)\, du - \frac{1}{2}\sigma^2 \int_0^{\tau} \mathrm{A}(u)^2\, du$$

where $\hat{\alpha}$ is the unique solution to:

$$\hat{\alpha} = \gamma\sigma^2 \left(\int_0^{\Lambda} \rho \mathrm{A}(s)^2\, ds \right) + \alpha$$

and

$$\hat{\kappa} = \mu + \frac{\frac{1}{2}\gamma\sigma^4 \left(\int_0^{\Lambda} \rho A(s) \left(\int_0^s A(u)^2\, du \right)\, ds \right) + \gamma\sigma^2(\beta - \mu) \left(\int_0^{\Lambda} s\rho A(s)\, ds \right)}{\hat{\alpha} \left(\gamma\sigma^2 \left(\int_0^{\Lambda} \rho A(s) \left(\int_0^s A(u)\, du \right)\, ds \right) + 1 \right)}$$

3.5.3 Model extension

The linear demand function in term for corporate financing given in (3.6) may be considered restrictive. One possible extension to the demand function for which there is an analytical solution is a function of the form:

$$k_{t,\tau} = \rho\tau(R_{t,\tau} - \beta(\tau))d\tau \tag{3.32}$$

In this case the relation in (3.20) and the boundary conditions (3.22)-(3.25) remain unchanged. (3.21) becomes:

$$\frac{C^{(4)}(\tau) - \alpha^2 C^{(2)}(\tau)}{-\sigma^2\gamma} = -\rho(C(\tau) - \beta(\tau)\tau) \tag{3.33}$$

and the boundary conditions (3.26) - (3.29) remain unchanged. (3.33) together with (3.26)

- (3.29) yields the solution:

$$C(\tau) = \sum_{j=1}^{2} \sum_{k=1}^{2} e^{c_{j,k}\tau} d_{2(j-1)+k} + \Psi_{j,k} \tag{3.34}$$

where:

$$c_{j,k} = \frac{(-1)^j \sqrt{\alpha^2 + (-1)^k \sqrt{\alpha^4 - 4\nu}}\tau}{\sqrt{2}}$$

$$\Psi_{j,k} = -e^{c_{j,k}\tau} \int_1^\tau e^{-c_{j,k}s} c_{j,k} \epsilon_{j,k} \mathrm{s} \beta(s) ds$$

$$\epsilon_{j,k} = \left(1 - (-1)^k \frac{\alpha^2}{\sqrt{(\alpha^4 - 4\nu)}}\right)$$

and d_i, $i \in \{1, 2, 3, 4\}$ are determined by the boundary conditions.

3.6 Figures and Tables

Figure 3.1: Credit spreads by rating class
A plot of the time series of option-adjusted spreads (OAS) on the Bank of America Merrill Lynch US corporate indices stratified by rating class over the period 1996 to 2013.

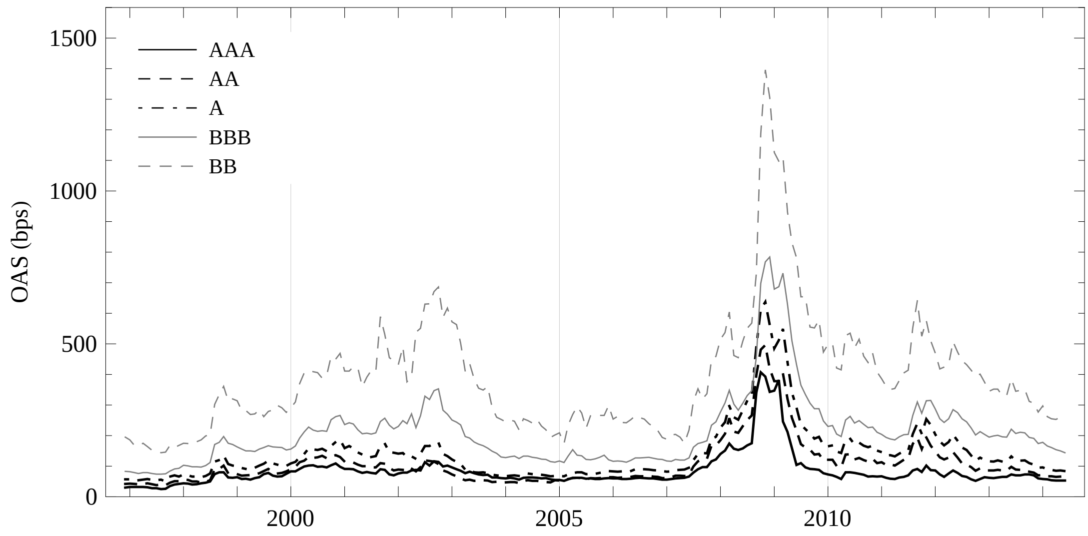

Figure 3.2: Credit spreads by maturity
Plot of the option-adjusted spreads (OAS) on the Bank of America Merrill Lynch US corporate indices stratified by maturity over the period 1996 to 2013.

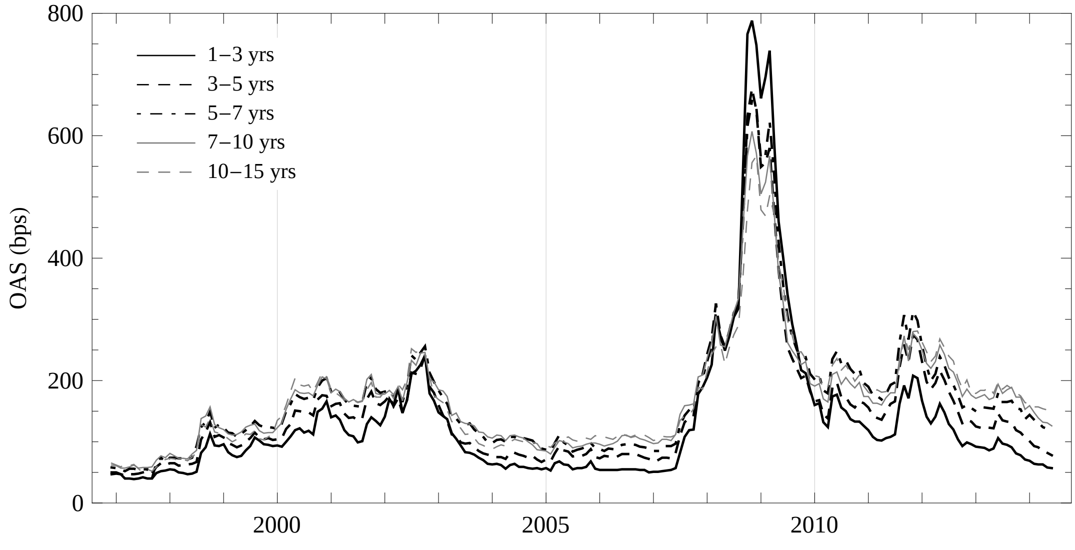

Figure 3.3: Credit spreads - term structure differences
Plot of the difference between option-adjusted spreads on the Bank of America Merrill Lynch US corporate indices stratified by maturity and the 1-3 yr index over the period 1996 to 2013.

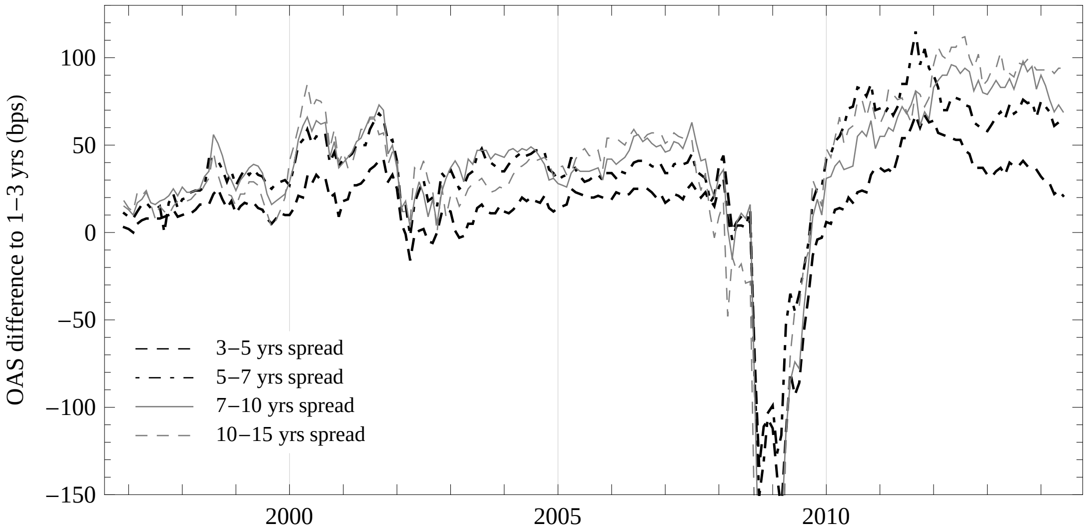

Figure 3.4: Credit spreads - rating differences
Plot of the difference between option-adjusted spreads on the Bank of America Merrill Lynch US corporate indices stratified by rating and the AAA index over the period 1996 to 2013.

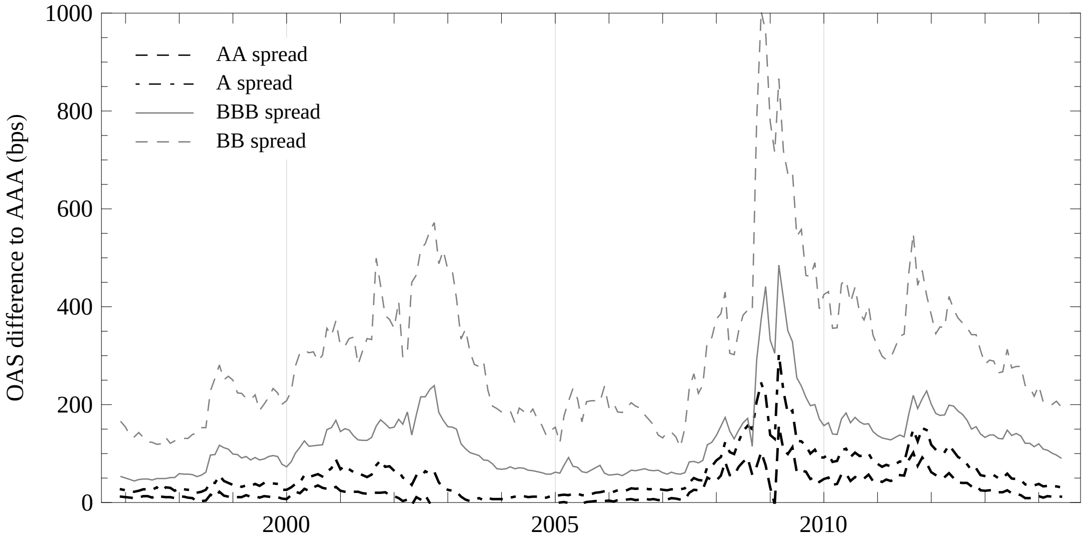

Figure 3.5: Equilibrium term structure of credit
Plot of $A(\tau)$ under the real world probability measure, in equilibrium with short-term agents (ST) and in equilibrium with long-term agents (LT). Panel B corresponds to the case where credit demand sensitivity is higher ($\rho = 3$). Credit sensitivity in Panel A is set at $\rho = 1$.

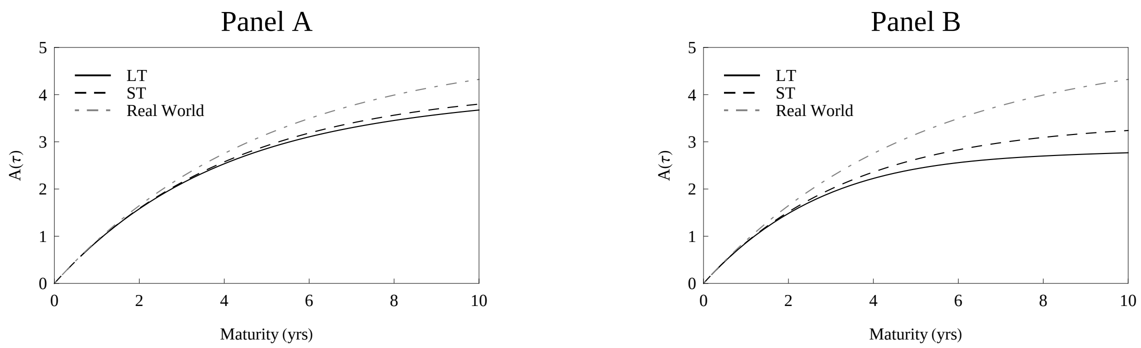

Figure 3.6: Difference between equilibrium long term and risk-neutral default sensitivities
The dependent variable in the contour plots below is the difference between the 10 year sensitivity to the hazard rate, $A(510)$, under (i) the expectations hypothesis and (ii) in the equilibrium with long-term investors. Panel A shows how the dependent variable varies with credit demand sensitivity (ρ) and risk aversion (γ). Panel B shows how the dependent variable varies with the volatility of the hazard rate (σ) and risk aversion (γ). Darker shading indicates larger differences.

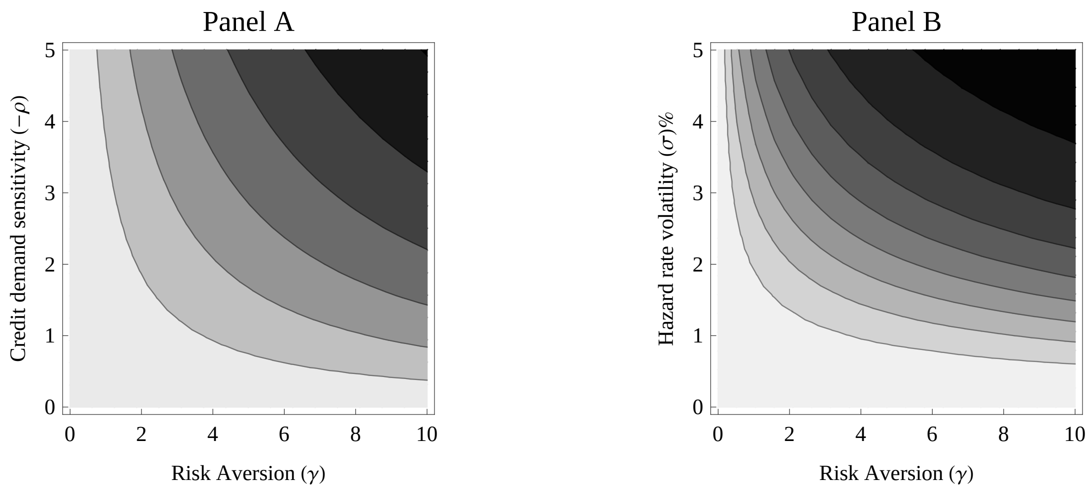

Figure 3.7: Equilibrium yields with varying credit financing demand β
The figure below shows the equilibrium yield term-structure in a market where there are (i) long-term (LT) investors and (ii) short-term (ST) investors; and compares these yields to (iii) the term-structure of yields if the expectations hypothesis were to hold (Real World). Panel A and B correspond to the case where there is a greater and lesser level of credit financing demand respectively ($\beta = 0.3$, $\beta = 0.15$).

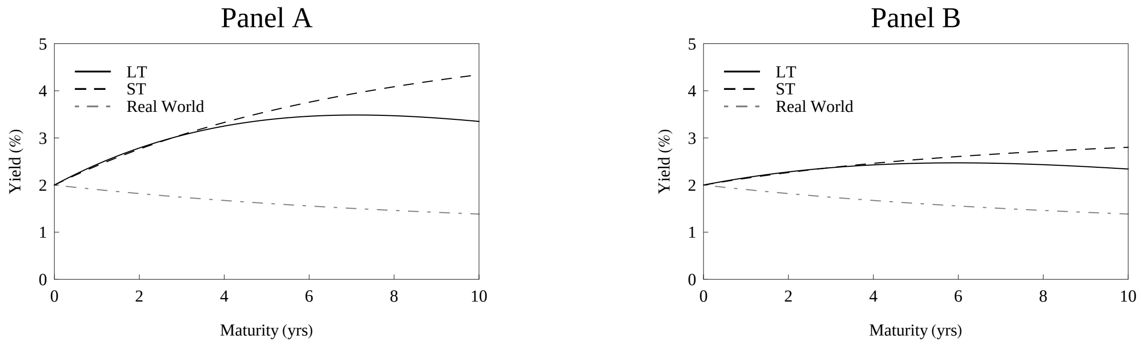

Figure 3.8: Equilibrium yields with short term and long term investors
The dependent variable in the contour plots below is the difference between the 10 year bond yield with short-term investors and the 10 year bond yield with long long-term investors. Panel A shows how the discrepancy in 10 year bond yields varies as a function of the spot hazard rate, λ_t, and credit demand, β. Panel B shows how the difference varies as a function of the volatility of credit, σ, and credit demand, β . Larger values are illustrated with darker shading.

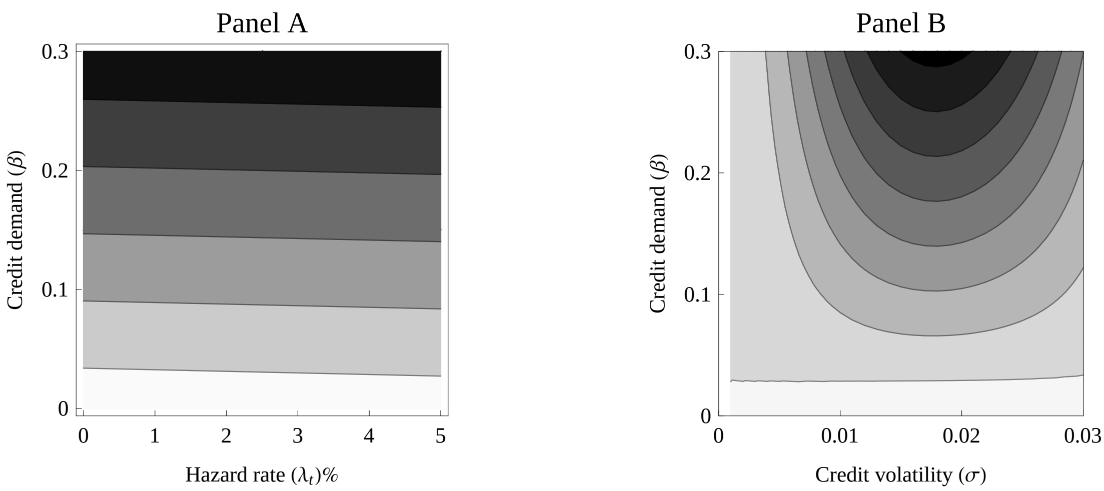

Figure 3.9: Equilibrium bond supply differences

The figure below shows the difference in the equilibrium term-structure of bond supply between the market with long-term investors and short-term investors. The difference in bond supply is calculated as a percentage of total equilibrium bond supply in the market with short-term investors. Panel A and B plot the difference in equilibrium supply for different values of credit volatility ($\sigma = 0.01$, $\sigma = 0.02$) and risk aversion ($\gamma = 5$, $\gamma = 8$) respectively.

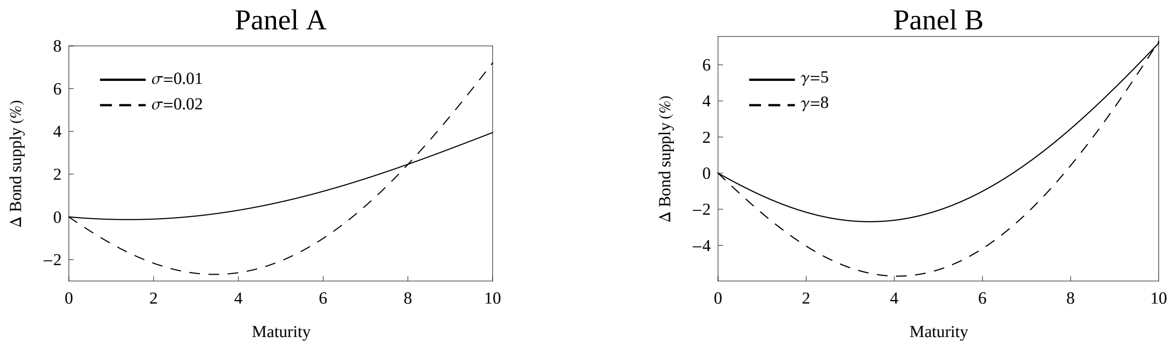

Figure 3.10: Equilibrium bond supply differences with different agent parameters

The figure below shows the difference in equilibrium bond supply when the agent volatility parameters are estimated differently (Panel A), or when agent's coefficient of risk aversion is different (Panel B). Parameter subscripts in the legend refer to the parameter pertaining to short-term investors (s) and long-term investors (l).

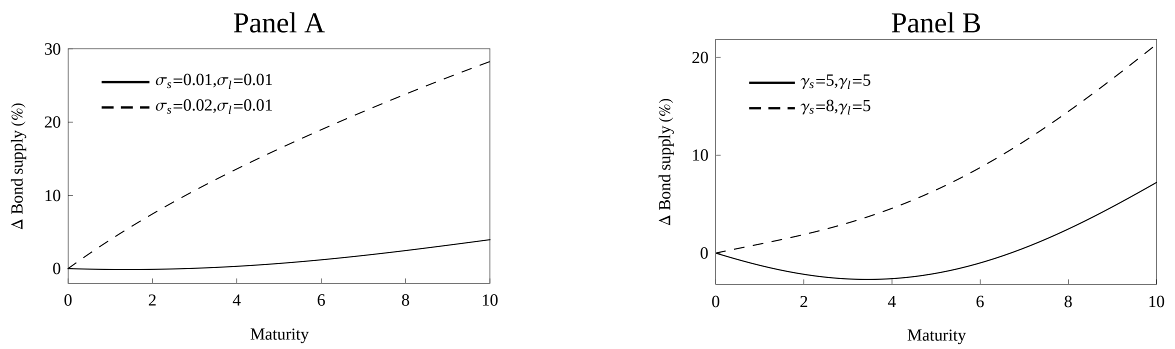

Table 3.1: Figure parameters

The table below documents the values of the parameters used to produce some of the figures in the paper. "-" indicates that the parameter is not appicable and "~" indicates that the parameter is varied within the figure.

$Figure$	ρ	α	σ	Λ	μ	γ	β	$\tilde{\lambda}_t$
$3.5\,(A)$	-1	0.2	0.01	10	$-$	5	$-$	$-$
$3.5(B)$	-3	0.2	0.01	10	$-$	5	$-$	$-$
$3.6(A)$	$\sim$	0.2	0.01	10	$-$	$\sim$	$-$	$-$
$3.6(B)$	-1	0.2	$\sim$	10	$-$	$\sim$	$-$	$-$
$3.7(A)$	-0.5	0.2	0.01	10	0.01	5	0.3	0.02
$3.7(B)$	-0.5	0.2	0.01	10	0.01	5	0.15	0.02
$3.8(A)$	-0.5	0.2	0.01	10	0.02	5	$\sim$	$\sim$
$3.8(B)$	-0.5	0.2	$\sim$	10	0.02	5	$\sim$	0.04
$3.9(A)$	-0.5	0.2	$\sim$	10	0.02	5	0.1	0.04
$3.9(B)$	-0.5	0.2	0.02	10	0.02	$\sim$	0.1	0.04
$3.10(A)$	-0.5	0.2	$\sim$	10	0.02	5	0.1	0.04
$3.10(B)$	-0.5	0.2	0.02	10	0.02	$\sim$	0.1	0.04

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