Editors' Suggestion

Instantons in ϕ^4 theories: Transseries, virial theorems, and numerical aspects

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We discuss numerical aspects of instantons in two- and three-dimensional ϕ^4 theories with an internal $O(N)$ symmetry group, the so-called N-vector model. By combining asymptotic transseries expansions for large arguments with convergence acceleration techniques, we obtain high-precision values for certain integrals of the instanton that naturally occur in loop corrections around instanton configurations. Knowledge of these numerical properties is necessary in order to evaluate corrections to the large-order factorial growth of perturbation theory in ϕ^4 theories. The results contribute to the understanding of the mathematical structures underlying the instanton configurations.

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I. ORIENTATION

The N-vector model (the self-interacting ϕ^4 field theory in $D = 2$ and $D = 3$ dimensions) gives rise to instanton configurations, whose structure is more complicated than the corresponding configurations in quantum mechanics (in one space dimension), which is equivalent to a $D = 1$-dimensional field theory (see Figs. 2 and 3 of Ref. [1]). The instantons provide nontrivial saddle points of the Euclidean action, about which we expand partition functions, and generating functions [2–6]. Instantons also constitute fundamental objects in statistical and optimization problems possessing hard phases (see Refs. [7–9]). Here, we derive a semianalytic representation which can be used to describe the instanton uniformly over the radial variable, to a relative accuracy of 10^{-22} or better.

In one dimension (1D), one canonically identifies the argument of the instanton as the Euclidean "time" t, with the notion that $-\infty < t < \infty$ (see Ref. [1]). In 2D and 3D, this is not so easy, because the angular symmetry dictates that one should choose a radial variable. The radial variable r, in turn, can only take values in the range $0 < r < \infty$. The connection to the 1D case [4] is found if we consider that in 1D, we can interpret the "radial" variable with the $\mathbb{Z}_2$ symmetry (positive and negative real numbers). The surface area of the zero-dimensional unit sphere embedded in one-dimensional space is $2\pi^{(D=1)/2}/\Gamma((D=1)/2) = 2$; the result confirms the $\mathbb{Z}_2$ symmetry of the (analytically known) instantons in one-dimensional theories [1,4].

In two-dimensional and three-dimensional ϕ^4 theories, the instanton is not known analytically. Here, we aim to demonstrate that the analytic structure of the instanton is linked to the concept of transseries and resurgent expansions (see Refs. [10–18]). Specifically, we derive an asymptotic representation of the instanton, for large argument, in the form of a transseries (resurgent expansion) in the variables $\chi = 1/r$ and $\exp(-1/\chi) = \exp(-r)$, where r is the distance from the origin. The transseries representation for large r is complemented by a power-series representation for small r, which is augmented by Padé approximants and nonlinear sequence transformations to enhance its applicability for intermediate values of the radial variable. The goal is to match the large-r and small-r representations at a suitable intermediate transition value of the radial variable, to obtain a uniform, high-precision representation of the instanton in 2D and 3D.

We organize the paper as follows. Fundamentals of instantons in ϕ^4 theories are discussed in Sec. II. The three-dimensional instanton in a three-dimensional ϕ^4 theory is analyzed in Sec. III. Our analysis of the instanton configuration in a two-dimensional field theory follows in Sec. IV. Virial theorems and the asymptotic behavior of the instanton are discussed in Sec. V. The high-precision evaluation of instanton integrals and of instanton actions is discussed in Sec. VI. Conclusions are drawn in Sec. VII.

II. FUNDAMENTALS OF INSTANTONS IN ϕ^4 THEORIES

A. Instanton equations

For the consideration of the instanton configuration, it is sufficient to consider the D-dimensional scalar theory, with the action

$$S[\phi] = \int \mathrm{d}^D x \left[\frac{1}{2}\left(\vec{\nabla}\phi(\vec{x})\right)^2 + \frac{1}{2}\phi(\vec{x})^2 + \frac{g}{4}\phi(\vec{x})^4\right], \tag{1}$$

where $\vec{x}$ is a D-dimensional vector. Consideration of the variation $\delta S[\phi]$ leads to the defining equation of the instanton,

$$-\vec{\nabla}^2\phi_{\mathrm{cl}}(\vec{x}) + \phi_{\mathrm{cl}}(\vec{x}) + g\phi_{\mathrm{cl}}(\vec{x})^3 = 0. \tag{2}$$

Differentiation with respect to a coordinate leads to the equation of the zero mode $\partial_\mu\phi_{\mathrm{cl}}(\vec{x})$,

$$\left(-\vec{\nabla}^2 + 1 + 3g\phi_{\mathrm{cl}}(\vec{x})^2\right)\partial_\mu\phi_{\mathrm{cl}}(\vec{x}) = 0, \tag{3}$$

where $\mu = 1, \ldots, D$. The zero mode constitutes an eigenstate (with zero eigenvalue) of the longitudinal fluctuation operator given in Eq. (47); the zero mode is included here in the discussion in order to illustrate that the instanton configuration is crucial in the exploration of several fundamental properties of the fluctuation operator of quartic theories [4,5].

In a quartic theory, the instanton solution exists only for negative g, because the tunneling can proceed only through a barrier. Therefore, with the scaling

$$\phi_{\mathrm{cl}}(\vec{x}) = \sqrt{-\frac{1}{g}}\xi_{\mathrm{cl}}(\vec{x}), \tag{4}$$

the equations for the instanton and the zero mode are, respectively,

$$\left(-\vec{\nabla}^2 + 1 - \xi_{\mathrm{cl}}(\vec{x})^2\right)\xi_{\mathrm{cl}}(\vec{x}) = 0, \tag{5}$$

$$\left(-\vec{\nabla}^2 + 1 - 3\xi_{\mathrm{cl}}(\vec{x})^2\right)\partial_\mu\xi_{\mathrm{cl}}(\vec{x}) = 0. \tag{6}$$

The presence of the prefactor $\sqrt{-1/g}$ in Eq. (4) illustrates the fact that the instanton solution exists only for negative values of the coupling parameter g, i.e., in the unstable sector of the theory where the self-interaction term $\frac{g}{4}\phi(\vec{x})^4$ becomes negative [4,5].

In a theory with an internal $O(N)$ symmetry group, one has the following instanton:

$$\underline{\phi}_{\mathrm{cl}}(\vec{x}) = \phi_{\mathrm{cl}}(\vec{x})\underline{u} = \sqrt{-\frac{1}{g}}\xi_{\mathrm{cl}}(\vec{x})\underline{u}, \tag{7}$$

where vectors in the internal space are designated by underlining, and we can choose

$$\underline{u} = \{1, 0, \ldots, 0\}^{\mathrm{T}}. \tag{8}$$

Hence, up to multiplication by a (constant) unit vector $\underline{u}$ in the internal $O(N)$ space of the theory, the instanton configuration, whose radial part is governed by Eq. (2), does not depend on the dimension N of the internal symmetry group.

The instanton equation (2) is invariant under the replacement $\phi_{\mathrm{cl}}(\vec{x}) \to -\phi_{\mathrm{cl}}(\vec{x})$. Hence, there is a sign ambiguity in the choice of the instanton, and the degeneracy under the operation $\xi_{\mathrm{cl}} \to -\xi_{\mathrm{cl}}$ needs to be taken into account when using dispersion relations. Indeed, via dispersion relations, one can establish that the instanton action A, defined via

$$S[\phi_{\mathrm{cl}}] = -A/g, \tag{9}$$

governs the large-order behavior of the perturbative coefficients G_K in the Kth order of the expansion in g of the n-point correlation functions in a D-dimensional $O(N)$ theory [4]. In the notation adopted in Eq. (1.9) of Ref. [4], we have

$$\mathsf{G}_K = \frac{c(N,D)}{\pi}\left(\frac{1}{A}\right)^{(n+N+D-1)/2}\left(-\frac{1}{A}\right)^K \tag{10}$$

$$\times\Gamma\left(K + \frac{n+N+D-1}{2}\right)[1 + \mathcal{O}(1/K)]. \tag{11}$$

Here, $c(N,D)$ is a constant coefficient to be determined separately for each N and D.

B. Instantons and large-order behavior

The connection between instantons and large-order behavior is usually obtained by saddle-point evaluations of contour integrals [19]. In the following, we will mention a less known derivation [20] that has the advantage of being simpler and more intuitive. The basic idea is that Feynman diagrams of the ϕ^4 theory at large orders $K \gg 1$ are essentially random regular graphs with connectivity 4 and size K. For a large number of vertices, it is known that random regular graphs have a locally tree-like structure (with the size of the loops growing as $\log K$). This allows us to write an iterative equation that turns out to be equivalent to the instanton equation (2). The Feynman rules imply that there is a factor $1/(\vec{k}_i^2 + m^2)$ for each line i in the graph and

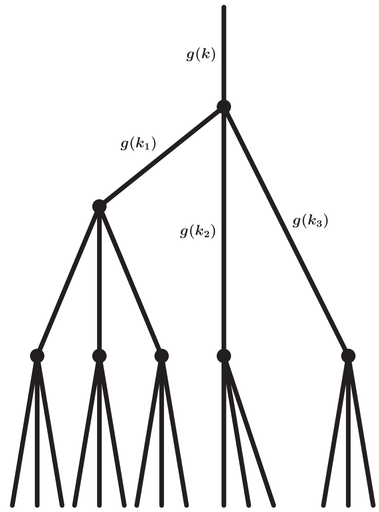

FIG. 1. Tree diagram illustrating the emergence of the instanton.

a Dirac-δ function on each vertex ensuring the conservation of momentum. Invoking the tree-like structure, one can then write the following equation (see also Fig. 1):

$$g(\vec{k}) = \frac{1}{\vec{k}^2 + m^2} \int \frac{\mathrm{d}^D k_1}{(2\pi)^D} \int \frac{\mathrm{d}^D k_2}{(2\pi)^D} \int \frac{\mathrm{d}^D k_3}{(2\pi)^D} \times g(\vec{k}_1) g(\vec{k}_2) g(\vec{k}_3) \delta^{(D)}(\vec{k}_1 + \vec{k}_2 + \vec{k}_3 - \vec{k}). \quad (12)$$

So, multiplying everything by $\vec{k}^2 + m^2$, the previous equation in real space is equivalent to

$$(-\vec{\nabla}^2 + m^2) g(\vec{x}) = g^3(\vec{x}), \quad (13)$$

which is the instanton equation for the ϕ^4 theory for $g(\vec{x}) = \xi_{\rm cl}(\vec{x})$.

Using a standard procedure (similar in spirit to the cavity method from spin-glass theory [21]), one can identify the action in Eq. (1) as the Bethe free energy of the problem. This allows us to derive in a simple way the instanton equation and the action; the $1/K$ correction about the instanton [4,5] corresponds to the $1/K$ finite size correction to the Bethe lattice random graphs.

III. THREE-DIMENSIONAL INSTANTON

A. Large argument

We use the fact that $\xi_{\rm cl}(\vec{x}) = \xi_{\rm cl}(|\vec{x}|) = \xi_{\rm cl}(r)$ is radially symmetric and this constitutes am "S state" in the formalism adopted in atomic physics [22,23]. The equation fulfilled by the instanton $\xi_{\rm cl}(r) = \xi_{\rm cl}^{(3)}(r)$ (including the dimension D in the superscript) in three dimensions is [see Eq. (5)],

$$-\frac{\partial^2}{\partial r^2} \xi_{\rm cl}^{(3)}(r) - \frac{2}{r}\frac{\partial}{\partial r} \xi_{\rm cl}^{(3)}(r) + \xi_{\rm cl}^{(3)}(r) - \left[\xi_{\rm cl}^{(3)}(r)\right]^3 = 0. \quad (14)$$

We are attempting to find a systematic expansion of the solution of Eq. (14), and do so for large argument $r \to \infty$ in the current section. A remark might be in order. Namely, linear second-order differential equations typically have solutions regular and irregular at infinite argument. The equations defining the instanton, by contrast, are highly nonlinear, and hence this consideration does not apply. In fact, the asymptotics for large argument uniquely determine the instanton solution. In this context, it is instructive to recall [24] that the uniqueness of the solution for a *nonlinear* differential equation, determined by a given initial condition or asymptotic behavior, constitutes a pivotal factor in the emergence of a range of complex phenomena, including chaos. This uniqueness leads to behaviors which are sensitive to the small variations in the initial conditions [25–27]. Furthermore, this intrinsic uniqueness in nonlinear differential equations is analogous to the sensitive dependence on initial conditions observed in fluid dynamics, particularly in the transition from laminar to turbulent flow, where even minor perturbations can drastically alter the flow patterns, echoing the underlying chaotic dynamics described in fluid mechanics research [28].

The instanton goes to zero as $r \to \infty$, and so one can neglect the term $[\xi_{\rm cl}^{(3)}(r)]^3 \ll \xi_{\rm cl}^{(3)}(r)$ in a first approximation. Combining Eqs. (2) and (4), and neglecting the term proportional to the third power of the instanton, for large r, one obtains the relation

$$-\frac{\partial^2}{\partial r^2} \xi_{\rm cl}^{(3)}(r) - \frac{2}{r}\frac{\partial}{\partial r} \xi_{\rm cl}^{(3)}(r) + \xi_{\rm cl}^{(3)}(r) \approx 0. \quad (15)$$

Our ansatz

$$\xi_{\rm cl}^{(3)}(r) = \frac{\exp(-r)}{r} \sum_{n=0}^{\infty} \frac{a_n}{r^n}, \quad (16)$$

is a nonanalytic expansion in the variable $1/r$, for large r. In fact, when expressed in terms of the variable $\chi = 1/r$, the expansion (16) constitutes a nonanalytic (resurgent, transseries) expansion in the variables χ and $\exp(-1/\chi)$ (see Refs. [10–12]), and $\chi = 0$ becomes a singular point of the differential equation. The importance of nonanalytic exponentials (resurgent expansions) in the solution of differential equations with singular points has been stressed in Ref. [12]. The substitution $\xi_{\rm cl}^{(3)}(r) = g(r)/r$ takes Eq. (15) into the form

$$-\frac{\partial^2}{\partial r^2} g(r) + g(r) = 0, \qquad g(r) = \mathcal{C} \exp(-r), \quad (17)$$

for which the solution regular at infinity is just $\exp(-r)$. Hence, the ansatz (16) collapses to a single term, with $\mathcal{C}$ being an overall constant, and reads

$$\xi_{\rm cl}^{(3)}(r) = \mathcal{C}\,\frac{\exp(-r)}{r}, \tag{18}$$

for which the approximate equality in Eq. (15) becomes an exact equality. Here, $\mathcal{C}$ is a coefficient which can be determined numerically. We have mapped the differential equation (15) onto a linear numerical grid with a lattice spacing that decreases as $1/\mathcal{N}$, where $\mathcal{N}$ is the number of the iteration. We use computer algebra [29,30] with a 128-decimal-digit internal precision, investigate the asymptotic behavior of the resulting solution in the regime of large radial argument of the instanton, where the convergence toward the exact solution can be written in terms of powers of $1/\mathcal{N}$, and employ suitable convergence acceleration techniques [31], in order to extrapolate to zero lattice spacing.

With this method, we obtain a 60-figure result for $\mathcal{C}$ which reads as follows:

$$\mathcal{C} = 2.712\,808\,360\,940\,844\,770\,465\,994\,573\,657\,808\,840\,265\,350\,950\,750\,281\,746\,458\,229(1). \tag{19}$$

One can now approximate, in Eq. (14),

$$-\frac{\partial^2}{\partial r^2}\xi_{\rm cl}^{(3)}(r) - \frac{2}{r}\frac{\partial}{\partial r}\xi_{\rm cl}^{(3)}(r) + \xi_{\rm cl}^{(3)}(r) - \xi_{\rm cl}^{(3)}(r)^3 \approx -\frac{\partial^2}{\partial r^2}\xi_{\rm cl}^{(3)}(r) - \frac{2}{r}\frac{\partial}{\partial r}\xi_{\rm cl}^{(3)}(r) + \xi_{\rm cl}^{(3)}(r) - \mathcal{C}^3\,\frac{\exp(-3r)}{r^3} = 0. \tag{20}$$

The structure of this equation justifies the ansatz

$$\xi_{\rm cl}^{(3)}(r) = \mathcal{C}\,\frac{\exp(-r)}{r} + \frac{\exp(-3r)}{r}\sum_{n=0}^{\infty}\frac{b_n}{r^n} + \ldots. \tag{21}$$

Matching of the b_n coefficients leads to the result,

$$\xi_{\rm cl}^{(3)}(r) = \mathcal{C}\,\frac{\exp(-r)}{r} - \mathcal{C}^3\,\frac{\exp(-3r)}{8r^3}\left(1 - \frac{3}{2r} + \frac{21}{8r^2} - \frac{45}{8r^3} + \frac{465}{32r^4} - \frac{2835}{64r^5} + \frac{40005}{256r^6} + \mathcal{O}(r^{-7})\right) + \ldots. \tag{22}$$

This expression, cubed, generates terms proportional to $[\frac{\exp(-r)}{r} + \frac{\exp(-3r)}{r^3}]^3 \to (\frac{\exp(-r)}{r})^2 \times \frac{\exp(-3r)}{r^3} = \frac{\exp(-5r)}{r^5}$. Now, we enter with the ansatz that also contains a term of the form $\frac{\exp(-5r)}{r}\sum_{n=0}^{\infty}\frac{c_n}{r^n}$ again into Eq. (14), match the coefficients c_n, and find

$$\begin{aligned}\xi_{\rm cl}^{(3)}(r) = {}& \mathcal{C}\,\frac{\exp(-r)}{r} - \mathcal{C}^3\,\frac{\exp(-3r)}{8r^3}\left(1 - \frac{3}{2r} + \frac{21}{8r^2} - \frac{45}{8r^3} + \frac{465}{32r^4} - \frac{2835}{64r^5} + \frac{40005}{256r^6} + \mathcal{O}(r^{-7})\right)\\ & + \mathcal{C}^5\,\frac{\exp(-5r)}{64r^5}\left(1 - \frac{19}{6r} + \frac{151}{18r^2} - \frac{815}{36r^3} + \frac{56921}{864r^4} - \frac{1094215}{5184r^5} + \frac{2592553}{3456r^6} + \mathcal{O}(r^{-7})\right).\end{aligned} \tag{23}$$

Finally, with the contribution of order $\exp(-7r)$ included, we have

$$\begin{aligned}\xi_{\rm cl}^{(3)}(r) = {}& \mathcal{C}\,\frac{\exp(-r)}{r} - \mathcal{C}^3\,\frac{\exp(-3r)}{8r^3}\left(1 - \frac{3}{2r} + \frac{21}{8r^2} - \frac{45}{8r^3} + \frac{465}{32r^4} - \frac{2835}{64r^5} + \frac{40005}{256r^6} + \mathcal{O}(r^{-7})\right)\\ & + \mathcal{C}^5\,\frac{\exp(-5r)}{64r^5}\left(1 - \frac{19}{6r} + \frac{151}{18r^2} - \frac{815}{36r^3} + \frac{56921}{864r^4} - \frac{1094215}{5184r^5} + \frac{2592553}{3456r^6} + \mathcal{O}(r^{-7})\right)\\ & - \mathcal{C}^7\,\frac{\exp(-7r)}{512r^7}\left(1 - \frac{29}{6r} + \frac{271}{16r^2} - \frac{3943}{72r^3} + \frac{614143}{3456r^4} - \frac{8322275}{13824r^5} + \frac{80215771}{36864r^6} + \mathcal{O}(r^{-7})\right).\end{aligned} \tag{24}$$

For the term proportional to $\exp(-3r)$, we find the compact formula,

$$-\mathcal{C}^3\,\frac{\exp(-3r)}{8r^3}\left(1 - \frac{3}{2r} + \frac{21}{8r^2} - \frac{45}{8r^3} + \frac{465}{32r^4} + \mathcal{O}(r^{-5})\right) = -\mathcal{C}^3\,\frac{\exp(-r)}{r}\,(\mathrm{Ei}(-2r) - 2\exp(2r)\mathrm{Ei}(-4r)), \tag{25}$$

where $\mathrm{Ei}(r)$ the exponential integral function, but we were unable to find general expressions for the terms in the series multiplying the exponential factors $\exp(-5r)$ and $\exp(-7r)$.

Transseries in the variable $\chi = 1/r$ have been encountered in the study of anharmonic oscillators [32–35]. They have also been investigated mathematically [10–12]. We see that only "odd-transseries" orders of the form $\exp[-(2n+1)/\chi]$ contribute. The "one-transseries" contribution to the instanton wave function is found to read as $\mathcal{C}\chi\exp(-1/\chi)$, without correction terms. The expansion (24) shows that the large-argument expansion of the instanton wave function $\xi_{\rm cl}^{(3)}(r)$ is determined by a single constant $\mathcal{C}$, whose numerical value is given in Eq. (19).

An inspection shows that the perturbative coefficients in the variable $\chi = 1/r$ grow factorially. In order to match the resurgent expansion for large argument with the Taylor expansion for small argument r, we have calculated terms up to the 13-instanton contribution and summed the series, starting from large values of r, down to $r = 3$,

$$\xi_{\rm cl}^{(3)}(r = 3.0) = 0.045\,013\,219\,071\ldots, \tag{26}$$

where a more precise result for $\xi_{\rm cl}^{(3)}$ at the matching point is given in Eq. (62b). In the summation process, we have used [40/40]-Padé approximations [36] in order to sum the divergent perturbative series decorating the instanton contributions of order $\exp(-nr)$, where n is an odd integer, and 77-order Weniger-Levin transformations [37], in order to verify the accuracy of the result (26) in the intermediate region near $r \approx 3.0$. The matching point is chosen heuristically, based on the requirement that both methods for the calculation should work in the intermediate region between the large-r transseries representation and the small-r power series, which will be discussed in the following.

B. Small argument

We recall the equation fulfilled by the instanton [see Eq. (14)]

$$-\frac{\partial^2}{\partial r^2}\xi_{\rm cl}^{(3)}(r) - \frac{2}{r}\frac{\partial}{\partial r}\xi_{\rm cl}^{(3)}(r) + \xi_{\rm cl}^{(3)}(r) - [\xi_{\rm cl}^{(3)}(r)]^3 = 0. \tag{27}$$

Plugging in a polynomial ansatz into Eq. (27), with

$$\xi_{\rm cl}^{(3)}(0) = \mathcal{F} = 4.337\,387\,679\,976\ldots \tag{28}$$

[see also Eq. (61c)], one finds

$$\xi_{\rm cl}^{(3)}(r) = \mathcal{F} + \frac{1}{6}(\mathcal{F} - \mathcal{F}^3)r^2 + \frac{1}{120}(\mathcal{F} - 4\mathcal{F}^3 + 3\mathcal{F}^5)r^4 + \frac{\mathcal{F} - 17\mathcal{F}^3 + 35\mathcal{F}^5 - 19\mathcal{F}^7}{5040}r^6 + \mathcal{O}(r^8). \tag{29}$$

Only even powers of r contribute. Using computer algebra [29], one can easily determine all coefficients up to order r^{80}, and write

$$\xi_{\rm cl}^{(3)}(r) = \sum_{n=0}^{\infty} a_{2n} r^{2n}. \tag{30}$$

A closer inspection reveals that the series of the a_{2n} is factorially divergent and alternating. Still, one can use summation techniques to confirm the result (26) at the matching point $\xi_{\rm cl}^{(3)}(r = 3.0)$ [see also Eq. (62b)]. In the summation process, we have used [62/62]-Padé approximations [36] in order to sum the divergent perturbative series at $r = 3$, or alternatively 117-order Weniger-Levin transformations [37]. This leads to the desired accuracy in the intermediate region. Improvements of the numerical accuracy are possible when one expands the instanton about additional reference points (e.g., where r assumes the value of a small integer) and concatenates the expansions in regions of overlap.

IV. TWO-DIMENSIONAL INSTANTON

A. Large argument

In two dimensions, the instanton is equally radially symmetric (see Fig. 2), and we can write $\xi_{\rm cl}^{(2)}(\vec{x}) = \xi_{\rm cl}^{(2)}(r)$. The equation fulfilled by the instanton is

$$-\frac{\partial^2}{\partial r^2}\xi_{\rm cl}^{(2)}(r) - \frac{1}{r}\frac{\partial}{\partial r}\xi_{\rm cl}^{(2)}(r) + \xi_{\rm cl}^{(2)}(r) - [\xi_{\rm cl}^{(2)}(r)]^3 = 0. \tag{31}$$

Just like in the three-dimensional case (see Sec. III A), the instanton goes exponentially to zero as $r \to \infty$, and so one can neglect the term $[\xi_{\rm cl}^{(2)}(r)]^3$ in a first approximation. Then, one obtains the relation [see also Eq. (15)]

$$-\frac{\partial^2}{\partial r^2}\xi_{\rm cl}^{(2)}(r) - \frac{1}{r}\frac{\partial}{\partial r}\xi_{\rm cl}^{(2)}(r) + \xi_{\rm cl}^{(2)}(r) \approx 0. \tag{32}$$

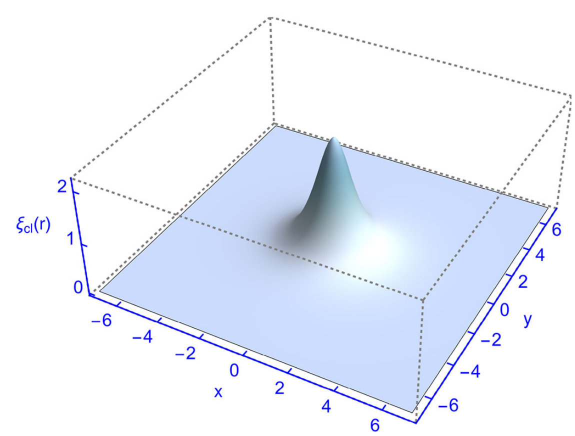

FIG. 2. The two-dimensional instanton $\xi_{\rm cl}(r) = \xi_{\rm cl}^{(2)}(r)$ is radially symmetric. Its value at the origin is $\xi_{\rm cl}^{(2)}(0) = \mathcal{G} = 2.206\,200\,864\,650\ldots$, according to Eqs. (38) and (61b).

By a similar analysis as described for the three-dimensional case, one obtains

$$\begin{aligned}\xi_{\rm cl}^{(2)}(r) &= \mathcal{D}\frac{\exp(-r)}{r^{1/2}}\left(1-\frac{1}{8r}+\frac{9}{128r^2}-\frac{75}{1024r^3}+\frac{3675}{32768r^4}-\frac{59535}{262144r^5}+\frac{2401245}{4194304r^6}+\mathcal{O}(r^{-7})\right)\\ &-\mathcal{D}^3\frac{\exp(-3r)}{8r^{3/2}}\left(1-\frac{9}{8r}+\frac{213}{128r^2}-\frac{3215}{1024r^3}+\frac{238563}{32768r^4}-\frac{5283711}{262144r^5}+\frac{273186513}{4194304r^6}+\mathcal{O}(r^{-7})\right)\\ &+\mathcal{D}^5\frac{\exp(-5r)}{64r^{5/2}}\left(1-\frac{53}{24r}+\frac{589}{128r^2}-\frac{96193}{9216r^3}+\frac{23551553}{884736r^4}-\frac{544320827}{7077888r^5}+\frac{85429251785}{339738624r^6}+\mathcal{O}(r^{-7})\right)\\ &-\mathcal{D}^7\frac{\exp(-7r)}{512r^{7/2}}\left(1-\frac{79}{24r}+\frac{10037}{1152r^2}-\frac{629833}{27648r^3}+\frac{55541473}{884736r^4}-\frac{1326870785}{7077888r^5}+\frac{23212812833}{37748736r^6}+\mathcal{O}(r^{-7})\right). \quad (33)\end{aligned}$$

Just as in the three-dimensional case, we can find a compact expression for the leading term, which for $D=2$ is of order $\exp(-r)/\sqrt{r}$,

$$\mathcal{D}\frac{\exp(-r)}{r^{1/2}}\left(1-\frac{1}{8r}+\frac{9}{128r^2}-\frac{75}{1024r^3}+\frac{3675}{32768r^4}+\mathcal{O}(r^{-5})\right) = \mathrm{i}\mathcal{D}H_0^{(1)}(\mathrm{i}r). \quad (34)$$

Here, $H_\alpha^{(1)}(r)$ is the Hankel function [38] of the first kind of order α. However, we were unable to find general expressions for the terms in the series multiplying the exponential factors $\exp(-3r)$, $\exp(-5r)$ and $\exp(-7r)$.

For $D=2$, one finds for the $\mathcal{D}$ coefficient the following 60-figure result [using the same method as previously employed for Eq. (19)]:

$$\begin{aligned}\mathcal{D} = 3.518\,062\,198\,025\,031\,180\,209\,129\,887\,741\\ 356\,933\,215\,813\,390\,992\,384\,663\,366\,560(1). \quad (35)\end{aligned}$$

Furthermore, it is clear that the perturbative coefficients in the variable $\chi = 1/r$ grow very fast, and in fact, they grow factorially. We have calculated terms up to the contribution of order $\exp(-13r)$ and summed the series, starting from large values of r, down to $r=3$, with the result

$$\xi_{\rm cl}^{(2)}(r=3.0) = 0.097\,418\,218\,653\ldots, \quad (36)$$

where the most precise result for the value of $\xi_{\rm cl}^{(2)}(r=3.0)$ at the matching point is given in Eq. (62a). Just as in the three-dimensional case, in the summation process, we have used [40/40]-Padé approximations in order to sum the divergent perturbative series decorating the instanton contributions of order $\exp(-nr)$, where $n \le 13$ is an odd integer, or 77-order Weniger-Levin transformations [37], in order to achieve the accuracy in the intermediate region.

B. Small argument

We now need to repeat the analysis from Sec. III B, for the two-dimensional case. Plugging a polynomial ansatz into Eq. (31),

$$\left(-\frac{\partial^2}{\partial r^2}-\frac{1}{r}\frac{\partial}{\partial r}+1-\xi_{\rm cl}^{(2)}(r)^2\right)\xi_{\rm cl}^{(2)}(r)=0 \quad (37)$$

with

$$\xi_{\rm cl}^{(2)}(0) = \mathcal{G} = 2.206\,200\,864\,650\ldots, \quad (38)$$

[see also Eq. (61b)], one finds

$$\begin{aligned}\xi_{\rm cl}^{(2)}(r) &= \mathcal{G}+\frac{1}{4}(\mathcal{G}-\mathcal{G}^3)r^2+\frac{1}{64}(\mathcal{G}-4\mathcal{G}^3+3\mathcal{G}^5)r^4\\ &+\frac{\mathcal{G}-19\mathcal{G}^3+39\mathcal{G}^5-21\mathcal{G}^7}{2304}r^6+\mathcal{O}(r^8). \quad (39)\end{aligned}$$

Using computer algebra [29], one can easily determine all coefficients up to order r^{80}, say, and write the divergent, asymptotic expansion

$$\xi_{\rm cl}^{(2)}(r) = \sum_{n=0}^{\infty} a_{2n} r^{2n}. \quad (40)$$

Here, too, a closer inspection reveals that the series is divergent, because of factorial divergence of the magnitude of the (alternating-in-sign) power series at about $r=0$. One confirms the result given in Eq. (36). In the summation process, we have used [62/62]-Padé approximations in order to sum the divergent perturbative series at $r=3$, or alternatively 117-order Weniger-Levin transformations [37]. This yields the desired accuracy in the intermediate region.

V. INSTANTONS AND VIRIAL THEOREMS

A. Derivation of the virial theorems

One of the goals of the current investigation is to explore possible analytic representations of integrals of powers of the instanton, for example, on the basis of the PSLQ algorithm [39–42]. In this endeavor, it helps to realize that integrals of different powers of the instanton are related to each other, and to the instanton action. Virial theorems of the instanton are instrumental in this regard, and we here present their derivation.

We consider the general action

$$\mathcal{S}(\phi) = \int \mathrm{d}^D x \left[\frac{1}{2}\left(\vec{\nabla}\phi(\vec{x})\right)^2 + \mathcal{V}(\phi(\vec{x}))\right], \qquad (41)$$

where in the case of the action (1), one has

$$\mathcal{V}(\phi(\vec{x})) = \frac{1}{2}\phi(\vec{x}) + \frac{1}{4}g\phi^4(\vec{x}). \qquad (42)$$

We assume that the field equation, obtained by variational calculus, has a finite action solution $\phi_{\mathrm{cl}}(\vec{x})$. If the action $\mathcal{S}(\phi_{\mathrm{cl}})$ is finite so is the action $\mathcal{S}(\phi_{\mathrm{cl}}, \lambda)$, obtained from $\mathcal{S}(\phi_{\mathrm{cl}}) = \mathcal{S}(\phi_{\mathrm{cl}}, \lambda = 1)$ by the replacement $\phi(\vec{x}) \to \phi_{\mathrm{cl}}(\lambda\vec{x})$ in the integrand of the action. One finds, after a suitable backsubstitution,

$$\mathcal{S}[\phi_{\mathrm{cl}}, \lambda] = \lambda^{2-D} \int \mathrm{d}^D x \left[\frac{1}{2}\left(\vec{\nabla}\phi_{\mathrm{cl}}(\vec{x})\right)^2\right] + \lambda^{-D} \int \mathrm{d}^D x \mathcal{V}(\phi_{\mathrm{cl}}(\vec{x})). \qquad (43)$$

Because $\phi_{\mathrm{cl}}(x)$ satisfies the field equation, the variation of the action vanishes for $\lambda = 1$, i.e., we have the equation $\frac{\mathrm{d}}{\mathrm{d}\lambda}\mathcal{S}[\phi_{\mathrm{cl}}, \lambda]|_{\lambda=1} = 0$,

$$\int \mathrm{d}^D x' \left[\frac{D-2}{2}\left(\vec{\nabla}\phi_{\mathrm{cl}}(\vec{x}')\right)^2 + D\mathcal{V}(\phi_{\mathrm{cl}}(\vec{x}'))\right] = 0. \qquad (44)$$

This relation allows us to express the kinetic term [integral of $(\vec{\nabla}\phi_{\mathrm{cl}})^2$] in terms of the potential term [integral of $\mathcal{V}(\phi_{\mathrm{cl}})$] and vice versa. The classical action $\mathcal{S}[\phi_{\mathrm{cl}}]$ can thus be expressed in terms of the kinetic term only:

$$\mathcal{S}[\phi_{\mathrm{cl}}] = \frac{1}{D}\int \mathrm{d}^D x \left(\vec{\nabla}\phi_{\mathrm{cl}}(\vec{x})\right)^2, \qquad (45)$$

a form that shows that $\mathcal{S}(\phi_{\mathrm{cl}})$ is always positive. The second derivative of $\mathcal{S}(\phi_{\mathrm{cl}}, \lambda)$ reads as

$$\frac{\mathrm{d}^2}{(\mathrm{d}\lambda)^2}\mathcal{S}[\phi_c, \lambda]\bigg|_{\lambda=1} = (2-D)\int [\partial_\mu \phi_c(\vec{x})]^2 \mathrm{d}^D x. \qquad (46)$$

For $D \geq 2$, this result shows that the solution is not a local minimum of the action and, thus, the so-called longitudinal fluctuation operator $\mathbf{M}_{\mathrm{L}}$, defined as

$$\mathbf{M}_{\mathrm{L}}(\vec{x}, \vec{x}') = \frac{\delta^2 \mathcal{S}}{\delta\phi(\vec{x})\delta\phi(\vec{x}')}\bigg|_{\phi=\phi_{\mathrm{cl}}}, \qquad (47)$$

has at least one negative eigenvalue [4,35]. A closer inspection [4,35] reveals one, and only one, negative eigenvalue. This fact is well known and it leads, within the path and field integral formalisms, to an imaginary part of the square root of the Fredholm determinant of the fluctuation operator, which corresponds to the product of the eigenvalues of $\mathbf{M}_{\mathrm{L}}$. This imaginary part, in turn, is instrumental in deriving the large-order estimate given in Eq. (10) for the perturbative expansions of correlation functions [4,5,35].

In the example of potentials of special form

$$\mathcal{V}(\phi) = \frac{1}{2}\phi^2 + \frac{1}{4}g\phi^M, \qquad (48)$$

one can derive an additional relation. If the action $\mathcal{S}[\phi_{\mathrm{cl}}]$ is finite, so is the following action obtained by the replacement $\phi_{\mathrm{cl}} \to \Lambda\phi_{\mathrm{cl}}$:

$$\mathcal{S}[\Lambda\phi_{\mathrm{cl}}] = \Lambda^2 \int \mathrm{d}^D x \frac{1}{2}\left[\left(\vec{\nabla}\phi_{\mathrm{cl}}(\vec{x})\right)^2 + \phi_{\mathrm{cl}}^2(\vec{x})\right] + \frac{1}{4}\Lambda^M g \int \mathrm{d}^D x \phi_{\mathrm{cl}}^M(\vec{x}). \qquad (49)$$

Again, if ϕ_{cl} is the instanton solution, then the derivative with respect to Λ must vanish for $\Lambda = 1$. One obtains further relations, in addition to (45),

$$\mathcal{S}(\phi_{\mathrm{cl}}) = -\frac{g}{8}(M-2)\int \mathrm{d}^D x \phi_{\mathrm{cl}}^M(\vec{x}) \overset{M=4}{=} -\frac{g}{4}\int \mathrm{d}^D x \phi_{\mathrm{cl}}^4(\vec{x}) \quad (M=4). \qquad (50)$$

This relation is consistent with the fact that the instanton exists only for negative g. Thus, one can express the instanton action as follows:

$$\mathcal{S}(\phi_{\mathrm{cl}}) = \frac{M-2}{2D - M(D-2)}\int \mathrm{d}^D x \phi_{\mathrm{cl}}^2(\vec{x}) \overset{M=4}{=} \frac{1}{4-D}\int \mathrm{d}^D x \phi_{\mathrm{cl}}^2(\vec{x}). \qquad (51)$$

We have used Eqs. (45) and (50). In particular, Eqs. (50) and (51) are consistent only if the denominator in the expression on the right-hand side of Eq. (51) is positive, which implies

$$M \leq \frac{2D}{D-2}, \qquad (52)$$

and, therefore, the field theory must be super-renormalizable or at least renormalizable. At the special dimension $D = 2M/(M-2)$, where the theory is renormalizable, one finds the paradoxical result $\int \mathrm{d}^D x \phi_{\mathrm{cl}}^2(\vec{x}) = 0$. This implies that only the massless equation (where the coefficient of ϕ^2 in the original action vanishes) has instanton solutions (see also Appendix A). Finally, one verifies that the second derivative of the scaled action (49) at $\Lambda = 1$ is negative, confirming the existence of a negative eigenvalue of $\mathbf{M}_{\mathrm{L}}$ for all dimensions [4,5,35].

B. Summary of the virial theorems

We summarize. From Eqs. (45), (50), and (51), we have for the ϕ^4 theory with the action (1),

$$\mathcal{S}(\phi_{\mathrm{cl}}) = \frac{1}{D}\int \mathrm{d}^D x \left(\vec{\nabla}\phi_{\mathrm{cl}}(\vec{x})\right)^2 = -\frac{g}{4}\int \mathrm{d}^D x \phi_{\mathrm{cl}}^4(\vec{x}) = \frac{1}{4-D}\int \mathrm{d}^D x \phi_{\mathrm{cl}}^2(\vec{x}). \tag{53}$$

With the scaling given by Eq. (4), the action of the instanton becomes ($g < 0$),

$$\phi_{\mathrm{cl}}(\vec{x}) = \sqrt{-\frac{1}{g}}\xi_{\mathrm{cl}}(r), \qquad r = |\vec{x}|, \qquad \mathcal{S}(\phi_{\mathrm{cl}}) = -\frac{A}{g} > 0. \tag{54a}$$

We have three equivalent representations of the action A,

$$A = \frac{1}{D}\int \mathrm{d}^D x \left(\vec{\nabla}\xi_{\mathrm{cl}}(r)\right)^2 = \frac{1}{4}\int \mathrm{d}^D x \xi_{\mathrm{cl}}^4(r) = \frac{1}{4-D}\int \mathrm{d}^D x \xi_{\mathrm{cl}}^2(r). \tag{54b}$$

We have numerically verified these relations on lattices with decreasing lattice spacing, using the method outlined in the discussion preceding Eq. (19). Using the radial symmetry of the solution, we can establish that

$$A = \frac{\Omega_D}{D}\int_0^\infty \mathrm{d}r\, r^{D-1}[\xi_{\mathrm{cl}}'(r)]^2, \tag{55}$$

where $\Omega_D = 2\pi^{D/2}/\Gamma(\frac{1}{2}D)$ is the generalized surface of the $(D-1)$-dimensional unit sphere embedded in D-dimensional space.

C. Asymptotic behavior

The asymptotic behavior of the radial instanton equation

$$\left[-\left(\frac{\mathrm{d}}{\mathrm{d}r}\right)^2 - \frac{D-1}{r}\frac{\mathrm{d}}{\mathrm{d}r} + 1\right]\xi_{\mathrm{cl}}(r) - \xi_{\mathrm{cl}}^3(r) = 0 \tag{56}$$

is of interest for large r. Asymptotically, one can show that, for $r \to \infty$,

$$\xi_{\mathrm{cl}}(r) = C\sqrt{\frac{2}{\pi}}r^{1-D/2}K_{D/2-1}(r) + \mathcal{O}(\mathrm{e}^{-3r}) \tag{57}$$

where K_ν is a modified Bessel function of the second kind normalized such that $K_\nu(r) \sim \sqrt{\pi/(2r)}\mathrm{e}^{-r}$ for large r. This implies that

$$\xi_{\mathrm{cl}}^{(D)}(r) \propto \sqrt{\frac{2}{\pi}}r^{1-D/2}\sqrt{\frac{\pi}{2r}}\mathrm{e}^{-r} = \frac{\mathrm{e}^{-r}}{r^{D/2-1/2}}. \tag{58}$$

Our formulas (24) and (33) confirm this asymptotic behavior.

VI. INSTANTON INTEGRALS

We give a collection of numerical results for integrals of the instanton in quartic theories, with enhanced accuracy. Our aim is to give, for 2D and 3D, results approaching the realm of applicability of the PSLQ algorithm [39–42] which is designed to search for analytic expressions of integrals in terms of known constants. We remember that the PSLQ algorithm requires as input data only high-precision numerical values of the quantity under investigation, as well as a guess of the mathematical constants in which the high-precision quantity could potentially be expressed, and then attempts to find a linear combination of mathematical constants, multiplied by rational fractions, as a candidate representation for the quantity under investigation [43]. First, for completeness, in one dimension, we recall that the instanton solution is [1]

$$\xi_{\mathrm{cl}}(r) = \frac{2}{\sqrt{\cosh(r)+1}}, \qquad \xi_{\mathrm{cl}}^{(D=1)}(r=0) = \sqrt{2}, \tag{59}$$

where the one-dimensional action is

$$S[\phi] = 2\int_0^\infty \mathrm{d}r\left[\frac{1}{2}(\partial_r\phi(r))^2 + \frac{1}{2}\phi(r)^2 + \frac{g}{4}\phi(r)^4\right]. \tag{60}$$

The prefactor 2 reflects on the angular factor $\Omega_D = 2\pi^{D/2}/\Gamma(\frac{1}{2}D)$ which evaluates to 2 for $D = 1$. The prefactor matters because the instanton action is normalized to $S[\phi_{\mathrm{cl}}(r)] = -A/g$, according to Eq. (9). Analytically known instantons in a four-dimensional ϕ^4 theory and in a six-dimensional ϕ^3 theory are given in Appendixes A and B, respectively.

We have mentioned that instanton solutions are determined by the value at the origin, given in Eqs. (28) and (38). It is of interest to obtain results of higher accuracy for the instanton at the origin, described by the constants $\mathcal{F}$ and $\mathcal{G}$, and for the instanton at the matching point. In order to obtain more accurate values for $\mathcal{F}$ and $\mathcal{G}$, one maps the differential equation (15) onto a linear numerical grid with a lattice spacing that decreases an $1/\mathcal{N}$, where $\mathcal{N}$ is the number of the iteration, while the origin is kept as the starting point of the lattice. One then uses computer arithmetic [29,30] with 128-decimal-digit internal precision, and extrapolates to $\mathcal{N} \to \infty$. In order to calculate the instanton at the matching point $r = 3.0$, one decreases the lattice spacing in such a way that the matching point remains a member of the numerical lattice in every iteration as $\mathcal{N}$ is being increased. The convergence toward the exact values of the instanton, in either instance, can be written in terms of powers of $1/\mathcal{N}$. One can thus employ suitable convergence acceleration techniques [31], in order to extrapolate to zero lattice spacing, and obtain numerically more accurate results. We obtain the following values, accurate to 78 decimal figures:

$$\xi_{\rm cl}^{(D=1)}(r=0) = \sqrt{2}, \tag{61a}$$

$$\xi_{\rm cl}^{(D=2)}(r=0) = \mathcal{G} = 2.206\,200\,864\,650\,746\,074\,783\,634\,064\,578\,940\,196\,610\,274\,520\,602\,192\,125\,757\,262\,456\,450\,184\,032\,518\,642(1), \tag{61b}$$

$$\xi_{\rm cl}^{(D=3)}(r=0) = \mathcal{F} = 4.337\,387\,679\,976\,994\,356\,522\,109\,173\,841\,761\,465\,745\,284\,082\,970\,785\,762\,761\,882\,558\,415\,947\,364\,399\,341(1). \tag{61c}$$

The values at the matching point $r = 3$ are interesting for $D = 2$ and $D = 3$,

$$\xi_{\rm cl}^{(D=2)}(r=3.0) = 0.097\,418\,218\,653\,642\,217\,741\,513\,024\,960\,584\,546\,095\,157\,618\,276\,680\,772\,556\,932\,915\,093\,354\,850\,219\,044(1). \tag{62a}$$

$$\xi_{\rm cl}^{(D=3)}(r=3.0) = 0.045\,013\,219\,071\,010\,523\,997\,989\,047\,723\,112\,322\,109\,014\,075\,244\,317\,789\,103\,014\,970\,206\,885\,072\,459\,490(1). \tag{62b}$$

Numerical results for the instanton action A are

$$A(D=1) = 4/3, \tag{63a}$$

$$A(D=2) = 5.850\,448\,262\,279\,826\,939\,326\,986\,338\,934\,453\,868\,499\,064\,115\,959\,470\,267\,545\,644\,043\,014\,800\,957\,116\,007\,(1). \tag{63b}$$

$$A(D=3) = 18.897\,251\,302\,546\,190\,505\,297\,247\,993\,763\,227\,763\,807\,178\,891\,316\,289\,857\,028\,151\,589\,245\,449\,182\,127\,167\,(1). \tag{63c}$$

These results are essential for large-order perturbation theory [see Eq. (10)]. For what follows, it is convenient to introduce the notation

$$I_n = \int {\rm d}^D x [\xi_{\rm cl}(r)]^n, \qquad I_2 = \frac{4-D}{4} I_4, \qquad I_4 = \frac{4}{D} \int {\rm d}^D x [\vec{\nabla} \xi_{\rm cl}(r)]^2, \qquad A = \frac{1}{4} I_4, \tag{64}$$

where we recall that the (generalized) surface area of the $(D-1)$-dimensional unit sphere, embedded in D dimensions, is $\Omega_D = 2\pi^{D/2}/\Gamma(D/2)$. Results for I_2 and I_4 follow from the above results for the instanton action A. Results for I_3 and I_6 are given as follows:

$$I_3(D=1) = \sqrt{2}\pi, \qquad I_6(D=1) = 128/15, \tag{65a}$$

$$I_3(D=2) = 15.109\,669\,726\,889\,195\,199\,613\,754\,001\,702\,125\,888\,865\,874\,563\,104\,430\,202\,476\,703\,241\,753\,965\,063\,516\,331(1). \tag{65b}$$

$$I_3(D=3) = 31.691\,521\,838\,323\,486\,451\,591\,907\,257\,120\,270\,170\,457\,351\,985\,790\,745\,758\,122\,769\,708\,466\,866\,938\,412\,287(1). \quad (65c)$$

$$I_6(D=2) = 71.080\,171\,542\,041\,917\,440\,792\,898\,285\,323\,353\,751\,992\,546\,589\,483\,197\,579\,092\,397\,461\,668\,330\,495\,246\,091(1). \quad (65d)$$

$$I_6(D=3) = 659.868\,351\,544\,567\,238\,188\,639\,540\,582\,544\,719\,267\,515\,748\,898\,263\,457\,303\,680\,826\,218\,470\,215\,371\,739\,493(1). \quad (65e)$$

VII. CONCLUSIONS

We have analyzed the properties of instanton solutions in $O(N)$-symmetric quartic field theories in $D = 2$ and $D = 3$ dimensions. The basic formulation for the quartic instanton has been given in Sec. II. We concentrate on the three-dimensional instanton ($D = 3$) in Sec. III, which is phenomenologically the most interesting case. We derive asymptotic expansions for large argument in the form of a transseries [Eq. (24)] and in the form of an asymptotic power series [Eq. (29)] for small argument. The quartic instanton in $D = 2$ is discussed in Sec. IV. Virial theorems are derived in Sec. V. Instanton integrals are given in Sec. VI, with a precision approaching the realm of applicability of the PSLQ algorithm [39–42] which is designed to search for analytic expressions of integrals in terms of known constants such as the Euler constant $\gamma_E = 0.57721\ldots$, various Riemann zeta functions, powers of π, and multiplicative combinations of these constants. We can report that we have carried out a limited set of searches with the same constants that were used in Eq. (A11) of Ref. [4] without success. A more detailed search might constituent a possible direction for the future.

In a quartic theory, the instanton solution exists only for negative g, because the tunneling can proceed only through a barrier, and the latter exists only negative coupling $g < 0$. The imaginary part of the partition function, and of correlation functions, obtained by expanding about the instanton solution, is proportional (see Ref. [4]) to $\exp(-(-A/g)) = \exp(A/g)$, where $\mathcal{S}[\phi_{\rm cl}] = -A/g$ and A is given for $D = 2$ in Eq. (63b) for $D = 3$ in Eq. (63c). The instanton action A universally enters large-order formulas for the perturbative coefficients of Green functions [see Eq. (10)].

Our calculations suggest that instanton solutions in quartic theories cannot be expressed in closed analytic form, except for the case $D = 1$. We also note that the general properties of the instanton in massive theories are valid only for dimensions $D < 4$. The dimension 4 is singular, as is evident, e.g., from Eq. (53). The cases of a massless quartic theory in four dimensions, and of a cubic theory in six dimensions, are treated in Appendixes A and B.

Our investigations indicate that, with the exception of known singular cases (see Appendixes A and B), instanton configurations cannot be calculated analytically for general field theories, notably, for the two- and three-dimensional ϕ^4 theories. Nevertheless, in view of the nonlinear nature of the defining differential equations, they admit transseries solutions and asymptotic expansions which can be used for accurate numerical calculations. These results are useful in expansions of partition and correlation functions about instanton configurations [6].

Let us conclude by mentioning open problems, which could inspire future research. The first of these concerns the possibility of analytic expressions for the 78-figure results reported here for particular function values and integrals of the two- and three-dimensional instantons, notably, those communicated in Eqs. (61b)–(65). As already mentioned, we have performed a limited search based on the PSLQ algorithm [39–42] using various Riemann zeta functions, logarithms, and polylogarithms, without finding suitable analytic formulas. Our inability to find fully analytic representations is mirrored in recent, somewhat related investigations [18]. The second open problem concerns the search for closed-form representations of the higher-order terms in the transseries solution (24) and (33), generalizing the results given in Eqs. (25) and (34) to higher orders of the exponential factor $\exp(-r)$.

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APPENDIX A: FOUR-DIMENSIONAL MASSLESS QUARTIC THEORY

The existence of instantons in the renormalizable (but not super-renormalizable) quartic theory in four dimensions has been anticipated in Sec. V A. We consider the action

$$\mathcal{S}[\phi] = \int \mathrm{d}^4x \left[\frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{1}{4}g\phi^4\right]. \tag{A1}$$

The corresponding field equation is $-\vec{\nabla}^2\phi_{\mathrm{cl}}(\vec{x}) + g\phi_{\mathrm{cl}}^3(\vec{x}) = 0$. We know that the solution of minimal action is spherically symmetric, thus we set

$$\phi_{\mathrm{cl}}(x) = \sqrt{-\frac{1}{g}}\xi_{\mathrm{cl}}(r), \tag{A2}$$

where $r = |\vec{x}|$. We then obtain a differential equation $[(\frac{\mathrm{d}}{\mathrm{d}r})^2 + \frac{3}{r}\frac{\mathrm{d}}{\mathrm{d}r} + \xi_{\mathrm{cl}}^2(r)]\xi_{\mathrm{cl}}(r) = 0$. The solution is

$$\xi_{\mathrm{cl}}(r) = \frac{2\sqrt{2}}{1+r^2}. \tag{A3}$$

The instanton action is

$$S[\phi_{\mathrm{cl}}] = -\frac{A}{g}, \qquad A = \frac{8\pi^2}{3}. \tag{A4}$$

The instanton integrals, $I_n = \int \mathrm{d}^4x\xi_{\mathrm{cl}}(r)^n$, for $n = 3, 4, 6$, are $I_3 = 8\sqrt{2}\pi^2$, $I_4 = \frac{32\pi^2}{3}$, $I_6 = \frac{128\pi^2}{5}$.

APPENDIX B: SIX-DIMENSIONAL MASSLESS CUBIC THEORY

Another example of the existence of analytically calculable instantons is the six-dimensional massless cubic theory [44]. We consider the action

$$\mathcal{S}[\phi] = \int \mathrm{d}^6x \left[\frac{1}{2}(\vec{\nabla}\phi)^2 + \frac{1}{3}g\phi^3\right]. \tag{B1}$$

The corresponding field equation for the instanton is $-\vec{\nabla}^2\phi_{\mathrm{cl}}(\vec{x}) + g\phi_{\mathrm{cl}}^2(\vec{x}) = 0$. We know that the solution of minimal action is spherically symmetric, thus we set

$$\phi_{\mathrm{cl}}(x) = \frac{1}{g}\xi_{\mathrm{cl}}(r), \tag{B2}$$

where $r = |\vec{x}|$ and we observe that the instanton exists for positive g. We then obtain a differential equation $[(\frac{\mathrm{d}}{\mathrm{d}r})^2 + \frac{5}{r}\frac{\mathrm{d}}{\mathrm{d}r} + \xi_{\mathrm{cl}}(r)]\xi_{\mathrm{cl}}(r) = 0$. The solution is

$$\xi_{\mathrm{cl}}(r) = -\frac{24}{(1+r^2)^2}. \tag{B3}$$

The instanton action is

$$S[\phi_{\mathrm{cl}}] = \frac{A}{g^2}, \qquad A = \frac{192\pi^3}{5}. \tag{B4}$$

The instanton integrals, $I_n = \int \mathrm{d}^4x\xi_{\mathrm{cl}}(r)^n$, for $n = 3, 4, 6$, are $I_3 = -\frac{1152\pi^3}{5}$, $I_4 = \frac{55296\pi^3}{35}$, $I_6 = \frac{10616832\pi^3}{55}$.

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