

DISTRIBUTED FUNCTION ESTIMATION: ADAPTATION USING MINIMAL COMMUNICATION

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We investigate whether in a distributed setting, adaptive estimation of a smooth function at the optimal rate is possible under minimal communication. It turns out that the answer depends on the risk considered and on the number of servers over which the procedure is distributed. We show that for the L_∞ -risk, adaptively obtaining optimal rates under minimal communication is not possible. For the L_2 -risk, it is possible over a range of regularities that depends on the relation between the number of local servers and the total sample size.

1. Introduction. Distributed methods have attracted a lot of attention in the statistics and machine learning communities recently. There are several reasons for this, the most prominent ones being that they provide a way of dealing with large datasets and with privacy considerations. The theoretical literature on distributed methods is still rather minimal at the moment. A number of papers have recently investigated fundamental performance limits in distributed models, in particular pointing out issues that occur in high-dimensional or nonparametric problems, see for instance [1, 2, 4, 8, 16, 17, 21, 24, 27]. For example, optimal rates in distributed function estimation depend on the amount of communication that is allowed, and the relation of that amount with the regularity of the unknown function. The lower bounds obtained in [25] and [28] and the subsequent adaptation results in [25] show that in particular, automatically adapting to the smoothness of the unknown function is a complicated issue in communication restricted distributed settings. In the present paper we study this problem from a different, we think relevant and interesting perspective, not restricting communication a priori, but asking for rate-optimal procedures that require minimal communication.

In distributed estimation problems it is of interest to achieve high estimation accuracy, while at the same time limiting communication between servers, or cores, since this may give rise to undesirable time loss, costs, or

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congestion. In this paper we investigate this problem for a basic distributed architecture, where we have m local servers over which the data is distributed and that each carry out a statistical procedure using their local data, independently of each other. They communicate their result to a central server that performs some aggregation and produces a final estimate of the quantity of interest. The two goals of high accuracy and little communication are conflicting in this setting. It is intuitively clear that to achieve high accuracy it is beneficial to have a lot of data in one server, which is only possible if the total number of local servers m is small, which we will not assume, or the local servers are allowed to communicate a lot of information to the central server, which we will consider to be undesirable.

The problem becomes most interesting if the unknown object is high-, or infinite-dimensional. To be specific, we will consider a distributed signal estimation problem in which the goal is to estimate a function $f \in L_2[0, 1]$ with (Besov) regularity $s > 0$. (A precise description of the model is given in Section 2.) The best accuracy that can be achieved with respect to the L_2 -norm can be described by minimax lower bounds. In the classical, non-distributed setting the minimax lower bound over Besov balls of regularity s is known to be of the order $n^{-s/(1+2s)}$, where n is the sample size, or signal-to-noise ratio (e.g. [12]). Recently established lower bounds for distributed nonparametric methods under communication constraints (see [25], [28], and Section 2 ahead) show that this optimal rate can also be achieved by distributed methods, but only if each local machine is allowed to communicate at least order $n^{1/(1+2s)}$ bits of information to the central machine. This is what the authors of [28] call the *sufficient regime*.

A distributed strategy that achieves the rate $n^{-s/(1+2s)}$ under the restriction that the local machines communicate at most the minimal order $n^{1/(1+2s)}$ bits is easily constructed (see Theorem 2.2). However, this simple strategy uses knowledge of the regularity s of the unknown signal. The real interesting question is whether this can be done adaptively, without knowing s . This greatly complicates the problem, since we do not only want adaptation to smoothness of the estimator, but we also require that the local machines determine the maximally allowed number of bits in a purely data-driven manner.

It turns out that whether or not this is possible for the L_2 -risk depends on the relation between the number of machines m and the total sample size, or signal-to-noise ratio n . We prove that if $m = n^p$ for some $p \in (0, 1/2)$, then:

- There exists a distributed estimator that is adaptive over any range of

regularities $[s_1, s_2]$ such that

$$0 < s_1 < s_2 < \frac{1}{4p} - \frac{1}{2},$$

achieving the optimal rate and transmitting the minimal amount of bits.

- If

$$s_2 > s_1 > \frac{1}{4p} - \frac{1}{2}$$

however, then there exists no distributed procedure that achieves the optimal rate for every signal f with regularity in $\{s_1, s_2\}$, while transmitting the minimal amount of bits.

Stated differently, when considering L_2 -risk, adaptively achieving the optimal rate using minimal communication over a range of regularities $[s_1, s_2]$ is possible if and only if

$$(2 + 4s_2) \log m < \log n.$$

This shows that it is problematic if either the number of machines is too large, or the range of regularities to which adaptation is required is too large.

The adaptive, minimal communication procedure that we propose in the first case implicitly exploits the fact that for the L_2 -risk, there is a difference between lower bounds for estimation and testing, see for instance [12, 15]. Indeed, we employ the testing result of [9] to extract sufficient information about the regularity of the unknown signal in the local servers, which we then use in the subsequent estimation procedure. This approach depends crucially on the fact that we consider the L_2 -risk. For the L_∞ -risk there is no difference between testing and estimation rates and this approach breaks down. In fact we prove that for the L_∞ -norm, adaptive estimation at the optimal rate under minimal communication is never possible!

The impossibility results all derive from the fact that in the local servers, sample size is too small to extract sufficient information about the regularity of a general signal. This suggests that if we restrict to a class of “nice” signals for which we do have access to such smoothness information from limited data, we should be able to obtain optimal rates and minimal communication adaptively. We prove that this is indeed the case if we consider the class of self-similar functions, first introduced in [5] in the context of nonparametric confidence regions, where closely related issues occur. See also for instance [6, 7, 13, 20, 22, 23].

The remainder of the paper is organized as follows. In the next section we first present the minimax lower bounds under communication restrictions

that show that if we want to attain the optimal rate $n^{-s/(1+2s)}$ for estimating s -smooth functions in the distributed setting, we need to transmit at least order $n^{1/(1+2s)}$ bits from the local machines to the central one. For completeness we show that it is easy to obtain the optimal rate under minimal communication if s is known. We also prove that if it is assumed that s belong to some known range (s_0, s_{\max}) , then adaptation to smoothness over that range is possible while transmitting order $n^{1/(1+2s_0)}$ bits. After this we present our main results. Theorems 2.4 and 2.5 and Corollary 2.6 assert that whether simultaneous adaptation over a range of regularities and minimal communication is possible for the L_2 risk, depends on the relation between the range of regularities and the number of local machines. Theorem 2.7 shows that simultaneous adaptation and minimal communication is not possible when L_∞ risk is considered. Finally, Theorem 2.8 asserts that it is possible under a self-similarity assumption. Proofs and auxiliary results are deferred to Section 3–5 and the appendices.

1.1. *Notations.* For two positive sequences a_n, b_n we use the notation $a_n \lesssim b_n$ if there exists an universal positive constant C such that $a_n \leq Cb_n$. Along the lines $a_n \asymp b_n$ denotes that $a_n \lesssim b_n$ and $b_n \lesssim a_n$ hold simultaneously. In the proofs we use the notation C and c for universal constants which value can differ from line to line and denote by $\#S$ or $|S|$ the cardinality of the finite set S . Furthermore, let $l(Y)$ denote the length of a binary string Y , and $\log x$ denote the logarithm with base 2, i.e. $\log_2 x$.

2. Main results. In our analysis we work with the distributed Gaussian white noise model also considered for instance in [24], [28], and which can be seen as an idealized version of the nonparametric regression model. Our results can in principle be derived in the regression context as well, similar as we did in [25]. However, since the additional technical issues would seriously lengthen the already long paper and would add no fundamental insight, we formulate everything in the signal in white noise setting in this paper.

We assume that we have m machines and in the i th machine we observe the random function $X_t^{(i)}$ given by the stochastic differential equation

$$(2.1) \quad dX_t^{(i)} = f_0(t)dt + \sqrt{\frac{m}{n}}dW_t^{(i)}, \quad t \in [0, 1], \quad i = 1, 2, \dots, m,$$

where $W^{(1)}, \dots, W^{(m)}$ are independent standard Wiener processes and f_0 is the unknown function of interest. It is common to assume that the unknown true function f_0 belongs to some regularity class. We work in our

analysis with Besov smoothness classes, more specifically we assume that $f_0 \in B_{2,\infty}^s(L)$ or $f_0 \in B_{\infty,\infty}^s(L)$, see Appendix B for a rigorous introduction of these smoothness classes. The first class is of Sobolev type, while the second one is Hölder type.

Parallel to each other, the local machines carry out a local statistical procedure and transmit the results to the central machine, which provides the final inference about the functional parameter of interest f_0 by somehow aggregating the local outcomes. There are however constraints on the communication between the local and global machines. Local machine i is allowed to send at most $B^{(i)}$ bits (on average) to the central machine. The central machine will then collect the transmitted bits from the local computers and combine them to a global, aggregated answer. More formally, for a target function class \mathcal{F} , we write $\hat{f} \in \mathcal{F}_{\text{dist}}(B^{(1)}, \dots, B^{(m)}; \mathcal{F})$ if \hat{f}_n is a measurable function of messages of length $\hat{B}^{(i)}$ sent from the local machines and for every $f_0 \in \mathcal{F}$ it holds that $E_{f_0} \hat{B}^{(i)} \leq B^{(i)}$ for every i . For simplicity, we will focus on the case $B^{(1)} = \dots = B^{(m)}$ that the communication restriction is the same for every local machine.

2.1. Distributed minimax rates. As a first step we give lower bounds for the minimax risk for the L_2 -norm. We assume that in each local machine we have the same communication budget, i.e. $B^{(1)} = \dots = B^{(m)} = B$. Then the corresponding minimax L_2 estimation rates are the following, see also [25, 28].

THEOREM 2.1. *Let $s, L > 0$.*

- If $B \geq n^{1/(1+2s)}/\log m$:

$$\inf_{\hat{f} \in \mathcal{F}_{\text{dist}}(B, \dots, B; B_{2,\infty}^s(L))} \sup_{f_0 \in B_{2,\infty}^s(L)} E_{f_0} \|\hat{f} - f_0\|_2^2 \gtrsim n^{-\frac{2s}{1+2s}}.$$

- If $(n \log(n)/m^{2+2s})^{1/(1+2s)} \leq B \leq n^{1/(1+2s)}/\log m$:

$$\inf_{\hat{f} \in \mathcal{F}_{\text{dist}}(B, \dots, B; B_{2,\infty}^s(L))} \sup_{f_0 \in B_{2,\infty}^s(L)} E_{f_0} \|\hat{f} - f_0\|_2^2 \gtrsim \left(\frac{B \log n}{n^{1/(1+2s)}} \right)^{-\frac{s}{1+s}} n^{-\frac{2s}{1+2s}}.$$

- If $B \leq (n \log(n)/m^{2+2s})^{1/(1+2s)}$:

$$\inf_{\hat{f} \in \mathcal{F}_{\text{dist}}(B, \dots, B)} \sup_{f_0 \in B_{2,\infty}^s(L)} E_{f_0} \|\hat{f} - f_0\|_2^2 \gtrsim \left(\frac{n}{m \log n} \right)^{-\frac{2s}{1+2s}}.$$

PROOF. See Section A.1

□

The result shows that it is indeed only possible to obtain the optimal rate $n^{-s/(1+2s)}$ over Besov balls of regularity s if, up to a logarithmic factor, every machine is allowed to transmit order $n^{1/(1+2s)}$ bits to the central machine. The following theorem shows that this result is indeed sharp (up to log-factors), i.e. if order $n^{1/(1+2s)}$ bits are allowed, then the optimal rate can indeed be achieved with some procedure. In fact, the theorem considers the first two cases of the preceding one, i.e. $(n \log(n)/m^{2+2s})^{1/(1+2s)} \leq B$. The third case is not interesting since in that case distributed methods do not perform better than any standard technique applied on a single, local server.

THEOREM 2.2. *Let $s, L > 0$, $m \leq n$. Then there exists a distributed estimator $\hat{f} \in \mathcal{F}_{\text{dist}}(B, \dots, B; B_{2,\infty}^s(L))$ satisfying:*

- for $B \geq n^{1/(1+2s)}/\log n$:

$$\sup_{f_0 \in B_{2,\infty}^s(L)} \mathbb{E}_{f_0} \|\hat{f} - f_0\|_2^2 \lesssim n^{-\frac{2s}{1+2s}} \vee (B/\log n)^{-2s},$$

- for $(n \log(n)/m^{2+2s})^{1/(1+2s)} \vee \log n \leq B \leq n^{1/(1+2s)}/\log n$:

$$\sup_{f_0 \in B_{2,\infty}^s(L)} \mathbb{E}_{f_0} \|\hat{f} - f_0\|_2^2 \lesssim M_n \left(\frac{n^{1/(1+2s)}}{B \log n} \right)^{\frac{2s}{2+2s}} n^{-\frac{2s}{1+2s}},$$

with $M_n = (\log n)^{2s}$.

PROOF. See Section [A.2](#)

□

One can also derive similar matching lower and upper bounds for the L_∞ -norm for $f_0 \in B_{\infty,\infty}^s(L)$ in case of the Gaussian white noise model, as in [\[25\]](#) where the nonparametric regression model was considered. Since our focus in this paper is not on deriving minimax rates, we have deferred this result to Section [A.3](#) in the appendix.

2.2. Simultaneous adaptation to smoothness and minimal communication.

In view of the preceding two theorems we can conclude that when the goal is to estimate s -smooth functions at the rate $n^{-s/(1+2s)}$, the optimal, minimal number of transmitted bits is $n^{1/(1+2s)}$ (up to a logarithmic factor). Transmitting less bits will result in (polynomially) sub-optimal convergence rate for any distributed method, while by transmitting at least the optimal amount of bits one can construct distributed estimators reaching the convergence rate of non-distributed techniques.

2.2.1. *Adaptation in L_2 .* The procedure \hat{f} exhibited in (the proof of) Theorem 2.2 has the desirable property that if $f_0 \in B_{2,\infty}^s(L)$, then, up to log-factors, using the minimal communication it achieves the optimal rate $n^{-s/(1+2s)}$. This procedure is, however, not adaptive: it uses the knowledge of the regularity level s of the unknown function. In this section we investigate the more relevant question under which conditions can we simultaneously achieve the optimal convergence rate and minimal communication without using any information about the smoothness of the truth.

If we are willing to assume that the true regularity is at least $s \geq s_0$ for some known $s_0 > 0$ and are in addition willing to allow order $n^{1/(1+2s_0)}$ bits to be communicated between the local and the central machines, then it is straightforward to achieve adaptation to smoothness.

PROPOSITION 2.3. *Let $s_{\max} > s_0 > 0$, $L > 0$, $m \leq n$, and $B_0 = n^{1/(1+2s_0)} \log n$. Then there exists a distributed estimator $\hat{f} \in \mathcal{F}_{\text{dist}}(B_0, \dots, B_0; B_{2,\infty}^s(L))$ for all $s \in [s_0, s_{\max}]$ satisfying that*

$$\sup_{s \in [s_0, s_{\max}]} \sup_{f_0 \in B_{2,\infty}^s(L)} n^{\frac{2s}{1+2s}} \mathbb{E}_{f_0} \|\hat{f} - f_0\|_2^2 = O(1).$$

PROOF. See Section 3.1 □

The problem with the above method is that it always transmits a multiple of $n^{1/(1+2s_0)} \log n$ bits, which can be substantially more than the optimal $n^{1/(1+2s)}$ if the true smoothness s happens to be larger than the assumed lower bound s_0 . The question naturally arises: is it possible to achieve adaptation to smoothness while at the same time automatically transmitting the minimal amount of bits?

We show that in case of the L_2 -norm one can only adapt up to a limited range of smoothness levels (depending on the number of local machines), and beyond that one will achieve sub-optimal rates (where the rate is sub-optimal by a polynomial factor).

THEOREM 2.4. *Suppose that $m = n^p$ for some $p \in (0, 1/2)$. Then for any regularity parameters $s_2 > s_1 > 1/(4p) - 1/2$ there does not exist a distributed method which adapts to the number of transmitted bits and at the same time achieves the minimax risk as well, i.e. it is not possible to simultaneously obtain a distributed method with $\hat{B}^{(i)} \leq n^{1/(1+2s_1)+\varepsilon_1} \log n$*

and for $l = 1, 2$ that

$$(2.2) \quad \sup_{i \in \{1, \dots, m\}} \sup_{f_0 \in B_{2, \infty}^{s_l}(L)} E_{f_0}^{(i)} \hat{B}^{(i)} \lesssim n^{\frac{1}{1+2s_l} + \varepsilon_1} \quad \text{and}$$

$$(2.3) \quad \sup_{f_0 \in B_{2, \infty}^{s_l}(L)} E_{f_0} \|\hat{f} - f_0\|_2^2 \lesssim n^{-\frac{2s_l}{1+2s_l} + \varepsilon_2},$$

for some small enough constants $\varepsilon_1, \varepsilon_2 > 0$ depending only on s_1, s_2 and p .

PROOF. See Section 3.2. \square

The above theorem tells us that considering even just two regularity classes (with regularities above some threshold level) there doesn't exist any distributed method, which transmits the optimal amount of bits multiplied by some (small) polynomial factor and reaches the minimax rate in both smoothness classes up to a (small) polynomial factor. The above negative results deliver a strong message, as the question of non-existence can not be resolved by allowing extra logarithmic factors, but is on the polynomial level.

The phenomenon behind the negative result is that in case of many local machines (large m) it is getting more difficult to test locally between the regularity classes (as the local "sample size" decreases in m) and also the "local regularity" of the function which one can judge at noise level m/n might be completely different than the "global regularity" of the truth which can be judged at a smaller noise level $1/n$.

Although full adaptation is not possible, it turns out that on a limited range of regularity levels it is possible to construct adaptive methods. Below we derive the complement of the preceding result and show that for regularities below the threshold $1/(4p) - 1/2$ we can adapt to smoothness and transmit the minimal number of bits at the same time.

THEOREM 2.5. *For arbitrary $0 < s_1 < s_2 \leq 1/(4p) - 1/2$ and $m \geq 5 \log n$ there exists a distributed estimator \hat{f} with number of transmitted bits $(\hat{B}^{(1)}, \dots, \hat{B}^{(m)})$, such that $\hat{B}^{(i)} \leq n^{\frac{1}{1+2s_1}} \log n$, $i = 1, \dots, m$, and for all $s \in \{s_1, s_2\}$*

$$\begin{aligned} \max_{i \in \{1, \dots, m\}} \sup_{f_0 \in B_{2, \infty}^s(L)} E_{f_0}^{(i)} \hat{B}^{(i)} &\lesssim C_2 n^{\frac{1}{1+2s}} \log n \quad \text{and} \\ \sup_{f_0 \in B_{2, \infty}^s(L)} E_{f_0} \|\hat{f} - f_0\|_2^2 &\lesssim n^{-\frac{2s}{1+2s}}. \end{aligned}$$

PROOF. See Section 3.3. \square

The proposed procedure has two stages. First we “estimate” the smoothness of the underlying functional parameter of interest in every local machine parallel to each other and based on that transmit the right amount of information to the central machine. In the second stage we aggregate the locally transmitted information and provide a “global” adaptive estimator. The difficulty, as also discussed above, arises from the higher noise level in the local problems which results in less accurate tests between the smoothness classes. The existence of an estimator which can achieve adaptation (in a limited range of smoothness classes) is a consequence of the difference between the nonparametric testing and estimation rates in the case of the L_2 -norm, see for instance [12, 15]. Since one can test between smoothness classes with a faster rate than the corresponding estimation rate, it can compensate (up to some extent) for the higher local noise level m/n .

The preceding result can be extended to a scale of smoothness classes as well.

COROLLARY 2.6. *Assume that $m = n^p$ for some $0 < p \leq 1/2$, then for arbitrary $0 < s_1 < s_2 < 1/(4p) - 1/2$ and $m \geq 5 \log n$ there exists a distributed estimator \hat{f} transmitting $\hat{B}^{(i)}$ bits in the local machines $i = 1, \dots, m$ satisfying that $\hat{B}^{(i)} \leq n^{\frac{1}{1+2s_1}} \log n$ and*

$$\begin{aligned} \max_{i=1, \dots, m} \sup_{s \in [s_1, s_2]} \sup_{f_0 \in B_{2, \infty}^s(L)} \frac{E_{f_0}^{(i)} \hat{B}^{(i)}}{n^{1/(1+2s)} \log n} &\lesssim 1 \quad \text{and} \\ \sup_{s \in [s_1, s_2]} \sup_{f_0 \in B_{2, \infty}^s(L)} \frac{E_{f_0} \|\hat{f} - f_0\|_2^2}{n^{-2s/(1+2s)}} &\lesssim 1. \end{aligned}$$

PROOF. See Section 4. \square

The idea of the proof of this corollary is to introduce a grid of regularities in the interval $[s_1, s_2]$ and test between which two grid points the true regularity lies. Then one can apply the distributed method introduced in the proof of Theorem 2.5 to derive the stated results.

2.2.2. Adaptation in L_∞ . Next we deal with the L_∞ -norm case. Here we show that in contrast to the L_2 -case, adaptation is not possible even on a limited range of smoothness classes. The reason behind it is that in this case the minimax testing and estimation rates are the same and hence there is no room left to compensate for the higher local noise level.

THEOREM 2.7. *Take any $0 < s_1 < s_2$ and assume that $m = n^p$, with $p \in (0, 1/2)$. Then there does not exist a distributed estimator \hat{f} with transmitted bits $\hat{B}^{(i)} \leq n^{\frac{1}{1+2s_1} + \varepsilon_1}$, $i = 1, \dots, m$, satisfying that for $\ell = 1, 2$*

$$(2.4) \quad \max_{i=1, \dots, m} \sup_{f_0 \in B_{\infty, \infty}^{s_\ell}(L)} E_{f_0}^{(i)} \hat{B}^{(i)} \lesssim n^{\frac{1}{1+2s_\ell} + \varepsilon_1}, \quad \text{and}$$

$$(2.5) \quad \sup_{f_0 \in B_{\infty, \infty}^{s_\ell}(L)} E_{f_0} \|\hat{f} - f_0\|_\infty \lesssim (n/\log n)^{-\frac{s_\ell}{1+2s_\ell} + \varepsilon_2},$$

for some sufficiently small $\varepsilon_1, \varepsilon_2 > 0$.

PROOF. See Section 4.1

□

Next we introduce some additional restriction on the true function of interest under which adaptation is possible in the distributed setting. To do so we consider the so-called self-similarity assumption, where loosely speaking we assume that the true function has similar smoothness at every resolution level. This will allow us to estimate the regularity s of the functional parameter of interest and therefore transmit the right amount of bits from the local machines to the central one.

We first introduce necessary notation. Let ψ_{jk} be the wavelet basis functions described in Appendix B. For $f \in L_2[0, 1]$ and natural numbers $j_1 \leq j_2$ we define

$$f_{[j_1, j_2]} = \sum_{j=j_1}^{j_2} \sum_{k=1}^{2^j} f_{jk} \psi_{jk}.$$

Then following [5] we say that the function $f \in B_{\infty, \infty}^s(L)$ belongs to the self-similar class $S_\infty^s(L, \varepsilon, j_0, \rho)$ if,

$$(2.6) \quad \|f_{[j, \rho j]}\|_{B_{\infty, \infty}^s} \geq \varepsilon L, \quad \text{for } j \geq j_0 \text{ and } \rho > 1.$$

The self-similarity property was introduced (amongst other places) in the context of adaptive confidence bands. It was shown that under self-similarity one can construct adaptive L_∞ confidence bands whose size also adapts to the level of regularity, see for instance [5, 13, 19]. The underlying idea is the same as here. Under this assumption one can provide a consistent estimator for the smoothness and based on that construct the band corresponding the function class.

The following theorem shows that under the self-similarity assumption there exists a distributed method which adapts to regularity and at the

same time transmits the minimal amount of bits (again up to logarithmic factors).

THEOREM 2.8. *Consider the distributed Gaussian white noise model with $m \leq n^\delta$, for some $\delta \in (0, 1)$ and assume that $f_0 \in B_{\infty, \infty}^s(L)$ for some $s \in [s_1, s_2]$ (where $0 < s_1 < s_2$ are arbitrary). Then there exists a distributed method such that the number of transmitted bits satisfies $\hat{B}^{(i)} \leq (n/\log n)^{1/(1+2s_1)} \log n$ and*

$$\begin{aligned} \max_{i \in \{1, \dots, m\}} \sup_{s \in [s_1, s_2]} \sup_{f_0 \in S_{\infty}^s(L, \varepsilon, j_0, \rho)} \frac{E_{f_0}^{(i)} \hat{B}^{(i)}}{n^{\frac{1}{1+2s}} (\log n)^{\frac{2s}{1+2s}}} &\lesssim 1 \quad \text{and} \\ \sup_{s \in [s_1, s_2]} \sup_{f_0 \in S_{\infty}^s(L, \varepsilon, j_0, \rho)} \frac{E_{f_0} \|\hat{f} - f_0\|_{\infty}}{(n/\log n)^{-\frac{s}{1+2s}}} &\lesssim 1. \end{aligned}$$

PROOF. See Section 4.2 □

3. Proofs for the adaptation results. In the proofs we will work with the wavelet decomposition of the functional parameter f_0 . In our analysis we consider the Daubechie wavelets $\psi_{jk}(t)$ for $j = 0, 1, \dots, k = 1, \dots, 2^j$, $t \in [0, 1]$ and denote by $f_{0,jk} = \int_0^1 \psi_{jk}(t) f_0(t) dt$ the corresponding wavelet coefficients. In Section B we have collected a few properties of Daubechie wavelets which we will apply throughout the proofs.

We note that following from the orthonormality of the Daubechie wavelets we have that the Gaussian white noise model can be written in the sequence representation

$$(3.1) \quad X_{jk}^{(i)} = f_{0,jk} + \sqrt{\frac{m}{n}} Z_{jk}^{(i)}, \quad j = 0, 1, 2, \dots; k = 1, \dots, 2^j; i = 1, \dots, m,$$

where $X_{jk}^{(i)}$, $j = 0, 1, \dots, k = 1, \dots, 2^j$ are the noisy observations $X_{jk}^{(i)} = \int_0^1 \psi_{jk}(t) dX^{(i)}(t)$ and $Z_{jk}^{(i)}$ are iid standard normal random variables.

3.1. *Proof of Proposition 2.3.* Consider the sequence representation of the distributed Gaussian white noise model, see (3.1) using at least s_{\max} regular Daubechie wavelets. Then by transmitting $n^{1/(1+2s_0)} \log n$ bits (the first $n^{1/(1+2s_0)}$ elements of the sequence representation of the model up to the first $0.5 \log n$ digits in the binary representation of the number, see Algorithm 1) to the central machine and averaging the transmitted local data we arrive to the global sequence model

$$Y_{jk} = f_{0,jk} + \sqrt{\frac{1}{n}} Z_{jk} + \varepsilon_{jk}, \quad j = 0, \dots, \lfloor \frac{\log n}{1+2s_0} \rfloor, k = 1, \dots, 2^j,$$

where Z_{jk} are iid standard Gaussian random variables and $|\varepsilon_{jk}| \leq n^{-1/2}$ are random variables representing the error term arising from transmitting only the first $0.5 \log n$ digits of the observations. These error terms are in fact negligible. Then using arbitrary adaptation technique (for instance Lepski's method [18]) one can construct an estimator \hat{f} achieving the minimax risk for every $f \in B_{2,\infty}^s(L)$, $s_0 \leq s \leq s_{\max}$.

3.2. Proof of Theorem 2.4. We argue by contradiction. We assume that the inequalities (2.2) and (2.3) hold. Then we construct a finite but large enough set $\mathcal{F}_0 \subset B_{2,\infty}^{s_1}(L)$ such that there does not exist a consistent test between the elements of the set and the zero function, which clearly belongs to the smoother class $B_{2,\infty}^{s_2}(L)$. Using this non-existence result we arrive to contradiction with our assumptions.

As a first step we construct the set \mathcal{F}_0 . Let us introduce the following notations

$$(3.2) \quad \begin{aligned} \tilde{\delta}_n &= \bar{\delta}_n \wedge (n/m)^{-\frac{1+2s_1}{1/2+2s_1}} n^{-\varepsilon_3}, \quad \text{with} \\ \bar{\delta}_n &= \min \left\{ \frac{m}{n \log m}, \frac{1}{n[\bar{\delta}_n^{1/(1+2s_1)} \beta_n \wedge 1] \log m} \right\}, \\ \beta_n &= (\Gamma_n \vee n^{\varepsilon_1 - \frac{(s_1+1/4)\varepsilon_3}{1+2s_1}} n^{\frac{1}{1+2s_1}}) \log n \quad \text{and} \quad \Gamma_n = n^{\frac{1/2}{1+2s_1} + \frac{1/2}{1+2s_2} + \varepsilon_1}, \end{aligned}$$

and constants $\varepsilon_3 \in (0, \frac{p(1+2s_1)-1/2}{1/2+2s_1})$, where $p(1+2s_1) - 1/2 > 0$ follows from the assumption $s_1 > 1/(4p) - 1/2$, and $\varepsilon_1 \in (0, \frac{s_2-s_1}{(1+2s_1)(1+2s_2)} \wedge \frac{(s_1+1/4)\varepsilon_3}{1+2s_1})$. Note that $\beta_n \leq n^{\frac{1}{1+2s_1} - \varepsilon_4} \log n$, with $\varepsilon_4 = (\frac{s_2-s_1}{(1+2s_1)(1+2s_2)} - \varepsilon_1) \wedge (\frac{(s_1+1/4)\varepsilon_3}{1+2s_1} - \varepsilon_1) > 0$. In view of the definition of $\bar{\delta}_n$ this implies that $\bar{\delta}_n \geq n^{\varepsilon_4 \frac{1+2s_1}{2+2s_1}} / (n \log n) \gg n^{-1+\varepsilon_4/2}$.

Furthermore,

$$(n/m)^{-\frac{1+2s_1}{1/2+2s_1}} n^{-\varepsilon_3} = n^{-(1-p)\frac{1+2s_1}{1/2+2s_1} - \varepsilon_3} = n^{-1} n^{\frac{p(1+2s_1)-1/2}{1/2+2s_1} - \varepsilon_3}.$$

Therefore we can conclude that for large enough n

$$(3.3) \quad \tilde{\delta}_n \geq n^{-1+\varepsilon_5} \quad \text{with} \quad \varepsilon_5 = (\varepsilon_4/2) \wedge \left(\frac{p(1+2s_1)-1/2}{1/2+2s_1} - \varepsilon_3 \right) > 0.$$

The elements $f \in \mathcal{F}_0$ are then defined with the wavelet coefficients as

$$(3.4) \quad f_{jk} = \begin{cases} \beta_k \tilde{\delta}_n^{1/2}, & \text{if } j = j_n := \lfloor \frac{\log \tilde{\delta}_n^{-1}}{1+2s_1} \rfloor, k = 1, \dots, 2^{j_n}, \\ 0, & \text{else,} \end{cases}$$

where $\beta_k \in \{-1, 1\}$. It is easy to check that $\mathcal{F}_0 \subset B_{2,\infty}^{s_1}(1)$ and besides, for every $f \in \mathcal{F}_0$, in view of the definition of $\tilde{\delta}_n$,

$$\|0 - f\|_2^2 = \sum_{j=0}^{\infty} \sum_{k=1}^{2^j} f_{jk}^2 = 2^{jn} \tilde{\delta}_n \leq \tilde{\delta}_n^{\frac{2s_1}{1+2s_1}} = o((n/m)^{-\frac{2s_1}{1/2+2s_1}}).$$

Next we take the average likelihood ratio over the class \mathcal{F}_0

$$Z = \frac{1}{|\mathcal{F}_0|} \sum_{f \in \mathcal{F}_0} \frac{dP_f^{(i)}}{dP_0^{(i)}}, \quad \text{where } |\mathcal{F}_0| = 2^{jn}.$$

In view of (6.23) of [12]

(3.5)

$$\inf_{\Psi^{(i)}} \{E_0^{(i)} \Psi^{(i)} + \frac{1}{|\mathcal{F}_0|} \sum_{f \in \mathcal{F}_0} E_f^{(i)} (1 - \Psi^{(i)})\} \geq (1 - \eta_n) \left(1 - \frac{\sqrt{E_0^{(i)} (Z - 1)^2}}{\eta_n}\right),$$

for every $\eta_n \in (0, 1)$, where the infimum is taken over all local tests in the local problems. Furthermore one can show by following the steps in the proof of Theorem 6.2.11 c) on pages 493-494 of [12] (with $\gamma'_{n/m} = c_0^2(n/m)\tilde{\delta}_n$ and $\gamma_{n/m} = (n/m)\tilde{\delta}_n^{\frac{1/2+2s_1}{1+2s_1}} \leq n^{-\frac{1/2+2s_1}{1+2s_1}\varepsilon_3}$) that

$$E_0^{(i)} (Z - 1)^2 \leq \exp\{c' \gamma_{n/m}^2\} - 1 \lesssim \gamma_{n/m}^2 \lesssim n^{-\frac{1+4s_1}{1+2s_1}\varepsilon_3}.$$

By choosing $\eta_n = n^{-\frac{(1/4+s_1)\varepsilon_3}{1+2s_1}}$ we get that

$$(3.6) \quad \inf_{\Psi^{(i)}} \{E_0^{(i)} \Psi^{(i)} + \frac{1}{|\mathcal{F}_0|} \sum_{f \in \mathcal{F}_0} E_f^{(i)} (1 - \Psi^{(i)})\} \geq (1 - C\eta_n)^2,$$

for some large enough constant $C > 0$, concluding the proof of the non-existence of consistent tests between \mathcal{F}_0 and the zero function.

Next we show that (3.6) contradicts our assumptions. Let us define the test

$$\Psi^{(i)} = 1_{\hat{B}^{(i)} \geq \Gamma_n}.$$

First note that following from Markov's inequality and assumption (2.2)

$$E_0^{(i)} \Psi^{(i)} = P_0^{(i)}(\hat{B}^{(i)} \geq \Gamma_n) \leq E_0^{(i)}(\hat{B}^{(i)})/\Gamma_n \leq n^{\frac{1/2}{1+2s_2} - \frac{1/2}{1+2s_1}} = o(1).$$

Therefore in view of (3.6) we have that

$$\begin{aligned} \frac{1}{|\mathcal{F}_0|} \sum_{f \in \mathcal{F}_0} P_f^{(i)}(\hat{B}^{(i)} < \Gamma_n) &= \frac{1}{|\mathcal{F}_0|} \sum_{f \in \mathcal{F}_0} E_f^{(i)}(1 - \Psi^{(i)}) \\ &\geq (1 - C\eta_n)^2 - n^{\frac{1/2}{1+2s_2} - \frac{1/2}{1+2s_1}}. \end{aligned}$$

As a consequence and in view of assumption $\hat{B}^{(i)} \leq Cn^{\frac{1}{1+2s_1} + \varepsilon_1} \log n$

$$\frac{1}{|\mathcal{F}_0|} \sum_{f \in \mathcal{F}_0} E_f^{(i)} \hat{B}^{(i)} \lesssim \Gamma_n + n^{\frac{1}{1+2s_1} + \varepsilon_1} (\log n) (\eta_n + n^{\frac{1/2}{1+2s_2} - \frac{1/2}{1+2s_1}}) \lesssim \beta_n.$$

This means that the expected number (with respect to the joint distribution of the variables F and P_f , $f \in \mathcal{F}_0$) of transmitted bits on the class \mathcal{F}_0 is bounded from above by a multiple of β_n . So the distributed estimator satisfies assertion (A.7) in the proof of Theorem A.1 with $B^{(i)}$ replaced by $C\beta_n$. Hence in view of the minimax lower bound derived in assertion (A.9) and the definition of $\tilde{\delta}_n$ (with $B^{(i)}$ replaced by β_n in the definition of δ_n in the proof of Theorem A.1)

$$\sup_{f_0 \in \mathcal{F}_0} E_{f_0} \|\hat{f} - f_0\|_2^2 \gtrsim \tilde{\delta}_n^{\frac{2s_1}{1+2s_1}} \gg n^{-\frac{2s_1}{1+2s_1} + \varepsilon_2},$$

with $\varepsilon_2 = 2\varepsilon_5 s_1 / (1 + 2s_1)$, where the last inequality follows from (3.3). This contradicts assumption (2.3), finishing the proof of our statement.

3.3. Proof of Theorem 2.5. In our proof we work with the equivalent sequence representation of the model (3.1). As a first step we split the data in all of the local models $i \in \{1, \dots, m\}$ into two subsets $X_{jk}^{(i,1)}, X_{jk}^{(i,2)}$ for $j = 0, 1, 2, \dots, k = 1, \dots, 2^j$, such that they are pairwise independent and their variance is $2m/n$ (this can be done by adding and subtracting $\tilde{Z}_{jk}^{(i)} \stackrel{iid}{\sim} N(0, m/n)$ from $X_{jk}^{(i)}$). Let us then denote by $P_{X^{(i,1)}}$ and $P_{X^{(i,2)}}$ the distribution of the first and second subset of observations, respectively, and by $P_{X^{(i,2)}|X^{(i,1)}}$ the conditional distribution of the second subset given the first. The corresponding expected values are denoted by $E_{X^{(i,1)}}$, $E_{X^{(i,2)}}$, and $E_{X^{(i,2)}|X^{(i,1)}}$, respectively. Finally let us introduce the notations $X_l = (X^{(1,l)}, \dots, X^{(m,l)})$, $l = 1, 2$ and denote by P_{X_l} and E_{X_l} the corresponding probability distributions and expected values.

Next note that it was shown in [9] that there exists a consistent composite test between the classes $B_{2,\infty}^{s_2}(L)$ and $B_{2,\infty}^{s_1}(L)$ in the local problem using the first subset of observations $X^{(i,1)}$ if they are at least $(n/m)^{-s_1/(1/2+2s_1)}$

separated. The test proposed in Section 3 of [9] takes the form (in the local machines using the first subset of observations $X^{(i,1)}$)

$$(3.7) \quad \Psi_{n/m}^{(i)} = \Psi_{n/m}^{(i)}(\alpha, s_1, s_2) = 1 - \prod_{0 \leq l \leq \lfloor \frac{\log(n/(2m))}{2s_1+1/2} \rfloor} 1_{\{T_{n/m}^{(i)}(l) \leq t_{n/m}(l, s_2, \alpha)\}},$$

where

$$\begin{aligned} t_{n/m}(l, s_2, \alpha) &= \frac{L^2}{2^{2ls_2}} + \frac{L}{2^{ls_2}} \tau_l + \frac{\tau_l^2}{4}, \\ \tau_l &= 24 \sqrt{\frac{z_0}{\alpha}} \frac{2^{l + \lfloor \frac{\log(n/(2m))}{1/2+2s_2} \rfloor}}{\sqrt{n/(2m)}}, \quad \text{for } l > 0, \\ \tau_0 &= 24 \sqrt{\frac{z_0}{\alpha}} \frac{1}{\sqrt{n/(2m)}}, \\ T_{n/m}^{(i)}(l) &= \|\Pi_l \hat{f}_{n/m}^{(i)}\|_2^2 - m2^{l+1}/n, \quad \text{for } l > 0, \\ T_{n/m}^{(i)}(0) &= \|\Pi_0 \hat{f}_{n/m}^{(i)}\|_2^2 - 2mz_0/n, \end{aligned}$$

where $\Pi_l f$ denotes the projection of the function f to the resolution level l , i.e. $\Pi_l f = \sum_{k=1}^{2^j} f_{lk} \psi_{l,k}$, see (3.1) and (3.2) of [9], $\hat{f}_{n/m}^{(i)}$ is the wavelet estimate of f in the i th local machine using observations $X^{(i,1)}$, see the top of page 6 of [9], and $z_0 = 1$ (since for notational convenience we take $J_0 = 0$, see Section B, we have $z_0 = 2^{J_0} = 1$). Let us introduce the notation $R_\alpha^{s_1}(L) = \{f \in B_{2,\infty}^{s_1}(L) : \|f - B_{2,\infty}^{s_2}(L)\|_2 \geq \tilde{C}_\alpha(n/m)^{-\frac{s_1}{1/2+2s_1}}\}$.

In view of Lemma 5.4 we have for all $\alpha \in (0, 1)$ and $0 < m \leq n$ that

$$(3.8) \quad \sup_{f \in B_{2,\infty}^{s_2}(L)} E_{X^{(i,1)}} \Psi_{n/m}^{(i)} + \sup_{f \in R_\alpha^{s_1}(L)} E_{X^{(i,1)}} (1 - \Psi_{n/m}^{(i)}) \leq ce^{-0.5/\sqrt{\alpha}},$$

with $\tilde{C}_\alpha = 24 \left(\frac{2^{s_1} L}{\sqrt{1-2^{-2s_1}}} + 19 \right) 2^{\frac{s_1}{1+2s_1}} / \sqrt{\alpha}$ and c not depending on α, n, m .

Let $M_n = n^{\frac{2s_1(1/2-p(1+2s_1))}{(1+2s_1)(1/2+2s_1)}}$ tending to infinity (where the positivity of the exponent follows from the assumption $s_1 < 1/(4p) - 1/2$). Then there exists a consistent test $\Psi_{n/m}^{(i)}$ (with $\alpha = M_n^{-1}$) in each local problem between the hypotheses

$$H_0 : f \in B_{2,\infty}^{s_2}(L) \quad \text{vs} \quad H_1 : f \in R_{M_n^{-1}}^{s_1}(L).$$

Using the test function above, we define the smoothness estimate as

$$\hat{s}_{n/m}^{(i)} = \begin{cases} s_2, & \text{if } \Psi_{n/m}^{(i)} = 0, \\ s_1, & \text{if } \Psi_{n/m}^{(i)} = 1. \end{cases}$$

In each local model we take the first $n^{1/(1+2\hat{s}_{n/m}^{(i)})}$ coefficients in the second subset of observations in the sequence representation, i.e. $X_{jk}^{(i,2)}$ with $2^j + k \leq n^{1/(1+2\hat{s}_{n/m}^{(i)})}$. Since these numbers might not have a finite binary representation we transmit their approximations $Y_{jk}^{(i)}$ following Algorithm 1. Note that in view of Lemma 5.2 (with $\mu = f_{0,jk}$) we have that $l(Y_{jk}^{(i)}) \leq \log n$ with approximation error $|\varepsilon_{jk}^{(i)}| = |X_{jk}^{(i,2)} - Y_{jk}^{(i)}| \leq n^{-1/2}$ on a set $\mathcal{E}_{jk}^{(i)}$ with $P_{X^{(i,2)}}((\mathcal{E}_{jk}^{(i)})^c) \leq e^{-c'n}$, for some $c' > 0$. Let us then introduce the notation

$$(3.9) \quad \mathcal{E} = \bigcap_{i=1}^m \bigcap_{j=0}^{\log n} \bigcap_{k=1}^{2^j} \mathcal{E}_{jk}^{(i)}$$

and note that $P_{X_2}(\mathcal{E}^c) \leq n^2 e^{-c'n} \lesssim e^{-cn}$, for any $0 < c < c'$. Hence the number of transmitted bits conditioned on the first subsample $X^{(i,1)}$ is bounded from above by $l(Y^{(i)}) \leq n^{1/(1+2\hat{s}_{n/m}^{(i)})} \log n$ almost surely.

Let us denote by \tilde{N} the median of the values $n^{1/(1+2\hat{s}_{n/m}^{(i)})}$, $i = 1, \dots, m$ and \hat{s} the corresponding regularity estimator. Then we construct our estimator \hat{f} as the average of the transmitted observations (for the first \tilde{N} coefficient), i.e.

$$\hat{f}_{n,jk} = \begin{cases} \frac{1}{|M_{jk}|} \sum_{i \in M_{jk}} Y_{jk}^{(i)}, & 2^j + k \leq \tilde{N}, \\ 0, & \text{for } 2^j + k > \tilde{N}, \end{cases}$$

where M_{jk} is the collection of local machines satisfying $2^j + k \leq n^{1/(1+2\hat{s}_{n/m}^{(i)})}$, i.e. the machines from which the local approximations $Y_{jk}^{(i)}$ are transmitted.

We show that this procedure achieves the minimax convergence rate and transmits the optimal amount of bits (up to a logarithmic factor). First note that $\hat{B}^{(i)} \lesssim n^{1/(1+2s_1)} \log n$ follows immediately by construction. Then recall that the test $\Psi_{n/m}^{(i)}$ is consistent, hence

$$\sup_{f \in B_{2,\infty}^{s_2}(L)} P_{X^{(i,1)}}(\hat{s}_{n/m}^{(i)} = s_1) \leq C e^{-M_n^{1/2}/2}$$

and

$$\begin{aligned} \sup_{f \in B_{2,\infty}^{s_2}(L)} E_{X^{(i,1)}, X^{(i,2)}} \hat{B}^{(i)} &\leq \sup_{f \in B_{2,\infty}^{s_2}(L)} E_{X^{(i,1)}} n^{1/(1+2\hat{s}_{n/m}^{(i)})} \log n \\ &\leq n^{1/(1+2s_2)} \log n + C e^{-M_n^{1/2}/2} n^{1/(1+2s_1)} \log n \\ &\leq (1 + o(1)) n^{1/(1+2s_2)} \log n, \end{aligned}$$

verifying that the number of transmitted bits is indeed optimal.

Next we provide optimal upper bounds for the risk. First let us consider the case $f \in B_{2,\infty}^{s_2}(L) \cup R_{M_n^{-1}}^{s_1}(L)$, where the estimator $\hat{s}_{n/m}^{(i)}$ is consistent, i.e. $\hat{s}_{n/m}^{(i)} = s_1$ for $f \in R_{M_n^{-1}}^{s_1}(L)$ and $\hat{s}_{n/m}^{(i)} = s_2$ for $f \in B_{2,\infty}^{s_2}(L)$, with $P_{X^{(i,1)}}$ -probability at least $1 - ce^{-M_n^{1/2}/2}$. Let us introduce the notation M for the number of machines in $\{1, \dots, m\}$, where the $\hat{s}_{n/m}^{(i)} \neq s_l$, $l = 2, 1$, for $f \in B_{2,\infty}^{s_2}(L)$ or $f \in R_{M_n^{-1}}^{s_1}(L)$, respectively. Note that M has a binomial distribution with parameters m and $p \leq ce^{-M_n^{1/2}/2}$. Then by Hoeffding's inequality

$$(3.10) \quad \sup_{f \in R_{M_n^{-1}}^{s_1}(L)} P_{X_1}(\tilde{N} \neq n^{\frac{1}{1+2s_1}}) + \sup_{f \in B_{2,\infty}^{s_2}(L)} P_{X_1}(\tilde{N} \neq n^{\frac{1}{1+2s_2}}) \leq P(M \geq m/2) < e^{-m/5}.$$

Then in view of the almost sure inequality $\tilde{N} \leq n^{1/(1+2s_1)}$ we have that

$$(3.11) \quad \begin{aligned} \sup_{f \in R_{M_n^{-1}}^{s_1}(L)} E_{X_1} \tilde{N}^{-2s_1} &= n^{-\frac{2s_1}{1+2s_2}} P_{X_1}(M \geq m/2) + n^{-\frac{2s_1}{1+2s_1}} P_{X_1}(M < m/2) \\ &\leq (1 + o(1)) n^{-\frac{2s_1}{1+2s_1}}, \\ \sup_{f \in B_{2,\infty}^{s_2}(L)} E_{X_1} \tilde{N} &= n^{1/(1+2s_2)} P_{X_1}(M < m/2) + n^{1/(1+2s_1)} P_{X_1}(M \geq m/2) \\ &\leq n^{1/(1+2s_2)} + n^{1/(1+2s_1)} e^{-m/5} \leq (1 + o(1)) n^{1/(1+2s_2)}, \end{aligned}$$

for $m \geq 5 \log n \geq \frac{10(s_2 - s_1)}{(2s_1 + 1)(2s_2 + 1)} \log n$.

Then similarly to the proof of Theorem 2.2 (with m replaced by $|M_{jk}|$) we get on the set \mathcal{E} (with $P_{X_2}(\mathcal{E}^c) \leq e^{-cn}$), that

$$\hat{f}_{n,jk} = f_{0,jk} + \frac{1}{\sqrt{n}} Z_{jk} + \varepsilon_{jk},$$

with $Z_{jk} \stackrel{iid}{\sim} N(0, \sqrt{2m/|M_{jk}|})$ and $|\varepsilon_{jk}| \leq n^{-1/2}$. Also note that $|\hat{f}_{n,k}| \leq \sqrt{n}$, since $|Y_{jk}^{(i)}| \leq \sqrt{n}$ for all i, j, k . Using this reformulation of the estimator and

the notation $\tilde{j}_n = \lfloor \log \tilde{N} \rfloor$ we get that

(3.12)

$$\begin{aligned}
\sup_{f \in B_{2,\infty}^{s_l}(L)} E_{X_2|X_1} \|\hat{f} - f_0\|_2^2 1_\mathcal{E} &\leq \sum_{j \geq \tilde{j}_n} \sum_{k=1}^{2^j} f_{0,jk}^2 + \sum_{j=0}^{\tilde{j}_n} \sum_{k=1}^{2^j} E\left(\frac{1}{\sqrt{n}} Z_{jk} + \varepsilon_{jk}\right)^2 1_\mathcal{E} \\
&\leq \sum_{j \geq \tilde{j}_n} 2^{-2js_l} \sup_{j \geq \tilde{j}_n} 2^{2js_l} \sum_{k=1}^{2^j} f_{0,jk}^2 + \sum_{j=0}^{\tilde{j}_n} \sum_{k=1}^{2^j} \frac{2E(Z_{jk}^2)}{n} + \frac{2}{n} \\
&\lesssim 2^{-2j_n s_l} + 2^{\tilde{j}_n} / n \asymp \tilde{N}^{-2s_l} + \tilde{N} / n, \\
\sup_{f \in B_{2,\infty}^{s_l}(L)} E_{X_2|X_1} \|\hat{f} - f_0\|_2^2 1_{\mathcal{E}^c} &\leq P_{X_2}(\mathcal{E}^c) 2^{\tilde{j}_n+1} (n + L^2) = o(n^{-1}),
\end{aligned}$$

for $l = 1, 2$. Therefore, in view of assertion (3.11)

$$\begin{aligned}
\sup_{f \in B_{2,\infty}^{s_2}(L)} E_{X_1, X_2} \|\hat{f} - f_0\|_2^2 &\lesssim \sup_{f \in B_{2,\infty}^{s_2}(L)} E_{X_1} (\tilde{N}^{-2s_2} + \tilde{N}/n) \lesssim n^{-2s_2/(1+2s_2)}, \\
\sup_{f \in R_{M_n}^{s_1}(L)} E_{X_1, X_2} \|\hat{f} - f_0\|_2^2 &\lesssim \sup_{f \in R_{M_n}^{s_1}(L)} E_{X_1} (\tilde{N}^{-2s_1} + \tilde{N}/n) \lesssim n^{-2s_1/(1+2s_1)}.
\end{aligned}$$

It remained to deal with the intermediate set, i.e. $f_0 \in B_{2,\infty}^{s_1}(L) \setminus R_{M_n}^{s_1}(L)$.

Our local estimator $\hat{s}_{n/m}^{(i)}$ will be either s_1 or s_2 , hence for each machine the amount of transmitted bits is bounded from above by $n^{1/(1+2\hat{s}_n^{(i)})} \log n \leq n^{1/(1+2s_1)} \log n$ $P_{X^{(i,2)}}$ -almost surely. Note that the median \tilde{N} also satisfies almost surely that $n^{1/(1+2s_1)} \geq \tilde{N} \geq n^{1/(1+2s_2)}$. Then, using the notation $f_{0,j \leq \tilde{j}_n} = \sum_{j=0}^{\tilde{j}_n} f_{0,jk} \psi_{jk}$, we get similarly as above, that

$$\begin{aligned}
E_{X_1, X_2} \|\hat{f} - f_{0,j \leq \tilde{j}_n}\|_2^2 &\leq E_{X_1} \sum_{j=0}^{\tilde{j}_n} \sum_{k=1}^{2^j} E_{X_2|X_1} \left(\frac{1}{\sqrt{n}} Z_{jk} + \varepsilon_{jk}\right)^2 + o(n^{-1}) \\
(3.13) \quad &\lesssim E_{X_1} \tilde{N} / n \leq n^{-\frac{2s_1}{1+2s_1}}.
\end{aligned}$$

To deal with the bias term let us denote by $\tilde{f} \in B_{2,\infty}^{s_2}(L)$ a function satisfying $\|f_0 - \tilde{f}\|_2^2 \lesssim \tilde{C}_{M_n}^{-1} (n/m)^{-2s_1/(1/2+2s_1)}$, then by recalling that

$$(n/m)^{1/(1/2+2s_1)} = n^{(1-p)/(1/2+2s_1)} = n^{\frac{1/2-p(1+2s_1)}{(1+2s_1)(1/2+2s_1)}} n^{1/(1+2s_1)},$$

we get that

(3.14)

$$\begin{aligned}
E_{X_1} \|f_{0,j \leq \tilde{j}_n} - f_0\|_2^2 &\leq E_{X_1} \sum_{j=\tilde{j}_n}^{\infty} \sum_{k=1}^{2^j} f_{0,jk}^2 \\
&\leq 2E_{X_1} \left(\sum_{j=\tilde{j}_n}^{\infty} \sum_{k=1}^{2^j} (f_{0,jk} - \tilde{f}_{jk})^2 + \sup_{j \geq \tilde{j}_n} (2^{2js_2} \sum_{k=1}^{2^j} \tilde{f}_{jk}^2) \sum_{j=\tilde{j}_n}^{\infty} 2^{-2js_2} \right) \\
&\lesssim \tilde{C}_{M_n^{-1}}^2 (n/m)^{-\frac{2s_1}{1/2+2s_1}} + E_{X_1} \tilde{N}^{-2s_2} \lesssim n^{-\frac{2s_1}{1+2s_1}},
\end{aligned}$$

where the last inequality follows from $\tilde{C}_{M_n^{-1}} \asymp n^{\frac{s_1(1/2-p(1+2s_1))}{(1+2s_1)(1/2+2s_1)}}$. Then by combining (3.13) and (3.14) we get that $E_{X_1, X_2} \|\hat{f} - f_0\|_2^2 \lesssim n^{-\frac{2s_1}{1+2s_1}}$, concluding the proof of the theorem.

4. Proof of Corollary 2.6. We adapt the method and proof of Theorem 2.5 to the collection of regularity classes $s_0 \in [s_1, s_2]$, where s_0 denotes the regularity of the truth we want to adapt to. Similarly to the discrete case we divide the data in each machine to two independent samples $X^{(i,1)}$ and $X^{(i,2)}$. Let \mathcal{S}_n denote a $1/\log n$ -grid of the interval $[s_1, s_2]$, i.e. $\mathcal{S}_n = \{s_1, s_1 + 1/\log n, \dots, s_2\}$, and denote by $\underline{s} = s_1 + \gamma_n/\log n$, for some $0 \leq \gamma_n \leq \lceil (s_2 - s_1) \log n \rceil$, $\gamma_n \in \mathbb{N}$, the lower bound of the $1/\log n$ -bin containing s_0 , i.e. $s_0 \in [\underline{s}, \underline{s} + 1/\log n]$. We will describe next a testing procedure for the regularity hyper-parameter s_0 . Let us compute the test $\Psi_{n/m}^{(i)}(M_{n,t}^{-1}, t, s)$ for all $t < s$, $s, t \in \mathcal{S}_n$ and take $\hat{s}_{n/m}^{(i)}$ to be the largest regularity s for which the null hypothesis was retained for every $t < s$, i.e.

$$\hat{s}_{n/m}^{(i)} = \max\{s \in \mathcal{S}_n : \Psi_{n/m}^{(i)}(M_{n,t}^{-1}, t, s) = 0, \forall t < s\}.$$

The aggregated regularity estimator \hat{s} and the distributed estimator \hat{f} is then constructed the same way as in the proof of Theorem 3.3, using the above defined $\hat{s}_{n/m}^{(i)}$.

The probability of under smoothing is bounded from above by $(\gamma_n - 1)^2 \leq (s_2 - s_1)^2 \log^2 n$ times the probability of rejecting the correct null-hypothesis. Hence in view of assertion (3.8) and the monotone decreasing property of the function $s \mapsto M_{n,s}$, we get that

$$P\left(\hat{s}_{n/m}^{(i)} < \underline{s}\right) \lesssim (s_2 - s_1)^2 (\log n)^2 e^{-M_{n,s_2}^{1/2}/2} = o(1).$$

This implies that for all $i \in \{1, \dots, m\}$

$$\begin{aligned} E_{X^{(i,1)}, X^{(i,2)}} \hat{B}^{(i)} &= E_{X^{(i,1)}} \hat{B}^{(i)} \leq E_{X^{(i,1)}} n^{\frac{1}{1+2\hat{s}^{(i)}}} \log n \\ &\lesssim n^{\frac{1}{1+2\underline{s}}} \log n + n^{\frac{1}{1+2s_1}} e^{-M_{n,s_2}^{1/2}/2} \log^2 n \lesssim n^{\frac{1}{1+2s_0}} \log n \end{aligned}$$

and similarly to assertions (3.10) and (3.11) that

$$(4.1) \quad \begin{aligned} P_{X_1}(\hat{s} < \underline{s}) &= P_{X_1}(\tilde{N} > n^{\frac{1}{1+2\underline{s}}}) \leq e^{-m/5} \quad \text{and} \\ E_{X_1} \tilde{N} < n^{\frac{1}{1+2\underline{s}}} + n^{\frac{1}{1+2s_1}} P_{X_1}(\tilde{N} > n^{\frac{1}{1+2\underline{s}}}) &\lesssim n^{\frac{1}{1+2\underline{s}}} \lesssim n^{\frac{1}{1+2s_0}}, \end{aligned}$$

for $m \geq 5 \log n$.

It remains to show that our procedure adapts to the minimax risk. First note that in view of assertion (3.13) and (4.1)

$$\sup_{f_0 \in B_{2,\infty}^{\underline{s}}} E_{X_1}(E_{X_2|X_1} \|\hat{f} - f_{0,j \leq \tilde{j}_n}\|_2^2) \leq E_{X_1} \tilde{N}/n \lesssim n^{-\frac{2s_0}{1+2s_0}}.$$

Next let $j_{n,s} = (1+2s)^{-1} \log n$, then for $\tilde{j}_n = \lfloor \log \tilde{N} \rfloor$

(4.2)

$$\begin{aligned} E_{X_1}(\|f_{0,j \leq \tilde{j}_n} - f_0\|_2^2) &= \left(\sum_{s < \underline{s}, s \in \mathcal{S}_n} + \sum_{s = \underline{s}} + \sum_{s > \underline{s}, s \in \mathcal{S}_n} \right) P_{X_1}(\hat{s} = s) E_{X_1}(\|f_{0,j \leq j_{n,s}} - f_0\|_2^2 | \hat{s} = s) \\ &= \left(\sum_{s < \underline{s}, s \in \mathcal{S}_n} + \sum_{s = \underline{s}} + \sum_{s > \underline{s}, s \in \mathcal{S}_n} \right) P_{X_1}(\hat{s} = s) \sum_{j=j_{n,s}}^{\infty} \sum_{k=1}^{2^j} f_{0,jk}^2. \end{aligned}$$

We deal with the three terms on the right hand side separately. In view of assertion (4.1) and $\|f_0\|_2^2 \leq L^2$ we have that

$$\sum_{s < \underline{s}} P_{X_1}(\hat{s} = s) \sum_{j=j_{n,s}}^{\infty} \sum_{k=1}^{2^j} f_{0,jk}^2 \leq L^2 e^{-m/5} \lesssim n^{-\frac{2s_0}{1+2s_0}}.$$

Then it is also easy to see that

$$\begin{aligned} P_{X_1}(\hat{s} = \underline{s}) \sum_{j=j_{n,\underline{s}}}^{\infty} \sum_{k=1}^{2^j} f_{0,jk}^2 &< \sum_{j=j_{n,\underline{s}}}^{\infty} 2^{-2j\underline{s}} \sup_{j \geq j_{n,\underline{s}}} 2^{2j\underline{s}} \sum_{k=1}^{2^j} f_{0,jk}^2 \\ &\leq L^2 n^{-\frac{2\underline{s}}{1+2\underline{s}}} \lesssim n^{-\frac{2s_0}{1+2s_0}}. \end{aligned}$$

Then for arbitrary $s > \underline{s}$, $s \in \mathcal{S}_n$, using the notation $R_{M_{n,\underline{s}}^{-1}}^{\underline{s},s}(L) := \{f \in B_{2,\infty}^s(L) : \|f - B_{2,\infty}^s(L)\|_2 \geq \tilde{C}_{M_{n,\underline{s}}^{-1}}(n/m)^{-\frac{s}{1/2+2\underline{s}}}\}$, we have that

$$\begin{aligned} \sup_{f_0 \in R_{M_{n,\underline{s}}^{-1}}^{\underline{s},s}(L)} P_{X^{(i,1)}}(\hat{s}_{n/m}^{(i)} \geq s) &\leq \sup_{f_0 \in R_{M_{n,\underline{s}}^{-1}}^{\underline{s},s}(L)} E_{X^{(i,1)}}\left(1 - \Psi_{n/m}^{(i)}(M_{n,\underline{s}}^{-1}, \underline{s}, s)\right) \\ &\lesssim e^{-M_{n,\underline{s}}^{1/2}/2}. \end{aligned}$$

Therefore, by Hoeffding's inequality,

$$(4.3) \quad \sup_{f_0 \in R_{M_{n,\underline{s}}^{-1}}^{\underline{s},s}(L)} P_{X_1}(\hat{s} \geq s) \leq e^{-m/5},$$

hence by combining the preceding two displays we get that

$$\sup_{f_0 \in R_{M_{n,\underline{s}}^{-1}}^{\underline{s},s}(L)} \sum_{j=j_{n,s}}^{\infty} \sum_{k=1}^{2^j} f_{0,jk}^2 P_{X_1}(\hat{s} = s) \leq L^2 e^{-m/5} = o(n^{-2s_0/(1+2s_0)}/\log n).$$

For any $f_0 \in \mathcal{F}_s := B_{2,\infty}^s(L) \setminus R_{M_{n,\underline{s}}^{-1}}^{\underline{s},s}(L)$ there exists an $\tilde{f}_0 \in B_{2,\infty}^s(L)$ such that $\|f_0 - \tilde{f}_0\|_2 \leq \tilde{C}_{M_{n,\underline{s}}^{-1}}(n/m)^{-\frac{s}{1/2+2\underline{s}}}$. Then similarly to assertion (3.14) we get that

$$\begin{aligned} \sup_{f_0 \in \mathcal{F}_s} \sum_{j=j_{n,s}}^{\infty} \sum_{k=1}^{2^j} f_{0,jk}^2 &\leq 2 \sup_{f_0 \in \mathcal{F}_s} \left(\sum_{j=j_{n,s}}^{\infty} \sum_{k=1}^{2^j} (f_{0,jk} - \tilde{f}_{0,jk})^2 + \sum_{j=j_{n,s}}^{\infty} 2^{-2js} \sup_{j \geq j_{n,s}} 2^{2js} \sum_{k=1}^{2^j} \tilde{f}_{0,jk}^2 \right) \\ &\lesssim \tilde{C}_{M_{n,\underline{s}}^{-1}}(n/m)^{-\frac{2s}{1/2+2\underline{s}}} + 2^{-2j_{n,s}} \\ &\lesssim n^{-\frac{2s}{1+2\underline{s}}} + n^{-\frac{2s}{1+2s}} \lesssim n^{-\frac{2s_0}{1+2s_0}}. \end{aligned}$$

Hence

$$\begin{aligned} \sup_{f_0 \in B_{2,\infty}^s(L)} \sum_{s > \underline{s}}^{s_2} P_{X_1}(\hat{s} = s) \sum_{j=j_{n,s}}^{\infty} \sum_{k=1}^{2^j} f_{0,jk}^2 \\ \lesssim \sum_{s > \underline{s}}^{s_2} (P_{X_1}(\hat{s} = s) + o(1/\log n)) n^{-\frac{2s_0}{1+2s_0}} \lesssim n^{-\frac{2s_0}{1+2s_0}}. \end{aligned}$$

Combining the upper bounds above we get that

$$\begin{aligned} \sup_{f_0 \in B_{2,\infty}^s(L)} E_{X_1, X_2} \|\hat{f} - f_0\|_2^2 &\leq 2 \sup_{f_0 \in B_{2,\infty}^s(L)} \left(E_{X_1} \|f_{0, j \leq \tilde{j}_n} - f_0\|_2^2 \right. \\ &\quad \left. + E_{X_1, X_2} \|\hat{f} - f_{0, j \leq \tilde{j}_n}\|_2^2 \right) \\ &\lesssim n^{-\frac{2s_0}{1+2s_0}}, \end{aligned}$$

concluding the proof of the corollary.

4.1. *Proof of Theorem 2.7.* The proof follows the same lines of reasoning as the proof of Theorem 2.4, here we highlight only the differences.

First of all the set of functions \mathcal{F}_0 is defined slightly differently. Let us introduce the notations

$$(4.4) \quad \begin{aligned} \tilde{\delta}_n &= \bar{\delta}_n \wedge (m/n), \quad \text{with} \\ \bar{\delta}_n &= \min \left\{ \frac{m}{n \log m}, \frac{1}{n[\bar{\delta}_n^{1/(1+2s_1)} \beta_n \wedge 1] \log m} \right\}, \\ \beta_n &= (\Gamma_n \vee n^{\frac{1}{1+2s_1}-\varepsilon_1}) \log n \quad \text{and} \quad \Gamma_n = n^{\frac{1/2}{1+2s_1} + \frac{1/2}{1+2s_2} + \varepsilon_1}, \end{aligned}$$

with $\varepsilon_1 \in (0, \frac{s_2-s_1}{(1+2s_1)(1+2s_2)} \wedge \frac{(1-p)/8}{1+2s_1})$. By elementary computations one can deduce that $\bar{\delta}_n \geq n^{\varepsilon_1/2-1}$ and therefore

$$(4.5) \quad \tilde{\delta}_n \geq n^{(\varepsilon_1/2 \wedge p)-1}.$$

Next, let us denote by K_j the largest set of Daubechies wavelets with disjoint supports at resolution level j . Note that $|K_j| \geq c_0 2^j$ (for large enough j and sufficiently small $c_0 > 0$). Then we consider the class of functions

$$(4.6) \quad \mathcal{F}_0 = \{f_k : k \in K_{j_n}\}, \quad \text{where} \quad f_k = \tilde{\delta}_n^{1/2} \psi_{j_n, k}.$$

Since the functions in \mathcal{F}_0 have disjoint supports we have

$$\begin{aligned} \sup_{f \in \mathcal{F}_0} \|0 - f\|_\infty &= \sup_{k \in K_{j_n}} \tilde{\delta}_n^{1/2} \|\psi_{j_n, k}\|_\infty \lesssim 2^{j_n/2} \tilde{\delta}_n^{1/2} \\ &\lesssim \tilde{\delta}_n^{s_1/(1+2s_1)} \ll (n/m)^{-s_1/(1+2s_1)}, \end{aligned}$$

following from the definition of $\tilde{\delta}_n$. Hence it is not possible to test between the zero function and the set \mathcal{F}_0 in the local servers.

Using the notation Z for the likelihood ratio introduced in the proof of Theorem 2.4 we note that in view of the proof of Theorem 6.2.11 b) on page 493 of [12] we have that

$$E(Z - 1)^2 \leq (e^{\bar{\gamma}_n^2} - 1)/|\mathcal{F}_0|, \quad \text{where } \bar{\gamma}_n = \sqrt{\tilde{\delta}_n n/m}.$$

Then the infimum of the tests given in (3.5) is bounded from below by $(1 - C\eta_n)^2$ for $\eta_n = \tilde{\delta}_n^{1/(4+8s_1)} \leq n^{-(1-p)/(4+8s_1)} \leq n^{-2\varepsilon_1}$. This leads to

$$\frac{1}{|\mathcal{F}_0|} \sum_{f \in \mathcal{F}_0} E_f^{(i)} \hat{B}^{(i)} \lesssim \Gamma_n + n^{\frac{1}{1+2s_1} + \varepsilon_1} (\log n) (\eta_n + n^{\frac{1/2}{1+2s_2} - \frac{1/2}{1+2s_1}}) \lesssim \beta_n.$$

This means that the expected number (with respect to the joint distribution of the variables F and P_f , $f \in \mathcal{F}_0$) of transmitted bits on the class \mathcal{F}_0 is bounded from above by a multiple of β_n . So the distributed estimator satisfies assertion (A.7) in with $B^{(i)}$ replaced by $C\beta_n$. Hence in view of the minimax lower bound derived in assertion (A.13) (with $B^{(i)}$ replaced by β_n in the definition of δ_n in the proof of Theorem A.3) and the definition of $\tilde{\delta}_n$

$$\sup_{f_0 \in \mathcal{F}_0} E_{f_0} \|\hat{f} - f_0\|_\infty \gtrsim \tilde{\delta}_n^{\frac{s_1}{1+2s_1}} \gg n^{-\frac{s_1}{1+2s_1} + \varepsilon_2},$$

with $\varepsilon_2 = (\varepsilon_1/2 \wedge p)s_1/(1+2s_1)$, where the last inequality followed from (4.5). This contradicts assumption (2.5), finishing the proof of our statement.

4.2. Proof of Theorem 2.8. First note that in Lemma 5.2 of [5] it was shown that the smoothness can be consistently estimated under the self-similarity condition, i.e. there exists an estimator $\hat{s}_{n/m^{(i)}}$ such that for every $i \in \{1, \dots, m\}$ and $c > 0$ there exists $C > 0$ satisfying

$$(4.7) \quad \inf_{s \in [s_1, s_2]} \inf_{f_0 \in S_{\infty}^s(L, \varepsilon, j_0)} P_{f_0}(s - C/\log(n/m) \leq \hat{s}_{n/m}^{(i)} \leq s) \lesssim (m/n)^c.$$

By choosing $c = 1/(1-p)$ we have $(m/n)^c = 1/n$. Then we propose a similar estimation method as in Theorem 2.5. First we split the data into $X^{(i,1)}$ and $X^{(i,2)}$ and use the first sample $X^{(i,1)}$ to construct the estimator $\hat{s}_{n/m}^{(i)}$ for the smoothness parameter s . Next transmit the approximation of the first $\tilde{N}^{(i)} = (n/\log n)^{1/(1+2\hat{s}_{n/m}^{(i)})}$ coefficients (instead of $n^{1/(1+2\hat{s}_{n/m}^{(i)})}$) as in Theorem 2.5) of the second subset of observations $X^{(i,2)}$, following Algorithm 1. Then $\hat{B}^{(i)} \leq (n/\log n)^{1/(1+2s_1)} \log n$ and

$$\begin{aligned} E_{X^{(i,1)}, X^{(i,2)}} \hat{B}^{(i)} &= E_{X^{(i,1)}} \hat{B}^{(i)} = E_{X^{(i,1)}} \tilde{N}^{(i)} \log n \\ &\leq (n/\log n)^{\frac{1}{1+2s}} \log n + n^{-1} (n/\log n)^{\frac{1}{1+2s_1}} \log n \\ &\lesssim n^{\frac{1}{1+2s}} (\log n)^{\frac{2s}{1+2s}}. \end{aligned}$$

Besides we also have that the median \tilde{N} of the values $\tilde{N}^{(i)}$ satisfy that

$$(4.8) \quad P_{X_1}(n^{1/(1+2s)} \leq \tilde{N} \leq C_1 n^{1/(1+2s)}) \geq 1 - C_2 e^{-m/5},$$

for some large enough constants $C_1, C_2 > 0$.

Similarly to before let $\tilde{j}_n = \lfloor \log \tilde{N} \rfloor$ and $f_{0,j \leq \tilde{j}_n} = \sum_{j \leq \tilde{j}_n} \sum_{k=1}^{2^j} f_{0,jk} \psi_{jk}$. Then using the notation \mathcal{E} introduced in (3.9) we get that

$$\begin{aligned} \|\hat{f} - f_0\|_\infty 1_\mathcal{E} &\leq \|\hat{f} - f_{0,j \leq \tilde{j}_n}\|_\infty 1_\mathcal{E} + \|f_{0,j \leq \tilde{j}_n} - f_0\|_\infty \\ &\leq \left\| \sum_{j \leq \tilde{j}_n} \sum_{k=1}^{2^j} \frac{1}{|M_{jk}|} \sum_{i \in M_{jk}} \left(\sqrt{\frac{m}{n}} Z_{jk}^{(i)} + \varepsilon_{jk}^{(i)} \right) \psi_{jk} \right\|_\infty 1_\mathcal{E} + \sum_{j=\tilde{j}_n}^{\infty} 2^{j/2} \sup_{k \in K_j} |f_{0,jk}| \\ &\lesssim \sup_{j \leq \tilde{j}_n} \left(\left| \frac{1}{|M_{jk}|} \sum_{i \in M_{jk}} \sqrt{\frac{m}{n}} Z_{jk}^{(i)} \right| + n^{-1/2} \right) \sum_{j=0}^{\tilde{j}_n} 2^{j/2} + \sum_{j=\tilde{j}_n}^{\infty} 2^{j/2} \sup_{k \in K_j} |f_{0,jk}| \\ &\lesssim \sqrt{\frac{\tilde{N}}{n}} \sup_{j \in \{1, \dots, \tilde{j}_n\}} \sup_{k \in K_j} (|Z_{j,k}| + 1) + 2^{-\tilde{j}_n s} \sum_{j=\tilde{j}_n}^{\infty} 2^{j(s+1/2)} \sup_{k \in K_j} |f_{0,jk}|, \end{aligned}$$

where $Z_{jk} := \frac{\sqrt{n}}{|M_{jk}|} \sum_{i \in M_{jk}} \sqrt{\frac{m}{n}} Z_{jk}^{(i)} \stackrel{iid}{\sim} N(0, \frac{m}{|M_{jk}|})$, $0 \leq \varepsilon_{jk}^{(i)} \leq 1/\sqrt{n}$ on \mathcal{E} . Therefore in view of (4.8)

$$\begin{aligned} E_{X_1, X_2} \|\hat{f} - f_0\|_\infty &\lesssim E_{X_1} \sqrt{\frac{\tilde{N}}{n}} \log \tilde{N} + E_{X_1} \tilde{N}^{-s} + o(n^{-1}) \\ &\lesssim (n/\log n)^{-\frac{s}{1+2s}} + e^{-m/5} \lesssim (n/\log n)^{-\frac{s}{1+2s}}. \end{aligned}$$

concluding the proof of our statement.

5. Technical lemmas. The first lemma extends slightly the results of Shannon's source coding theorem by allowing also non-prefix codes, see Lemma 5.1 of [25].

LEMMA 5.1. *Let Y be a random finite binary string. Its expected length satisfies the inequality*

$$H(Y) \leq 2El(Y) + 1.$$

Let us take an arbitrary $x \in \mathbb{R}$ and write it in a scientific binary representation, i.e. $|x| = \sum_{k=-\infty}^{\log_2 |x|} b_k 2^k$, with $b_k \in \{0, 1\}$, $k \in \mathbb{Z}$. Then let us take y consisting the same digits as x up to the $(D \log_2 n)$ th digits, for some $D > 0$,

after the binary dot (and truncated there), i.e. $|y| = \sum_{k=-D \log_2 n}^{\log_2 |x|} b_k 2^k$, unless $|x| \geq \sqrt{n}$, in which case we set y to zero, see also Algorithm 1, a slightly modified version of Algorithm 1 from [25]. In the algorithm the function $x \mapsto \text{sign}(x)$ is one if $x \geq 0$ and zero otherwise.

Algorithm 1 Transmitting a finite-bit approximation of a number

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1: procedure TRANSAPPROX( $x$ )
2:   if  $|x| \geq n$  then
3:     Transmit:  $\text{sign}(x), b_{- \lfloor D \log n \rfloor + 1}, \dots, b_{\lfloor \log |x| \rfloor}$ .
4:     Construct:  $y = (2\text{sign}(x) - 1) \sum_{k=-D \log n + 1}^{\log |x|} b_k 2^k$ .
5:   else
6:     Transmit: 0.
7:     Construct:  $y = 0$ .

```

The next lemma gives an upper bound for the number of transmitted bits and the accuracy of the procedure described in Algorithm 1. It is a slightly reformulated version of Lemma 2.3 of [25] to accommodate almost sure upper bound on the code length.

LEMMA 5.2. *For $X \sim N(\mu, \sigma^2)$, with $|\mu| \leq M$ and $\sigma \leq 1$ let the approximation Y of X given in Algorithm 1 and denote by \mathcal{E}_X the event that $|X| \leq \sqrt{n}$. Then for large enough n ,*

$$P_X(\mathcal{E}_X^c) = O(e^{-cn}) \quad , \quad |X - Y| 1_{\mathcal{E}_X} < 2n^{-D}, \text{ and } l(Y) \leq (D + 1/2) \log n,$$

for some $c > 0$.

PROOF. It is straightforward to see that the last two inequalities of the statement hold. To prove the first one note that

$$P_X(\mathcal{E}_X^c) \leq P_X(|X| \geq \sqrt{n}) \leq P_X(|X - \mu| \geq \sqrt{n} - M) \lesssim e^{-cn}.$$

□

Next we provide an extended version of Lemma 4.2 of [9] with tighter upper bounds for small $\Delta > 0$. The main difference in the proof is that instead of Chebyshev's inequality we apply a more accurate concentration inequality, see Lemma 8.1 of [3].

LEMMA 5.3. *Let $\Delta > 0$. Then*

$$P\left\{\forall l : J_0 \leq l \leq j, |T_n(l) - \|\Pi_l f\|_2^2| \geq 4\sqrt{\frac{3z_0}{\Delta} \left(\frac{2^{(j+l)/2}}{n^2} + 2^{l/4} \frac{\|\Pi_l f\|_2^2}{n}\right)}\right\} \leq 2e^{-c/\sqrt{\Delta}},$$

for $c = \sqrt{3/2}$ and $z_0 = 2^{J_0}$ the number of father wavelets (at resolution level J_0) and $\Pi_l f = \sum_{k=1}^{2^l} f_{lk} \psi_{lk}$ the projection of f into the wavelet resolution level l .

PROOF. Note that for the wavelet estimator \hat{f} with signal-to-noise ration n we get that $\|\Pi_l \hat{f}\|_2^2 = \sum_k \hat{f}_{lk}^2$, where $\hat{f}_{lk} - f_{lk} \stackrel{iid}{\sim} N(0, 1/n)$.

Hence in view of Lemma 8.1 of [3] (with degree of freedom $D = 2^l$, non-centrality parameter $B = n \sum_{k=1}^{2^l} f_{lk}^2$ and $x = 1/(2\sqrt{\delta_l})$) we get for $\delta_l \leq 1/4$ that

$$\begin{aligned} P\left\{\left|\|\Pi_l \hat{f}\|_2^2 - \frac{2^l}{n} - \|\Pi_l f\|_2^2\right| \geq \sqrt{\frac{4}{\delta_l} \left(\frac{2^l}{n^2} + \frac{\|\Pi_l f\|_2^2}{n}\right)}\right\} \\ = P\left\{\left|\sum_{k=1}^{2^l} \hat{f}_{lk}^2 - \frac{2^l}{n} - \sum_{k=1}^{2^l} f_{lk}^2\right| \geq \sqrt{\frac{4}{\delta_l} \left(\frac{2^l}{n^2} + \frac{\sum_{k=1}^{2^l} f_{lk}^2}{n}\right)}\right\} \\ \leq P\left\{\left|\sum_{k=1}^{2^l} n \hat{f}_{lk}^2 - 2^l - n \sum_{k=1}^{2^l} f_{lk}^2\right| \geq 2\sqrt{\left(2^l + 2n \sum_{k=1}^{2^l} f_{lk}^2 \frac{1}{2\sqrt{\delta_l}}\right) + 2\frac{1}{2\sqrt{\delta_l}}}\right\} \\ \leq 2e^{-0.5/\sqrt{\delta_l}}. \end{aligned}$$

Similarly

$$P\left\{\left|\|\Pi_{J_0} \hat{f}\|_2^2 - \frac{z_0}{n} - \|\Pi_{J_0} f\|_2^2\right| \geq \sqrt{\frac{4}{\delta_{J_0}} \left(\frac{z_0}{n^2} + \frac{\|\Pi_{J_0} f\|_2^2}{n}\right)}\right\} \leq 2e^{-0.5/\sqrt{\delta_{J_0}}}.$$

By the definition of $T_n(l)$ and union bound these results imply that

$$\begin{aligned} P\left\{\forall l : J_0 < l \leq j, |T_n(l) - \|\Pi_l f\|_2^2| \geq \sqrt{\frac{4}{\delta_l} \left(\frac{2^l}{n^2} + \frac{\|\Pi_l f\|_2^2}{n}\right)}, \right. \\ \left. |T_n(J_0) - \|\Pi_{J_0} f\|_2^2| \geq \sqrt{\frac{4}{\delta_{J_0}} \left(\frac{z_0}{n^2} + \frac{\|\Pi_{J_0} f\|_2^2}{n}\right)}\right\} \leq \sum_{J_0 \leq l \leq j} e^{-0.5/\sqrt{\delta_l}}. \end{aligned}$$

Setting similarly to Lemma 4.2 of [9] the parameters $\delta_l = (2^{-(j-l)/2} + 2^{-l/4})\Delta/12$ and $\delta_{J_0} = \Delta/12$ we get in view of

$$\sum_{l=J_0}^j e^{-0.5/\sqrt{\delta_j}} \leq \sum_{l=J_0}^j \left(e^{-\sqrt{3/2}\Delta^{-1/22}2^{(j-l)/4}} + e^{-\sqrt{3/2}\Delta^{-1/22}2^{l/8}} \right) \lesssim e^{-\sqrt{3/2}\Delta^{-1/2}}$$

which implies together with $z_0 \geq 1$ that

$$\begin{aligned} P\left\{\forall l : J_0 \leq l \leq j, |T_n(l) - \|\Pi_l f\|_2^2| \geq 4\sqrt{\frac{3z_0}{\Delta} \left(\frac{2^{(j+l)/2}}{n^2} + 2^{l/4} \frac{\|\Pi_l f\|_2^2}{n}\right)}\right\} \\ \lesssim e^{-\sqrt{3/2}\Delta^{-1/2}}, \end{aligned}$$

concluding the proof of the lemma. \square

The next lemma is a slightly rewritten version of Theorem 3.1 of [9] with tighter error bounds (for small $\alpha > 0$).

LEMMA 5.4. *Let $\alpha > 0$. The test $\Psi_n(\alpha)$ satisfies that for all $\alpha > 0$ and $n > 0$*

$$\sup_{f \in H_0} E_f \Psi_n + \sup_{f \in H_1} E_f (1 - \Psi_n) \leq 2e^{-1/\sqrt{\alpha}},$$

where

$$H_0 : f \in B_{2,\infty}^{s_2}(L) \quad \text{and} \quad H_1 : f \in \{B_{2,\infty}^{s_1}(L) : \|f - B_{2,\infty}^{s_2}(L)\|_2 \geq \rho_n\},$$

$$\text{with } \rho_n = \tilde{C}_\alpha n^{-s_1/(1/2+2s_1)} \quad \text{and} \quad \tilde{C}_\alpha = 24 \left(\frac{2^{s_1} L}{\sqrt{1-2^{-2s_1}+19}} \sqrt{1/\alpha} \right).$$

PROOF. The proof goes the same way as of Theorem 3.1 of [9], with the only difference that we apply Lemma 5.3 instead of Lemma 4.2 of [9]. \square

We also recall a slight modification of Fano's inequality, see Corollary 1 of [11] or Theorem A.6. of [25]. Given a finite set $\mathcal{F}_0 \subset \mathcal{F}$, we use the notations

$$\begin{aligned} N_t^{\max} &= \max_{f \in \mathcal{F}_0} \left\{ \#\{\tilde{f} \in \mathcal{F}_0 : d(f, \tilde{f}) \leq t\} \right\}, \\ N_t^{\min} &= \min_{f \in \mathcal{F}_0} \left\{ \#\{\tilde{f} \in \mathcal{F}_0 : d(f, \tilde{f}) \leq t\} \right\}. \end{aligned}$$

THEOREM 5.5. *If \mathcal{F} contains a finite set \mathcal{F}_0 and $|\mathcal{F}_0| - N_t^{\min} > N_t^{\max}$, then for all $p, t > 0$,*

$$\inf_{\hat{f} \in \mathcal{E}(Y)} \sup_{f \in \mathcal{F}} \mathbb{E}_f d^p(\hat{f}, f) \geq t^p \left(1 - \frac{I(F; Y) + \log 2}{\log(|\mathcal{F}_0|/N_t^{\max})} \right),$$

where $\mathcal{E}(Y)$ denotes the set of all estimators depending only on Y and the function class \mathcal{F} , and F is a uniformly distributed random variable on \mathcal{F}_0 .

The next lemma gives an upper bound for the mutual information between the uniform random variable F on $\mathcal{F}_0 \subset \mathbb{R}^d$ and the set of observations on all local machines $Y = (Y^{(1)}, \dots, Y^{(m)})$ in the d -dimensional many normal means model.

LEMMA 5.6. *Let $F = \delta\beta$, with $\delta^2 \leq 2^{-10}m/(n \log(md))$ and β a uniformly distributed random variable over $\{-1, 1\}^d$. Furthermore, suppose that $X = (X^{(1)}, \dots, X^{(m)})$, where $X^{(i)}$ s are d -dimensional random variables satisfying that $X_j^{(i)} | F_j$ and F_j are independent of F_{-j} , and $X_j^{(i)} | (F = f) \sim \mathbb{P}_{f_j}^{(i)} = N(f_j, m/n)$. Then*

$$I(F; Y) \leq \sum_{i=1}^m \frac{2\delta^2}{m/n} \min \left\{ 2^{10} \log(md) H(Y^{(i)}), d \right\} + 4 \log 2,$$

where $I(F; Y)$ is the mutual information between F and Y in the Markov chain $F \rightarrow X \rightarrow Y$.

PROOF. Let us introduce the notation $a^2 = 2^4 \log(md)m/n$ and note that

$$\sup_{|x| \leq a} \frac{\varphi_{\delta, m/n}(x)}{\varphi_{-\delta, m/n}(x)} \leq \sup_{|x| \leq a} e^{\frac{n(x-\delta)^2 - (x+\delta)^2}{2m}} \leq \sup_{|x| \leq a} e^{\frac{2n\delta|x|}{m}} \leq e^{\frac{2an\delta}{m}},$$

where φ_{μ, σ^2} denotes the density function of a normal distribution with mean μ and variance σ^2 . Furthermore, let us introduce the notation $B_j = \{|x_j| \leq a\}$, $j = 1, \dots, d$. Then by Theorem 5.7 (with $\mathcal{F}_0 = \{f = \delta\beta : \beta \in \{-1, 1\}^d\}$) we have that

$$(5.1) \quad \begin{aligned} I(F; Y^{(i)}) &\leq d(\log 2) \sqrt{P_{X_j^{(i)}}(X_j^{(i)} \notin B_j) + d^2 P_{X_j^{(i)}}(X_j^{(i)} \notin B_j)} \\ &\quad + 2C^2(C-1)^2 I(X^{(i)}; Y^{(i)}), \end{aligned}$$

with $C = e^{2^3|\delta|\sqrt{\log(md)n/m}}$. Next note that for $Z \sim N(0, m/n)$

$$P_{X_j^{(i)}}(X_j^{(i)} \notin B_j) \leq P(|Z| \geq a - \delta) \leq 2e^{-\frac{(a-\delta)^2 n}{2m}} \leq 2e^{-\frac{a^2 n}{4m}} \leq 2(md)^{-4},$$

and the inequality $I(X^{(i)}; Y^{(i)}) \leq H(Y^{(i)})$ holds. Then by plugging in the above inequalities into (5.1) and using the inequalities $e^x \leq 1+2x$ for $x \leq 0.4$ and $C^2 \leq 2$ we get that

$$I(F; Y^{(i)}) \leq \sqrt{2}(\log 2)m^{-2}d^{-1} + 2(\log 2)m^{-4}d^{-2} + 2^{11}\delta^2 \frac{\log(md)n}{m} H(Y^{(i)}).$$

Furthermore, from the data-processing inequality and the convexity of the KL divergence

$$\begin{aligned} I(F; Y^{(i)}) &\leq I(F; X^{(i)}) \leq \frac{1}{|\mathcal{F}_0|^2} \sum_{f, f' \in \mathcal{F}_0} K(\mathbb{P}_f^{(i)} \|\mathbb{P}_{f'}^{(i)}) \\ &= \frac{\delta^2}{2m/n} \frac{1}{|\mathcal{F}_0|^2} \sum_{f, f' \in \mathcal{F}_0} \|\beta - \beta'\|_2^2 \leq 2(n/m)d\delta^2. \end{aligned}$$

We conclude our statement by noting that

$$I(F; Y) \leq \sum_{i=1}^m I(F; Y^{(i)})$$

□

The next theorem provide an upper bound for the mutual information, see Theorem A.9 in [25] or Lemma 3 of [26].

THEOREM 5.7. *Let us consider the Markov chain $F \rightarrow X^{(i)} \rightarrow Y^{(i)}$, where F is the uniform distribution on $\mathcal{F}_0 \subset \mathbb{R}^d$ and $X^{(i)} | (F = f) \sim P_{X^{(i)}|F=f}$ is a d -dimensional random variable. Assume that $X_j^{(i)} | F_j$ and F_j are independent of F_{-j} . For $C \geq 1$, define*

$$B_j = \left\{ x_j : \max_{f \neq f'} \frac{p(x_j | f_j)}{p(x_j | f'_j)} \leq C \right\}$$

for a constant $C \geq 1$ and density $p(x_j | f_j)$. Then

$$\begin{aligned} I(F; Y^{(i)}) &\leq \sum_{j=0}^d \left((\log 2) \sqrt{P_{X_j^{(i)}}(X_j^{(i)} \notin B_j)} + \log |\mathcal{F}_0| P_{X_j^{(i)}}(X_j^{(i)} \notin B_j) \right) \\ &\quad + 2C^2(C-1)^2 I(X^{(i)}; Y^{(i)}), \end{aligned}$$

where $I(X^{(i)}; Y^{(i)})$ is the mutual information between $X^{(i)}$ and $Y^{(i)}$.

APPENDIX A: PROOFS FOR THE MINIMAX RATES IN THE GAUSSIAN WHITE NOISE MODEL

A.1. Proof of Theorem 2.1. The proof of the theorem follows from the following, more general theorem with taking $B^{(1)} = \dots = B^{(m)} = B$. The proof is slight extension for a larger set of estimators and adaptation to the Gaussian white noise setting of the proof of Theorem 2.1 [25].

THEOREM A.1. *Let the sequence $\delta_n = o(1)$ be defined as*

$$(A.1) \quad \delta_n = \min \left\{ \frac{m}{n \log n}, \frac{m}{n \sum_{i=1}^m [\delta_n^{\frac{1}{1+2s}} B^{(i)} \log n \wedge 1]} \right\}.$$

Then in the distributed Gaussian white noise model (2.1) we have for any $s > 0$ that

$$\inf_{\hat{f} \in \mathcal{F}_{dist}(B^{(1)}, \dots, B^{(m)})} \sup_{f_0 \in B_{2,\infty}^s(L)} \mathbb{E}_{f_0} \|\hat{f} - f_0\|_2^2 \gtrsim \delta_n^{\frac{2s}{1+2s}}.$$

Proof of Theorem A.1. Note that without loss of generality we can multiply δ_n with an arbitrary constant. In the proof we define δ_n as the solution to

$$(A.2) \quad \delta_n = 2^{-15} L^{-2} \min \left\{ \frac{m}{n \log n}, \frac{m}{n \sum_{i=1}^m [\delta_n^{\frac{1}{1+2s}} \log(n) B^{(i)} \wedge 1]} \right\}.$$

We note, however, that all the computations below hold for arbitrary $\delta'_n \leq \delta_n$ as well.

We prove the desired lower bound for the minimax risk using a modified version of Fano's inequality, given in Theorem 5.5. As a first step we construct a finite subset $\mathcal{F}_0 \subset B_{2,\infty}^s(L)$. We use the wavelet notation outlined in Appendix B and define $j_n = \lfloor (\log \delta_n^{-1}) / (1 + 2s) \rfloor$. For $\beta \in \{-1, 1\}^{2^{j_n}}$, let $f_\beta \in L_2[0, 1]$ be the function with wavelet coefficients

$$(A.3) \quad f_{\beta,jk} = \begin{cases} L \beta_k \delta_n^{1/2}, & \text{if } j = j_n, k = 1, \dots, 2^{j_n}, \\ 0, & \text{else.} \end{cases}$$

Now define $\mathcal{F}_0 = \{f_\beta : \beta \in \{-1, 1\}^{2^{j_n}}\}$. Note that $\mathcal{F}_0 \subset B_{2,\infty}^s(L)$, since

$$\|f_\beta\|_{B_{2,\infty}^s}^2 = \sup_j 2^{2sj} \sum_{k=1}^{2^j} f_{\beta,jk}^2 = L^2 2^{(2s+1)j_n} \delta_n \leq L^2.$$

Therefore, for an arbitrary set of estimators $\hat{\mathcal{F}}$ we have that

$$\inf_{\hat{f} \in \hat{\mathcal{F}}} \sup_{f_0 \in B_{2,\infty}^s(L)} \mathbb{E}_{f_0} \|\hat{f} - f_0\|_2^2 \geq \inf_{\hat{f} \in \hat{\mathcal{F}}} \sup_{f_0 \in \mathcal{F}_0} \mathbb{E}_{f_0} \|\hat{f} - f_0\|_2^2.$$

To prove the statement of the theorem we take the set of distributed estimators $\hat{\mathcal{F}} = \mathcal{F}_{dist}(B^{(1)}, \dots, B^{(m)}; B_{2,\infty}^s(L))$, but the inequality holds more generally.

For this set of functions \mathcal{F}_0 , the maximum and minimum number of elements in balls of radius $t > 0$, given by

$$N_t^{\max} = \max_{f_\beta \in \mathcal{F}_0} \left\{ \#\{f_{\beta'} \in \mathcal{F}_0 : \|f_\beta - f_{\beta'}\|_2 \leq t\} \right\},$$

$$N_t^{\min} = \min_{f_\beta \in \mathcal{F}_0} \left\{ \#\{f_{\beta'} \in \mathcal{F}_0 : \|f_\beta - f_{\beta'}\|_2 \leq t\} \right\},$$

satisfy $N_t^{\max} = N_t^{\min}$ and $N_t^{\max} = \sum_{i=0}^{\tilde{t}} \binom{2^{j_n}}{i} < |\mathcal{F}_0|/2$ for $\tilde{t} := \frac{t^2}{4\delta_n L^2} < 2^{j_n-1}$ (and therefore $N_t^{\max} < |\mathcal{F}_0| - N_t^{\min}$).

Recall the notations $X = (X^{(1)}, \dots, X^{(m)})$ for the data available at the local machines and $Y = (Y^{(1)}, \dots, Y^{(m)})$ for the binary messages transmitted to the central machine satisfying the distribution protocol, and consider the Markov chain $F \rightarrow X \rightarrow Y$, where F is a uniform random element in \mathcal{F}_0 . It then follows from Theorem 5.5 (with $t^2 = L^2 \delta_n 2^{j_n+1}/3$ and $d(f, g) = \|f - g\|_2$) that

$$(A.4) \quad \inf_{\hat{f} \in \hat{\mathcal{F}}} \sup_{f_0 \in \mathcal{F}_0} \mathbb{E}_{f_0} \|\hat{f} - f_0\|_2^2 \gtrsim L^2 \delta_n 2^{j_n} \left(1 - \frac{I(F; Y) + \log 2}{\log(|\mathcal{F}_0|/N_t^{\max})} \right),$$

where $I(F; Y)$ is the mutual information between the random variables F and Y .

To lower bound the right-hand side, first note that $N_t^{\max} = \sum_{i=1}^{\tilde{t}} \binom{2^{j_n}}{i} < 2 \binom{2^{j_n}}{\tilde{t}} \leq 2(e 2^{j_n}/\tilde{t})^{\tilde{t}}$ and therefore, for $\tilde{t} = 2^{j_n-1}/3$ (i.e. $t^2 = L^2 \delta_n 2^{j_n+1}/3$),

$$\log(|\mathcal{F}_0|/N_t^{\max}) \geq 2^{j_n} \log(2(6e)^{-1/6} 2^{-2^{-j_n}}) \geq 2^{j_n-1}/3.$$

Hence, recalling that $2^{j_n} = \delta_n^{-\frac{1}{1+2s}}$ we see that to prove

$$(A.5) \quad \inf_{\hat{f} \in \hat{\mathcal{F}}} \sup_{f_0 \in \mathcal{F}_0} \mathbb{E}_{f_0} \|\hat{f} - f_0\|_2^2 \gtrsim \delta_n^{2s/(1+2s)}$$

and as a consequence to derive the statement of the theorem it is sufficient to show that

$$(A.6) \quad I(F; Y) \leq \delta_n^{-1/(1+2s)}/8 + O(1).$$

Observe that for the class of distributed estimators $\hat{\mathcal{F}} = \mathcal{F}_{\text{dist}}(B^{(1)}, \dots, B^{(m)}; B_{2, \infty}^s(L))$, by definition the following inequality holds

$$(A.7) \quad E^{(i)} l(Y^{(i)}) = \frac{1}{|\mathcal{F}_0|} \sum_{f \in \mathcal{F}_0} E_f^{(i)} l(Y^{(i)}) \leq B^{(i)},$$

where the expectation is taken over the joint distribution of the random variable F and $P_f^{(i)}$, $f \in \mathcal{F}_0$. Next note that for $\delta_n \leq m/(2^{11}L^2n \log n)$ the conditions of Lemma 5.6 are satisfied hence by applying the lemma (with $\delta^2 = L^2\delta_n$ and $d = \delta_n^{-\frac{1}{1+2s}}$) we get

$$\begin{aligned} I(F; Y) &\leq 2L^2n\delta_n m^{-1} \sum_{i=1}^m \min \left\{ 2^{10} \log(m\delta_n^{-\frac{1}{1+2s}}) H(Y^{(i)}, \delta_n^{-\frac{1}{1+2s}}) \right\} + 4 \log 2 \\ (A.8) \quad &\leq 2L^2n\delta_n m^{-1} \delta_n^{-\frac{1}{1+2s}} \sum_{i=1}^m \left(2^{11} \log(n) \delta_n^{\frac{1}{1+2s}} B^{(i)} \wedge 1 \right) + O(1), \end{aligned}$$

where the last inequality follows from Lemma 5.1 and assertion (A.7). Since from the definition of δ_n it follows that

$$\delta_n \leq \frac{2^{-4}L^{-2}mn^{-1}}{\sum_{i=1}^m [2^{11} \log(n) \delta_n^{\frac{1}{1+2s}} B^{(i)} \wedge 1]},$$

the right-hand side of (A.8) is further bounded by $2^{-3}\delta_n^{-\frac{1}{1+2s}} + O(1)$, finishing the proof of assertion (A.6) and concluding the proof of the theorem.

Note that we have used the properties of the distributed estimation class $\hat{\mathcal{F}}$ only in assertion (A.7), hence for any distributed method satisfying this inequality we have that

$$(A.9) \quad \inf_{\hat{f} \in \hat{\mathcal{F}}} \sup_{f_0 \in B_{2,\infty}^s(L)} E_{f_0} \|\hat{f} - f_0\|_2^2 \gtrsim \delta_n^{\frac{2s}{1+2s}}.$$

A.2. Proof of Theorem 2.2. First we give the algorithm achieving the upper bound. Let us introduce the notation $\eta = (\lfloor (n^{\frac{1}{1+2s}} \log(n)/B)^{(1+2s)/(2+2s)} \rfloor \vee 1) \wedge m$. Then we group the local machines into η groups and let the different groups work on different parts of the signal as follows: the machines with indexes $1 \leq i \leq m/\eta$ each transmit the approximations $Y_{jk}^{(i)}$ of the observations $X_{jk}^{(i)}$ for $1 \leq 2^j + k \leq (B/\log n) \wedge n^{1/(1+2s)}$ using Algorithm 1. If $\eta > 1$ then the next machines, with indexes $m/\eta < i \leq 2m/\eta$, each transmit the approximations $Y_{jk}^{(i)}$ for $B/\log n < 2^j + k \leq 2B/\log n$, and so on. The last machines with numbers $(\eta - 1)m/\eta < i \leq m$ transmit $Y_{jk}^{(i)}$ for $(\eta - 1)B/\log n < 2^j + k \leq \eta B/\log n$. Then in the central machine we average the corresponding transmitted approximated noisy coefficients in the obvious way. Formally, using the notation $\mu_{jk} = \lfloor (2^j + k) \log(n)/B \rfloor - 1$, the aggregated estimator \hat{f} is the function

with wavelet coefficients given by

$$\hat{f}_{jk} = \begin{cases} \text{mean}\{Y_{jk}^{(i)} : \frac{\mu_{jk}m}{\eta} < i \leq \frac{(\mu_{jk}+1)m}{\eta}\}, & \text{if } 2^j + k \leq \frac{\eta B}{\log n}, \\ 0, & \text{else.} \end{cases}$$

The procedure is summarized as Algorithm 2.

Algorithm 2 Algorithm for the L_2 -norm

- 1: **In the local machines:**
 - 2: **for** $\ell = 1$ to η **do**
 - 3: **for** $i = \lfloor (\ell - 1)m/\eta \rfloor + 1$ to $\lfloor \ell m/\eta \rfloor$ **do**
 - 4: **for** $2^j + k = \lfloor (\ell - 1)B/\log n \rfloor + 1$ to $\lfloor \ell B/\log n \rfloor$ **do**
 - 5: $Y_{jk}^{(i)} := \text{TransApprox}(X_{jk}^{(i)})$
 - 6: **In the central machine:**
 - 7: **for** $2^j + k = 1$ to $\lfloor (\eta B/\log n) \wedge n^{1/(1+2s)} \rfloor$ **do**
 - 8: $\hat{f}_{jk} := \text{mean}\{Y_{jk}^{(i)} : \mu_{jk}m/\eta < i \leq (\mu_{jk} + 1)m/\eta\}$
 - 9: Construct: $\hat{f} = \sum \hat{f}_{jk} \psi_{jk}$.
-

In the algorithm described above each machine transmits the approximations of at most $n^{1/(1+2s)} \wedge (B/\log n)$ noisy coefficients. Note that for any $f \in B_{2,\infty}^s(L)$ we have that $f_{jk}^2 \leq \sup_j 2^{js} \sum_k f_{jk}^2 \leq L^2$, hence in view of Lemma 5.2 (with $|\mu| = |f_{0,jk}| \leq L$) the approximation satisfies

$$0 \leq |X_{jk}^{(i)} - Y_{jk}^{(i)}| \mathbf{1}_{\mathcal{E}} \leq 1/\sqrt{n}, \quad |Y_{jk}^{(i)}| \leq \sqrt{n}, \quad \text{and} \quad l(Y_{jk}^{(i)}) \leq \log n,$$

where the set \mathcal{E} was defined in (3.9) and satisfies that $P_X(\mathcal{E}) \leq e^{-cn}$, for some $c > 0$. Therefore we need at most B bits to transmit $n^{1/(1+2s)} \wedge (B/\log n)$ coefficients, hence $\hat{f} \in \mathcal{F}_{\text{dist}}(B, \dots, B; B_{2,\infty}^s(L))$.

Next for convenience we introduce the notation $A_{jk} = \{\lfloor \mu_{jk}m/\eta \rfloor + 1, \dots, \lfloor (\mu_{jk} + 1)m/\eta \rfloor\}$ for the collection of machines transmitting the (j, k) th coefficient and note that $\#(A_{jk}) \asymp m/\eta$. Then our aggregated estimator \hat{f} on the set \mathcal{E} satisfies for $2^j + k \leq \eta B/\log n$ (i.e. the total number of different coefficients transmitted) that

$$\hat{f}_{jk} = \frac{1}{\#(A_{jk})} \sum_{i \in A_{jk}} Y_{jk}^{(i)} = f_{0,jk} + \sqrt{\frac{m}{n\#(A_{jk})}} Z_{jk} - \varepsilon_{jk},$$

where $\varepsilon_{jk} = \frac{1}{\#(A_{jk})} \sum_{i \in A_{jk}} \varepsilon_{jk}^{(i)} \in [0, n^{-1/2}]$ and $Z_{jk} \stackrel{iid}{\sim} N(0, 1)$.

Let $j_n = \lfloor \log(n^{1/(1+2s)} \wedge (\eta B/\log n)) \rfloor$. Then the risk of the aggregated

estimator is bounded as

$$\begin{aligned}
\mathbb{E}_{f_0} \|\hat{f} - f_0\|_2^2 1_{\mathcal{E}} &\leq \sum_{j=j_n}^{\infty} \sum_{k=1}^{2^j} f_{0,jk}^2 + \sum_{j=0}^{j_n} \sum_{k=1}^{2^j} \mathbb{E}_{f_0} \left(\frac{m}{n \#(A_{jk})} Z_{jk}^2 + \varepsilon_{jk}^2 \right) 1_{\mathcal{E}} \\
&\lesssim \sum_{j=j_n}^{\infty} 2^{-2js} \sup_{j \geq j_n} 2^{2js} \sum_{k=1}^{2^j} f_{0,jk}^2 + \sum_{j=0}^{j_n} \sum_{k=1}^{2^j} \eta/n \\
&\lesssim \left(\frac{\eta B}{\log_2 n} \wedge n^{1/(1+2s)} \right)^{-2s} + \frac{\eta}{n} \left(\frac{\eta B}{\log_2 n} \wedge n^{1/(1+2s)} \right) \\
&\asymp \left\{ (\log n)^{\frac{2s}{1+s}} \left(\frac{n^{1/(1+2s)}}{B \log n} \right)^{\frac{s}{1+s}} \vee 1 \right\} n^{-\frac{2s}{1+2s}} \vee \left(\frac{mB}{\log n} \right)^{-2s} \\
\text{(A.10)} \quad &\lesssim \left\{ (\log n)^{2s} \left(\frac{n^{1/(1+2s)}}{B \log n} \right)^{\frac{s}{1+s}} \vee 1 \right\} n^{-\frac{2s}{1+2s}},
\end{aligned}$$

where we have used that for $f_0 \in B_{2,\infty}^s(L)$ we have $|f_{0,jk}| \leq L$ for any $j \geq 0, k = 1, \dots, 2^j$. The above inequality together with

$$\mathbb{E}_{f_0} \|\hat{f} - f_0\|_2^2 1_{\mathcal{E}^c} \lesssim n \mathbb{P}_{f_0}(\mathcal{E}^c) \lesssim n e^{-cn} = o(n^{-1})$$

concludes the proof of the theorem.

A.3. Minimax bounds for distributed methods in L_{∞} -norm.

Similarly to the L_2 -case we consider the situation where all communication budgets are the same, i.e. $B^{(1)} = \dots = B^{(m)} = B$.

THEOREM A.2. *Consider $s, L > 0$, communication constraint $B^{(1)} = \dots = B^{(m)} = B > 0$, then*

(ib) *if $B \geq (n/(\log n)^{3+4s})^{1/(1+2s)}$, then*

$$\inf_{\hat{f} \in \mathcal{F}_{dist}(B, \dots, B; B_{\infty, \infty}^s(L))} \sup_{f_0 \in B_{\infty, \infty}^s(L)} \mathbb{E}_{f_0} \|\hat{f} - f_0\|_{\infty} \gtrsim (n/\log n)^{-\frac{s}{1+2s}}.$$

(iib) *if $(n \log(n)/m^{2+2s})^{1/(1+2s)} \leq B < (n/(\log n)^{3+4s})^{1/(1+2s)}$, then*

$$\inf_{\hat{f} \in \mathcal{F}_{dist}(B, \dots, B; B_{\infty, \infty}^s(L))} \sup_{f_0 \in B_{\infty, \infty}^s(L)} \mathbb{E}_{f_0} \|\hat{f} - f_0\|_{\infty} \gtrsim \left(\frac{n^{\frac{1}{1+2s}}}{B(\log n)^{\frac{3+4s}{1+2s}}} \right)^{\frac{s}{2+2s}} \left(\frac{n}{\log n} \right)^{-\frac{s}{1+2s}}.$$

(iiib) *if $(n \log(n)/m^{2+2s})^{1/(1+2s)} > B$, then*

$$\inf_{\hat{f} \in \mathcal{F}_{dist}(B, \dots, B; B_{\infty, \infty}^s(L))} \sup_{f_0 \in B_{\infty, \infty}^s(L)} \mathbb{E}_{f_0} \|\hat{f} - f_0\|_{\infty} \gtrsim \left(\frac{n \log n}{m} \right)^{-\frac{s}{1+2s}}.$$

This theorem is actually a direct consequence of the following more general theorem where the communication thresholds can vary between the machines.

THEOREM A.3. *Consider $s, L > 0$, communication constraints $B^{(1)}, \dots, B^{(m)} > 0$ and let the sequence $\delta_n = o(1)$ be defined as the solution to the equation (A.1). Then in the distributed Gaussian white noise model (2.1) we have that*

$$\inf_{\hat{f} \in \mathcal{F}_{\text{dist}}(B^{(1)}, \dots, B^{(m)}; B_{\infty, \infty}^s(L))} \sup_{f_0 \in B_{\infty, \infty}^s(L)} \mathbb{E}_{f_0} \|\hat{f} - f_0\|_{\infty} \gtrsim \left(\frac{n}{\log n} \right)^{-\frac{s}{1+2s}} \vee \delta_n^{\frac{s}{1+2s}}.$$

PROOF. First of all we note that in the non-distributed case where all the information is available in the global machine the minimax L_{∞} -risk is $(n/\log n)^{-\frac{s}{1+2s}}$. Since the class of distributed estimators is clearly a subset of the class of all estimators this will be also a lower bound for the distributed case. The rest of the proof goes similarly to the proof of Theorem A.1.

First we construct a finite subset $\mathcal{F}_0 \subset B_{\infty, \infty}^s(L)$ and then give a lower bound for the minimax risk over it. Let us denote by K_j the largest set of Daubechies wavelets at resolution level j with disjoint supports. Note that $|K_j| \geq c_0 2^j$ (for large enough j and sufficiently small $c_0 > 0$). Let us again multiply δ_n with a sufficiently small constant and work with this δ_n in the rest of the proof

$$(A.11) \quad \delta_n := c_0 2^{-13} L^{-2} \min \left\{ \frac{m}{n \log n}, \frac{m}{n \sum_{i=1}^m [\delta_n^{\frac{1}{1+2s}} \log(n) B^{(i)} \wedge 1]} \right\}.$$

Let $j_n = \lfloor (\log \delta_n^{-1}) / (1 + 2s) \rfloor$ and for $\beta \in \{-1, 1\}^{|K_{j_n}|}$ let $f_{\beta} \in L_{\infty}[0, 1]$ be the function with wavelet coefficients

$$f_{\beta, jk} = \begin{cases} L \delta_n^{1/2} \beta_k, & \text{if } j = j_n, k \in K_{j_n}, \\ 0, & \text{else.} \end{cases}$$

Now let $\mathcal{F}_0 = \{f_{\beta} : \beta_k \in \{-1, 1\}, k \in K_{j_n}\}$.

Note that each function $f_{\beta} \in \mathcal{F}_0$ belongs to the set $B_{\infty, \infty}^s(L)$, since

$$\|f_{\beta}\|_{B_{\infty, \infty}^s} = \sup_{j, k} 2^{(s+1/2)j} f_{\beta, jk}^2 = 2^{(s+1/2)j_n} \sup_{k \in K_{j_n}} L \delta_n^{1/2} = L 2^{(s+1/2)j_n} \delta_n^{1/2} \leq L.$$

Furthermore, if $f_{\beta} \neq f_{\beta'}$, then there exists a $k' \in K_{j_n}$ such that $\beta_{k'} \neq \beta'_{k'}$. Then due to the disjoint support of the corresponding Daubechies' wavelets

$\psi_{j_n, k}$, $k \in K_{j_n}$ the L_∞ -distance between the two functions is bounded from below by

$$\|f_\beta - f_{\beta'}\|_\infty \geq |f_{j_n k'} - f'_{j_n k'}| \cdot \|\psi_{j_n, k'}\|_\infty \gtrsim 2^{j_n/2+1} \delta_n^{1/2} \geq \delta_n^{\frac{s}{1+2s}}.$$

Next observe that for an arbitrary set of estimators $\hat{\mathcal{F}}$

$$\inf_{\hat{f} \in \hat{\mathcal{F}}} \sup_{f_0 \in B_{\infty, \infty}^s(L)} \mathbb{E}_{f_0} \|\hat{f} - f_0\|_\infty \geq \inf_{\hat{f} \in \hat{\mathcal{F}}} \sup_{f_0 \in \mathcal{F}_0} \mathbb{E}_{f_0} \|\hat{f} - f_0\|_\infty.$$

Now let F be a uniform random variable on the set \mathcal{F}_0 . Then in view of Fano's inequality (see Theorem 5.5 with $t = \delta_n^{s/(1+2s)}$ and $p = 1$) we get that

$$\inf_{\hat{f} \in \hat{\mathcal{F}}} \sup_{f_0 \in \mathcal{F}_0} \mathbb{E}_{f_0} \|\hat{f} - f_0\|_\infty \gtrsim \delta_n^{\frac{s}{1+2s}} \left(1 - \frac{I(F; Y) + \log 2}{\log |\mathcal{F}_0|} \right).$$

Hence, since $\log |\mathcal{F}_0| \geq |K_{j_n}| \geq c_0 2^{j_n} = c_0 \delta_n^{-1/(1+2s)}$, it remains to show that $I(F; Y) \leq (c_0/2) \delta_n^{-1/(1+2s)} + O(1)$.

In view of Lemma 5.6 (applied with $\delta = \delta_n^{1/2}$, $d = |K_{j_n}| = c_0 \delta_n^{-\frac{1}{1+2s}}$, $X = X^{(i)}$, $Y = Y^{(i)}$, $i = 1, \dots, m$, and noting that $\delta_n \leq m/(2^{11} L^2 n \log n)$ hence the conditions are fulfilled)

$$\begin{aligned} I(F; Y) &\leq 2L^2 n \delta_n m^{-1} \delta_n^{-\frac{1}{1+2s}} \sum_{i=1}^m \left(2^{10} \log(n) \delta_n^{\frac{1}{1+2s}} H(Y^{(i)}) \wedge c_0 \right) + 4 \log 2, \\ &\leq 2^{12} L^2 n \delta_n m^{-1} \delta_n^{-\frac{1}{1+2s}} \sum_{i=1}^m \left(\log(n) \delta_n^{\frac{1}{1+2s}} B^{(i)} \wedge 1 \right) + O(1) \\ &\leq (c_0/2) \delta_n^{-\frac{1}{1+2s}} + O(1), \end{aligned}$$

where the second inequality follows from Theorem 5.1 and assertion (A.7) for $\hat{\mathcal{F}} = \mathcal{F}_{\text{dist}}(B^{(1)}, \dots, B^{(m)}; B_{\infty, \infty}^s(L))$ and the third by the definition of δ_n , see (A.11). Hence we can conclude that

$$(A.12) \quad \inf_{\hat{f} \in \mathcal{F}_{\text{dist}}(B^{(1)}, \dots, B^{(m)}; \mathcal{F}_0)} \sup_{f_0 \in \mathcal{F}_0} \mathbb{E}_{f_0} \|\hat{f} - f_0\|_\infty \gtrsim \delta_n^{\frac{s}{1+2s}}.$$

Note that we have used the properties of the distributed estimation class $\hat{\mathcal{F}}$ only in assertion (A.7), hence for any class distributed estimator $\hat{\mathcal{F}}$ satisfying this inequality we have that

$$(A.13) \quad \inf_{\hat{f} \in \hat{\mathcal{F}}} \sup_{f_0 \in B_{\infty, \infty}^s(L)} \mathbb{E}_{f_0} \|\hat{f} - f_0\|_\infty \gtrsim \delta_n^{\frac{s}{1+2s}}.$$

□

Next we give an algorithm providing matching upper bounds in the first two cases. Note that the last case, similarly to the L_2 -norm is less relevant as using the data available only on a single machine would provide at least as good an estimator as any distributed algorithm. The algorithm is very similar to the L_2 -case, i.e. Algorithm 2, and is basically the rewrite of Algorithm 4 of [25] tailored to the Gaussian white noise model. Here we just highlight the differences compared to Algorithm 2. We divide the machines into $\eta = (\lfloor (L^2 n (\log_2 n)^{2s} / B^{1+2s})^{\frac{1}{2+2s}} \rfloor \wedge m) \vee 1$ equal sized groups ($\eta = 1$ corresponds to case (ib), while $\eta > 1$ corresponds to case (iib)). Similarly to before machines with indexes $1 \leq i \leq m/\eta$ transmit the approximations $Y_{jk}^{(i)}$ for $1 \leq 2^j + k \leq \lfloor B/\log_2 n \rfloor \wedge (n/\log_2 n)^{\frac{1}{1+2s}}$, and so on, the last machines with numbers $(\eta - 1)m/\eta < i \leq m$ transmit the approximations $Y_{jk}^{(i)}$ for $((\eta - 1)\lfloor B/\log_2 n \rfloor) \wedge (n/\log_2 n)^{\frac{1}{1+2s}} < 2^j + k \leq (\eta \lfloor B/\log_2 n \rfloor) \wedge (n/\log_2 n)^{\frac{1}{1+2s}}$. Then in the central machine we average the corresponding transmitted coefficients in the obvious way, similarly to the L_2 -norm case. The procedure is summarized as Algorithm 3 and the (up to a logarithmic factor) optimal behaviour is given in Theorem A.4 below.

Algorithm 3 Nonadaptive L_∞ -method, combined

```

1: In the local machines:
2: for  $\ell = 1$  to  $\eta$  do
3:   for  $i = \lfloor (\ell - 1)m/\eta \rfloor + 1$  to  $\lfloor \ell m/\eta \rfloor$  do
4:     for  $2^j + k = (\ell - 1)\lfloor B/\log_2 n \rfloor + 1$  to  $\ell \lfloor B/\log_2 n \rfloor$  do
5:        $Y_{jk}^{(i)} := \text{TransApprox}(X_{jk}^{(i)})$ .
6: In the central machine:
7: for  $2^j + k = 1$  to  $\eta \lfloor B/\log_2 n \rfloor$  do
8:    $\hat{f}_{jk} := \text{mean}\{Y_{jk}^{(i)} : \mu_{jk}m/\eta < i \leq (\mu_{jk} + 1)m/\eta\}$ .
9: Construct:  $\hat{f} = \sum \hat{f}_{jk} \psi_{jk}$ .

```

THEOREM A.4. *Let $s, L > 0$, then the distributed estimator \hat{f} described in Algorithm 3 belongs to $\mathcal{F}_{\text{dist}}(B, \dots, B; B_{\infty, \infty}^s(L))$ and satisfies*

- for $B \geq n^{1/(1+2s)} (\log_2 n)^{2s/(1+2s)}$,

$$\sup_{f_0 \in B_{\infty, \infty}^s(L)} \mathbb{E}_{f_0} \|\hat{f} - f_0\|_\infty \lesssim (n/\log_2 n)^{-\frac{s}{1+2s}};$$

- for $(n(\log_2 n)/m^{2+2s})^{1/(1+2s)} \vee \log_2 n \leq B < n^{1/(1+2s)} (\log_2 n)^{2s/(1+2s)}$,

$$\sup_{f_0 \in B_{\infty, \infty}^s(L)} \mathbb{E}_{f_0} \|\hat{f} - f_0\|_\infty \lesssim M_n \left(\frac{n^{\frac{1}{1+2s}}}{B(\log_2 n)^{\frac{3+4s}{1+2s}}} \right)^{\frac{s}{2+2s}} (n/\log_2 n)^{-\frac{s}{1+2s}},$$

with $M_n = (\log_2 n)^{s \vee \frac{3s}{2+2s}}$.

The proof of the theorem follows the same reasoning as the proof of Theorem A.2 but for the L_∞ -norm and it basically follows from the proof of Theorem 2.8 of [25] tailored to the Gaussian white noise model.

APPENDIX B: DEFINITIONS AND NOTATIONS FOR WAVELETS

In this section we collect some notations and definitions about wavelets, a more detailed description can be found for instance in [12, 14].

We consider the Cohen, Daubechies and Vial construction of compactly supported, orthonormal, N -regular wavelet basis of $L_2[0, 1]$, see for instance [10] and let us use the notation $\{\psi_{jk} : j = 0, 1, \dots, k = 1, \dots, 2^j\}$. For arbitrary function $f \in L_2[0, 1]$ we can consider the wavelet representation

$$f = \sum_{j=0}^{\infty} \sum_{k=1}^{2^j} f_{jk} \psi_{jk},$$

with $f_{jk} = \langle f, \psi_{jk} \rangle$. Following from the orthonormality of the wavelet basis we have that

$$\|f\|_2^2 = \sum_{j=0}^{\infty} \sum_{k=1}^{2^j} f_{jk}^2.$$

In our analysis we work with the Besov spaces $B_{2,\infty}^s$ and $B_{\infty,\infty}^s$. The corresponding Besov norms for $s \in (0, N)$ are defined as

$$\|f\|_{B_{2,\infty}^s}^2 = \sup_{j \geq j_0} 2^{2js} \sum_{k=0}^{2^j-1} f_{jk}^2 \quad \text{and} \quad \|f\|_{B_{\infty,\infty}^s} = \sup_{j \geq 0, k} \{2^{j(s+1/2)} |f_{jk}|\}.$$

Then the Besov spaces $B_{2,\infty}^s, B_{\infty,\infty}^s$ and the corresponding Besov balls $B_{2,\infty}^s(L), B_{\infty,\infty}^s(L)$ of radius $L > 0$ are defined as

$$\begin{aligned} B_{2,\infty}^s &= \{f \in L_2[0, 1] : \|f\|_{B_{2,\infty}^s} < \infty\}, \\ B_{2,\infty}^s(L) &= \{f \in L_2[0, 1] : \|f\|_{B_{2,\infty}^s} < L\}, \\ B_{\infty,\infty}^s &= \{f \in L_2[0, 1] : \|f\|_{B_{\infty,\infty}^s} < \infty\} \quad \text{and} \\ B_{\infty,\infty}^s(L) &= \{f \in L_2[0, 1] : \|f\|_{B_{\infty,\infty}^s} < L\}, \end{aligned}$$

respectively. We note that the Besov space $B_{2,\infty}^s$ is larger than the standard Sobolev space where instead of the supremum one would take the sum over the resolution levels j . For $s \neq N$ $B_{\infty,\infty}^s$ is equivalent to the classical Hölder space with regularity s , while for integer s they are equivalent to the so called Zygmund spaces, see [10].

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