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NO SEMIMARTINGALES AND CLUSTERS IN FINANCIAL MARKETS

A dissertation presented

by

Domenico Tarzia

In partial fulfillment of the requirements for the Degree of
Doctor in Finance

Dissertation Committee:

Pietro Muliere, Francesco Corielli
and Gennady Samorodnitsky

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Abstract

Despite the prediction of standard models, recurring crises happen every decade or so, with sufficient ferocity to overwhelm the statistical assumptions embedded in traditional models used for trading, portfolio management and derivative pricing. These crises seldom have discernible economic causes or warnings, and tend to propagate across the world with little regard for economic fundamentals in the affected markets. The idea of independent and stationary increments proposed by the usual class of driving process employed in the literature, the Lévy processes, seems not be a valid assumption for capturing these phenomena in certain markets: stationary stochastic processes with independent increments, indeed, even if discontinuous, do not allow for any type of dependence. Other scenarios, where the hypotheses of independence and stationarity hold, might be characterized by a particular jump structure, inconsistent with the compound Poisson process, main reference in the literature for modelling jumps. Discontinuities in prices can be clustered in space and time: a previous jump might affect, for instance, the probability of a future jump in asset returns. To model all these different scenarios, the widely applied class of Lévy jumps should be abandoned, since it seems to fail in explaining all this type of financial propagation; and different approaches and models proposed as an attempt to describe and

capture some particular features of the financial markets. This dissertation enriches the class of stochastic processes, used to model securities, by following a dual path. The first trajectory consists in relaxing the idea of "independent increments" and "semimartingale" by proposing random paths, whose principal difference with respect to previous literature stems in considering the "Non-Markovian" property as an embedded in the financial time series. The route is crossed into two different ways. The first road proposes a stochastic differential equation for the continuous-time framework, where the innovation term belongs to the class of Gaussian processes. Differently with respect to the existing literature that picks, as benchmark for this particular group, the Brownian motion, the Gaussian process might not be a semimartingale, allowing the dynamics of the stock return to incorporate the idea of non-stationary and dependent increments. The other way departs from the "Normality" and leads into the "Leptorkutosis" world. A linear fractional stable model is proposed as innovation term in the stochastic differential equations. A fractional process is chosen, because of the possibility of modelling some particular phenomenon, arising in the analysis of spatial or time series data: different values of its Hurst parameter characterize the dependence structure of the process, ranging from long-range dependence to negative dependence. The self-similarity property is combined with heavy-tailed distributions. Even if fat tails are considered undesirable because of the additional risk they imply, traumatic "real-world" events (such as an oil shock, a large corporate bankruptcy, or an abrupt change in a political situation) might be better explained by a process characterized by infinite variance: historical examples include the Black Monday (1987), Dot-com bubble, Late-2000s financial crisis, and the unpegging of some currencies. The second path, instead, aims at the jump structure of the stochastic differential equation, driving the stock returns. The compound Poisson process, as innovation term able to capture point of discontinuity in the prices, is generalized and substituted in the stock's dynamics by an additive process, whose integral form is defined with respect to a compensated Poisson random measure. The kernel function inside the stochastic integral provides enough flexibility in modelling the jumps, particularly with respect its dependence structure. The jump structure is allowed to be "self-excite" itself: the best process belongs to the class "self-exciting point processes". They are an extension of temporal Poisson processes, being dependent on the past event history, i.e. possessing after-effects of previous point events, unlike Poisson processes. Their intensity process takes a constant value for a homogeneous Poisson process and a deterministic function for a not-homogeneous Poisson path; and the probability of point event occurrences can be influenced by past point events. Similar to territory characterized by seismic quiescence, a scenario where financial primary shocks generate secondary after-shocks is modelled, deriving a theoretical pricing formula for the contingent claims.

Acknowledgements

First I would like to thank my advisor Pietro Muliere. It has been an honour to be his Ph.D. student. I appreciate all his contributions of time and ideas. The joy and enthusiasm he has for his research was contagious and motivational for me, even during tough times in the Ph.D. pursuit. I would also like to thank the other two members of my oral defence committee, Francesco Corielli, for his time and insightful questions; and Gennady Samorodnitsky, for his constructive feedback and the given opportunity of visiting the School of Operations Research and Information Engineering at Cornell. It is no easy task, reviewing a thesis, and I am grateful for their thoughtful and detailed comments. This thesis is the end of my journey in obtaining my Ph.D. It is a special feeling to bring a huge story to the end. I have not travelled alone in the desert during this journey. This thesis has been kept on track and been seen through to completion with the support and encouragement of my beloved friends, without whom life would be bleak. I have been lucky to have them being part of my life: my time at Milan was made enjoyable in large part thanks to them. A special thanks to my family. I am grateful to my uncles, aunts, cousins and grandparents for having provided a loving environment to me, and all of the sacrifices that they have made on my behalf. I wish to thank my brother, Antonello, and my sister in law, Wilma.

Words cannot express how thankful I am to them for their support; and I am indebted to them for having brought into my life two little angels, Giuseppe Gabriel and Greta Maria. I would not have contemplated this road if not for my mom, Maria Eleonora, who instilled within me a love of science and knowledge, all of which finds a place in this thesis. She has sacrificed her live and provided unconditional love and care. I thank her for giving me the chance of becoming who I am. Lastly, and most importantly, I would like to remember my father, Giuseppe. His understanding, the faith, advise, intellectual spirit and indescribable support to me throughout my whole life were invaluable. I appreciated his honesty, including compliments and criticisms. I am very grateful about his care, his love and trust in me. To him I dedicate this thesis.

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Contents

Introduction

The First Fundamental Theorem of Asset Pricing states that "No arbitrage[1] holds if and only if there is a probability measure Q with $Q(\omega) > 0\ \forall\omega \in \Omega$, such that every discounted price process $S^*(t)$ is a $Q-martingale$. Such a measure Q is called an equivalent martingale measure (EMM)". A contingent claim Y is a contract between a seller and a buyer. Since the seller promises to pay the buyer the amount Y at time T, the buyer normally pays some money to the seller at a certain time $t < T$, when they make the agreement. Y is said to be marketable or attainable if there exists a self-financing trading strategy h whose value at T satisfies $V(T) = Y$. In this case, h is said to replicate or generate Y. The Risk neutral valuation principle claims that "Assuming no arbitrage, the time t value of a marketable contingent claim Y is equal to $V(t)$, the time t value of the portfolio that replicates Y. Moreover, the discounted value process $V^*(t)$

$$V^*(t) = E^Q[Y|B(T)|\mathcal{F}_t]$$

[1]A portfolio is self-financing if there is no exogenous infusion or withdrawal of money; the purchase of a new asset must be financed by the sale of an old one. An arbitrage opportunity is said to exist if there is a self-financing strategy h whose value function $V(\cdot)$ satisfies
(a) $V(0) = 0$;
(b) $V(T) \geq 0$;
(c) $P(V(T) > 0) > 0$.

where $\left(B(T)^{-1} = \exp(-\int_t^T r(s)ds\right)$ is the discount factor. In 1973, Fisher Black and Myron Scholes published in the Journal of Political Economy an innovative article whose title was “The Pricing of Options and Corporate Liabilities" to govern the prices of the securities in the continuous-time framework. The Black–Scholes model of the market for a particular stock makes the following explicit assumptions:

- no arbitrage opportunity, i.e. no opportunity to make a riskless profit;
- it is possible to borrow and lend cash at a known constant risk-free rate;
- it is possible to buy and sell any amount, even fractional, of stock (this includes short selling);
- The above transactions do not incur any fees or costs (i.e., frictionless market);
- The stock price follows a geometric Brownian motion with constant drift and volatility;
- The underlying security does not pay a dividend.

From a mathematical point of view, they assumed the following diffusion, under the historical probability, for the stock return

$$dX_t = \mu dt + \sigma dB_t$$

where B_t is a one dimensional Brownian motion, and its pathwise uniqueness is guaranteed by the boundedness of σ and its Holder continuity of order greater than or equal to $\frac{1}{2}$. A change of measure by means of Girsanov theorem defined the dynamics under the risk-neutral probability; and a closed-form expression for the stock, by means of Ito's lemma. Once derived a risk-neutral expression for the stock, they derived the price π_t of a contingent claim with payoff function $\Phi(\cdot)$ as

$$\pi_t(S, K, r, \sigma) = E^Q \left[e^{-r(T-t)} \Phi(S_T) \,|\mathcal{F}_t \right]$$

Many doubted the validity of the model; in particular criticism picked up the following drawbacks:

1. the risk-free rate and the stock's volatility are constant; while the risk free rate and volatility fluctuates according to market conditions;

2. the stock prices are continuous and that large changes (such as those seen after a merger announcement) do not occur;

3. the stock pays no dividends until after expiration;

4. the stock's volatility is not directed observed, but estimated;

5. the model overvalues far out-of-the-money calls and undervalues deep in-the-money calls; it tends to misprice options that involve high-dividend stocks

Among all these flaws, the constant volatility is the most relevant issue. Take a European call option, whose payoff function is $\Phi\left(S_T\right) = \left(S_T - K\right)_+$, the pricing functional becomes

$$\pi_t(S, K, r, \sigma) = S(t)N(d_1) - e^{-r(T-t)}KN(d_2)$$

where $N(\cdot)$ is a standard normal cumulative distribution function, and

$$\begin{aligned} d_1 &= \frac{\ln\frac{S_t}{K} + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{(T-t)}} \\ d_2 &= \frac{\ln\frac{S_t}{K} + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{(T-t)}} \end{aligned}$$

The theoretical value of a vanilla option is a function of the following set of parameters $\Theta = \{T, K, r, S, \sigma\}$, while $N(\cdot)$ stems from using the Brownian motion as innovation term in the stochastic differential equation. All the parameters in the model (other than the volatility σ) — the time to maturity T, the strike price K, the risk-free interest rate r, and the current underlying price S – are unequivocally observable: all other things being equal, an option’s theoretical value is a monotonic increasing function of implied volatility and is possible to compute a unique implied volatility from a given market price for an option. If the assumption of constant volatility of the underlying is correct, by plotting the implied volatility against the strike price, a flat line must describe the volatility, parallel to the X-axis, the strike price. Empirical results are, instead, different:

- equity markets have typically downward sloping graph of the implied volatility, and the more familiar term"volatility skew" is often used;
- currency markets are characterized by valley-shaped implied volatility;
- in other markets, such as foreign exchange options, the typical

graph turns up at either end, justifying the term "volatility smile".

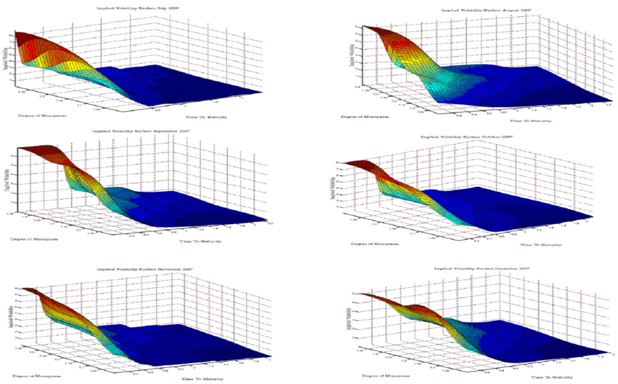

Figure A: Implied volatility surfaces of the S&P500 Index Options [July 2007-December 2007]

It is often useful to plot implied volatility as a function of both strike price and time to maturity. The result is a three-dimensional curved surface whereby the current market implied volatility ($Z-$axis) for all options on the underlying is plotted against strike price and time to maturity (X & $Y-$axes). If the assumption of constant volatility holds, the implied volatility surface must be a flat plane with respect to K and T. Figure (A) plots the implied volatility surfaces of the S&P500 Index Options for six different days, ranging from July 2007 to December 2007. Not only does the implied volatility surface represent a flat with respect to strike price and time to maturity, but it shows both volatility skew and term structure of volatility. Large-scale effort in the continuous-time framework has been model in order to propose more accurate stochastic differential equation, able to capture and better explain the implied volatility structure. Innovations have been proposed in the structure of the drift and diffusive coefficient. Constant parameters $\theta = [\mu, \sigma]$ have been replaced first with time-varying parameters $\theta(t) = \mu(t), \sigma(t)$; and later with stochastic drift and random diffusive coefficient $\mu(t, X_t)$ and $\sigma(t, X_t)$. Researchers and practitioners have been focused also on proposing different innovation terms with respect to the Brownian Motion; discontinuities have been modelled through compound Poisson processes, mainly. Even if agreement on the mechanism driving the historical data has not been reached, many models make their strongest assumption in imposing random paths belonging to the class of semimartingales: particularly the class of Lévy processes. This specific class imposes a particular structure in the dynamics of the financial securities: the innovation term has independent and stationary increments. This dissertation points at proposing a generalization of the problem by applying different classes of random path in the option pricing fields . It has been shown that often the semimartingale assumption is not reflected in the historical data, and this anomaly in pricing contingent claims must be considered. The work is structured in the following way. Chapter 1 describes some statistical properties of the financial time series, by pointing out some stylized facts, quite common to wide markets. After a short review of the existing literature in the continuous-time framework, two different paths are suggested to be followed, in order to generalize the problem, without imposing, necessarily, as starting point, the semimartingale property. The two paths are not necessarily mutually exclusive, since there might be some space in future for combining them in a more general approach; and some possible theoretical explanations for generalizing the problem are provided, ranging from economic and financial reasons to cognitive mechanisms. The appendix briefly summarizes the three most relevant theories about financial markets: the efficient market hypothesis, the most dominant in the literature; the fractal market hypothesis and the coherent market hypothesis. The first path is tackled in Chapter 2 and 3, where different stochastic differential equations with respect to the literature are proposed to model the dynamics of financial securities. Two classes of random paths are taken into consideration for modelling the innovation term in the stochastic differential equation; their main difference stems in their dispersion measure: the first group has finite variance; while the second unit is characterized by infinite variance. The finite-variance processes are introduced in Chapter 2, where defines the class of Gaussian processes, their canonical representation and main properties. An integral representation with a Gaussian process is then provided; and some restriction on the kernel function of the integral are introduced, in order to use all the powerful machinery provided by the stochastic calculus: the differential of a time-dependent function of a stochastic process can be computed, by means of Ito's lemma. An entropy argument, combined with the tightness property, establishes a connection among different families of Gaussian processes; and derives a closed-form formula for the stock's expression and contingent claims, even if the stochastic differential equation, under the historical probability, is not a semimartingale. A specific random element of the Gaussian class, characterized by long-range dependent and nonstationarity increments, the sub-fractional Brownian motion ($SFBM$), is then considered to model the stock's behavior and price some contingent claims: a closed-form formula for a European call option is provided. An empirical analysis on the Standard & Poor's (S&P) 500 Index Options concludes the chapter by comparing theoretical prices and market quotes and regressing the pricing errors on some explanatory variable, suggested by the existing option-pricing literature. Chapter 3 selects a specific stochastic process, inside a particular class of infinite-variance process, the alpha-stable family. Stationary alpha stable processes ($S\alpha S$) are characterized by means of conservative and dissipative flows, and a fundamental decomposition of $S\alpha S$ is provided. After defining the properties of long-range and negative dependence, a specific process inside the class of stable processes is selected: the linear fractional stable motion ($LFSM$). This specific random path combines heavy tails and dependence: its Hurst parameters H models different dependence structures, ranging from negative to long-range dependence. class of self-similar processes with infinite variance by defining their main properties. A stochastic differential equation where the innovation term is the linear fractional stable motion is proposed to model the stock's return under the historical probability. A minimum distance technique is used to provide a link between the linear fractional stable motion and a family of stable motions; and by means of Fourier analysis a theoretical pricing formula for contingent claims is provided. An empirical analysis on the S&P100 Index Options suggests the existence of negative dependence in the financial data. The last part of the analysis compares theoretical prices and market quotes and tries to explain the pricing errors, as in the previous chapter. The second path very relevant is to generalize the discontinuities by moving from the class of Compound Poisson processes to the group of Poisson random measures. Chapter 4 defines an additive process, whose integral representation form might be associated to a certain Poisson random measure and its compensated measure. Poisson processes are characterized by constant intensity function; while in more general point processes the probability of a jump is not constant over time, but time-varying. The class of self-exciting process, i.e. random paths where the stochastic intensity is an increasing function of the past evolution of the process, is defined and a particular element of this group, the Hawkes process, is described by pointing out its main differences with respect to the Poisson process. The massive machinery of the stochastic calculus with respect to point processes, permits to compute the differential of a time-dependent function of the process. By Ito's lemma, a closed-form expression for the stock price is derived when its dynamics is described by a stochastic differential equation whose its jump are modelled through a Poisson random measures. A specific example is provided by using a Hawkes process as innovative term, and simulation of the process are shown at the end of Chapter 4. Chapter 5 concludes the work, summarizes the main results and draws some final conclusions regarding theoretical and empirical approach, adopted in this work. It has also the intent to suggest some possible ways to extend the analysis, either by taking the two paths and moving further, or suggesting alternative routes to be followed.

References

[1] Black, F., and M. S. Scholes, "The pricing of Options and Corporate Liabilities", *Journal of Political Economy*, 81 (1973), 637-654.

Chapter 1

The two paths

The study of statistical properties of financial time series has revealed a wealth of interesting stylized facts which seem to be common to a wide markets, instruments and periods:

- Excess volatility: many empirical studies point out to the fact that it is difficult to justify the observed level of variability in asset return by variations in "fundamental" economic variables. In particular, the occurrence of large (negative or positive) returns is not always explainable by the arrival of the new information on the market.
- Heavy tails: the (unconditional) distribution of returns displays a heavy tail with positive excess kurtosis.
- Absence of autocorrelation in returns: (linear) autocorrelation of asset returns is often insignificant, except for very small intraday time scale ($\simeq$ 20 minutes) where microstructure effects come into play.

- Volatility clustering: as noted by Mandelbrot, "large changes tend to be followed by large changes, of either sign, and small changes to be followed by small changes". A quantitative manifestation of this fact is that, while returns themselves are not correlated, absolute returns $|r_t|$ or their squares displays a positive, significant and slowly decaying autocorrelation function: $corr\left(|r_t|, |r_{t+\tau}|\right) > 0$ for τ ranging from a few minutes to a several weeks.

- Volume/volatility correlation: trading volume is positively correlated with market. Moreover, trading volume and volatility show the same type of "long memory" behavior.

This empirical evidence has intrigued many researchers and oriented in a major way the development of stochastic models in finance, primarily intended to model all these phenomena. A large number of empirical studies have investigated the failure of the idea of "independent assets returns". The concept of self-similarity, scaling, leptokurtosis have been repeatedly used to describe properties of financial time series such as stock prices, foreign exchange rates, market indices and commodity prices. A typical display of daily log-returns is shown in Figure (1): the volatility clustering-feature is seen graphically from the presence of sustained periods of high or low volatility. While there is a vast literature on dependence in asset prices, most authors tackle the questions ei-

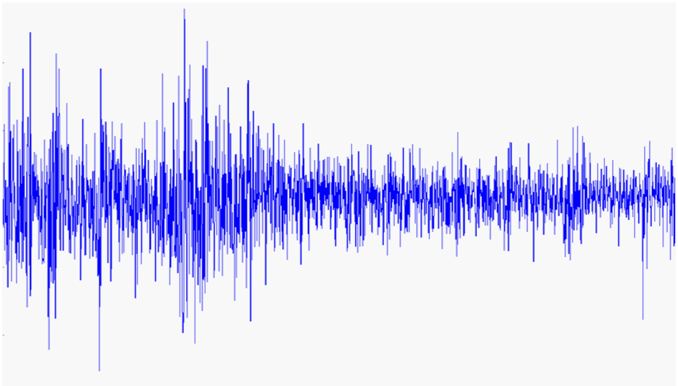

Figure 1: Standard & Poor's (S&P) 500 index daily log returns. Δ= 1 day

ther from a purely theoretical perspective or from a purely empirical one, rarely both. This work attempts to discuss the relevance of these notion in the context of financial modelling, in the particular area of continuous-time processes and stochastic differential equations, in relation with the basic principles of financial theory, and empirically, by comparing them to properties of market data. Step by step, different types of random processes very relevant are proposed in order to model different scenarios in the field of option pricing. Section 1.1. briefly summarizes the option pricing literature; while the next section defines some possible paths to follow in order to perform the analysis. Section 1.3. discusses some possible explanations for the presence of such properties in financial time series. The appendix provides some alternative theories to the efficient market hypothesis (EMF).

1.1. Lévy processes as benchmark for the option pricing literature

Standard models for portfolio allocation and option pricing both assume that continuously compounded returns are normally distributed. The central limit theorem is often invoked as a primary motivation for this assumption. By this theorem, the normal distribution arises as the limiting distribution for the sum of n independent random variables, when the sum is divided by $\sqrt{n}$. A normal distribution arises by assuming financial returns, as a sum of a large number of independent influences. Unfortunately, it is well documented that normality of the return distribution is violated in both time-series and in option prices. The Brownian motion emerged as benchmark process for describing asset returns in continuous time, given that the key to developing successful strategies for managing risk and pricing assets is to parsimoniously describe the stochastic process governing asset dynamics. However, many studies of the time series of asset returns and derivatives prices have concluded that there are at least three systematic and persistent departures from this benchmark for both statistical and risk-neutral process.

First, prices jump, leading to non-normal return innovations. Second, return volatility varies stochastically over time. Third, returns and their volatilities are correlated, often negatively for equities. New sophisticated processes have been proposed in the option pricing literature to capture all three facts. Merton (1976) has adopted a compound Poisson processes to model jumps; while Heston (1993) a mean-reverting square-root to model stochastic volatility. Duffie et al. (2000) has proposed a model where the asset return and variance are driven by a finite number of potentially correlated state variables. More general jump structure occurring within any finite time interval have been investigated by the inverse Gaussian model of Barndorff-Nielsen (1998), the generalized hyperbolic class of Eberlein et al. (1998), the variance-gamma (VG) model of Madan et al. (1998), the generalization of VG in Carr et al. (2002), the finite moment log-stable model of Carr and Wu (2003). Empirical work by these authors has been generally supportive of the use of infinite-activity process as a way to model returns in a parsimonious way. A further step has been done by Carr and Wu (2004) where time-changed Lévy process have been used in order to address simultaneously these three issues. The classical Black&Scholes model, step by step, has been substituted by more general Lévy processes, relatively easy to manage because of their structure and, more important, the martingale property, not violating the first fundamental theorem of finance.

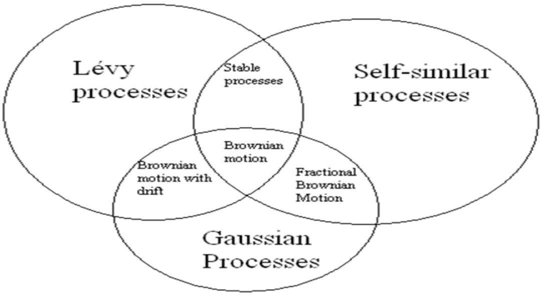

Figure 2: Relation between Self-similar processes, Lévy processes and Gaussian Processes

Carr et al (2005) have proposed the additive processes: a slight generalization of Lévy processes, with independent but time-inhomogeneous increments. Figure (2) describes the relationship among different classes of stochastic processes; while Figure (3) points out what has been the most relevant class in the option pricing literature.

1.2. Beyond Lévy processes: possible departures

A common element among all these processes considered in the literature lies in generating process whose increments are independent. Phe-

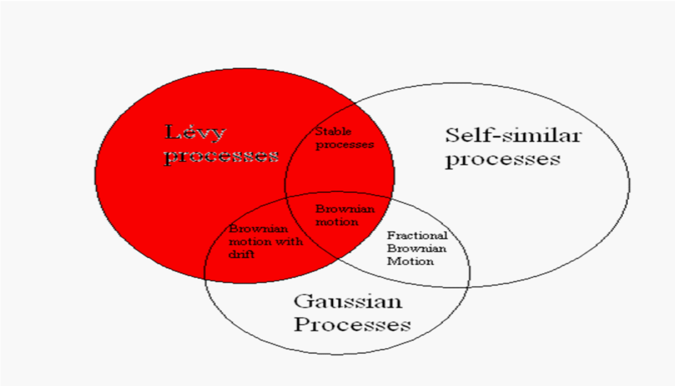

Figure 3: Most relevant class in the option pricing literature

nomena such a stochastic volatilities and correlation are taken into consideration by these processes but clustered inside independent blocks. Independent increments are the main core of the theory of rational expectations theory; but this assumption can be released to verify the existence of some deviations. The idea of this dissertation is to overcome the barrier created by the all these processes by working with random processes belonging to different classes. The first idea, presented in the second chapter, consists in relaxing the assumption of independent and stationary increments in order to verify the existence of some distortions in prices. Stochastic processes, not belonging to the Lévy classes, are used to deliver a theoretical pricing formula in the option pricing context and capture the existence of deviation from rationality. The sto-

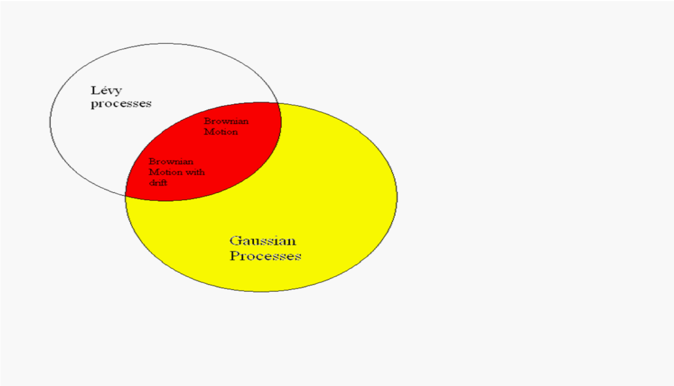

Figure 4: From Brownian motion to Gaussian processes

chastic process proposed keeps the Gaussian properties but it is not strictly characterized by independent and stationary increments. Figure (4) shows how to move from the class to the Lévy process to the generic class of Gaussian processes, where stationarity and independence of increments are not anymore relevant properties.

Financial data might be also self-similar. Self-similarity can have very different origins: it can arise from high variability, in situations where increments are independent and heavy tailed (stable Lévy processes) or from strong dependence between increments even in absence of high variability. These two mechanisms for self-similarity have been called the "Noah effect" and the "Joseph effect" by Mandelbrot. By mixing these effects, one can construct self-similar processes where both long range

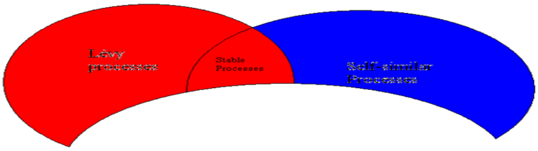

Figure 5: From the Stable class to Self-similar processes

dependence and heavy tails are present: the third chapter tries to analyze this particular scenario by picking inside the class of fractional stable processes. Figure (5) outlines the path to followed in order to move from the Lévy-stable class to the group of self-similar processes. Economic crisis, war, decisions of the central banks, credit crunches might affect the financial data. All these unexpected events might cause a sudden shift in the stock's price, so evident and impossible to be justified by continuous diffusion like the following

$$dX_t = \mu\left(t, X_t\right) + \sigma\left(t, X_t\right) dB_t$$

where B_t is a one dimensional Brownian motion, even if its pathwise uniqueness is guaranteed by the boundedness of σ and its Holder continuity of order greater than or equal to $\frac{1}{2}$. The classical stochastic differential equation with respect to Brownian motion has been, therefore, replaced by a stochastic differential equation so defined

$$dX_t = a\left(X_{t-}\right) dZ_t$$

where Z_t is a Lévy process. Z_t might be for instance a symmetric stable process of index α. A further step has been made by considering the following dynamics

$$dX_t = \mu\left(t, X_t\right) + \sigma\left(t, X_t\right) dB_t + a\left(X_{t-}\right) dZ_t$$

where the stock's price is driven by a continuous component and a jump component. The main critique to the previous dynamics is that the jump size might not be appropriate for certain markets. The previous diffusion assumes that if Z_t has a jump of size z, then the jump of X_t would be $a\left(X_{t-}\right) z$; whereas the jump size could depend on X_{t-} and z, but not necessarily linear in z: the behavior of X might be qualitatively different from the case where the underlying randomness has a small jump. Figure (6) shows how to face this particular process in the fourth chapter of this dissertation.

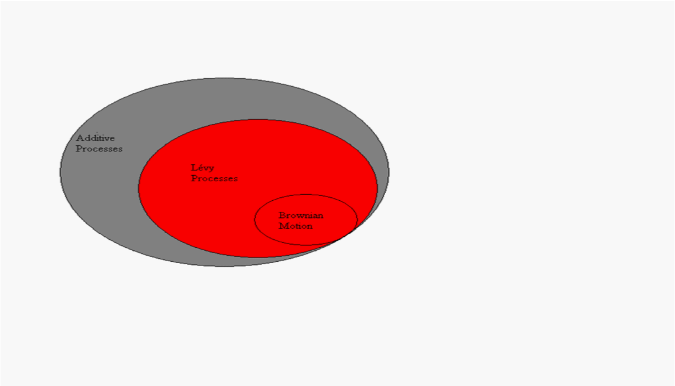

Figure 6: From Lévy processes to the group of Additive processes

1.3. Economic, financial and cognitive mechanisms for relaxing the idea of independence

Many stochastic processes mimic volatility clustering and dependence in financial time series but do not provide any economic explanations for it. The fact that these observations are common to a wide variety of markets and time periods suggest that common mechanisms may be at work in these markets. Many attempts have been made to trace back the phenomenon of dependence, particularly long-range dependence, to some mechanisms present in the market generating this volatility.

1.3.1. Heterogeneity in time horizon of economic agents

Heterogeneity in agent's time scale has been considered as a possible origin for various stylized facts. Long-term investors, for instance, focus on long-term behavior of prices, whereas traders aim to exploit short-term fluctuations. Long memory in economic time series might be due to the aggregation of a cross section of time series with different perspective levels. This phenomenon can explain volatility clustering in terms of aggregation of different information flows.

1.3.2. Evolutionary models

Several studies have considered modelling financial markets by analogy with ecological systems where various trading strategies coexist and evolve via a "natural selection" mechanism, according to their relative profitability. A financial market can be seen as set of agents, characterized by their (set of) decision rules. A decision rule is defined as a mapping from an agent information set (price history, trading volume, other economic indicators) to the set of actions (buy, sell, no trade)

1.3.3. Switching between trading strategies

Dependence might be caused by switching agents trading behavior between two or more strategies. Examples where switching of economic agents between two behavioral patterns are quite common in the literature: in the context of financial markets, these behavioral patterns can be seen as trading rule and the resulting aggregate fluctuations as large movements in the market price, i.e. heavy tails.

1.3.4. Behavioral finance

Many studies investigated how people depart from rationality, not whether consumers depart from rationality. "Observed departures from perfect rationality in judgment under uncertainty" have been conceptualized by Kahneman and Tversky by noting that "people rely in "heuristic principles" which reduce the complex task of assessing probabilities and predicting values to simpler judgmental operations. These heuristics are quite useful, in general, but sometimes they lead to severe and systematic errors. Many judgments are all based on data of limited validity, which are processed according to heuristic rules. For example, the apparent distance of an object is determined in part by its clarity, the more sharply the object is seen, the closer it appears to be. This rule has some validity, because in any given scene the more distant objects are seen less sharply than nearer objects. However, the reliance on this rule leads to systematic errors in the estimations of distance. Specifically, distances are often overestimated when visibility is poor because the contours of objects are blurred. On the other hand, distances are often underestimated when visibility is good because the objects are seen sharply. Thus, the reliance on clarity as an indication of distance leads to common biases. Such biases are also found in the intuitive judgment of probability. An economic agent could indeed infer:

1. "too much" because she thinks she knows more than she does (Overinference);
2. "too little", not absorbing all the evidence that should tell her things with high confidence(Underinference);
3. the wrong things altogether (Misinference).

Failure in inference might be caused singularly or jointly by:

- errors in processing some information (Lonely Errors);
- the thinking and behavior of other volitional agents involved in reasoning (Social Errors);
- from data engaged with (Active Errors)
- inattentiveness to the world, not extracting info that you should extract (Passive Errors).

Phenomena like "Overconfidence", where (private) signals are overweighted, can be easily associated with "Under-extraction", where an economic agent gather too little information from the environment around you because she does not notice relevant events ("Inattention") or is not able to extract the right information content ("Cursedness"). She might perceive more streaks of good or bad performance that there really is, in the sense that she perceives long streaks of recent performance signal repetition of the same performance (Hot-Hand Fallacy").[1] Once she forms a strong assumption, she simply stop being attentive to relevant new information that contradicts or support her assumptions ("Anchoring"): she becomes convinced that one investment strategy is more lucrative than another and does not pay attention to every freely available additional information.[2] The same agent might, once adopted an opinion, draw all things else to support and agree with it ("Confirmatory Bias"); or mispredicts her future tastes ("Projection Bias"). Other sampling errors might also arise because she erroneously believes that:

1. a fair coin that has come up 3 heads in a row is "due" for a tails ("Gambler's Fallacy");

2. a mutual fund she is absolutely confident is only average but has

[1] Some might believe that if a basketball player has made several shots in a row that he is "hot"/ in the zone/ on fire, and is likely to continue making shots (till he turns cold).

[2] People who use weak evidence to form initial hypotheses have difficulty correctly interpreting subsequent better information that contradicts those initial assumptions. Perkins (1981) supports for the perspective that fresh thinkers may be better at seeing solutions to problems than people who have meditated at length on the problems, because the fresh thinkers are not overwhelmed by the inference of old hypotheses.

outperformed the market 3 quarters in a row is "due" for underper-fomance ("Law of Averages");

3. short subsequences of long sequences are more likely that they are to resemble proportions of the overall sequence ("Local Representativeness"), i.e. she is flipping a coin 200 times,and exaggerates the frequency of 4-flip sequences containing 2 heads and 2 tails (and hence she underestimates the frequency of streaks);

4. unusual events might be followed by usual events ("Misuderstanding regression to the mean");[3]

5. there is more variance across in performance from different sources than there really is ("Fictitious Variation"), f.i. an economic agent might think that there is more variation in the long-run or expect performance of different mutual funds than there really is.

Most of the time, a rational agent does not get just how much she learns from large data sets (Non Belief in the Law of Large Numbers) and believes she is learning more from small samples that they really are ("Law of Small Numbers"). Not only does she believe that large-sample means might not be close to population proportion; but she also

[3]"Israeli flight instructors observed that after praising trainess for unusually good landings, the trainess tended to do worse on their next landing. So they stopped praising." (Kahneman and Tversky).

exaggerates how likely it is that the proportions in small random samples closely resembles the proportions of the overall population: she over-infers from small samples and under-infers from large samples by exaggerating, f.i., the likelihood that 4 flips of fair coin will be 2 heads/2 tails; or the likelihood that 6 quarters of an average mutual fund will beat the market 3 times and underperform 3 times. So often, then, she might face situations where she infers too much from his friend's bad experience with a car; or is not convinced by falseness of a theory she believes, even with infinite evidence. These are all clearly related: compared to how truly i.i.d. the world around us is, she expects too much balance/negative autocorrelation in the small sample she observes and either she fails to figure out the truth despite large sample of signals revealed in behavior ("Bernoulli Failure"); or thinks she knows the truth even though she is wrong ("Socratic Failure").

Appendix

A. The efficient market hypothesis

Definition 1 *A financial market is (informational) efficient when market prices reflect all available information about value*

The observations have to be independent or, at best, must have a short-memory: that is, the current change in prices could not be inferred from previous changes. This could occur only if price changes are a random walk and if the best estimate of the future price is the current price. The process should be "a martingale", or fair game. Future price changes could not be inferred from past price changes. The current prices reflect this information because all investors have equal access to it, and being "rational", they would, in their collective wisdom, value the security accordingly. Thus investors, in aggregate, could not profit from the market because the market "efficiently" valued securities at a price that reflected all known information.

B. The fractal market hypothesis

The fractal market hypothesis emphasizes the impact of liquidity and investment horizons on the behavior of investors. To make the hypothesis as general as possible, it does place no statistical requirements on the process. Its purpose is to give a model of investor behavior and market price movements that fits the observations. Market exists to provide a stable, liquid environment for trading. Investors wish to get a good price, but that would not necessarily be a "fair" price in the economic sense. Short covering rarely occurs at a fair price. Market remains stable when many investors participate and have many different horizons. When a five-minute trader experiences a six-sigma event, an investor with a longer investment horizon must step in and stabilize the market. As long as another investor has a longer trading horizon than the investor in crisis, the market stabilizes itself. For this reason, the investor must share the same risk levels (once an adjustment is made for the scale of investment horizon) and the shared risk explains why the frequency distribution of returns looks at the same at different investment horizons. This proposal is called the fractal market hypothesis because of this self-similar statistical structure. Market becomes unstable when the fractal structure breaks down. A break-down occurs when investor with long investment horizons either stop participating in the market or become short-investor themselves. The fractal market hypothesis proposes the following:

1. The market is stable when it consists of investors covering a large number of investment horizons. This ensures that there is ample liquidity for traders.

2. The information set is more related to market sentiment and technical factors in the short term than in the longer term. As investor horizons increase, longer-term fundamental information dominates. Thus, price changes may reflect information important only to that

investment horizon.

3. If an event occurs that makes the validity of fundamental questionable, long-term investors either stop participating in the market or begin trading based on the short-term information set. When the overall investment horizon of the market shrinks to a uniform level, the market becomes unstable. There are no long-term investors to stabilize the market by offering liquidity to short-term investors.

4. Prices reflect a combination of short-term technical trading and long-term fundamental valuation. Thus, short-term price changes are likely to be more volatile, or "noisier", than long-term trades. The underlying trend in the market is reflective of changes in expected earning, based on the changing economic environment. Short-term trends are more likely the result of crowd behavior. There is no reason to believe that the length of the short-term trends is related to the long-term economic trend.

5. If a security has no tie to the economic cycle, then there will be no long-term trend. Trading, liquidity, and short-term information will dominate.

C. The coherent market hypothesis

This capital market theory looks at a market's progressions as complex dynamic processes similar to those explained by chaos theory. According to this model, as government policy, investor expectation and sentiment, technological and financial innovation, and other such factors change over time, four types of markets emerge during different phases of economic cycle:

(a) steady state random walk;

(b) unstable transition;

(c) chaotic dynamics;

(d) coherent cycles.

Investor sentiment and the prevailing bias in economic fundamentals control the state of the market. When investor sentiment is not conducive to "group think" or crowd behavior, the market is likely to be in a random-walk state (efficient market): the stock market is least likely to outperform fixed dollar and fixed income alternatives. The second step would be an unstable transition, where the random-walk model would be no longer valid. As time passes, the market will reach the third step, i.e. the chaotic dynamics where there is a possibility of abrupt sentiment shifts from bullish to bearish. Then, the coherent-cycle stage, where depending on polarization, comes up on his bullish or bearish form: a coherent bull market can be thought of as a chaotic market in which the bearish side of potential well is high, and the associate lobe of the probability distribution becomes small; while coherent bear is the mirror image.

References

[1] Barndorf-Nielsen, O. E., "Processes of Normal Inverse Gaussian Type", *Finance and Stochastics*, 2 (1997), 41-68.

[2] Bates, D. S., "Post Crash Moneyness Bias in S&P 500 Futures Options", *Rodney L. White Center working paper* (1991), Wharton School, University of Pennsylvania, Philadelphia, PA.

[3] Black, F., and M. S. Scholes, "The pricing of Options and Corporate Liabilities", *Journal of Political Economy*, 81 (1973), 637-654.

[4] Carr, P., and D. Madan, "Option valuation using the fast Fourier transform", *Journal of Computational Finance*, 2 (1999), 61–73.

[5] Carr, P., and L. Wu, "Finite moment log stable process and option pricing", *Journal of Finance*, 58 (2003) , 753–777.

[6] Carr, P., and L. Wu,"Time-changed Lévy processes and option pricing", *Journal of Financial Economics*, 71 (2004), 113–141.

[7] Carr, P., and L. Wu, "Stochastic skew in currency options", *Journal of Financial Economics*, 86, 1 (2007), 213-247.

[8] Carr, P., Geman, H., Madan, D., and M. Yor, "The fine structure of asset returns: an empirical investigation", *Journal of Business*, 75 (2002), 305–332.

[9] Cochrane, J., *"Asset Pricing",* Princeton University Press, Princeton, NJ, (2004).

[10] Cont, R., "*Long range dependence in financial time series"*, Fractals in Engineering, Springer, (2005).

[11] Duffie, D., Pan, J., and K. Singleton, "Transform analysis and asset pricing for affine jump-diffusions", *Econometrica*, 68 (2000), 1343–1376.

[12] Eberlein, E., and U. Keller, "Hyperbolic distributions in Finance", *Bernoulli*, 1 (1995), 281-299.

[13] Eberlein, E., and U. Keller, and K. Prause "New insights into smile, mispricing and value at risk: the hyperbolic model", *Journal of Business*, 71 (1998), 371-405.

[14] Eberlein, E., and K. Prause, "The generalized hyperbolic model: Financial derivatives and risk measures", *FDM preprint*, 56, (1998) University of Freiburg.

[15] Heston, S., "A Closed-Form Solution of Options with Stochastic Volatility with Applications to Bond and Currency Options", *The Review of Financial Studies*, 6 (1993), 327-343.

[16] Kahneman, D., and A. Tversky, "Prospect Theory: An Analysis of Decision Under Risk", *Econometrica*, 47 (1979), 263-291.

[17] Kahneman, D., Slovic, P. and A. Tversky, *"Judgment Under Uncertainty: Heuristics and Biases",* New York: Cambridge University Press, (1982).

[18] Kahneman, D., and A. Tversky, "Choices, Values and Frames", *American Psychologist*, 39 (1984), 341-350.

[19] Madan, D. B., and F. Milne, "Option Pricing with VG Martingale Components", *Mathematical Finance*, 1 (1991), 39-55.

[20] Madan, D. B., and E. Seneta, "The Variance Gamma (V.G.) Model for Share Market Returns", *Journal of Business*, 63 (1990), 4, 511-524.

[21] Vaga, T., "The coherent market hypothesis", *Financial Analysts Journal*, November–December (1990), 36–49.

[22] Weron, A., and R. Weron, "Fractal market hypothesis and two power-laws", *Chaos, Solitons and Fractals,* 11 (2000), 289-296.

Chapter 2

Markovian approximation of Gaussian processes

2.1. Introduction

Chapter 2 suggests a generalization of Brownian motion as a model for the dynamics of the logarithm of the stock prices. A new class of processes, termed the class of the generic Gaussian pricing processes ($CGGPP$) is obtained by assuming, under the historical probability, a generic Gaussian process, where dependence and nonstationarity across the increments are taken into consideration. By choosing a process inside this class, a "not-necessarily-Markovian process for pricing" is derived by generalizing the Ito's formula, under some restrictions on the kernel function. The process is then approximated by a Markovian random element in order for the market to derive a no-arbitrage pricing formula of a contigent claim at some fixed time: the aim of adjustment is to correct the dependence structure of the true process, generated by some distortions and the consequent failure in the No-Arbitrage theory. A generic Gaussian process should improve accuracy in pricing with respect to a Brownian motion and, eventually, explain the "implied volatility bias" in pricing options. The parameters of the market-approximating Markovian process, derived by minimizing the distance between the two process, depend on the dependence structure of the process taken into consideration under the historical measure. Section 2.2. defines the statistical and mathematical framework defining the main features of the Gaussian family; and provides specific conditions on the kernel function in order to generalize the Ito's formula to a general Gaussian process. An information theory argument completes the section; while the long-range dependence is defined in the section 2.3. The sub-fractional Brownian motion is characterized in the section 2.3.1. A generic Gaussian pricing formula is provided in the section 2.4. and the specific case of sub-fractional innovation term applied to derive the theoretical price for a contingent claim. The data are described in the next section ; while the section 2.6. provides some empirical findings. Section 2.7. concludes and suggests some future research.

2.2. Gaussian processes: some basic elements

A canonical Gaussian process is defined through its canonical representation and its main properties. A generalized version of Ito's formula is derived in order to show how possible it is to transform a Gaussian path, assuming some regularity condition on the kernel function. Once the process and his probability space are defined and it is possible derive some (proper) transformation of its, the framework can be applied to investigate an investor's problem who might face some rationality failures. After a small digression on tightness and information theory, the section is concluded by exploiting the relationship between a long-range dependent process and the Gaussian measures.

2.2.1. Canonical representation

A system of random variable $X = \{X_\lambda : \lambda \in \Lambda\}$ is called Gaussian, if any finite linear combination $\sum a_k X_\lambda$, $a_k \in \mathbb{R}$ is a Gaussian random variable.[4] If, in particular, X is a stochastic process, it is called a Gaussian process, where the mean vector μ is the mean function, and the covariance matrix Σ the covariance function.

Definition 2 *Let* $X = \{X_t : t \in I\}$, $I \subseteq \mathbb{R}$ *be a Gaussian process. Denote by* $\mathcal{F}_t^X = \sigma(X_s, s \leq t)$ *the* $\sigma - field$ *generated by* $X_s, s \leq t$. *Set* $\mathcal{F}_{t+}^X = \bigcap_{u>t} \mathcal{F}_u^X$ *and* $\mathcal{F}_{t-}^X = \bigvee_{u>t} \mathcal{F}_u^X$. $\bigvee_{u>t} \mathcal{F}_u^X$ *denotes the smallest* $\sigma - field$

[4] A n-dimensional real-valued random element $X(\omega)$ is called Gaussian if its probability distribution is absolutely continuous and is defined by the density function

$$p(x) = \frac{1}{(2\pi)^{\frac{n}{2}} \sqrt{|\Sigma|}} \exp\left[-\frac{1}{2}(x-\mu)' \Sigma^{-1} (x-\mu)\right] \quad x \in \mathbb{R}^n$$

where is $\mu\,(\in \mathbb{R}^n)$ the mean vector and where Σ is a positive definite $n \times n$ matrix called the covariance matrix.

containing all the $\mathcal{F}_t^X$ *with* $u < t$. *Assume that there exists a Gaussian process* $B = \{B_t : t \in I\}$ *with independent increments such that*

$$B_t = \left(B_i\left(t\right),\ 1 \leq i \leq N;\ B_j^l\left(t\right), 1 \leq l \leq L_j, 1 \leq j \leq J\right)$$
$$B_0 = 0, \quad N \leq \infty, \ \ L_j \leq \infty, \ \ J \leq \infty$$

for which the following conditions are satisfied:

1. *Each* $B_i\left(t\right)$ *has independent increments and* $E\left(\left|dB_i\left(t\right)\right|^2\right) = m_i\left(dt\right)$ *defines a continuous measure. In addition,* m_{i+1} *is absolutely continuous with respect to* $m_i : m_i\left(dt\right) >> m_{i+1}\left(dt\right)$ *for every* i.

2. *Each* $B_j^l\left(t\right)$ *is a process of the form*

$$B_j^l\left(t\right) = \begin{cases} 0, & t \leq t_j \quad (\text{or } t < t_j) \\ B_j^l, & t > t_j \quad (\text{or } t \geq t_j) \end{cases}$$

   where each B_j^l *is subject to the standard Gaussian distribution.*

3. *The Gaussian process* B_i *and the* B_i^j *are independent*

4. *For every* t, $X(t)$ *has the same distribution as* $\widetilde{X}(t)$ *given by*

$$\widetilde{X}(t) = \sum_{i=1}^{N} \int_0^t K_i\left(t,u\right) dB_i\left(u\right) + \sum_{t_j \leq t} \sum_{l=1}^{L_j} b_j^l\left(t\right) B_j^l\left(t\right) \qquad (2.1)$$

where the kernel functions $K_i(t,u)$ satisfy, for every t, the conditions

$$\sum_{i=1}^{N}\int_0^t K_i(t,u)\,m_i(du)<\infty \quad i=1,2,...,N$$

and where the function $b_j^l(t)$ vanishes for $t_j<t$ and satisfy, for every t

$$\sum_{t_j\le t}\sum_{l=1}^{L_j} b_j^l(t)^2<\infty$$

Then $\left\{K_i(t,u), B_i(u); b_j^l(t)\, B_j^l(t)\right\}$ is called representation of X.[5]

Furthermore, if the equalities $\mathcal{F}_t^{\widetilde{X}}=\mathcal{F}_t^B$, $\mathcal{F}_{t-}^{\widetilde{X}}=\mathcal{F}_{t-}^B$ and $\mathcal{F}_{t+}^{\widetilde{X}}=\mathcal{F}_{t+}^B$ hold for any t, then X is said to be represented canonically or X has the canonical representation relative to B. If X_t itself can be represented canonically in the form (2.1) (i.e. $\widetilde{X}(t)=X(t)$ a.s.), then the process B is called the innovation process for X. The number N in the above representation is called the continuous multiplicity of X. L_j is the discrete multiplicity at t_j and $L=\sup_{1\le j\le J} L_j$ is simply called the discrete multiplicity of X. The (total) multiplicity M of X is defined by $M=\max\{N,L\}$ .The measure $m_i(dt)=E\left(|dB_i(t)|^2\right)$ is called a continuous spectral measure of X and the system $m_i(dt)$ is called the system of continuous spectral measures. The following theorem states

[5] A Brownian motion, f.i., is canonically represented by itself. Indeed $B(t)=\int_0^t 1dB(u), t\in I$ with $N=1, L=0, m_1(dt)=dt$.

the uniqueness of the canonical representation.

Theorem 3 *For a Gaussian process, a canonical representation of the form (2.1) exists. If X has another canonical representation of the form*

$$\overline{X}(t)=\sum_{i=1}^{\overline{N}}\int_0^t \overline{K}_i(t,u)\,d\overline{B}_i(u)+\sum_{s_k\le t}\sum_{l=1}^{\overline{L}_k}\overline{b}_k^l(t)\,\overline{B}_k^l(t)$$

then $N=\overline{N}$, $m_i\sim\overline{m}_i$ with $\overline{m}_i(du)=E\left(\left|d\overline{B}_i(u)\right|^2\right)$, the set $\{t_j\}$ is the same as $\{s_k\}$ and $L_j=\overline{L}_k$ if $t_j=s_k$.[6]

Proof. See Hida, T. and M. Hitsuda, 1993, pages 59-65. ∎

Let $E(X(t))=0$ and take the covariance function $R(t,s)$. Since $R(t,s)$ is nonnegative definite, there exists a reproducing kernel Hilbert space H with reproducing kernel $R(t,s)$. That is, H is a vector space involving functions $(f=f(t);t\in I)$ satisfying the conditions:

1. $R(\cdot,s)\in H$ for every $s\in I$;
2. For any $f\in H$ and any $s\in I$, $\langle f(\cdot),R(\cdot,s)\rangle=f(s)$, where $\langle,\rangle$ denotes the inner product in H;[7]
3. H is generated by the family $\{R(\cdot,s);s\in I\}$.

[6] In the case $I=(-\infty,\infty)$, the statement needs a slight modification: the innovation process is and the integral sign has to be replaced with noting square integrability of the kernels over $(-\infty,\infty)$, and so forth.

[7] By taking $f(\cdot)=R(\cdot,s)$ the equation $\langle f(\cdot),R(\cdot,s)\rangle$ implies the interesting relation $\langle R(\cdot,t),R(\cdot,s)\rangle=R(t,s)$ where R looks like a factor of itself.

Theorem 4 *Let $R(t,s)$ be the covariance function of a Gaussian process having canonical representation, and let H be the reproducing kernel Hilbert space with kernel $R(t,s)$. Then, an element f of H is expressed in the form*

$$f(t) = \sum_{i=1}^{N} \int_0^t K_i(t,u)\,\alpha_i(u)\,m_j(du) + \sum_{t_j \le t} \sum_{l=1}^{L_j} b_j^l(t)\,\beta_j^l$$

where

$$\sum_{i=1}^{N} \int_0^\infty \alpha_j(u)\,m_i(du) + \sum_j \sum_{l=1}^{L_j} \left(\beta_j^l\right)^2 < \infty$$

The α_i, $1 \le i \le N$, and β_j^l, $j \ge 1$, $1 \le l \le L_j$ are uniquely determined by f.

Proof. See Hida, T. and M. Hitsuda, 1993, page 68. ■

2.2.2. Stochastic calculus with respect to Gaussian processes

Suppose that $X = \{X_t, t \in [0,T]\}$ is a centered continuous Gaussian process, of multiplicity equal to one, with the following integral representation

$$X_t = \int K(t,s)\,dB_s \tag{2.2}$$

where $B = \{B_t, t \in [0,T]\}$ is a Brownian motion and $K(t,s)$ is a square integrable kernel. Its covariance function can be be expressed as

$$R(t,s) = \int_0^{t \wedge s} K(t,r) K(s,r) \, dr \tag{2.3}$$

Suppose now that $K(t,s)$ satisfies the following condition:

(a) $K(\cdot, s)$ has bounded variation on any interval $(u,T]$, $u > s$.

Consider the following seminorm on Ξ, the set of the step functions

$$\|\varphi\|_K^2 = \int_0^T \varphi(s)^2 K(T,s)^2 ds + \int_0^T \left(\int_s^T |\varphi(t) - \varphi(s)| \, |K| (dt, s) \right)^2 ds$$

The completion of Ξ with respect to this seminorm is denoted by H_K. The space H_K is the class of functions φ on $[0,T]$ such that $\|\varphi\|_K^2$ and it is included in $L^2\left([0,T], K(T,s)^2 ds\right)$. Moreover H_K is continuously embedded in H because $\|\varphi\|_H \leq \sqrt{2} \|\varphi\|_K$. Let $u = \{u_t, t \in [0,T]\}$ be a stochastic process in $D^{1,2}(H_K)$, i.e. u verifies the following conditions

(i) $E\|u\|_K^2 = E \int_0^T u_s K(T,s)^2 ds$

$+E \int_0^T \left(\int_s^T |u_t - u_s| \, |K| (dt, s) \right)^2 ds < \infty;$

(ii) $E \int_0^T \|D_r u\|_K^2 dr = E \int_0^T \int_0^T (D_r u_s)^2 K(T,s)^2 ds dr$

$$+E\int_0^T\int_0^T\left(\int_s^T |D_r u_t - D_r u_s|\,|K|\,(dt,s)\right)^2 dsdr < \infty.$$

These conditions imply that the adjoint operator K^*u belong to $L^{1,2}$ and, as a consequence, u belongs to the domain of δ^X and $\delta^X(u) = \int_0^T (K^*u)_s\,\delta B_s$. For a process u in $D^{1,2}(H_K)$ the notation $\delta^X(u) = \int_0^T u_s\delta X_s$ is used, and, therefore, it is possible to write

$$\int_0^T u_s\delta X_s = \int_0^T (K^*u)_s\,\delta B_s$$

Notice that if u satisfies conditions (i) and (ii), then $u1_{[0,t]}$ also satisfies these conditions for any $t \in [0,T]$. Moreover for $s \le t$

$$\begin{aligned} K^*\left(u1_{[0,t]}\right) &= u_sK(T,s) + \int_{(s,t]}(u_r - u_s)\,K(dr,s) - \int_{(t,T]} u_sK(dr,s) \\ &= u_sK(t,s) + \int_{(s,t]}(u_r - u_s)\,K(dr,s) \end{aligned}$$

and for $s > t$, clearly, $K^*\left(u1_{[0,t]}\right) = 0$. Denote $K^*\left(u1_{[0,t]}\right)_s$ by $(K_t^*u)_s$ where K_t^* is the adjoint operator K in the interval $[0,t]$. So, for a process u in $D^{1,2}(H_K)$ the indefinite integral $Z_t = \int_0^t u_s\delta X_s$ is given by

$$\begin{aligned} \int_0^t u_s\delta X_s &= \int_0^t u_sK(t,s)\delta B_s + \int_0^t\left(\int_s^t (u_r - u_s)\,K(dr,s)\right)\delta B_s \\ &= \int_0^t (K^*u)_s\,\delta B_s \end{aligned}$$

In order to show an Ito's formula for the Gaussian process X_t the following conditions must be satisfied:

(a) $K(\cdot, s)$ has bounded variation on any interval $(u, T]$, $u > s$;

(b) $\int_0^T (\int_s^T \|B_t - B_s\|_2 |K| (dt, s))^2 ds < \infty$;

(c) The function $R(s, s)$ and $\int_{t \wedge s}^{s} K(s, r) dr$ have bounded variation in $s \in [0, T]$ for any $t \in [0, T]$.

Let F be twice continuously differentiable function satisfying the growth condition

$$\max\{|F(x)|, |F'(x)|, |F''(x)|\} \leq c \exp\left(\lambda |x|^2\right) \tag{2.4}$$

where c and λ are positive constants such that $\lambda < \frac{1}{4}\left(\sup_{0 \leq t \leq T} R_t\right)^{-1}$. This condition implies

$$E\left(\sup_{0 \leq t \leq T} |F(X_t)|^p\right) \leq c^p E e^{p\lambda \sup_t |X_t|^2} < \infty$$

for all $p < \frac{1}{2\lambda}\left(\sup_{0 \leq t \leq T} R_t\right)^{-1}$ and the same property holds for F' and F''. As a consequence of condition (b), for any function F of this type, the process $F'(X_t)$ belongs to the space $L^2(\Omega, H_K)$.

Indeed, if $2 < p < \frac{1}{2\lambda}\left(\sup_{0\le t\le T} R_t\right)^{-1}$, applying Holder's inequality

$$\begin{aligned}
E\left\|F'(X_t)\right\|_K^2 &= E\int_0^T F'(X_t)^2 K(T,s)^2\,ds \\
&\quad + E\int_0^T\left(\int_s^T \left|F'(X_t)-F'(X_s)\right|\left|K\right|(dt,s)\right)^2 ds \\
&< E\left(\sup_{0\le t\le T}\left|F'(X_t)\right|^2\right) R(T,T) \\
&\quad + c\left(E\left(\sup_{0\le t\le T}\left|F'(X_t)\right|^p\right)\right)^{\frac{1}{p}} \\
&\quad \cdot \int_0^T\left(\int_s^T \left\|X_t - X_t\right\|_2 \left|K\right|(dt,s)\right)^2 ds \\
&< \infty
\end{aligned}$$

Theorem 5 *Let F be a function of class $C^2(\mathbb{R})$satisfying (2.4). Suppose that $X = \{X_t, t\in[0,T]\}$ is a zero mean continuous Gaussian process whose covariance function $R(t,s)$ is of the form (2.3) with a kernel $K(t,s)$ satisfying condition (a), (b) and (c).*[8] *Then for each $t\in[0,T]$ the process $F'(X_t)1_{[0,t]}(s)$ belongs to $Dom(\delta^X)$ and the following for-*

[8]Note that any condition on the regularity of the kernel function is imposed. If the Gaussian process has a regular kernel, the following condition is sufficient to implement Ito's formula is the following:

1. For all $s \in [0,T)$ $|K|(\cdot,s)$ has bounded variation on interval $(s,T]$ and $\int_0^T |K|((s,T],s)^2\,ds < \infty$.

mula holds:

$$F(X_t) = F(0) + \int_0^t F'(X_t)\delta X_s + \frac{1}{2}\int_0^t F''(X_t)dR_s$$

where $R_s = R(s,s)$.

Proof. See Alos, E. , Mazet, O. and D. Nualart, 2001, pages 775-776.

■

2.2.3. Tightness and information theory

An important property for certain family of probability measures is the tightness. A family Π of probability measure on a metric space (S, X) is tight, if for every ε, there exists a compact set K such that $P(K) > 1 - \varepsilon$ for every P in Π. Take the space of the continuous function C, and endow it with the uniform topology, defining the distance between two point x and y as

$$\rho(x, y) = \|x - y\| = \sup_t |x(t) - y(t)|$$

to make C a complete and separable metric space.

Theorem 6 *Let* P_n, P *be two probability measures on the complete metric space* (C, X). *If the finite-dimensional distributions of* P_n *converges* *weakly to those of P , and if $\{P_n\}$ is tight then $P_n \Longrightarrow P$.*

Proof. See Billingsley, P., 1999, pages 80-86 . ■

Information theory deals with a basic challenge in communication: how to transmit information efficiently? In addressing that issue, information theorists have created a rich mathematical framework to describe communication processes with tools to characterize so-called fundamental limits of data compression and transmission. When it goes to statistics, it provides a constructive criterion for setting up probability distributions on the basis of partial knowledge and leads to a type of statistical inference which is called the maximum entropy estimate. Not only is it concerned with the measure of the uncertainty associated with a random variable; but also focuses on the statistical problem of discrimination, by considering a measure of the "distance" or "divergence" between statistical populations. For the statistician two populations differ more or less according as to how difficult it is to discriminate between them with the best test. Suppose it is given the space V of all densities of probability measures on the real line equipped with its Borel field, which are absolutely continuous w.r.t. the Lebesgue measure. The most well-known information criterion regarding theoretic divergence between two probability density functions $p_i(x)$, $i = 1, 2,$ is the Kullback-Leibler information divergence $(KLID)$

$$\begin{aligned} D(p_1 \| p_2) \quad &: \quad = \int p_1(x) \log \frac{p_1(x)}{p_2(x)} dx \\ &: \quad = E_{p_1} \{\log p_1 - \log p_2\} \geq 0, \quad p_1, p_2 \in H \end{aligned}$$

where in general

$$E_{p_i} \{\phi\} = \int \phi(x) p_i(x) dx \quad p \in H$$

In statistics, the $KLID$ arises as an expected logarithm of the likelihood ratio and is a measure of the inefficiency of assuming that the distribution is p_2 when the true distribution is p_1. The $KLID$ is always non-negative and is zero if and only if $p_1 = p_2$ $a.e.$ Even if it is a true metric, being not symmetric and not satisfying the triangle inequality, it is often used as the quantification of "how close are" two density functions. The $KLID$ assumes a particular form in case of an exponential family. A family of probability measure Φ is said to be an exponential family provided there exists a $\sigma - finite$ measure μ on Ψ, a positive integer k, real-valued functions on and real-valued measurable functions b, $t_1....t_k$ on Ψ such that Φ is dominated by μ, $b \geq 0$ and for every $P \in \Phi$

$$\frac{dP}{d\mu}(x) = a(P)b(x)e^{\alpha(P)t(x)} \tag{2.5}$$

where $\alpha = (\alpha_1, ...\alpha_k)$, $t = (t_1, ...t_k)$. In this case equation (2.5) is called an exponential representation of the densities of Φ with respect to μ. Consider a finite dimensional manifold of exponential probabilities densities such as

$$EM(c) = \{p(\cdot, \theta) : \theta \in \Theta \subset \mathbb{R}^m\}, \quad \Theta \text{ open in } \mathbb{R}^m$$
$$p(\cdot, \theta) = \exp[\theta_1 c_1(\cdot) + ... + \theta_m c_m(\cdot) - \psi(\theta)]$$

expressed w.r.t. the expectation parameters η defined by

$$\eta_i(\theta) = E_{p(\cdot,\theta)}\{c_i\} = \partial_{\theta_i}\psi(\theta) \quad i = 1, ...m$$

Define $p(x; \eta(\theta)) = p(x; \theta)$; and, given a density $p \in H$, and approximate it by a density of the finite dimensional manifold $EM(c)$. It seems reasonable to find a density $p(\cdot, \theta)$ in $EM(c)$ which minimizes the Kullback-Leibler information

$$\begin{aligned}
\min_\theta D(p, p(\cdot, \theta)) &= \min_\theta E_p \{\log p - \log p(\cdot, \theta)\} \\
&= E_p \log p - \max_\theta \{\theta_1 E_p c_1 + ... + \theta_m E_p c_m - \psi(\theta)\} \\
&= E_p \log p - \max_\theta V(\theta) \\
V(\theta) &:= \theta_1 E_p c_1 + ... + \theta_m E_p c_m - \psi(\theta)
\end{aligned}$$

It follows immediately that a necessary condition for the minimum to be attained at θ^* is

$$\partial_{\theta_i} V(\theta^*) = 0 \quad i = 1, ...m$$

which yields

$$\begin{aligned} E_p c_i - \partial_{\theta_i}\psi(\theta^*) &= E_p c_i - E_{p(\cdot,\theta^*)} c_i = 0 \quad i = 1, ...m \\ E_p c_i &= \eta_i(\theta^*) \qquad\qquad\qquad\qquad i = 1, ...m \end{aligned}$$

i.e. the best approximation of p in the manifold $EM(c)$ is given by the density of $EM(c)$ which shares the same c_i expectations (c_i-moments) as the given density p: it means that in order to approximate p only its c_i-moments, $i = 1, 2..., m$ are needed.

2.3. Long-range dependence

Assuming that a process X_t has finite variance, the long range dependence is defined by looking at autocorrelation function of the process

Definition 7 *A stationary process X_t is said to have long-range dependence if its autocorrelation function $C(\tau) = corr(X_t, X_{t+\tau})$ decays as a power of the lag τ :*

$$C(\tau) = corr(X_t, X_{t+\tau}) \underset{\tau\to\infty}{\sim} \frac{L(\tau)}{\tau^{1-2d}} \quad 0 < d < \frac{1}{2} \tag{2.6}$$

where L is a slowly varying function at infinity, i.e. $\forall a > 0, \lim_{t\to\infty} \frac{L(at)}{L(t)} = 1$. By contrast, a stationary process is said to have short-range dependence if the autocorrelation function decreases at a geometric rate

$$\exists K > 0, c \in \left]0,1\right[, |C(\tau)| \leq K c^{\tau}$$

The long-range dependence property depends on the behavior of the autocorrelation function at large lags, a quantity which may be difficult to estimate empirically. Models with long-range dependence are often formulated in terms of self-similar processes, which allow to extrapolate across time scales and deduce long time behavior from short time behavior.

Definition 8 *A stochastic process X_t is said to be self-similar if there exists $H > 0$ such that for any scaling factor $c > 0$, the processes have the same law*

$$(X_{ct})_{t\geq 0} \stackrel{d}{=} \left(c^H X_t\right)_{t\geq 0}$$

where H is called the self-similarity exponent of the process.

Fractional Brownian motion is a typical example of self-similar process whose increments exhibit long range dependence: it is a real centered

Gaussian process with stationary increments with covariance function

$$corr\left(B_t^H, B_s^H\right) = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right)$$

where, depending on the value of the Hurst parameter H, some particular dependence might arise. A Hurst parameter greater than $\frac{1}{2}$ leads to long-range dependence; whereas $H < \frac{1}{2}$ pops up short-range dependence. For $H = \frac{1}{2}$, B_t^H becomes a Brownian motion. Note that a self-similar process cannot be stationary, but the definition of long-range dependence, even if it cannot hold for the process itself, might hold for its increments (if they are stationary). By removing the idea of stationarity, the above definitions might become inappropriate. The next subsection addresses the idea of long-range dependence with a non-stationary random process.

2.3.1. Non-stationarity and long-range dependence in the Gaussian class: the sub-fractional Brownian motion

Define a stochastic process $X^h = \left\{X^h(t), t \geq 0\right\}, h \in (0,2)$ obtained by the following transformation

$$X^h = \frac{1}{\sqrt{2}}\left(B^h(t) + B^h(-t)\right) \quad t \geq 0 \tag{2.7}$$

where $B^h(t)$ is a fractional Brownian motion on the whole real line with covariance function $R_h(s,t) = \frac{1}{2}\left(s^h + t^h - |s-t|^h\right)$. The Fractional Brownian motion in (2.7) is defined in terms of h, and not H. The relationship between h and H is the following

$$h = 2H$$

The following theorem summarizes the main of properties of X^h.

Theorem 9 *The process X^h is called sub-fractional Brownian motion and whose main properties are:*

1. *Self-similarity*

$$\left\{X^h(at), t \geq 0\right\} \stackrel{d}{=} \left\{a^{\frac{h}{2}} X^h(t), t \geq 0\right\} \quad \text{for each } a > 0$$

2. *Covariance: for all $s, t > 0$ with $0 \leq s \leq t$, its covariance function $C_h(s,t) = s^h + t^h - \frac{1}{2}\left[(s+t)^h + |s-t|^h\right]$. Moreover*

$$C_h(s,t) > R_h(s,t) \quad \text{if } h < 1$$

$$C_h(s,t) < R_h(s,t) \quad \text{if } h > 1$$

$$C_h(s,t) > 0$$

3. *Second moment of increments: for all* $s, t > 0$

$$E\left(X^h(t) - X^h(s)\right)^2 = -2^{h-1}\left(t^h + s^h\right) + (t+s)^h - (t-s)^h$$

$$\left(2 - 2^{h-1}\right)(t-s)^h \leq E\left(X_h(t) - X_h(s)\right)^2 \leq (t-s)^h \quad \text{if } h > 1 \tag{2.8}$$

$$(t-s)^h \leq E\left(X_h(t) - X_h(s)\right)^2 \leq \left(2 - 2^{h-1}\right)(t-s)^h \quad \text{if } h < 1 \tag{2.9}$$

4. *Holder continuity:* X^h *has a continuous version for each* h*, and for each* $0 \leq \xi \leq \frac{h}{2}$*, and each* $T > 0$ *there exists a random variable* $K_{\xi,T}$ *such that*

$$\left|X^h(t) - X^h(s)\right| \leq K_{\xi,T}\,|t-s|^{\frac{h}{2}-\xi} \quad s, t \in [0, T] \;\; \text{a.s.}$$

5. *Correlation of increments: For* $0 \leq u < v \leq s < t$*, let*

$$R_{u,v,s,t} = E\left(B^h(v) - B^h(u)\left(B^h(t) - B^h(s)\right)\right)$$

and

$$C_{u,v,s,t} = E\left(X^h(v) - X^h(u)\left(X^h(t) - X^h(s)\right)\right)$$

then

$$C_{u,v,s,t} = \frac{1}{2}\left[\begin{array}{c} (t+u)^h + (t-u)^h + (s+v)^h + (s-v)^h \\ -(t+v)^h + (t-v)^h + (s+u)^h + (s-u)^h \end{array}\right]$$

$$C_{u,v,s,t} > 0 \text{ if } h > 1$$

$$C_{u,v,s,t} < 0 \text{ if } h < 1$$

Defined $D_{u,v,s,t}$ *as*

$$C_{u,v,s,t} = R_{u,v,s,t} + D_{u,v,s,t}$$

then

$$\begin{array}{lcl} D_{u,v,s,t} & < & 0 \text{ if } h > 1 \\ D_{u,v,s,t} & > & 0 \text{ if } h < 1 \end{array}$$

For $u \geq 0, r > 0$ *let* $\rho_{u,r}^{B^h}$*and* $\rho_{u,r}^{X^h}$ *denote the correlation coefficients of* $B_{u+r}^h - B_u^h, B_{u+2r}^h - B_{u+r}^h$ *and* $X_{u+r}^h - X_u^h, X_{u+2r}^h - X_{u+r}^h$*, re-*

spectively. Then

$$\left|\rho_{u,r}^{X^h}\right| \leq \left|\rho_{u,r}^{B^h}\right|$$

$$\lim_{s,t\to\infty} C_{u,v,s,t} = 0$$

$$C_{u,v,s,t} < 2^{h-2}v^h \text{ if } h > 1$$

$$C_{u,v,s,t} > -\frac{1}{2}v^h \text{ if } h < 1$$

$$C_{u,v,s+\tau,t+\tau} \sim \frac{h(h-1)(2-h)}{2}(t-s)\left(v^2-u^2\right)\tau^{h-3}$$

$$\text{as } \tau \to \infty \text{ if } h \neq 1$$

6. *X^h is not a Markov process if $h \neq 1$;*[9]

7. *X^h is not a semimartingale if $h \neq 1$;*

8. *Integral representation (for $h \neq 1$):*

[9]Let be a probability space $(\Omega, \mathcal{F}, P)$ with a filtration $(\mathcal{F}_t,\ t \in T)$, for some (totally ordered) index set T; and let $(S,\ S)$ be a measurable space. An *S*-valued stochastic process $X = \{X_t, t \geq 0\}$, adapted to the filtration, is said to possess the Markov property with respect to the $\{\mathcal{F}_t\}$ if, for each $A \in S$ and each $s, t \in T$ with $s < t$,

$$P(X_t \in A|\mathcal{F}_s) = P(X_t \in A|X_s)$$

A Markov process is a stochastic process which satisfies the Markov property with respect to its natural filtration.

(a) *Moving average representation*

$$X^h(t) = \frac{1}{C_1(h)} \int_R K(t,s)\,dB(s)$$

where B *is the Brownian measure on* $\mathbb{R}$ *and*

$$K(t,s) = \left[\left((t-s)^+\right)^{\frac{(h-1)}{2}} + \left((t+s)^-\right)^{\frac{(h-1)}{2}} - 2\left((-s)^+\right)^{\frac{(h-1)}{2}} \right]$$

$$C_1(h) = \left[2\int_0^\infty \left((1+s)^{\frac{(h-1)}{2}} - s^{\frac{(h-1)}{2}}\right)^2 ds + \frac{1}{h} \right]^{\frac{1}{2}}$$

(b) *Spectral representation:*

$$X^h(t) = \frac{1}{C_2(h)} \int_R \frac{\cos(ts) - 1}{is} |s|^{\frac{(1-h)}{2}} d\widetilde{B}(s)$$

where $\widetilde{B} = B^{(1)} + iB^{(2)}$ *is a complex Gaussian measure on* $\mathbb{R}$ *such*

$$\begin{aligned} B^{(1)}(A) &= B^{(1)}(-A) \\ B^{(1)}(A) &= B^{(1)}(-A) \\ E\left(B^{(1)}(A)\right)^2 &= E\left(B^{(2)}(A)\right)^2 = \frac{1}{2}|A| \end{aligned}$$

where $|\cdot|$ denotes Lebesgue measure on $\mathbb{R}$), and

$$C_2(h) = \left(\frac{\pi}{h\Gamma(h)\sin\left(\frac{\pi h}{2}\right)} \right)^{\frac{1}{2}}$$

Proof. See Bojdecki, T., Gorostiza, L.G., and A.Talarczyk, 2004, pages 3–8. ■

The derived process X^h is characterized by not being either a semimartingale or a Markov process. It arises from occupation time fluctuations of branching particle systems for $h \geq 1$ and exhibits the long memory effect of the initial condition. Its increments are neither self-similar nor stationary but these properties are replaced by the inequalities (2.8) and (2.9). Its long-range dependence decays faster than a FBM: its decay rate, for increments separated by distance τ, is τ^{h-3}, instead of τ^{h-2} as τ goes to infinity. Figure (7) visualizes its main properties by showing how the sub-fractional Brownian stands in the intersection among Gaussian processes, long-range dependence random paths and stochastic processes whose increments are non-stationary.

2.4. Generic Gaussian pricing formula

Fix a filtered probability space $\{\Omega, \mathcal{F}, \mathbf{P}\}$, $\mathcal{F}_t^X = \sigma(X_s, s \leq t)$. Assume that a representative agent under the historical probability $\mathbf{P}$ considers

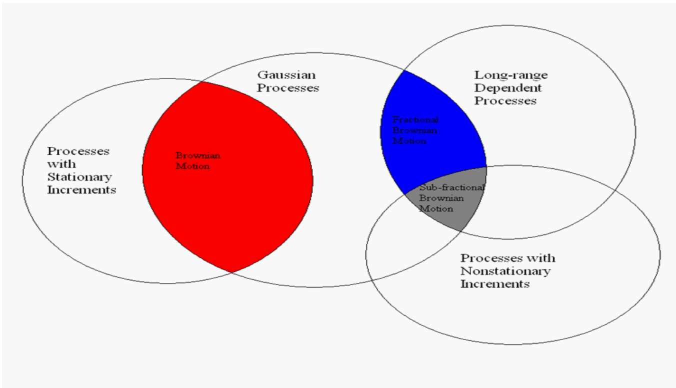

Figure 7: Sub-fractional Brownian Motion as possible synthesis between Gaussianity, Long-range dependence and Nonstationarity

the following dynamics for the stock return:

$$dS_t = S_t \mu(t) \, dt + S_t \sigma(t) \, dX_t \tag{2.10}$$

where the innovation process $X = \{X_t, t \geq 0\}$ is a centered Gaussian process, with the following integral representation

$$X_t = \int_0^t K(t,s) \, dB_s$$

where $B = \{B_t, t \geq 0\}$ is a Brownian motion, $K(t,s)$ is a square integrable kernel, and $R(s,t) = \int_0^{t \wedge s} K(t,r) \, K(s,r) \, dr$. By applying the Ito's Lemma to the diffusion process, the dynamics of the stock can be easily derived. Indeed taking $Y_t = F(S_t) = \ln S_t$,

$$\begin{aligned} dY_t &= \frac{\partial f}{\partial S_t} dS_t + \frac{1}{2} \frac{\partial^2 f}{(\partial S_t)^2} (dS_t)^2 = \\ &= \frac{1}{S_t} (S_t \mu(t)\, dt + S_t \sigma(t)\, dX_t) - \frac{1}{2S_t^2} \left(S^2 \sigma(t)^2 \, dR_t \right) \\ &= (\mu(t)\, dt + \sigma(t)\, dX_t) - \frac{1}{2} \left(\sigma(t)^2 \, dR_t \right) \end{aligned}$$

$$Y_t = Y_0 + \int_0^t \mu(s)\, ds + \int_0^t \sigma(s)\, dX_s - \int_0^t \frac{1}{2} \sigma(s)^2 \, dR_s$$

and

$$\begin{aligned} S_t &= S_0 \exp\left(\int_0^t \mu(s)\, ds + \int_0^t \sigma(s)\, dX_s - \int_0^t \frac{1}{2} \sigma(s)^2 \, dR_s \right) \\ &= S_0 \exp\left(\int_0^t \mu(s)\, ds + \int_0^t \sigma(s)\, dX_s - \int_0^t \frac{1}{2} \sigma(s)^2 \frac{\partial R_s}{\partial s} ds \right) \\ &= S_0 \exp\left(\int_0^t \mu(s)\, ds + \int_0^t \frac{1}{2} \sigma(s)^2 \frac{\partial R_s}{\partial s} ds + \int_0^t \sigma(s)\, dX_s \right) \\ &= S_0 \exp\left(\int_0^t \left[\mu(s)\, dt - \frac{1}{2} \sigma(s)^2 \frac{\partial R_s}{\partial s} \right] ds + \int_0^t \sigma(s)\, K(t,s)\, dB_s \right) \end{aligned}$$

given initial condition $Y(0) = Y_0$ and $S(0) = S_0$. Assuming an innovation process with not-necessarily stationary and independent increments gives a process where the semimartingale properties are not granted.

Therefore given a contingent claim whose final payoff is $\Phi(S_T)$, it is possible conjecture that its (generalized) pricing formula can be

$$\pi_t(S_T, r, \sigma') = g[m_T \Phi(S_T) | \mathcal{F}_t]. \tag{2.11}$$

for some functional g, with m_T being the stochastic discount factor. If the market prices do not allow for profitable arbitrage, the prices are said to constitute an arbitrage equilibrium or arbitrage-free market. An arbitrage equilibrium is a precondition for a general economic equilibrium. The existence of arbitrage opportunities, instead, pushes the market away from equilibrium, since the arbitrage drives the demand up to infinity. The following doubts might, then, arise:

(QA) Is this an arbitrage-free price?

(QB) Is the market aware of the existence of some distortions?

(QC) What if the market tries to correct these fallacies?[10]

Finding a functional g, able to clear all the market inefficiencies, in the equation (2.11) would answer, immediately, from a normative point of view, the three previous questions; this work, instead, wants to contribute to the debate if the market is arbitrage-free or correct the inefficiencies, by suggesting an alternative approach, more practical and less normative.

[10] This might be similar to the idea of "paternalism" in economics; but here the market tries to correct some rationality failures to avoid arbitrage.

Consider another probability space $\left\{\Omega', \mathcal{F}_t^B, \mathbf{M}\right\}, \mathcal{F}_t^B = \left\{\sigma\left(B_t^\alpha, t \leq T\right)\right\}$. The following family of diffusion has been defined in this probability space to describe the dynamics of a fictitious stock:

$$dS_t'^\delta = S_t'^\delta \mu'\left(t, \delta\right) dt + S_t'^\delta \sigma'\left(t, \delta\right) dB_t$$

where B_t is a Brownian motion, and all the possible sample paths are function of $\delta\left(t\right) = \left[\mu'\left(t, \delta\right), \sigma'\left(t, \delta\right)\right]$. The log-transformation of the diffusion gives the classical dynamics of the stock's return

$$\begin{aligned} dY_t'^\delta &= \frac{\partial f}{\partial S_t'^\delta} dS_t + \frac{1}{2} \frac{\partial^2 f}{\left(\partial S_t'^\delta\right)^2} \left(dS_t'^\delta\right)^2 = \\ &= \frac{1}{S_t'^\delta} \left(S_t'^\delta \mu'\left(t, \delta\right) dt + S_t'^\delta \sigma'\left(t, \delta\right) dB_t\right) - \frac{1}{2 S_t^{2\prime\delta}} \left(S_t^{2\prime\delta} \sigma'\left(t, \delta\right)^2 dt\right) \\ &= \left(\mu'\left(t, \delta\right) dt + \sigma'\left(t, \delta\right) dB_t\right) - \frac{1}{2} \left(\sigma'\left(t, \delta\right)^2 dt\right) \end{aligned}$$

that gives

$$Y_t'^\delta = Y_0 + \int_0^t \mu'\left(s, \delta\right) ds + \int_0^t \sigma'\left(s, \delta\right) dB_s - \int_0^t \frac{1}{2} \sigma'\left(s, \delta\right)^2 ds$$

and

$$\begin{aligned} S_t'^{\delta} &= S_0 \exp\left(\int_0^t \mu'(s,\delta)\,ds + \int_0^t \sigma'(s,\delta)\,dB_s - \int_0^t \frac{1}{2}\sigma'(t,\delta)^2\,ds\right) \\ &= S_0 \exp\left(\int_0^t \mu'(s,\delta)\,ds - \int_0^t \frac{1}{2}\sigma'(s,\delta)^2\,ds + \int_0^t \sigma'(t,\delta)\,dB_s\right) \\ &= S_0 \exp\left(\int_0^t \left(\mu'(s,\delta)\,ds - \frac{1}{2}\sigma'(s,\delta)^2\right)ds + \int_0^t \sigma'(s,\delta)\,dB_s\right) \end{aligned}$$

Once a family of diffusions is defined, this work proposes the following procedure to answer (QA), (QB) and (QC):

1. The market knows that the true dynamics of the stock's return is represented by the evolution of Y_t. Its only concern is that, Y_t is not necessarily a semimartingale;

2. a No-arbitrage (NA) pricing formula is not always possible;

3. a good strategy to overcome the problem is the following:

   (a) among all the possible sample paths of $S_t'^{\delta}$, all semimartingales, the market selects the "optimal trajectory", i.e. that one, given the events $A_1, A_2, ...A_t \in B(\mathbb{R})$, $A_1', A_2', ...A_t' \in B(\mathbb{R})$, whose joint probability density function is the closest to the dynamics of the equation (2.10), i.e.

$$P\left(S_1 \in A_1, ..., S_t \in A_t\right) \simeq M_{\delta^*(t)}\left(S_1'^{\delta} \in A_1', ...S_t'^{\delta} \in A_t'\right) \tag{2.12}$$

(b) Once the optimal parameters able approximate the process S_t up to time t with $S_t'^{\delta}$ and correct the distortions; the markets might use them to simulate $S_t'^{\delta}$ from t onward, and provide No-Arbitrage price[11] for contingent claims.

How to determine the optimal drift and diffusive coefficient able to approximate the non-Markovian process with a Markov random path? S_t and $S_t'^{\delta}$ are log-normal distributed, while

$$Y_t = \left(Y_0 + \int_0^t \left(\mu(s) - \frac{1}{2}\sigma(s)^2 \frac{\partial R_s}{\partial s}\right) ds + \int_0^t \sigma(s) K(t,s) dB_s\right)$$

[11] By moving into the diffusion world, the functional g that rules out arbitrage opportunities in the market is the expected value $E[\cdot]$. Equation (2.11) becomes, under the historical probability P,

$$\begin{aligned} \pi_t\left(S_T, r, \sigma'\right) &= g\left[m_T \Phi\left(S_T\right) | \mathcal{F}_t\right] \\ &= E^P\left[m_T \Phi\left(S_T\right) | \mathcal{F}_t\right] \end{aligned}$$

A change of measure into the risk-neutral measure Q, assuming a deterministic interest rate r, gives the following pricing formula

$$\pi_t\left(S_T, r, \sigma'\right) = E^Q\left[e^{-\int_t^T r(s)ds} \Phi\left(S_T\right) | \mathcal{F}_t\right].$$

and

$$Y_t^{\prime\delta} = \left(\int_0^t \left(\mu\left(s,\delta\right) - \frac{1}{2}\sigma\left(s,\delta\right)^2 \right) ds + \int_0^t \sigma\left(s,\delta\right) dB_s \right)$$

are Gaussian processes. Indeed their marginal distributions are

$$\begin{aligned} \log S_t &\sim N\left(\log s_0 + \int_0^t \left[\mu\left(s\right) - \frac{1}{2}\sigma\left(s\right)^2 \frac{\partial R_s}{\partial s} \right] ds, \int_0^t \sigma\left(s\right)^2 \frac{\partial R_s}{\partial s} ds \right) \\ \log S_t^{\prime\delta} &\sim N\left(\log s_0 + \int_0^t \left[\mu'\left(s,\delta\right) - \frac{1}{2}\sigma'\left(s,\delta\right)^2 \right] ds, \int_0^t \sigma'\left(s,\delta\right)^2 ds \right) \end{aligned}$$

Information theory provides the link to connect the true dynamics of the stock and its fictitious counterpart. An entropy argument suggests that, minimizing the distance between the two log-normal distribution S_t and $S_t^{\prime\delta}$ is the same as minimizing the distance between Y_t and $Y_t^{\prime\delta}$. Since Gaussian measures are element of the exponential family, the approach suggested by the section 2.2.3. holds: the approximation can be easily done by matching the moments. Figure (8) compares a non-Markovian Gaussian process with a family of Brownian motion: the idea is to approximating step by step a certain Gaussian process by selecting the best fit in the class of the Brownian motions. The probability density of

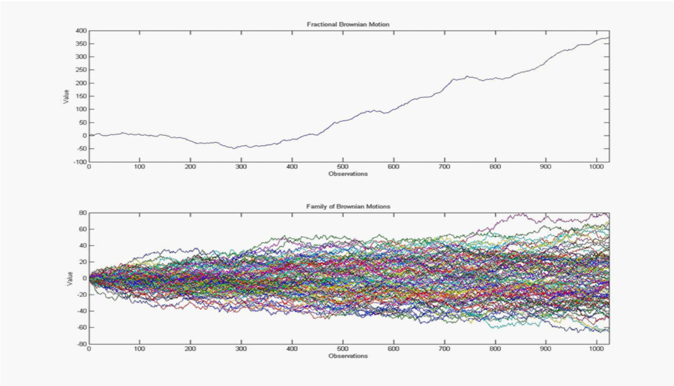

Figure 8: Generic Gaussian Process (above) vs Family of Brownian Motions (below)

S_t at any time t, can be then expressed as

$$p_{S_t}(x) = p(x, \theta(t)) = \exp\left\{\theta_1(t)\ln\frac{x}{s_0} + \theta_2(t)\ln^2\frac{x}{s_0} - \psi(\theta_1(t), \theta_2(t))\right\}$$

where

$$\begin{aligned}
\theta_1(t) &= \frac{\int_0^t \mu'(s,\delta)\,ds}{\int_0^t \sigma'(s,\delta)^2\,ds} - \frac{3}{2} \\
\theta_2(t) &= -\frac{1}{2\int_0^t \sigma'(t,\delta)^2\,dt} \\
\psi(\theta_1(t), \theta_2(t)) &= -\frac{(\theta_1(t)+1)^2}{4\theta_2(t)} + \frac{1}{2}\ln\left(\frac{-\pi s_0^2}{\theta_2(t)}\right)
\end{aligned}$$

where x is clearly in the exponential class with $c_1(x) = \ln\left(\frac{x}{s_0}\right)$, $c_2 = c_1^2$.

The expectation parameters can be easily computed as follows

$$\eta_1 = E_\theta \ln \frac{x}{s_0} = \partial_{\theta_1}\psi(\theta_1,\theta_2) = -\frac{\theta_1+1}{2\theta_2}$$
$$\eta_2 = E_\theta \ln^2 \frac{x}{s_0} = \partial_{\theta_2}\psi(\theta_1,\theta_2) = \left(\frac{\theta_1+1}{2\theta_2}\right)^2 - \frac{1}{2\theta_2}$$

The parameters θ can be computed back from the η parameters by inverting the above formulae

$$\begin{aligned}\theta_1 &= \frac{\eta_1}{\eta_2-\eta_1^2} - 1\\ \theta_2 &= -\frac{1}{2\left(\eta_2-\eta_1^2\right)}\\ \psi(\theta_1,\theta_2) &= \frac{1}{2}\left[\frac{\eta_1^2}{\left(\eta_2-\eta_1^2\right)} + \ln\left(2\pi\left(\eta_2-\eta_1^2\right)s_0^2\right)\right]\end{aligned}$$

i.e.

$$\theta_1 = \frac{\log s_0 + \int_0^t \left[\mu(s) - \frac{1}{2}\sigma(s)^2 \frac{\partial R_s}{\partial s}\right] ds}{\int_0^t \sigma(s)^2 \frac{\partial R_s}{\partial s} ds} - 1$$
$$\theta_2 = -\frac{1}{2\int_0^t \sigma(s)^2 \frac{\partial R_s}{\partial s} ds}$$

and (denoting by * the best approximating parameters)

$$\frac{\int_0^t \mu'^*(s,\delta)\,ds}{\int_0^t \sigma'^*(s,\delta)^2\,ds} - \frac{3}{2} = \frac{\log s_0 + \int_0^t \left[\mu(s) - \frac{1}{2}\sigma(s)^2 \frac{\partial R_s}{\partial s}\right] ds}{\int_0^t \sigma(s)^2 \frac{\partial R_s}{\partial s} ds} - 1$$

$$\int_0^t \mu'^*(s,\delta)\,ds - \frac{1}{2}\int_0^t \sigma'^*(s,\delta)^2\,ds = \left(\int_0^t \sigma'(s,\delta)^2\,ds\right) \cdot \left(\frac{\log s_0 + \int_0^t \left[\mu(s) - \frac{1}{2}\sigma(s)^2 \frac{\partial R_s}{\partial s}\right] ds}{\int_0^t \sigma(s)^2 \frac{\partial R_s}{\partial s} ds}\right)$$

$$\int_0^t \mu'^*(s,\delta)\,ds - \frac{1}{2}\int_0^t \sigma'^*(s,\delta)^2\,ds = \log s_0 + \int_0^t \left[\mu(s) - \frac{1}{2}\sigma(s)^2 \frac{\partial R_s}{\partial s}\right] ds \tag{2.13}$$

provided that $\left(\int_0^t \sigma'(s,\delta)^2\,ds\right)$ is finite and, given

$$\begin{aligned} -\frac{1}{2\int_0^t \sigma'(s,\delta)^2\,ds} &= -\frac{1}{2\int_0^t \sigma(s)^2 \frac{\partial R_s}{\partial s} ds} \\ \int_0^t \sigma'^*(s,\delta)^2\,ds &= \int_0^t \sigma(s)^2 \frac{\partial R_s}{\partial s} ds \end{aligned} \tag{2.14}$$

Then at time t, the best approximating marginal distribution is

$$\log S_t'^{\delta^*} \sim N\left(\log s_0 + \int_0^t \left[\mu(s)\,ds - \frac{1}{2}\sigma(s)^2 \frac{\partial R_s}{\partial s}\right] ds, \int_0^t \sigma(s)^2 \frac{\partial R_s}{\partial s} ds\right)$$

and

$$\begin{aligned} S_t^{\prime\delta^*} &= s_0 \exp\left[\int_0^t \left[\mu'(s)\,dt - \frac{1}{2}\sigma'(s)^2\right] ds + \int_0^t \sigma'(s)^2\,ds\right] \\ &= s_0 \exp\left[\int_0^t \left[\mu(s)\,dt - \frac{1}{2}\sigma(s)^2 \frac{\partial R_s}{\partial s}\right] ds + \int_0^t \sigma(s)^2 \frac{\partial R_s}{\partial s} ds\right] \end{aligned}$$

At time t, the finite-dimensional joint distribution of the two processes S_t and $S_t^{\prime\delta^*}$are the same, but what is going to happen at time $t_i > t$, with $t_0 = t, ...t_N = T$? Assume that the market selects the coefficient $\theta(t) = \left[\int_0^t \mu(s)\,dt; \int_0^t \sigma(s)^2 \frac{\partial R_s}{\partial s} ds\right]$ in order to simulate the dynamics of $S_t^{\prime\delta^*}$ until time T. The expression of the stock at time T would be

$$S'_T = S_t \exp\left[\int_t^T \left[\int_0^t \mu'(s)\,ds - \frac{1}{2}\sigma'(s)^2\right] du + \int_t^T \left(\int_0^t \sigma(s)^2 \frac{\partial R_s}{\partial s} ds\right) du\right]$$

with $S_t = S'_t$.[12] By Girsanov theorem the risk neutral dynamics Q of the process $S_t^{\prime *}$ can be determined: a change of measure gives the following closed-form formula:

$$S'_T = S_t \exp\left[\int_t^T \left[\int_0^t r(s)\,ds - \frac{1}{2}\sigma'(s)^2\right] du + \int_t^T \left(\int_0^t \sigma(s)^2 \frac{\partial R_s}{\partial s} ds\right) du\right]$$

[12] The general expression will be, if we approximate from t_{i-1} to t_i

$$S_{t_i}^{\prime\delta^*} = s_{t_{i-1}} \exp\left[\begin{array}{c} \int_{t_{i-1}}^{t_i} \left[\int_0^{t_{i-1}} \mu'(s)\,ds - \frac{1}{2}\sigma'(s)^2\right] du + \\ \int_{t_{i-1}}^{t_i} \left(\int_0^{t_{i-1}} \sigma(s)^2 \frac{\partial R_s}{\partial s} ds\right) du \end{array}\right].$$

where

$$dW_Q = dW_P - v_t dq$$

and

$$v_s = \frac{\mu'_s - r_s}{\sigma'_s}$$

and μ'_s and σ'_s are given by expressions (2.13) and (2.14). The value of a contigent claim, expiring at time T would be given by

$$\pi_t\left(S'_T, r, \sigma'\right) = E^Q\left[\Phi\left(S'_T\right)|\mathcal{F}_t\right]$$

where $\Phi\left(S'_T\right)$ is the payoff function of the contigent claim; Q is the risk-neutral probability, where the optimal parameters $[\mu', \sigma']$ were taken by the market for approximating the non-Markovian process S. The price of a European call option c_t is the same as the standard Black and Scholes framework with time-dependent coefficient, i.e.

$$\begin{aligned} c_t(t, S'_T, r, \sigma') &= E^Q\left[\left(S'_T - K\right)_+ |\mathcal{F}_0\right] = \\ &= S_t N\left(d_1\right) - K N\left(d_2\right) e^{-\int_t^T r(u)du} \end{aligned}$$

where

$$d_1 = \frac{\ln \frac{S_t}{K} + \left(\int_t^T \left[\int_0^t r(s)ds + \frac{1}{2}\int_0^t \sigma'(s)^2 ds\right] du\right)}{\left(\int_t^T \left(\int_0^t \sigma'(s)^2 ds\right) du\right)^{\frac{1}{2}}}$$

$$d_2 = \frac{\ln \frac{S_t}{K} + \left(\int_t^T \left[\int_0^t r(s)ds - \frac{1}{2}\int_0^t \sigma'(s)^2 ds\right] du\right)}{\left(\int_t^T \left(\int_0^t \sigma'(s)^2 ds\right) du\right)^{\frac{1}{2}}}$$

The volatility used in the option prices is an adjustment made in order to approximate the dynamics of a non-Markovian process by a diffusion: a correction done to avoid arbitrage opportunities. The implied volatility profile extrapolated by the No-Arbitrage price $c_t(t, S, r, \sigma')$ should assume a different form with respect to the constant flat curve in the Black & Scholes model, since it depends on the extremes of integration inside the No-Arbitrage pricing formula. Its behavior should depend on the effort done by the market to clear the dependence bias, generated by a process whose true behavior is not Markovian.

2.4.1. Stochastic differential equation driven by a sub-fractional innovation term

Imagine a scenario the stock's movements is strongly influenced either by relying too heavily on a past reference or one trait or piece of information when making decision, i.e. "anchoring"; or by some "availability heuristic", in other words estimating what is more likely by what is more available in memory. Clearly, in this scenario, a Markovian model, with independent and stationary increments, might not be considered as best representation of the real world. A long-range dependent process, with increments not-necessarily stationary, might represents a better choice to model this situation. A good candidate as innovation term in a stochastic differential equations might be represented by the sub-fractional Brownian motion, because of its nonstationarity and quasi-self similarity. Under the historical probability $\mathbf{P}$, equation (2.10) assumes the following dynamics:

$$dS_t = S_t \mu(t)\, dt + S_t \sigma(t)\, dX_t^h$$

The Ito's Lemma provides the tools to derive the dynamics of the stock, under the assumption that the process is driven by the sub-fractional Brownian motion. The transformation $Y_t = F(S_t) = \ln S_t$

$$\begin{aligned} dY_t &= \frac{\partial f}{\partial S_t} dS_t + \frac{1}{2} \frac{\partial^2 f}{(\partial S_t)^2} (dS_t)^2 = \\ &= \frac{1}{S_t} \left(S_t \mu(t)\, dt + S_t \sigma(t)\, dX_t^h \right) - \frac{1}{2S_t^2} \left(S^2 \sigma(t)^2\, dR_t \right) \\ &= \left(\mu(t)\, dt + \sigma(t)\, dX_t^h \right) - \frac{1}{2} \left(\sigma(t)^2\, dR_t \right) \end{aligned}$$

$$Y_t = Y_0 + \int_0^t \mu(s)\,ds + \int_0^t \sigma(s)\,dX_s^h - \int_0^t \frac{1}{2}\sigma(s)^2\,dR_s$$

and

$$\begin{aligned} S_t &= S_0 \exp\left(\int_0^t \mu(s)\,ds + \int_0^t \sigma(s)\,dX_s^h - \int_0^t \frac{1}{2}\sigma(s)^2\,dR_s\right) \\ &= S_0 \exp\left(\int_0^t \mu(s)\,ds + \int_0^t \sigma(s)\,dX_s^h - \frac{h}{2}(2-2^{h-1})\int_0^t \sigma(s)^2 s^{h-1} ds\right) \\ &= S_0 \exp\left(\int_0^t \mu(s)\,ds - \frac{h}{2}(2-2^{h-1})\int_0^t \sigma(s)^2 s^{h-1} ds + \int_0^t \sigma(s)\,dX_s^h\right) \\ &= S_0 \exp\left(\begin{array}{c} \int_0^t \left[\mu(s)\,dt - \frac{1}{2}\sigma(s)^2 h(2-2^{2h-1})s^{h-1}\right] ds + \\ \frac{1}{C_1(h)} \int_0^t \sigma(s) \left[\left((t-s)^+\right)^{\frac{(h-1)}{2}} + \left((t+s)^-\right)^{\frac{(h-1)}{2}}\right] dB_s \\ -\frac{1}{C_1(h)} \int_0^t \sigma(s) \left[2\left((-s)^+\right)^{\frac{(h-1)}{2}}\right] dB_s \end{array}\right) \end{aligned}$$

The process is not a semimartingale, then it fails to satisfy the No-Arbitrage assumptions. Proposing a No-Arbitrage price for contingent claims expiring at time $T > t$, based on this model, would be impossible under this dynamics. How does the market act? Among the approximating family of diffusions

$$S_t'^{\delta} = S_0 \exp\left(\int_0^t \left[\mu'(s,\delta) - \frac{1}{2}\sigma'(s,\delta)^2\right] ds + \int_0^t \sigma'(s,\delta)\,dB_s\right)$$

the optimal drift and diffusive coefficients are

$$\begin{aligned}\int_0^t \mu'^*(s,\delta)\,ds - \frac{1}{2}\int_0^t \sigma'^*(s,\delta)^2\,ds &= \log s_0 + \int_0^t \mu(s)\,ds \\ &\quad -\frac{h}{2}\int_0^t \sigma(s)^2\,(2-2^{h-1})\sigma(s)^2\,s^{h-1}ds\end{aligned}$$

and

$$\int_0^t \sigma'^*(s,\delta)^2\,ds = h(2-2^{h-1})\int_0^t \sigma(s)^2\,s^{h-1}ds \qquad (2.15)$$

Then at time t, the best approximating marginal distribution of a sub-fractional BM driven model is

$$\log S_t'^{\delta^*} \sim N\begin{pmatrix} \log s_0 + \int_0^t \left[\mu(s) - \frac{h}{2}\sigma(s)^2\,(2-2^{h-1})s^{h-1}\right] ds, \\ h(2-2^{h-1})\int_0^t \sigma(s)^2\,s^{h-1}ds \end{pmatrix}$$

From time t onward, using the best-approximation parameter at time t, a fictitious stock's return is simulated in order to derive a price for time T. $S_T'^{\delta}$ would be given by

$$S_T'^{\delta} = S_t \exp\left(\int_t^T \left(\int_0^t \left[\mu'(s) - \frac{1}{2}\sigma'(s,\delta)^2\right] ds\right) du + \int_t^T \left(\int_0^t \sigma'(s,\delta)^2\,ds\right) dB_u\right)$$

and in a risk-neutral scenario

$$S_T'^{\delta} = S_t \exp\left(\int_t^T \left(\int_0^t \left[r(s) - \frac{1}{2}\sigma'(s,\delta)^2\right] ds\right) du + \int_t^T \left(\int_0^t \sigma'(s,\delta)^2\,ds\right) dB_u\right)$$

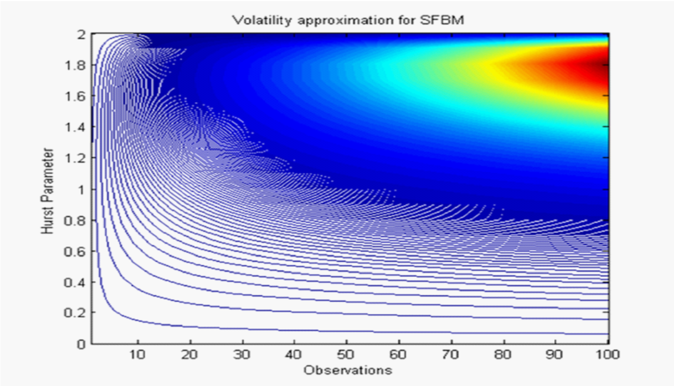

Figure 9: Volatility approximation for sub-fractional Brownian motion.

given $S_t = S_t'$. Figure (9) showing how the approximating volatility might increase or decrease, when both diffusive coefficients are assumed to be constant. The price of a European call option would be under this model

$$\begin{aligned} c_t(T, S_T', r, \sigma') &= E^Q\left[(S_T - K)_+ |\mathcal{F}_T\right] = \\ &= S_t N(d_1) - KN(d_2)\, e^{-\int r(u)du} \end{aligned}$$

where $N(\cdot)$ is a standard normal cumulative distribution with

$$d_1 = \frac{\ln \frac{St}{K} + \int_t^T \left(\int_0^t r(s)\, ds \right) du + \int_t^T \frac{1}{2} \left(\int_0^t \sigma'(s,\delta)^2\, ds \right) du}{\left(\int_t^T \left(\int_0^t \sigma'(s)^2 ds \right) du \right)^{\frac{1}{2}}}$$

$$= \frac{\ln \frac{St}{K} + \int_t^T \left(\int_0^t r(s) ds \right) du + \int_t^T \left(\frac{h}{2}(2 - 2^{h-1}) \int_0^t \sigma(s)^2 s^{h-1} ds \right) du}{\left(\int_t^T \left(h(2 - 2^{h-1}) \int_0^t \sigma(s)^2 s^{h-1} ds \right) du \right)^{\frac{1}{2}}}$$

and

$$d_2 = \frac{\ln \frac{St}{K} + \int_t^T \left(\int_0^t r(s)\, ds \right) du - \int_t^T \frac{1}{2} \left(\int_0^t \sigma'(s,\delta)^2\, ds \right) du}{\left(\int_t^T \left(\int_0^t \sigma'(s,\delta)^2 ds \right) du \right)^{\frac{1}{2}}}$$

$$= \frac{\ln \frac{St}{K} + \int_t^T \left(\int_0^t r(s) ds \right) du - \int_t^T \left(\frac{h}{2}(2 - 2^{h-1}) \int_0^t \sigma(s)^2 s^{h-1} ds \right) du}{\left(\int_t^T \left(h(2 - 2^{h-1}) \int_0^t \sigma(s)^2 s^{h-1} ds \right) du \right)^{\frac{1}{2}}}$$

2.5. The option data

The prices used in this study are for the Standard & Poor's (S&P) 500 Index Options listed at the Chicago Board of Option Prices (CBOE). The data for the study were obtained from the Option Metrics, whose access is provided by the Wharton Research Data Services (WRDS); and include all index prices from September 2000 to September 2007. The index behavior and its daily return are shown in Figure (10). Option Metrics contains also bid and ask European option quotes on the S&P 500

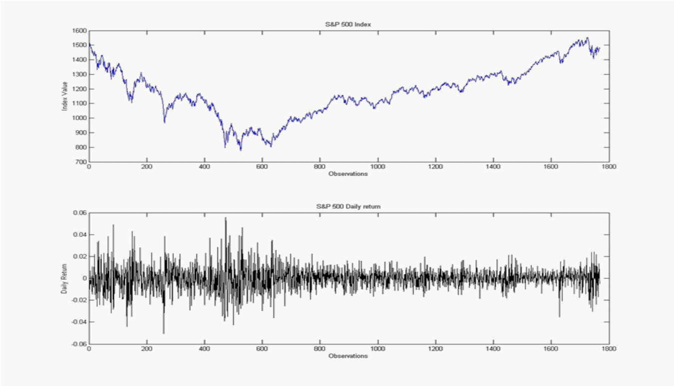

Figure 10: S&P 500 Index (above) vs Daily Return (below)

index, across different strikes and maturities and dividend yield inferred from the theoretical no-arbitrage relationship between the spot and future index r. All options contract are viewed as written on the underlying spot index. The WRDS interface has "market" section where it is possible to obtain daily data on the three month Treasury Bill rate. At the 16th of September, the total number of contract is 2014; and universe of option prices is composed by thirteen different maturities and one-hundred and twenty-two strike prices. As summarized in Figure (11), a significant amount of contract is characterized by short time to maturity and a strike price close to the index value. To enlarge the analysis to the last quarter of the year, option prices are selected at October 17, November 15 and December 17. The sample is quite homogenous for the entire period.

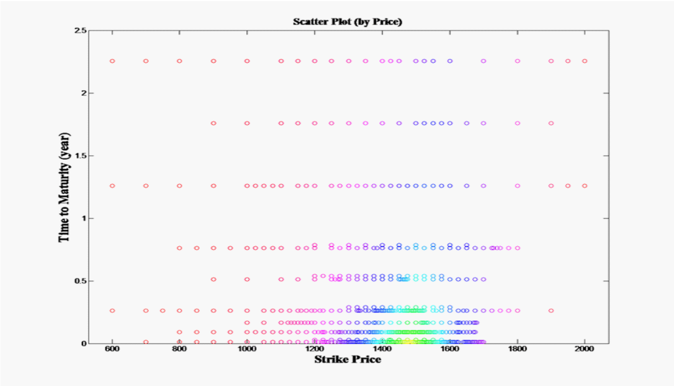

Figure 11: Scatter plot of option prices

The main differences stems in November 2007, period with 26 different expiration period, instead of thirteen. The final sample consists of 10382 bid-ask quotes.

2.6. Empirical analysis

2.6.1. Parameter estimation and simulations

Parameter values of the statistical densities underlying the Geometrical Brownian Motion are estimated, in order to have some reference value. The data employed were the 1768 daily observation of log spot price relatives covering the period from September 2000 to September 2007.

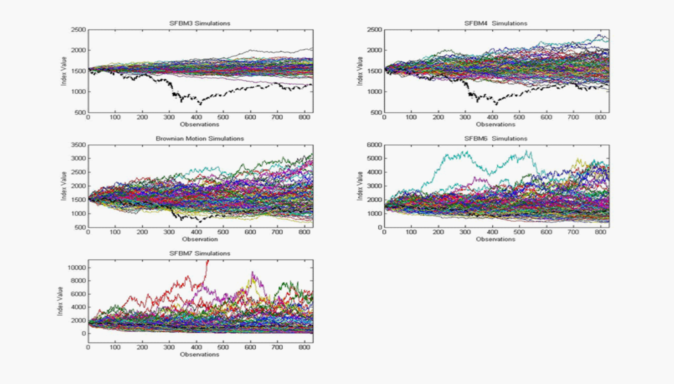

Figure 12: Simulated Index Value (solid lines) vs True Market Behavior (black dotted line)

For the stock price dynamics a maximum-likelihood estimation of the drift and diffusive coefficient is computed; and the same procedure repeated for the month of October, November and December.

Table 1: Parameter estimation for GBM.

Month	Nobs	Drift	Volatility
September	1768	-0.00018	0.0341
October	1790	0.000075	0.0341
November	1812	-0.000234	0.0341
December	1832	-0.000242	0.0341

Assume that the diffusive coefficient is right but there is some uncertainty regarding X_t: the true data might be driven by an innovation process whose increments are dependent and nonstationary. Particularly

$$dS_t = S_t\mu(t)\,dt + S_t\sigma(t)\,dX_t$$

is such that X_t :

1. independent and stationary increments $X_t = B_t$, i.e. a Brownian motion (BM);
2. short-range dependent nonstationary increments $X_t = SB_t^3$, i.e. a sub-Fractional Brownian motion with Hurst parameter $H = 0.3$ $(SFBM3)$;
3. short-range dependent and nonstationary increments $X_t = SB_t^4$, i.e. sub-Fractional Brownian motion with Hurst parameter $H = 0.4$ $(SFBM4)$;
4. long-range dependent and nonstationary increments $X_t = SB_t^6$, i.e. sub-Fractional Brownian motion with Hurst parameter $H = 0.6$ $(SFBM6)$;
5. long-range dependent and nonstationary increments $X_t = SB_t^7$, i.e. sub-Fractional Brownian motion with Hurst parameter $H = 0.7$ $(SFBM7)$.

By the No-Arbitrage assumption, dependence/nonstationarity in delivering option prices are corrected by (2.15).

Table 2: Volatility approximation for different models.

Month	X_t=B_t	X_t=SB_t^3	X_t=SB_t^4	X_t=SB_t^6	X_t=SB_t^7
September	0.0341	0.00851	0.01715	0.0664	0.12550
October	0.0340	0.00847	0.01708	0.06635	0.12544
November	0.0342	0.00850	0.01716	0.06681	0.12649
December	0.0343	0.00851	0.01719	0.06708	0.12714

Once the approximation of diffusive coefficient is done, $N\!-\!Geometrical$ Brownian motion are simulated, with the approximating diffusive coefficient; and the fictitious data compared with the true market movements. Data ranges from September 2007 to December 2010.

Figure (12) plots the five different simulated scenarios against the true behavior of the index. The first scenario, where a sub-fractional Brownian motion with short-term dependence, $H = 0.3$, has been corrected in order to satisfy the No-Arbitrage assumption, does not capture the true market behavior. The second set of simulated paths still approximates short-range dependence, but H increases to 0.4: the majority of the paths are still above the market data. The third figure describes a situation where no approximation is required, i.e. a simulation N-Brownian Motion with constant volatility. By increasing H, long-range dependence starts to be taken in consideration. The fourth graph provides simulations of long-range dependent approximated processes with Hurst Parameter equal to 0.6. The last figure compares the empirical data with fictitious data where long-range dependence, $H = 0.7$, and nonstationarity have been approximated. The previous set of figures shows how accuracy in simulations improves with an increasing Hurst parameter. By approximating a long-range dependence process with nonstationary increments, theoretical values are more in line with empirical data, with a smaller probability of mispricing contingent claims with longer time to maturities. By simulating fictitious data for the month of October, November and December, nothing changes: longer dependence, higher volatility approximation, smaller probability of mispricing contingent claim with higher time to expiration.

2.6.2. Option prices and regression analysis

2.6.2.1 Option prices

For the five different scenarios above assumed, the theoretical prices of European call and put models are computed by the "corrected" Black and Scholes formula, where volatility is the parameter in charge of correcting dependence and nonstationarity in data, for all different time to maturities and strike prices. Figure (13) compares the different theoretical prices of two European contingent claims. Data are sorted by expiration date and, then, keeping maturity fixed, by strike price. For call options, the models deliver quite similar values for short maturities.

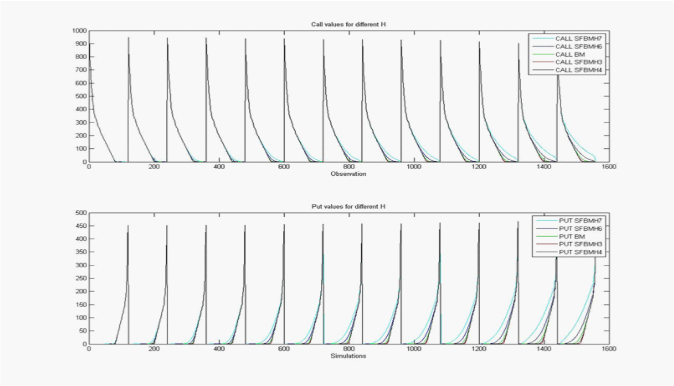

Figure 13: Theoretical Call Option Prices (above) vs Theoretical Put Option Prices (below)

As expiration time increases, some differences arise. Dissimilarities in pricing are evident when, for longer maturities, the strike price is bigger. Processes with high volatility, due to the approximation of long-range dependence, deliver higher call option prices. Similar behavior is shown by put options with the main difference that, for longer maturities, smaller the strike price, higher the difference in pricing. The section is completed by comparing the theoretical option prices with the bid-ask quote. Figure (14) plots the theoretical values of our approximating models against the bid-offer quotes. The right part of the plot, corresponding to options with long time to maturity, shows no consistency between theoretical prices and true data: the market prices positively long-term contingent claims;

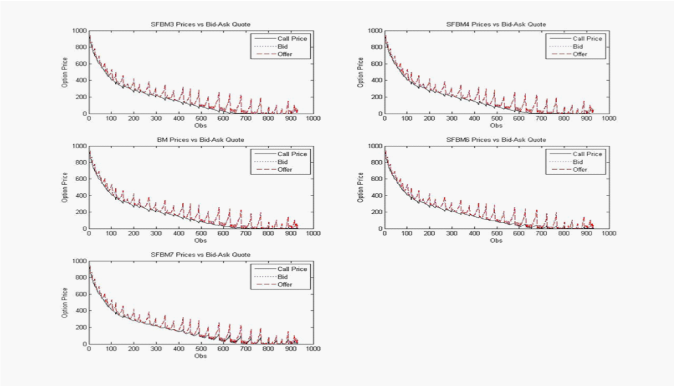

Figure 14: Theoretical Call Option Prices vs Bid-Ask Quotes.

whereas our model delivers long-term contingent claims with zero value. The scenario does not change if the sub-fractional motion is defined with Hurst parameter equal to 0.4: the failure of the model in the long-term is evident. Long-term failure occurs also when no correction is required. Some negligible improvement in pricing accuracy arises around "middle-term" maturities; while in the long-term the model delivers zero-value option prices. The situation starts to improve, when the diffusive coefficient of the Black and Scholes formula corrects long-range dependence. Approximating a SFBM6 might deliver positive prices in the long-term. As H increases, so does the long-range dependence to be corrected, prices become more and more accurate. The SFBM7 prices shown are, indeed, in line with bid-ask quotes given by the market. Similar conclu-

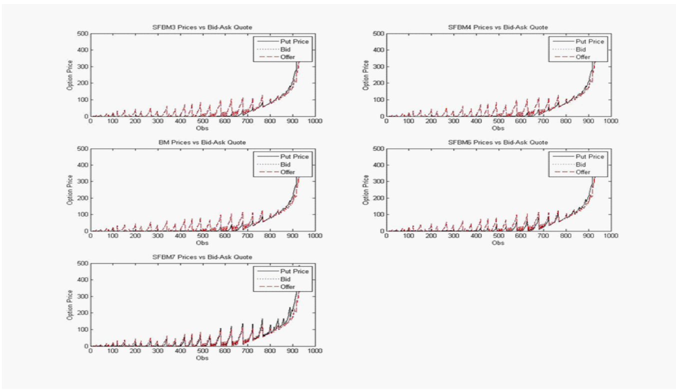

Figure 15: Theoretical Put Option Prices vs Bid-Ask Quotes.

sions can be derived for the put options. Figure (15) shows no consistent between prices derived from a model correcting short-term dependence and market data. No significant improvement is seen when considering a Brownian Motion; while accuracy in prices increases if correcting for long-range dependence: better results, especially in "medium term" (median values in the plot). Similar results occur when the same analysis is extended to the other periods of the sample.

2.6.2.2. Regression analysis

The quality of the correcting models is investigated by comparing the pricing errors, computed as relative distance between model price and market midquote, in the different scenarios. Deviations in pricing should not exhibit any consistent patterns and not be predictable. With a view to assessing the estimated models in this way a regression analysis is performed on the pricing errors obtained from each model. The explanatory variables for the regression summarize the characteristics of the option. The presence of implied volatility smiles suggests that pricing errors are systematically related to the degree of moneyness, measured by the ratio of the spot index level to the option strike. To allow for the possibility that both-out-of-the money puts and calls may have higher implied volatilities, both the degree of moneyness and its square are introduced in the set of the explanatory variables. Implied volatility is also know to rise with the option maturity as an explanatory variable. In addition, the level of interest rates is added as an additional regressor. Consistency with the implied volatility surface suggests that the coefficient of the degree of moneyness should be negative, while the coefficients for the square of moneyness and the option maturity should be positive. The perfect model should have a low R^2, with lack of predictability, i.e. not significant $F - stat$. Even if $SFBM6$ has a higher R^2 with respect to the other models, it is the only one with a not significant $F - stat$. It has a negative moneyness bias; while the coefficient of the square of the moneyness is positive and the smallest among the different models. Together with the other models, it has no maturity bias. Based on these orthogonality tests, $SFBM6$ appears to be the best model about delivering acceptable option prices.

Table 3: OLS results on the predictability of the S&P500 pricing errors.

Regressors	BM	SFBM3	SFBM4	SFBM6	SFBM7
Const	-0.1637 (0.7275)	-0.2149 (0.6513)	-0.2012 (0.6713)	-0.0522 (0.9094)	0.3521 (0.4042)
Moneyness	-0.7022 (0.0023)	-0.6979 (0.0027)	-0.6988 (0.0026)	-0.7216 (0.0013)	-1.0128 (0.0000)
Moneyness^2	0.2643 (0.0012)	0.2687 (0.0012)	0.2678 (0.0012)	0.2542 (0.0015)	0.2930 (0.001)
TimetoMaturity	-0.0341 (0.3972)	-0.0387 (0.3420)	-0.0380 (0.3499)	-0.0169 (0.6661)	0.0464 (0.1998)
InterestRate	0.0048 (0.9532)	0.0112 (0.8920)	0.0093 (0.9096)	-0.0063 (0.9372)	-0.0176 (0.8094)
R^2	0.0047	0.0058	0.0055	0.0032	0.0270
F-stat	3.3783 (0.0092)	3.9156 (0.0036)	3.7927 (0.0045)	2.6395 (0.0323)	14.9465 (0.000)

2.7. Conclusions and future extensions

This Chapter investigates possible departures from rationality in the field of continuous option pricing. The stock's returns might deviate from the classical finance idea of independent and stationarity increments, mostly because of the existence of some distorsions. The structure of its dynamics might be different: the innovation term in the stochastic differential equation could not necessarily belong to the class of Lévy processes. A stochastic differential equation is taken into account; and its random error, even if normally distributed, is not necessary a Brownian motion.

By assuming some regularity condition of the kernel function, it is possible to derive a closed-form expression for the stock, by means of Ito's Lemma. Prices are not necessarily martingales; and arbitrage opportunities might arise in the setting. At this point, a dual solution to the problem is possible: either by means of random functions, the process is transformed into a "fundamental semimartingale", as assumed by Norros et al. , for the fractional Brownian motion framework; or the market selects from a family of Markovian processes, Brownian motion family in the Gaussian case, the best fit to the stock dynamics. The correction is done to clear biases in the market and deliver no-arbitrage contingent claims prices. An entropy argument shows that for the Gaussian family the approximation is done by matching the moments. For European options, this implies that the volatility used in the Black & Scholes formula should be function of the process approximation. The sub-fractional Brownian motion case, where long-range dependence comes along with nonstationarity in the increments, is then studied. Data on the S&P500 Index option from the last decade are used for the empirical analysis. Three different scenarios and five different subcases are imagined for the true dynamics of the index: short-range and nonstationarity, independence and stationarity of increment, long-range dependence. The volatility is corrected, assuming that the market knows and corrects the distortions in the data; and then theoretical prices of European contingent claims are computed. Option prices seems to have higher volatility with respect to that one inferred from the market in the Brownian Motion framework: most accurate prices arise when the market corrects the long-range dependence bias, especially in the long-term, as suggested by comparing them with bid-ask spread. A final regression analysis on the pricing error, in order to check the existence of any consistent patterns and their predictability, seems to give additional strength to the idea that distorsions might exist and be corrected by the market. This idea of market-correction should be further investigated because it can represent a good starting point to develop an alternative explanation to the implied volatility bias of options; moreover, it is possible to refine it either by extending it to more complex contingent claims, where the correction might be taken not only modifying the volatility; or by relaxing the normal scenario, moving to more complicated stochastic processes, like infinite-variance random vectors.

References

[1] Ait-Sahalia, Y., “Maximum-Likelihood Estimation of Discretely-Sampled Diffusions: A Closed-Form Approximation Approach”, *Econometrica*, 70 (2002), 223–262.

[2] Alos, E., Mazet, O., and D. Nualart, "Stochastic calculus with respect to Gaussian processes", *Annals of Probability*, 29 (2001), 766-801.

[3] Barndorf-Nielsen, O. E., "Processes of Normal Inverse Gaussian Type", *Finance and Stochastics*, 2 (1997), 41-68.

[4] Bates, D. S., "Post Crash Moneyness Bias in S&P 500 Futures Options", *Rodney L. White Center working paper* (1991), Wharton School, University of Pennsylvania, Philadelphia, PA.

[5] Billingsley, P., *"Convergence of Probability Measures",* New York, Wiley, (1999).

[6] Black, F., and M. S. Scholes, "The pricing of Options and Corporate Liabilities", Journal of Political Economy, 81 (1973), 637-654.

[7] Bojdecki, T., Gorostiza, L.G., and A.Talarczyk, "Sub-fractional Brownian motion and its relation to occupation times", .Statistics & Probability Letters, 69 (2004), 405–419.

[8] Carr, P., and D. Madan, "Option valuation using the fast Fourier transform", *Journal of Computational Finance,* 2 (1999), 61–73.

[9] Carr, P., Geman, H., Madan, D., and M. Yor, "The fine structure of asset returns: an empirical investigation", *Journal of Business*, 75 (2002), 305–332.

[10] Carr, P., and L. Wu, "Finite moment log stable process and option pricing", *Journal of Finance*, 58 (2003) , 753–777.

[11] Carr, P., and L. Wu,"Time-changed Lévy processes and option pricing", *Journal of Financial Economics*, 71 (2004), 113–141.

[12] Carr, P., and L. Wu, "Stochastic skew in currency options", *Journal of Financial Economics*, 86, 1, (2007), 213-247.

[13] Cochrane, J., *"Asset Pricing",* Princeton University Press, Princeton, NJ, (2004).

[14] Cochrane, J., "Production-based asset pricing and the link between stock returns and economic fluctuations", *Journal of Finance*, 46 (1991), 209–237.

[15] Cont, R., "*Long range dependence in financial time series"*, Fractals in Engineering, Springer, (2005).

[16] Duffie, D., Pan, J., and K. Singleton, "Transform analysis and asset pricing for affine jump-diffusions", *Econometrica*, 68 (2000), 1343–1376.

[17] Eberlein, E., and U. Keller, "Hyperbolic distributions in Finance", *Bernoulli*, 1 (1995), 281-299.

[18] Eberlein, E., and U. Keller, and K. Prause "New insights into smile, mispricing and value at risk: the hyperbolic model", *Journal of Business*, 71 (1998), 371-405.

[19] Eberlein, E., and K. Prause, "The generalized hyperbolic model: Financial derivatives and risk measures", *FDM preprint*, 56, (1998) University of Freiburg.

[20] Heston, S., "A Closed-Form Solution of Options with Stochastic Volatility with Applications to Bond and Currency Options", *The Review of Financial Studies*, 6 (1993), 327-343.

[21] Hida, T., "Canonical representations of Gaussian processes and their applications", *Memoirs of the College of Science*, *University of Kyoto, Series A,* 33.(1960),109-155.

[22] Hida, T., and M. Hitsuda, *"Gaussian Processes",* American Mathematical Society, (1993).

[23] Kahneman, D., and A. Tversky, "Prospect Theory: An Analysis of Decision Under Risk", *Econometrica*, 47 (1979), 263-291.

[24] Kahneman, D., Slovic, P., and A. Tversky, *"Judgment Under Uncertainty: Heuristics and Biases",* New York: Cambridge University Press, (1982).

[25] Kahneman, D., and A. Tversky, "Choices, Values and Frames", *American Psychologist*, 39 (1984), 341-350.

[26] Madan, D. B., and F. Milne, "Option Pricing with VG Martingale Components", *Mathematical Finance*, 1 (1991), 39-55.

[27] Madan, D. B., and E. Seneta, "The Variance Gamma (V.G.) Model for Share Market Returns", *Journal of Business*, 63 (1990), 4, 511-524.

[28] Norros, I., Valkeila, E., and J. Virtamo, "An elementary approach to a Girsanov formula and other analytical results on fractional Brownian motions", *Bernoulli*, 5, 4 (1999), 571-587.

[29] Skorohod, A. V., "On a generalization of a stochastic integral", *Theory of Probability and its Application*, 20 (1975), 219-233.

Chapter 3

Approximation of infinite-variance processes

3.1. Introduction

The normal distribution arises as the limiting distribution for the sum of n independent random variables, when the sum is divided by $\sqrt{n}$. Translating it in financial terms, by assuming returns, as a sum of a large number of independent influences, a Gaussian distribution should describe accurately empirical data. Unfortunately, it is well documented that normality of the return distribution is violated in both time-series and in option prices. Recall that the pointwise Holder exponent of a stochastic process $\{X(t)\}_{t\in\mathbb{R}^d}$ whose trajectories are continuous and nowhere differentiable is the stochastic process $\{\alpha_X(t)\}_{t\in\mathbb{R}^d}$ defined for every t as

$$\alpha_X(t) = \sup\left\{\alpha, \limsup_{h\to 0} \frac{|X(t+h)-X(t)|}{|h|^\alpha} = 0\right\}$$

It allows to measure the local variations of regularity of $\{X(t)\}_{t\in\mathbb{R}^d}$. Figure (16) describes the behavior of the Holder exponent for the daily return of the Standard & Poor's (S&P)100 index in the last ten years. The nice, continuous version of Brownian motion actually satisfies a

Holder condition of exponent α so long as $0 < \alpha < \frac{1}{2}$; but the threshold is exceeded by the exponent in Figure (16). A stationary alpha-stable process should be more accurate for pricing options given its leptokurtosis. Its skewness and kurtosis, joint with a specific dependence structure, must be implemented in order to study the effects of some distortions in the stock. Rosinski (1995) decomposed a stationary alpha stable process in three different components: a part capturing the short-term behavior; a second component describing the long-range dependence and a residual process. The flexibility, implied by the decomposition, could be really helpful in modelling all these departures from normality. This chapter proposes a stationary alpha stable process as building block of the stock dynamics. A stochastic differential for the stock's return is assumed to be driven, under the historical probability, for a representative agent, by a stable innovation term, where dependence across increments is assumed to exist: the specific case of linear fractional stable motion is introduced to model self-similarity. The infinite variance of the process causes some huge problems in deriving a closed-form expression for the stock, by means of stochastic calculus; and the non-Markovian structure violates the No-Arbitrage conditions. The path, suggested by this work, attempts to solve the previous issues by defining on an alternative probability space a family of stable motions, in charge of describing the dynamics of a fictitious securities. The idea is to approximate the true stock with some imaginary securities, and to derive a No-Arbitrage pricing formula, under the risk-neutral probability, by posing some restriction on the drift coefficient of the process. The aim of this work is to show how self-similarity is considered and corrected by the market in pricing contingent claim. The correction should improve accuracy in pricing: the parameters of the approximating Markovian process, derived by minimizing the distance between the two processes, depend on the dependence structure of the process taken into consideration under the historical measure. The distance between the characteristic functions of the two random paths is minimized for a dual reason: stable process might not have density functions; and the set of financial instruments to be priced is restricted to European contingent claims. Section 3.2. defines a stationary alpha stable process, introducing also the Rosinski's decomposition and the definition of conservative and dissipative flow. The idea of long and short memory is presented in the section 3.3.; and is discussed in scenarios where variance is infinite and for a selected class of deterministic functions f under which stochastic integrals with respect to a linear fractional stable are well-defined. Section 3.4. derives a pricing formula for a generic asset whose dynamics is driven by a not-necessarily Markovian process, the fractional-stable motion. Data are described in the section 3.5.; while some empirical findings are presented in the section 3.6. Section 3.7. concludes and

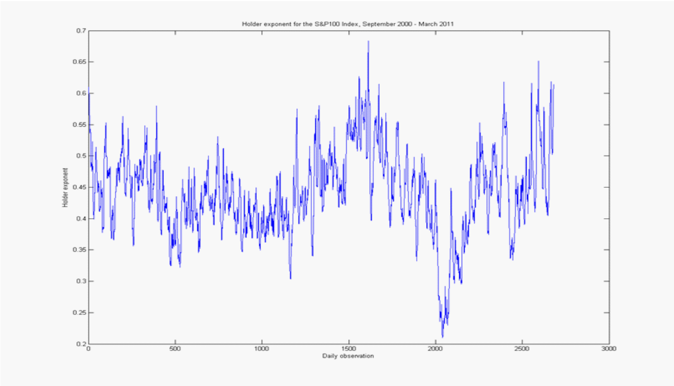

Figure 16: Holder exponent of the Standard & Poor's (S&P)100 index, September 2000- August 2008

suggests some future research paths.

3.2. Stationary alpha stable processes

3.2.1. Preliminaries

A non-degenerate distribution X is a stable distribution if it satisfies the following property:

Definition 10 *Let X_1 and X_2 be independent copies of a random variable X. Then X is said to be stable if for any constants $a > 0$ and $b > 0$ the random variable $aX_1 + bX_2$ has the same distribution as $cX + d$ for* *some constants $c > 0$ and d. The distribution is said to be strictly stable if this holds with $d = 0$.*

The stable random variable X is characterized by the a specific characteristic function has the following form

$$E\exp\{iuX\} = \begin{cases} \exp\left\{-\sigma^{\alpha}|u|^{\alpha}\left(1 - i\beta\,(sign\ u)\tan\frac{\pi\alpha}{2}\right) + i\mu u\right\} & \text{if } \alpha \neq 1 \\ \exp\left\{-\sigma|u|\left(1 + i\beta\frac{2}{\pi}\,(sign\ u)\ln|u|\tan\frac{\pi\alpha}{2}\right) + i\mu u\right\} & \text{if } \alpha = 1 \end{cases}$$

The parameter $\alpha, 0 < \alpha \leq 2$ is the index of stability and

$$sign\ u = \begin{cases} 1 & \text{if } u > 0 \\ 0 & \text{if } u = 0 \\ -1 & \text{if } u < 0 \end{cases}$$

The parameter σ, β, μ are unique and they capture covariation, skewness and location, respectively with $\sigma \geq 0, -1 \leq \beta \leq 1$. X is characterized by four parameters and is possible to denote stable distributions by $S_{\alpha}(\sigma, \beta, \mu)$ and write $X \sim S_{\alpha}(\sigma, \beta, \mu)$. Denote $X \sim S\alpha S$ when X is symmetric $\alpha - stable$, i.e. when $\mu = \beta = 0$.

3.2.2. Characterization

A real (complex, resp) stochastic process $\{X_t\}_{t \in T}$ is said to be a $S\alpha S$ process if any finite linear combination $\sum a_j X_j$ has a $S\alpha S$ (isotropic α-

stable, resp.) distribution ($\mathbf{T}$ is an arbitrary parameter set). Let (S, B) be a Borel space equipped with a σ-finite measure μ. A countably additive set function $M : B_0 \longmapsto L^0(\Omega, P)$ is said to a $S\alpha S$ random measure if M is independently scattered and

$$E \exp\{iuM(A)\} = \exp\{-|u|^\alpha \mu(A)\} \qquad u \in \mathbb{R}, \quad A \in B_0$$

where $B_0 = \{A \in B_0 : \mu(A) < \infty\}$.[13] Every (separable) $S\alpha S$ process $\{X_t^\alpha\}_{t\in T}$ admits an integral representation[14]

$$I(f) = \{X_t^\alpha, t \in T\} \stackrel{d}{=} \left\{\int_S f(s,t) M^\alpha(ds)\right\}_{t\in T}$$

where M is a $S\alpha S$ random measure with control Lebesgue measure λ and $\{f_t\}_{t\in T} \subset L^\alpha(X, \lambda)$ is a family of deterministic functions.

Theorem 11 (Minimal Integral representation) *Let $\mathbf{T}$ be a separable metric space and $\alpha \in (0, 2)$. Suppose that*

$$\left\{\int_{S_1} f_t^{(1)} dM_1\right\} = \left\{\int_{S_2} f_t^{(2)} dM_2\right\} \tag{3.1}$$

where for $\{i = 1, 2\}$, M_i is a $S\alpha S$ random measure on a Borel space

[13] A complex $S\alpha S$ random measure is defined analogously with the symmetry assumption replaced by isotropy.

[14] This representation can be extended to an enlarged probability space.

(S_i, B_{S_i}) with a σ-finite control measure μ_i and $\left\{f_t^{(i)}\right\}_{t\in T} \subset L^{\alpha}(S_i, \mu_i)$ is such that the map $\mathbf{T}\times S \ni (t,s) \to f_t^{(i)} \in \mathbb{R}$ (or $\mathbb{C}$) is Borel measurable ($i = 1, 2$). Then , for every σ-finite Borel measure v on $\mathbf{T}$, there exists Borel functions $\phi : S_2 \longrightarrow S_1$ and $h : S_2 \longrightarrow \mathbb{R}$ (or $\mathbb{C}$) such that

$$f_t^{(2)}(s) = h(s) f_t^{(1)}(\phi(s)) \qquad v \otimes \mu_2 - a.e.$$

Proof. Rosinski, J., 1995, 1167-1172 ■

Any separable in probability $S\alpha S$ process admits an integral representation (3.1). The stationarity property implies

$$\left\{\int_S f_{u+t} dM\right\}_{u\in\mathbb{R}} = \left\{\int_S f_u dM\right\}_{u\in\mathbb{R}}$$

for every $t \in \mathbb{R}$. By the minimal integral representation theorem, for each fixed t

$$f_{u+t}(s) = h_t(s)\left(f_u \circ \phi_t(s)\right) \qquad \lambda \otimes \mu - a.e. \tag{3.2}$$

Suppose equation (3.2) holds for all s, t, u. and define $f \circ g\,(t) = f\,(g\,(t))$. Putting $u = 0$

$$f_t(s) = h_t(s)\left(f_0 \circ \phi_t(s)\right) \qquad t \in \mathbb{R},\ s \in S$$

Is it possible to derive an integral representation in such a way to recognize almost sure a stationary $S\alpha S$ process? Define additive flow and cocycle.[15]

Definition 12 (Additive Flow) *A measurable additive flow on a Borel space S is a jointly measurable map $\mathbb{R}\times S \ni (t,s) \mapsto \phi_t(s) \in S$ such that for every $t_1, t_2 \in \mathbb{R},$ and $s \in S$*

$$\phi_{t_1+t_2}(s) = \phi_{t_1} \circ \phi_{t_2}(s)) \tag{3.3}$$

$$\phi_0(s) = s \tag{3.4}$$

A flow is said to be nonsingular if $\mu \circ \phi_t^{-1}$ is absolutely continuous with respect to μ, for every t.

Let A be a locally compact second countable group.[16]

Definition 13 (Additive Cocycle) *A measurable map $\mathbb{R}\times S \ni (t,s) \mapsto a_t(s) \in A$ is said to be a cocycle[17] for the flow $\{\phi_t\}_{t\in\mathbb{R}}$ if*

$$a_{t_1+t_2}(s) = a_{t_1}(s)\left(a_{t_2} \circ \phi_{t_1}(s)\right) \quad \forall t_1, t_2 \in \mathbb{R}, s \in S$$

[15] Please note that, from now onward, $\mathbf{T} = \mathbb{R}$.

[16] A group is said to be second-countable if its topology has a countable base.

[17] For istance, the Radon-Nikodyn derivative $m_t(s) = \frac{d(\mu\circ\phi_t)}{d\mu}$ is a cocycle (taking values in $A = (0,\infty)$) for a nonsingular flow $\{\phi_t\}_{t\in\mathbb{R}}$.

Once defined additive flows and cocycle, it is possible to characterize a stationary $S\alpha S$ process

Theorem 14 (Representation of stationary $S\alpha S$ processes) *Let*

$\{f_t\}_{t\in\mathbb{R}} \subset L^{\alpha}(S,\mu)$ be a measurable spectral representation of a measurable stationary $S\alpha S$ process $\{X_t\}_{t\in\mathbb{R}}$. Then there exist a unique modulo μ nonsingular measurable flow $\{\phi_t\}_{t\in\mathbb{R}}$ on some Borel σ-finite measure space (S,μ), a cocycle $\{a_t\}_{t\in\mathbb{R}}$ for $\{\phi_t\}_{t\in\mathbb{R}}$, with values in $\{-1,1\}$ [18] such that, for each $t\in\mathbb{R}$

$$f_t = a_t\left\{\frac{d(\mu\circ\phi_t)}{d\mu}\right\}^{\frac{1}{\alpha}}(f_0\circ\phi_t) \quad \mu - a.e. \tag{3.5}$$

Proof. See Rosinski, J. , 1995, pag.1167-1169. ■

The stable integral of a stationary $S\alpha S$ process $\{X_t\}_{t\in\mathbb{R}}$ can be rewritten as

$$X_t = \int_S a_t m_t^{\frac{1}{\alpha}} f\circ\phi_t dM \quad a.s.\ t\in\mathbb{R}$$

In ergodic theory there are some classical decomposition of measure generated by nonsingular flows. The most trivial one is obtained by considering the set of fixed points

$$S_1 = \{s\in S : \phi_t(s) = s \quad \forall t\in\mathbb{R}\}$$

[18] $\{|z|=1\}$ in the complex case.

The flow instantaneously moves the points on $S \setminus S_1$ and is constant on S_1. Both these sets are invariant under the flow. Now, $S \setminus S_1$ admits the Hopf decomposition into the dissipative S_2 and conservative S_3 parts, which are invariant under the flow

$$
\begin{aligned}
S_2 &= \left\{ x \in S : \int_T |f \circ \phi_t|^\alpha m_t \lambda(dt) < \infty \right\} \quad a.e. \\
S_3 &= \left\{ x \in S : \int_T |f \circ \phi_t|^\alpha m_t \lambda(dt) = \infty \right\} \quad a.e
\end{aligned}
$$

where $\lambda\left(dt\right)$ is the Lebesgue measure if $T = R$, and a counting measure if $T = Z$. Thus it is possible to decompose S into three invariant sets

$$S = S_1 \cup S_2 \cup S_3$$

This decomposition yields a decomposition by Rosinski of the stochastic integral in expression (3.5) into three independent parts, say $X_t^{(1)}, X_t^{(2)}$ and $X_t^{(3)}$

Theorem 15 (Decomposition of stationary $S\alpha S$ processes) *Every stationary $S\alpha S$ process $\{X_t^\alpha\}_{t\in\mathbb{R}}$ has a unique decomposition into the sum of three mutually independent stationary $S\alpha S$ processes*

$$X_t^\alpha \overset{d}{=} X_t^1 + X_t^2 + X_t^3 \quad t \in \mathbb{R}$$

(on a possibly enlarged probability space) such that $\{X_t^1\}_{t\in\mathbb{R}}$ *is a harmonizable process,* $\{X_t^2\}_{t\in\mathbb{R}}$ *is mixed moving average, and* $\{X_t^3\}_{t\in\mathbb{R}}$ *does not admit harmonizable or mixed moving average components.*

Proof. See Rosinski, J. , 1995, pag.1186. ■

A specific representation for the independent processes X_t^1 and X_t^2, is characterized as follows:

Definition 16 (Harmonizable process) *A* $S\alpha S$ *process is said to be harmonizable if there exists a complex random measure* M *on* $\mathbb{R}$ *with finite control measure* μ *such that*

$$\{X_t^\alpha\}_{t\in\mathbb{R}} = \left\{\int_{\mathbb{R}} e^{its} M^\alpha(ds)\right\}_{t\in\mathbb{R}}$$

with $f = 1$, $\phi_t(s) = s$, $a_t(s) = e^{its}$ $\forall t, s \in \mathbb{R}$.

Definition 17 (Moving average process) *A* $S\alpha S$ *is a moving average if there exists a function* $f \in L^\alpha(\mathbb{R}, \lambda)$ *with Lebesgue control measure* λ *such that*

$$\{X_t^\alpha\}_{t\in\mathbb{R}} = \left\{\int_{\mathbb{R}} f(t-s) M^\alpha(ds)\right\}_{t\in\mathbb{R}}$$

with $\phi_t(s) = s - t$, $a_t(s) = 1$, $\mu = \lambda$ $\forall t, s \in \mathbb{R}$.

A stationary $S\alpha S$ process can be easily decomposed into a dissipative part and a conservative one, i.e.

$$X_t^\alpha = X_t^D + X_t^C$$

where X_t^D, the mixed moving average process, is generated by a dissipative flow, while X_t^C, given by the harmonizable process X_t^1 and the residual process X_t^3, is generated by infinitely recurrent flow: an identity flow for the "never-ergodic" harmonizable process; a conservative flow without fixed points for the residual process. The decomposition into dissipative and conservative flow permits to study the dependence of the process along time. The conservative part X_t^C must capture the long-range dependence structure of the process; whereas X_t^D should account for the short-range dependence or i.i.d. observations.

3.3. $S\alpha S$ processes and kernel functions

Let X be a stationary process with spectral density f. X is said to be a long-memory process only if, for some number d in the interval $(0, 1)$, $f(\omega)$ is asymptotically equivalent to ω^{-d} times a slowly varying function

$L(\omega)$,[19] as ω approaches zero, i.e.

$$\lim_{\omega\to 0}\frac{f(\omega)}{L(\omega)\,\omega^{-d}} = 1 \quad \forall\, some\ slowly\ varying\ function\ L$$

Consider a process X_t with the following integral representation

$$X_t = \int_S f_t(x)\, M(dx)$$

where $M(dx)$ is a a stable Lévy motion. Is it possible to assess something about the dependence-structure of the process?

1. If the kernel function is the indicator function, i.e. $f_t(x) = 1_{(x\in S)}$, X_t would be still a stable Lévy motion;

2. a different kernel would plug-in some particular form of dependence into the process.

The importance of the kernel function is evident looking at its relationship with the exceedance probability. Let a claim process $\mathbf{X} = \{X(t), t \geq 0\}$ be a continuous-time measurable stationary $S\alpha S$ process

$$X(t) = \int_{\mathbb{R}} f(x-t)M(dx) \quad t \geq 0 \tag{3.6}$$

[19] A slowly varying function of ω is a function L such that $\lim_{\omega\to 0}\frac{L(t\omega)}{L(\omega)} = 1\ \forall t > 0$.

where $f \in L^{\alpha}(\mathbb{R}, B,\lambda)$ and M is a $S\alpha S$ random measure on $(\mathbb{R}, B)$ with Lebesgue control measure λ, and $\alpha \in (1, 2)$. Also for a positive constant $\mu > 0$ let the cumulative premium process be a non-random linear drift $\mu = \{\mu(t) = t\mu, t \geq 0\}$, and define

$$S(t) := \int_0^t X(s)ds \quad s \geq 0 \tag{3.7}$$

as the total amount claimed until time t. Set

$$h_t\,(x) = \int_0^t f(x - s)ds.$$

The process $\mathbf{S} = \{S(t), t \geq 0\}$ described by (3.7) is well defined and

$$S(t) := \int_0^t h_t(x)M(dx) \quad t \geq 0$$

Now, for $u > 0$ write the ruin probability as

$$\psi(u) = P\left(\sup_{t\geq 0}\left(S(t) - \mu(t)\right) > u\right) \quad u > 0$$

The following theorem proves that, as far as the ruin probability is concerned, there is a strong relation between a specific dependence in continuous-time stationary $S\alpha S$ processes generated by ergodic dissipative flows and integrability of the kernel function in their integral representation.

Theorem 18 *Let $f : \mathbb{R} \to \mathbb{R}$ given in (3.6) be a nonnegative function.*

(1) Let $f \notin L^1(\mathbb{R}, B, \lambda)$, then

$$\lim_{u \to \infty} u^{\alpha - 1} \psi(u) = \infty$$

i.e. the claim process X has long-range dependence

(2) Suppose for some positive real L and R, the function

$$f^*(x) := \begin{cases} \sup_{t \geq 0} f(x - t), & x \leq -L \\ f(x), & x \in (-L, R) \\ \sup_{t \geq 0} f(x + t), & x \geq R \end{cases}$$

is in $L^1(\mathbb{R}, B, \lambda) \cap L^\alpha(\mathbb{R}, B, \lambda)$. Then,

$$\lim_{u \to \infty} u^{\alpha - 1} \psi(u) = \frac{C_\alpha \|f\|^\alpha_{L^1(\mathbb{R}, B, \lambda)}}{2(\alpha - 1)\mu} < \infty$$

In particular, the claim process X is short-range dependence.

Proof. Alparslan, U.T., and G. Samorodnitsky, 2007, pages 7-19 ■

Once shown the importance of the kernel functions, it is immediate to study the features of a fractional process belonging to the class of stable processes: the linear fractional stable motion.

3.3.1. Fractional $S\alpha S$ process: the linear fractional stable motion

3.3.1.1. Characterization

For $0 < H < 1$ and $0 < \alpha \leq 2$ such that $H \neq \frac{1}{\alpha}$, define a linear fractional stable motion $\{\Delta_{H,\alpha}(t), t \in R\}$ as a stochastic process characterized by the following integral representation

$$\begin{aligned}\Delta_{H,\alpha}(t) &= \int_{-\infty}^{\infty} f_{\alpha,H} dZ_{\alpha}(s) \qquad (3.8)\\ &= \int_{-\infty}^{\infty} a\left(\left((t-s)_{+}\right)^{H-\frac{1}{\alpha}} - \left((-s)_{+}\right)^{H-\frac{1}{\alpha}}\right)\\ &\quad +b\left(\left((t-s)_{-}\right)^{H-\frac{1}{\alpha}} - \left((-s)_{-}\right)^{H-\frac{1}{\alpha}}\right) dZ_{\alpha}(s)\end{aligned}$$

where a, b are real constants, $|a| + |b| > 0$, $0 < \alpha < 2$, $0 < H < 1$, $H \neq \frac{1}{\alpha}$ and $Z_{\alpha}(s)$ is an $\alpha - stable$ random measure on $\mathbb{R}$ with Lebesgue control measure and skewness intensity $\beta(x), -\infty < x < \infty$ satisfying:

1. $\beta(\cdot) = 0$ if $\alpha = 1$;
2. for all integers $d \geq 1$ and real $\theta_j, t_j, j = 1, ..., d$

$$\int_{-\infty}^{\infty} \left(\sum_{j=1}^{d} \theta_j \left(f_{\alpha,H}(a, b; t_j, x) - a, b; 0, x\right)\right)^{\langle\alpha\rangle} \beta(cx + h)\, dx$$

is independent of $c > 0$ and -$\infty < h < \infty$.

The previous representation of the linear fractional Stable motion (LFSM) is quite similar to the characterization of the fractional Brownian motion: the main difference is that here the exponent is $H-\frac{1}{\alpha}$. Therefore, when $\alpha = 2$, expression (3.8) reduces to a fractional Brownian motion.

Theorem 19 *The process $\Delta_{H,\alpha}(t)$ is:*

1. *well defined;*
2. *a symmetric alpha stable ($S\alpha S$) process when $\beta(\cdot) = 0$;*
3. *is self-similar with index H and has stationary increments;*
4. *its increments have long-range dependence if $H > \frac{1}{\alpha}$ and negative dependence when $H < \frac{1}{\alpha}$.*

Proof. See Samorodnitsky, G., and M. S. Taqqu, 1994, chapter 7. ■

Figure (17) describes the relationship between the Linear Fractional Stable Motion (LFSM), the fractional Brownian Motion (FBM), the Brownian Motion (BM) and the Lévy-Stable Motion (LSM). By restricting $a = b = 1,\ L_{H,\alpha}(t)$ becomes the (well-balanced) linear fractional stable motion

$$\Delta_{H,\alpha}(t) = \int_{-\infty}^{\infty} \left(\left((t-s)_+ \right)^{H-\frac{1}{\alpha}} - \left((-s)_+ \right)^{H-\frac{1}{\alpha}} \right) dZ_\alpha(s)$$

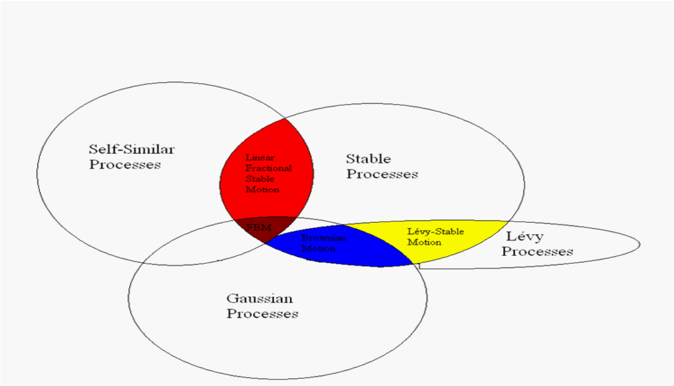

Figure 17: Relations between the class of Gaussian processes, the Lévy class, the set of Stable processes and the group of Self-Similar processes.

Figure (18) shows different behavior of LFSM, depending on the Hurst parameter and the stability index (H, α). As shown in the plot, the behavior of the process changes dramatically by varying the couple of parameters.

3.3.1.2. Linear fractional stable motion as a convolution with respect to a stable motion

Denote with $*$ the following operation $(f * g)(s) = \int_{-\infty}^{\infty} f(s-u)\, g(u)\, du$ for measurable function f and g. Cambanis and Maejima (1989) have proposed a class of deterministic function for which is possible to deter-

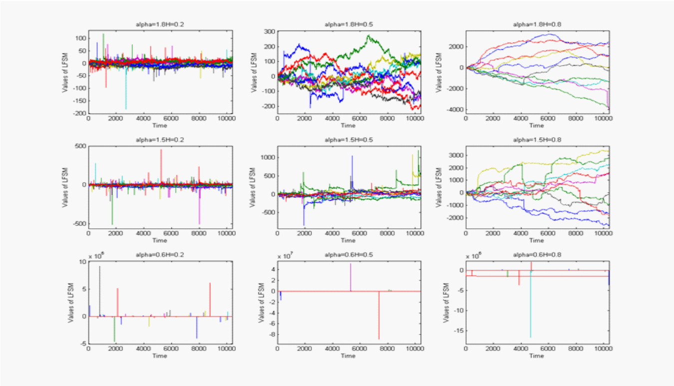

Figure 18: Simulation of different Linear Fractional Stable Motion as function of (H, α)

mine the stochastic integral with respect to $d\Delta_{H,\alpha}(t)$. Define for $\gamma > 0$

$$\rho_{\gamma}(x) = \gamma |x|^{\gamma-1} 1_{(-\infty,0)}(x)$$

Then since

$$\begin{aligned} (t-s)_+^{H-\frac{1}{\alpha}} - (-s)_+^{H-\frac{1}{\alpha}} &= \left(H - \frac{1}{\alpha}\right) \int_s^{\infty} 1_{[0,t)}(u) * (u-s)^{\left(H-\frac{1}{\alpha}\right)-1} du \\ &= \left(1_{[0,t)} * \rho_{\beta}\right)(s) \end{aligned}$$

Define $\gamma = H - \frac{1}{\alpha}$, the previous characterization of the process can be rewritten as

$$\Delta_{H,\alpha}(t) = \int_{-\infty}^{\infty} \left(1_{[0,t)} * \rho_\gamma\right)(s)\, dZ_\alpha(s)$$

On the other hand, for the simple function of the form

$$f(u) = \sum_{k=1}^{n} f_k 1_{[u_k,u_{k+1})}(u) \quad f_k \in R \quad u_k < u_{k+1} \quad k = 1, ..., n$$

Define the following identity

$$\int_{-\infty}^{\infty} f(u) d\Delta_{H,\alpha}(t) = \sum_{k=1}^{n} f_k \left(\Delta_{H,\alpha}(u_{k+1}) - \Delta_{H,\alpha}(u_k)\right)$$

However the right-hand side can be rewritten as

$$\int_{-\infty}^{\infty} \left(f * \rho_\gamma\right)(s)\, dZ_\alpha(s)$$

Thus the definition of $I(f)$, linear with respect to f, is the following:

Definition 20 *Let*

$$\Lambda_{\alpha,\beta} = \left\{f : |f| * \rho_\gamma \in L^\alpha\right\}$$

For $f \in \Lambda_{\alpha,\beta}$ *define* $I(f)$ *to be*

$$I(f) = \int_{-\infty}^{\infty} f(u)\, d\Delta_{H,\alpha}(u) = \int_{-\infty}^{\infty} \left(f * \rho_\gamma\right)(s)\, dZ_\alpha(s)$$

Here, notice that the right-hand side is well defined thanks to the assumption that $|f| * \rho_\beta \in L^\alpha$ and it holds

$$E\left[e^{i\theta I(f)}\right] = \exp\left\{|\theta|^\alpha : \left\|f * \rho_\gamma\right\|_{L^\alpha}^\alpha\right\}, \quad \theta \in R$$

i.e.

$$E\left[\exp\left\{i\theta \int_{-\infty}^{\infty} f(u)\, d\Delta_{H,\alpha}(u)\right\}\right] = \exp\left\{-|\theta|^\alpha \gamma^\alpha \int_{-\infty}^{\infty} \left|\int_s^\infty f(u)(u-s)^{\gamma-1} du\right|^\alpha ds\right\}$$

3.4. Stochastic differential equation driven by a linear fractional stable motion

Fix a filtered probability space $\{\Omega, \mathcal{F}, \mathbf{P}\}$, $\mathcal{F}_t^\Delta = \sigma(X_t^\alpha, t \leq T)$. Consider a single investor, and assume that she is the representative agent. Under the historical probability $\mathbf{P}$, the true dynamics of the the stock's return is governed by the following stochastic differential equation up to time t :

$$dR_t = \frac{dS_t}{S_t} = \mu(t, X(t,\omega))\, dt + \sigma(t, X(t,\omega))\, d\Delta_{H,\alpha}(t) \tag{3.9}$$

A restriction on the drift and diffusive coefficients is considered: they are not stochastic but time-dependent. Equation (3.9) becomes

$$dR_t = \mu(t)\,dt + \sigma(t)\,d\Delta_{H,\alpha}(t)$$

By following the approach of the previous section, the stochastic differential equation can be rewritten as

$$\begin{aligned} dR_t &= \mu(t)\,dt + \left(\sigma(s) * \rho_\gamma\right)(s)\,dZ_s^\alpha \\ &= \mu(t)\,dt + \left(\gamma \int_s^t \sigma(u)\,(u-s)^{\gamma-1}\,du\right) dZ_s^\alpha \end{aligned}$$

i.e.

$$R_t = \int_0^t \mu(s)\,ds + \int_0^t \left(\int_s^t \gamma\sigma(u)\,(u-s)^{\gamma-1}\,du\right) dZ_s^\alpha$$

with $R(0) = 0$ and $dZ_\alpha(s) \sim S_\alpha\left((ds)^{\frac{1}{\alpha}}, 0, 0\right)$, $\sigma(u)$ is a positive compactly supported function. Its characteristic function is given by

$$\begin{aligned} E\left[e^{i\theta R_t}\right] &= E\left[\int_0^t \mu(s)\,ds + \int_0^t \left(\int_s^t \gamma\sigma(u)\,(u-s)^{\gamma-1}\,du\right) dZ_s^\alpha\right] \\ &= \exp\left[i\theta \int_0^t \mu(s)\,ds - \left\{-\left|\theta\right|^\alpha \gamma^\alpha \int_0^t \left|\int_s^t \sigma(u)\,(u-s)^{\gamma-1}\,du\right|^\alpha ds\right\}\right] \\ &= \left[i\theta\mu_t - \left|\theta\sigma_t\right|^\alpha\right] \end{aligned}$$

i.e. $R_t \sim S_\alpha(\sigma_t, \mu_t, \beta_t)$ with the following parameters

$$\begin{aligned}
\sigma_t &= \left(\gamma^\alpha \int_0^t \left|\int_s^t \sigma(u)(u-s)^{\gamma-1} du\right|^\alpha ds\right)^{\frac{1}{\alpha}} \\
\mu_t &= \int_0^t \mu(s)\, dt \\
\beta_t &= 0
\end{aligned}$$

The infinite variance of the process might cause some troubles in delivering a closed-form expression of the the stock price S_t. The powerful machinery of Ito's lemma, in stochastic calculus, cannot be applied, given the statistical structure of the process. Not only is the process characterized by infinite variance, but it does not have independent increments: the semimartingale properties are not granted. The price of a contingent claim whose final payoff is $\Phi(S_T)$, even if a closed-form expression S_t might exist, should be conjectured with (generalized) pricing formula can be

$$\pi_t(S_T, r, \sigma') = g\left[m_T \Phi(S_T) | \mathcal{F}_t\right]. \tag{3.10}$$

for some functional g, with m_T being the stochastic discount factor. If the market prices do not allow for profitable arbitrage, the prices are said to constitute an arbitrage equilibrium or arbitrage-free market. An arbitrage equilibrium is a precondition for a general economic equilibrium. The existence of arbitrage opportunities, instead, moves the market away from equilibrium, since the arbitrage drives the demand up to infinity. At the end, the following doubts might be faced:

(Q1) Is this an arbitrage-free price?

(Q2) Is the market aware of the existence of some distortions?

(Q3) What if the market tries to correct these fallacies?

A normative approach, delivering a functional g, when a closed-form expression of S_T exists, would provide unambiguous answers to the previous questions. The main concern is the existence of both a closed-form expression for the stock, and a functional g, able to clear all arbitrage opportunities. The approach suggested by this work follows an alternative route. The idea is to contribute to introduce infinite-variance in modelling financial securities, with a more positive approach and less rigorous way. Consider another probability space $\left\{\Omega', \mathcal{F}_t^Z, \mathbf{M}\right\}$, $\mathcal{F}_t^Z = \sigma\left(Z_t^\alpha, t \leq T\right)$.The following family of diffusion has been defined in this probability space:

$$dT_t^\delta = \mu'\left(t, \delta\right) dt + \sigma'\left(t, \delta\right) dZ_t^\alpha$$

i.e.

$$T_t^\delta = \int_0^t \mu\left(s\right) ds + \int_0^t \sigma\left(s\right) dZ_s^\alpha$$

given $T_0^\delta = 0$, and $dZ_\alpha\left(t\right) \sim S_\alpha\left(\left(dt\right)^{\frac{1}{\alpha}}, 0, 0\right)$; and all the possible sample paths are function of $\delta\left(s\right) = \left[\mu'\left(s, \delta\right), \sigma'\left(s, \delta\right)\right]$. Its characteristic function is given by

$$\begin{aligned} E\left[e^{i\theta T_t^\delta}\right] &= E\left[\exp\int_0^t \mu'(s,\delta)\,ds + \int_0^t \sigma'(s,\delta)\,dZ_s^\alpha\right] \\ &= \exp\left[i\theta\int_0^t \mu'(s,\delta)\,dt - \left\{-|\theta|^\alpha \int_0^t |\sigma'(s,\delta)|^\alpha\,ds\right\}\right] \\ &= \exp\left[i\theta\mu_t'^\delta - \left|\theta\sigma_t'^\delta\right|^\alpha\right] \end{aligned}$$

i.e. $T_t^\delta \sim S_{\alpha'}\left(\sigma_A^\delta, \mu_A^\delta, \beta_A^\delta\right)$ and the following parameters

$$\sigma_t'^\delta = \left(\int_0^t |\sigma'(s,\delta)|^\alpha\,ds\right)^{\frac{1}{\alpha}} \quad (3.11)$$

$$\mu_t'^\delta = \left(\int_0^t \mu'(s,\delta)\,ds\right) \quad (3.12)$$

$$\beta_t^\delta = 0$$

Figure (19) compares the dynamics of a linear fractional stable motion (upper part) and a family of stable motions (lower part). Assume the following path to clear all the doubts in (Q1), (Q2) and (Q3):

1. The market knows that the true dynamics of the stock's return is represented by the evolution of R_t. The only concern is that, R_t is not a semimartingale;

2. Taking its log to derive a no-arbitrage pricing formula for the stock is not possible.

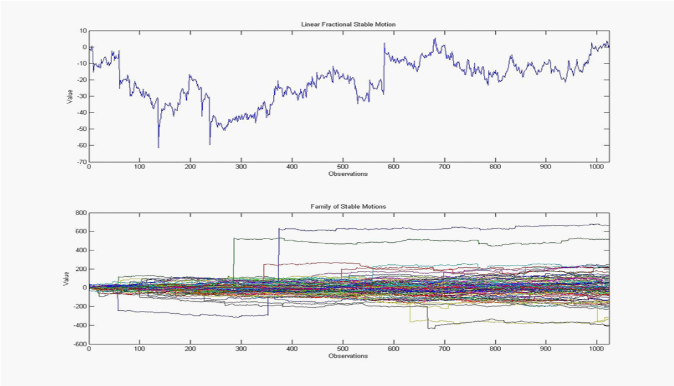

Figure 19: Linear Fractional Stable Motion (above) vs Family of Stable Motions (below)

3. A good strategy to overcome the problem is the following:

   (a) among all the possible sample paths of T_t^{δ}, the market selects the "optimal trajectory", i.e. that one, given the events $A_1, A_2, ...A_t \in B(\mathbb{R})$, $A'_1, A'_2, ...A'_t \in B(\mathbb{R})$, whose joint probability density function is the closest to the LFSM, i.e.

$$P(R_1 \in A_1, ...R_t \in A_t) \simeq M_{\delta^*(t)}(T_1 \in A'_1, ...T_t \in A'_t) \quad (3.13)$$

   (b) Given the optimal parameters, the markets uses them to provide a no arbitrage price for a contigent claim and correct for dependence across increments.

The approximation, (3.13) given the structure of the increments of the two processes, might not exist but, if restricting the attention to European contingent claims, i.e. without path-dependence, the third point becomes much easier. Really important is to find the closest marginal distribution between R_t and T_t at time t, i.e. the optimal parameters $\mu_t'^{\delta^*}, \sigma_t'^{\delta^*}$. A good way to find the optimal parameters might be to compute the minimum distance method between theoretical characteristic function $\phi(T)$ and the sample characteristic function $\widehat{\phi}(t)$. Define

$$D(\alpha, \beta, \sigma, \mu) = \left\| \phi(t) - \widehat{\phi}(t) \right\| \tag{3.14}$$

the method finds

$$\widehat{\theta} = \arg\min_{\theta \in \Theta} D(\theta)$$

where $\theta = (\alpha, \beta, \sigma, \mu)$ is a point in the parametric space, $\|\cdot\|$ is a norm, Θ is the parameter space. By choosing as norm the L^r weighted norm (alternatively it's possible to choose L^∞)

$$L^r(\theta) = \int_0^t \left| \phi(s) - \widehat{\phi}(s) \right|^r J(s)\, ds$$

where $J(s)$ is the weight making the integral convergent. Set $r = 2$ and $J(s) = \exp\left(-s^2\right)$ .Once the optimal μ'^*, σ'^* are computed, the following proposition can be applied to derive a close formula for the the approximating process from time t onward. Since A_t is a semimartingale, S_T[20] is an exponential Lévy model

$$S_T = S_t \exp T_t^* = S_t \exp \int \mu'^{*} (u)\, du + \int \sigma'^{*} (u)\, dZ_u^{\alpha}$$

The fundamental theorem of finance states that, under the risk-neutral measure, the discounted stock price, must be a martingale. Sato (1999) provides the necessary and sufficient condition in terms of generating triplets for a density transformation.

Theorem 21 *Let* $\left(\{X_t\}_{t\leq T}, P\right)$ *and* $\left(\{X_t\}_{t\leq T}, P^{\#}\right)$ *be Lévy processes on* $\mathbb{R}^d$ *with generating triplets* (A, v, γ) *and* $\left(A^{\#}, v^{\#}, \gamma^{\#}\right)$ *respectively. Then the following two statements are equivalent:*

1. $[P]_{\mathcal{F}_t} \approx \left[P^{\#}\right]_{\mathcal{F}_t}$ *for every* $t \in (0, \infty)$

[20] The property that expected payoffs on assets and call options are infinite under most log-stable distributions led both Paul Samuelson and Robert Merton (1976) to conjecture that assets and derivatives could not be reasonably priced under these distributions, despite their attractive feature as limiting distributions under the Generalized Central Limit Theorem. Carr and Wu (2003) are able to price options under log-stable uncertainty, but only by making the extreme assumption of maximally negative skewness, i.e. $\beta = 1$. This work does not impose any restriction on the skewness coefficient β. A paper by McCulloch (2003) show that the restriction is not necessary and that the concerns of Samuelson and Merton were in fact unfounded. The work demonstrates that when the observed distribution of prices is log-stable, the Risk Neutral Measure (RNM), under which asset and derivative prices may be computed as expectations, is not itself log-stable in the problematic cases. The RNM is determined by the convolution of two densities, one negatively skewed stable, and the other an exponentially tilted positively skewed stable. The resulting RNM gives finite expected payoffs for all parameter values and enables options on log-stable assets to be computed easily by means of the Fast Fourier Transform (FFT) methodology of Carr and Madan (1999).

2. *The generating satisfy*

$$\begin{aligned} A &= A^{\#} \\ v &\approx v^{\#} \end{aligned}$$

with the function $\varphi(x)$ *defined by* $\frac{dv^{\#}}{dv} = e^{\varphi(x)}$ *satisfying*

$$\int_{\mathbb{R}^d} \left(e^{\frac{\varphi(x)}{2}} - 1 \right)^2 v(dx) < \infty \tag{3.15}$$

and

$$\gamma^{\#} - \gamma - \int_{|x| \leq 1} x \left(v^{\#} - v \right)(dx) \in \Re(A)$$

where $\Re(A) = \left\{ Ax : x \in \mathbb{R}^d \right\}$

Proof. See Sato, K., 1999, page 230. ■

The discounted stock price has generating triplet

$$\left(0,\ \rho \frac{1}{|x|^{\alpha+1}} dxdu,\ \int_t^T (\mu^*(u) - r(u))\, du \right)$$

The existence of an equivalent martingale measure depends on the existence of $\varphi^*(x)$ such that

$$\int_t^T \sigma^*(u) \int_{\mathbb{R} \setminus \{0\}} \left(e^{\varphi^*(x,u)} - 1 \right) x \rho \frac{1}{|x|^{\alpha+1}} dxdu = -\int_t^T (\mu^*(u) - r(u))\, du \tag{3.16}$$

provided that condition (3.15) is satisfied. The existence of $\varphi^*(x,u)$ is, then, strictly dependent on the expression of $\gamma(u) = \mu^*(u) - r(u)$, i.e. it should exist $\varphi^*(x,u)$ such that the integral on the left-hand side of (3.16) would be equal to the right-hand side. By assuming $\gamma(u) = -\lambda\rho^{\frac{1}{\alpha}} f'(u)$, then

$$\varphi(x,u) = \log\left(1 + \lambda\rho^{-\frac{\alpha-1}{\alpha}} \frac{2-\alpha}{2} f'(u) x 1_{\{|x|\le 1\}}\right) \tag{3.17}$$

with $\varphi(x,s)$ well defined since

$$\lambda\rho^{-\frac{\alpha-1}{\alpha}} \frac{2-\alpha}{2} |f'(u)| < 1 \text{ for almost all } u \in [0,1]$$

A change of measure from the historical probability P into the risk-neutral measure Q is possible and the risk neutral dynamics of the stock price is

$$S_T = S_t \exp\left(\int_t^T r(u)\,du + \int_t^T \sigma^*(u)\,dZ_u^Q\right)$$

where Z_u^Q is a transformed process, with Lévy triplet is $\left(0, 0, \rho e^{\varphi^*(x,u)} \frac{dx}{|x|^{\alpha+1}} du\right)$, and $\varphi^*(x,u)$ is given by (3.17). The price π of a contingent claim is then given by

$$\pi(t,r,\sigma) = E_t^Q\left[e^{-\int_t^T r(u)du} g(S_T)\right] \tag{3.18}$$

where $g(S_T)$ is the payoff function of the financial security. The payoff

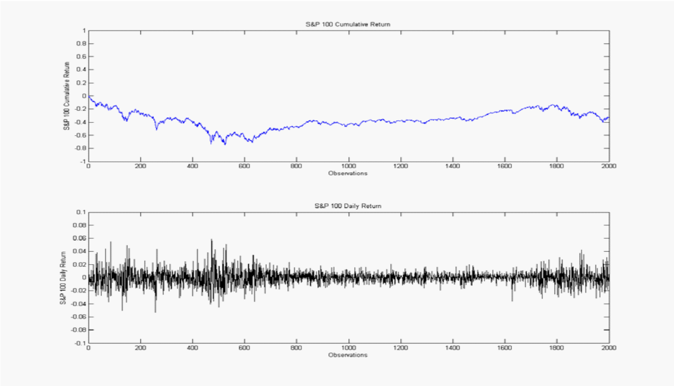

Figure 20: S&P 100 cumulative return, Sept. 2000- Aug. 2008 (above) vs S&P 100 daily return, Sept. 2000- Aug. 2008 (below)

structure of the contingent claim could simplify expression (3.18). For a European call option, it is relatively easy to compute the Fourier transform of an option price, as suggested by Carr et al. (1999). Let k denote the log of the strike price K, and let $C_T(k)$ be the desired value of a T maturity call option with strike $\exp(k)$. Let the risk-neutral density of the log price s_T be $q_T(s)$. The characteristic function of this density is defined by

$$\phi_T(s) \equiv \int_R e^{ius} q_T(s) ds$$

and

$$
\begin{aligned}
C_T(k) &= \exp\frac{(-\alpha k)}{\pi}\int_0^\infty \exp(-ivk)\psi(v)dv \\
&= \exp\frac{(-\alpha k)}{\pi}\int_0^\infty \exp(-ivk)\left(\frac{e^{-rT}\phi_T(v-(\alpha+1)\,i)}{\alpha^2+\alpha-v^2+i\,(2\alpha+1)\,v}\right)
\end{aligned}
$$

where $\exp(-\alpha k)$ is introduced to obtain a square integrable function, with $\alpha > 0$.[21]

3.5. The option data

The prices used in this study are for the Standard & Poor's (S&P) 100 Index Options listed at the Chicago Board of Option Prices (CBOE). The data for the study were obtained from the Option Metrics, whose access is provided by the Wharton Research Data Services (WRDS); and include all index prices from September 2000 to August 2008. The index behavior and its daily return are shown in Figure (20). Option Metrics contains also bid and ask European option quotes on the S&P 100 index, across different strikes and maturities and dividend yield inferred

[21] By the Put-Call parity it is possible to derive the price for the put options

$$C_t(K) - P_t(K) = S(t) - K * B(t,T)$$

where $C_T(K)$ is the value of the call at time t; $P_t(K)$ is the value of the put of the same expiration date; $S(t)$ is the forward price of the underlying asset; K is the strike price; $B(t,T)$, the present value factor for K, the present value of a zero-coupon bond that matures to \$1 at time T.

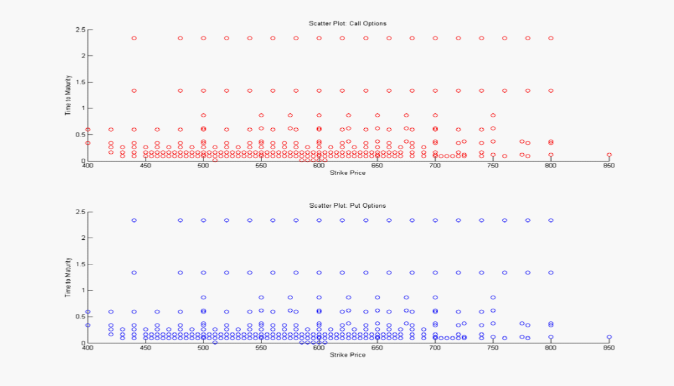

Figure 21: Call Options (above) vs Put Options (below)

from the theoretical no-arbitrage relationship between the spot and future index. r. All options contract were viewed as written on the underlying spot index. The WRDS interface has "market" section where it is possible to obtain daily data on the three month Treasury Bill rate. At the 18th of August, 2008, the total number of contract is 520 and the universe of option prices is composed by thirteen different maturities and one-hundred and twenty-two strike prices. As summarized in Figure (21), a significant amount of contract is characterized by short time to maturity and a strike price close to the index value.

3.6. Parameter estimation and empirical performance of the linear fractional stable motion

The Hurst parameter, the index of stability and the covariation of the S&P100 index are computed, in order to find if the index is driven by the linear fractional stable motion and verify its dependence structure.

Table 1: Parameter estimation for LFSM.

Month	Nobs	Value
Hurst Exponent H	2600	0.48
Index of Stability α	2600	1.6
Location μ	2600	-0.41
Covariation σ	2600	0.0092

A stochastic process is normally distributed if the index of stability α is equal to 2.[22] A stability index α, lower than 2, and equal to 1.6, suggests that the S&P100 index is not normally distributed: it has a finite-first moment, since $\alpha > 1$,[23] but infinite variance. Some strange peaks are evident in Figure (20) Having a Hurst exponent equal to 0.48 suggests a negative dependence, being $H < \frac{1}{\alpha} = \frac{1}{1.6} = 0.62$.

[22] A normal distributions X is a special case of stable distributions: $X \sim S_2(\sigma, 0, \mu)$, where the index of stability is equal to 2 and the skewness to 0.

[23] A stable distribution $X \sim S_\alpha(\sigma, \beta, \mu)$ has finite first moment if the index of stability is not lower than 1. Otherwise, also, his expected valued is not finite.

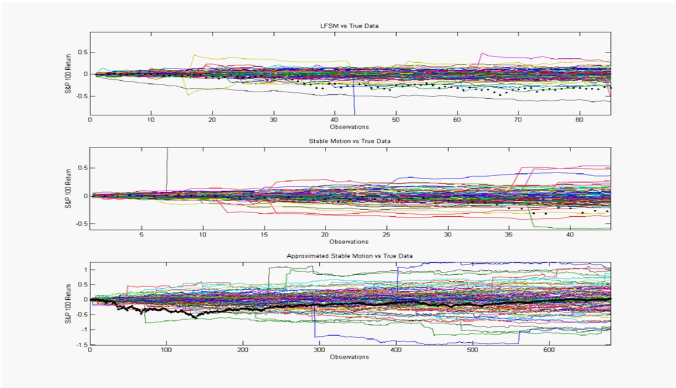

Figure 22: Linear fractional Stable motion (above) vs family of Lévy motions (below)

Once estimated the parameters, three different Monte Carlo simulations can be run on the index behavior and compared with the true data. The following scenarios have been considered:

1. The market knows the underlying process is a linear fractional stable motion; and maintains, regardless of arbitrage opportunities, a process with negative dependence to price contingent claims ("No Approximation" Scenario): its pricing formula is still the expected value, instead of a generalized pricing kernel.

2. The market, instead, does not recognize the underlying process as a linear fractional stable motion; it does consider, instead, a Lévy motion as process driving the index ("Independent increments"

Scenario).

3. The market is aware of some distortions in the prices; but it is conscious as well of possible arbitrage opportunities. Therefore, it decides to pick the best approximating $Lévy$ motion in a family of stable process to approximate the past behavior of the index: this is done by the minimization (3.14) ("Approximation" Scenario).

Figure (22) describes the three different scenarios against the cumulative return of the S&P 100 from August 2008 to April 2011. The solid lines represent the simulated data in the three different scenarios, while the dotted line describes the market data. The upper part describes 100 simulated linear fractional stable motions: the simulated processes do not seem to capture very well the index behavior, especially in the last period. The middle plot, 100 simulated $Lévy$-stable motions, seems to capture much better the index behavior, especially in the first part of the graph. A better scenarios arises in the third scenario, where the market is aware of the negative dependence structure of the index, but decides to clear the distortions in the security. The lower figure provides better result, after an initial period: the middle and the last period are very well captured by the fictitious data.

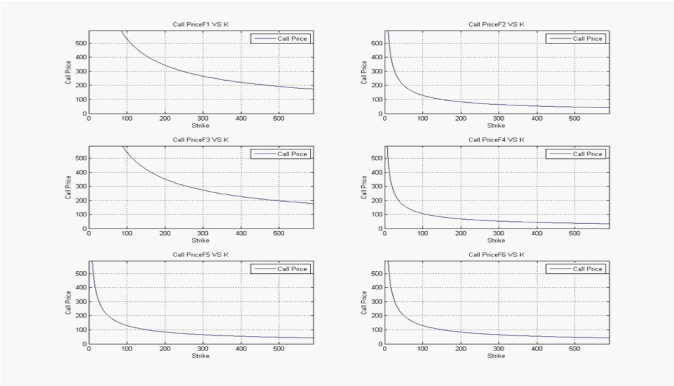

Figure 23: Theoretical Call Prices in the "No Approximation Scenario" for *T*=$\{0.011, 0.0888, 0.1162, 0.1667, 0.2610, 0.338\}$

3.6.1. Option prices

For the three different scenarios the theoretical prices of European call options are derived by applying the Fast Fourier Transform (FFT), as suggested by Carr et al. The technique is able to deliver different values for different maturities and strike prices. Call options on the index have twelve different time to maturities T, from immediate execution, T equal to 0.011 years, to a longer expiration time, T equal to 2.333 years. Given that, the three different situations are divided into six subgroups, differentiating between short and long time-horizon. Strike prices are varying from 0 to 800. Figure (23) describes the different theoretical prices in the "No Approximation Scenario" with respect to different expiration dates,

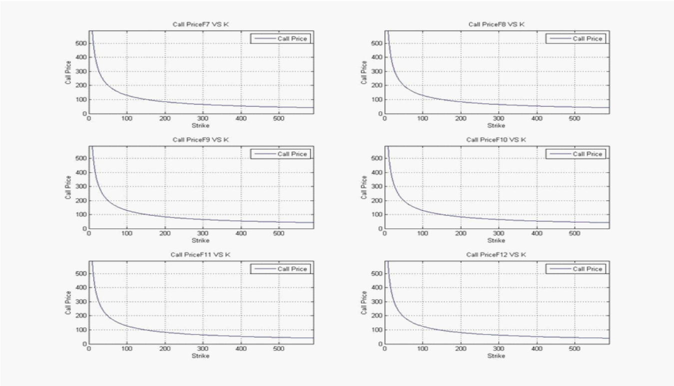

Figure 24: Theoretical Call Prices in the "No Approximation Scenario" for *T*=$\{0.3689, 0.5916, 0.6189, 0.8662, 1.3361, 2.333\}$

traded by the market for short maturities. Negative dependence is detected in this scenario, but the market does not correct it. The value of the call options is clearly negatively correlated to the strike price K: it does decrease when K increases. Even if the time to maturity is really small, the contingent claim reaches very fast its lower boundary, zero, especially when T is bigger or equal to 0.1162 years. An immediate time to maturity, i.e. T equal to 0.0011 or 0.0888 years, produces a slight different behavior of the call value: the two lines, representing it in Figure (23) have a bigger starting point when K is equal to 0. Figure (24) plots theoretical call prices in the same scenario for longer time to maturities. Longer the time to maturity T, the call prices converges fast to its zero

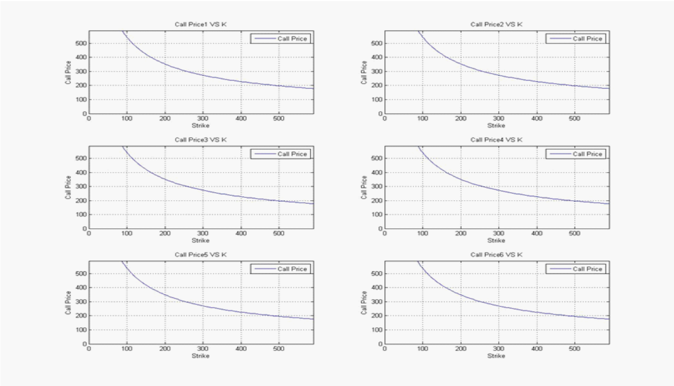

Figure 25: Theoretical Call Prices in the "Independent Increments Scenario" for *T*=$\{0.011, 0.0888, 0.1162, 0.1667, 0.2610, 0.338\}$

boundary, as the strike prices increases. The behavior of the call is quite similar to previous graph, by the way, even with a longer expiration time, i.e. T equal to 2.333 years. Excluding the two immediate times to maturity Ts equal to 0.0011 or 0.0888 years, the call values follow a homogeneous behavior for the other ten cases, as described by the curves, especially when pushing the strike price to the limit. The second scenario is detected in Figure (25) for short time to maturities, and in Figure (26) for longer time horizons. The market does not detect the negative dependence of the index, considering a Lévy-stable motion as innovation term. The stock is a semimartingale, even in the historical probability space, and no correction on the parameter is required, since there is no

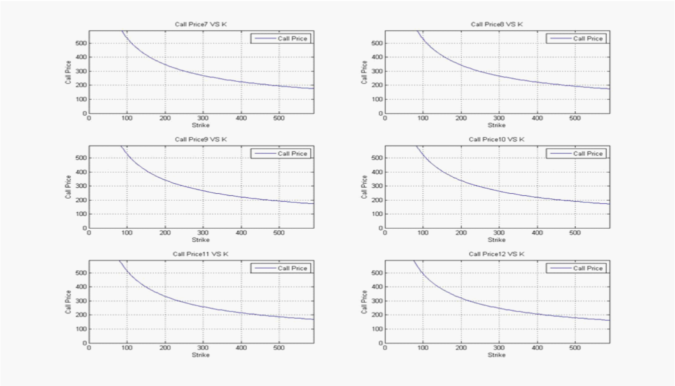

Figure 26: Theoretical Call Prices in the "Independent Increments Scenario" for *T*=$\{0.3689, 0.5916, 0.6189, 0.8662, 1.3361, 2.333\}$

need to consider another probability space, with fictitious stock's values. The threshold between short and long horizon is set to be, as above, T equal 0.338 years. Figure (25) and (26) are quite similar, in the sense that the behavior of the values is quite homogeneous. No significance different values between short and long time horizon is evident from the two plots, even when the time to maturity is immediate, i.e. T equal to 0.0011 or 0.0888 years. The speed at which the call option approaches its lower boundary for the call price, zero value, is lower with respect to the previous scenario. The starting points are different, the convexity level of the graphs are different, and the curves less steeper. Moreover, the behavior of Figure (24) and (26), as strike prices increase, are com-

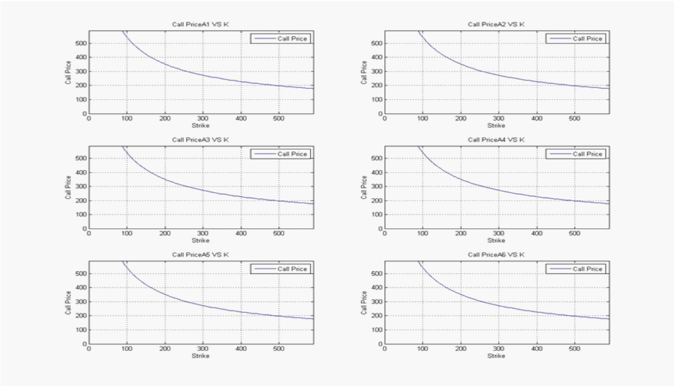

Figure 27: Theoretical Call Prices in the "Approximation Scenario" for *T*=$\{0.011, 0.0888, 0.1162, 0.1667, 0.2610, 0.338\}$

pletely different; by pushing K to the limit, it is immediate to notice that, while the call price in Figure (26) is above 100, the value in Figure (24) is really close to zero. The third scenario is where the market is aware of the negative dependence, but, constrained by the No-Arbitrage assumption, decides to clear all the distortions by approximating it with a stable motion. The parameters of stocks' return, driven a linear fractional stable motion innovative term, are modified in such a way to create a connection with a family of Lévy motion, and derive a No-Arbitrage formula. As before, two subperiods are considered in the simulation of fictitious data. Figure (27) plots theoretical call prices for short-term time horizon. The behavior of the simulated data is quite similar to the second scenario and

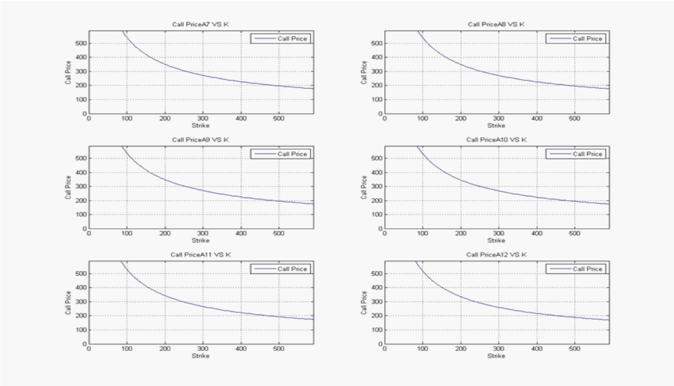

Figure 28: Theoretical Call Prices in the "Approximation Scenario" for *T*=$\{0.3689, 0.5916, 0.6189, 0.8662, 1.3361, 2.333\}$

completely different with the first situation. Convexity is less marked, and all the theoretical values are quite homogeneous, even with an immediate expiration time. Convergence is less fast with respect to the "No approximation" scenario. Figure (28) considers longer horizon. The plot is consistent with what said above. Dissimilarities arise by comparing it with respect to the first scenario. The convergence to the lower boundary is much lower as T increases; while the convexity of the theoretical prices is quite similar to the previous scenario. At the end, it is possible to conclude that the call values follows a homogeneous and similar behavior for the second and third scenario; while the first scenarios provides different and heterogeneous result, when considering immediate

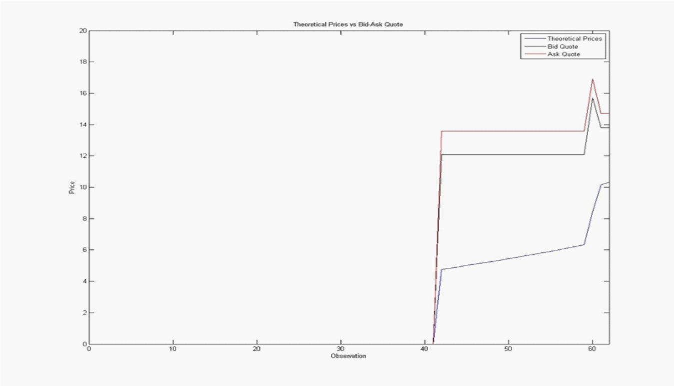

Figure 29: "No Approximation Scenario": Theoretical Prices vs Bid-Ask Quotes

maturity. In order to assess what is the true behavior of the market, it is necessary to compare theoretical data and market quotes. The entire spectrum of strike prices for the theoretical prices consists of 4096 different values; while in the market data 68 different strike prices are given, ranging from 400 to 850. A unique database, containing between fictitious values and market quotes, is created, by using the strike prices as connection point between the two different sets of data; then the graph of theoretical prices against bid-ask quotes is drawn. Data are sorted by time to maturity, and then strike prices. The number of observations and the path in the following figures are significant different, especially the first one, because theoretical prices, whose value is higher or lower

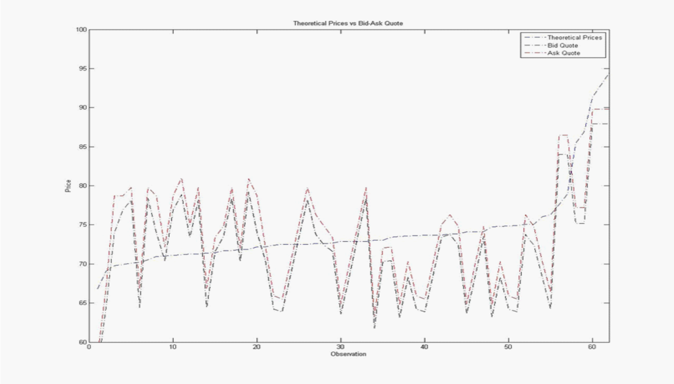

Figure 30: "Independent Increments Scenario": Theoretical Prices vs Bid-Ask Quotes

than the bid-ask quote with respect to a certain threshold, are not to shown graphically. The first case to be considered is the "No Approximation Scenario", i.e. when the underlying process is a linear fractional stable motion; and the market maintains a process with negative dependence to price contingent claims. Figure (29) plots theoretical values against the bid-ask quotes. Not only are prices affected by arbitrage opportunities, but the model is not able to capture the market behavior. The model delivers a relevant number of call prices close to zero, and is not consistent with market prices, even when increasing the threshold to 100: only 20 observations seem to be in line with the market and the distance between them and market quotes is quite evident. The theoretical price is always below the bid-ask quote; and the gap is too marked to be easily explained by the commission cost or other market frictions. The situation does improve in the second scenario, i.e. when the market does not recognize the underlying process as a linear fractional stable motion; it does consider, instead, a Lévy motion as process driving the index. As shown in Figure (30), the number of observations increases and the distance between the path of the theoretical values and the market quotes decreases significantly. The threshold has been reduced to $10,$ and, differently than the previous scenario, "out-of-the market" call prices have been ruled out of the sample. Not only have observations increased in numbers, but also the amount of zero prices is smaller: the difference between bid-ask quote is less marked with respect to the previous scenario. In the "Approximation Scenario", the market is aware of some distortions in the prices; but is conscious as well of possible arbitrage opportunities. In order to overcome the problem, it chooses to pick the best approximating $Lévy$ motion in a family of stable process to approximate the past behavior of the index by the minimizing the distance between characteristic functions. Figure (31) plots the theoretical prices of the approximating models against the bid-offer quotes. The threshold is set to be 10 again. The scenario is quite similar to the "Independent Increments Scenario". Differently than the previous scenario, it seems to follow more the market's quotes for short maturities: the gap between

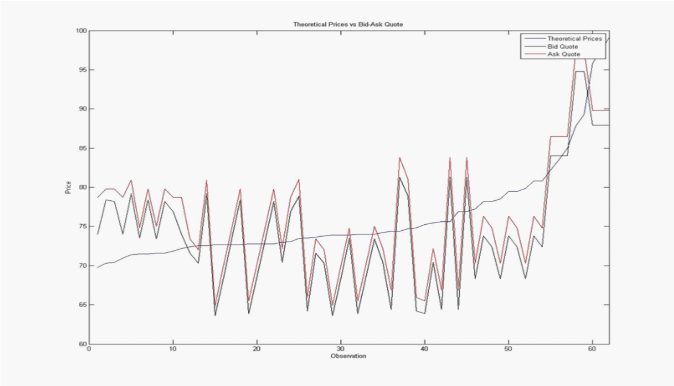

Figure 31: "Approximation Scenario": Theoretical Prices vs Bid-Ask Quotes

theoretical prices and market quotes is smaller than above. In the long-term no significative differences arise. It should be said, by the way, that some differences in the two graphs might be due to the threshold and the way data are sorted. The previous set of figures represent the market situation at August 2008, 18. The analysis is repeated for an entire semester, in order to increase the database. At the end, from a descriptive point of view, the paths, drawn by the second and the third scenario, seem to be consistent. A predictive analysis is needed in order to see and assess what really the market does. The next subsection presents the regression results based on the three models' pricing errors.

3.6.2. Regression analysis

The quality of the correcting models is investigated by comparing the pricing errors, computed as distance between model price and market midquotes, in the different scenarios. Deviations in pricing should not exhibit any consistent patterns and not be predictable. With a view to assessing the estimated models in this way, a regression analysis is performed on the pricing errors obtained from each model. The explanatory variables for the regression summarized the characteristics of the option. The presence of implied volatility smiles suggests that pricing errors are systematically related to the degree of moneyness, measured by the ratio of the spot index level to the option strike. The set of regressors is composed by the degree of moneyness and its square as explanatory variables. Implied volatilities are also know to rise with the option maturity as an explanatory variable. In addition the level of interest rates is included as an additional regressor. Table 3 shows the beta coefficients and their p-values of the ordinary least squares (OLS) estimator for the three different scenarios. The table is completed by the coefficient of determination R^2 and the $F-$statistics. The perfect model should have a low R^2, with lack of predictability, i.e. not significant $F - stat$. The "No Approximation Scenario" is characterized by having a really high R^2. Moreover the beta coefficient are significantly different than zero, show-

Table 3: OLS results on the predictability of the S&P100 pricing errors.

Regressors	"No Approximation Scenario"	Independent Increments Scenario"	"Approximation Scenario"
Const	166.55 (0.00)	-93.84 (0.131)	-89.24 (0.1469)
Moneyness	-6.91 (0.00)	0.2959 (0.91)	0.3991 (0.891)
Moneyness^2	-368.29 (0.00)	171.68 (0.18)	163.43 (0.1981)
TimetoMaturity	183.09 (0.00)	-75.051 (0.2540)	-71.23 (0.23)
InterestRate	4.534 (0.00)	0.7143 (0.7818)	0.6461 (0.81)
R^2	0.6294	0.0103	0.0084
F-stat	149.13 (0.000)	1.90 (0.1085)	1.74 (0.141)

ing that there exists a consistent pattern and predictability in the errors. As suspected in the descriptive analysis of the previous section, it is not consistent with what the market does. The second situation, "Independent Increments Scenario" delivers more appropriate results. The beta coefficients are not significant and the coefficient of determination of the linear regression R^2 is only 0.0103. Better results arise in the "Approximation Scenario", where the pricing errors do not exhibit any consistent patters, and the coefficient of determination is even lower than 0.01. The results suggest that the market is aware of the negative dependence in the index, and, driven by the no-arbitrage constraints, decide to correct this bias by selecting a fictitious process to price contingent claims.

3.7. Conclusions and future extensions

This Chapter investigates possible departures from rationality in the field of continuous option pricing. The stock's returns might deviate from the classical finance idea of independent increments, mostly because of the existence of some distortions, f.i. cognitive biases. The structure of its dynamics, under its historical probability, might be different, and the innovation term in the stochastic differential equation could not necessarily belong to the class of Lévy processes: normality might not even hold, as suggested in Figure (16). A stochastic differential equation, whose random error is stable distributed, is selected to model securities; and, by considering a linear fractional stable motion as error term, a particular dependence structure is imposed. Prices are not necessarily martingales; and arbitrage opportunities might arise in the setting. At this point, a dual solution to the problem is possible: either by means of random functions, the process is transformed into a "fundamental semimartingale"; or the market selects from a family of Markovian processes the best fit to the stock dynamics. The correction is done to clear biases in the market and deliver no-arbitrage contingent claims prices. By minimizing the distance between the characteristic function of the stochastic processes, the connection between the linear fractional stable motion and the Lévy motion is established. By imposing specific conditions on the drift coefficient of the stochastic differential equation, a change of measure into the risk-neutral world is performed; and the Fourier analysis is employed to deliver theoretical option prices for European contingent claims. Data on the S&P 100 Index option from the last decade are used for the empirical analysis. As suggested by the Hurst parameters, the index has negative dependence and three different scenarios for the true dynamics of the index: "No Approximation scenario" where the negative dependence is considered; "Independent increments"; "Approximation scenario" where the negative dependence is corrected. A descriptive analysis seems to rule out the "negative dependence" scenario; while a predictive analysis based on the pricing errors picks the "corrected" negative dependence as the best model in the stable family to deliver accurate option prices. This idea of market-correction should be further investigated because it can represent a good starting point to develop an alternative explanation to the implied volatility bias of options; moreover, it is possible to refine it either by extending it to more complex contingent claims, where the correction might be taken not only modifying the volatility; or by applying generalized pricing formulas instead of using expected values as prices.

References

[1] Alparslan, U. T., and G. Samorodnitsky, "Ruin probability with certain stationary stable claims generated by conservative flows", *Advances in Applied Probability*, 39, 2 (2007), 360-384.

[2] Ayache, A., and J. L. Vehel, "On the identification of the pointwise Holder exponent of the generalized multifractional Brownian motion", *Stochastic Processes and their Applications,* 111, 1 (2004) 119-156.

[3] Barndorf-Nielsen, O. E., "Processes of Normal Inverse Gaussian Type", *Finance and Stochastics*, 2 (1997), 41-68.

[4] Bates, D. S., "Post Crash Moneyness Bias in S&P 500 Futures Options", *Rodney L. White Center working paper* (1991), Wharton School, University of Pennsylvania, Philadelphia, PA.

[5] Black, F., and M. S. Scholes, "The pricing of Options and Corporate Liabilities", *Journal of Political Economy*, 81 (1973), 637-654.

[6] Cambanis, S., and M. Maejima, "Two classes of self-similar stable processes with stationary increments", *Stochastic Processes and their Application*, 32 (1989), 305–329

[7] Carr, P., and D. Madan, "Option valuation using the fast Fourier transform", *Journal of Computational Finance,* 2 (1999), 61–73.

[8] Carr, P., Geman, H., Madan, D., and M. Yor, "The fine structure of asset returns: an empirical investigation", *Journal of Business*, 75 (2002), 305–332.

[9] Carr, P., and L. Wu, "Finite moment log stable process and option pricing", *Journal of Finance*, 58 (2003) , 753–777.

[10] Carr, P., and L. Wu,"Time-changed Lévy processes and option pricing", *Journal of Financial Economics*, 71 (2004), 113–141.

[11] Carr, P., and L. Wu, "Stochastic skew in currency options", *Journal of Financial Economics*, 86, 1, (2007), 213-247.

[12] Cochrane, J., *"Asset Pricing",* Princeton University Press, Princeton, NJ, (2004).

[13] Cochrane, J., "Production-based asset pricing and the link between stock returns and economic fluctuations", *Journal of Finance*, 46 (1991), 209–237.

[14] Cont, R., "*Long range dependence in financial time series"*, Fractals in Engineering, Springer, (2005).

[15] Duffie, D., Pan, J., and K. Singleton, "Transform analysis and asset pricing for affine jump-diffusions", *Econometrica*, 68 (2000), 1343–1376.

[16] Eberlein, E., and U. Keller, "Hyperbolic distributions in Finance", *Bernoulli*, 1 (1995), 281-299.

[17] Eberlein, E., and U. Keller, and K. Prause "New insights into smile, mispricing and value at risk: the hyperbolic model", *Journal of Business*, 71 (1998), 371-405.

[18] Eberlein, E., and K. Prause, "The generalized hyperbolic model: Financial derivatives and risk measures", *FDM preprint*, 56, (1998) University of Freiburg.

[19] Heston, S., "A Closed-Form Solution of Options with Stochastic Volatility with Applications to Bond and Currency Options", *The Review of Financial Studies*, 6 (1993), 327-343.

[20] Kahneman, D., and A. Tversky, "Prospect Theory: An Analysis of Decision Under Risk", *Econometrica*, 47 (1979), 263-291.

[21] Kahneman, D., Slovic, P., and A. Tversky, *"Judgment Under Uncertainty: Heuristics and Biases",* New York: Cambridge University Press, (1982).

[22] Kahneman, D., and A. Tversky, "Choices, Values and Frames", *American Psychologist*, 39 (1984), 341-350.

[23] Madan, D. B., and F. Milne, "Option Pricing with VG Martingale Components", *Mathematical Finance*, 1 (1991), 39-55.

[24] Madan, D. B., and E. Seneta, "The Variance Gamma (V.G.) Model for Share Market Returns", *Journal of Business*, 63 (1990), 4, 511-524.

[25] McCulloch, J. H., "The Risk-Neutral Measure and Option Pricing under Log-Stable distribution", *Technical Report* (2003), Department of Economics, Ohio State University.

[26] Rosinski, J., "On uniqueness of the spectral representation of stable processes", *Journal of Theoretical Probabilities*, 7(3) (1994) 615-634.

[27] Rosinski, J., "On the structure of stationary stable processes", *The Annals of Probability*, 23, (1995) 1163-1187.

[28] Samorodnitsky, G., and M. S. Taqqu, *"Stable Non-Gaussian Random Process: Stochastic Models with Infinite Variance",* Chapman and Hall, New York, (1994).

[29] Sato, K., *"Lévy Processes and Infinitely Divisible Distributions",* Cambridge University Press, (1999).

Chapter 4

Generalization of the jumps: additive processes

4.1. Introduction

The peculiar properties of financial transaction data, such as the irregular spacing in time, the discreteness of price changes, the bid-ask bounce as well as the presence of serial dependence, induce the surge of new approaches. One important string of the literature deals with the irregular spacing of data in time. It is indispensable whenever the full amount in financial transaction data has to exploited and no loss of information due to fixed-interval aggregation schemes can be accepted. Moreover, it has been realized that the timing of trading events, such as the arrival of particular orders and trades, and the frequency, in which the latter occur, have information value for the state of the market; and play an important role in market microstructure analysis, for the modelling of intraday volatility as well as the measurement of liquidity and implied liquidity risks. The irregular occurrence of transaction data requires to consider it as a point process, a so-called financial point process. Depending on the type of financial "event" under consideration, it is possible to distinguish between different types of financial point processes or processes of so-called financial durations. The most common types are trade durations and quote durations as defined by the time between two consecutive trades or quote arrivals, respectively. Price durations correspond to the time between absolute cumulative price changes of given size and can be used as an alternative volatility measure. Similarly, a volume duration is defined as the time until a cumulative order volume of given size is traded and captures an important dimension of market liquidity. One important property of transaction data is that market events are clustered over time, implying that financial durations follow positively autocorrelated processes with strong persistence. Actually, it turns out that the dynamic properties of financial durations are quite similar to those of daily volatilities. All these properties cannot be captured by continuous diffusion process. Modelling them through compound Poisson processes might fail in exploiting the full amount of information embedded in the stock's movement. An additive process, defined as an integral form with respect to compensated Poisson random measure, is then considered in the stochastic differential equation, with the specific intent to describe the jump's activity. Section 4.2. defines the class of additive and its integral representation through compensated Poisson random measures. The class of additive process, who might be represented as integrals with respect to the compensated Poisson random measures, is selected to model the points of discontinuity. The group of self-exciting processes is then described by means of its conditional intensity function, that is time-dependent, differently than Poisson process. Section 4.3. gives some technical conditions for computing the differential of a time-dependent function of an additive process; while section 4.4. provided a closed-form expression for the stock, under the condition that its points of discontinuity are described by an additive process. Hawkes process are used to model a specific scenario where the intensity of jumps is similar to the earthquakes activity, since its conditional intensity function increases over time. Simulations of the stock, under the hypothesis that kernel function of the Hawkes process decays exponentially, are performed in Section 4.5. Section 4.6. concludes and suggests some possible extensions.

4.2. Additive processes

Assume a filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq +\infty}\right)$, such that

1. $\mathcal{F}_t$ contains all null set of $\mathcal{F}$, for all t such that $0 \leq t < +\infty$
2. $\mathcal{F}_t = \mathcal{F}_t^+$, where $\mathcal{F}_t^+ = \cap_{u>t}\mathcal{F}_u$, for all t such that $0 \leq t < +\infty$,i.e. the filtration is right continuous.

An additive process on $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq+\infty}, P\right)$ with values in $(E, B(E))$ can be defined as

Definition 22 *A process* $(X_t)_{t\geq 0}$ *with state space* $(E, B(E))$ *is an*

$\mathcal{F}_t-$additive process on $(\Omega, \mathcal{F}, P)$ if:

1. $X_{t\geq 0}$ is adapted to $(\mathcal{F}_t)_{0\leq t\leq+\infty}$
2. $X_0 = 0$ a.s. ;
3. $(X_t)_{t\geq 0}$ has increments independent of the past, $i.e.$ is independent of $X_t - X_s$ is independent of $\mathcal{F}_s$;
4. if $0 \leq s < t; (X_t)_{t\geq 0}$ is stochastically continuous;
5. $(X_t)_{t\geq 0}$ is càdlàg.

The intent of this section is to present and construct compensated Poisson random measure; and define additive processes as integral forms with respect to them.[24] The class of self-exciting processes would be then characterized, and, inside this group, the Hawkes processes selected.

[24] See Rudinger (2004) for more details.

4.2.1 Compensated Poisson random measure

Let $A \in B(E), 0 \in \left(\overline{A}\right)^c$, and define

$$N_t^A := \sum_{0<s\leq t} 1_A(\Delta X_s) = \sum_{n\geq 1} 1_{t\geq T_n^A}$$

where

$$T_1^A := \inf\{s > 0 : \Delta X_s \in A\}$$

$$T_{n+1}^A := \inf\left\{s > T_n^A : \Delta X_s \in A\right\}, \quad n \in \mathbb{N}$$

N_t^A is an adapted counting process without explosions and, moreover, is a Poisson random measure:

1. $P\left(N_t^A = k\right) = \exp(-v_t(A)) \frac{(v_t(A))^k}{k!}$ where $v_t(A) := E\left[N_t^A\right]$;
2. if $A_1, A_2, ...A_k \in B(E)$ with $v(A_i) < \infty$ are disjoint, then $N([0,t] \times A_i)$ are independent.

An important predictable measure associated to the Poisson random measure N_t^A is its compensator[25]

$$v_t(A) := E\left[N_t^A(\omega)\right]$$

[25] The compensator encodes the stochastic structure of a point process: it describes the dynamics of event arrivals relative to the available information.

Theorem 23 *Let $B(E\backslash\{0\})$ be the trace $\sigma-$algebra on $E\backslash\{0\}$ of the Borel $\sigma-$algebra $B(E)$ on E, and let*

$$\mathcal{F}(E\backslash\{0\}) := \left\{A \in B(E\backslash\{0\}) : 0 \in \left(\overline{A}\right)^c\right\}$$

then $\mathcal{F}(E\backslash\{0\})$ is a ring and for all $\omega \in \Omega$ the set function

$$N_t := N_t(\omega, \cdot) : \mathcal{F}(E\backslash\{0\}) \to \mathbb{R}_+ \;\; A \to N_t^A(\omega)$$

is a $\sigma-$finite pre-measure. The set function $v_t(A) := E\left[N_t^A(\omega)\right] \in \mathbb{R}$, $A \in \mathcal{F}(E\backslash\{0\}), \omega \in \Omega$ satisfies:

$$v_t : \mathcal{F}(E\backslash\{0\}) \to \mathbb{R}_+ \;\; A \to E\left[N_t^A(\omega)\right]$$

and is a $\sigma-$finite pre-measure on $((E\backslash\{0\}), \mathcal{F}(E\backslash\{0\}))$.[26]

Proof. See Albeverio, S. and B. Rudinger, 2004. ■

Once the Poisson random measure and the compensator are defined, it is possible to characterize the compensated Poisson random measure. Define $S(\mathbb{R}_+)$ as the semi-rings of sets $(t_1, t_2], 0 \leq t_1 < t_2$;

[26] Both measure are unique on $B(E\backslash\{0\})$ and $v_t(A)$ is a Lévy measure. A $\sigma-$finite positive measure v on $((E\backslash\{0\}), B(E\backslash\{0\}))$ is a Lévy measure if there is a probability measure μ on $(E, B(E))$ such that the Fourier transform $\widehat{\mu}(F), F \in E'$ satisfies

$$\widehat{\mu}(F) = \exp\int_{E\backslash\{0\}} \exp\left(iF(x) - 1 - iF(x)1_{\|x\|\leq 1}\right) v(dx).$$

and $S(\mathbb{R}_+) \times B(E\backslash\{0\})$ as the semi-ring of the product sets $(t_1, t_2] \times A, A \in B(E\backslash\{0\})$. Let $N((t_1, t_2] \times A)(\omega) = N_{t_2}(A)(\omega) - N_{t_1}(A)(\omega)$ $\forall A \in B(E\backslash\{0\})$ be the adapted counting process, $\forall\omega \in \Omega$. For all $\omega \in \Omega$ fixed $N(dt, dx)(\omega)$ is a $\sigma-$finite pre-measure on the product semi-ring $S(\mathbb{R}_+) \times B(E\backslash\{0\})$: define with $N(dt, dx)(\omega)$ the measure which is the unique extension of the pre-measure to the $\sigma-$finite algebra $B(\mathbb{R}_+) \times B(E\backslash\{0\})$ generated by $S(\mathbb{R}_+) \times B(E\backslash\{0\})$. Let the compensator $v_t(A)$ to be characterized as $v((t_1, t_2] \times A) = v_{t_2}(A) - v_{t_1}(A)$ $\forall A \in B(E\backslash\{0\})$, $v(dt, dx)$ is a $\sigma-$finite pre-measure on the product semi-ring $S(\mathbb{R}_+) \times B(E\backslash\{0\})$: denote by $v(dt, dx)$ the $\sigma-$finite measure which is the unique extension of this pre-measure on $B(\mathbb{R}_+) \times B(E\backslash\{0\})$. The unique extension of $N(dt, dx)(\omega)$ and $v(dt, dx)$ gives the opportunity to characterized the compensated Poisson random measure in the following way:

Definition 24 *Call $N(dt, dx)(\omega)$ the Poisson random measure associated to the additive process $(X_t)_{t\geq 0}$ and $v(dt, dx)$ as its compensator. Define*

$$q(dt, dx) = N(dt, dx)(\omega) - v(dt, dx)$$

as the compensated Poisson random measure associated to the additive process $(X_t)_{t\geq 0}$.

4.2.2. Self-exciting processes

A natural class of point processes, for which the stochastic intensities are increasing functions of the past evolution of the processes, is considered; that is, for such processes larger (in some sense) realizations before t imply that the corresponding intensities of having a point just after t are also larger. Let.$\aleph = \aleph(E)$ be the space of Radon (i.e. locally finite) measures on a locally compact second countable Hausdorff space E. Let $B = B(E)$ be the class of Borel sets generated by the topology of E, and let $F_C = F_C(E)$ be the class of nonnegative continuous functions $E \to \mathbb{R}$ with compact supports. The space $\aleph$ can be endowed with the vague topology, for which the class of all finite intersections of sets of the form $\{\mu \in \aleph : s < \int_E f d\mu < t\}$ for $s, t \in \mathbb{R}$ and $f \in F_C$, may serve as a base. The space $\aleph$ with the vague topology is metrizable as a Polish space. Call any .$\aleph$-valued random element M a random measure on E. The distributions of vectors $(M(B_1), ..., M(B_n))$, $n > 1$, for arbitrary bounded sets $B_1, ..., B_n \in B$, entirely determine the distribution of a random measure M. Define a point process N on the space E as a random measure confined with probability 1 to the subset $N = N(E)$ of $\aleph$ consisting of all integer Radon measures on the space E. Assume $E = \mathbb{R}_+$. Denote the following partial order $\prec_\aleph$ in $\aleph(E)$. For $\mu, r \in \aleph$ let $\mu \prec_\aleph r$ if and only if $\mu(B) \leq r(B)$ for all (topologically) bounded sets

$B \in B(E)$. The ordering $\prec_{\aleph}$ordering restricted to N, often called thinning, is be denoted by $\mu \prec_N r$ for $\mu, r \in N$. Denote $D(\mathbb{R}_+)$ as the space of real-valued functions on $\mathbb{R}_+$, which are right-continuous with left-hand limits. The natural ordering in $D(\mathbb{R}_+)$ is $(\mu_t) \prec_D (r_t)$ iff $\mu_t < r_t \ \forall t > 0$. Consider, however, a point process as a random element of $N(\mathbb{R}_+)$ and for $\forall \ \mu, r \in N(\mathbb{R}_+)$ define $\mu \prec_D r$ iff $(\mu_t) \prec_D (r_t)$ for the corresponding functions $\mu_t, r_t \in D(\mathbb{R}_+)$. Define $\mathcal{F}$s, as the history of $N(s)$ at time t.

Definition 25 *Let N be a simple point process with a compensator ν and a stochastic intensity λ, where $\lambda(t) = \lim\limits_{\Delta t \to 0} \frac{E[dN(t)|\mathcal{F}s]}{\Delta t}$*

1. *N is positively self-exciting w.r.t. $\prec_N$ if, $\forall \ \mu, r \in N(\mathbb{R}_+), n \geq 1$*

$$\mu \prec_N r \Longrightarrow \left(\forall_{t \in \mathbb{R}_+} \lambda(t, \mu) \leq \lambda(t, r)\right)$$

2. *N is positively self-exciting w.r.t. $\prec_D$ if, $\forall \ \mu, r \in N(\mathbb{R}_+), n \geq 1$*

$$\mu \prec_D r \Longrightarrow \left(\forall_{t \in \mathbb{R}_+} v(t, \mu) \leq v(t, r)\right)$$

or equivalently.

4.2.2.1. Hawkes processes

Kwiecinski and Szekli (1996) have shown that positively self-exciting processes are "increasing" transformations of Poisson processes, where monotonicity is related to the underlying ordering of point processes. Quite used in seismology to model the occurrence of earthquakes are the Hawkes process, belonging to the class of self-exciting processes, since their intensity is driven by a weighted function of the time distance to previous point of the process. A general class of univariate Hawkes processes is given by the following intensity function

$$\begin{aligned} \lambda(t) &= \lim_{\Delta t \to 0} \frac{E\left[dN(t)|\mathcal{F}s\right]}{\Delta t} \\ &= \varphi\left(\mu(t) + \sum_{t_i < t} w(t - t_i)\right) \end{aligned}$$

where φ denotes a possibly nonlinear function, $\mu(t)$ denotes a deterministic function of time and $w(s)$ is a weight function. Its most basic form, in the one dimensional case, is characterized by being a linear function and

$$\lambda(t) = \mu + \sum_{t_i < t} w(t - t_i) = \mu + \int_{-\infty}^{t} \phi_{t-s} dN_s \tag{4.1}$$

where $\mu > 0$ is a constant background rate, N_s is the cumulative counting process and ϕ is a positive real function called decay kernel. Hawkes processes can be seen as clusters of Poisson processes, where each

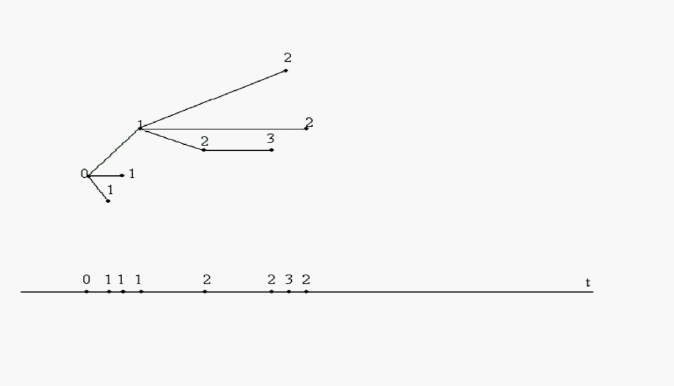

Figure 32: The branching structure of the various generations of events in a cluster (top), and the events on the time axis (bottom)

event is one of two types: an immigrant process or an offspring process. Its clustering size is

$$c = \frac{1}{1 - \int_0^\infty g(t)dt}$$

The immigrants follow a Poisson process and define the centers of so-called Poisson clusters. Conditioning on the arrival time, say t_i, of an immigrant, then, independently of the previous history, t_i is the center of a Poisson process, $\Psi(t_i)$, of offspring on (t_i, ∞) with intensity function $\lambda(t_i) = \lambda(t - t_i)$ where λ is a non-negative function. The process $\Psi(t_i)$ defines the first generation offspring process with respect to t_i. Furthermore, by conditioning on the process $\Psi(t_i)$, then each of the events in $\Psi(t_i)$, say t_i, generates a Poisson process with intensity

$\lambda(t_i) = \lambda(t - t_i)$. These independent processes build the second generation of offspring with respect to t_i. Similarly, further generations arise. The set of all offspring points arising from one immigrant are called a Poisson cluster. The immigrants and offsprings can be referred to as "main shocks" and "after shocks" respectively. This admits an interesting interpretation which is useful not only in seismology but also in high-frequency finance: the cluster structure implied by the self-exiciting nature of Hawkes processes seem to be a reasonable description of the timing structure of events on financial markets. Figure (32) describes the branching structure of the process. The main difference with respect to a simple Poisson process are summarized by the following set of figures. Figure (33) compares the different arrival times of a Hawkes process with respect to a homogeneous Poisson process. As you can observe, the number arrivals is more uniform for a Poisson process; while it is clustered for a linear Hawkes process, due to dual effect of main shocks and after shocks. Figure (34) describes the different behavior of the conditional intensities for the two processes. The intensity in the homogeneous Poisson process is a flat line, while it has some spikes and peaks for the linear Hawkes process, due to the integral term in the right-hand side of (4.1). The different behavior is evident by looking at the particular form of the conditional intensity (4.1). The conditional intensity of a

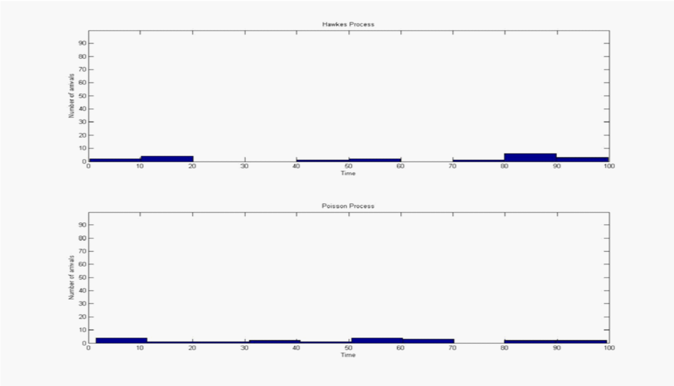

Figure 33: Number of arrivals in time for the point processes: linear Hawkes processes (top), Poisson process (down)

linear Hawkes process can be decomposed into two parts

$$\begin{aligned} \lambda(t) &= \mu + \sum_{t_i < t} w(t - t_i) \\ &= \lambda_c(t) + \lambda_a(t) \end{aligned}$$

where

$$\lambda_c = \mu$$

is the conditional intensity of the primary events, Poisson distributed with rate μ; while λ_a is the conditional intensity of an aftershock. Before closing this section, it is important to recall the Doob-Meyer decomposition. In martingale-based point process theory, an $\mathcal{F}_t-$adapted point process

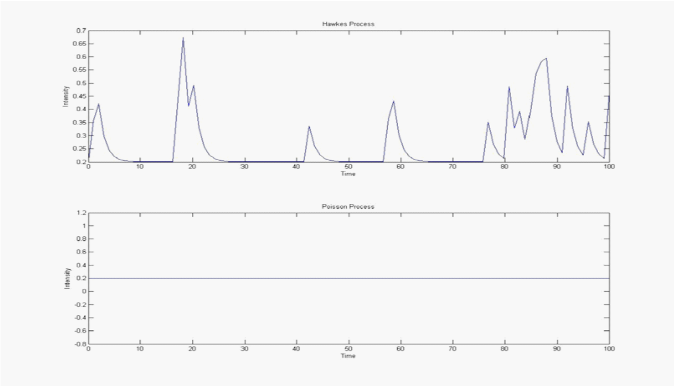

Figure 34: Conditional intensity for the occurrence times of the point processes: linear Hawkes processes (top); Poisson processes (down)

$N(t)$ is a submartingale, it can be decomposed into a zero mean martingale $M(t)$ and a (unique) $\mathcal{F}_t-$predictable increasing process, $v(t)$,which is called the compensator of $N(t)$ and can be interpreted as the local conditional mean of $N(t)$ given the past. Define $\lambda(t)$ as a scalar, positive, $\mathcal{F}_t-$predictable process, i.e. is adapted, and left-continuous with right hand limits. Then, $\lambda(t)$ is called the ($\mathcal{F}_t-$conditional) intensity of $N(t)$ if

$$v(t) = \int_0^t \lambda(u)\, du \tag{4.2}$$

where $v(t)$ is the (unique) compensator of $N(t)$. For a linear Hawkes (4.2) becomes

$$v(t)=\int_0^t\left(\mu+\int_{-\infty}^{s}\phi_{t-s}dN_s\right)du$$

4.3. Stochastic calculus for additive processes

Let F be a separable Banach space with norm $\|\cdot\|_F$. Let $F_t := B(\mathbb{R}_+)\times B(E\backslash\{0\})\otimes\mathcal{F}_t$ be the product $\sigma-$algebra generated by the semi-ring $B(\mathbb{R}_+)\times B(E\backslash\{0\})\times\mathcal{F}_t$ of the product sets $A\times F$, $A\in B(\mathbb{R}_+)\times B(E\backslash\{0\})\,,\ F\in\mathcal{F}_t$ where $\mathcal{F}_t=\sigma(X_s, s\le t)$ .Let $T>0$, and define the class:

$$M^T(E/F) := \left\{\begin{array}{c} f:\mathbb{R}_+\times(E\backslash\{0\})\times\Omega\to F,\ f \text{ is } F_T/B(F) \text{ measurable} \\ \text{and } f(t,x,\omega) \text{ is } \mathcal{F}_t-\text{adapted } \forall x\in E\backslash\{0\}\,,\ \forall t\in(0,T] \end{array}\right\}$$

The section defines the stochastic integration of random functions $f(t,x,\omega)\in M^T(E/F)$ with respect to the compensated Poisson random measures $q(dt,dx)\times(\omega) = N(dt,dx)(\omega)-v(dt,dx)$ associated to an additive process $(X_t)_{t\geq 0}$.

4.3.1. Stochastic integral

In order to define an integral with respect to a compensated Poisson point process, it should be provided, first, the definition of Bochner integral, applied to any generic $\sigma-$finite measure v on $B(\mathbb{R}_+ \times E\backslash\{0\})$ of a function $f:\mathbb{R}_+ \times E \to F$ which is $B(\mathbb{R}_+ \times E\backslash\{0\})/B(F)$ measurable. Let $f:\mathbb{R}_+ \times (E\backslash\{0\}) \to F$ be $B(\mathbb{R}_+ \times E\backslash\{0\})/B(F)$ measurable and finite valued, i.e.

$$f(t,x) := \sum_{k=1}^{N} a_k 1_{A_k}(t,x) \quad a_k \in F, \ \ A_k \in B(\mathbb{R}_+ \times E\backslash\{0\})$$

Then $\forall\ A \in B(\mathbb{R}_+ \times E\backslash\{0\})$ the Bochner integral of f w.r.t. v on A is

$$\int_A f(t,x)\, v(dt,dx) = \sum_{k=1}^{N} a_k v(A_k \cap A)$$

The following condition is sufficient for Bochner-integrability:

Definition 26 *Let $f:\mathbb{R}_+ \times (E\backslash\{0\}) \to F$ be $B(\mathbb{R}_+ \times E\backslash\{0\})/B(F)$ measurable; f is said to be Bochner-integral w.r.t. v, if there exists a sequence of finite valued functions $\{f_n\}_{n\in N}$ such that $f_n \to f \ \ v-a.s.$ and*

$$\lim_{n\to\infty} \int \|f_n - f\|\, dv = 0$$

the Bochner-integral of f on any $A \in B(\mathbb{R}_+ \times E\backslash\{0\})$ is

$$\int_A f(t,x)\, v(dt,dx) = \lim_{n\to\infty} \int_A f_n(t,x)\, v(dt,dx)$$

and is independent of the sequence $f_n \to f$ as long as the previous limit *holds.*

A natural integral with respect to a compensated Poisson process is defined by:

Definition 27 *Let* $t \in (0,T]$, $\Lambda \in \mathcal{F}(E\backslash\{0\})$. *Assume that* $f(\cdot,\cdot,\omega)$ *is Bochner integrable on* $(0,T]\times\Lambda$ *w.r.t.* λ, *for all* $\omega\in\Omega$. *The natural integral of* f *on* $(0,T]\times\Lambda$ $w.r.t.$ *the compensated Poisson random measure* $q(dt,dx)$ *is,* $\forall\omega\in\Omega$

$$\begin{aligned}&\int_0^t\int_\Lambda f(s,x,\omega)\left(N(ds,dx)(\omega)-v(ds,dx)\right)\\ :&=\sum_{0<s\leq t} f(s,(\Delta X_s)(\omega),\omega)\,1_A(\Delta X_s(\omega))-\int_0^t\int_\Lambda f(s,x,\omega)\,v(ds,dx)\end{aligned}$$

The stochastic integral could be interpreted as follows: if N assigns mass one to the point (t,x) then the additive process X_t jumps at this time t and the size of the jumps is $f(t,x)$. It is more difficult to define stochastic integral on those sets $(0,t]\times A$, $A\in B(E\backslash\{0\})$ such that $v((0,t]\times A)=\infty$. Therefore for this particular case additional condition for the stochastic integral must be provided.

Definition 28 *Let* $p\geq 1, t>0$. f *is strong* p*-integrable on* $(0,t]\times A$, $A\in B(E\backslash\{0\})$, *if there is a sequence* $\{f_n\}_{n\in N}$ *belonging to the set* $\Sigma(E/F)$ *of simple function s.t.* f_n *is* L^p-*approximating* f *on* $(0,t]\times A\times\Omega$

$w.r.t.\ \lambda \otimes P$, and for any such sequence the limit of the natural integrals of f_n $w.r.t.\ q(dt,dx)$ exists in $\in L_p^F(\Omega,\mathcal{F},P)$ for $n \to \infty$, i.e.

$$\int_0^t \int_A f(s,x,\omega)\, q(ds,dx)(\omega) := \lim_{n\to\infty}^{p} \int_0^t \int_A f_n(s,x,\omega)\, q(ds,dx)(\omega)$$

i.e.

$$\lim_{n\to\infty} \int_0^t \int_A E\left[\left\|f_n(s,x,\omega) - f(s,x,\omega)\right\|^p\right] dv = 0$$

The previous limit is the strong p-integral of f w.r.t. $q(dt,dx)$ on $(0,t]\times A$.

Define

$$M_v^{T,2}(E/F) := \left\{ f \in M^T(E/F) : \int_0^T \int_A E\left[\|f(s,x)\|^2\right] v(dt,dx) < \infty \right\}$$

Theorem 29 *Suppose that F is a separable Banach space of type 2.*[27] *Let $f \in M_v^{T,2}(E/F)$, and f be a deterministic function, i.e. $f(t,x,\omega) = f(t,x)$, then f is strongly 2-integrable w.r.t.$q(dt,dx)$ on $(0,t]\times A$ for any $0 < t \le T$, $A \in B(E\backslash\{0\})$. Moreover*

$$E\left\|\int_0^t \int_A f(s,x) q(dt,dx)\right\|^2 \le 4K_2 \int_0^t \int_A E\left\|f(s,x)\right\|^2 v(ds,dx)$$

[27] A separable Banach space F is of type 2, if there is a constant K_2, such that if $\{X_i\}_{i=1}^n$ is any finite set of centered independent $F-$valued random variables, such that $E\|X_i\|^2 < \infty$, then

$$E\left\|\sum_{i=1}^n X_i\right\|^2 \le K_2 \sum_{i=1}^n E\|X_i\|^2 .$$

holds with K_2 being defined in the definition of type 2-Banach spaces.

Proof. See Rudinger, B., 2004, page 231. ■

4.3.2. Stochastic differential equation: Ito's lemma

Consider a stochastic differential equation that has a jump component

$$dX_t = b(X_t)\,dt + \sigma(X_t)\,dB_t + + \int f(X_{t-}, z)\,dq \tag{4.3}$$

i.e.

$$X_t = x_0 + \int_0^t b(X_s)\,ds + \int_0^t \sigma(X_s)\,dB_s + + \int_0^t \int f(X_{s-}, z)\,dq(s,z) \tag{4.4}$$

with B_t being a standard Brownian motion and $X_0 = x_0$. If the non-random measure is defined as $v([0,t] \times A_i) = t\lambda(A)$ where λ is the Lebesgue measure, N would become the classical Poisson process, and if $\lambda(A)$, $q(dt, dx) = N(dt, dx)(\omega) - t\lambda(dx)$ is the same as a Poisson process minus its mean, hence is a (square integrable) martingale. Bass (1998) defines the conditions for a unique pathwise solution of the previous stochastic differential equation. If σ and b are bounded and Lipschitz, and the two following conditions hold

$$\int \sup_x |f(x,s)|^2 \lambda(dz) < \infty$$

$$\int \sup_x |f(x,z) - f(y,z)|^2 \lambda(dz) \le c_1 |x-y|^2$$

for all x, y, then the standard Picard iteration procedure works to prove there exists a solution to the previous SDE and that solution is pathwise unique. By setting $b = 0$ and $\sigma = 0$, X_t is be a pure-jump process, i.e. a stochastic integral defined with respect to a compensated Poisson process. Expression (4.4) is a semimartingale. The following theorem defines Ito's formula for this particular class of processes.

Theorem 30 *Suppose X is a semimartingale and $f \in C^2$ is function. Then $f(X_t)$ is also a semimartingale and*

$$f(X_t) = f(X_0) + \int_0^t f'(X_{s-})dX_s + \sum_{s \le t} [f(X_s) - f(X_{s-}) - f'(X_{s-})\Delta X_s] \tag{4.5}$$

Proof. See Meyer, P. A., 1976. ■

Note that is $f'(X_{s-})$ a left continuous process, hence predictable. For $d-$dimensional processes, each of whose components is a semimartingale, the obvious generalization holds. A transformation of the semimartingale is obtained by applying the previous equation. Expression (4.5) can be refined for pure-jump process.[28]

Proposition 31 *Suppose that $f \in C^2$ with bounded first and second*

[28] It's possible to generalize this approach with the deterministic part and the stochastic continuous component of the process, i.e.$b \neq 0$ and $\sigma \neq 0$.

derivatives and that X_t is the solution to

$$dX_t = \int F\left(X_{t-}, z\right) dq\left(s, z\right)$$

then

$$f(X_t) = f(X_0) + \int_0^t f''(X_s) dX_s + \int_0^t \mathcal{L}f\left(X_s\right) ds \tag{4.6}$$

where the stochastic term is a martingale and

$$\mathcal{L}f\left(x\right) = \int_0^t \left[f\left(x + F\left(x, z\right)\right) - f\left(x\right) - f'(x) F\left(x, z\right)\right] \lambda\left(dz\right)$$

Proof. By Ito's formula

$$f(X_t) = f(X_0) + \int_0^t f'(X_{s-}) dX_s + \sum_{s \leq t} \left[f(X_s) - f(X_{s-}) - f'(X_{s-}) \Delta X_s\right]$$

The stochastic integral term is a martingale. Rewrite $f(X_s)$ as

$$f(X_s) = f(X_{s-} + \Delta X_s)$$

The jump of X_t at time s is given by $F(X_{s-}, z)$ if N assigns mass one to the point (s, x), $f(X_s) - f(X_0)$ is equal to a martingale plus

$$\int_0^t \int \left[f\left(X_{s-} + F\left(X_{s-}, z\right)\right) - f\left(X_{s-}\right) - f'(X_{s-}) F\left(X_{s-}, z\right)\right] N\left(ds, dz\right)$$

i.e. a martingale plus

$$\begin{aligned} & \int_0^t \int \left[f\left(X_{s-} + F\left(X_{s-}, z\right)\right) - f\left(X_{s-}\right) - f'(X_{s-}) F\left(X_{s-}, z\right)\right] v\left(ds, dz\right) \\ = & \int_0^t \int \left[f\left(X_{s-} + F\left(X_{s-}, z\right)\right) - f\left(X_{s-}\right) - f'(X_{s-}) F\left(X_{s-}, z\right)\right] \lambda\left(dz\right) ds \end{aligned}$$

Define

$$\mathcal{L}F\left(x\right) = \int_0^t \left[f\left(x + F\left(x, z\right)\right) - f\left(x\right) - \frac{\partial f\left(x\right)}{\partial x} F\left(x, z\right)\right] \lambda\left(dz\right)$$

so that

$$f(X_t) - f(X_0) - \int_0^t \mathcal{L}f\left(X_{s-}\right) ds$$

is a martingale. Since X has only countably many jumps, then the Lebesgue measure of the set of times where $X_s \neq X_{s-}$ is zero and hence

$$f\left(X_t\right) - f\left(X_0\right) - \int_0^t \mathcal{L}f\left(X_s\right) ds = f\left(X_t\right) - f\left(X_0\right) - \int_0^t \mathcal{L}f\left(X_{s-}\right) ds$$

Therefore

$$f\left(X_t\right) - f\left(X_0\right) - \int_0^t \mathcal{L}f\left(X_s\right) ds$$

is a martingale. ■

4.4. Stochastic differential equation driven by an additive process

Fix a filtered probability space $\{\Omega, \mathcal{F}, \mathbf{P}\}$, $\mathcal{F}_t = \sigma(X_s, s \leq t)$. Assume the following dynamics, under the risk-neutral probability $\mathbf{P}$, for the stock return:

$$\frac{dS_t}{S_t} = \mu(t)\, dt + dX_t^P \tag{4.7}$$

where $X_t = \{X_t, t \geq 0\}$ defined as

$$dX_t = \int_0^t f(X_{t-}, s)\, dq(z, s)$$

is an additive process defined with respect to a compensated Poisson point process, that describes the jumps. By applying the Ito's Lemma to the stochastic differential equation the dynamics of the stock can be easily derived. Indeed taking $Y_t = F(S_t) = \ln S_t$,

$$\begin{aligned} Y_t &= Y_0 + \int_0^t \frac{\partial F(S_{s-})}{\partial S_{s-}} dS_s + \sum_{s \leq t} \left[\ln(S_s) - \ln(S_{s-}) - \frac{\partial F(S_{s-})}{\partial S_{s-}} \Delta S_s \right] \\ &= Y_0 + \int_0^t \frac{1}{S_s} S_s (\mu(s)\, ds + dX_s) + \sum_{s \leq t} \left[\ln(S_s) - \ln(S_{s-}) - \frac{1}{S_s} \Delta S_s \right] \end{aligned}$$

i.e.

$$\begin{aligned} Y_t &= Y_0 + \int_0^t \mu(s)\,ds + \int_0^t dX_s \\ &\quad + \int_0^t \int \left[\ln\left(S_{s-} + f\left(X_{s-}, z\right)\right) - \ln\left(S_{s-}\right) - \frac{1}{S_s} f\left(X_{s-}, z\right)\right] dq\left(s, z\right) \\ &\quad + \int_0^t \int \left[\ln\left(S_{s-} + f\left(X_{s-}, z\right)\right) - \ln\left(S_{s-}\right) - \frac{1}{S_s} f\left(X_{s-}, z\right)\right] v\left(ds, dz\right) \\ &= Y_0 + \int_0^t \mu(s)\,ds + \int_0^t \int f\left(X_{s-}, s\right)\left(N\left(ds, dz\right) - v\left(ds, dz\right)\right) \\ &\quad + \int_0^t \int \left[\ln\left(S_{s-} + f\left(X_{s-}, s\right)\right) - \ln\left(S_{s-}\right) - \frac{1}{S_s} f\left(X_{s-}, s\right)\right] dq\left(s, z\right) \\ &\quad + \int_0^t \int \left[\ln\left(S_{s-} + f\left(X_{s-}, s\right)\right) - \ln\left(S_{s-}\right) - \frac{1}{S_s} f\left(X_{s-}, s\right)\right] \lambda\left(dz\right) ds \\ &= Y_0 + \int_0^t \mu(s)\,ds + M_t + \int_0^t \mathcal{L} \ln\left(S_{s-}\right) ds \end{aligned}$$

and

$$\begin{aligned} S_t &= S_0 \exp\left(\int_0^t \mu(s)\,ds + M_t + \int_0^t \mathcal{L} \ln\left(S_{s-}\right) ds\right) \\ &= S_0 \exp\left(\int_0^t \mu(s)\,ds + \int_0^t \mathcal{L} \ln\left(S_{s-}\right) ds + M_t\right) \qquad (4.8) \end{aligned}$$

given some initial condition $Y(0) = Y_0$ and $S(0) = S_0$. The first fundamental theorem of finance states that, in order to avoid arbitrage, the discounted-stock price should be a martingale. Assume that the interest rate is not random but time-dependent, i.e. $R(t) = \int_0^t r(s)\,ds$; and define the discounted-stock price as

$$\begin{aligned} \widetilde{S}_t &= S_t \exp R(t) = \\ &= S_t \exp \int_0^t r(s)\,ds \end{aligned}$$

Its dynamics under the historical probability is represented by the following stochastic differential equation

$$\frac{d\widetilde{S}_t}{\widetilde{S}_t} = (\mu(t) - r(t))\,dt + dX_t$$

The process is an additive process with the following triplet

$$(\mu(t) - r(t), 0, \lambda(t))$$

Theorem 32 *Let* $\left(\{X_t\}_{t \leq T}, P\right)$ *and* $\left(\{X_t\}_{t \leq T}, P^{\#}\right)$ *be Lévy processes on* $\mathbb{R}^d$ *with generating triplets* (A, v, γ) *and* $\left(A^{\#}, v^{\#}, \gamma^{\#}\right)$ *respectively. Then the following two statements are equivalent:*

1. $[P]_{\mathcal{F}_t} \approx \left[P^{\#}\right]_{\mathcal{F}_t}$ *for every* $t \in (0, \infty)$
2. *The generating triplet satisfies*

$$A = A^{\#}$$

$$v \approx v^{\#}$$

with the function $\varphi(x)$ *defined by* $\frac{dv^{\#}}{dv} = e^{\varphi(x)}$ *satisfying*

$$\int_{\mathbb{R}^d} \left(e^{\frac{\varphi(x)}{2}} - 1 \right)^2 v(dx) < \infty$$

and

$$\gamma^{\#} - \gamma - \int_{|x| \leq 1} x \left(v^{\#} - v \right)(dx) \in \Re(A) \tag{4.9}$$

where $\Re(A) = \left\{ Ax : x \in \mathbb{R}^d \right\}$

Proof. See Sato, K., 1999, page 230. ■

The existence of $\varphi(x)$ guarantees a change of measure into the risk-neutral world, and a no-arbitrage price for contigent claims would be possible. The stochastic differential equation has no continuous diffusive component, i.e. $\sigma = 0$. Introducing a non-zero diffusive coefficient into the stochastic differential equation does not change dramatically the analysis. A closed-form expression for the stock would be still possible to derive: a continuous integral with respect to the Brownian motion and its correction should be added to (4.8). The process would have generating triplet $(\mu(t), \sigma(t), \lambda(t))$. Some difficulties may arise when moving into the risk-neutral world. Infinite equivalent martingale measures (EMM) exist since equation (4.9) might have infinite solutions. By the second fundamental theorem of asset pricing, the market would be free of arbitrage and incomplete. By imposing some structure on $\varphi(x)$, a particular EMM could be selected and no-arbitrage prices for contingent claims derived.[29] The expression (4.6) could be easily applied in a stochastic differential equation where the innovation term is not a semimartingale, but a submartingale by means of the compensator function. Take $L_t \sim S\alpha S,\ \alpha \in (1,2)$. The operator $\mathcal{L}F(\cdot)$ is defined by

$$\mathcal{L}F(x) = \int \left[F(x+z) - F(x) - 1_{(|z|\leq 1)} \frac{\partial f(x)}{\partial x} s \right] v(x, ds)$$

where $v(x, ds) = \frac{a}{|s|^{1+\alpha}} ds$.[30] Assume $f \in C^2$ with bounded first and second derivatives, and

$$X_t = \int_0^t H_s dL_s$$

[29] The change of measure for the jump-diffusion model, proposed by Merton, for istance, depends on two parameter (θ, ϕ), where θ is the risk premium for the diffusive risk, and ϕ describes the risk aversion of the investors to the jump risk. Merton select a particular EMM among the infinitely many by claiming that no risk premium for the jumps are required: $\phi = 1$ and the jump intensity is the same under the historical probability and the risk neutral world. Therefore

$$\theta = \frac{\mu - r + (\mu_Y - 1)\lambda^P}{\sigma}.$$

[30] $\mathcal{L}F(\cdot)$ is the infinitesimal generator of the stable process.

where H_s is a bounded predictable process. Then

$$f(X_t) = f(X_0) + M_t + \int_0^t |H_s|^\alpha \mathcal{L} f(Z_{s-})\, ds$$

where

$$M_t = \int_0^t f'(X_{s-})\, dX_s + \int_0^t \int K(s,y)\,(N(dy,ds) - v(dy)\,ds)$$

$$K(s,y) = [f(X_{s-} + H_s y) - f(X_{s-}) - f'(X_{s-})\, H_s y]$$

$$N(A \times [0,t]) = \sum_{s \le t} 1_A(\Delta L_s)$$

$$\lambda(A) = E[N(A \times [0,t])] = \int_A \frac{1}{|z|^{1+\alpha}} dz$$

Set $H_s = \sigma_s$ and consider the following stochastic differential equation

$$\begin{aligned} d\frac{S_t}{S_t} &= \mu_t dt + dX_t = \\ &= \mu_t dt + \sigma_t dL_t \end{aligned}$$

and $f = \ln S_t$, then

$$\begin{aligned}
\ln S_t &= \ln S_0 + \int_0^t \frac{1}{S_s} dS_s + \sum_{s \leq t} \left[\ln (S_s) - \ln (S_{s-}) - \frac{1}{S_{s-}} \Delta S_s \right] \\
&= \ln S_0 + \int_0^t \mu(s)\, ds + \int_0^t dZ_s + \\
&\quad + \sum_{s \leq t} \left[\ln (S_s) - \ln (S_{s-}) - \frac{1}{S_{s-}} \Delta S_s \right] \\
&= \ln S_0 + \int_0^t \mu(s)\, ds + \int_0^t \sigma(s) dL_s + \int_0^t \int K(s,y) N(dy, ds) \\
&= \ln S_0 + \int_0^t \mu(s)\, ds + \int_0^t \sigma(s) dL_s + \\
&\quad + \int_0^t \int K(s,y) \left(N(dy, ds) - v(dy, ds) \right) + \int_0^t \int K(s,y) \lambda(dy)\, ds \\
&= \ln S_0 + \int_0^t \mu(s)\, ds + \int_0^t \sigma(s) dL_s + \\
&\quad \int_0^t \int K(s,y) \left(N(dy, ds) - v(dy, ds) \right) + \int_0^t |\sigma(s)|^{\alpha} \mathcal{L} f(S_{s-})\, ds
\end{aligned}$$

4.4.1. Pricing formula with respect to Hawkes processes

Imagine a market where the number of large jumps is finite in number over any finite interval, and the large jumps do not affect the existence or uniqueness of solutions.[31] In the filtered probability space $\{\Omega, \mathcal{F}, \mathbf{P}\}$, $\mathcal{F}_t = \sigma(X_s, s \leq t)$, the stock dynamics (4.7) becomes, under the risk-

[31] In this scenario, one wants not to worry about the integrability of $\int_0^t |s| f(x) \lambda(ds)$.

neutral probability $\mathbf{P}$:

$$\frac{dS_t}{S_t} = \mu\left(t\right)dt + dX_t^P$$

where $X_t = \{X_t, t \geq 0\}$ is a stochastic process defined with respect to a compensated Hawkes process

$$\begin{aligned} dX_t^P &= \int_0^t f\left(X_{t-}, s\right) dq^P(u, s) \\ &= \int_0^t g\left(s\right) dq(u, s) \\ &= \int g\left(s\right)\left[dN(u, s) - \lambda^P(u)ds\right] \\ &= \int g\left(s\right) dN(u, s) - \int g\left(s\right)\left[\mu + \int_{-\infty}^{u} \phi_{s-u} dN_u\right] ds \end{aligned}$$

By applying the Ito's Lemma to the stochastic differential equation the dynamics of the stock is easily derived. Indeed taking $Y_t = F(S_t) = \ln S_t$,

$$
\begin{aligned}
Y_t &= Y_0 + \int_0^t \frac{\partial F(S_{s-})}{\partial S_{s-}} dS_s + \sum_{s \le t} \left[\ln(S_s) - \ln(S_{s-}) - \frac{\partial F(S_{s-})}{\partial S_{s-}} \Delta S_s \right] \\
&= Y_0 + \int_0^t \frac{1}{S_s} S_s \left(\mu(s)\, ds + dX_s\right) + \sum_{s \le t} \left[\ln(S_s) - \ln(S_{s-}) - \frac{1}{S_s} \Delta S_s \right] \\
&= Y_0 + \int_0^t \mu(s)\, ds + \int_0^t dX_s \\
&\quad + \int_0^t \int \left[\ln(S_{s-} + f(X_{s-}, z)) - \ln(S_{s-}) - \frac{1}{S_s} f(X_{s-}, z) \right] v(ds, dz) \\
&= Y_0 + \int_0^t \mu(s)\, ds + \int_0^t \int g(s)\, dN_s - \int g(s) \left[\mu + \int_{-\infty}^s \phi_{t-s} dN_s \right] ds \\
&\quad + \int_0^t \int \left[\ln(S_{s-} + g(s)) - \ln(S_{s-}) - \frac{1}{S_s} g(s) \right] \left[\mu + \int_{-\infty}^u \phi_{s-u} dN_u \right] ds
\end{aligned}
$$

and

$$
\begin{aligned}
S_t &= S_0 \exp \begin{pmatrix} \int_0^t \mu(s)\, ds + \int_0^t \int g(s)\, dN_s - \int g(s) \left[\mu + \int_{-\infty}^u \phi_{s-u} dN_u \right] ds \\ + \int_0^t \mathcal{L} \ln(S_{s-})\, ds \end{pmatrix} \\
&= S_0 \exp \left(\int_0^t \mu(s)\, ds + \int_0^t \mathcal{L} \ln(S_{s-})\, ds + M_t \right)
\end{aligned}
$$

where

$$
M_t = \int_0^t \int g(s)\, dN_s - \int g(s) \left[\mu + \int_{-\infty}^s \phi_{t-s} dN_s \right] ds
$$

given some initial condition $Y(0) = Y_0$ and $S(0) = S_0$. The first fundamental theorem of finance states that, in order to avoid arbitrage, the discounted-stock price should be a martingale. By considering the interest rate being not stochastic, but time-dependent, i.e. $R(t) = \int_0^t r(s)\,ds$; the discounted-stock price is

$$\begin{aligned}\widetilde{S}_t &= S_t \exp R(t) = \\ &= S_t \exp \int_0^t r(s)\,ds\end{aligned}$$

Its dynamics under the historical probability is represented by the following stochastic differential equation

$$\frac{d\widetilde{S}_t}{\widetilde{S}_t} = (\mu(t) - r(t))\,dt + dX_t^P$$

The process is an additive process with the following triplet

$$\left(\mu(t) - r(t), 0, \lambda^P(t)\right)$$

The expression (2.9) defines the condition for a change of measure into a probability measure where the discounted stock price is characterized by the following triplet

$$\left(0, 0, \lambda^Q(t)\right)$$

The condition to be satisfied for a moving into the risk-neutral probability measure Q is then

$$\int_0^t g(s) \int_{\mathbb{R}\setminus\{0\}} u \left(e^{\varphi^*(x,u)} - 1\right) u\lambda^P(du)ds = -\int_0^t (\mu(s) - r(s))\, ds \tag{4.10}$$

i.e.

$$\int_0^t \mu(s)\, ds = \int_0^t r(s)ds - \int_0^t g(s) \int_{\mathbb{R}\setminus\{0\}} u \left(e^{\varphi(x,u)} - 1\right) \lambda^P(du)ds \tag{4.11}$$

The dynamics of the return in the risk-neutral world is therefore described by

$$\frac{dS_t}{S_t} = \left(r(t) - g(s) \int_{\mathbb{R}\setminus\{0\}} u \left(e^{\varphi(x,u)} - 1\right) \lambda^P(du)\right) dt + dX_t^Q$$

The stock expression, under the risk neutral measure, is

$$S_t = S_0 \exp \begin{pmatrix} \int_0^t \left(r(s) - g(s) \int_{\mathbb{R}\setminus\{0\}} u \left(e^{\varphi(x,u)} - 1\right) \lambda^P(du)\right) ds \\ + \int_0^t \int g(s)\, dN_s - \int g(s) \left[\mu + \int_{-\infty}^u \phi_{s-u} dN_u\right] ds \\ + \int_0^t \mathcal{L} \ln(S_{s-})\, ds \end{pmatrix}$$

i.e.

$$S_t = S_0 \exp\left(\begin{array}{c} \int_0^t \left(r(s) - g(s)\int_{\mathbb{R}\setminus\{0\}} u\left(e^{\varphi(x,u)} - 1\right)\lambda^P(du) + \mathcal{L}\ln(S_{s-})\right) ds \\ +M_t \end{array}\right) \tag{4.12}$$

The price π of a contingent claim written on S_t is then given by

$$\pi(0, r, g) = E_0^Q\left[e^{-\int_0^t r(s)ds} g(S_t)\right]$$

where $g(S_T)$ is the payoff function of the financial security.

4.5. Simulations of an exponential Hawkes process

Monte Carlo experiments are run to describe the behavior of the equation (4.12). Hawkes processes are restricted to belong the case of exponential decay, where the kernel ϕ_{t-s} is given by

$$\phi(x) = \alpha e^{-\beta x}$$

The expression (4.1) for the intensity of the process becomes

$$\begin{aligned} \lambda(t) &= \mu + \int_{-\infty}^t \phi_{t-s} dN_s \\ &= \mu + \sum_{t_i < t} \alpha e^{-\beta(t-t_i)} \end{aligned} \tag{4.13}$$

The choice of the parameters $\theta = [\alpha, \beta]$ is fundamental in modelling the chances of point event occurrence, and how this probability is influenced by past point events. The intensity process increases or decreases simultaneously where a point event occurs and then recovers exponentially as time proceeds, if the magnitude $\alpha \geq -1$ and the recovery time $\beta \geq 0$ are set appropriately. If α is positive, the intensity process increases after point event occurrences and the process belongs to the class of the self-exciting point processes; while if $0 > \alpha \geq -1$ the intensity process decreases after point event occurrences, characterizing a self-inhibiting point process. Self-exciting point processes are an extension of temporal Poisson processes, having a dependency on the past event history, i.e. possessing after-effects of previous point events, unlike Poisson processes. The intensity process takes a constant value for a homogeneous Poisson process and a deterministic function for a non-homogeneous Poisson path. The connection is immediate and if $\alpha = 0$, the intensity of the stochastic process becomes

$$\lambda(t) = \mu$$

i.e. the standard Poisson case. The intent is to show how flexible is the closed-formula provided in the previous section and how by modifying the parameters $\theta = [\mu, \alpha, \beta]$, it is possible to model different scenario. A

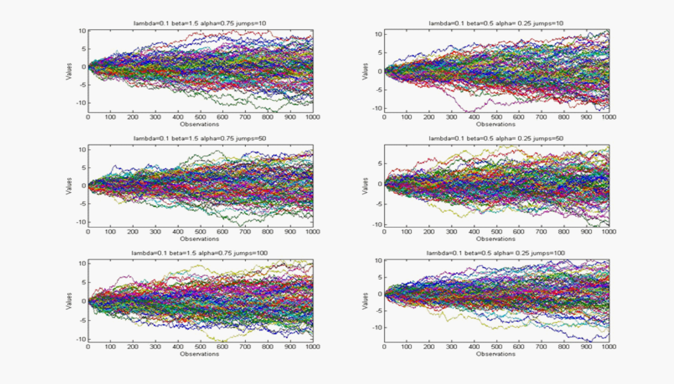

Figure 35: Low Lambda $\lambda = 0.1$

continuous innovation term has been added to the stochastic differential equation (4.7), $\sigma'(s)\, dB_s$, where dB_s is the classical Brownian Motion and $g(s)$ set equal to the indicator function. $N-$random paths, whose intensity is provided by (4.13) have been simulated. The starting point is 0, and the drift coefficient and diffusive are kept constant in the simulations. The magnitude α, the recovery time β are allowed to vary; whereas the intensity of the Poisson process μ is kept constant. The number of jumps are not fixed; and the final time T is set to be equal to 30, with a number of grid point equal to 1000. Figure (35) describes a situation where the intensity is low. Depending on the number of jumps, recovery rate β and magnitude α, the behavior of the fictitious values is different. Higher recovery rate implies a less trendy process, especially

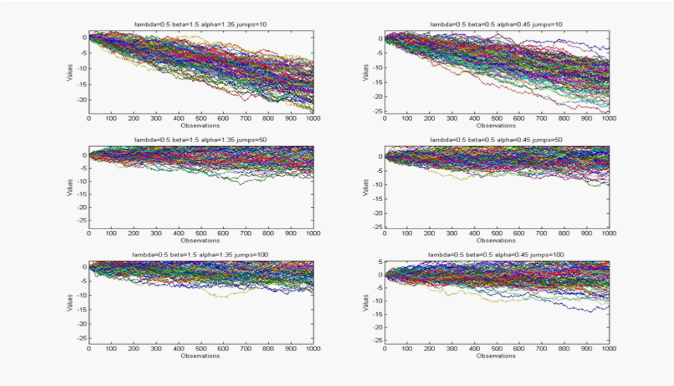

Figure 36: High Lambda $\lambda = 0.8$

when the number of jumps increases significantly. Differences become much more evident as the conditional intensity increases. Figure (36) plots a scenario where the probability of a jump occurring is significantly high. In this scenario, the number of jump is clearly fundamental in describing the process behavior. A small number of jumps, associated with a high probability of occurring, implies that they occur immediately and the process is not anymore affected by these discontinuities. Its behavior depends on the deterministic part and the continuos innovation term: the parameters have been set in such a way to drive downward the path. A small recovery rate β exacerbates the behavior. By increasing the number of jumps, the path fluctuates more around zero, especially when β is high. Figure (37) describes the behavior a stochastic differential equa-

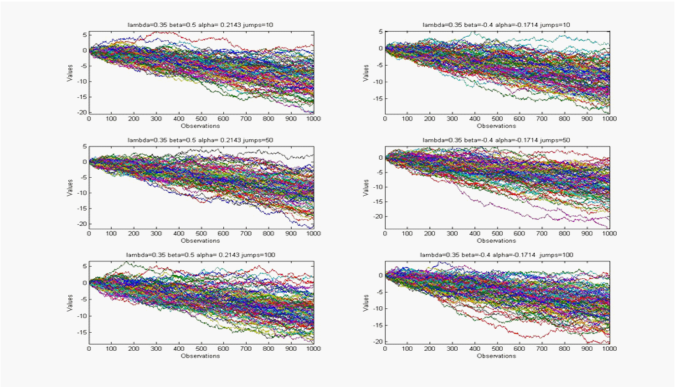

Figure 37: Self-exciting process ($\alpha > 0$) vs Self-inhibiting process ($\alpha < 0$)

tion, driven by a Hawkes process as innovation term for its discontinuity points. As before, the initial seed is 0, and the drift coefficient and diffusive are kept constant in the simulation. The intensity function λ is kept constant, $\lambda = 0.35$, while the magnitude α can vary: the process is self-exciting or self-inhibiting, depending on the magnitude value α. The number of jumps is important for the behavior of the random paths. As before, the diffusive and drift coefficient pushes down the process; the self-inhibiting process goes down much faster than the self-exciting in case of a large number of jumps; the behavior is the opposite with large jumps.

4.6. Conclusions and future extensions

This Chapter investigates alternative point processes for modelling discontinuities in the continuous-time framework. It has been testified that unexpected events like economic crisis, war, have a huge impact in the stock's price, leading financial data to irregularity of spacing in time. Not only is the impact so evident for being modelled by a continuous diffusion; but also the classical jump-diffusion model, where the compound Poisson process is in charge of modelling discontinuities, fails to exploit the full amount of information embedded in the stock's movement. A stochastic differential equation is proposed, with the peculiarity of having jumps' behavior captured by a particular additive process, associated to a compensated Poisson random measure. After defining the Poisson random measure and its compensator, additive processes are defined as integral forms with respect to the compensated Poisson random measure. A particular class of Poisson measures is defined and proposed for its particular properties: the class of self-exciting processes. These random paths are characterized by having a conditional intensity function, whose increases or decreases along time. Its intensity can be decomposed into a constant term, the intensity of homogeneous Poisson process, and an integral term with respect to a counting process: jumps are now clustered over time, similarly to earthquakes. Stochastic calculus with respect to Poisson random measure derives the differential of a time-dependent function of a stochastic process. By means of Ito's lemma, a closed-form expression for the stock's price is computed, under the conditions that its dynamics is described by expression (4.7). The stochastic differential equation does not have a continuous innovation term, but introducing it does not make the analysis much more difficult: the only difference is that, when changing the measure to move into a risk-neutral world, the market would be not complete. An additive process, defined with respect to a compensated Hawkes process is assumed to the random error in expression (4.7); and a closed-form formula for a stock's price, where jumps are affected by a self-exciting process, is computed. The Chapter is closed by simulating diffusions, driven by a specific Hawkes process, the random path with exponential decay, to show the flexibility of the stochastic equations in modelling financial data. The idea of jumps clustering over time, similarly to seismic quiescence, should be further investigated either by an empirical analysis, or by proposing alternative Poisson random measure. Expected-maximization (EM) literature has been developed, recently, parametric and non-parametric approach to estimate the conditional intensity of Hawkes process. Estimating the parameters from financial data and check the pricing performance of the model might be a good way to attest the validity of the closed-form stock's expression. Alternatively, it is possible to model jumps with additive process,defined with respect to more sophisticated compensated Poisson random measures, like the epidemic-type aftershock sequence model, whose intensity function

$$\lambda(t) = \mu + \sum_{t_i < t} g(t - t_i, m_i)$$

the space–time–magnitude distribution of earthquake occurrence, since where the history of the process also includes earthquake magnitudes m_i.

References

[1] Albeverio, S., and B. Rudinger, "The Lévy-Ito decomposition theorem and stochastic integrals, on separable Banach spaces", *Stochastic Analysis and Application*, 22, 5 (2004), 217–253.

[2] Bacry, E., Dayri, K. and J. F. Muzy, "Nonparametric kernel estimation for symmetric Hawkes process. Application to high frequency financial data", *Journal of Econometrics*, 66 (1995), 251-87.

[3] Barndorf-Nielsen O. E., "Processes of Normal Inverse Gaussian Type", *Finance and Stochastics*, 2 (1997), 41-68.

[4] Bass, R. F., "Stochastic differential equations driven by symmetric stable processes", *Séminaire de Probabilités, 36, Springer, New York, (2003), 302–313*.

[5] Bass, R. F., "Stochastic differential equations with jumps", *Probability Surveys*, 1 (2004), 1.19.

[6] Bass, R. F., "Stochastic Calculus for Discontinuous Process", *Lectures Notes, Department of Mathematics, University of Connecticut, October 1998.*

[7] Bates, D. S., "Post Crash Moneyness Bias in S&P 500 Futures Options", *Rodney L. White Center working paper* (1991), Wharton School, University of Pennsylvania, Philadelphia, PA.

[8] Bauwens, L., and N. Hautsch, *"Modelling Financial High Frequency Data using Point Processes",* Handbook of Financial Time Series, Springer, New York (2009), Part 6, 953-979.

[9] Black, F., and M. S. Scholes, "The pricing of Options and Corporate Liabilities", *Journal of Political Economy*, 81 (1973), 637-654.

[10] Bowsher, C. G., "Modelling security market events in continuous time: Intensity based, multivariate point process models", *Journal of Econometrics*, 141 (2007), 876-912.

[11] Brillinger, D. R., "Some statistical methods for random process data from seismology and neurophysiology", *The Annals of Statistics,* (1988), 16, 1, 1-54

[12] Carr, P., and D. Madan, "Option valuation using the fast Fourier transform", *Journal of Computational Finance,* 2 (1999), 61–73.

[13] Carr, P., Geman, H., Madan, D., and M. Yor, "The fine structure of asset returns: an empirical investigation", *Journal of Business*, 75 (2002), 305–332.

[14] Carr, P., and L. Wu, "Finite moment log stable process and option pricing", *Journal of Finance*, 58 (2003) , 753–777.

[15] Carr, P., and L. Wu,"Time-changed Lévy processes and option pricing", *Journal of Financial Economics*, 71 (2004), 113–141.

[16] Carr, P., and L. Wu, "Stochastic skew in currency options", *Journal of Financial Economics*, 86, 1, (2007), 213-247.Carr, P., Madan, D., "Option valuation using the fast Fourier transform. Journal of Computational Finance, 2 (1999), 61–73.

[17] Carr, P., Geman, H., Madan, D., and M. Yor, "The fine structure of asset returns: an empirical investigation", *Journal of Business*, 75 (2002), 305–332.

[18] Cochrane, J., "Production-based asset pricing and the link between stock returns and economic fluctuations", *Journal of Finance*, 46 (1991), 209–237.

[19] Cochrane, J., *"Asset Pricing",* Princeton University Press, Princeton, NJ, (2004).

[20] Eberlein, E., and U. Keller, "Hyperbolic distributions in Finance", *Bernoulli*, 1 (1995), 281-299.

[21] Eberlein, E., and U. Keller, and K. Prause "New insights into smile, mispricing and value at risk: the hyperbolic model", *Journal of Business*, 71 (1998), 371-405

[22] Eberlein, E., and K. Prause, "The generalized hyperbolic model: Financial derivatives and risk measures", *FDM preprint*, 56 (1998) University of Freiburg.

[23] Hawkes, A. G., "Spectra of some self-exciting and mutually exciting point process". *Biometrika*, 58 (1971), 83-90.

[24] Hawkes, A. G., "Point Spectra of Some Mutually exciting point process", *Journal of the Royal Statistical Society*, 33 (1971) 438-443.

[25] Heston, S., “A Closed-Form Solution of Options with Stochastic Volatility with Applications to Bond and Currency Options”, *The Review of Financial Studies*, 6 (1993), 327-343.

[26] Kahneman, D., and A. Tversky, "Prospect Theory: An Analysis of Decision Under Risk", *Econometrica*, 47 (1979), 263-291.

[27] Kahneman, D., Slovic, P., and A. Tversky, *"Judgment Under Uncertainty: Heuristics and Biases",* New York: Cambridge University Press, (1982).

[28] Kahneman, D., and A. Tversky, "Choices, Values and Frames", *American Psychologist*, 39 (1984), 341-350.

[29] Kwiecinski, A., and R. Szekli, "Some Monotonicity and Dependence Properties of Self-Exciting Point Processes", *The Annals of Applied Probability*, 6 (1996), 1211–1231.

[30] Madan, D. B., and F. Milne, "Option Pricing with VG Martingale Components", *Mathematical Finance*, 1 (1991), 39-55.

[31] Madan, D. B. and E. Seneta, "The Variance Gamma (V.G.) Model for Share Market Returns", *Journal of Business*, 63 (1990), 4, 511-524.

[32] Meyer, P. A., "Un Cours sur les Intégrales Stochastiques", *Séminaire de Probabilités, X, Lecture Notes 511, Springer Verlag, (1976).*

[33] Moller, J., and J. K. Rasmussen "Perfect Simulation of Hawkes Processes", *Advances in Applied Probability*, 37 (2005), 629-646.

[34] Ogata, Y., "Statistical Models for Earthquake Occurrence and Residual Analysis for Point Processes", *Journal of the American Statistical Association*, 83 (1988), 9-27.

[35] Rudiger, B., "Stochastic integration with respect to compensated Poisson random measures on separable Banach spaces", *Stochastics and Stochastics Reports*, 76, 3, (2004), 213-242.

[36] Sato, K., *"Lévy Processes and Infinitely Divisible Distributions",* Cambridge University Press, (1999*).*

[37] Schoenberg, F. P., Brillinger, D. R., and P. Guttorp, *"Point process, Spatial-temporal",* Encyclopedia of Environmetrics, Wiley, New York (2002), Vol.3, 1573–1577.

[38] Schoenberg, F. P., "Transforming spatial point processes into Poisson processes", *Stochastic Processes and their Application*, 81(2) (1999), 155-164.

[39] Veen, A., and F. Schoenberg, "Estimation of spacetime branching process models in seismology using an EM-type algorithm", *Journal of the American Statistical Association*, 103 (482) (2008), 614-624.

[40] Wolpert, R. L. and K. Ickstadt, "Poisson/gamma random field models for spatial statistics", *Biometrika*, 85, 2 (1998), 251-267.

Chapter 5

Conclusions

The primary intent of this work has been to discuss some stylized facts in financial and propose alternative approach to capture and model them. Statistical methods have been combined with financial modelling in order to propose different stochastic differential equations in the continuous-time framework. This need stems from the empirical evidence that financial data might be affected by excess volatility, serial dependence, clusters or leptokurtosis. The demand to combine statistics and finance has led to follow a dual approach in this study. From a statistical point, a normative approach has been considered: a generic class of processes has been selected, by defining its main properties and providing its integral representation. Mathematical techniques and information theory have been used to perform specific modifications of our random paths. This rigorous approach has been substituted by a more positive view when moving into the financial world. A specific random path, inside the generic class defined previously, has been selected to model some specific scenarios: particular emphasis has been focused on long-range dependence, heavy tails and seismic quiescence. New innovation terms have been proposed to change the structure of the stochastic differential equation, driving the underlying under the historical probability. This Chapter concludes the work by summarizing what done previously and suggesting alternative approaches. Section 5.1. summarizes and reviews, briefly, the statistical and mathematical framework used to deliver a closed-form expression for the stock and some specific contingent claim. The work cannot explain the entire spectrum of anomalies existing in the financial markets; and needs to be extended in order to cover a bigger set of financial irregularities. Section 5.2. points at providing some possible ways to extend and generalize the work, either by taking the two suggested paths and moving further; or by proposing alternative and specific routes to be followed.

5.1. Summary

The Black–Scholes model is a mathematical model of a financial market containing certain derivative investment instruments. The Black and Scholes market consists in imposing that the stock's movements are described by a continuous diffusion, with constant drift and diffusive coefficient. Everything is in line with the first fundamental theorem of finance: the discounted stock is a martingales and there are no arbitrage opportunities existing in the market. The existence of replicating portfolios gives the opportunity to price contingent claims in a risk-neutral scenarios. Normality is assumed to hold and completes this perfect world, where all the theoretical assumptions, implied by the literature, are perfectly satisfied. This idyllic environment, unfortunately, fails when moving into the real world. Phenomena, like excess volatility, heavy tails, volatility clustering and jumps affect every day the financial markets. A continuous diffusion with constant coefficients, whose innovation term is a standard Brownian motion does not capture all this empirical evidence, because all these distortions cannot be recognized by its statistical structure. The literature tried to provide some answers and innovation to this puzzle either from a purely theoretical perspective or from a purely empirical one. A minority group, doubting efficient market hypothesis (EMF), proposed alternative theories: the fractal market hypothesis and the coherent market theories. The majority, instead, did not deviate from the idea of efficient markets and no-arbitrage theory, and modelled all these phenomenons, step by step, by Lévy processes, relatively easy to manage because of their structure and, more important, their martingales and Markovian properties. Stochastic differential equation, characterized by having a continuous innovation term and a discontinuous error term, modelled through a compound Poisson process, have been proposed to described the dynamics of securities; and imposing more sophisticated drift and diffusive coefficient, often random functions, has led to more general conditional Lévy processes. There is no clear empirical evidence on which is the best stochastic process in absolute terms; and the intent of this work is to suggest two possible alternative trajectories to be considered in financial modelling. Innovations are possible by:

1. Relaxing the idea of an innovation term, characterized by being martingale and Markovian, and proposing a different random error;
2. Generalizing the jumps structure by modelling discontinuities with more general additive process than the compound Poisson process.

The first departures from the field of Lévy processes consists in proposing finite-variance processes, not martingales: specifically, the class of Gaussian processes. A canonical representation is provided an their main properties are derived. Restrictions on the kernel functions make possible to derive the differential of a time-dependent function by Ito's lemma. Financial securities have their own dynamics described by a stochastic differential equation, whose is not necessarily a Markovian process. An entropy argument provides conditions to approximate this stochastic differential equation with a Markovian process and price contingent claims, under the risk-neutral probability. A sub-fractional Brownian motion innovation term, where long-range dependence comes along with nonstationarity in the increments, is selected to model situation where serial dependence might affect the market. A theoretical price for European contingent claims is derived; and empirical analysis on the Standard & Poor's (S&P) 500 Index option performed. Empirical results suggest that the market knows that the true dynamics of the security is affected by long-range dependence, but corrects this bias in delivering prices for contingent claims. The Gaussian distribution is, then, replaced, in Chapter 3, by heavy tails. An innovation term, belonging to the class of stationary alpha stable processes ($S\alpha S$), substitutes the Brownian motion in the stochastic differential equation, describing the dynamics of the securities. By imposing the random error to be a linear fractional stable motion, the security is characterized by having serial dependence, under its historical measures. A distance-minimization problem between the true process and a class of semimartingales permits, combined with the Fourier techniques, to deliver no-arbitrage prices for contingent claims under this specific scenario. An empirical analysis on S&P 100 Index option is performed and three different scenarios for the index are considered: negative dependence; independent increments; corrected negative dependence. Empirical results suggest that the market rules out the "negative dependence" scenario; while correcting negative dependence seems to best way to deliver more accurate option prices. The second path consists in modelling points of discontinuity through more general additive processes. Financial data are irregular in space and time and the magnitude of unexpected event like economic crisis might be different. Modelling them through compound Poisson processes might fail in exploiting the full amount of information embedded in the stock's movement. An additive process, defined as an integral form with respect to compensated Poisson random measure, is then added to a continuous diffusion, with the specific intent to describe the jump's activity. By means of stochastic calculus for point processed, a closed-form expression for the stock is derived. A specific scenarios, where the intensity of jumps is similar to the earthquakes activity, is modelled by means of self-exciting processes, Hawkes process, whose conditional intensity function increases over time. Simulations of the stock, under the hypothesis that jumps decays exponentially, are performed to show how flexible might be the use of compensated Poisson random measures in the option pricing field.

5.2. Extensions

This work can be extended by following multiple directions. Some improvements can be done to the two paths simply by extending the analysis to some particular data or areas. Changing the time-horizon of the empirical data or selecting a different security would check the robustness of the work. Challenging tasks for future research would be to use the framework to:

1. Extend the analysis towards other asset classes like bonds, plain-vanilla options, variance contracts, or pension funds;

2. Quantify the magnitude of the tails of the asset returns and evaluate, f.i. the joint tails over the typical ten-day horizon that is employed in Value-at-Risk (VaR) calculations

3. Derive the optimal composition of an investor's portfolio or price much more sophisticated contingent claims.

Other extensions, instead, are specific, because they depend on the structure of the processes. The next subsections propose alternative and relative routes that must be taken into account. For the first path, i.e. possible departure from Lévy process by using a "not-necessarily" Markovian stochastic process, a different solution to approximating the true process with a family of fictitious processes would consist in modifying the structure of the process and its main properties. The class of Poisson random measure, focal point of our second path, might be generalized by:

- modelling discontinuities through alternative additive processes, defined with respect to different compensated Poisson measures

- extending the work to a multi-dimensional setting in order to capture cross-effects.

5.2.1. First path: departure from Lévy processes

The following stochastic differential equation has been considered, under the historical probability, to model the stock's dynamics

$$\frac{dS_t}{S_t} = \mu(t)\, dt + \sigma(t)\, dZ_t$$

where Z_t could have been a generic Gaussian process X_t, or a linear Fractional Stable motion $\Delta_{H,\alpha}(t)$. The process, not necessarily Markovian, has been then approximated by a family of semimartingales. The most immediate extension would be to enlarge the analysis into a multi-dimensional setting, taking into the account many difficulties arising in a $N-$dimensional scenario. An alternative way, much more fascinating, would be, instead, to consider a specific diffusive coefficient, most likely a random function, such that the product between this specific-form coefficient and the innovation term would be a martingale. Its existence depends on the structure of the stochastic processes, andmight be really hard to derive it, when dealing with infinite-variance process. The scenario becomes much easier in case of a Gaussian process, especially when the random path has an integral representation with a "nice" kernel function. The way has been suggested by Norros, when defining a fundamental martingale M for a fractional Brownian motion. The intent of next subsection is to provide some intuition for deriving a fundamental martingale for a generic Gaussian process.

5.2.1.1. Fundamental martingale M

The Gaussian process X_t is characterized by having a variance-covariance function $R(t,s)$. Assume that X_t is differentiable and denote the integral operator

$$\Gamma f(t) = \int_R f(s) \frac{\partial^2 R(t,s)}{\partial t \partial s} ds$$

By using the central Hilbert spaces isomorphism in the integration of Gaussian process, the usual inner product is defined as

$$\begin{aligned} \langle\langle f, g \rangle\rangle_\Gamma &= \langle\langle f, \Gamma g \rangle\rangle = E\left(\int_R f(s) dX_s \int_R g(t) dX_t \right) \\ &= \int\int_{R\times R} f(s) g(t) d^2 R(t,s) \\ &= \int\int_{R\times R} f(s) g(t) \frac{\partial^2 R(t,s)}{\partial t \partial s} ds dt \end{aligned}$$

A generic Gaussian process is not a semimartingale unless it is a Brownian motion. Its paths are continuous with locally unbounded variation. The fundamental martingale M_t is defined to be the (centred) Gaussian process

$$M_t = \int_0^t w\left(t,s\right) dX_s \tag{5.1}$$

with independent increments and variance function

$$EM_t^2 = f(t) \tag{5.2}$$

It turns out that the process M_t satisfies condition (5.2) if and only if there exists a function $w\left(t,s\right)$ such that

$$\Gamma w\left(t,s\right) = \begin{cases} 1 & \text{for } s \in [0,t) \\ k \times \int_0^t h\left(s,u\right) du & \text{for } s > t \end{cases}$$

Let $\Gamma w\left(t,s\right)$ be defined as previously, then for $s < t$

$$\begin{aligned} Cov(M_s, M_t) &= \langle\langle w(s,\cdot), \Gamma w\left(t,\cdot\right)\rangle\rangle \\ &= \left\langle\left\langle w(s,\cdot), 1_{[0,t)}(s)\right\rangle\right\rangle \\ &= \int_0^s w(s,u)du \\ &= f(s) \end{aligned}$$

Vice versa, let $M_t = \int_0^t w(t,s)\, dX_s$, and $s < t$, be a centred Gaussian process.

$$
\begin{aligned}
Cov(M_s, M_t) &= E\left(\int_R w(s,u)dX_u \int_R w(t,u)dX_u\right) \\
&= \int_0^s \int_0^t w(s,u)w(t,u)\frac{\partial^2 R(t,s)}{\partial t \partial s} dsdt
\end{aligned}
$$

The independent increment property and the normality imply that the $Cov(M_s, M_t)$ is only function of s. Then

$$
f(s) = \int_0^s \int_0^t w(s,u)w(t,u)\frac{\partial^2 R(t,s)}{\partial t \partial s} dsdt
$$

is possible if $\int_0^t w(t,u)\frac{\partial^2 R(t,s)}{\partial t \partial s} dt = 1$, i.e. $\Gamma w(t,\cdot) = 1$ for $u \in [0,t)$. The existence and the expression of $w(t,s)$ is strictly connected to the structure of the variance-covariance function $R(t,s)$ and thus to the kernel. Norros has derived the random function $w(t,s)$ such that

$$
\begin{aligned}
M_t &= \int_0^t w(t,s)\, dX_s \\
&= \int_0^t w(t,s)\, dB_s^H
\end{aligned}
$$

is a fundamental martingale, when it is represented as an integral form with respect to the fractional Brownian motion B_s^H. The random function should have the following expression

$$w\left(t,s\right)=\frac{1}{2HB\left(\frac{3}{2}-H,H+\frac{1}{2}\right)}s^{\frac{1}{2}-H}\left(t-s\right)^{\frac{1}{2}-H}1_{(0,t)}\left(s\right)$$

where $B\left(\frac{3}{2}-H,H+\frac{1}{2}\right)$ is the beta function.[32]

5.2.2. Second path: Poisson random paths

The second path has been to consider the following stochastic differential equation, under the historical probability, to model the stock's dynamics

$$\frac{dS_t}{S_t}=\mu\left(t\right)dt+dX_t$$

where X_t is an additive process, defined as an integral from with respect to a compensated Poisson random measure. A financial world with jumps similar to earthquakes has been considered, and a closed-form expression for the stock proposed, by imposing the compensated Poisson measure to belong to the class of self-exciting point processes. This particular class is an extension of temporal Poisson processes, being dependent on the past event history and possessing after-effects of previous point events. New technologies, globalization raised up the in-

[32] This subsection has the intent of "only" suggesting the approach that must be followed. Other theoretical conditions must be satisfied in order to claim M_t is a fundamental martingale: the most importan is about the filtration generated by the process. Norros proves all of them for the fractional Brownian motion, and suggest possible applications of M_t: the maximum-likelihood estimation (MLE) theory; the Black-Scholes market.

terdependence among the financial markets. Unexpected events could be not self-excited and restricted to a specific segment; the magnitude of jumps might be along time and space. Departing from a seismic-quiescence scenario would be then a crucial point in extending the work.

5.2.2.1. Different additive processes

From a theoretical and computational point of view, extending the procedure to an integral form with respect to a compensated measure does not introduce too much complexity, as long as our integral form satisfies the conditions imposed in Section 4.3., i.e. being an element of a specific group: an epydemic-type aftershock sequence, for instance, has been suggested at the end of the chapter as alternative point process. Difficulties might arise when it goes to estimating the parameters. Due to the fact that neither the stochastic volatilities nor the jumps and jump intensities are directly observable, the corresponding inference problem is particularly challenging, and some restrictions should be imposed. Assume that the probability of a jump occurring is function of a set of parameter θ and some covariates $x = \left(x^{(1)}, x^{(2)}, ..., x^{(N)}\right)$, i.e.

$$\Pr\left(\Delta N_k|\theta, x^{(1)}, x^{(2)}, ..., x^{(N)}\right) = f\left(\Delta N_k, x^{(1)}, x^{(2)}, ..., x^{(N)}\right)$$

It is possible to restrict the relationship between λ_k, θ,and $x^{(i)}$ to follow the GLM framework for a Poisson distribution. This means simply the following

$$\log(\lambda_k) = \beta_0 + \sum_{j=1}^{J} \alpha_j f_j \left(x \left(k \right) \right)$$

where $\theta = \{\beta_0, \alpha_1, \alpha_2 ..., \alpha_j\}$, $x(k) = \left\{ x^{(1)}(k), x^{(2)}(k), ..., x^{(N)}(k) \right\}$, $f = \{f_1, f_2, ..., f_j\}$ is a set of general functions of the covariates. Note the log of λ_k is linear in θ, and that the functions can be arbitrary nonlinear function. This makes the GLM model class quite broad. In addition, if different types of covariates (e.g. extrinsic vs intrinsic) independently impact the firing probability, then the GLM can model these separately as follows:

$$\log(\lambda_k) = \beta_0 + \sum_{j=1}^{J} \alpha_j f_j \left(extrinsic \left(k \right) \right) + \sum_{m=1}^{M} \gamma_m g_m \left(intrisic \left(k \right) \right)$$

5.2.2.2. Contagion and cross excitation

Claessens, Dornbusch, and Park (2000), used the following sentence to define contagion: "The causes of contagion can be divided conceptually into two categories. The first category emphasizes spillover from the normal interdependence among market economies. This interdependence means that shocks, whether of a global or a local nature, can be transmitted across countries because of their real and financial linkage. [...] The second category involves a financial crisis that is not linked to observed changes in macroeconomic or other fundamentals but is solely the result of behavior of investors or other financial agents. Under this definition, contagion arises when a co-movement occurs, even when there are no global shocks and interdependence and fundamentals are not factors.[...] This type of contagion is often said to be caused by "irrational" phenomena, such as financial panics, hear behavior, loss of confidence and increased risk aversion". Imagine the following scenario:

1. Contagion can (but need not) be triggered by an initial shock. This may be a reduction in global economic growth, a change in commodity prices, or a change in interest rates or currency values. Any of these shocks can lead to increased co-movement in capital flows and asset prices.

2. Through a fundamental transmission mechanism, other institutions in the system are affected in the aftermath of the initial shocks. Shocks may also be transmitted without any obvious linkages, e.g. due to investor irrationality. Investor behavior may be individually rational ex-ante, but can lead to excessive co-movement in market prices, in the sense that market prices are not explained by real fundamentals.

3. Whether or not propagated through market linkages, systemic risk goes along with a change in the (perceived) risk-return trade-off in the economy. This implies that contagion has a time dimension, the risk of adverse events in the financial markets is increased for a certain time period: systemic risk, stated differently in terms of the asset allocations and asset pricing literature, influences the investment opportunity set.

4. Contagion does not only manifest in the degree of risk in the economy, but may also be accompanied by a large amount of uncertainty. Market participants do not only fear subsequent losses in some asset prices, but there is also a large dispersion about the magnitude of a current crisis. Investors may have different beliefs about future growth rates in the economy so that very pessimistic traders seem to behave irrationally from the point of view of rather optimistic traders.

5. The link between time series of stock prices and systemic risk is worked out properly, by looking at probability of jumps occurring in the same region. Due to region interdependence, the jump in that specific region might affect the intensity of jumps occurring in other region (cross-sectional dependence).

A suitable stochastic model for asset return dynamics should capture the cross-sectional and serial dependence observed across stock markets around the world, in such a way that the type of jump clustering is observable empirically. Mutually jumps should be employed to capture discontinuities, caused by specific crises, while the remaining components are there to represent the evolution of the asset returns in normal times. Mutually exciting jump process, like the Hawkes processes, for instance, might be natural candidates for modelling the case where a jump in a specific segment increases the intensity of jumps occurring both in the same segment (self-excitation) as well as in other areas (cross-excitation).

References

[1] Brillinger, D. R., "Maximum likelihood analysis of spike trains of interacting nerve cells", *Biological Cybernetics*, 59 (1988), 189–200.

[2] Brown, E. N., Barbieri, R., Eden, U. T. and L. M. Frank, "Likelihood methods for neural data analysis", *in Feng, J.*, *"Computational neuroscience: A comprehensive approach"*, CRC Press, London, (2003), Chapter 9, pp 253-286.

[3] Brown, E. N., Frank, L. M., Tang, D., Quirk, M. C., and M. A.Wilson, "A statistical paradigm for neural spike train decoding applied to position prediction from ensemble firing patterns of rat hippocampal place cells", *Journal of Neuroscience*, 18 (1988), 7411-7425.

[4] Claessens, S., Dornbusch, R., and Y. Park, "Contagion: Understanding how it spreads", *The World Bank Researcher Observer*, 15 (2) (2000).177-197.

[5] McCullagh, P., (1989) *"Generalized linear models",* Chapman and Hall, London (1989).

[6] Norros, I., Valkeila, E., and J. Virtamo,"An elementary approach to a Girsanov formula and other analytical results on fractional Brownian motions", *Bernoulli*, 5, 4 (1999), 571-587.

[7] Truccolo, W., Eden U. T., Fellow, M. R., Donoghue, J.P. and E.N. Brown, "A point process framework for relating neural spiking activity for spiking history, neural ensemble and extrinsic covariate effects", *Journal of Neurophysiology*, 93 (2005), 1074-1089.

[8] Veen, A., and F. Schoenberg, "Estimation of spacetime branching process models in seismology using an EM-type algorithm", *Journal of the American Statistical Association*, 103 (482) (2008), 614-624.

References

[1] Ait-Sahalia, Y., “Maximum-Likelihood Estimation of Discretely-Sampled Diffusions: A Closed-Form Approximation Approach”, *Econometrica*, 70 (2002), 223–262.

[2] Albeverio, S., and B. Rudinger, "The Lévy-Ito decomposition theorem and stochastic integrals, on separable Banach spaces", *Stochastic Analysis and Application*, 22, 5 (2004), 217–253.

[3] Alos, E., Mazet, O., and D. Nualart, "Stochastic calculus with respect to Gaussian processes", *Annals of Probability*, 29 (2001), 766-801.

[4] Alparslan, U. T., and G. Samorodnitsky, "Ruin probability with certain stationary stable claims generated by conservative flows", *Advances in Applied Probabability*, 39, 2 (2007), 360-384.

[5] Ayache, A., and J. L. Vehel, "On the identification of the pointwise Holder exponent of the generalized multifractional Brownian motion", *Stochastic Processes and their Applications,* 111, 1 (2004) 119-156.

[6] Bacry, E., Dayri, K. and J. F. Muzy, "Nonparametric kernel estimation for symmetric Hawkes process. Application to high frequency financial data", *Journal of Econometrics*, 66 (1995), 251-87.

[7] Barndorf-Nielsen O. E., "Processes of Normal Inverse Gaussian Type", *Finance and Stochastics*, 2 (1997), 41-68.

[8] Bass, R. F., "Stochastic differential equations driven by symmetric stable processes*", Séminaire de Probabilités, 36, Springer, New York, (2003), 302–313*.

[9] Bass, R. F., "Stochastic differential equations with jumps", *Probability Surveys*, 1 (2004), 1.19.

[10] Bass, R. F., "Stochastic Calculus for Discontinuous Process", *Lectures Notes, Department of Mathematics, University of Connecticut, October 1998.*

[11] Bates, D. S., "Post Crash Moneyness Bias in S&P 500 Futures Options", *Rodney L. White Center working paper* (1991), Wharton School, University of Pennsylvania, Philadelphia, PA.

[12] Bauwens, L., and N. Hautsch, *"Modelling Financial High Frequency Data using Point Processes",* Handbook of Financial Time Series, Springer, New York (2009), Part 6, 953-979.

[13] Billingsley, P., *"Convergence of Probability Measures",* New York, Wiley, (1999).

[14] Black, F., and M. S. Scholes, "The pricing of Options and Corporate Liabilities", *Journal of Political Economy*, 81 (1973), 637-654.

[15] Bojdecki, T., Gorostiza, L.G., and A.Talarczyk, "Sub-fractional Brownian motion and its relation to occupation times", .Statistics & Probability Letters, 69 (2004), 405–419.

[16] Bowsher, C. G., "Modelling security market events in continuous time: Intensity based, multivariate point process models", *Journal of Econometrics*, 141 (2007), 876-912.

[17] Brillinger, D. R., "Some statistical methods for random process data from seismology and neurophysiology", *The Annals of Statistics,* (1988), 16, 1, 1-54.

[18] Brillinger, D. R., "Maximum likelihood analysis of spike trains of interacting nerve cells", *Biological Cybernetics*, 59 (1988), 189–200.

[19] Brown, E. N., Barbieri, R., Eden, U. T. and L. M. Frank, "Likelihood methods for neural data analysis", *in Feng, J.*, *"Computational neuroscience: A comprehensive approach"*, CRC Press, London, (2003), Chapter 9, pp 253-286.

[20] Brown, E. N., Frank, L. M., Tang, D., Quirk, M. C., and M. A.Wilson, "A statistical paradigm for neural spike train decoding applied to position prediction from ensemble firing patterns of rat hippocampal place cells", *Journal of Neuroscience*, 18 (1988), 7411-7425.

[21] Cambanis, S., and M. Maejima, "Two classes of self-similar stable processes with stationary increments", *Stochastic Processes and their Application*, 32 (1989), 305–329.

[22] Carr, P., and D. Madan, "Option valuation using the fast Fourier transform", *Journal of Computational Finance*, 2 (1999), 61–73.

[23] Carr, P., and L. Wu, "Finite moment log stable process and option pricing", *Journal of Finance*, 58 (2003) , 753–777.

[24] Carr, P., and L. Wu,"Time-changed Lévy processes and option pricing", *Journal of Financial Economics*, 71 (2004), 113–141.

[25] Carr, P., and L. Wu, "Stochastic skew in currency options", *Journal of Financial Economics*, 86, 1 (2007), 213-247.

[26] Carr, P., Geman, H., Madan, D., and M. Yor, "The fine structure of asset returns: an empirical investigation", *Journal of Business*, 75 (2002), 305–332.

[27] Claessens, S., Dornbusch, R., and Y. Park, "Contagion: Understanding how it spreads", *The World Bank Researcher Observer*, 15 (2) (2000).177-197.

[28] Cochrane, J., "Production-based asset pricing and the link between stock returns and economic fluctuations", *Journal of Finance*, 46 (1991), 209–237.

[29] Cochrane, J., *"Asset Pricing",* Princeton University Press, Princeton, NJ, (2004).

[30] Cont, R., "*Long range dependence in financial time series"*, Fractals in Engineering, Springer, (2005).

[31] Duffie, D., Pan, J., and K. Singleton, "Transform analysis and asset pricing for affine jump-diffusions", *Econometrica*, 68 (2000), 1343–1376.

[32] Eberlein, E., and U. Keller, "Hyperbolic distributions in Finance", *Bernoulli*, 1 (1995), 281-299.

[33] Eberlein, E., and U. Keller, and K. Prause "New insights into smile, mispricing and value at risk: the hyperbolic model", *Journal of Business*, 71 (1998), 371-405.

[34] Eberlein, E., and K. Prause, "The generalized hyperbolic model: Financial derivatives and risk measures", *FDM preprint*, 56, (1998) University of Freiburg.

[35] Hawkes, A. G., "Spectra of some self-exciting and mutually exciting point process". *Biometrika*, 58 (1971), 83-90.

[36] Hawkes, A. G., "Point Spectra of Some Mutually exciting point process", *Journal of the Royal Statistical Society*, 33 (1971) 438-443.

[37] Heston, S., "A Closed-Form Solution of Options with Stochastic Volatility with Applications to Bond and Currency Options", *The Review of Financial Studies*, 6 (1993), 327-343.

[38] Hida, T., "Canonical representations of Gaussian processes and their applications", *Memoirs of the College of Science*, *University of Kyoto, Series A,* 33.(1960),109-155.

[39] Hida, T., and M. Hitsuda, *"Gaussian Processes",* American Mathematical Society, (1993).

[40] Kahneman, D., and A. Tversky, "Prospect Theory: An Analysis of Decision Under Risk", *Econometrica*, 47 (1979), 263-291.

[41] Kahneman, D., Slovic, P. and A. Tversky, *"Judgment Under Uncertainty: Heuristics and Biases",* New York: Cambridge University Press, (1982).

[42] Kahneman, D., and A. Tversky, "Choices, Values and Frames", *American Psychologist*, 39 (1984), 341-350.

[43] Kwiecinski, A., and R. Szekli, "Some Monotonicity and Dependence Properties of Self-Exciting Point Processes", *The Annals of Applied Probability*, 6 (1996), 1211–1231.

[44] Madan, D. B., and F. Milne, "Option Pricing with VG Martingale Components", *Mathematical Finance*, 1 (1991), 39-55.

[45] Madan, D. B., and E. Seneta, "The Variance Gamma (V.G.) Model for Share Market Returns", *Journal of Business*, 63 (1990), 4, 511-524.

[46] McCullagh, P., (1989) *"Generalized linear models",* Chapman and Hall, London (1989).

[47] McCulloch, J. H., "The Risk-Neutral Measure and Option Pricing under Log-Stable distribution", *Technical Report* (2003), Department of Economics, Ohio State University.

[48] Meyer, P. A., "Un Cours sur les Intégrales Stochastiques"*, Séminaire de Probabilités, X, Lecture Notes 511, Springer Verlag, (1976).*

[49] Moller, J., and J. K. Rasmussen "Perfect Simulation of Hawkes Processes", *Advances in Applied Probability*, 37 (2005), 629-646.

[50] Norros, I., Valkeila, E., and J. Virtamo,"An elementary approach to a Girsanov formula and other analytical results on fractional Brownian motions", *Bernoulli*, 5, 4 (1999), 571-587.

[51] Ogata, Y., "Statistical Models for Earthquake Occurrence and Residual Analysis for Point Processes", *Journal of the American Statistical Association*, 83 (1988), 9-27.

[52] Rosinski, J., "On uniqueness of the spectral representation of stable processes", *Journal of Theoretical Probabilities*, 7(3) (1994) 615-634.

[53] Rosinski, J., "On the structure of stationary stable processes", *The Annals of Probability*, 23, (1995) 1163-1187.

[54] Rudiger, B., "Stochastic integration with respect to compensated Poisson random measures on separable Banach spaces", *Stochastics and Stochastics Reports*, 76, 3, (2004), 213-242.

[55] Samorodnitsky, G., and M. S. Taqqu, *"Stable Non-Gaussian Random Process: Stochastic Models with Infinite Variance", C*hapman and Hall, New York, (1994).

[56] Sato, K., *"$Lévy$ Processes and Infinitely Divisible Distributions",* Cambridge University Press, (1999*).*

[57] Schoenberg, F. P., Brillinger, D. R., and P. Guttorp, *"Point process, Spatial-temporal",* Encyclopedia of Environmetrics, Wiley, New York (2002), Vol.3, 1573–1577.

[58] Schoenberg, F. P., "Transforming spatial point processes into Poisson processes", *Stochastic Processes and their Application*, 81(2) (1999), 155-164.

[59] Skorohod, A. V, "On a generalization of a stochastic integral", *Theory of Probability and its Application*, 20 (1975), 219-233.

[60] Truccolo, W., Eden U. T., Fellow, M. R., Donoghue, J.P. and E.N. Brown, "A point process framework for relating neural spiking activity for spiking history, neural ensemble and extrinsic covariate effects", *Journal of Neurophysiology*, 93 (2005), 1074-1089.

[61] Vaga, T., "The coherent market hypothesis", *Financial Analysts Journal*, November–December (1990), 36–49.

[62] Veen, A., and F. Schoenberg, "Estimation of spacetime branching process models in seismology using an EM-type algorithm", *Journal of the American Statistical Association*, 103 (482) (2008), 614-624.

[63] Weron, A., and R. Weron, "Fractal market hypothesis and two power-laws", *Chaos, Solitons and Fractals,* 11 (2000), 289-296.

[64] Wolpert, R. L. and K. Ickstadt, "Poisson/gamma random field models for spatial statistics", *Biometrika*, 85, 2 (1998), 251-267.