# ABSTRACT ACTION SPACES AND THEIR TOPOLOGICAL AND DYNAMIC PROPERTIES 

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#### Abstract

We introduce the concept of action space, a set $\boldsymbol{X}$ endowed with an action cost a : $(0,+\infty) \times \boldsymbol{X} \times \boldsymbol{X} \rightarrow[0,+\infty)$ satisfying suitable axioms, which turn out to provide a 'dynamic' generalization of the classical notion of metric space. Action costs naturally arise as dissipation terms featuring in the Minimizing Movement scheme for gradient flows, which can then be settled in general action spaces.

As in the case of metric spaces, we will show that action costs induce an intrinsic topological and metric structure on $\boldsymbol{X}$. Moreover, we introduce the related action functional on paths in $\boldsymbol{X}$, investigate the properties of curves of finite action, and discuss their absolute continuity. Finally, under a condition akin to the approximate mid-point property for metric spaces, we provide a dynamic interpretation of action costs.


Dedicated to Pierluigi Colli on the occasion of his 65th birthday

## 1. Introduction

Since the pioneering work by E. De Giorgi DG93], which was inspired the approach by Almgren, Taylor and Wang to the mean curvature and other geometric flows, Minimizing Movements have become a paradigmatic tool for constructing solutions to a large class of evolutionary problems.

In its full generality, the Minimizing Movement scheme consists in finding, for a given time interval $[0, T]$ and a time step $\tau>0$ inducing the uniform grid $0<\tau<2 \tau<\cdots<N_{\tau} \tau$, with $N_{\tau} \in \mathbb{N}$ such that $\left(N_{\tau}-1\right) \tau<T \leq N_{\tau} \tau$, discrete solutions $\left(U_{\tau}^{n}\right)_{n=0}^{N_{\tau}}$ in some topological space $\boldsymbol{X}$, as solutions of the following recursive family of minimum problems

$$
\begin{equation*}
U_{\tau}^{n} \in \underset{V \in \boldsymbol{X}}{\operatorname{argmin}} \mathscr{F}\left(\tau, n, U_{\tau}^{n-1} ; V\right), \quad n=1, \ldots, N_{\tau}, \tag{1.1}
\end{equation*}
$$

with $U_{\tau}^{0}=u_{0}$ a given initial datum, where $\mathscr{F}:(0,+\infty) \times \mathbb{N} \times \boldsymbol{X} \times \boldsymbol{X} \rightarrow \mathbb{R} \cup\{+\infty\}$ is a suitable functional. Minimizing Movements are limits as $\tau \downarrow 0$ of the piecewise constant interpolations $U_{\tau}$ of the values $U_{\tau}^{n}$, viz. $U_{\tau}:[0, T] \rightarrow \boldsymbol{X}, U_{\tau}(t):=U_{\tau}^{n}$ if $t \in((n-1) \tau, n \tau]$.

A particularly significant example arises in the case of gradient flows of a time-dependent functional $\mathscr{E}:[0, T] \times \boldsymbol{X} \rightarrow \mathbb{R} \cup\{+\infty\}$, with respect to a metric $d$ on $\boldsymbol{X}$ : the related Minimizing Movement scheme corresponds to a functional $\mathscr{F}$ of the form

$$
\begin{equation*}
\mathscr{F}(\tau, n, U ; V):=\frac{1}{2 \tau} d^{2}(U, V)+\mathscr{E}(n \tau, V) . \tag{1.2}
\end{equation*}
$$

Under suitable assumptions on $\mathscr{E}$, the approximate solutions defined by interpolation of the discrete values $\left(U_{\tau}^{n}\right)_{n=0}^{N_{\tau}}$ converge to a curve of maximal slope Amb95, AGS08, RMS08].

[^0]In the applications, gradient flows model processes whose temporal evolution results from the trade-off of energy conservation and energy dissipation. Dissipative mechanisms are then encoded in the metric $d$. In the general case, dissipation may be mathematically modelled by functionals of the type $\psi(d)$, with $\psi:[0,+\infty) \rightarrow[0,+\infty)$ a convex function null at 0 (the gradient-flow case corresponding to $\left.\psi(r)=\frac{1}{2} r^{2}\right)$. The corresponding functional generating the Minimizing Movement scheme, i.e.

$$
\begin{equation*}
\mathscr{F}(\tau, n, U ; V):=\tau \psi\left(\frac{d(U, V)}{\tau}\right)+\mathscr{E}(n \tau, V), \tag{1.3}
\end{equation*}
$$

for a general (convex) $\psi$ with superlinear growth at infinity, has been tackled in RMS08 (cf. also [CRZ09]). The linear-growth case $\psi(r)=r$ falls into the realm of rate-independent evolution MM05, MR15].

It is then natural (see PRST22) to study more general MM-functionals $\mathscr{F}$ of the form

$$
\begin{equation*}
\mathscr{F}(\tau, n, U ; V):=\mathrm{a}(\tau, U, V)+\mathscr{E}(n \tau, V), \tag{1.4}
\end{equation*}
$$

where a : $(0,+\infty) \times \boldsymbol{X} \times \boldsymbol{X} \rightarrow[0,+\infty)$ can be interpreted as a sort of action functional, measuring the cost for moving from the point $U$ to the point $V$ in the amount of time $\tau>0$. The structure (1.4) still preserves the natural splitting between a driving energy functional $\mathscr{E}$ and a metric-like dissipation functional a, which however is not derived from a given metric $d$ on $\boldsymbol{X}$.

The structural property which takes into account the heuristic interpretation of $a$ is the concatenation inequality

$$
\begin{equation*}
\mathrm{a}\left(\tau_{1}+\tau_{2}, u_{1}, u_{3}\right) \leq \mathrm{a}\left(\tau_{1}, u_{1}, u_{2}\right)+\mathrm{a}\left(\tau_{2}, u_{2}, u_{3}\right) \quad u_{i} \in \boldsymbol{X}, \tau_{i}>0, \tag{1.5}
\end{equation*}
$$

which may also be interpreted as a dynamic version of the triangle inequality for a metric. Still inspired by the axioms of metrics, we will focus on actions that vanish only on the diagonal of $\boldsymbol{X} \times \boldsymbol{X}$ and are symmetric, i.e. for all $\tau>0, u_{0}, u_{1} \in \boldsymbol{X}$,

$$
\begin{gather*}
\mathrm{a}\left(\tau, u_{0}, u_{1}\right) \geq 0 ; \quad \mathrm{a}\left(\tau, u_{0}, u_{1}\right)=0 \Leftrightarrow u_{0}=u_{1},  \tag{1.6}\\
\mathrm{a}\left(\tau, u_{1}, u_{2}\right)=\mathrm{a}\left(\tau, u_{2}, u_{1}\right) . \tag{1.7}
\end{gather*}
$$

Hereafter, we will term action cost any function complying with (1.5), (1.6), \& (1.7) and action space a pair $(\boldsymbol{X}, \mathrm{a})$ given by a set $\boldsymbol{X}$ and an action cost a. Under further compatibility conditions between a and the driving energy functional $\mathscr{E}$, in PRST22 it has been shown that the approximate solutions arising from the Minimizing Movement scheme generated by the functional $\mathscr{F}$ from (1.4) converge to a curve fulfilling a suitable Energy-Dissipation (in)equality that in fact generalizes the metric formulation of gradient flows. In particular, the Minimizing Movement scheme (1.4) can be set up in the general framework of action spaces, in which a is not induced by an underlying metric on the ambient space $\boldsymbol{X}$.

It is easy to check that properties (1.5), (1.6), \& (1.7) are satisfied by all functions of the form $\mathrm{a}(\tau, u, v)=\tau \psi\left(\frac{1}{\tau} d(u, v)\right)$, where $\psi:[0,+\infty) \rightarrow[0,+\infty)$ is a convex function vanishing only at 0 . The class of action costs, however, is much larger and includes diverse functionals. Paradigmatic examples are provided by costs arising from the minimization of a suitable action functional: when $\boldsymbol{X}=\mathbb{R}^{d}$ we can define

$$
\begin{equation*}
\mathrm{a}(\tau, u, v):=\inf \left\{\int_{0}^{\tau} \mathcal{R}\left(\Theta(r), \Theta^{\prime}(r)\right) \mathrm{d} r: \Theta \in \mathrm{AC}\left([0, \tau] ; \mathbb{R}^{d}\right) \Theta(0)=u, \Theta(\tau)=v\right\} \tag{1.8}
\end{equation*}
$$

with $\mathcal{R}: \mathbb{R}^{d} \times \mathbb{R}^{d} \rightarrow[0,+\infty)$ such that $\mathcal{R}(\Theta, \cdot)$ is convex, superlinear and vanishes only at 0 (cf. Sec. 2 for all details). We emphasize that, while Riemannian-Finsler metrics are defined
by infimizing an action integral which is positively 1-homogeneous with respect to the velocity variable, here we allow for general convex integrands.

The other main motivating example is provided by the so-called Dynamical-Variational Transport (DVT) costs, which generalize transport distances between measures. Their definition in PRST22 has been indeed inspired by the well-known dynamic reformulation of the Wasserstein distance $W_{2}$ advanced by Benamou \& Brenier BB00, cf. also DNS09, Maa11, Mie13. Another notable and inspiring construction for the present paper has been proposed in [FGY11], where the authors extended the Minimizing Movement scheme, and De Giorgi's interpolation techniques [DG93, Amb95], to carry out a variational analysis of PDEs that are not gradient flows but, still, possess an entropy functional and an underlying Lagrangian. It is in terms of this Lagrangian, which also depends on the spatial variable, that they defined an action integral and, ultimately, an action cost for the Minimizing Movement scheme.

Now, in the same way as the 'action integral cost' a (1.8) generalizes the standard construction of Riemann-Finsler metrics, so do DVT costs extend transport distances between measures. In fact, given two positive finite measures $\mu_{0}, \mu_{1}$ on $\mathbb{R}^{d}$ (for simplicity; more general state spaces have been considered in PRST22]), the cost for connecting them over a certain interval $[0, \tau]$ is defined by minimizing a suitable action integral over curves of measures joining $\mu_{0}$ to $\mu_{1}$ and solving the continuity equation on $(0, \tau)$ with flux $\boldsymbol{j}$ :

$$
\begin{equation*}
\mathrm{a}\left(\tau, \mu_{0}, \mu_{1}\right):=\inf \left\{\int_{0}^{\tau} \mathscr{R}(\rho(r), \boldsymbol{j}(r)) \mathrm{d} r:(\rho, \boldsymbol{j}) \in \mathcal{C} \mathcal{E}(0, \tau), \rho(0)=\mu_{0}, \rho(1)=\mu_{1}\right\} \tag{1.9}
\end{equation*}
$$

(where $\mathcal{C} \mathcal{E}(0, \tau)$ denotes the family of solutions to the continuity equation on $(0, \tau)$ ).
Because of the flexibility and frequent occurrence of action structures in the variational approach to evolutionary problems, we believe that action spaces deserve to be studied in their own right: like in the case of metric spaces, we will show that they induce a natural (metrizable) topology, an intrinsic notion of completeness, and canonical action functionals on $\boldsymbol{X}$-valued paths.

Plan of the paper. In this note we develop a systematic analysis of a space $\boldsymbol{X}$ endowed with an action cost a. We introduce the main definitions with relevant examples in Section 2. In Section 3 we will show that an action cost a induces a canonical topology $\mathfrak{D}$ on $\boldsymbol{X}$, cf. Proposition 3.1, In fact, it even generates a uniform structure on $\boldsymbol{X}$, namely a topological structure (whose precise definition is postponed to Proposition 3.3 ahead) by means of which it is possible to render the concept that two points in $\boldsymbol{X}$ are 'close' and to define an intrinsic notion of completeness. Since this uniform structure has a countable base, the associated topology $\mathfrak{O}$ is metrizable. Indeed, in Section 4 with Theorem4.5 we explicitly provide a family of equivalent metrics $d_{a, \lambda}$ metrizing the topology $\mathfrak{O}$ and inducing the uniform structure.

As shown in PRST22, the Minimizing Movement scheme (1.4) leads to limiting curves $u$ with finite a-action on $[0, T]$, i.e. such that

$$
\mathbb{A}(u ;[0, T]):=\sup \left\{\sum_{j=1}^{M} \mathrm{a}\left(t^{j}-t^{j-1}, u\left(t^{j-1}\right), u\left(t^{j}\right)\right):\left(t^{j}\right)_{j=0}^{M} \in \mathscr{P}_{f}([0, T])\right\}<+\infty
$$

(where $\mathscr{P}_{f}([0, T])$ denotes the set of all partitions of $[0, T]$ ). In Section 5 we focus on these curves and show that they indeed have BV-like properties. In particular, when ( $\boldsymbol{X}, \mathrm{a})$ is complete they are regulated in the $\mathfrak{O}$-topology, hence their jump set is well defined. We then
turn to a-absolutely continuous curves $u:[0, T] \rightarrow \boldsymbol{X}$ in Section 6, for which an action density $\mathfrak{a}\left[u^{\prime}\right]$ can be introduced, fulfilling

$$
\mathrm{a}(t-s, u(s), u(t)) \leq \int_{s}^{t} \mathfrak{a}\left[u^{\prime}\right](r) \mathrm{d} r \quad \text { for all } 0 \leq s \leq t \leq T
$$

and

$$
\mathbb{A}(u ;[0, T])=\int_{0}^{T} \mathfrak{a}\left[u^{\prime}\right](r) \mathrm{d} r .
$$

In Section 7 we provide a sufficient condition on the metric cost ensuring that all finite-action curves on $[0, T]$ are in fact a-absolutely continuous on $[0, T]$, cf. Theorem [7.3, Finally, in Section 8 we demonstrate that, under an additional condition on a which amounts to the existence of 'approximate mid-points', a dynamic characterization for a is available, cf. Thm. 8.2. Thm. 8.3 and Corollary 8.4. In this way, we somehow 'close the circle' by providing, for general costs a, a dynamic interpretation akin to (1.8) for action integral costs, and to (1.9) for Dynamical-Variational Transport costs.

This paper is dedicated to Pierluigi Colli: it is a privilege for us to have him as a valuable colleague and, more importantly, as a loyal friend.

## 2. Action spaces

In this section we introduce the main definitions we will deal with and we show some important examples.

Definition 2.1 (Action cost). We say that a function a : $(0,+\infty) \times \boldsymbol{X} \times \boldsymbol{X} \rightarrow[0,+\infty)$ is an action cost on the set $\boldsymbol{X}$ if it satisfies the following properties:
(1) Strict positivity off the diagonal: For all $\tau>0, u_{0}, u_{1} \in \boldsymbol{X}$,

$$
\begin{equation*}
\mathrm{a}\left(\tau, u_{0}, u_{1}\right)=0 \Leftrightarrow u_{0}=u_{1} . \tag{2.1a}
\end{equation*}
$$

(2) Symmetry: For every $\tau>0, u_{1}, u_{2} \in \boldsymbol{X}$

$$
\begin{equation*}
\mathrm{a}\left(\tau, u_{1}, u_{2}\right)=\mathrm{a}\left(\tau, u_{2}, u_{1}\right) \quad \text { for all } \tau \in(0,+\infty), u_{1}, u_{2} \in \boldsymbol{X} . \tag{2.1b}
\end{equation*}
$$

(3) Concatenation inequality: For all $u_{1}, u_{2}, u_{3} \in \boldsymbol{X}$ and $\tau_{1}, \tau_{2} \in(0,+\infty)$

$$
\begin{equation*}
\mathrm{a}\left(\tau_{1}+\tau_{2}, u_{1}, u_{3}\right) \leq \mathrm{a}\left(\tau_{1}, u_{1}, u_{2}\right)+\mathrm{a}\left(\tau_{2}, u_{2}, u_{3}\right) . \tag{2.1c}
\end{equation*}
$$

We call action space a pair ( $\boldsymbol{X}$, a) consisting of a set $\boldsymbol{X}$ endowed with an action cost a.
As mentioned in the Introduction, $\mathrm{a}\left(\tau, u_{0}, u_{1}\right)$ represents the cost to reach $u_{1}$ from $u_{0}$ in the amount of time $\tau>0$.

A first important consequence of (2.1a) and (2.1c) is the monotonicity property w.r.t. $\tau$ :

$$
\begin{equation*}
0<\tau^{\prime}<\tau^{\prime \prime} \quad \Rightarrow \quad \mathrm{a}\left(\tau^{\prime}, u_{0}, u_{1}\right) \geq \mathrm{a}\left(\tau^{\prime \prime}, u_{0}, u_{1}\right) \quad \text { for every } u_{0}, u_{1} \in \boldsymbol{X} \tag{2.2}
\end{equation*}
$$

In order to check (2.2) it is sufficient to notice that

$$
\mathrm{a}\left(\tau^{\prime \prime}, u_{0}, u_{1}\right) \leq \mathrm{a}\left(\tau^{\prime}, u_{0}, u_{1}\right)+\mathrm{a}\left(\tau^{\prime \prime}-\tau^{\prime}, u_{1}, u_{1}\right)=\mathrm{a}\left(\tau^{\prime}, u_{0}, u_{1}\right),
$$

so that the map $\tau \mapsto \mathrm{a}\left(\tau, u_{0}, u_{1}\right)$ is decreasing. Estimate (2.2) renders the intuitive property that the 'cost for connecting' $u_{0}$ and $u_{1}$ decreases if they are joined over a longer time interval,
and it is a consequence of the positivity of a and the fact that "staying" at the same point is costless. In particular, we can define

$$
\begin{align*}
& \mathrm{a}_{+}\left(\tau, u_{0}, u_{1}\right):=\inf _{\tau^{\prime}<\tau} \mathrm{a}\left(\tau^{\prime}, u_{0}, u_{1}\right)=\lim _{\tau^{\prime} \uparrow \tau} \mathrm{a}\left(\tau^{\prime \prime}, u_{0}, u_{1}\right),  \tag{2.3a}\\
& \mathrm{a}_{-}\left(\tau, u_{0}, u_{1}\right):=\sup _{\tau^{\prime \prime}>\tau} \mathrm{a}\left(\tau^{\prime \prime}, u_{0}, u_{1}\right)=\lim _{\tau^{\prime \prime} \downarrow \tau} \mathrm{a}\left(\tau^{\prime \prime}, u_{0}, u_{1}\right), \tag{2.3b}
\end{align*}
$$

observing that

$$
\mathrm{a}_{-}\left(\tau, u_{0}, u_{1}\right) \leq \mathrm{a}\left(\tau, u_{0}, u_{1}\right) \leq \mathrm{a}_{+}\left(\tau, u_{0}, u_{1}\right) \quad \text { for every } \tau>0, u_{0}, u_{1} \in \boldsymbol{X}
$$

It is easy to check that
Proposition 2.2. If a is an action cost on $\boldsymbol{X}$ then the functions $\mathrm{a}_{-}$and $\mathrm{a}_{+}$defined by (2.3a,b) are action costs as well.

Proof. Let us just check the concatenation inequality for $a_{+}$, as the corresponding property for $a_{-}$follows by a similar argument. For $\tau_{1}, \tau_{2}>0, u_{1}, u_{2}, u_{3} \in \boldsymbol{X}, \varepsilon \in\left(0, \tau_{1} \wedge \tau_{2}\right)$ we have

$$
\mathrm{a}\left(\tau_{1}+\tau_{2}-2 \varepsilon, u_{1}, u_{3}\right) \leq \mathrm{a}\left(\tau_{1}-\varepsilon, u_{1}, u_{2}\right)+\mathrm{a}\left(\tau_{2}-\varepsilon, u_{2}, u_{3}\right) .
$$

Passing to the limit as $\varepsilon \downarrow 0$ we obtain

$$
\mathrm{a}_{+}\left(\tau_{1}+\tau_{2}, u_{1}, u_{3}\right) \leq \mathrm{a}_{+}\left(\tau_{1}, u_{1}, u_{2}\right)+\mathrm{a}_{+}\left(\tau_{2}, u_{2}, u_{3}\right) .
$$

We also set

$$
\begin{equation*}
\mathrm{a}_{\text {sup }}\left(u_{0}, u_{1}\right):=\sup _{\tau>0} \mathrm{a}\left(\tau, u_{0}, u_{1}\right)=\lim _{\tau \downarrow 0} \mathrm{a}\left(\tau, u_{0}, u_{1}\right) \in[0,+\infty] . \tag{2.4}
\end{equation*}
$$

Definition 2.3 (Continuity and superlinearity). We say that the action cost

- a is continuous if for every $u_{0}, u_{1} \in \boldsymbol{X}$ the map $\tau \mapsto \mathrm{a}\left(\tau, u_{0}, u_{1}\right)$ is continuous. Equivalently, if

$$
\begin{equation*}
\mathrm{a}_{+}\left(\tau, u_{0}, u_{1}\right)=\mathrm{a}_{-}\left(\tau, u_{0}, u_{1}\right) \quad \text { for every } \tau>0, u_{0}, u_{1} \in \boldsymbol{X} ; \tag{2.5}
\end{equation*}
$$

- a is metric-like if $\mathrm{a}_{\text {sup }}\left(u_{0}, u_{1}\right)<+\infty$ for every $u_{0}, u_{1} \in \boldsymbol{X}$;
- a has a local superlinear growth if

$$
\begin{equation*}
\mathrm{a}_{\text {sup }}\left(u_{0}, u_{1}\right)=+\infty \quad \text { for every } u_{0}, u_{1} \in \boldsymbol{X}, u_{0} \neq u_{1} . \tag{2.6}
\end{equation*}
$$

It is immediate to check that if a is metric-like then the function $\mathrm{a}_{\text {sup }}$ is a metric in $\boldsymbol{X}$. In turn, we refer to (2.5) as a continuity property on the grounds of Proposition 4.10 ahead. In what follows, we will mostly focus on the superlinear case (2.6).
2.1. Examples. We illustrate the above definitions in some examples.

Example 2.4 (Metrics). If $d$ is a metric on $\boldsymbol{X}$ then the $\tau$-independent cost a $(\tau, u, v):=d(u, v)$ is an action cost. In fact, an action cost a is $\tau$-independent if and only if it is a metric on $\boldsymbol{X}$.

Example 2.5 (Rescaling). If b is an action cost on $\boldsymbol{X}$ and $\lambda, \theta>0$ then also the rescaled function

$$
\begin{equation*}
\mathrm{a}(\tau, u, v):=\theta \mathrm{b}(\tau / \lambda, u, v) \tag{2.7}
\end{equation*}
$$

is an action cost.

Example 2.6 (The convex construction). If b is an action cost on $\boldsymbol{X}$ and $\psi:[0,+\infty) \rightarrow$ $[0,+\infty)$ is a convex function with $0=\psi(0)<\psi(a)$ for every $a>0$, then

$$
\begin{equation*}
\mathrm{a}(\tau, u, v):=\tau \psi\left(\tau^{-1} \mathrm{~b}(\tau, u, v)\right) \quad \text { is an action cost. } \tag{2.8}
\end{equation*}
$$

It is immediate to check strict positivity and symmetry. The concatenation property follows by the convexity and the monotonicity of $\psi$ : for every $\tau_{i}>0, i=1,2$, with $\tau:=\tau_{1}+\tau_{2}$ and $\alpha_{i}:=\tau_{i} / \tau$, we have

$$
\begin{aligned}
\mathrm{a}(\tau, u, w) & =\tau \psi\left(\tau^{-1} \mathbf{b}\left(\tau_{1}+\tau_{2}, u, w\right)\right) \\
& \leq \tau \psi\left(\tau^{-1}\left(\mathbf{b}\left(\tau_{1}, u, v\right)+\mathbf{b}\left(\tau_{2}, v, w\right)\right)\right) \\
& \left.=\tau \psi\left(\alpha_{1} \tau_{1}^{-1} \mathbf{b}\left(\tau_{1}, u, v\right)+\alpha_{2} \tau_{2}^{-1} \mathbf{b}\left(\tau_{2}, v, w\right)\right)\right) \\
& \left.\leq \alpha_{1} \tau \psi\left(\tau_{1}^{-1} \mathbf{b}\left(\tau_{1}, u, v\right)\right)+\alpha_{2} \tau \psi\left(\tau_{2}^{-1} \mathbf{b}\left(\tau_{2}, v, w\right)\right)\right) \\
& =\mathrm{a}\left(\tau_{1}, u, v\right)+\mathbf{a}\left(\tau_{2}, v, w\right) .
\end{aligned}
$$

Example 2.7 (Action cost induced by a metric). Recalling the metric case of Example 2.4, as a particular case of the construction set up in Ex. [2.6 we get that the functional

$$
\begin{equation*}
\mathrm{a}(\tau, u, v)=\tau \psi\left(\frac{d(u, v)}{\tau}\right) \text {, for a given metric } d \text {, on } \boldsymbol{X} \text { is an action cost. } \tag{2.9}
\end{equation*}
$$

Concerning the properties stated in Definition [2.3, we immediately see that the action cost defined by (2.9) is continuous; moreover, setting

$$
\begin{equation*}
\psi_{\infty}^{\prime}:=\lim _{r \rightarrow+\infty} \frac{\psi(r)}{r}=\sup _{r>0} \frac{\psi(r)-\psi\left(r_{0}\right)}{r-r_{0}} \in(0,+\infty] \tag{2.10}
\end{equation*}
$$

we have two cases:
(1) If $\psi_{\infty}^{\prime}<+\infty$, then a is metric-like;
(2) If $\psi_{\infty}^{\prime}=+\infty$ (i.e. $\psi$ has superlinear growth at infinity), then a has a local superlinear growth.

Example 2.8 (Linear combination of action costs). It is immediate to check that if $\mathrm{a}_{i}, i$ running in a finite set $\mathcal{J}$, are action costs on $\boldsymbol{X}$ and $\theta_{i}>0$ are positive real numbers, then also a $:=\sum_{i} \theta_{i} \mathrm{a}_{i}$ is an action cost.

In particular, if $\boldsymbol{X}$ is endowed with two metrics $d_{1}$ and $d_{2}$ we may consider the action cost

$$
\begin{equation*}
\mathrm{a}(\tau, u, w)=\tau \psi_{1}\left(\frac{d_{1}(u, w)}{\tau}\right)+\tau \psi_{2}\left(\frac{d_{2}(u, w)}{\tau}\right) \tag{2.11}
\end{equation*}
$$

where $\psi_{i}$ are as in Example 2.7. We mention that an action cost induced by two metrics as in (2.11) occurs in the Minimizing-Movement scheme for the generalized gradient system ( $\boldsymbol{X}, d_{1}, \psi_{1}, d_{2}, \psi_{2}$ ) providing the vanishing-viscosity approximation of the rate-independent $\operatorname{system}\left(\boldsymbol{X}, d_{1}, \psi_{1}\right)$, cf. e.g. MRS09, MRS16.
Example 2.9 (Concave compositions). Let $h:(0,+\infty) \times[0,+\infty)^{I} \rightarrow[0,+\infty)$ be a concave function such that $h(\tau, 0)=0$ for every $\tau>0$ and $h(\tau, \boldsymbol{a})>0$ for every $\tau>0, \boldsymbol{a} \neq 0$. If $\mathrm{a}_{i}$, $i=1, \cdots, I$, are action costs on $\boldsymbol{X}$ then also

$$
\begin{equation*}
\mathrm{a}(\tau, u, v):=h\left(\tau, \mathrm{a}_{1}(\tau, u, v), \cdots, \mathrm{a}_{I}(\tau, u, v)\right) \quad \tau>0, u, v \in \boldsymbol{X} \tag{2.12}
\end{equation*}
$$

is an action cost. We just check the concatenation property, by using the facts that

$$
\begin{gathered}
0 \leq a_{i} \leq a_{i}^{\prime}, i=1, \cdots, I \quad \Rightarrow \quad h\left(\tau, a_{1}, \cdots, a_{I}\right) \leq h\left(\tau, a_{1}^{\prime}, \cdots, a_{I}^{\prime}\right) \\
h\left(\tau^{\prime}+\tau^{\prime \prime}, a_{1}+b_{1}, \cdots, a_{I}+b_{I}\right) \leq h\left(\tau^{\prime}, a_{1}, \cdots, a_{I}\right)+h\left(\tau^{\prime \prime}, b_{1}, \cdots, b_{I}\right) .
\end{gathered}
$$

If $\tau=\tau_{1}+\tau_{2}$ we have

$$
\begin{aligned}
\mathrm{a}(\tau, u, w) & =h\left(\tau, \mathrm{a}_{1}(\tau, u, w), \cdots, \mathrm{a}_{I}(\tau, u, w)\right) \\
& \leq h\left(\tau_{1}+\tau_{2}, \mathrm{a}_{1}\left(\tau_{1}, u, v\right)+\mathrm{a}_{1}\left(\tau_{2}, v, w\right), \cdots, \mathrm{a}_{I}\left(\tau_{1}, u, v\right)+\mathrm{a}_{I}\left(\tau_{2}, v, w\right)\right) \\
& \leq h\left(\tau_{1}, \mathrm{a}_{1}\left(\tau_{1}, u, v\right), \cdots, \mathrm{a}_{I}\left(\tau_{1}, u, v\right)\right)+h\left(\tau_{2}, \mathrm{a}_{1}\left(\tau_{2}, v, w\right), \cdots, \mathrm{a}_{I}\left(\tau_{2}, v, w\right)\right) \\
& =\mathrm{a}\left(\tau_{1}, u, v\right)+\mathrm{a}\left(\tau_{2}, v, w\right)
\end{aligned}
$$

Example 2.10 (Supremum of a directed family). Let $\mathcal{A}$ be a directed family of actions costs on $\boldsymbol{X}$, i.e. for every $\mathrm{a}_{1}, \mathrm{a}_{2} \in \mathcal{A}$ there exists $\mathrm{a} \in \mathcal{A}$ such that $\mathrm{a}_{1} \vee \mathrm{a}_{2} \leq \mathrm{a}$. If

$$
\begin{equation*}
\bar{a}:=\sup _{\mathrm{a} \in \mathcal{A}} \mathrm{a} \tag{2.13}
\end{equation*}
$$

is finite in $(0,+\infty) \times \boldsymbol{X} \times \boldsymbol{X}$, then $\overline{\bar{a}}$ is an action cost as well.
Example 2.11 (Supremum of truncated metrics). Let $d_{\lambda}, \lambda>0$, be a family of metrics on $\boldsymbol{X}$, increasing w.r.t. $\lambda$, such that $\sup _{\lambda>0} \mathrm{~d}_{\lambda}(u, v)<\infty$ for every $u, v \in \boldsymbol{X}$. Then

$$
\begin{equation*}
\mathrm{a}(\tau, u, v):=\sup _{\lambda>0} \mathrm{~d}_{\lambda}(u, v) \wedge \lambda \tau \tag{2.14}
\end{equation*}
$$

is an action cost. In fact, each term $\mathrm{a}_{\lambda}:=\mathrm{d}_{\lambda} \wedge \lambda \tau$ is an action cost, thanks Examples 2.4 and [2.9, Moreover, the set $\mathcal{A}:=\left\{\mathrm{a}_{\lambda}\right\}_{\lambda>0}$ is obviously directed: by Example 2.10 it is sufficient to prove that a is finite.
2.2. Action integral costs. Let $\boldsymbol{X}$ be, in addition, a separable and reflexive Banach space with norm $\|\cdot\|$, and let us consider an integrand $\mathcal{R}: \boldsymbol{X} \times \boldsymbol{X} \rightarrow[0,+\infty)$ which is bounded on bounded sets and fulfils the following properties:
$\left(\mathcal{R}_{1}\right)$ for all $\left(\boldsymbol{\vartheta}_{n}\right)_{n},\left(\zeta_{n}\right)_{n} \subset \boldsymbol{X}$

$$
\left.\begin{array}{rll}
\boldsymbol{\vartheta}_{n} & \rightharpoonup & \boldsymbol{\vartheta},  \tag{2.15}\\
\zeta_{n} & \rightharpoonup & \zeta
\end{array}\right\} \Rightarrow \liminf _{n \rightarrow \infty} \mathcal{R}\left(\boldsymbol{\vartheta}_{n}, \zeta_{n}\right) \geq \mathcal{R}(\boldsymbol{\vartheta}, \zeta)
$$

$\left(\mathcal{R}_{2}\right)$ for all $\boldsymbol{\vartheta} \in \boldsymbol{X}$ the functional $\mathcal{R}(\boldsymbol{\vartheta}, \cdot)$ is convex, even, and

$$
\begin{equation*}
\mathcal{R}(\boldsymbol{\vartheta}, \zeta)=0 \text { if and only if } \zeta=0 \tag{2.16}
\end{equation*}
$$

$\left(\mathcal{R}_{3}\right)$ there exists $\Phi_{\mathcal{R}}: \boldsymbol{X} \rightarrow[0,+\infty)$ with $\lim _{\|\zeta\| \uparrow+\infty} \frac{\Phi_{\mathcal{R}}(\zeta)}{\|\zeta\|}=+\infty$ such that

$$
\begin{equation*}
\forall(\boldsymbol{\vartheta}, \zeta) \in \boldsymbol{X} \times \boldsymbol{X}: \quad \mathcal{R}(\boldsymbol{\vartheta}, \zeta) \geq \Phi_{\mathcal{R}}(\zeta) \tag{2.17}
\end{equation*}
$$

For later use, we point out that property $\left(\mathcal{R}_{3}\right)$ is indeed equivalent to the existence of a convex and increasing function $\phi_{\mathcal{R}}:[0,+\infty) \rightarrow[0,+\infty)$ such that

$$
\begin{equation*}
\forall(\boldsymbol{\vartheta}, \zeta) \in \boldsymbol{X} \times \boldsymbol{X}: \quad \mathcal{R}(\boldsymbol{\vartheta}, \zeta) \geq \phi_{\mathfrak{R}}(\|\zeta\|) \tag{2.18}
\end{equation*}
$$

Relying on properties $\left(\mathcal{R}_{1}\right)-\left(\mathcal{R}_{3}\right)$ we are in a position to prove the following result.
Proposition 2.12. Let a: $(0,+\infty) \times \boldsymbol{X} \times \boldsymbol{X} \rightarrow[0,+\infty)$ be defined by

$$
\begin{equation*}
\mathrm{a}(\tau, u, v):=\inf \left\{\int_{0}^{\tau} \mathcal{R}\left(\Theta(r), \Theta^{\prime}(r)\right) \mathrm{d} r: \Theta \in \operatorname{AC}([0, \tau] ; \boldsymbol{X}), \Theta(0)=u, \Theta(\tau)=v\right\} \tag{2.19}
\end{equation*}
$$

Then, the infimum in (2.19) is attained, a is a continuous action cost, with local superlinear growth.

Proof. It is easy to check that a is an action cost in the sense of Def. 2.1. We now show that the inf in (2.19) is attained for all $(\tau, u, v) \in(0,+\infty) \times \boldsymbol{X} \times \boldsymbol{X}$. Indeed, let $\left(\Theta_{n}\right)_{n}$ be a minimizing sequence: thanks to (2.17) we have $\sup _{n} \int_{0}^{\tau} \Phi_{\mathcal{R}}\left(\Theta_{n}^{\prime}(r)\right) \mathrm{d} r \leq C$. Combining this with the fact that $\Phi_{\mathcal{R}}$ has superlinear growth, and taking into account that $\Theta_{n}(0)=u$ and $\Theta_{n}(1)=v$, we conclude that
$\left(\Theta_{n}\right)_{n}$ is bounded in $L^{\infty}(0, \tau ; \boldsymbol{X})$ and $\left(\Theta_{n}^{\prime}\right)_{n}$ is uniformly integrable in $L^{1}(0, \tau ; \boldsymbol{X})$.
Therefore, there exist a (not relabeled) subsequence and $\Theta \in A C([0, \tau] ; \boldsymbol{X})$ such that

$$
\begin{cases}\Theta_{n} \rightharpoonup \Theta & \text { in } W^{1,1}(0, \tau ; \boldsymbol{X}) \\ \Theta_{n}(t) \rightharpoonup \Theta(t) & \text { in } \boldsymbol{X} \text { for all } t \in[0, T]\end{cases}
$$

so that $\Theta$ connects $u$ to $v$. By a variant (cf. Val90, Theorem 21]) of the Ioffe Theorem we gather that

$$
\mathrm{a}(\tau, u, v)=\liminf _{n \rightarrow \infty} \int_{0}^{\tau} \mathcal{R}\left(\Theta_{n}(r), \Theta_{n}^{\prime}(r)\right) \mathrm{d} r \geq \int_{0}^{\tau} \mathcal{R}\left(\Theta(r), \Theta^{\prime}(r)\right) \mathrm{d} r
$$

In order to prove continuity of $\mathrm{a}(\cdot, u, v)$ with fixed $u, v \in \boldsymbol{X}$, let us take $\tau>0$ and a sequence $\tau_{n} \downarrow \tau$; let $\Theta_{n}$ and $\Theta$ optimal curves for $\mathrm{a}\left(\tau_{n}, u, v\right)$ and $\mathrm{a}(\tau, u, v)$, respectively. Extend $\Theta$ to a curve $\bar{\Theta}$ on $\left[0, \tau_{n}\right]$ by setting $\bar{\Theta}(t):=\Theta(\tau)=v$ for all $t \in\left(\tau, \tau_{n}\right]$. Then, $\bar{\Theta}$ is an admissible competitor for the minimum problem defining $a\left(\tau_{n}, u, v\right)$, and we thus have for all $n \in \mathbb{N}$

$$
\mathrm{a}\left(\tau_{n}, u, v\right) \leq \int_{0}^{\tau_{n}} \mathcal{R}\left(\bar{\Theta}(r), \bar{\Theta}^{\prime}(r)\right) \mathrm{d} r \stackrel{(1)}{=} \int_{0}^{\tau} \mathcal{R}\left(\bar{\Theta}(r), \bar{\Theta}^{\prime}(r)\right) \mathrm{d} r=\mathrm{a}(\tau, u, v)
$$

where (1) follows from the fact that $\bar{\Theta}^{\prime} \equiv 0$ on $\left[\tau, \tau_{n}\right]$. Therefore,

$$
\begin{equation*}
\limsup _{n \rightarrow \infty} \mathrm{a}\left(\tau_{n}, u, v\right) \leq \mathrm{a}(\tau, u, v) \tag{2.21}
\end{equation*}
$$

We now aim to show

$$
\begin{equation*}
\liminf _{n \rightarrow \infty} \mathrm{a}\left(\tau_{n}, u, v\right)=\liminf _{n \rightarrow \infty} \int_{0}^{\tau_{n}} \mathcal{R}\left(\Theta_{n}(r), \Theta_{n}^{\prime}(r)\right) \mathrm{d} r \geq \mathrm{a}(\tau, u, v) \tag{2.22}
\end{equation*}
$$

Let $\left(\Theta_{n}\right)_{n}$ be a (non-relabeled) subsequence for which the above liminf is a lim. It follows from (2.21) that for $\left(\Theta_{n}\right)_{n}$ estimates (2.20) hold. In particular, from the uniform integrability of $\left(\Theta_{n}^{\prime}\right)_{n}$ we gather that

$$
\forall \epsilon>0 \exists \bar{n} \in \mathbb{N} \forall n \geq \bar{n}:\left\|\Theta_{n}\left(\tau_{n}\right)-\Theta_{n}(\tau)\right\| \leq \int_{\tau}^{\tau_{n}}\left\|\Theta_{n}^{\prime}(r)\right\| \mathrm{d} r \leq \epsilon
$$

Choosing $\epsilon=\frac{1}{k}$, we thus extract subsequences $\left(\tau_{n_{k}}\right)_{k}$ and $\left(\Theta_{n_{k}}\right)_{k}$ such that for all $k \geq 1$

$$
\left\|\Theta_{n_{k}}\left(\tau_{n_{k}}\right)-\Theta_{n_{k}}(\tau)\right\| \leq \frac{1}{k}
$$

Now, the same compactness arguments as in the above lines apply to the sequence $\left(\left.\Theta_{n_{k}}\right|_{[0, \tau]}\right)_{k}$, yielding convergence, along a non-relabeled subsequence, to a curve $\widehat{\Theta} \in A C([0, \tau] ; \boldsymbol{X})$ connecting $u$ to $v$, since

$$
\widehat{\Theta}(\tau)=\lim _{k \rightarrow \infty} \Theta_{n_{k}}(\tau)=\lim _{k \rightarrow \infty} \Theta_{n_{k}}\left(\tau_{n_{k}}\right)=v
$$

Therefore, we gather that

$$
\mathrm{a}(\tau, u, v) \leq \int_{0}^{\tau} \mathcal{R}\left(\widehat{\Theta}(r), \widehat{\Theta}^{\prime}(r)\right) \mathrm{d} r \leq \liminf _{k \rightarrow \infty} \int_{0}^{\tau_{n_{k}}} \mathcal{R}\left(\Theta_{n_{k}}(r), \Theta_{n_{k}}^{\prime}(r)\right) \mathrm{d} r=\liminf _{n \rightarrow \infty} \mathrm{a}\left(\tau_{n}, u, v\right)
$$

and (2.22) ensues. All in all, we have shown that $\lim _{n \rightarrow \infty} \mathrm{a}\left(\tau_{n}, u, v\right)=\mathrm{a}(\tau, u, v)$ whenever $\tau_{n} \downarrow \tau$. With analogous arguments, we show that $\lim _{n \rightarrow \infty} \mathrm{a}\left(\tau_{n}, u, v\right)=\mathrm{a}(\tau, u, v)$ if $\tau_{n} \uparrow \tau$.

In order to show that

$$
\begin{equation*}
\mathrm{a}_{\text {sup }}(u, v)=+\infty \quad \text { if } u \neq v, \tag{2.23}
\end{equation*}
$$

let us observe that, for any fixed $\tau>0$, with $\Theta$ an optimal curve for a $(\tau, u, v)$, the following estimates hold:

$$
\begin{aligned}
\mathrm{a}(\tau, u, v)=\tau f_{0}^{\tau} \mathcal{R}\left(\Theta(t), \Theta^{\prime}(t)\right) \mathrm{d} t \stackrel{(1)}{\geq} \tau f_{0}^{\tau} \phi_{\mathcal{R}}\left(\left\|\Theta^{\prime}(t)\right\|\right) \mathrm{d} t & \stackrel{(2)}{\geq} \tau \phi_{\mathfrak{R}}\left(\left\|f_{0}^{\tau} \Theta^{\prime}(t) \mathrm{d} t\right\|\right) \\
& =\tau \phi_{\mathcal{R}}\left(\frac{\|v-u\|}{\tau}\right)
\end{aligned}
$$

where (1) ensues from (2.18), and (2) from Jensen's inequality, as $\phi_{\mathcal{R}}$ is convex. Property (2.23) then follows, taking into account that $\phi_{\mathcal{R}}$ has superlinear growth at infinity.

## 3. The topology induced by an action cost

Our first step is to show that an action cost a induces a natural Hausdorff topology in $\boldsymbol{X}$ satisfying the first countability axiom. We start by introducing a fundamental system of neighborhoods of every $u \in \boldsymbol{X}$ : it is the collection of sets $\mathrm{U}(u ; \tau, c)$ indexed by the real parameters $\tau, c>0$

$$
\mathrm{U}(u ; \tau, c):=\{v \in \boldsymbol{X}: \mathrm{a}(\tau, u, v)<c\} \text { for } u \in \boldsymbol{X}, \tau \in(0,+\infty), c \in(0,+\infty)
$$

Proposition 3.1. Let ( $\boldsymbol{X}$, a) be an action space according to Definition 2.1 and consider for every $u \in \boldsymbol{X}$ the family of sets

$$
\begin{equation*}
\mathscr{U}(u):=\{\mathrm{U} \subset \boldsymbol{X}: \mathrm{U} \supset \mathrm{U}(u ; \tau, c) \text { for some } \tau, c>0\} . \tag{3.1}
\end{equation*}
$$

The collection $\mathscr{U}(u)$ satisfies the axioms of a (Hausdorff) neighborhood system, i.e.
(1) If $\mathrm{U} \in \mathscr{U}(u)$ and $\mathrm{U} \subset \mathrm{U}^{\prime}$ then $\mathrm{U}^{\prime} \in \mathscr{U}(u)$;
(2) Every finite intersection of elements of $\mathscr{U}(u)$ belongs to $\mathscr{U}(u)$;
(3) The element $u$ belongs to every set of $\mathscr{U}(u)$;
(4) If $\mathrm{U} \in \mathscr{U}(u)$ then there is $\mathrm{V} \in \mathscr{U}(u)$ such that $\mathrm{U} \in \mathscr{U}(v)$ for every $v \in \mathrm{~V}$;
(5) If $u_{1} \neq u_{2}$ then there exist $\mathrm{U}_{i} \in \mathscr{U}\left(u_{i}\right), i=1,2$, such that $\mathrm{U}_{1} \cap \mathrm{U}_{2}=\varnothing$.

In particular, there exists a unique topology $\mathfrak{D}$ such that for every $u \in \boldsymbol{X} \mathscr{U}(u)$ is the system of neighborhoods in the topology $\mathfrak{O}$. Moreover, $\mathfrak{O}$ is a Hausdorff topology satisfying the first countability axiom.

Proof. Property (1): obvious by the definition of (3.1).
Property (2): if $\mathrm{U}_{n} \in \mathscr{U}(u), n=1, \cdots, N$, then we can find $\tau_{n}, c_{n}>0, n=1, \cdots N$, such that $\mathrm{U}_{n} \supset \mathrm{U}\left(u ; \tau_{n}, c_{n}\right)$. Setting $\tau:=\max _{n} \tau_{n}$ and $c_{n}:=\min _{n} c_{n}$, (2.2) shows that $\mathrm{U}(u ; \tau, c) \subset \mathrm{U}_{n}$ for every $n$, so that $\bigcap_{n=1}^{N} \mathrm{U}_{n} \in \mathscr{U}(u)$.
Property (3) is obvious since a $(\tau, u, u)=0$ for all $\tau>0$.

Property (4): since $\mathrm{U} \in \mathscr{U}(u)$ there exists $\tau, c>0$ such that $\mathrm{U}(u ; \tau, c) \subset \mathrm{U}$. We then define $\mathrm{V}:=\mathrm{U}(u ; \tau / 2, c / 2)$ and we observe that for every $v \in \mathrm{~V}$ and every $z \in \mathrm{U}(v ; \tau / 2, c / 2)$

$$
\mathrm{a}(\tau, z, u) \leq \mathrm{a}(\tau / 2, z, v)+\mathrm{a}(\tau / 2, v, u)<c / 2+c / 2=c
$$

so that $z \in \mathrm{U}$. Therefore $\mathrm{U} \supset \mathrm{U}(v ; \tau / 2, c / 2)$, whence $\mathrm{U} \in \mathscr{U}(v)$.
Property (5): For some $\tau>0$ we define $c:=\mathrm{a}\left(\tau, u_{1}, u_{2}\right)>0$ and we set $\mathrm{U}_{i}:=\mathrm{U}\left(u_{i} ; \tau / 2, c / 2\right)$. Clearly $\mathrm{U}_{1} \cap \mathrm{U}_{2}=\emptyset$, since otherwise, there would exist $y$ such that $\mathrm{a}\left(\frac{\tau}{2}, u_{i}, y\right)<c / 2$ for $i=1,2$, and thus by the triangle inequality

$$
\mathrm{a}\left(\tau, u_{1}, u_{2}\right) \leq \mathrm{a}\left(\frac{\tau}{2}, u_{1}, y\right)+\mathrm{a}\left(\frac{\tau}{2}, y, u_{2}\right)<c=\mathrm{a}\left(\tau, u_{1}, u_{2}\right),
$$

which is a contradiction.
In order to show that $\mathfrak{O}$ satisfies the first countability axiom, it is sufficient to observe that for every $u \in \boldsymbol{X}$ the sets $\left(\mathrm{U}\left(u ; 2^{-n}, 2^{-n}\right)\right)_{n \in \mathbb{N}}$ form a countable fundamental system of neighborhoods.

Remark 3.2. The last statement in Proposition 3.1, and in particular the fact that sets $\left(\mathrm{U}\left(u ; 2^{-n}, 2^{-n}\right)\right)_{n \in \mathbb{N}}$, parametrized by the sole index $n$, provide a fundamental system of neighborhoods, suggest that, to generate the topology $\mathfrak{O}$, it would be sufficient to work with a family of neighborhoods parametrized by a single real parameter. Interestingly, this approach is equivalent to the one in which the topology is generated by the sets $\mathrm{U}(u ; \tau, c)$, which naturally arise from an action cost.

We can considerably refine the previous Proposition by showing, more or less with the same argument, that a even induces a uniform structure on $\boldsymbol{X}$ (see Bou98a, Chapter II, §1]), which allows us to formalize the concept of 'closeness' of two points. More precisely, for every $\tau, c>0$ we define

$$
\begin{equation*}
\mathrm{V}(\tau, c):=\left\{\left(u_{1}, u_{2}\right) \in \boldsymbol{X} \times \boldsymbol{X}: \mathrm{a}\left(\tau, u_{1}, u_{2}\right)<c\right\} . \tag{3.2}
\end{equation*}
$$

We will also use the following notation for subsets $\mathrm{V}, \mathrm{V}_{i}$ of $\boldsymbol{X} \times \boldsymbol{X}$ :

$$
\begin{gather*}
\mathrm{V}^{-1}:=\left\{\left(u_{2}, u_{1}\right):\left(u_{1}, u_{2}\right) \in \mathrm{V}\right\}  \tag{3.3}\\
\mathrm{V}_{2} \circ \mathrm{~V}_{1}:=\left\{\left(u_{1}, u_{3}\right) \in \boldsymbol{X} \times \boldsymbol{X}: \exists u_{2} \in \boldsymbol{X} \text { such that }\left(u_{1}, u_{2}\right) \in \mathrm{V}_{1},\left(u_{2}, u_{3}\right) \in \mathrm{V}_{2}\right\} . \tag{3.4}
\end{gather*}
$$

Proposition 3.3. Let $(\boldsymbol{X}, \mathrm{a})$ be an action space. The family of sets

$$
\begin{equation*}
\mathfrak{U}:=\{\mathrm{U} \subset \boldsymbol{X} \times \boldsymbol{X}: \mathrm{U} \supset \mathrm{~V}(\tau, c) \text { for some } \tau, c>0\} . \tag{3.5}
\end{equation*}
$$

satisfies the axioms of a uniform structure, i.e.
(1) If $\mathrm{U} \in \mathfrak{U}$ and $\mathrm{U} \subset \mathrm{U}^{\prime}$ then $\mathrm{U}^{\prime} \in \mathfrak{U}$;
(2) Every finite intersection of elements of $\mathfrak{U}$ belongs to $\mathfrak{U}$;
(3) Every $\mathrm{U} \in \mathfrak{U}$ contains the diagonal $\Delta:=\{(u, u): u \in \boldsymbol{X}\}$ in $\boldsymbol{X} \times \boldsymbol{X}$;
(4) If $\mathrm{U} \in \mathfrak{U}$ then also $\mathrm{U}^{-1}$ belongs to $\mathfrak{U}$;
(5) if $\mathrm{U} \in \mathfrak{U}$ then there is $\mathrm{V} \in \mathfrak{U}$ such that $\mathrm{V} \circ \mathrm{V} \subset \mathrm{U}$.

Proof. The proof of properties (1), (2), (3) follows by the same arguments for the corresponding properties of Proposition 3.1.

Property (4) is an immediate consequence of (2.1b) and the symmetry of a, yielding that $\left(u_{1}, u_{2}\right) \in \mathrm{V}(\tau, c)$ if and only if $\left(u_{2}, u_{1}\right) \in \mathrm{V}(\tau, c)$.

In order to prove property (5) we argue as for claim (4) of Proposition 3.1] we first select $\tau, c>0$ such that $\mathrm{V}(\tau, c) \subset \mathrm{U}$ and we set $\mathrm{V}:=\mathrm{V}(\tau / 2, c / 2)$, observing that if $\left(u_{1}, u_{3}\right) \in \mathrm{V} \circ \mathrm{V}$ we can find $u_{2}$ such that $\mathrm{a}\left(\tau / 2, u_{1}, u_{2}\right)<c / 2$ and $\mathrm{a}\left(\tau / 2, u_{2}, u_{3}\right)<c / 2$. Applying (2.1C) we get $\mathrm{a}\left(\tau, u_{1}, u_{2}\right)<c$, i.e. $\left(u_{1}, u_{3}\right) \in \mathrm{U}$.

Following the terminology of Bou98a, we shall refer to the sets of $\mathfrak{U}$ as entourages; if V is an entourage in $\mathfrak{U}$ and $\left(u, u^{\prime}\right) \in \mathrm{V}$, we may say that $u$ and $u^{\prime}$ are ' V -close'. Likewise, the structure $\mathfrak{U}$ induces a topology on $\boldsymbol{X}$ such that the neighborhoods of a point $u \in \boldsymbol{X}$ are given by the sets

$$
\begin{equation*}
\mathrm{V}(u):=\{y \in \boldsymbol{X}:(u, y) \in \mathrm{U}\} \quad \text { for some } \mathrm{U} \in \mathfrak{U} . \tag{3.6}
\end{equation*}
$$

It is straightforward to check that
Proposition 3.4. The topology $\mathfrak{O}$ coincides with the topology induced by the uniform structure $\mathfrak{U}$.

Having a uniform structure at our disposal, we can define a corresponding notion of Cauchy sequence and completeness.
Definition 3.5 (Cauchy sequences and complete action spaces). Let ( $\boldsymbol{X}, \mathrm{a}$ ) be an action space. We say that a sequence $\left(u_{n}\right)_{n} \subset \boldsymbol{X}$ is a Cauchy sequence in the $\mathfrak{U}$-uniform structure if it enjoys the following property:

$$
\begin{equation*}
\forall \tau, c>0 \quad \exists \bar{n} \in \mathbb{N}: \quad \forall n, m \geq \bar{n}: \quad\left(u_{n}, u_{m}\right) \in \mathrm{V}(\tau, c) . \tag{3.7}
\end{equation*}
$$

We say that $(\boldsymbol{X}, \mathrm{a})$ is complete if every Cauchy sequence is convergent.
As in Proposition 3.1 we immediately see that the filter of the entourages $\mathfrak{U}$ of the uniform structure has a countable base, given by the collection $\mathrm{V}\left(2^{-n}, 2^{-n}\right)$. We can thus apply Bou98b, Theorem 1, Chap. IX, §2] to obtain the metrizability of $\mathfrak{U}$. In our setting, we can be even more precise by introducing a metric which is induced by a and induces the same uniform and topological structure: we will discuss this issue in the next section.

## 4. Metric structures induced by an action cost

Definition 4.1. For every $u_{1}, u_{2} \in \boldsymbol{X}$ and every $\lambda>0$ we set

$$
\begin{align*}
\mathrm{d}_{\mathrm{a}, \lambda}\left(u_{1}, u_{2}\right) & :=\lambda \inf \left\{r \geq 0: \mathrm{a}\left(r, u_{1}, u_{2}\right) \leq \lambda r\right\}  \tag{4.1}\\
& =\inf \left\{s \geq 0: \mathrm{a}\left(s / \lambda, u_{1}, u_{2}\right) \leq s\right\} . \tag{4.2}
\end{align*}
$$

In the particular case $\lambda=1$ we also set $d_{a}:=d_{a, 1}$.
Recalling (2.2) we have

$$
0<\lambda^{\prime}<\lambda^{\prime \prime} \quad \Rightarrow \quad\left\{s \geq 0: \mathrm{a}\left(s / \lambda^{\prime \prime}, u_{1}, u_{2}\right) \leq s\right\} \subset\left\{s \geq 0: \mathrm{a}\left(s / \lambda^{\prime}, u_{1}, u_{2}\right) \leq s\right\} .
$$

It is then immediate to check from (4.2) that the family $d_{a, \lambda}$ is increasing w.r.t. $\lambda$ :

$$
\begin{equation*}
0<\lambda^{\prime}<\lambda^{\prime \prime} \quad \Rightarrow \quad d_{a, \lambda^{\prime}} \leq d_{a, \lambda^{\prime \prime}} . \tag{4.3}
\end{equation*}
$$

On the other hand, using (4.1), we see that $\lambda^{-1} d_{a, \lambda}$ is decreasing w.r.t. $\lambda$ :

$$
\begin{equation*}
0<\lambda^{\prime}<\lambda^{\prime \prime} \quad \Rightarrow \quad \frac{d_{a}, \lambda^{\prime \prime}}{\lambda^{\prime \prime}} \leq \frac{d_{a, \lambda^{\prime}}}{\lambda^{\prime}} \tag{4.4}
\end{equation*}
$$

so that the metrics $d_{a, \lambda}$ are equivalent and there holds

$$
\begin{equation*}
(1 \wedge \lambda) \mathrm{d}_{\mathrm{a}} \leq \mathrm{d}_{\mathrm{a}, \lambda} \leq(1 \vee \lambda) \mathrm{d}_{\mathrm{a}} . \tag{4.5}
\end{equation*}
$$

Lemma 4.2 (Equivalent characterizations). For every $\lambda>0$ we have

$$
\begin{equation*}
\mathrm{d}_{\mathrm{a}, \lambda}\left(u_{1}, u_{2}\right) \leq \lambda r \quad \Leftrightarrow \quad \mathrm{a}_{-}\left(r, u_{1}, u_{2}\right) \leq \lambda r \quad \Leftrightarrow \quad \mathrm{a}\left(\tau, u_{1}, u_{2}\right) \leq \lambda r \quad \text { for every } \tau>r \tag{4.6}
\end{equation*}
$$

and

$$
\begin{equation*}
\mathrm{d}_{\mathrm{a}, \lambda}\left(u_{1}, u_{2}\right) \geq \lambda r \quad \Leftrightarrow \quad \mathrm{a}_{+}\left(r, u_{1}, u_{2}\right) \geq \lambda r \quad \Leftrightarrow \quad \mathrm{a}\left(\tau, u_{1}, u_{2}\right) \geq \lambda r \quad \text { for every } \tau<r . \tag{4.7}
\end{equation*}
$$

In particular, $\mathrm{d}_{\mathrm{a}, \lambda}$ can also be characterized by

$$
\begin{equation*}
\mathrm{d}_{\mathrm{a}, \lambda}\left(u_{1}, u_{2}\right)=\lambda r \quad \Leftrightarrow \quad \mathrm{a}_{-}\left(r, u_{1}, u_{2}\right) \leq \lambda r \leq \mathrm{a}_{+}\left(r, u_{1}, u_{2}\right), \tag{4.8}
\end{equation*}
$$

and by the variational formulae

$$
\begin{align*}
& \mathrm{d}_{\mathrm{a}, \lambda}\left(u_{1}, u_{2}\right)=\min _{\tau>0} \mathrm{a}_{-}\left(\tau, u_{1}, u_{2}\right) \vee \lambda \tau,  \tag{4.9}\\
& \mathrm{d}_{\mathrm{a}, \lambda}\left(u_{1}, u_{2}\right)=\max _{\tau>0} \mathrm{a}_{+}\left(\tau, u_{1}, u_{2}\right) \wedge \lambda \tau . \tag{4.10}
\end{align*}
$$

Proof. To check (4.6) we observe that if $\mathrm{d}_{\mathrm{a}, \lambda}\left(u_{1}, u_{2}\right) \leq \lambda r$ then for every $r^{\prime}$ with $\tau>r^{\prime}>r$ we get $\mathrm{a}\left(\tau, u_{1}, u_{2}\right) \leq \mathrm{a}\left(r^{\prime}, u_{1}, u_{2}\right) \leq \lambda r^{\prime}$ so that $\mathrm{a}\left(\tau, u_{1}, u_{2}\right) \leq \lambda r$. Conversely, if for every $\tau>r$ we have $\mathrm{a}\left(\tau, u_{1}, u_{2}\right) \leq \lambda r$ we also get $\mathrm{a}\left(\tau, u_{1}, u_{2}\right) \leq \lambda \tau$, so that $\mathrm{d}_{\mathrm{a}, \lambda}\left(u_{1}, u_{2}\right) \leq \lambda \tau$ and eventually $\mathrm{d}_{\mathrm{a}, \lambda}\left(u_{1}, u_{2}\right) \leq \lambda r$.

Concerning characterization (4.7), if $\mathrm{d}_{\mathrm{a}, \lambda}\left(u_{1}, u_{2}\right) \geq \lambda r$ then for $\tau<r^{\prime}<r$ we have $\mathrm{a}\left(\tau, u_{1}, u_{2}\right) \geq \mathrm{a}\left(r^{\prime}, u_{1}, u_{2}\right)>\lambda r^{\prime}$, so that $\mathrm{a}\left(\tau, u_{1}, u_{2}\right) \geq \lambda r$. Conversely, if for $\tau<r$ we have $\mathrm{a}\left(\tau, u_{1}, u_{2}\right) \geq \lambda r>\lambda \tau$, then we conclude that $\mathrm{d}_{\mathrm{a}, \lambda}\left(u_{1}, u_{2}\right) \geq \lambda \tau$ : since $\tau<r$ is arbitrary we eventually get $\mathrm{d}_{\mathrm{a}, \lambda}\left(u_{1}, u_{2}\right) \geq \lambda$.

Let us now check (4.9) (notice that the minimum in (4.9) is attained since $a_{-}$is decreasing and lower semicontinuous). If $r:=\mathrm{a}_{-}\left(\tau, u_{1}, u_{2}\right) \leq \lambda \tau$ then clearly $\mathrm{d}_{\mathrm{a}, \lambda}\left(u_{1}, u_{2}\right) \leq \lambda \tau$ by (4.6). If $r>\lambda \tau$ then $a_{-}\left(r / \lambda, u_{1}, u_{2}\right) \leq r=\lambda(r / \lambda)$ since $\tau \mapsto a_{-}\left(\tau, u_{1}, u_{1}\right)$ is decreasing, so that $\mathrm{d}_{\mathrm{a}, \lambda}\left(u_{1}, u_{2}\right) \leq r$. This argument shows that $\mathrm{d}_{\mathrm{a}, \lambda}\left(u_{1}, u_{2}\right) \leq \min _{\tau>0} \mathrm{a}_{-}\left(\tau, u_{1}, u_{2}\right) \vee \lambda \tau$. On the other hand (4.8) yields that, when $u_{1} \neq u_{2}$, setting $\tau:=\mathrm{d}_{\mathrm{a}, \lambda}\left(u_{1}, u_{2}\right) / \lambda$ we have $\mathrm{d}_{\mathrm{a}, \lambda}\left(u_{1}, u_{2}\right)=\mathrm{a}_{-}\left(\tau, u_{1}, u_{2}\right) \vee \lambda \tau$, which provides the equality in (4.9).

A similar argument yields (4.10).
A simple consequence of (4.9) and (4.10) is the joint continuity of the map $\left(\lambda, u_{1}, u_{2}\right) \mapsto$ $\mathrm{d}_{\mathrm{a}, \lambda}\left(u_{1}, u_{2}\right)$ in $(0,+\infty) \times \boldsymbol{X} \times \boldsymbol{X}$.
Remark 4.3. In view of the above discussion, we can also define $d_{a, \lambda}$ using a strict inequality in (4.1):

$$
\begin{equation*}
\mathrm{d}_{\mathrm{a}, \lambda}\left(u_{1}, u_{2}\right)=\lambda \inf \left\{r \geq 0: \mathrm{a}\left(r, u_{1}, u_{2}\right)<\lambda r\right\} . \tag{4.11}
\end{equation*}
$$

Example 4.4. It is interesting to compute $\mathrm{d}_{\mathrm{a}, \lambda}$ in the case a is induced by a metric $d$ on $\boldsymbol{X}$ as in Example 2.7. We have

$$
\mathrm{d}_{\mathrm{a}, \lambda}\left(u_{1}, u_{2}\right)=\lambda \inf \left\{r \geq 0: \psi\left(\frac{1}{r} d\left(u_{1}, u_{2}\right)\right) \leq \lambda\right\} .
$$

Therefore, if $\psi$ is invertible, we conclude that

$$
\begin{equation*}
\mathrm{d}_{\mathrm{a}, \lambda}\left(u_{1}, u_{2}\right)=\frac{\lambda}{\psi^{-1}(\lambda)} d\left(u_{1}, u_{2}\right) . \tag{4.12}
\end{equation*}
$$

Theorem 4.5. If $(\boldsymbol{X}, \mathrm{a})$ is an action space, then for all $\lambda>0$ the function $\mathrm{d}_{\mathrm{a}, \lambda}: \boldsymbol{X} \times \boldsymbol{X} \rightarrow$ $[0,+\infty)$ defined by (4.11) is a metric in $\boldsymbol{X}$ which satisfies

$$
\begin{equation*}
\mathrm{d}_{\mathrm{a}, \lambda}\left(u_{1}, u_{2}\right) \leq \lambda \tau \vee \mathrm{a}\left(\tau, u_{1}, u_{2}\right) \leq \lambda \tau+\mathrm{a}\left(\tau, u_{1}, u_{2}\right) \quad \text { for every } \tau>0, u_{1}, u_{2} \in \boldsymbol{X} \tag{4.13}
\end{equation*}
$$

and

$$
\begin{equation*}
\mathrm{d}_{\mathrm{a}, \lambda}\left(u_{1}, u_{2}\right)<\lambda r \quad \Rightarrow \quad \mathrm{a}\left(r, u_{1}, u_{2}\right)<\lambda r . \tag{4.14}
\end{equation*}
$$

In particular $\mathrm{d}_{\mathrm{a}, \lambda}$ metrizes the uniform structure $\mathfrak{U}$ (and a fortiori the topology $\mathfrak{O}$ ): setting

$$
\begin{equation*}
\mathrm{B}_{\lambda}(r):=\left\{\left(u_{1}, u_{2}\right) \in \boldsymbol{X} \times \boldsymbol{X}: \mathrm{d}_{\mathrm{a}, \lambda}\left(u_{1}, u_{2}\right)<\lambda r\right\} \tag{4.15}
\end{equation*}
$$

we have:
(1) for every $\tau, c>0$ choosing $r \in(0, \tau \wedge(c / \lambda)]$ we have $\mathrm{B}_{\lambda}(r) \subset \mathrm{V}(\tau, c)$;
(2) for every $r>0$ choosing $\tau, c \in(0, r]$ we have $\mathrm{V}(\tau, c) \subset \mathrm{B}_{\lambda}(r)$.

Proof. (4.13) is an trivial consequence of (4.9). In particular, the above estimate shows that $d_{a, \lambda}$ takes finite values. Similarly, (4.14) follows immediately from (4.10).

Properties (1) and (2) immediately follow by (4.14) and (4.13) respectively.
Let us now check that $d_{a}$ is a metric. Clearly $d_{a}$ is symmetric and satisfies $d_{a}(u, u)=0$. Property (4.14) also shows that $\mathrm{d}_{\mathrm{a}}\left(u_{1}, u_{2}\right)=0$ implies $u_{1}=u_{2}$.

Concerning the triangle inequality, let $r_{1}:=\mathrm{d}_{\mathrm{a}, \lambda}\left(u_{1}, u_{2}\right)$ and $r_{2}:=\mathrm{d}_{\mathrm{a}, \lambda}\left(u_{2}, u_{3}\right)$, so that $a_{-}\left(r_{1}, u_{1}, u_{2}\right) \leq \lambda r_{1}, a_{-}\left(r_{2}, u_{2}, u_{3}\right) \leq \lambda r_{2}$ by (4.6). Recalling Proposition 2.2 we get a- $\left(r_{1}+\right.$ $\left.r_{2}, u_{1}, u_{3}\right) \leq \lambda r_{1}+\lambda r_{2}$ and therefore $\mathrm{d}_{\mathrm{a}, \lambda}\left(u_{1}, u_{3}\right) \leq r_{1}+r_{2}$.

It is interesting to notice that passing to the limit as $\tau \downarrow 0$ in (4.13) we get for all $\lambda>0$ and $u_{1}, u_{2} \in \boldsymbol{X}$

$$
\begin{equation*}
\mathrm{d}_{\mathrm{a}, \lambda}\left(u_{1}, u_{2}\right) \leq \lambda \mathrm{a}_{\text {sup }}\left(u_{0}, u_{1}\right), \tag{4.16}
\end{equation*}
$$

(cf. (2.4) for the definition of $\mathrm{a}_{\text {sup }}$ ).
Thanks to the previous Theorem we can use the metric $d_{a}$ (or, equivalently, any of the equivalent metrics $d_{a, \lambda}$ ) to characterize completeness.
Corollary 4.6. The action space $(\boldsymbol{X}, \mathrm{a})$ is complete if and only if $\left(\boldsymbol{X}, \mathrm{d}_{\mathrm{a}}\right)$ is a complete metric space.

Example 4.7 (Completeness for action costs induced by a metric). Let a be induced by a metric $d$ as in Example 2.7, with an invertible $\psi$ : then, by virtue of (4.12) it is straightforward to see that $\left(\boldsymbol{X}, \mathrm{d}_{\mathrm{a}}\right)$ is complete if and only if $(\boldsymbol{X}, d)$ is complete.

Example 4.8 (Completeness for linear combinations of actions). Let $\boldsymbol{X}$ be endowed with finitely many costs $\mathrm{a}_{i}, i \in I$. Recall (cf. Example 2.8) that, for any family $\left(\boldsymbol{\vartheta}_{i}\right)_{i \in I}$ of positive coefficients, $\mathrm{a}=\sum_{i} \theta_{i} \mathrm{a}_{i}$ is again an action cost on $\boldsymbol{X}$. It is immediate to check that the action space $(\boldsymbol{X}, \mathrm{a})$ is complete if and only if the spaces $\left(\boldsymbol{X}, \mathrm{a}_{i}\right)$ are complete for all $i \in I$.

In particular, in view of Example 4.7, if $\boldsymbol{X}$ is endowed with two metrics $d_{1}$ and $d_{2}$, and complete w.r.t. both, then the action cost $\mathrm{a}(\tau, u, w)=\tau \psi_{1}\left(\frac{d_{1}(u, w)}{\tau}\right)+\tau \psi_{2}\left(\frac{d_{2}(u, w)}{\tau}\right)$ gives rise to a complete space.

In fact, in view of Theorem 4.5 all the topological notions such as convergence of sequences or continuity of functions can be given in terms of $d_{a}$. We collect the obvious definitions in the following statements,

Corollary 4.9. A sequence $\left(u_{n}\right)_{n} \subset \boldsymbol{X}$ converges to some $u$ in the $\mathfrak{O}$-topology, and write

$$
u_{n} \xrightarrow{\text { a }} u \text { as } n \rightarrow \infty, \quad \text { or a- } \lim _{n \rightarrow \infty} u_{n}=u,
$$

if

$$
\forall \tau>0, c>0 \quad \exists \bar{n} \in \mathbb{N}: \quad \forall n \geq \bar{n} \quad u_{n} \in \mathrm{U}(u ; \tau, c),
$$

or, equivalently, if

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \mathrm{a}\left(\tau, u_{n}, u\right)=0 \quad \text { for every } \tau>0 \tag{4.17}
\end{equation*}
$$

A sequence $\left(u_{n}\right)_{n} \subset \boldsymbol{X}$ is a Cauchy sequence in the $\mathfrak{U}$-uniform structure if it enjoys the following property:

$$
\begin{equation*}
\forall \tau>0, c>0 \quad \exists \bar{n} \in \mathbb{N}: \quad \forall n, m \geq \bar{n}: \quad\left(u_{n}, u_{m}\right) \in \mathrm{V}(\tau, c) ; \tag{4.18}
\end{equation*}
$$

equivalently, if

$$
\begin{equation*}
\lim _{m, n \rightarrow \infty} \mathrm{a}\left(\tau, u_{m}, u_{n}\right)=0 \quad \text { for every } \tau>0 \tag{4.19}
\end{equation*}
$$

We can now easily discuss the continuity of a and the lower-upper semicontinuity of $a_{ \pm}$ (recall (2.3) for the definition of $a_{ \pm}$).

Proposition 4.10. Let $\left(u_{n}\right)_{n},\left(v_{n}\right)_{n}$ be two sequences in $\boldsymbol{X}$ converging to $u, v$ in the $\mathfrak{O}$ topology and let $\left(\tau_{n}\right)_{n}$ be a sequence in $(0,+\infty)$ converging to $\tau>0$. We have

$$
\begin{equation*}
\mathrm{a}_{-}(\tau, u, v) \leq \liminf _{n \rightarrow \infty} \mathrm{a}_{-}\left(\tau_{n}, u_{n}, v_{n}\right) \leq \limsup _{n \rightarrow \infty} \mathrm{a}_{+}\left(\tau_{n}, u_{n}, v_{n}\right) \leq \mathrm{a}_{+}(\tau, u, v) . \tag{4.20}
\end{equation*}
$$

In particular, $a_{-}\left(\right.$resp. $\left.a_{+}\right)$is lower (resp. upper) semicontinuous in $(0,+\infty) \times \boldsymbol{X} \times \boldsymbol{X}$ with respect to the canonical product topology; if a satisfies property (2.5), then it is continuous.

Proof. Let us select $\tau^{\prime}<\tau$ and $\varepsilon<\frac{1}{2}\left(\tau-\tau^{\prime}\right)$. For $n$ sufficiently large we can assume that $\tau_{n}>\tau^{\prime}+2 \varepsilon$ so that the concatenation property and the monotonicity of $a_{+}$yield

$$
\mathrm{a}_{+}\left(\tau_{n}, u_{n}, v_{n}\right) \leq \mathrm{a}\left(\tau^{\prime}+2 \varepsilon, u_{n}, v_{n}\right) \leq \mathrm{a}\left(\varepsilon, u_{n}, u\right)+\mathrm{a}\left(\tau^{\prime}, u, v\right)+\mathrm{a}\left(\varepsilon, v, v_{n}\right)
$$

Taking the lim sup of the left hand side as $n \rightarrow \infty$ and using the convergence property (4.17) we get

$$
\limsup _{n \rightarrow \infty} \mathrm{a}_{+}\left(\tau, u_{n}, v_{n}\right) \leq \mathrm{a}\left(\tau^{\prime}, u, v\right)
$$

Since $\tau^{\prime}<\tau$ is arbitrary we obtain $\lim \sup _{n \rightarrow \infty} \mathrm{a}_{+}\left(\tau_{n}, u_{n}, v_{n}\right) \leq \mathrm{a}_{+}(\tau, u, v)$.
Similarly, selecting $\tau^{\prime \prime}>\tau$ and $\varepsilon>0$ such that $\tau^{\prime \prime}-2 \varepsilon>\tau$ we get for $n$ sufficiently large

$$
\mathrm{a}_{-}\left(\tau^{\prime \prime}, u, v\right) \leq \mathrm{a}\left(\tau_{n}+2 \varepsilon, u, v\right) \leq \mathrm{a}\left(\varepsilon, u, u_{n}\right)+\mathrm{a}\left(\tau_{n}, u_{n}, v_{n}\right)+\mathrm{a}\left(\varepsilon, v_{n}, v\right) .
$$

Taking the liminf of the right-hand side and then the supremum with respect to $\tau^{\prime \prime}$ we obtain $\liminf _{n \rightarrow \infty} \mathrm{a}_{-}\left(\tau_{n}, u_{n}, v_{n}\right) \geq \mathrm{a}_{-}(\tau, u, v)$.

It is interesting to consider the interaction of action costs with an auxiliary topology $\sigma$ on $\boldsymbol{X}$. This is useful when metric costs are involved in a Minimizing Movement scheme and, typically, the driving energy functional has compact sublevels w.r.t. $\sigma$. Our next result shows that (sequential) lower semicontinuity of a- w.r.t. $\sigma$ leads to (sequential) lower semicontinuity of $d_{a}$.

Proposition 4.11. Let $(\boldsymbol{X}$, a) be an action space and let $\sigma$ be a Hausdorff topology on $\boldsymbol{X}$ for which $a_{-}$is sequentially lower semicontinuous, i.e. for all sequences $\left(u_{i}^{n}\right)_{n \in \mathbb{N}}, \sigma$-converging to $u_{i}, i=0,1$, we have

$$
\begin{equation*}
\liminf _{n \rightarrow+\infty} \mathrm{a}_{-}\left(\tau, u_{0}^{n}, u_{1}^{n}\right) \geq \mathrm{a}_{-}\left(\tau, u_{0}, u_{1}\right) \quad \text { for every } \tau>0 \tag{4.21}
\end{equation*}
$$

Then also $\mathrm{d}_{\mathrm{a}}$ is $\sigma$-sequentially lower semicontinuous, i.e. for all sequences $\left(u_{i}^{n}\right)_{n \in \mathbb{N}}, \sigma$-converging to $u_{i}, i=0,1$, we have

$$
\begin{equation*}
\liminf _{n \rightarrow+\infty} \mathrm{d}_{\mathrm{a}}\left(u_{0}^{n}, u_{1}^{n}\right) \geq \mathrm{d}_{\mathrm{a}}\left(u_{0}, u_{1}\right) . \tag{4.22}
\end{equation*}
$$

In particular, every $\sigma$-sequentially compact subset $K \subset \boldsymbol{X}$ is a-complete and every $\mathfrak{O}$-convergent sequence is also $\sigma$-convergent.
Proof. It is sufficient to prove that (4.21) implies (4.22). For that, let us show that if $\mathrm{d}_{\mathrm{a}}\left(u_{0}^{n}, u_{1}^{n}\right) \leq r$ definitely, then also $\mathrm{d}_{\mathrm{a}}\left(u_{0}, u_{1}\right) \leq r$. From characterization (4.6) it follows that $a_{-}\left(r, u_{0}^{n}, u_{1}^{n}\right) \leq r$. Passing to the limit as $n \rightarrow \infty$ and using (4.21) we deduce that $a_{-}\left(r, u_{0}, u_{1}\right) \leq r$, so that $\mathrm{d}_{\mathrm{a}}\left(u_{0}, u_{1}\right) \leq r$ as well.

Remark 4.12. It is worth noticing that sequential lower semicontinuity of a w.r.t. $\sigma$ implies the same property for $a_{-}$.

## 5. Curves with finite action

In this section we assume that $(\boldsymbol{X}, \mathrm{a})$ is an action space.
Definition 5.1 (Action of a curve). Let $u:[a, b] \rightarrow \boldsymbol{X}$. The a-action of $u$ is defined by

$$
\begin{equation*}
\mathbb{A}(u ;[a, b]):=\sup \left\{\sum_{j=1}^{M} \mathrm{a}\left(t^{j}-t^{j-1}, u\left(t^{j-1}\right), u\left(t^{j}\right)\right):\left(t^{j}\right)_{j=0}^{M} \in \mathscr{P}_{f}([a, b])\right\} \tag{5.1}
\end{equation*}
$$

where $\mathscr{P}_{f}([a, b])$ is the set of all finite partitions of the interval $[a, b]$.
This definition is clearly reminiscent of that of the total variation $\operatorname{Var}_{d}$ induced by a metric d, i.e.

$$
\begin{equation*}
\operatorname{Var}_{\mathrm{d}}(u ;[a, b]):=\sup \left\{\sum_{j=1}^{M} \mathrm{~d}\left(u\left(t^{j-1}\right), u\left(t^{j}\right)\right):\left(t^{j}\right)_{j=0}^{M} \in \mathscr{P}_{f}([a, b])\right\} . \tag{5.2}
\end{equation*}
$$

Now, thanks to (4.13) we have a simple estimate of the $\operatorname{Var}_{\mathrm{d}_{\mathrm{d}, \lambda}}$-variation in terms of $\mathbb{A}(u)$.
Lemma 5.2. For every curve $u:[a, b] \rightarrow \boldsymbol{X}$

$$
\begin{equation*}
\operatorname{Var}_{d_{\mathrm{a}, \lambda}}(u ;[a, b]) \leq \lambda(b-a)+\mathbb{A}(u ;[a, b]) . \tag{5.3}
\end{equation*}
$$

In particular, if $u$ has finite action $\mathbb{A}(u ;[a, b])<\infty$ then it also has finite $d_{\mathrm{a}, \lambda}$-variation.
Hence, assuming completeness of ( $\boldsymbol{X}, \mathrm{a}$ ) we show that curves of finite a-action are regulated, and indeed have BV-like properties.

Proposition 5.3. If the action space $(\boldsymbol{X}, \mathrm{a})$ is complete then every curve $u:[a, b] \rightarrow \boldsymbol{X}$ with finite action $\mathbb{A}(u ;[a, b])<+\infty$ satisfies

$$
\begin{array}{ll}
\forall t \in(a, b] & \exists u_{-}(t):=\mathfrak{O}-\lim _{s \uparrow t} u(s), \\
\forall t \in[a, b) & \exists u_{+}(t):=\mathfrak{O}-\lim _{s \downarrow t} u(s) \tag{5.4b}
\end{array}
$$

(we also adopt the convention $u_{-}(a):=u(a)$ and $u_{+}(b):=u(b)$ ). Furthermore, the pointwise jump set

$$
\mathrm{J}_{u}:=\mathrm{J}_{u}^{+} \cup \mathrm{J}_{u}^{-} \quad \text { with }\left\{\begin{array}{l}
\mathrm{J}_{u}^{-}:=\left\{t \in[a, b]: u_{-}(t) \neq u(t)\right\}, \\
\mathrm{J}_{u}^{-}:=\left\{t \in[a, b]: u(t) \neq u_{+}(t)\right\}
\end{array}\right.
$$

consists of at most countably many points, and the function $\mathrm{A}_{u}:[a, b] \rightarrow[0,+\infty), \mathrm{A}_{u}(t):=$ $\mathbb{A}(u ;[a, t])$, has bounded variation.

Proof. From $\mathbb{A}(u ;[a, b])<+\infty$ we infer that $\operatorname{Var}_{d_{a}}(u ;[a, b])<+\infty$ via (5.3). Since the metric space $\left(\boldsymbol{X}, \mathrm{d}_{\mathrm{a}}\right)$ is complete, any $u \in \mathrm{BV}_{\mathrm{d}_{\mathrm{a}}}([a, b] ; \boldsymbol{X})$ admits left- and right-limits w.r.t. the topology induced by $d_{a}$, whence (5.4). Furthermore, $J_{u}$ coincides with the (analogously defined) jump set of the bounded variation function $\mathrm{V}_{u}:[a, b] \rightarrow[0,+\infty), \mathrm{V}_{u}(t):=\operatorname{Var}_{\mathrm{d}_{\mathrm{a}}}(u ;[a, t])$, and thus $u$ has countably many jump points.

Our next result shows that, if the cost a has local superlinear growth according to (2.6) for any curve $u$ with finite action $u$ is $\mathfrak{O}$-continuous at any point.
Proposition 5.4. Suppose that $(\boldsymbol{X}$, a) is complete and a has superlinear local growth. Then, every curve $u:[a, b] \rightarrow \boldsymbol{X}$ with finite action $\mathbb{A}(u ;[a, b])<+\infty$ is continuous.

Proof. We fix $t \in(a, b)$ (similar arguments can be carried out for $t=a$ and $t=b$ ), $\tau>0$, and a strictly positive vanishing sequence $\left(\eta_{n}\right)_{n}$ with $\eta_{n} \downarrow 0$. For $n$ sufficiently large we have

$$
\begin{aligned}
\mathrm{a}\left(\tau, u\left(t-\eta_{n}\right), u(t)\right)+\mathrm{a}\left(\tau, u(t), u\left(t+\eta_{n}\right)\right) & \leq \mathrm{a}\left(\eta_{n}, u\left(t-\eta_{n}\right), u(t)\right)+\mathrm{a}\left(\eta_{n}, u(t), u\left(t+\eta_{n}\right)\right) \\
& \leq \mathbb{A}(u ;[a, b])<+\infty .
\end{aligned}
$$

Taking the limit inferior as $n \rightarrow \infty$ we get

$$
\mathrm{a}_{-}\left(\tau, u_{-}(t), u(t)\right)+\mathrm{a}_{-}\left(\tau, u(t), u_{+}(t)\right) \leq \mathbb{A}(u ;[a, b]) .
$$

We can now pass to the limit in the above estimate along a vanishing sequence $\left(\tau_{n}\right)_{n}$ such that $\mathrm{a}_{-}\left(\tau_{n}, u_{-}(t), u(t)\right)=\mathrm{a}\left(\tau_{n}, u_{-}(t), u(t)\right)$ and $\mathrm{a}_{-}\left(\tau_{n}, u(t), u_{+}(t)\right)=\mathrm{a}\left(\tau_{n}, u(t), u_{+}(t)\right)$ obtaining that $\mathrm{a}_{\text {sup }}\left(u_{-}(t), u(t)\right)+\mathrm{a}_{\text {sup }}\left(u(t), u_{+}(t)\right)<\infty$. The local superlinearity of a then yields $u_{-}(t)=$ $u(t)=u_{+}(t)$.

## 6. Absolute continuity

We now introduce a notion of absolute continuity for curves with values in an action space.
Definition 6.1 (Absolutely continuous curves). We say that a curve $u:[a, b] \rightarrow \boldsymbol{X}$ is aabsolutely continuous if there exists $g \in L^{1}(a, b)$ such that

$$
\begin{equation*}
\mathrm{a}(t-s, u(s), u(t)) \leq \int_{s}^{t} g(r) \mathrm{d} r \quad \text { for all } a \leq s \leq t \leq b \tag{6.1}
\end{equation*}
$$

Theorem 6.2 (Action density). Let $u \in \mathrm{AC}_{\mathrm{a}}([a, b] ; \boldsymbol{X})$. Then, the limit

$$
\begin{equation*}
\lim _{\sigma \downarrow t} \frac{\mathrm{a}(\sigma-t, u(t), u(\sigma))}{\sigma-t}=\lim _{\sigma \uparrow t} \frac{\mathrm{a}(t-\sigma, u(\sigma), u(t))}{t-\sigma}=: \mathfrak{a}\left[u^{\prime}\right](t) \quad \text { exists at almost all } t \in(a, b) \tag{6.2}
\end{equation*}
$$

and it fulfills

$$
\begin{align*}
& \mathrm{a}(t-s, u(s), u(t)) \leq \int_{s}^{t} \mathfrak{a}\left[u^{\prime}\right](r) \mathrm{d} r \quad \text { for all } a \leq s \leq t \leq b,  \tag{6.3a}\\
& \mathfrak{a}\left[u^{\prime}\right](t) \leq g(t) \quad \text { for a.e. } t \in(a, b) \tag{6.3b}
\end{align*}
$$

for every $g \in L^{1}(a, b)$ such that (6.1) holds. Therefore, $\mathfrak{a}\left[u^{\prime}\right] \in L^{1}(a, b)$. We shall refer to $\mathfrak{a}\left[u^{\prime}\right]$ as the action density for $u$.
Proof. The proof closely follows the argument for RMS08, Prop. 3.2]. For every fixed $s \in$ $[a, b)$, we introduce the function

$$
\begin{equation*}
\ell_{s}:(s, b] \rightarrow[0, \infty) \quad t \mapsto \ell_{s}(t):=\mathrm{a}(t-s, u(s), u(t)) \tag{6.4}
\end{equation*}
$$

and observe that, by the triangle inequality (2.1c), there holds

$$
\left(\ell_{s}\left(t_{2}\right)-\ell_{s}\left(t_{1}\right)\right)^{+} \leq \mathrm{a}\left(t_{2}-t_{1}, u\left(t_{1}\right), u\left(t_{2}\right)\right) \leq \int_{t_{1}}^{t_{2}} g(r) \mathrm{d} r \quad \text { for all } s<t_{1}<t_{2} \leq b
$$

Therefore, the map $t \mapsto \ell_{s}(t)-\int_{s}^{t} g(r) \mathrm{d} r$ is non-increasing on $(s, b]$, and thus it is a.e. differentiable, with

$$
\left(\ell_{s}^{\prime}(t)\right)^{+} \leq \mathfrak{a}_{-}(t):=\liminf _{\sigma \downarrow t} \frac{\mathrm{a}(\sigma-t, u(t), u(\sigma))}{\sigma-t} \quad \text { for a.e. } t \in(s, b),
$$

Observe that $\mathfrak{a}_{-}$is itself a measurable function, fulfilling $0 \leq \mathfrak{a}_{-}(t) \leq g(t)$ for almost all $t \in(a, b)$ (with the second inequality due to (6.1)). Thus, $\mathfrak{a}_{-} \in L^{1}(a, b)$. With the very same argument as in the proof of [RMS08, Prop. 3.2] we deduce that

$$
\begin{equation*}
\mathrm{a}(t-s, u(s), u(t))=\ell_{s}(t) \leq \int_{s}^{t}\left(\ell_{s}^{\prime}(r)\right)^{+} \mathrm{d} r \leq \int_{s}^{t} \mathfrak{a}_{-}(r) \mathrm{d} r \quad \text { for all }[s, t] \subset(a, b] . \tag{6.5}
\end{equation*}
$$

Finally, we consider the function

$$
\mathfrak{a}_{+}:(a, b) \rightarrow[0, \infty), \quad t \mapsto \mathfrak{a}_{+}(t):=\underset{\sigma \downarrow t}{\lim \sup } \frac{\mathrm{a}(\sigma-t, u(t), u(\sigma))}{\sigma-t}
$$

and observe that

$$
\begin{equation*}
\mathfrak{a}_{+}(t) \leq g(t) \quad \text { for almost all } t \in(a, b) \text {, for any function } g \text { for which (6.1) holds. } \tag{6.6}
\end{equation*}
$$

In view of (6.5), we may choose $g=\mathfrak{a}_{-}$and we thus conclude that

$$
\underset{\sigma \downarrow t}{\limsup } \frac{a(\sigma-t, u(t), u(\sigma))}{\sigma-t}=\mathfrak{a}_{+}(t) \leq \liminf _{\sigma \downarrow t} \frac{a(\sigma-t, u(t), u(\sigma))}{\sigma-t}=\mathfrak{a}_{-}(t) \quad \text { for a.e. } t \in(a, b) .
$$

Therefore, we ultimately conclude that $\lim _{\sigma \downarrow t} \frac{\mathrm{a}(\sigma-t, u(t), u(\sigma))}{\sigma-t}$ exists at almost all $t \in(a, b)$, and, also in view of (6.5), that it fulfills the minimality properties (6.3). The proof of the assert for $\lim _{\sigma \uparrow t} \frac{\mathrm{a}(t-\sigma, u(\sigma), u(t))}{t-\sigma}$ can be trivially adapted from the analogous argument in RMS08, to which we refer the reader for all details.
Theorem 6.3. For every $u \in \mathrm{AC}_{\mathrm{a}}([a, b] ; \boldsymbol{X})$ we have $\mathbb{A}(u ;[a, b])<\infty$ and

$$
\begin{equation*}
\mathbb{A}(u ;[a, b])=\int_{a}^{b} \mathfrak{a}\left[u^{\prime}\right](t) \mathrm{d} t \tag{6.7}
\end{equation*}
$$

Proof. Let us fix an arbitrary partition $\left(t^{j}\right)_{j=0}^{M} \in \mathscr{P}_{f}([a, b])$ and observe that, for every $j=$ $1, \ldots, M$,

$$
\mathrm{a}\left(t^{j}-t^{j-1}, u\left(t^{j-1}\right), u\left(t^{j}\right)\right) \leq \int_{t^{j-1}}^{t^{j}} \mathfrak{a}\left[u^{\prime}\right](r) \mathrm{d} r .
$$

Therefore,

$$
\sum_{j=1}^{M} \mathrm{a}\left(t^{j}-t^{j-1}, u\left(t^{j-1}\right), u\left(t^{j}\right)\right) \leq \int_{a}^{b} \mathfrak{a}\left[u^{\prime}\right](r) \mathrm{d} r
$$

and, by the arbitrariness of $\left(t^{j}\right)_{j=0}^{M}$, we conclude that

$$
\begin{equation*}
\mathbb{A}(u ;[a, b]) \leq \int_{a}^{b} \mathfrak{a}\left[u^{\prime}\right](t) \mathrm{d} t \tag{6.8}
\end{equation*}
$$

In order to show the converse of inequality (6.8), let now $\left(\mathcal{P}_{k}\right)_{k \in \mathbb{N}} \subset \mathscr{P}_{f}([a, b])$ be a sequence of uniform partitions of size $\tau(k):=(b-a) / k, \mathcal{P}_{k}=\left(t_{k}^{j}\right)_{j=0}^{k}$ where $t_{k}^{j}=a+j \tau(k)$. Let us introduce the piecewise constant functions associated with $\mathcal{P}_{k}$

$$
\begin{align*}
& \overline{\mathfrak{t}}_{k}:[a, b] \rightarrow[a, b],  \tag{6.9}\\
& \overline{\mathfrak{t}}_{k}(a):=a, \\
& \underline{\mathrm{t}}_{k}:\left[\begin{array}{ll} 
& (a, b] \rightarrow[a, b], \\
\mathrm{t}_{k}(b):=b, & \overline{\mathrm{t}}_{k}(t):=t_{k}^{j-1} \quad \text { if } t \in\left[t_{k}^{j-1}, t_{k}^{j}\right], \\
k-1
\end{array}, t_{k}^{j}\right),
\end{align*}
$$

and hence associate with $u$ the functions

$$
\begin{array}{ll}
\overline{\mathfrak{u}}_{k}:[a, b] \rightarrow \boldsymbol{X}, & \overline{\mathfrak{u}}_{k}(t):=u\left(\overline{\mathrm{t}}_{k}(t)\right), \\
\underline{\mathfrak{u}}_{k}:[a, b] \rightarrow \boldsymbol{X}, & \underline{\mathfrak{u}}_{k}(t):=u\left(\underline{\mathrm{t}}_{k}(t)\right) . \tag{6.10}
\end{array}
$$

We also introduce the functions

$$
\Upsilon_{k}(t):=\frac{\mathrm{a}\left(t-\underline{\mathrm{t}}_{k}(t), \underline{\mathfrak{u}}_{k}(t), u(t)\right)+\mathrm{a}\left(\overline{\mathrm{t}}_{k}(t)-t, u(t), \overline{\mathfrak{u}}_{k}(t)\right)}{\tau(k)}, \quad t \in[a, b],
$$

and observe that for every $t \notin \mathcal{P}_{k}$

$$
\Upsilon_{k}(t)=\alpha_{k}(t) \frac{\mathrm{a}\left(t-\underline{\mathrm{t}}_{k}(t), \underline{\mathfrak{u}}_{k}(t), u(t)\right)}{t-\underline{\mathrm{t}}_{k}(t)}+\beta_{k}(t) \frac{\mathrm{a}\left(\overline{\mathrm{t}}_{k}(t)-t, u(t), \overline{\mathfrak{u}}_{k}(t)\right)}{\tau(k)}
$$

where

$$
\alpha_{k}(t):=\frac{t-\underline{\mathrm{t}}_{k}(t)}{\tau(k)}, \quad \beta_{k}(t):=\frac{\overline{\mathrm{t}}_{k}(t)-t}{\tau(k)}, \quad \alpha_{k}(t), \beta_{k}(t) \geq 0, \quad \alpha_{k}(t)+\beta_{k}(t)=1 .
$$

We have that

$$
\lim _{k \rightarrow \infty} \Upsilon_{k}(t)=\mathfrak{a}\left[u^{\prime}\right](t) \quad \text { for a.e. } t \in(a, b) .
$$

Hence, by the Fatou Lemma we find

$$
\int_{a}^{b} \mathfrak{a}\left[u^{\prime}\right](t) \mathrm{d} t \leq \liminf _{k \rightarrow \infty} \int_{a}^{b} \Upsilon_{k}(t) \mathrm{d} t
$$

On the other hand we observe that

$$
\begin{aligned}
& \int_{a}^{b} \Upsilon_{k}(t) \mathrm{d} t=\sum_{j=0}^{k-1} \int_{t_{k}^{j}}^{t_{k}^{j+1}} \Upsilon_{k}(t) \mathrm{d} t \\
&=\sum_{j=0}^{k-1} \int_{0}^{\tau(k)} \Upsilon_{k}(a+j \tau(k)+s) \mathrm{d} s \\
&=\int_{0}^{\tau(k)}\left(\sum_{j=0}^{k-1} \Upsilon_{k}(a+j \tau(k)+s)\right) \mathrm{d} s \\
&=\frac{1}{\tau(k)} \int_{0}^{\tau(k)}\left(\sum_{j=0}^{k-1} \mathrm{a}(s, u(a+j \tau(k)), u(a+j \tau(k)+s))\right. \\
&\quad+\mathrm{a}(\tau(k)-s, u(a+j \tau(k)+s), u(a+(j+1) \tau(k)))) \mathrm{d} s \\
& \leq \frac{1}{\tau(k)} \int_{0}^{\tau(k)} \mathbb{A}(u ;[a, b]) \mathrm{d} s=\mathbb{A}(u ;[a, b]),
\end{aligned}
$$

which finishes the proof.

Corollary 6.4. If $u \in \mathrm{AC}_{\mathrm{a}}([a, b] ; \boldsymbol{X})$ then $u \in \mathrm{AC}\left([a, b] ;\left(\boldsymbol{X}, \mathrm{d}_{\mathrm{a}, \lambda}\right)\right)$ for every $\lambda>0$ and

$$
\begin{equation*}
\left|u^{\prime}\right|_{d_{a}, \lambda} \leq \lambda \vee \mathfrak{a}\left[u^{\prime}\right] \tag{6.11}
\end{equation*}
$$

Proof. Combining (4.13) with (6.1) one immediately sees that $u$ is absolutely continuous with respect to $d_{a, \lambda}$. We can then use (4.13) and (6.2) to deduce (6.11).

## 7. SUfFICIENT CONDITIONS FOR ABSOLUTE CONTINUITY

In this section we address the converse of Theorem6.3hold, namely we examine the validity of the implication

$$
\begin{equation*}
\mathbb{A}(u ;[a, b])<\infty \Rightarrow u \in \mathrm{AC}_{\mathrm{a}}([a, b] ; \boldsymbol{X}) \tag{7.1}
\end{equation*}
$$

It is immediate to realize that, in the case of the action integral costs from Section 2.2, the existence of an action minimizing curve (cf. Proposition 2.12) guarantees the validity of (7.1).

In a different spirit, we propose the following property of a, cf. (7.2) below, as a sufficient condition for (7.1).

Definition 7.1 (Uniform superlinearity). We say that the action cost a on $\boldsymbol{X}$ is uniformly superlinear if there exists an action cost b on $\boldsymbol{X}$, a convex superlinear function $\psi:[0,+\infty) \rightarrow$ $[0,+\infty)$, and a constant $\lambda \geq 1$ such that

$$
\begin{equation*}
\lambda^{-1} \tau \psi\left(\tau^{-1} \mathrm{~b}(\tau, u, v)\right) \leq \mathrm{a}(\tau, u, v) \leq \lambda \tau \psi\left(\tau^{-1} \mathrm{~b}(\tau, u, v)\right) \tag{7.2}
\end{equation*}
$$

for every $\tau>0, u, v \in \boldsymbol{X}$.
Remark 7.2. Definition 7.1 somehow mirrors the convex construction of Example 2.6 and it is clearly satisfied whenever $\psi$ is superlinear. In particular, action costs induced by a metric as in Example 2.7 comply with (7.2).

We have the following result.
Theorem 7.3. If a is a uniformly superlinear action cost on $\boldsymbol{X}$ then for every $u:[a, b] \rightarrow \boldsymbol{X}$ such that $\mathbb{A}(u ;[a, b])<\infty$ we have $u \in \mathrm{AC}_{\mathrm{a}}([a, b] ; \boldsymbol{X})$.

Proof. Since $\psi$ is superlinear, there exists a constant $\beta \geq 0$ such that $\psi(r) \geq r-\beta$ for all $r \in[0,+\infty)$, so that $\tau \psi\left(\tau^{-1} r\right) \geq r-\tau \beta$ and

$$
\begin{equation*}
\mathrm{b}(\tau, u, v) \leq \lambda \mathrm{a}(\tau, u, v)+\tau \beta \quad \text { for all }(\tau, u, v) \in(0,+\infty) \times \boldsymbol{X} \times \boldsymbol{X} \tag{7.3}
\end{equation*}
$$

From the above inequality it follows that any $u:[a, b] \rightarrow \boldsymbol{X}$ satisfies

$$
\mathbb{B}(u ;[a, b]) \leq \lambda \mathbb{A}(u ;[a, b])+\beta(b-a)
$$

where $\mathbb{B}$ denotes the action functional induced by $b$. In particular

$$
\mathbb{A}(u ;[a, b])<\infty \quad \Rightarrow \quad \mathbb{B}(u ;[a, b])<\infty .
$$

Therefore, the function $\mathrm{B}_{u}:[a, b] \rightarrow[0,+\infty), \mathrm{B}_{u}(t):=\mathbb{B}(u ;[a, t])$ has bounded variation. Let $\nu_{u}$ be its distributional derivative. Let now $\left(\mathcal{P}_{k}\right)_{k \in \mathbb{N}} \subset \mathscr{P}_{f}([a, b])$ be the sequence of uniform partitions considered in the proof of Theorem 6.3. With the same notation as in (6.9) and (6.10), we consider now the piecewise constant functions

$$
\Gamma_{k}(t):=\frac{\mathbf{b}\left(\overline{\mathfrak{t}}_{\mathcal{P}_{k}}(t)-\underline{\mathrm{t}}_{\mathcal{P}_{k}}(t), \underline{\mathfrak{u}}_{\mathcal{P}_{k}}(t), \overline{\mathfrak{u}}_{\mathcal{P}_{k}}(t)\right)}{\overline{\mathrm{t}}_{\mathcal{P}_{k}}(t)-\underline{\mathrm{t}}_{\mathcal{P}_{k}}(t)}, \quad t \in[a, b] .
$$

Then, the measures $\nu_{k}:=\Gamma_{k} \mathscr{L}^{1}$ (where $\mathscr{L}^{1}$ denotes the Lebesgue measure on $[a, b]$ ) weakly* converge to $\nu_{u}$. In turn, we observe that for every $t \in(a, b)$

$$
\psi\left(\Gamma_{k}(t)\right) \leq \lambda \frac{1}{\overline{\mathrm{t}}_{\mathcal{P}_{k}}(t)-\underline{\mathrm{t}}_{\mathcal{P}_{k}}(t)} \mathrm{a}\left(\overline{\mathrm{t}}_{\mathcal{P}_{k}}(t)-\underline{\mathrm{t}}_{\mathcal{P}_{k}}(t), \underline{\mathfrak{u}}_{\mathcal{P}_{k}}(t), \overline{\mathfrak{u}}_{\mathcal{P}_{k}}(t)\right),
$$

so that

$$
\begin{align*}
\sup _{k \in \mathbb{N}} \int_{a}^{b} \psi\left(\Gamma_{k}(t)\right) \mathrm{d} t & \leq \lambda \sup _{k \in \mathbb{N}} \int_{a}^{b} \frac{1}{\overline{\mathrm{t}}_{\mathcal{P}_{k}}(t)-\underline{\mathrm{t}}_{\mathcal{P}_{k}}(t)} \mathrm{a}\left(\overline{\mathrm{t}}_{\mathcal{P}_{k}}(t)-\underline{\mathrm{t}}_{\mathcal{P}_{k}}(t), \underline{\mathfrak{u}}_{\mathcal{P}_{k}}(t), \overline{\mathfrak{u}}_{\mathcal{P}_{k}}(t)\right) \mathrm{d} t  \tag{7.4}\\
& \leq \lambda \mathbb{A}(u ;[a, b])<+\infty
\end{align*}
$$

Since the convex function $\psi$ has superlinear growth at infinity, by the well known De ValléePoussin criterion we conclude that the bounded sequence $\left(\Gamma_{k}\right)_{k} \subset L^{1}(a, b)$ admits a (nonrelabeled) subsequence weakly converging to some $\Gamma \in L^{1}(a, b)$, so that $\nu_{u}=\Gamma \mathcal{L}^{1}$. It then follows from (7.4) that

$$
\int_{a}^{b} \psi(\Gamma(t)) \mathrm{d} t \leq \liminf _{k \rightarrow \infty} \int_{a}^{b} \psi\left(\Gamma_{k}(t)\right) \mathrm{d} t \leq \lambda \mathbb{A}(u ;[a, b])<+\infty
$$

Now we have for all $a \leq s \leq t \leq b$

$$
\begin{aligned}
\mathrm{a}(t-s, u(s), u(t)) & \leq \lambda(t-s) \psi\left(\frac{1}{t-s} \mathrm{~b}(t-s, u(s), u(t))\right) \leq \lambda(t-s) \psi\left(\frac{1}{t-s} \nu_{u}([s, t])\right) \\
& =\lambda(t-s) \psi\left(\frac{1}{t-s} \int_{s}^{t} \Gamma(r) \mathrm{d} r\right) \stackrel{(1)}{\leq} \lambda \int_{s}^{t} \psi(\Gamma(r)) \mathrm{d} r
\end{aligned}
$$

where (1) due to the Jensen inequality. Since $\psi \circ \Gamma \in L^{1}(a, b)$, we conclude that $u \in \mathrm{AC}_{\mathrm{a}}([a, b] ; \boldsymbol{X})$ and that, in fact,

$$
\begin{equation*}
\mathfrak{a}\left[u^{\prime}\right](t) \leq \lambda \psi(\Gamma(t)) \quad \text { for a.e. } t \in(a, b) \tag{7.5}
\end{equation*}
$$

(cf. the minimality property (6.3b)). This finishes the proof.
Remark 7.4. Revisiting the proof of Theorem 7.3 we observe that, a fortiori, $u \in \mathrm{AC}_{\mathrm{b}}([a, b] ; \boldsymbol{X})$, since $u \in \mathrm{AC}_{\mathrm{a}}([a, b] ; \boldsymbol{X})$ and a dominates b, cf. (7.3). Now, we immediately check that

$$
\begin{equation*}
\lambda^{-1} \psi\left(\mathfrak{b}\left[u^{\prime}\right](t)\right) \leq \mathfrak{a}\left[u^{\prime}\right](t) \leq \lambda \psi\left(\mathfrak{b}\left[u^{\prime}\right](t)\right) \quad \text { for a.e. } t \in(a, b) \tag{7.6}
\end{equation*}
$$

Indeed, since for all $[s, t] \subset[a, b]$ we have

$$
\begin{aligned}
\int_{s}^{t} \Gamma(r) \mathrm{d} r & =\lim _{k \rightarrow \infty} \int_{s}^{t} \Gamma_{k}(r) \mathrm{d} r \\
& \leq \lim _{k \rightarrow \infty} \int_{s}^{t}\left(\frac{1}{\overline{\mathrm{t}}_{\mathcal{P}_{k}}(r)-\underline{\mathrm{t}}_{\mathcal{P}_{k}}(r)} \int_{\underline{\mathrm{t}}_{\mathcal{P}_{k}}}(r)\right. \\
\overline{\mathrm{t}}_{\mathcal{P}_{k}}(r) & \left.\mathfrak{b}\left[u^{\prime}\right](\omega) \mathrm{d} \omega\right) \mathrm{d} r=\int_{s}^{t} \mathfrak{b}\left[u^{\prime}\right](r) \mathrm{d} r
\end{aligned}
$$

we conclude that

$$
\Gamma(t) \leq \mathfrak{b}\left[u^{\prime}\right](t) \quad \text { for a.e. } t \in(a, b)
$$

and thus

$$
\mathfrak{a}\left[u^{\prime}\right](t) \leq \lambda \psi\left(\mathfrak{b}\left[u^{\prime}\right](t)\right) \quad \text { for a.e. } t \in(a, b)
$$

The other inequality in (7.6) follows by a similar argument.

Example 1: Cost induced by a metric. If a has the structure a $(\tau, u, v)=\tau \psi\left(\frac{1}{\tau} d(u, v)\right)$ for a strictly increasing convex superlinear $\psi$, we immediately infer from Theorem 7.3 and Remark 7.4 the following
Corollary 7.5. For every $u:[a, b] \rightarrow \boldsymbol{X}$ with $\mathbb{A}(u ;[a, b])<\infty$ we have $u \in \mathrm{AC}_{\mathrm{a}}([a, b] ; \boldsymbol{X}) \subset$ $\mathrm{AC}_{d}([a, b] ; \boldsymbol{X})$ and there holds

$$
\begin{equation*}
\mathfrak{a}\left[u^{\prime}\right](t)=\psi\left(\left|\mathfrak{u}^{\prime}\right|_{d}(t)\right) \quad \text { for a.e. } t \in(a, b) \tag{7.7}
\end{equation*}
$$

Proof. It suffices to observe that, in the construction set up in the proof of Theorem 7.3, we have in this case (cf. Remark (7.2) $\mathrm{b}(\tau, u, v)=d(u, v)$ for all $u, v \in \boldsymbol{X}, \lambda=1$ and $\Gamma(t)=\left|\mathfrak{u}^{\prime}\right|_{d}(t)$ for almost all $t \in(a, b)$.

Example 2: cost induced by two metrics. We now address the case in which $\boldsymbol{X}$ is endowed with two distances $d_{1}$ and $d_{2}$ satisfying

$$
\begin{equation*}
\exists K>0 \quad \forall u, v \in \boldsymbol{X} \quad d_{1}(u, v) \leq K d_{2}(u, v) \tag{7.8}
\end{equation*}
$$

Since by the latter condition $d_{2}$ 'dominates' $d_{1}$, in this case we obviously have

$$
\mathrm{AC}_{d_{2}}([a, b] ; \boldsymbol{X}) \subset \mathrm{AC}_{d_{1}}([a, b] ; \boldsymbol{X})
$$

As in Section 2, we consider metric costs of the form

$$
\begin{equation*}
\mathrm{a}(\tau, u, w)=\tau \psi_{1}\left(\frac{d_{1}(u, w)}{\tau}\right)+\tau \psi_{2}\left(\frac{d_{2}(u, w)}{\tau}\right) \quad \text { for all }(\tau, u, w) \in(0,+\infty) \times \boldsymbol{X} \times \boldsymbol{X} \tag{7.9}
\end{equation*}
$$

with $\psi_{1}, \psi_{2}:[0,+\infty) \rightarrow[0,+\infty)$ convex, such that $0=\psi_{i}(0)<\psi_{i}(a)$ for all $a>0$ (cf. Example 2.6). We have the following result.

Proposition 7.6. Assume (7.9) with $\psi_{2}$ invertible. Then, for every $u:[a, b] \rightarrow \boldsymbol{X}$ such that $\mathbb{A}(u ;[a, b])<\infty$ we have $u \in \mathrm{AC}_{\mathrm{a}}([a, b] ; \boldsymbol{X})$ and

$$
\begin{equation*}
\mathfrak{a}\left[u^{\prime}\right](t)=\psi_{1}\left(\left|u^{\prime}\right|_{\mathrm{d}_{1}}(t)\right)+\psi_{2}\left(\left|u^{\prime}\right|_{\mathrm{d}_{2}}(t)\right) \quad \text { for a.e. } t \in(a, b) \tag{7.10}
\end{equation*}
$$

Proof. From $\mathbb{A}(u ;[a, b])<\infty$ we now gather, in particular, that $\mathbb{A}_{2}(u ;[a, b])<\infty$, where $\mathbb{A}_{2}$ is the action functional associated with $\mathrm{a}_{2}(\tau, u, v)=\tau \psi_{2}\left(\frac{1}{\tau} d_{2}(u, v)\right)$. Then, by Corollary 7.5 we have $u \in \mathrm{AC}_{d_{2}}([a, b] ; \boldsymbol{X})$, and hence $u \in \mathrm{AC}_{d_{1}}([a, b] ; \boldsymbol{X})$. Then, for almost all $t \in(a, b)$

$$
\begin{aligned}
\mathfrak{a}\left[u^{\prime}\right](t)=\lim _{\sigma \downarrow t} \frac{\mathrm{a}(\sigma-t, u(t), u(\sigma))}{\sigma-t} & =\lim _{\sigma \downarrow t} \psi_{1}\left(\frac{d_{1}(u(t), u(\sigma))}{\sigma-t}\right)+\lim _{\sigma \downarrow t} \psi_{2}\left(\frac{d_{2}(u(t), u(\sigma))}{\sigma-t}\right) \\
& =\psi_{1}\left(\left|\mathfrak{u}^{\prime}\right|_{d_{1}}(t)\right)+\psi_{2}\left(\left|\mathfrak{u}^{\prime}\right|_{d_{2}}(t)\right)
\end{aligned}
$$

This finishes the proof.

## 8. A DYNAMIC INTERPRETATION OF ACTION COSTS

The goal of this section is to retrieve, for the action cost $a$, a dynamic interpretation akin to (1.8) and (1.9) for action integral costs and Dynamical-Variational Transport costs, respectively. We will obtain this if in addition a fulfills the property from Definition 8.1 below.

Definition 8.1. We say that the action $\operatorname{space}(\boldsymbol{X}, a)$ has the approximate mid-point property if

$$
\begin{equation*}
\forall \rho>0 \forall u, v \in \boldsymbol{X} \forall 0<\epsilon \ll 1 \exists w \in \boldsymbol{X}: \quad \mathrm{a}\left(\frac{\rho}{2}, u, w\right)+\mathrm{a}\left(\frac{\rho}{2}, w, v\right) \leq \mathrm{a}(\rho, u, v)+\epsilon \tag{8.1}
\end{equation*}
$$

Condition (8.1) mimicks the usual approximate mid-point property for metric spaces: for any $\rho \in(0,+\infty)$ and every couple of points $u, v \in \boldsymbol{X}$ and for any assigned threshold $\epsilon$, we find an 'intermediate' point between $u$ and $v$ such that the sum of the costs for connecting $u$ to $w$ and $w$ to $v$ over intervals of half-length $\frac{\rho}{2}$ does not exceed of $\epsilon$ the cost for connecting $u$ to $v$ over the whole interval of length $\rho$.
Theorem 8.2. Let us suppose that the action space ( $\boldsymbol{X}, \mathrm{a})$ is complete and has the approximate mid-point property. Then, for all $\tau, \eta>0$ and for all $u_{0}, u_{1} \in \boldsymbol{X}$

$$
\begin{align*}
& \text { there exists } \omega:[0, \tau] \rightarrow \boldsymbol{X} \text { with } \omega(0)=u_{0}, \omega(1)=u_{1} \text { such that } \\
& \qquad \mathrm{a}\left(\tau, u_{0}, u_{1}\right) \leq \mathbb{A}(\omega ;[0, \tau]) \leq \mathrm{a}\left(\tau, u_{0}, u_{1}\right)+\eta . \tag{8.2}
\end{align*}
$$

In particular

$$
\begin{equation*}
\mathrm{a}\left(\tau, u_{0}, u_{1}\right)=\inf \left\{\mathbb{A}(\Theta ;[0, \tau]): \Theta:[0, \tau] \rightarrow \boldsymbol{X}, \Theta(0)=u_{0}, \Theta(\tau)=u_{1}\right\} \tag{8.3}
\end{equation*}
$$

Proof. Without loss of generality, we may assume that $\tau=1$. Let us fix a threshold $\eta>0$ : to construct the curve $\omega$ we will resort to diadic partitions

$$
\mathcal{P}_{n}=\left\{0, \frac{1}{2^{n}}, \ldots, \frac{j}{2^{n}}, \ldots 1\right\}, \quad n \in \mathbb{N}
$$

of the interval $[0,1]$. Indeed, we start by defining 'discrete' curves defined on $\mathcal{P}_{n}$, in the following way: we pick a sequence $\left(\bar{\eta}_{n}\right)_{n}$ such that

$$
\sum_{n=0}^{\infty} 2^{n} \bar{\eta}_{n}=\eta
$$

and perform the following steps:
Step 0: We apply (8.1) with $\rho=1=\frac{1}{2^{0}}, u=u_{0}, v=u_{1}$, and $\epsilon:=\bar{\eta}_{0}$, thus finding a point $w \doteq w_{1 / 2}$ such that

$$
\mathrm{a}\left(\frac{1}{2}, u_{0}, w_{1 / 2}\right)+\mathrm{a}\left(\frac{1}{2}, w_{1 / 2}, u_{1}\right) \leq \mathrm{a}\left(1, u_{0}, u_{1}\right)+\bar{\eta}_{0} .
$$

Then, we define $\omega_{0}: \mathcal{P}_{1}=\left\{0, \frac{1}{2}, 1\right\} \rightarrow \boldsymbol{X}$ by

$$
\omega_{0}(0):=u_{0}, \quad \omega_{0}\left(\frac{1}{2}\right):=w_{1 / 2}, \quad \omega_{0}(1):=u_{1}
$$

Clearly, we have

$$
\begin{equation*}
\mathrm{a}\left(\frac{1}{2}, \omega_{0}(0), \omega_{0}\left(\frac{1}{2}\right)\right)+\mathrm{a}\left(\frac{1}{2}, \omega_{0}\left(\frac{1}{2}\right), \omega_{0}(1)\right) \leq \mathrm{a}\left(1, u_{0}, u_{1}\right)+\bar{\eta}_{0} . \tag{8.4}
\end{equation*}
$$

Step 1: We apply (8.1) with $\rho=\frac{1}{2}, u=\omega_{0}(0)=u_{0}, v=\omega_{0}\left(\frac{1}{2}\right)$, and $\epsilon:=\bar{\eta}_{1}$, thus obtaining a point $w \doteq w_{1 / 4}$ such that

$$
\mathrm{a}\left(\frac{1}{2^{2}}, \omega_{0}(0), w_{1 / 4}\right)+\mathrm{a}\left(\frac{1}{2^{2}}, w_{1 / 4}, \omega_{0}\left(\frac{1}{2}\right)\right) \leq \mathrm{a}\left(\frac{1}{2}, \omega_{0}(0), \omega_{0}\left(\frac{1}{2}\right)\right)+\bar{\eta}_{1}
$$

We also apply (8.1) with $\rho=\frac{1}{2}, u=\omega_{0}\left(\frac{1}{2}\right), v=\omega_{0}(1)=u_{1}$, and $\epsilon:=\bar{\eta}_{1}$, thus obtaining a point $w \doteq w_{3 / 4}$ such that

$$
\mathrm{a}\left(\frac{1}{2^{2}}, \omega_{0}\left(\frac{1}{2}\right), w_{3 / 4}\right)+\mathrm{a}\left(\frac{1}{2^{2}}, w_{3 / 4}, \omega_{0}(1)\right) \leq \mathrm{a}\left(\frac{1}{2}, \omega_{0}\left(\frac{1}{2}\right), \omega_{0}(1)\right)+\bar{\eta}_{1}
$$

We then define $\omega_{1}: \mathcal{P}_{2}=\left\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1\right\} \rightarrow \boldsymbol{X}$ by

$$
\omega_{1}(0):=u_{0}, \quad \omega_{1}\left(\frac{1}{4}\right):=w_{1 / 4}, \quad \omega_{1}\left(\frac{1}{2}\right)=\omega_{0}\left(\frac{1}{2}\right)=w_{1 / 2}, \quad \omega_{1}\left(\frac{3}{4}\right):=w_{134}, \quad \omega_{1}(1):=u_{1}
$$

By construction, we have

$$
\begin{align*}
& \mathrm{a}\left(\frac{1}{2^{2}}, \omega_{1}(0), \omega_{1}\left(\frac{1}{4}\right)\right)+\mathrm{a}\left(\frac{1}{2^{2}}, \omega_{1}\left(\frac{1}{4}\right), \omega_{1}\left(\frac{1}{2}\right)\right)+\mathrm{a}\left(\frac{1}{2^{2}}, \omega_{1}\left(\frac{1}{2}\right), \omega_{1}\left(\frac{3}{4}\right)\right)+\mathrm{a}\left(\frac{1}{2^{2}}, \omega_{1}\left(\frac{3}{4}\right), \omega_{1}(1)\right) \\
& \leq \mathrm{a}\left(\frac{1}{2}, \omega_{0}(0), \omega_{0}\left(\frac{1}{2}\right)\right)+\mathrm{a}\left(\frac{1}{2}, \omega_{0}\left(\frac{1}{2}\right), \omega_{0}(1)\right)+2 \bar{\eta}_{1}  \tag{8.5}\\
& \leq \mathrm{a}\left(1, u_{0}, u_{1}\right)+\bar{\eta}_{0}+2 \bar{\eta}_{1} .
\end{align*}
$$

Step $n$ : Let $t_{j-1}=\frac{j-1}{2^{n}}$ and $t_{j}=\frac{j}{2^{n}}$, for $j \in\left\{1, \ldots, 2^{n}\right\}$, be two nodes of the partition $\mathcal{P}_{n}$. Clearly, $t_{j-1}=\frac{m-2}{2^{n+1}} \in \mathcal{P}_{n+1}$ and $t_{j}=\frac{m}{2^{n+1}} \in \mathcal{P}_{n+1}$ with $m=2 j$. Applying (8.1) with $\rho=\frac{1}{2^{n}}$ and $\epsilon:=\bar{\eta}_{n}$, we find $w \doteq w_{(m-1) / 2^{n+1}}$ such that

$$
\begin{aligned}
& \mathrm{a}\left(\frac{1}{2^{n+1}}, \omega_{n-1}\left(\frac{m-2}{2^{n+1}}\right), w_{(m-1) / 2^{n+1}}\right)+\mathrm{a}\left(\frac{1}{2^{n+1}}, w_{(m-1) / 2^{n+1}}, \omega_{n-1}\left(\frac{m}{2^{n+1}}\right)\right) \\
& \leq \mathrm{a}\left(\frac{1}{2^{n}}, \omega_{n-1}\left(\frac{m-2}{2^{n+1}}\right), \omega_{n-1}\left(\frac{m}{2^{2+1}}\right)\right)+\bar{\eta}_{n} \\
& \mathrm{a}\left(\frac{1}{2^{n}}, \omega_{n-1}\left(\frac{j-1}{2^{n}}\right), \omega_{n-1}\left(\frac{j}{2^{n}}\right)\right)+\bar{\eta}_{n} .
\end{aligned}
$$

Repeating this construction for every pair $\left(t_{j-1}, t_{j}\right)$ of consecutive nodes of the partition $\mathcal{P}_{n+1}$, we define $\omega_{n}: \mathcal{P}_{n+1}=\left\{0, \ldots, \frac{k}{2^{n+1}}, \ldots 1\right\} \rightarrow \boldsymbol{X}$ by

$$
\begin{cases}\omega_{n}(0):=u_{0}, & \\ \omega_{n}\left(\frac{k}{2^{n+1}}\right):=\omega_{n-1}\left(\frac{j}{2^{n}}\right) & \text { if } k \text { is even with } k=2 j \\ \omega_{n}\left(\frac{k}{2^{n+1}}\right):=w_{k} & \text { if } k \text { is odd } \\ \omega_{n}(1):=u_{1} & \end{cases}
$$

The function $\omega_{n}$ satisfies

$$
\begin{align*}
\sum_{k=1}^{2^{n+1}} \mathrm{a}\left(\frac{1}{2^{n+1}}, \omega_{n}\left(\frac{k-1}{2^{n+1}}\right), \omega_{n}\left(\frac{k}{2^{n+1}}\right)\right) & \leq \sum_{j=1}^{2^{n}} \mathrm{a}\left(\frac{1}{2^{n}}, \omega_{n-1}\left(\frac{j-1}{2^{n}}\right), \omega_{n-1}\left(\frac{j}{2^{n}}\right)\right)+2^{n} \bar{\eta}_{n}  \tag{8.6}\\
& \leq \mathrm{a}\left(1, u_{0}, u_{1}\right)+\sum_{j=0}^{n} 2^{j} \bar{\eta}_{j}
\end{align*}
$$

Let now $\mathcal{P}_{\text {diad }}:=\bigcup_{n=0}^{\infty} \mathcal{P}_{n}$, and let us define $\omega_{\text {diad }}: \mathcal{P}_{\text {diad }} \rightarrow \boldsymbol{X}$ by

$$
\omega_{\mathrm{diad}}\left(t_{\ell}\right):=\omega_{n}\left(t_{\ell}\right) \quad \text { if } t_{\ell} \in \mathcal{P}_{n} .
$$

It follows from our construction that $\omega_{\text {diad }}$ is well defined; from (8.6), and the fact the map a $(\cdot, u, v)$ is non-increasing for every $u, v \in \boldsymbol{X}$, we gather that

$$
\begin{equation*}
\sum_{\ell=0}^{\infty} \mathrm{a}\left(\tau, \omega_{\text {diad }}\left(t_{\ell}\right), \omega_{\text {diad }}\left(t_{\ell+1}\right)\right) \leq \mathrm{a}\left(1, u_{0}, u_{1}\right)+\sum_{j=0}^{\infty} 2^{j} \bar{\eta}_{j} \quad \text { for all } \tau>0 \tag{8.7}
\end{equation*}
$$

We are now in a position to construct the curve $\omega$ by extending $\omega_{\text {diad }}$ to $[0,1] \backslash \mathcal{P}_{\text {diad }}$ in the following way: for every $t \in[0,1] \backslash \mathcal{P}_{\text {diad }}$ we pick the sequence $\left(t_{\ell_{h}}\right)_{h} \subset \mathcal{P}_{\text {diad }}$ with $t_{\ell_{h}} \rightarrow t$ as $h \rightarrow \infty$. It follows from (8.7) that

$$
\begin{aligned}
& \forall \tau>0 \quad \forall \varepsilon>0 \quad \exists \bar{h} \in \mathbb{N} \\
& \qquad \begin{array}{l}
\forall k, h \geq \bar{h}: \quad \mathrm{a}\left(\tau, \omega_{\text {diad }}\left(t_{h}\right), \omega_{\text {diad }}\left(t_{k}\right)\right) \\
\\
\quad \leq \sum_{\ell=h}^{k-1} \mathrm{a}\left(\tau, \omega_{\text {diad }}\left(t_{\ell}\right), \omega_{\text {diad }}\left(t_{\ell+1}\right)\right) \leq \varepsilon
\end{array},
\end{aligned}
$$

namely the sequence $\left(\omega_{\text {diad }}\left(t_{\ell_{h}}\right)\right)_{h}$ is Cauchy in the $\mathfrak{O}$-uniform structure. Since $(\boldsymbol{X}, \mathrm{a})$ is complete, $\left(\omega_{\text {diad }}\left(t_{\ell_{h}}\right)\right)_{h}$ admits a limit w.r.t. the $\mathfrak{O}$-topology. Let

$$
\omega_{\infty}(t):=\mathrm{a}-\lim _{h \rightarrow \infty} \omega_{\mathrm{diad}}\left(t_{\ell_{h}}\right)
$$

Therefore, we define $\omega:[0,1] \rightarrow \boldsymbol{X}$ via

$$
\omega(t):= \begin{cases}\omega_{\text {diad }}(t) & \text { if } t \in \mathcal{P}_{\text {diad }}  \tag{8.8}\\ \omega_{\infty}(t) & \text { if } t \in[0,1] \backslash \mathcal{P}_{\text {diad }}\end{cases}
$$

Now, let $\left(s_{m}\right)_{m=1}^{M} \in \mathscr{P}_{f}([0,1])$ be an arbitrary partition of $[0,1]$ : we have that

$$
\exists \bar{n} \in \mathbb{N} \forall m \in\{1, \ldots, M\} \forall n \geq n \quad \exists t_{n}^{m} \in \mathcal{P}_{n}: \quad\left\{\begin{array}{l}
\left|s_{m}-t_{n}^{m}\right| \leq \frac{1}{2^{n}} \\
s_{m}-s_{m-1} \geq \frac{1}{2^{n+1}}
\end{array}\right.
$$

Therefore,

$$
\begin{aligned}
\mathrm{a}\left(s_{m}-s_{m-1}, \omega\left(s_{m-1}\right), \omega\left(s_{m}\right)\right) & =\lim _{n \rightarrow \infty} \mathrm{a}\left(s_{m}-s_{m-1}, \omega_{\text {diad }}\left(t_{n}^{m-1}\right), \omega_{\text {diad }}\left(t_{n}^{m}\right)\right) \\
& \leq \liminf _{n \rightarrow \infty} \mathrm{a}\left(\frac{1}{2^{n+1}}, \omega_{n}\left(t_{n}^{m-1}\right), \omega_{n}\left(t_{n}^{m}\right)\right)
\end{aligned}
$$

where we have used that $\mathrm{a}(\cdot, u, v)$ is non-increasing for all $u, v \in \boldsymbol{X}$ and the fact that $\omega_{n}\left(t_{n}^{l}\right)=$ $\omega_{\text {diad }}\left(t_{n}^{l}\right) \xrightarrow{\text { a }} \omega\left(s_{l}\right)$ for $l \in\{m-1, m\}$. All in all, we have

$$
\begin{aligned}
& \sum_{m=1}^{M} \mathrm{a}\left(s_{m}-s_{m-1}, \omega\left(s_{m-1}\right), \omega\left(s_{m}\right)\right) \\
& \leq \liminf _{n \rightarrow \infty} \sum_{m=1}^{M} \mathrm{a}\left(\frac{1}{2^{n+1}}, \omega_{n}\left(t_{n}^{m-1}\right), \omega_{n}\left(t_{n}^{m}\right)\right) \\
& \leq \mathrm{a}\left(1, u_{0}, u_{1}\right)+\sum_{j=0}^{\infty} 2^{j} \bar{\eta}_{j}=\mathrm{a}\left(1, u_{0}, u_{1}\right)+\eta
\end{aligned}
$$

where the second estimate follows from (8.7). By the arbitrariness of $\left(s_{m}\right)_{m=1}^{M} \in \mathscr{P}_{f}([0,1])$, we ultimately conclude that $\mathbb{A}(\omega ;[0,1]) \leq \mathrm{a}\left(1, u_{0}, u_{1}\right)+\eta$. In turn, since $\omega(0)=u_{0}$ and $\omega(1)=u_{1}$, we obviously have $\mathrm{a}\left(1, u_{0}, u_{1}\right) \leq \mathbb{A}(\omega ;[0,1])$, and thus (8.2) follows. This finishes the proof.
8.1. Existence of curves with minimal action. Suppose now that $\boldsymbol{X}$ is compact with respect to a topology $\sigma$ w.r.t. which a is lower semicontinuous (cf. (4.21)). Then, we can exploit Theorem 8.2 to construct for all $\tau>0$ and $u_{0}, u_{1} \in \boldsymbol{X}$ an optimal curve $\omega_{\text {opt }}$ for the minimum problem (8.3).

Theorem 8.3. Let a be, in addition, uniformly superlinear in the sense of Definition 7.1 and suppose that $\boldsymbol{X}$ is endowed with a topology $\sigma$ such that $(\boldsymbol{X}, \sigma)$ is compact and a is $\sigma$-lower semicontinuous (4.21).

Then, for every $\tau>0$ and $u_{0}, u_{1} \in \boldsymbol{X}$ there exists $\omega_{\mathrm{opt}}:[0, \tau] \rightarrow \boldsymbol{X}$ such that

$$
\begin{equation*}
\mathrm{a}\left(\tau, u_{0}, u_{1}\right)=\mathbb{A}\left(\omega_{\text {opt }} ;[0, \tau]\right)=\min \left\{\mathbb{A}(\Theta ;[0, \tau]): \Theta:[0, \tau] \rightarrow \boldsymbol{X}, \Theta(0)=u_{0}, \Theta(\tau)=u_{1}\right\} \tag{8.9}
\end{equation*}
$$

Proof. Applying Thm. 8.2 with $\eta=\eta_{n}=\frac{1}{n}, n \geq 1$, we construct a sequence $\left(\omega^{n}\right)_{n}$ of finiteaction curves such that

$$
\lim _{n \rightarrow \infty} \mathbb{A}\left(\omega^{n} ;[0, \tau]\right)=\mathrm{a}\left(\tau, u_{0}, u_{1}\right)
$$

Due to (5.3), we have that

$$
\sup _{n} \operatorname{Var}_{\mathrm{d}_{\mathrm{a}}}\left(\omega^{n} ;[0, \tau]\right)<+\infty .
$$

Therefore, the sequence $\left(\omega^{n}\right)_{n}$ satisfy the conditions of MM05, Thm. 3.2], and we infer that there exist a subsequence $\left(\omega^{n_{k}}\right)_{k}$ and $\omega^{\infty}:[0, \tau] \rightarrow \boldsymbol{X}$ such that

$$
\begin{equation*}
\omega^{n_{k}}(t) \xrightarrow{\sigma} \omega^{\infty}(t) \quad \text { for every } t \in[0, T] \tag{8.10}
\end{equation*}
$$

(where $\xrightarrow{\sigma}$ indicates convergence in the $\sigma$-topology). We now show that

$$
\begin{equation*}
\mathbb{A}\left(\omega^{\infty} ;[0, \tau]\right) \leq \liminf _{n \rightarrow \infty} \mathbb{A}\left(\omega^{n} ;[0, \tau]\right) \tag{8.11}
\end{equation*}
$$

For this, we proceed as in the proof of Theorem 8.2 and fix any partition $\left(s_{m}\right)_{m=1}^{M} \in \mathscr{P}_{f}([0, \tau])$. Due to (8.10) and the $\sigma$-lower semicontinuity (4.21) of a, we find that for every $m=1, \ldots, M$

$$
\mathrm{a}\left(s_{m}-s_{m-1}, \omega^{\infty}\left(s_{m-1}\right), \omega^{\infty}\left(s_{m}\right)\right) \leq \liminf _{k \rightarrow \infty} \mathrm{a}\left(s_{m}-s_{m-1}, \omega^{n_{k}}\left(s_{m-1}\right), \omega^{n_{k}}\left(s_{m}\right)\right) .
$$

Then, (8.11) follows from adding the above estimate over $m=1, \ldots, M$ and using the arbitrariness of $\left(s_{m}\right)_{m=1}^{M} \in \mathscr{P}_{f}([0, \tau])$. We have thus shown that

$$
\mathbb{A}\left(\omega^{\infty} ;[0, \tau]\right) \leq \mathrm{a}\left(\tau, u_{0}, u_{1}\right)
$$

On the other hand, since by (8.10) we have $\omega^{\infty}(0)=u_{0}$ and $\omega^{\infty}(\tau)=u_{1}$, the converse of the above inequality hold, so that $\mathbb{A}\left(\omega^{\infty} ;[0, \tau]\right)=\mathrm{a}\left(\tau, u_{0}, u_{1}\right)$ and we may set $\omega_{\text {opt }}:=\omega^{\infty}$.

Ultimately, under the additional property from Definition 7.1 we obtain the existence of geodesics.

Corollary 8.4. In addition to the assumptions of Thm. 8.3, suppose that a is uniformly superlinear in the sense of Definition 7.1. Then, for every $\tau>0$ and $u_{0}, u_{1} \in \boldsymbol{X}$ there exists $\omega_{\text {opt }} \in \mathrm{AC}_{\mathbf{a}}([0, \tau] ; \boldsymbol{X})$ such that

$$
\begin{align*}
\mathrm{a}\left(\tau, u_{0}, u_{1}\right) & =\int_{0}^{\tau} \mathfrak{a}\left[\omega_{\text {opt }}^{\prime}\right](t) \mathrm{d} t  \tag{8.12}\\
& =\min \left\{\int_{0}^{\tau} \mathfrak{a}\left[\Theta^{\prime}\right](t) \mathrm{d} t: \Theta \in \operatorname{AC}_{\mathrm{a}}([0, \tau] ; \boldsymbol{X}), \Theta(0)=u_{0}, \Theta(\tau)=u_{1}\right\} .
\end{align*}
$$

Proof. It suffices to combine Theorem 8.3 with Theorem 7.3 and Theorem 6.3 ,

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[^0]:    R.R. acknowledges support from the PRIN project PRIN 2020: "Mathematics for Industry 4.0". G.S. acknowledges support from the PRIN project "PRIN 202244A7YL: Gradient Flows and Non-Smooth Geometric Structures with Applications to Optimization and Machine Learning".

