PhD THESIS DECLARATION

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PhD Registration Number *1690601*

Thesis title:

Option-Based Measures of Co-Skewness and Co-Kurtosis Risk

PhD in *Economics & Finance*

Cycle *29th*

Candidate's tutor *Professor Massimo Guidolin*

Year of thesis defence *2018*

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Date *31 January 2018*

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Abstract

Empirical research has argued that option returns are anomalous based on standard return metrics, such as average returns or Sharpe ratios. Other studies treat this return anomaly as evidence that skewness preference is priced. Recent theoretical developments predict a negative relationship between total skewness and average returns. In Chapter 1, based on the newly developed β-Heston model, I study the cross-section of equity option returns to investigate the out-of-the-money option mispricing issue. I find that by comparing historical statistics to those generated by the model, the puzzling out-of-the-money put returns are consistent with the β-Heston model estimation. I also find that the well documented skewness preference is not priced in equity options.

The specification and estimation of factor models is of paramount importance for research and practice. There is a growing amount of literature about extracting information from the option market to forecast asset returns and volatilities. In Chapter 2, I derive a new formula that expresses the measures of covariance, co-skewness, and co-kurtosis risk in terms of market risk-neutral moments and co-moments between stock and index returns. I then use the forward-looking information contained in the option prices to estimate option-implied moments and higher-moment correlations in order to construct market risk betas, co-skewness betas, and co-kurtosis betas. The empirical analysis suggests the following findings: compared to regression-based standard competitors, such as CAPM, the method that I have devised performs better in terms of mean squared error and R^2. An out-of-sample analysis of factor models incorporating co-skewness and co-kurtosis risk premium indicates that the new risk measures improve the return prediction.

In Chapter 3, I propose a percentile range estimator (PRE) for SPY weekly returns based on the intra-day minute data. It is a direct measure of return confidence intervals. Then, I use *Monday VIX open price* as a variable to forecast the future realized ranges, the correlation between VIX price and realized range is 0.73 for the whole sample period (2010-2017). The out-of-the-sample forecast R-square is 0.51, indicating a very strong predictive power. Panel regressions also show that the option-implied method dominates the historical-based ones.

Chapter 1

The Cross-Section of Equity Option Returns

Abstract

Empirical research has argued that option returns are anomalous based on standard return metrics, such as average returns or Sharpe ratios. Other studies treat this return anomaly as evidence that skewness preference is priced. Recent theoretical developments predict a negative relationship between total skewness and average returns. In Chapter 1, based on the newly developed β-Heston model, I study the cross-section of equity option returns to investigate the out-of-the-money option mispricing issue. I find that by comparing historical statistics to those generated by the model, the puzzling out-of-the-money put returns are consistent with the β-Heston model estimation. I also find that the well documented skewness preference is not priced in equity options. Additionally, I provide evidence that casts doubt on the hypothesis of market imperfections and constrained financial intermediaries.

1.1 Introduction

Recent studies have concluded that options are mispriced in the sense that certain option returns are excessive relative to their risks. For instance, Bondarenko (2003) reports that average at-the-money (ATM) put returns are -40% per month, and deep out-of-the-money (OTM) put returns are -95% per month for the S&P 500 index. Furthermore, standard return-based measures such as CAPM alphas or Sharpe ratios are statistically significant and larger than those of the underlying index.

However, we should pay attention to certain conditions when applying these metrics. Option returns are highly non-normal, and these metrics assume normality, which is inappropriate. Additionally, average put returns should be negative due to the leverage inherent in options and the presence of higher moment risk premium. To alleviate these issues, Broadie, Chernov, and Johannes (2009) use option pricing models as a benchmark to assess evidence for index option mispricing. They find that average returns, CAPM alpha, and Sharpe ratios for deep OTM put returns are statistically insignificant when compared to the Black-Scholes model.

Another strand of the literature treats this mispricing (overpriced put options) phenomenon as evidence that skewness preference is priced. Recent studies show that standard rational asset pricing models have difficulty explaining many of the basic empirical facts about financial markets. Experimental economists find that individuals deviate from standard utility theory when making choices in the face of uncertainty. For instance, investors prefer skewness or lottery-like features in asset return distributions, and these preferences influence asset prices in equilibrium. Based on these theories, Boyer and Vorkink (2014) find that total skewness is priced: portfolios of short-term options with high ex ante skewness lose approximately 10% to 50% per week on average compared to those with low ex ante skewness. Bali and Murray (2013) investigate the pricing of risk-neutral skewness in the stock options market by creating skewness assets (comprised of options and underlying equities). They find a strong negative relation between risk-neutral skewness and asset returns, which is consistent with a positive skewness preference.

Cao and Han (2012) present a robust finding that delta-hedged equity option return decreases monotonically with an increase in the idiosyncratic volatility of the underlying stock. The intuition behind this finding relates to market imperfections and constrained financial intermediaries: dealers charge a higher premium for options with high idiosyncratic volatility of the underlying stock due to their higher arbitrage costs. This hypothesis is motivated by the theory of option pricing in an imperfect market that emphasizes the role of constrained financial intermediaries. Shleifer and Vishny (1997) argue that idiosyncratic volatility is the most important proxy of arbitrage costs, as it is correlated with transaction costs and imposes a significant holding cost for arbitrageurs. Thus, financial intermediaries would charge extra compensation for supplying these options, which leads to higher prices and lower returns.

The return of the equity options has been a popular topic in the literature. Hu and Jacobs (2014) provide a theoretical and empirical analysis of the relationship between expected option returns and the volatility of the underlying assets. They find the raw call option return is a decreasing function of the volatility of the underlying assets, while the raw put option return is increasing with the volatility of the underlying assets. Aramonte (2014) finds that macroeconomic uncertainty is priced in the cross-section of option returns, even after controlling for a number of relevant factors.

One of the crucial issues in empirical option pricing is model specification. Christoffersen, Fournier, and Jacobs (2013) find that the principal component analysis of equity options on Dow-Jones firms reveals a strong factor structure. They further develop an equity option valuation model that captures the cross-sectional market factor structure as well as stochastic volatility through time. The model assumes a Heston (1993) style stochastic volatility model for the market return but additionally allows for stochastic idiosyncratic volatility for each firm; thus, it is referred as β-Heston model.

In this paper, I follow the methodology from Broadie, Chernov, and Johannes (2009) to investigate the cross-sectional returns of 29 individual equity options (from Dow Jones Industrial Average index (DJIA)). The β-Heston model is used as a benchmark to assess the evidence for equity option mispricing. Option returns computed from formal option pricing models automatically reflect the leverage and kinked payoffs of options, and anchor hypothesis tests at null values, provide a framework for assessing statistical uncertainty via simulations. Furthermore, option returns are more straightforward to interpret economically than pricing errors. Returns represent actual gain or losses on purchased securities.

First, I find that, compared to index option returns, individual equity option returns are highly volatile; their patterns are less clear and not easily traceable. Thus, comparing each of the equity option average returns to those generated by the model found few interesting results. However, if we take these 29 equity options' average returns as a whole, and then compare this distribution to that generated by the β-Heston model, I find that the two distributions do not differ significantly from each other, indicating that the β-Heston model could provide key insights for understanding and evaluating equity put returns. The overall performance of equity options is consistent with model estimation.

Meanwhile, the hypothesis that skewness preference influences asset prices is also tested in this paper. Recent studies have found that total skewness is priced in stocks. However, this literature also concludes that estimating ex ante skewness for option returns is quite difficult because the correct set of predictive instruments is not known. Boyer and Vorkink (2014) introduce an ex ante option return skewness measurement that is simple to construct. It only relies on three variables: moneyness, underlying asset expected return and volatility. Compared to previous studies, I modify their methods to compute the ex ante skewness in order to exploit the information embedded in the model. First, I find that this new parametric expected measurement is able to replicate the results from Boyer and Vorkink (2014). There is a negative and robust relationship between expected skewness and equity option returns. The spread between the Low Skewness portfolio and the High Skewness portfolio is positive and significant across all maturities. Again, I apply the simulation procedure to test the null hypothesis that skewness is not priced. However, the simulations under the β-Heston model produce very similar patterns as the actual data; indeed, they are statistically insignificant when compared to each other. Consequently, we cannot reject the null hypothesis that skewness is not priced in the cross-section of individual equity option returns.

Furthermore, I also provide evidence that the negative relationship between delta-hedged equity option returns and idiosyncratic volatility of the underlying stocks can be replicated by model simulations, which casts doubt on the hypothesis of market imperfections and constrained financial intermediaries.

The remainder of this paper is structured as follows. Section 2 introduces the β-Heston Model. Section 3 discuss the dataset and estimation methods. Section 4 investigates the mispricing issue based on the β-Heston model. Section 5 shows how to construct skewness portfolios and compares the actual returns with artificial returns generated from the model. Section 6 provides evidence that casts doubt on the hypothesis of market imperfections and constrained financial intermediaries. Conclusions are given in Section 7.

1.2 Model

In the option pricing literature, it is typical to assume a stochastic process for each underlying equity price. Option pricing based on this stochastic process ignores any links the underlying equity prices may have with other equity prices through common factors. When considering a single stock option, ignoring an underlying equity factor structure maybe be harmless. However, it is crucial in portfolio management to understand links between the underlying stocks or options.

1.2.1 Physical Measure

In this paper, following Christoffersen et al. (2013), I consider an equity market consists of n firms driven by a single market factor, I_t (index). The individual stock prices are denoted by S_t^j, for j=1,2,...,n. The market factor has the following dynamics:

$$\frac{dI_t}{I_t} = (r + \mu_I)dt + \sigma_{I,t} dW_t^{I,1} \tag{1.1}$$

$$d\sigma_{I,t}^2 = \kappa_v(\theta_v - \sigma_{I,t}^2)dt + \delta_I \sigma_{I,t} dW_t^{I,2} \tag{1.2}$$

where μ_I is the instantaneous market risk premium, θ_I denotes the long-run variance, κ_I captures the speed of mean reversion of $\sigma_{I,t}^2$ to θ_I, and δ_I measures volatility of volatility. The innovations to the market factor return and volatility are correlated with coefficient ρ_I.

Individual equity prices are driven by the market factor as well as an idiosyncratic term which also has stochastic volatility:

$$\frac{dS_t^j}{S_t^j} - rdt = \alpha_j dt + \beta_t^j (\frac{dI_t}{I_t} - rdt) + \sigma_{j,t} dW_t^{j,1} \tag{1.3}$$

$$d\sigma_{j,t}^2 = \kappa_j(\theta_j - \sigma_{j,t}^2)dt + \delta_j \sigma_{j,t} dW_t^{j,2} \tag{1.4}$$

where α_j denotes the excess return (equity premium) and β_j is the market beta of firm j.

It is true that the relationship between the mean and volatility of returns is a critical issue in finance. Many asset pricing models imply a proportional relationship between expected excess returns and risk in the cross-section and through time. For instance, it is common to assume that the Brownian contribution to the equity premium is linear in volatility.[1] However, the empirical findings on this relationship are mixed. Using volatility-in-mean models (essentially ARCH models in which the variance enters contemporaneously in the mean return), French, Schwert, and Stambaugh (1987) and Campbell and Hentschel (1992) estimate a positive correlation, while Campbell (1987), Breen, Glosten, and Jagannathan (1989), Nelson (1991), and Glosten, Jagannathan, and Runkle (1993) all report a negative correlation. Koopman and Uspensky (1999) find evidence of a weak negative relationship with an SV-in-mean model but a weak positive relationship with an ARCH-based volatility-in-mean model. Harrison and Zhang (1999) employ Gallant and Tauchen's (1989) semi-parametric method to document a positive relationship between the conditional mean and volatility at long horizons but none at short horizons (such as a month). The evidence on the time-series relationship between the conditional mean and volatility of stock returns is inconclusive and depends on the model and exogenous predictors used to draw inferences (e.g., Harvey, 2001). Consequently, Broadie et al. (2009) assume a constant equity premium (that matches observed historical sample), to investigate the expected index option returns.

1.2.2 Risk-Neutral Measure

According to the equivalent martingale measure, the $\mathbb{Q}$-process of the market factor is given by:

$$\frac{dI_t}{I_t} = rdt + \sigma_{I,t} dW_t^{\mathbb{Q}(I,1)} \tag{1.5}$$

$$d\sigma_{I,t}^2 = \kappa_v^{\mathbb{Q}}(\theta_v^{\mathbb{Q}} - \sigma_{I,t}^2)dt + \delta_I \sigma_{I,t} dW_t^{\mathbb{Q}(I,2)} \tag{1.6}$$

[1] I thank Prof. Alessandro Sbuelz for this comment.

and the $\mathbb{Q}$-processes of the individual equities are given by:

$$\frac{dS_t^j}{S_t^j} = rdt + \beta_t^j(\frac{dI_t}{I_t} - rdt) + \sigma_{j,t}dW_t^{\mathbb{Q}(j,1)} \tag{1.7}$$

$$d\sigma_{j,t}^2 = \kappa_j(\theta_j - \sigma_{j,t}^2)dt + \delta_j\sigma_{j,t}dW_t^{\mathbb{Q}(j,2)} \tag{1.8}$$

Note that the market factor structure is preserved under $\mathbb{Q}$. The market beta is the same under both risk-neutral and physical measure. This is consistent with Serban, Lehoczky, and Seppi (2008), who document that the risk-neutral and objective betas are economically and statistically close for most stocks.

It should also be noticed that κ_j and θ_j are the same under both $\mathbb{P}$ and $\mathbb{Q}$ measure, indicating that the idiosyncratic variance risk is not priced. Put it differently, all of the risk premium (except for the equity premium) which is defined as the difference between physical and risk neutral measure is explained by the market factor through β.

1.2.3 Closed-Form Option Price

The model discussed before is affine. It implies that the characteristic function for the log equity price can be derived analytically. The characteristic function for the market index will be exactly identical to that in Heston (1993). While for the individual equity options, the risk-neutral conditional characteristic function $\phi_{t,T}^{\mathbb{Q},j}(u)$ is given by

$$\phi_{t,T}^{\mathbb{Q},j}(u) = (S_t^j)^{iu}exp(iur(T-t) - A^I(\Lambda^S,u) - B^I(\Lambda^S,u)\sigma_{I,t}^2 - A^j(\Lambda^S,u) - B^j(\Lambda^S,u)\sigma_{j,t}^2). \tag{1.9}$$

The expression for A^I, B^I, A^j, and B^j can be found in the appendix. Given the spot price characteristic function under $\mathbb{Q}$, the price of a European equity call option with strike price K and maturity $T-t$ is

$$C_t^j(K, T-t) = S_t^j\Pi_1^j - Ke^{-r(T-t)}\Pi_2^j \tag{1.10}$$

where the risk-neutral probabilities Π_1^j and Π_2^j are defined by:

$$\Pi_1^j = \frac{1}{2} + \frac{e^{-r(T-t)}}{\pi S_t^j} \int_0^\infty Re\Big[\frac{e^{-iulnK}\phi_{t,T}^{\mathbb{Q}}(u-i)}{iu}\Big]du \tag{1.11}$$

$$\Pi_2^j = \frac{1}{2} + \frac{1}{\pi} \int_0^\infty Re\Big[\frac{e^{-iulnK}\phi_{t,T}^{\mathbb{Q}}(u)}{iu}\Big]du \tag{1.12}$$

with conditional characteristic function $\phi_{t,T}^{\mathbb{Q}}$ defined before.

1.3 Estimation Methods

Several methods have been proposed in the literature for estimating stochastic volatility model with latent variables, including MCMC, EMM, IS-GMM (Pan (2002) proposed Implied-State GMM estimation strategy. The author approximates the unobserved volatility, V_t, with an option-implied volatility which is inverted from the time-t spot price and a near-the-money short-dated option price) and so on. Another approach treats the latent variable as a parameter to be estimated and thus avoids filtering problem.

1.3.1 Data

I collect historical daily data on S&P500 and 29 equity options from January 1996 to August 2014. S&P500 index are used to proxy for the market factor. The individual equities are selected from the Dow Jones Industrial Average index (DJIA). Of the 30 firms in the index, Kraft Foods is excluded for which OptionMetrics only has data from 2001. I filter out options having more than 365 days to maturity. Following Bakshi, Cao and Chen (1997), I use mid-quotes (average bid-ask spread) in all computations, and eliminate options with moneyness (K/S) less than 0.9 or greater than 1.1. I also filter out quotes with implied Black-Scholes Vega equal to zero. The S&P500 index options are European, but the individual equity options are American style. As a result, their prices are influence by early exercise premium. To circumvent possible biases due to the presence of early exercise premium, I eliminate in-the-money (ITM) options for which the early exercise premium matters most (See also Bakshi, Kapadia, and Madan (2003)).

1.3.2 Volatility Series

In this model, two vectors of latent variables $\{\sigma^2_{I,t}, \sigma^2_{I,t}\}$ and two sets of structural parameters $\{\Theta_I, \Theta_j\}$ need to be estimated where $\Theta_I \equiv \{\kappa_I^{\mathbb{Q}}, \theta_I^{\mathbb{Q}}, \delta_I, \rho_I\}$, and $\Theta_j \equiv \{\kappa_j, \theta_j, \delta_j, \rho_j \, \beta_j\}$. The parameters Θ_I and Θ_j are taken from Christoffersen et al. (2013), the details of estimation procedure can be found in the appendix. While the volatility series of market index and equities are estimated using the least square method.

$$\hat{\sigma}^2_{I,t} = arg \min_{\sigma^2_{I,t}} \sum_{m=1}^{N_{I,t}} (C_{I,t,m} - C_m(\Theta_I, \sigma^2_{I,t}))^2 / Vega^2_{I,t,m}, t = 1, 2, ...T \tag{1.13}$$

Where $C_{I,t,m}$ is the market price of index option contract m quoted at t, $C_m(\Theta_I, \sigma^2_{I,t})$ is the model index option price, $N_{I,t}$ is the number of index contracts available on day t, and $Vega_{I,t,m}$ is the Black-Scholes sensitivity of the option price with respect to volatility evaluated at the implied volatility. These vega-weighted price errors are good approximation to implied volatility errors and they are much more quickly computed. This method has been used in Carr and Wu (2007).

Given an initial value Θ_j and using the estimated $\hat{\sigma}^2_{I,t}$ and $\hat{\Theta}_I$, we can estimate the spot equity variance each day by sequentially solving

$$\hat{\sigma}^2_{j,t} = arg \min_{\sigma^2_{j,t}} \sum_{m=1}^{N_{j,t}} (C_{j,t,m} - C_m(\Theta^0_j, \sigma^2_{j,t}))^2 / Vega^2_{j,t,m}, t = 1, 2, ...T \tag{1.14}$$

1.3.3 GMM

Since I treat the latent variable as parameters, indeed $\sigma_{I,t}$ and $\sigma_{j,t}$ are estimated in the previous step, then it is natural to use standard GMM method to estimate $\mathbb{P}$-parameters. Given the characteristic function solved before, we can write conditional moment generating function based on the following relationship:

$$M_X(t) = \phi(-it) \tag{1.15}$$

Letting

$$y_n = lnS_n - lnS_{n-1} \tag{1.16}$$

denote the date-n return. We can construct n moment conditions by

$$G_N(\theta) = \frac{1}{N}\sum_{n\le N} h\big(y_n, \sigma_n^2, \theta\big), \tag{1.17}$$

h is some test function be to chosen, such that

$$E_{n-1}^{\theta_0}\big[h(y_n), \sigma_n, \theta_0\big] = 0 \tag{1.18}$$

where θ_0 is the true model parameters, E_{t-1}^{θ} denotes the conditional expectation associated with parameter set θ. Define the GMM estimator $\hat{\theta}_N$ by

$$\hat{\theta}_N = arg\min_{\theta\in\Theta} G_N(\theta)^\top W_N G_N(\theta). \tag{1.19}$$

Given the explicitly known moment-generating function $M_X(t)$ defined before, the conditional moments of the log returns (setting $S_t = 1$ in ϕ) can be derived by

$$E_n(y_{n+1}^i) = \left.\frac{\partial^i M_X(u)}{\partial^i u}\right|_{u=0}, i \in \{0, 1, ...\}. \tag{1.20}$$

Let $m_i(\theta, \sigma_n^2) = E_n^{\theta}(y_{n+1}^i)$ and $h(y_n) = y_n^i - m_i(\theta, \sigma_n^2)$.

1.3.4 Parameters: Result

Broadie, Chernov and Johannes (2007) argued that the absolute continuity requirement implies that certain model parameters, are the same under both measures. In this model, a comparison of the dynamics of S_t under physical and risk-neutral measure demonstrate that $\kappa_j, \theta_j, \delta_j$ and ρ_j are the same under both measures. This implies that these parameters can be estimated either by index/equity returns or option prices, however, the estimates should be the same from either data source. As advocated by Bates (2000), in order to impose this theoretical restriction, we should constrain these parameters to be equal under both measures. For the parameters that are theoretically constrained to be equal across measures, I use $\mathbb{Q}$-measure parameters estimated by Christoffersen, Fournier and Jacobs (2013). The spot volatility is estimated in order to perform the standard GMM method. The results are summarized in Table 1.

[Insert Table 1 about here]

1.4 Expected option return

In light of Broadie et al. (2009), hold-to-expiration put returns are defined as

$$r^p_{t,T} = \frac{(K - S_{t+T})^+}{P_{t+T}(K, S_t)} - 1 \tag{1.21}$$

where $x^+ \equiv$ max(x,0) and $P_{t,T}(K, S_t)$ is the time-t price of a put option written on S_t, struck at K, and expiring at time t+T. Hold-to-expiration returns are widely analyzed in both academic studies and in practice given the fact that option trading involves significant cost, for example, ATM (OTM) index option bid-ask spreads are currently on the order of 3-5% (10%) of the option price. The goal of this section is to assess whether or not equity option returns are excessive, either in absolute terms or relative to their risks. It is common to compute average returns or Sharpe ratios to measure the performance of the asset returns. Strategies that writing put options generally deliver higher average returns than the underlying asset, have economically and statistically higher Sharpe ratios than the market.

It is well known that options are leveraged positions in the underlying asset, so call (put) options have higher (lower) expected returns than the underlying. The precise magnitude of expected returns depends on a number of factors that include the specific model, the parameters, and factor risk premiums. Previous studies concluded that EORs (expected option returns) are very sensitive to both the equity premium and volatility.

The experiment performed in this section is straightforward: I compare the observed values of these intuitive metrics (average returns and Sharpe ratios) in the data to those generated by the β-Heston model. The formal model provides an appropriate null value for anchoring hypothesis test.

1.4.1 Theoretical Predication Under the β-Heston Model

The β-Heston model provides theoretical background to study the equity option returns, more precisely, we are interesting in whether/how much the idiosyncratic return can be

explained by the systematic risk factor. Christoffersen et al. (2013) provide an expression for the expected equity option returns as a function of the expected market return.
For a derivative f_j written on the stock price , S_t^j, the expected excess return on the derivative contract is given by:

$$\frac{1}{dt}E_p^{\mathbb{P}}[\frac{df^j}{f^j} - rdt] = \frac{\partial f^j}{\partial S_t^j}\frac{S_t^j}{f^j}\alpha_j + \frac{\partial f^j}{\partial I_t}\frac{I_t}{f^j}\mu_I = \frac{\partial f^j}{\partial S_t^j}\frac{S_t^j}{f^j}(\alpha_j + \beta_j\mu_I) \tag{1.22}$$

where $\frac{\partial f_j}{\partial I_t}$ is the sensitivity of derivative contract f^j with respect to the index level, I_t (the market delta). It is given by

$$\frac{\partial f^j}{\partial I_t} = \frac{\partial f^j}{\partial S_t^j}\frac{S_t^j}{I_t}\beta_j. \tag{1.23}$$

This result reveals that the beta of the stock provides a simple link between the expected return on the market index and the expected return on the equity option via the delta of the option. The model thus decomposes the excess return on the options into two parts: The delta of the equity option and the beta of the stock. In other words, equity options provide investors with two sources of leverage: first, the beta with respect to the market, and second, the elasticity of the option prices with respect to changes in the stock price.

[**Insert Figure 1 about here**]

In Figure 1, I plot the expected hold-to-expiration returns on equity call options (top panel) and on put options (bottom panel) in percent per month against moneyness for firms with different betas. The top panel of Figure 1 shows that the difference in expected call returns across firms with different betas can be substantial for OTM calls where option leverage in general is high. The bottom panel of Figure 1 shows that put option expected returns (which are always negative) also vary across firms with different betas, when the put options are OTM. As the formula implied, the betas play different roles in expected return for call and put options since the delta of call and put options is opposite. For call options, given a moneyness group, higher beta indicates higher return. While for put options, the relationship is reversed.

1.4.2 Analytical expected option returns

Expected put option returns are given by

$$E_t^{\mathbb{P}}(r_{t,T}^p) = \frac{E_t^{\mathbb{P}}[(K - S_{t+T})^+]}{P_{t+T}(S_t, K)} - 1 = \frac{E_t^{\mathbb{P}}[(K - S_{t+T})^+]}{E_t^{\mathbb{Q}}[e^{-rT}(K - S_{t+T})^+]} - 1 \tag{1.24}$$

It is clear that any model that admits analytical option prices, such as affine models, will allow EORs to be computed explicitly since the numerator and denominator are known analytically. EORs do not depend on S_t. To see this, define the initial moneyness of the option as $\kappa = K/S_t$. Option homogeneity implies that

$$E_t^{\mathbb{P}}(r_{t,T}^p) = \frac{E_t^{\mathbb{P}}[(\kappa - R_{t+T})^+]}{E_t^{\mathbb{Q}}[e^{-rT}(\kappa - R_{t+T})^+]} - 1, \tag{1.25}$$

where $R_{t,T} = S_{t+T}/S_t$ is the gross index return. EORs depend only on the moneyness, maturity, interest rate, and distribution of the underlying returns.

These analytical results are primarily useful as they allow us to assess the exact quantitative impact of risk premiums or parameter configurations. Equation (32) implies the gap between the $\mathbb{P}$ and $\mathbb{Q}$ probability measures determines EORs, and the magnitude of the returns is determined by the relative shape and location of the two probability measures. In models without jump or stochastic volatility risk premiums, the gap is determined by the equity risk premium. When we take stochastic volatility or jump risk premium into consideration, both the shape and location of the distribution can change, leading to more interesting patterns of expected returns across different moneyness categories.

1.4.3 Summary for Equity Option Returns

Options analyzed in this section are one month time-to-maturity OTM put options. Hold-to-expiration returns are computed for fixed moneyness, measured by strike divided by the underlying (K/S_t), ranging from 0.92 to 1.00 (in 2% increments).

Table 2 shows the average hold-to-expiration returns for 29 equity options divided into five moneyness groups. As we can see from the table, the equity option returns are highly volatile, for moneyness equal to 0.92, returns range from -88% for JNJ (Johnson & Johnson) to 35% for TRV (Travellers) per month. The mean returns for each moneyness group are negative and increasing with moneyness. These patterns are consistent with the prediction derived in Coval and Shumway (2001) under general assumptions.

[**Insert Table 2 about here**]

We can find similar results from Table 3, the distribution of Sharpe ratios for each moneyness group is highly volatile. Generally, the Sharpe ratios of put options are larger (in absolute value) than those of the underlying market. For instance, the monthly Sharpe ratio for the market is about 0.1, and the put return Sharpe ratios are several times larger.

[**Insert Table 3 about here**]

1.4.4 Return Distribution via Monte Carlo simulation

To assess statistical significance, I use Monte Carlo simulation to compute the distribution of various returns statistics, including average returns and Sharpe ratios. I simulate $N = 10000$ times of index and 29 equities levels using Milstein scheme simulation. For each equity j and underlying simulation trial g, put returns for a fixed moneyness κ are

$$r_{t,T}^{j,\kappa,(g)} = \frac{(\kappa - R_{t,T}^{(g)})^+}{P_T(\kappa)} - 1 \tag{1.26}$$

where

$$P_T(\kappa) \equiv \frac{P_{t,T}(S_t, K)}{S_t} = e^{-rT} E_t^{\mathbb{Q}}[(\kappa - R_{t,T})^+] \tag{1.27}$$

$g = 1, ..., N$. Average simulated returns for the equity option j, from moneyness group κ are

$$\bar{r}_{t,T}^{j,\kappa} = \frac{1}{N} \sum_{g=1}^{N} r_{t,T}^{j,\kappa,(g)} \tag{1.28}$$

Similarly, we can construct finite-sample distributions for Sharpe ratios. The following subsection illustrates the simulation techniques.

1.4.5 Results From Simulations

Table 4 summarizes EORs (expected option returns) corresponding to each equities for various strikes. It is assumed that all risk premiums (except for the equity premium) are equal to zero. The simulated returns are relatively stable, compared to those generated by real data. For each of the equities, EORs are increasing with respect to moneyness. This pattern becomes clear in the simulated returns. It is worthy of note that under certain pricing kernel specification where the expected index return, μ_I, is linear in the market volatility, α_j is null[2]. In an unreported robustness test, I set all the α_js (Table 1, last column) equal to zero, however, it does not alter the conclusion.

[**Insert Table 4 about here**]

As we already seen, the equity option returns are volatile and their patterns are less clear. Thus, comparing each of the equity option average returns to those generated by the model could hardly find any interesting results. However, if we take these 29 equities options as a whole, which means their average returns constitute a distribution of individual equity option average returns. Then compare this distribution to the one generated by the β-Heston model, I find that the p-value is quite high, indicating that two distributions are not significantly different from each other. Similar result is also found for Sharpe Ratios in Table 5. The results are summarized in Table 6.

[**Insert Table 5 about here**]

The top panel of Table 6 reports population average returns for put options of 29 equities for various strikes. We first note that all the metrics except 8% OTM option return are statistically insignificant when compared to the model. Based on the β-Heston model, we can conclude that: generally, the β-Heston model could provide key insights for understanding and evaluating equity put returns. This result is interesting since the existing literature concludes that OTM put options are most anomalous or mispriced. The results for Sharpe ratios are similar, with none of the strikes statistically different from those generated by the model. These two findings indicate that the overall performance of the equity options is consistent with model estimation. This result is particularly useful when we are evaluating the performance of portfolios consist of equity options. It is proved indirectly in the next section.

[**Insert Table 6 about here**]

[2] I thank Prof. Alessandro Sbuelz for this comment.

1.5 Skewness Preference and Option Returns

Recent research shows that individuals deviate from standard utility theory when making choices in the face of uncertainty. For instance, investors prefer skewness or lottery-like features in asset return distributions, and these preferences influence asset prices in equilibrium. Asset returns have a strong negative relationship with skewness. The individual equity options market offers an ideal platform to study the skewness preference on asset returns. The unusual dramatic lottery-like features in option returns due to the implicit leverage in an option contract combined with a nonlinear payoff. Empirically, the ex ante return skewness of equity options can be more than 10 times higher than equity return skewness. Previous studies suggest that total skewness is priced. Boyer and Vorkink (2014) find a significant and economically large effect of total skewness preference on option prices in both call and put option markets.

1.5.1 Ex ante skewness measurement

To understand whether differences in the lottery-like characteristics of options help explain cross-sectional variation in their expected returns, it is assumed that skewness is a proxy for the lottery-like characteristics of options. In light of Boyer and Vorkink (2014), I construct closed-form ex ante skewness measures for the physical distribution of option returns by integrating the appropriate PDF under the assumption that stock returns are lognormal.

It is obvious that the lognormal assumption does not perfectly characterize the distribution of the underlying stocks. However, it allows for a simple approach to estimate the physical ex ante skewness of an option contract that uses only information available at the time of purchase. Lien (1985) provided a closed-form moments for options returns (under this assumption) by integrating the truncated lognormal PDF.

The ex ante skewness for option j over hirizon t to T is defined as

$$sk_{j,t:T} = \frac{E_t[R_{j,t:T} - \mu_{j,t:T}]^3}{[\sigma_{j,t:T}]^3} \tag{1.29}$$

where $R_{j,t:T}$ denotes option j's hold-to-expiration return defined before, $E_t[.]$ denotes the expectation given information known at time t, $\mu_{j,t:T} = E_t[R_{j,t:T}]$, and $\sigma_{j,t:T} = (E_t[R^2_{j,t:T}] - \mu^2_{j,t:T})^{1/2}$. By rewriting previous equation in terms of its raw moments,

$$sk_{j,t:T} = \frac{E_t[R^3_{j,t:T}] - 3E_t[R^2_{i,t:T}]\mu_{i,t:T} + 2\mu^3_{i,t:T}}{[E_t[R^2_{i,t:T}] - \mu^2_{i,t:T}]^{1.5}}, \tag{1.30}$$

note that only the first three raw moments of the option return are required to calculate $sk_{i,t:T}$. Given the definition of hold-to-expiration return, we can write the m^{th} raw moment for put option j as

$$E_t\big[(R^p_{j,t:T})^m\big] = E_t\bigg[(\frac{K_j - S_{j,T}}{P_{j,t}})^m | K_j > S_{j,T}\bigg] P_t(K_j > S_{j,T}) \tag{1.31}$$

where $P_t(.)$ indicates the probability given information at time t. Under the assumption of lognormality, equation (42) illustrates the raw moments for a put option are a function of the raw moments of a truncated lognormal distribution. The following section demonstrates how to construct the expected skewness measure, $sk_{i,t:T}$.

1.5.2 Closed form of raw raw moments

Let $r = ln(S_T/S_t)$, the log stock return, and assume that r is distributed $N(\mu, \sigma^2)$. Under this assumption, the stock return, S_T/S_t, is lognormal. In the original paper of Boyer and Vorkink (2014), μ_j and σ^2_j are estimated using six months of daily data prior to t. While in this paper, I estimate these two variables in a parametric way in order to absorb the firm specific information contained in the model. According to the β-Heston model, for equity j with options expire at T,

$$\mu_j = (r + \alpha_j + \beta_j \mu_I) \times (T - t) \tag{1.32}$$

and

$$\sigma^2_j = \big(E_t\big[V_{j,T}\big] + \beta^2_j E_t\big[V_{I,T}\big]\big) \times (T - t), \tag{1.33}$$

with

$$E_t\big[V_{i,T}\big] = e^{-\kappa_i (T-t)} V_{i,t} + \theta_i (1 - e^{-\kappa_i (T-t)}) \tag{1.34}$$

where $i = I, j$ stands for the index process and equity process, respectively. $\alpha_j, \beta_j, \mu_I, \kappa_i$, and θ_i are the parameters estimated before. $V_{i,t}$ is set equal to average spot variance of the underlying.

It should be emphasized that estimating μ and σ^2 in this parametric way does not rely on the distribution of the β-Heston model, instead, it provides an approximation for these two variables. Given the raw moments for put options, we can construct $sk_{j,t:T}$ for put options for any level of moneyness and maturity.

1.5.3 Option Characteristics and Skewness

In order to understand how different option characteristics could influence the expected skewness measure, $sk_{i,t:T}$, Figure 2 plots $sk_{i,t:T}$ as a function of moneyness K/S_t, for three different time to maturities. Although the way to estimate $sk_{i,t:T}$ is slightly different from the one used in Boyer and Vorkink (2014), we both find that there is a relationship between moneyness and ex ante skewness, especially for short maturity options. Generally, out-of-the-money options offer higher skewness than in-the-money options. For instance, the ex ante skewness of short-term, out-of-the-money options is well over 10, several times large than the ex ante skewness of equity returns (See Boyer, Mitton, and Vorkink (2010) and Conrad, Dittmar, and Ghysels (2013)). Comparing top panel and bottom panel, we can find that call options display similar patterns as their corresponding put options. Figure 3 plots $sk_{i,t:T}$ as a function of moneyness K/S_t, for three different betas. We can see that Beta plays the same role for both call options and put options: All things equal, as the beta increase, the ex ante skewness decrease for both call and put options.

[Insert Figure 2 about here]

[Insert Figure 3 about here]

1.5.4 Option Portfolio Formation and Returns

Based on the parameters and equations given in the previous sections, we can now compute the ex ante skewness for each of the equity options. Table 7 illustrate the distribution of put option ex ante skewness for fixed moneyness with different maturities. Note that each of the cell (skewness value) in the table is attached to one real return and one simulated return corresponding to its parameters. Similar procedures are repeated for each of the five moneyness groups (OTM 8% to ATM). Next, for each portfolio maturity, I sort options within each expiration bin into ex ante skewness quintiles.

[**Insert Table 7 about here**]

Table 8 reports average of portfolio returns generated by data and simulations for each ex ante skewness/maturity bin. Both actual returns and simulated returns decrease dramatically across skewness bins for every maturity group. We first focus on the return generated by data. For example, put options that expire in two weeks, the actual average hold-to-maturity return is monotonically decreasing from -17% for the low skewness bin to -54% for the high skewness bin. The paired t-statistic for the difference is 7.738. Furthermore, the average difference in spreads between the low and high skewness portfolios is positive and significant in all cases. These results (real data) are similar to those reported by Boyer and Vorkink (2014). Based on these results (and some control tests), they claimed that skewness preference is priced in the equilibrium. Put differently, these results indicate that individual equity option investors give up average returns on the order of 50% monthly for exposure to the lottery opportunities that options with high ex ante skewness offer.

[**Insert Table 8 about here**]

However, the results from simulations provide evidence for an opposite conclusion. The simulated portfolio returns exhibit very similar patterns as actual returns (the monotonicity feature is even more clear). Again, the difference in spreads between the low and high skewness portfolios is positive and significant across all maturity groups, with substantially higher t-statistics. The p-values between simulated and real portfolio returns are generally high for each ex ante skewness/maturity bin. It indicates that the two distributions are not statistically different from each other. Consequently, we can not reject the null that skewness is not priced in the cross-section of individual equity option returns.

1.6 Stock Volatility and Option Returns

Cao and Han (2012) presents a robust finding that delta-hedged equity option return decreases monotonically with an increase in the idiosyncratic volatility of the underlying stock. The result is still significant even after controlling for standard risk factors. The intuition behind this finding could be market imperfections and constrained financial intermediaries: Dealers charge a higher premium for options with high idiosyncratic volatility underlying due to their higher arbitrage costs. This hypothesis is motivated by theory of option pricing in imperfect market that emphasizes the role of constrained financial intermediaries.

Option prices are affected by demand and supply from the markets when there are limits to arbitrage and it is costly to hedge or replicate the options. Shleifer and Vishny (1997) argue that the idiosyncratic volatility is the most important proxy of arbitrage costs, as it is correlated with transaction costs and imposes a significant holding cost for arbitrageurs. On the one hand, options with high idiosyncratic volatility attract high demand from speculators. On the other hand, such options are more difficult to hedge. Thus, financial intermediaries would charge extra compensation for supplying these options, which leads to a higher price and lower return. Hu and Jacobs (2014) provide a theoretical and empirical analysis of the relationship between expected option returns and the volatility of the underlying. They find the raw call option return is a decreasing function of the volatility of the underlying, while the raw put option return is increasing with the volatility of the underlying.

1.6.1 Delta-hedged option returns

I study the delta-hedged put option returns in this paper, using the Black-Scholes delta as approximation. Following Goyal and Saretto (2009), the strategy return is defined as hold-to-expiration, the position of stocks and options is fixed after the strategy is built. The details follows: For each of the month during the sample period, I long one unit of at-the-money put option with maturity equal to one month (if available). Then, hedge the put with a long position of Δ (Black-Scholes delta) unit of underlying stock. The one month hold-to-expiration return Γ of this strategy is defined as

$$\Gamma(t,T) = \frac{max(0, K - S_T) + \Delta_t \cdot S_T}{P_t + \Delta_t \cdot S_t} - 1 \tag{1.35}$$

Where t is the time when we build the strategy, P_t is the put price at that time, Δ_t is the Black-Scholes delta of put option at time t.

1.6.2 Underlying volatility and portfolio construction

The total volatility (VOL_{tot}) of the underlying is computed based on the daily log-return of the underlying price over the previous month, then annualized. Similarly, the market volatility (VOL_{mkt}) is computed based on the daily log-return of the S&P500 index over the previous month, then annualized. The idiosyncratic volatility is defined as follows:

$$VOL_{idio} = \sqrt{VOL_{tot}^2 - \beta_j^2 \cdot VOL_{mkt}^2} \tag{1.36}$$

Where β_j is the parameter estimated before for stock j.

The portfolio is constructed by sorting the underlying total/idiosyncratic volatility. At the maturity of the put option, I rank the strategy returns into five quintiles based on the underlying idiosyncratic volatility (same procedures are repeated for total volatility). Note that the simulated portfolio returns are sorting based on total long-term volatility $\sqrt{\theta_j + \beta_j^2 \cdot \theta_i}$ and idiosyncratic long-term volatility $\sqrt{\theta_j}$.

1.6.3 Results summary

Table 9 summarize the delta-hedged portfolio returns from actual data and simulations based on two sorting criteria: Total volatility and Idiosyncratic volatility. We first look at the actual data, it shows that the returns of delta-hedged put options are always negative (long position), furthermore, the average return on high total/idiosyncratic volatility stocks is significantly higher than that on low total/idiosyncratic volatility stocks. For instance, the average difference in returns between the portfolio of long positions in delta-hedged puts for stocks ranked in the top volatility quintile and that for stocks ranked in the bottom volatility quintile is 1.6%, with a $t - statistic$ equal to 5.14. Similar result

(1.05 % with $t-statistic$ equal to 2.96) can be found when we sort stocks by their idiosyncratic volatility. Although in Cao and Han (2012), they use daily-hedge strategy (while in this paper, the position is hedged and fixed at the initiative), the negative relationship between portfolio returns and underlying volatility is also confirmed here.

[Insert Table 9 about here]

However, this pattern can be also found qualitatively in the corresponding simulation portfolios. As we can see from the table, the difference in simulated returns between high volatility quintile and low volatility quintile is 0.43% for sorting total volatility and 0.31% for sorting idiosyncratic volatility. The difference is statistically significant, however, it is much smaller than the one from actual data. This is due to the fact that the simulated average returns are more flat across different quintiles, compare to those from real data. The mean of the average returns from different quintiles is almost the same for both Data portfolios and Simulated portfolios, with a $P-value$ equal to 0.71 and 0.68 for total and idiosyncratic volatility portfolio, respectively. As we can expected, when comparing data with simulations (measured by $P-value$), the portfolios from top/bottom quintile are significantly different from each other. While it is not the case in the middle quintiles, indeed, we can not reject the null that the average return is the same for Data and Simulation portfolio for quintile 2, 3 and 4.

Although the results from the actual data illustrate that there are more extreme returns in the top/bottom quintile, the simulations still provide qualitatively similar pattern: the negative relationship between delta-hedged return and underlying volatility, cast doubts on the market imperfection and constrained financial intermediaries hypothesis.

1.7 Conclusion

In this paper, I study the cross-section of equity option returns to investigate the out-of-the-money option mispricing issue. The newly developed β-Heston model is used to construct sample distributions of average option returns and Sharpe ratios using Monte Carlo simulation. First, I find that the most puzzling, the very large (in absolute value) returns to OTM options is consistent with the β-Heston model. Second, I find little added benefit from using Sharpe ratios as diagnostic tools since the result is similar to those from average option returns.

Recent studies show that standard rational asset pricing models have difficult explaining many of the basic empirical facts about the financial markets. For instance, investors prefer skewness or lottery-like features in asset return distributions, and these preferences influence asset prices in equilibrium. I modify the method to compute the ex ante skewness (in a parametric way) in order to exploit the information from the model. First, I find that this new parametric ex ante skewness measurement is able to replicate the results from Boyer and Vorkink (2014). There is a negative and robust relationship between ex ante skewness and equity option returns. Then, I apply the simulation procedure to test the null that skewness is not priced. However, different from previous studies, the simulation under the β-Heston model produces very similar patterns as the actual data, indeed, they are not statistically different from each other. Consequently, we can not reject the null that skewness is not priced in the cross-section of individual equity option returns.

Furthermore, I also provide evidence that the negative relationship between delta-hedged equity option returns and idiosyncratic volatility of the underlying stocks can be replicated by model simulations, which casts doubts on the hypothesis of market imperfections and constrained financial intermediaries.

Note that these findings should not be interpret as the evidence that the β-Heston model is correct, but rather as highlighting the statistical difficulties present when analyzing option returns. Indeed, a natural extension to the β-Heston model is to incorporate jumps in the underlying as well as volatility process. It would be interesting to study how would expected option returns change due to these innovations. I leave these questions for future work.

1.8 Appendix

1.8.1 Appendix A: Ex Ante Skewness

In the appendix, following Boyer and Vorkink (2014), I demonstrate how the ex ante skewness measure, $sk_{j,t:T}$ is constructed based on the assuming lognormal stock prices. In light of Lien's (1985) theorem regarding truncated lognormal distributions, theorem A.1 is presented here.

Theorem A.1: *Let* $(u_1, u_2)'$ *be a normal random vector with mean (0,0)' and covariance matrix=* $\begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{bmatrix}$. *Then*

$$E(exp(ru_1 + su_2)|u_1 > a) = N(\frac{h-a}{\sigma_1})\frac{exp[-D/2Q]}{N(\frac{-a}{\sigma_1})}, \tag{1.37}$$

where $h = r\sigma_1^2 + s\sigma_{12}, D = -Q(r^2\sigma_1^2 + 2rs\sigma_{12} + s^2\sigma_2^2), Q = \sigma_2^2 sigma_1^2 - sigma_{12}^2, andN(\cdot)$ is the CDF of the normal.

Note first that Lien's (1985) theorem A.1 can be used to derive closed-form solutions for the raw moments of option returns given by equation (42). These raw moments can be substituted into equation (41) to construct $sk_{j,t:T}$. For $m = 1$, equation (42) can be written as

$$E[R_{t:T}^c] = [\frac{S_t}{C_t}E(\frac{S_T}{S_t}|\frac{S_T}{S_t} > \frac{X}{S_t}) - \frac{X}{C_t}]P(\frac{S_T}{S_t} > \frac{X}{S_t}), \tag{1.38}$$

where S_t is the value of the underlying asset at time $t < T$. Let $\hat{r} = ln(S_T/S_t)$, the log stock return, and define A as $A = lon(X/S_t)$. Then equation (52) can be written as

$$E[R_{t:T}^c] = [\frac{S_t}{C_t}E(e^{\hat{r}}|\hat{r} > A) - \frac{X}{C_t}]P(\hat{r} > A). \tag{1.39}$$

Now assume that $\hat{r}$ is distributed $N(\hat{\mu}, \hat{\sigma}^2)$, where in general $\hat{\mu}$ can be nonzero. Under this assumption, the stock return, S_T/S_t, is lognormal. Furthermore, define $z = \hat{r} - \hat{\mu}$, so that z is distributed $N(0, \hat{\sigma}^2)$. Then note that

$$E(e^{\hat{r}}|\hat{r} > A) = E(e^{z+\hat{\mu}}|z > A - \hat{\mu}) = e^{\hat{\mu}}E(e^z|z > A - \hat{\mu}) \tag{1.40}$$

Then applying Lien's (1985) theorem implies that equation (54) can be written as

$$E(e^{\hat{r}}|\hat{r} > A) = \frac{exp[\hat{\mu} + \frac{\hat{\sigma}^2}{2}]N(d_1)}{N(d_2)} \tag{1.41}$$

Then we can plug equation (55) in to equation (53) to get the first moment of the call option return, following the similar approach, the corresponding raw moments for put options are

$$E\big[R^p_{t:T}\big] = \frac{KN(-d_2) - S_t exp\big[\frac{\sigma^2}{2} + \mu\big]N(-d_1)}{P_t} \tag{1.42}$$

where $d_1 = \frac{\sigma^2 + ln(S_t/K) + \mu}{\sigma}$ and $d_2 = d_1 - \sigma$.

$$\begin{aligned} E\big[(R^p_{t:T})^2\big] = & \frac{K^2N(-d_2) - 2XS_t exp\big[\frac{\sigma^2}{2} + \mu\big]N(-d_1)}{P_t^2} \\ & + \frac{S_t^2 exp\big[2\sigma^2 + 2\mu\big]N(-d_3)}{P_t^2} \end{aligned} \tag{1.43}$$

with $d_3 = d_1 + \sigma$, and $d_4 = d_1 + 2\sigma$.

$$\begin{aligned} E\big[\big(R^p_{t:T}\big)^3\big] = & \frac{3KS_t^2 exp\big[2\sigma^2 + 2\mu\big]N(-d_3) - S_t^3 exp\big[\frac{9}{2}\sigma^2 + 3\mu\big]N(-d_4)}{P_t^3} \\ & + \frac{K^3N(d_2) - 3K^2 S_t exp\big[\frac{\sigma^2}{2} + \mu\big]N(-d_1)}{P_t^3} \end{aligned} \tag{1.44}$$

where P_t is the put price at time t and K is the strike price.

1.8.2 Appendix B: Closed-Form Option Price

Christoffersen et al. (2013) provide the closed-form option price for the β-Heston model, the proof of the following result can be found in their paper.

The risk-neutral conditional characteristic function $\phi^{\mathbb{Q},j}_{t,T}(u)$ is given by

$$\phi^{\mathbb{Q},j}_{t,T}(u) = (S^j_t)^{iu} exp(iur(T-t) - A^I(\Lambda^S, u) - B^I(\Lambda^S, u)\sigma^2_{I,t} - A^j(\Lambda^S, u) - B^j(\Lambda^S, u)\sigma^2_{j,t}). \tag{1.45}$$

Where

$$A^i(\Lambda, u) = \frac{\kappa_i^{\mathbb{Q}}\theta_i^{\mathbb{Q}}}{\delta_i^2}\Big\{2ln(1 - \frac{\Psi^i(\Lambda^S, u) - \kappa_i^C}{2\Psi^i(\Lambda^S, u)}(1 - e^{-\Psi^i(\Lambda^S,u)(T-t)}) \\ +(\Psi^i(\Lambda^S, u) - \kappa_i^C)(T-t)\Big\} \tag{1.46}$$

$$B^i(\Lambda^S, u) = \frac{2g_h(u)(1 - e^{-\Psi^i(\Lambda^S,u)(T-t)})}{2\Psi^i(\Lambda^S, u) - (\Psi^i(\Lambda^S, u) - \kappa_i^C)(1 - e^{-\Psi^i(\Lambda^S,u)(T-t)})} \tag{1.47}$$

with

$$\Psi^i(\Lambda^S, u) = \sqrt{(\kappa_i^C)^2 + 2\delta_i^2 g_i(u)} \tag{1.48}$$

$$g_1(u) = \frac{iu}{2}\beta_j^2(1 - iu) \;\; and \;\; g_2(u) = \frac{iu}{2}(1 - iu) \tag{1.49}$$

$$\kappa_I^C = \kappa_I^{\mathbb{Q}} - iu\rho_I\beta_j\delta_I, \;\; \theta_I^C = \frac{\kappa_I^{\mathbb{Q}}\theta_I^{\mathbb{Q}}}{\kappa_I^C}, \;\; \kappa_j^C = \kappa_j - iu\rho_j\delta_j, \;\; \theta_j^C = \frac{\kappa_j\theta_j}{\kappa_j^C} \tag{1.50}$$

Note $i = I, j$ for index and equity, respectively. $h = 1$ if $i = I$ and $h = 2$ if $i = j$.

1.8.3 Appendix C: Estimation Procedure

In this model, two vectors of latent variables $\{\sigma_{I,t}^2, \sigma_{I,t}^2\}$ and two sets of structural parameters $\{\Theta_I, \Theta_j\}$ need to be estimated where $\Theta_I \equiv \{\kappa_I^{\mathbb{Q}}, \theta_I^{\mathbb{Q}}, \delta_I, \rho_I\}$, and $\Theta_j \equiv \{\kappa_j, \theta_j, \delta_j, \rho_j\, \beta_j\}$. This involves two main steps. In the first step, the market index dynamic $\{\Theta_I, \sigma_{I,t}^2\}$ is estimated based on index option prices alone. In the second step, I take the market index dynamic as given, then estimate the firm-specific dynamics $\{\Theta_j, \sigma_{j,t}^2\}$. This step-wise estimation procedure (while not fully efficient) enables us to estimate the model for equities while ensuring that the same index dynamic is imposed for each of the individual equities. Christoffersen (2013) confirmed that this estimating technique has good finite sample properties in a Monte Carlo study.

Step 1: Market Index Volatility and Parameter Estimation Given a set of starting values, Θ_I^0, for the index structural parameters, I first estimate the spot market variance each day by sequentially solving

$$\hat{\sigma}^2_{I,t} = arg \min_{\sigma^2_{I,t}} \sum_{m=1}^{N_{I,t}} (C_{I,t,m} - C_m(\Theta^0_I, \sigma^2_{I,t}))^2 / Vega^2_{I,t,m}, t = 1, 2, ...T \tag{1.51}$$

where $C_{I,t,m}$ is the market price of index option contract m quoted at t, $C_m(\Theta_I, \sigma^2_{I,t})$ is the model index option price, $N_{I,t}$ is the number of index contracts available on day t, and $Vega_{I,t,m}$ is the Black-Scholes sensitivity of the option price with respect to volatility evaluated at the implied volatility. These vega-weighted price errors are good approximation to implied volatility errors and they are much more quickly computed. This method has been used in Carr and Wu (2007).

Once the set of T market spot variances have be obtained, we can solve for the set of market parameters as follows

$$\hat{\Theta}_I = arg \min_{\Theta_I} \sum_{m,t}^{N_I} (C_{I,t,m} - C_m(\Theta_I, \hat{\sigma}^2_{I,t}))^2 / Vega^2_{I,t,m}. \tag{1.52}$$

Iteration is needed between (20) and (21) until the improvement in fit is negligible.

Step 2: Equity Volatility and Parameter Estimation Given an initial value Θ^0_j and using the estimated $\hat{\sigma}^2_{I,t}$ and $\hat{\Theta}_I$, we can estimate the spot equity variance each day by sequentially solving

$$\hat{\sigma}^2_{j,t} = arg \min_{\sigma^2_{j,t}} \sum_{m=1}^{N_{j,t}} (C_{j,t,m} - C_m(\Theta^0_j, \sigma^2_{j,t}))^2 / Vega^2_{j,t,m}, t = 1, 2, ...T \tag{1.53}$$

Once the set of T individual equity spot variance have be obtained, the set of individual equity parameters can be estimated as follows

$$\hat{\Theta}_j = arg \min_{\Theta_j} \sum_{m,t}^{N_j} (C_{j,t,m} - C_m(\Theta_j, \hat{\sigma}^2_{j,t}))^2 / Vega^2_{j,t,m}. \tag{1.54}$$

1.8.4 Appendix D: Simulation Methods

The Euler scheme and the Milstein discretization are widely used in model simulation. The Euler scheme is a first-order method, it is the most basic explicit method for numerical integration of ordinary differential equations (ODE). While the disadvantage of the Euler scheme is its slow convergence. In this paper, I choose the Milstein scheme, which is a second-order method.

The corresponding scheme of discrete time stepping for index I_t is

$$I(t_{i+1}) = I(t_i) + I(t_i)\mu_I \Delta t + I(t_i)\sigma_I(t_i)\sqrt{\Delta t} W_i + \frac{1}{2}\sigma_I^2(t_i) I^2(t_i)\Delta t (W_i^2 - 1) \tag{1.55}$$

$$\sigma_I^2(t_{i+1}) = \sigma_I^2(t_i) + \kappa_I(\theta_I - \sigma_I^2(t_i))\Delta t_i + \delta_I \sigma_I(t_i)\sqrt{\Delta t} Z_i + \frac{1}{4}\delta_I^2 \Delta t (Z_i^2 - 1) \tag{1.56}$$

where W_i and Z_i are samples from a standard normal distribution with correlation equal to ρ_I. Note that $\mu_I, \kappa_I, \theta_I, \rho_I$ and δ_I are the parameters for the index process defined before.

The corresponding scheme of discrete time stepping for equity j is

$$\begin{aligned} S_j(t_{i+1}) = S_j(t_i) + S_j(t_i)(\alpha_j + r)\Delta t + \beta_j(\frac{I(t_{i+1}) - I(t_i)}{I(t_i)} - r\Delta t) \\ + S_j(t_i)\sigma_j(t_i)\sqrt{\Delta t} W_j + \frac{1}{2}\sigma_j^2(t_i) S_j^2(t_i)\Delta t (W_j^2 - 1) \end{aligned} \tag{1.57}$$

$$\sigma_j^2(t_{i+1}) = \sigma_j^2(t_i) + \kappa_j(\theta_j - \sigma_j^2(t_i))\Delta t_i + \delta_j \sigma_j(t_i)\sqrt{\Delta t} Z_j + \frac{1}{4}\delta_j^2 \Delta t (Z_j^2 - 1) \tag{1.58}$$

where W_j and Z_j are samples from a standard normal distribution with correlation equal to ρ_j. Note that $\alpha_j, \kappa_j, \theta_j, \beta_j, \rho_j$ and δ_j are the parameters for the equity process defined before.

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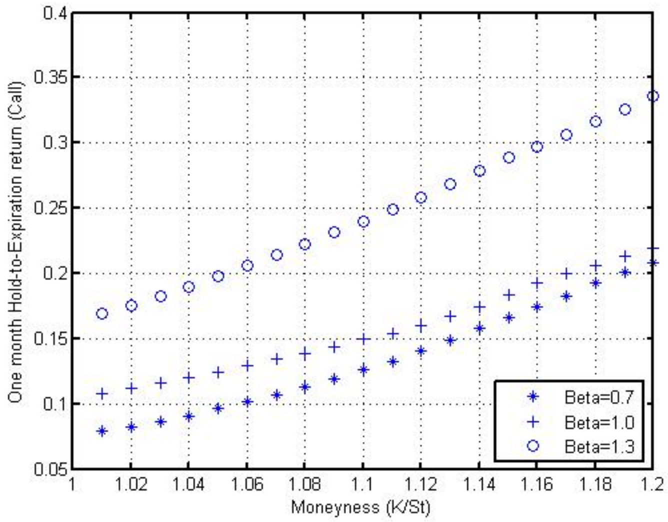
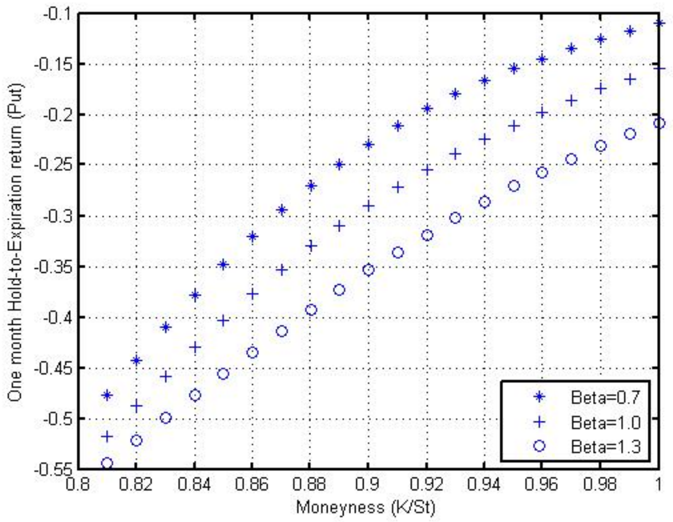
Figure 1.1: Expected Returns on Equity Options

Panel A: Call Options

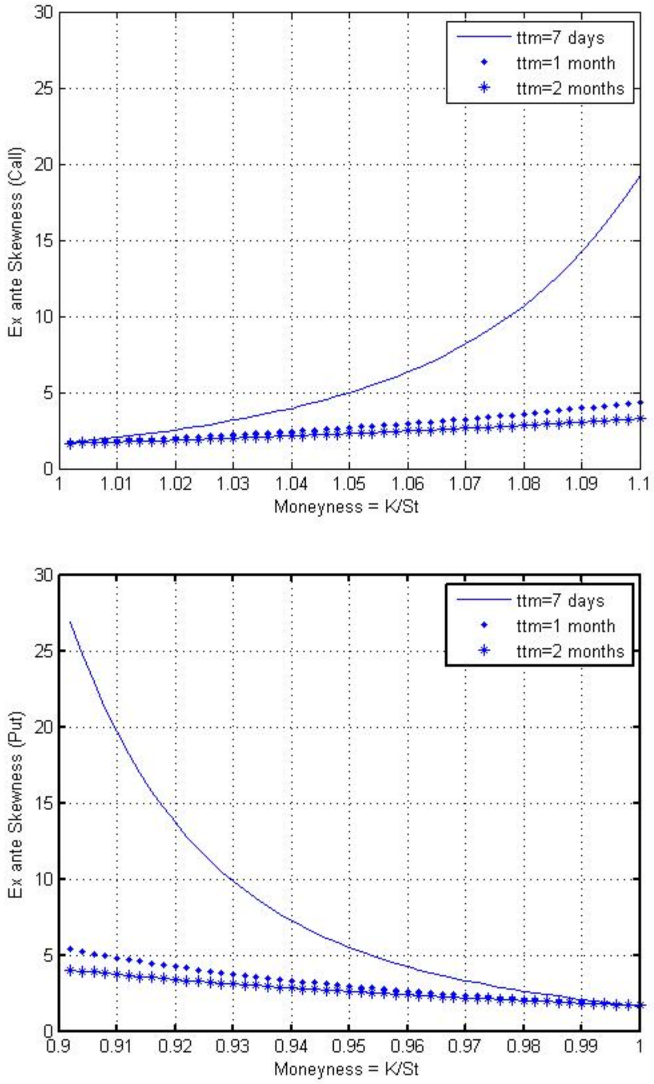


Panel B: Put Options



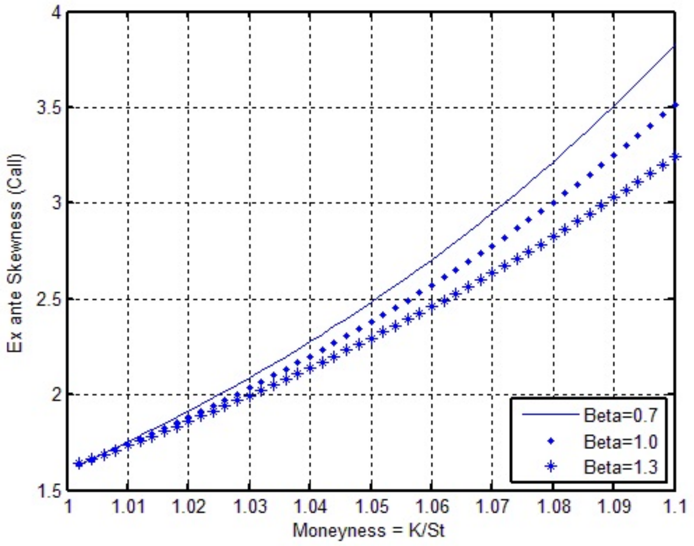
Note to Figure: In this figure, I plot the expected Hold-to-Expiration returns against betas on call and put using the model. Each line has a different beta. The parameters for the market index are $\kappa_I = 3.81$, $\theta_I = 0.0279$, $\delta_I = 0.456$, $\mu_I = 0.08$ and $\rho_I = -0.715$. The parameters for individual equity are $\kappa_j = 1.14$, $\theta_j = 0.0072$, $\delta_j = 0.128$, $\mu_j = 0.01$ and $\rho_j = -0.656$. The risk-free rate is set to 0.03. All the parameters are in annual basis.

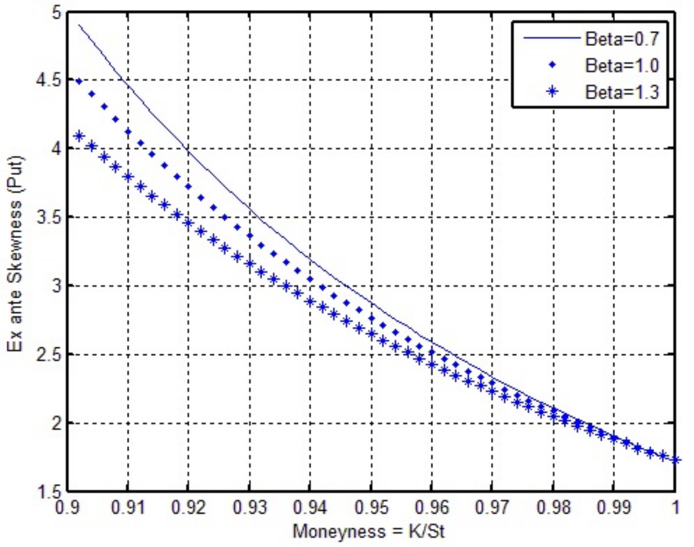
Figure 1.2: Option skewness against moneyness



Note to Figure: This figure plots option ex ante skewness against moneyness (K/S). The parameters for the market index are $\kappa_I = 3.81$, $\theta_I = 0.0279$, $\delta_I = 0.356$, $\mu_I = 0.08$ and $\rho_I = -0.715$. The parameters for individual equity are $\kappa_j = 1.29$, $\theta_j = 0.042$, $\delta_j = 0.329$, $\alpha_j = 0.013$ and $\rho_j = -0.474$. The risk-free rate is set to 0.03. All the parameters are in annual basis.

Figure 1.3: Option skewness against Beta





Note to Figure: This figure plots option holding return skewness against betas. The parameters for the market index are $\kappa_I = 3.81$, $\theta_I = 0.0279$, $\delta_I = 0.356$, $\mu_I = 0.05$ and $\rho_I = -0.715$. The parameters for individual equity are $\kappa_j = 1.14$, $\theta_j = 0.0072$, $\delta_j = 0.128$, $\alpha_j = 0.01$ and $\rho_j = -0.656$. The risk-free rate is set to 0.03. All the parameters are in annual basis.

Table 1.1: Physical Parameters. Index and Equities

Ticker	Beta	Kappa	Theta	Delta	Rho	Alpha/Mu
SPX		2.83	0.0383	0.371	-0.855	0.056
JNJ	0.72	0.8	0.0219	0.187	-0.566	-0.006
KO	0.75	0.9	0.0252	0.213	-0.571	0.000
PG	0.78	0.85	0.0317	0.233	-0.346	0.006
MCD	0.78	1.01	0.0451	0.302	-0.426	0.017
WMT	0.81	0.54	0.0494	0.231	-0.549	0.010
PFE	0.89	0.96	0.0323	0.248	-0.574	0.020
MMM	0.91	0.99	0.0153	0.174	-0.478	-0.006
TRV	0.92	0.54	0.0256	0.16	-0.565	0.053
VZ	0.89	0.73	0.0323	0.217	-0.545	-0.020
UTX	0.91	1.04	0.0247	0.226	-0.376	0.078
MRK	0.92	1.28	0.033	0.291	-0.495	0.010
IBM	0.97	1.24	0.0126	0.177	-0.598	0.049
CVX	0.88	0.85	0.0272	0.078	-0.458	0.008
DD	0.99	0.76	0.0113	0.126	-0.542	0.000
T	0.97	0.52	0.0229	0.055	-0.434	-0.058
XOM	0.97	0.5	0.0267	0.008	0.297	0.007
BA	0.99	1.07	0.0323	0.263	-0.523	0.041
HPQ	1.06	1.29	0.042	0.329	-0.474	0.013
BAC	1.11	0.15	0.0159	0.068	-0.724	0.020
DIS	1.08	0.95	0.0119	0.15	-0.496	0.012
MSFT	1.11	0.99	0.0131	0.14	-0.523	0.036
CSCO	1.17	0.96	0.0586	0.333	-0.529	0.015
INTC	1.16	1.24	0.023	0.23	-0.492	0.053
CAT	1.16	0.87	0.006	0.102	-0.466	0.065
GE	1.11	0.99	0.0022	0.029	-0.561	-0.004
HD	1.16	1.04	0.0142	0.17	-0.611	0.035
AA	1.18	1.04	0.0135	0.14	-0.37	0.000
AXP	1.24	0.81	0.0018	0.054	-0.6	0.094
JPM	1.21	1.14	0.0072	0.128	-0.656	0.072

For the parameters that are theoretically constrained to be equal across measures, I use $\mathbb{Q}$-measure parameters estimated by Christoffersen, Fournier and Jacobs (2013). The physical parameters for index process are estimated by Chambers el al. (2014). The rest of the equity parameters and the spot volatility are estimated based on the methods discussed before.

Table 1.2: Average Hold-to-Expiration Put Returns

	Moneyness: K/St				
Ticker	0.92	0.94	0.96	0.98	1.00
JNJ	-0.88	-0.92	-0.42	-0.57	-0.38
KO	-0.68	-0.50	-0.39	-0.27	0.00
PG	-0.73	-0.64	-0.35	-0.35	-0.19
MCD	-0.41	-0.59	-0.24	-0.16	-0.12
WMT	-0.40	-0.54	-0.69	-0.33	-0.36
PFE	-0.61	-0.80	-0.43	-0.12	0.01
MMM	-0.53	-0.48	-0.79	-0.37	-0.30
TRV	0.35	-0.40	-0.17	-0.18	-0.56
VZ	-0.50	-0.72	-0.54	-0.30	-0.15
UTX	-0.82	-0.28	-0.37	-0.47	-0.12
MRK	-0.43	0.31	-0.46	-0.65	-0.52
IBM	-0.17	-0.36	-0.05	-0.25	-0.27
CVX	-0.22	-0.39	-0.29	-0.10	-0.09
DD	-0.53	-0.55	-0.42	-0.34	-0.25
T	-0.25	-0.48	-0.50	-0.51	-0.16
XOM	-0.64	-0.23	-0.54	-0.48	-0.27
BA	-0.69	-0.22	-0.39	-0.17	-0.07
HPQ	-0.56	-0.14	-0.21	-0.34	-0.12
BAC	-0.19	-0.50	-0.29	-0.53	-0.16
DIS	-0.55	-0.81	-0.79	-0.57	-0.37
MSFT	-0.85	-0.55	-0.59	-0.55	-0.34
CSCO	-0.79	-0.12	-0.29	-0.42	-0.40
INTC	-0.47	-0.37	-0.18	-0.29	0.01
CAT	-0.85	-0.38	-0.68	-0.32	-0.61
GE	-0.66	-0.71	-0.29	-0.15	-0.21
HD	-0.54	-0.64	-0.53	-0.47	-0.03
AA	0.15	-0.14	0.22	-0.87	-0.30
AXP	0.00	-0.45	-0.35	-0.32	-0.17
JPM	-0.35	-0.46	-0.55	-0.19	-0.32
Mean	-0.48	-0.45	-0.40	-0.37	-0.24

The table reports the population average hold-to-expiration returns for 29 equity put options divided into five moneyness groups, from 8% OTM to ATM.

Table 1.3: Average Sharpe Ratios for Put Returns

	Moneyness: K/St				
Ticker	0.92	0.94	0.96	0.98	1.00
JNJ	-5.30	-4.87	-0.22	-0.58	-0.39
KO	-0.56	-0.24	-0.24	-0.21	-0.02
PG	-0.66	-0.42	-0.16	-0.24	-0.18
MCD	-0.22	-0.35	-0.13	-0.13	-0.11
WMT	-0.22	-0.32	-0.86	-0.25	-0.32
PFE	-0.59	-1.70	-0.42	-0.10	-0.01
MMM	-0.26	-0.32	-1.06	-0.28	-0.28
TRV	0.09	-0.40	-0.09	-0.12	-0.70
VZ	-0.33	-0.87	-0.43	-0.23	-0.16
UTX	-1.25	-0.13	-0.22	-0.37	-0.11
MRK	-0.23	0.04	-0.31	-0.72	-0.49
IBM	-0.04	-0.20	-0.04	-0.14	-0.23
CVX	-0.10	-0.18	-0.14	-0.08	-0.09
DD	-0.27	-0.34	-0.33	-0.24	-0.21
T	-0.07	-0.39	-0.48	-0.60	-0.18
XOM	-0.48	-0.10	-0.42	-0.39	-0.22
BA	-0.61	-0.08	-0.32	-0.14	-0.08
HPQ	-0.42	-0.05	-0.14	-0.23	-0.09
BAC	-0.11	-0.39	-0.20	-0.51	-0.17
DIS	-0.35	-1.29	-1.01	-0.48	-0.33
MSFT	-1.71	-0.37	-0.55	-0.54	-0.32
CSCO	-0.74	-0.07	-0.20	-0.40	-0.37
INTC	-0.28	-0.27	-0.12	-0.19	-0.02
CAT	-1.53	-0.21	-0.91	-0.24	-0.73
GE	-0.76	-0.69	-0.15	-0.11	-0.17
HD	-0.22	-0.56	-0.44	-0.43	-0.03
AA	0.03	-0.11	0.11	-3.08	-0.27
AXP	-0.01	-0.32	-0.23	-0.21	-0.16
JPM	-0.18	-0.30	-0.52	-0.14	-0.32
Mean	-0.60	-0.53	-0.35	-0.39	-0.23

The table reports the population Sharpe ratios for 29 equity put options divided into five moneyness groups, from 8% OTM to ATM.

Table 1.4: Simulated Hold-to-Expiration Put Returns

	Moneyness K/St				
Ticker	0.92	0.94	0.96	0.98	1.00
JNJ	-0.57	-0.47	-0.38	-0.29	-0.22
KO	-0.53	-0.43	-0.34	-0.27	-0.21
PG	-0.59	-0.49	-0.40	-0.32	-0.24
MCD	-0.53	-0.44	-0.36	-0.29	-0.23
WMT	-0.52	-0.43	-0.35	-0.28	-0.23
PFE	-0.48	-0.41	-0.34	-0.28	-0.23
MMM	-0.50	-0.41	-0.33	-0.26	-0.20
TRV	-0.51	-0.44	-0.37	-0.31	-0.26
VZ	-0.51	-0.42	-0.34	-0.27	-0.21
UTX	-0.53	-0.45	-0.39	-0.33	-0.28
MRK	-0.51	-0.44	-0.36	-0.30	-0.25
IBM	-0.48	-0.41	-0.34	-0.29	-0.24
CVX	-0.59	-0.52	-0.43	-0.35	-0.29
DD	-0.44	-0.36	-0.29	-0.23	-0.19
T	-0.44	-0.36	-0.29	-0.23	-0.18
XOM	-0.59	-0.51	-0.44	-0.38	-0.32
BA	-0.46	-0.39	-0.33	-0.28	-0.23
HPQ	-0.35	-0.30	-0.26	-0.22	-0.19
BAC	-0.34	-0.29	-0.24	-0.20	-0.16
DIS	-0.38	-0.31	-0.26	-0.21	-0.17
MSFT	-0.43	-0.37	-0.31	-0.26	-0.22
CSCO	-0.36	-0.31	-0.27	-0.23	-0.19
INTC	-0.34	-0.30	-0.26	-0.22	-0.19
CAT	-0.43	-0.37	-0.31	-0.27	-0.23
GE	-0.41	-0.34	-0.28	-0.23	-0.18
HD	-0.41	-0.36	-0.30	-0.26	-0.22
AA	-0.29	-0.24	-0.20	-0.17	-0.14
AXP	-0.37	-0.32	-0.28	-0.24	-0.21
JPM	-0.39	-0.34	-0.30	-0.26	-0.22
Mean	-0.46	-0.39	-0.32	-0.27	-0.22

I use Monte Carlo simulation to compute the distribution of average returns for equity put options. I simulate $N = 10000$ times of index and 29 equities levels using Milstein scheme simulation. It is assumed that all risk premiums (except for the equity premium) are equal to zero.

Table 1.5: Simulated Sharpe Ratios for Put Returns

	MoneynessK/St				
Ticker	0.92	0.94	0.96	0.98	1.00
JNJ	-0.36	-0.31	-0.27	-0.24	-0.21
KO	-0.33	-0.29	-0.25	-0.22	-0.19
PG	-0.40	-0.35	-0.30	-0.26	-0.23
MCD	-0.36	-0.32	-0.28	-0.24	-0.22
WMT	-0.35	-0.31	-0.28	-0.24	-0.21
PFE	-0.32	-0.28	-0.26	-0.23	-0.21
MMM	-0.31	-0.27	-0.24	-0.21	-0.19
TRV	-0.35	-0.31	-0.29	-0.27	-0.24
VZ	-0.34	-0.30	-0.26	-0.23	-0.20
UTX	-0.36	-0.33	-0.30	-0.28	-0.26
MRK	-0.35	-0.32	-0.28	-0.26	-0.23
IBM	-0.32	-0.29	-0.26	-0.24	-0.22
CVX	-0.43	-0.39	-0.35	-0.31	-0.28
DD	-0.27	-0.24	-0.21	-0.19	-0.17
T	-0.28	-0.25	-0.22	-0.19	-0.17
XOM	-0.41	-0.38	-0.36	-0.33	-0.31
BA	-0.31	-0.28	-0.26	-0.24	-0.22
HPQ	-0.24	-0.22	-0.20	-0.19	-0.18
BAC	-0.22	-0.20	-0.18	-0.17	-0.16
DIS	-0.23	-0.21	-0.19	-0.17	-0.16
MSFT	-0.29	-0.26	-0.24	-0.22	-0.21
CSCO	-0.26	-0.24	-0.22	-0.20	-0.19
INTC	-0.23	-0.21	-0.20	-0.19	-0.18
CAT	-0.27	-0.26	-0.24	-0.22	-0.21
GE	-0.25	-0.23	-0.21	-0.19	-0.17
HD	-0.27	-0.25	-0.23	-0.22	-0.20
AA	-0.18	-0.17	-0.16	-0.15	-0.14
AXP	-0.23	-0.22	-0.21	-0.20	-0.19
JPM	-0.26	-0.24	-0.23	-0.22	-0.21
mean	-0.30	-0.27	-0.25	-0.23	-0.21

I use Monte Carlo simulation to compute the distribution of Sharpe ratios for equity put options. I simulate $N = 10000$ times of index and 29 equities levels using Milstein scheme simulation. It is assumed that all risk premiums (except for the equity premium) are equal to zero.

Table 1.6: Average Put Returns, Sharpe Ratios, and P-values

		Moneyness: K/St				
		0.92	0.94	0.96	0.98	1.00
	Data	-0.48	-0.45	-0.40	-0.37	-0.24
Returns	Model	-0.46	-0.39	-0.32	-0.27	-0.22
	P-value %	75.3	19.7	7.2	0.5	56.4
	Data	-0.60	-0.53	-0.35	-0.39	-0.23
Sharpe Ratios	Model	-0.30	-0.27	-0.25	-0.23	-0.21
	P-value %	11.8	12.9	5.8	10.6	42.3

The top panel of Table 6 summarizes the average returns for put options of 29 equities for various strikes. The p-values are computed based on the distributions of simulations and actual data. The distributions were constructed from 10,000 simulations for each of the stocks. The bottom panel reports the similar metrics for Sharpe Ratios.

Table 1.7: Ex Ante Skewness Measure

	Moneyness = 0.96		
Maturity	2 weeks	1 month	2 months
JNJ	3.57	2.88	2.49
KO	3.41	2.79	2.45
PG	3.56	2.90	2.53
MCD	3.19	2.67	2.38
WMT	3.12	2.62	2.33
PFE	3.08	2.62	2.37
MMM	3.34	2.78	2.47
TRV	3.11	2.68	2.46
VZ	3.11	2.60	2.30
UTX	3.26	2.81	2.61
MRK	3.12	2.64	2.38
IBM	3.07	2.66	2.47
CVX	3.42	2.83	2.51
DD	3.08	2.62	2.37
T	2.98	2.49	2.18
XOM	3.53	2.92	2.61
BA	2.95	2.57	2.37
HPQ	2.57	2.27	2.10
BAC	2.62	2.31	2.12
DIS	2.91	2.52	2.32
MSFT	2.89	2.53	2.36
CSCO	2.47	2.19	2.03
INTC	2.61	2.34	2.21
CAT	2.91	2.58	2.44
GE	2.93	2.53	2.32
HD	2.86	2.52	2.35
AA	2.54	2.25	2.08
AXP	2.78	2.50	2.40
JPM	2.74	2.46	2.34

This table illustrates the distribution of put option ex ante skewness for fixed moneyness ($K/S = 0.96$) with five different maturities. The ex ante skewness is computed based on the parameters from β-Heston model

Table 1.8: Average Returns and P-values

	Days to Expiration								
Skew Quintile	14			30			60		
	Dat	Sim	p-value	Dat	Sim	p-value	Dat	Sim	p-value
Low	-0.18	-0.19	0.90	-0.24	-0.22	0.54	-0.25	-0.25	0.98
2	-0.28	-0.25	0.48	-0.36	-0.26	0.01	-0.28	-0.28	0.94
3	-0.37	-0.32	0.43	-0.38	-0.31	0.09	-0.35	-0.32	0.63
4	-0.44	-0.43	0.87	-0.44	-0.38	0.19	-0.32	-0.36	0.51
High	-0.55	-0.60	0.18	-0.51	-0.48	0.64	-0.53	-0.44	0.15
Low-High	0.37	0.41		0.27	0.27		0.28	0.19	
(t-stat)	8.02	29.33		4.32	27.89		3.27	18.08	

Table 8 reports average portfolio returns from data and simulations for each ex ante skewness/maturity bin. The portfolios are constructed by sorting on ex ante skewness and the returns are Hold-to-Expiration returns, using the midpoint of the bid and ask prices as the proxy for price. The final two rows report differences in average returns across the high and low skewness quintiles along with t-statistics that test whether these differences are equal to zero.

Table 1.9: Hold-to-Expiration delta-hedged return

	Total Volatility			Idiosyncratic Volatility		
Volatility Quintile	Dat %	Sim %	p-value	Dat %	Sim %	p-value
Low	-0.34	-0.74	0.02	-0.63	-0.82	0.03
2	-0.69	-0.88	0.97	-0.51	-0.96	0.62
3	-0.82	-0.94	0.55	-0.82	-0.99	0.39
4	-0.62	-1.17	0.09	-0.74	-0.98	0.76
High	-1.94	-1.18	0.00	-1.72	-1.14	0.00
Mean	-0.89	-0.82	0.71	-0.89	-0.85	0.68
Low-High	1.61	0.44		1.09	0.32	
(t-stat)	(5.04)	(12.68)		(2.96)	(9.65)	

Table 9 summarize the delta-hedged portfolio returns from actual data and simulations based on two sorting criteria: Total volatility and Idiosyncratic volatility. The portfolio is constructed by sorting the underlying total/idiosyncratic volatility. At the maturity of the put option, I rank the strategy returns into five quintiles based on the underlying idiosyncratic volatility (same procedures are repeated for total volatility). Note that the simulated portfolio returns are sorting based on total long-term volatility $\sqrt{\theta_j + \beta_j^2 \cdot \theta_i}$ and idiosyncratic long-term volatility $\sqrt{\theta_j}$.

Chapter 2

Option-Based Measures of Co-Skewness and Co-Kurtosis Risk

Abstract

The specification and estimation of factor models is of paramount importance for research and practice. There is a growing amount of literature about extracting information from the option market to forecast asset returns and volatilities. In Chapter 2, I derive a new formula that expresses the measures of covariance, co-skewness, and co-kurtosis risk in terms of market risk-neutral moments and co-moments between stock and index returns. I then use the forward-looking information contained in the option prices to estimate option-implied moments and higher-moment correlations in order to construct market risk betas, co-skewness betas, and co-kurtosis betas. The empirical analysis suggests the following findings: compared to regression-based standard competitors, such as CAPM, the method that I have devised performs better in terms of mean squared error and R^2. An out-of-sample analysis of factor models incorporating co-skewness and co-kurtosis risk premium indicates that the new risk measures improve the return prediction. My results suggest that using option market information improves asset pricing in terms of model fit as well as out-of-sample forecasting power.

2.1 Introduction

The specification and estimation of factor models is of paramount importance for research and practice, and the methods of doing so are still debatable. The linear form of the risk-return relation suggested by the Capital Asset Pricing Model (CAPM) has been criticized based from different perspectives. The empirical findings in the literature contradict one of the fundamental principles in finance—that higher risk is associated with a higher expected return—posing as one of the major puzzles in finance literature. Consequently, there is a widespread consensus that models with better explanatory power are badly needed. Kraus and Litzenberger (1976) show that if investors care about portfolio skewness, co-skewness (which is the cornerstone of skewness) is relevant for asset pricing along with co-variation with the market portfolio. Similarly, if investors care about portfolio kurtosis, co-kurtosis is also relevant. I propose a new strategy to estimate the measures of co-skewness and co-kurtosis risk from option prices, along with the price of the corresponding risk, which can be used to forecast asset returns. The method is naturally forward-looking; thus, it avoids problems inherent to the use of cross-sectional regressions.

Risk premium is obviously a forward-looking concept. In essence, it compensates investors for holding an asset that will yield an uncertain return. In practice, however, the most commonly used method for estimating the risk premium is based on time series data. Conceptually, using historical excess return relies on the belief that noise will be canceled out in the long run. Thus, using historical risk premium is subject to the trade off between reflecting recent market condition and estimation accuracy. Merton (1980) argues that the historical risk premium fails to account for the effect of changes in the market risk.

Christofferssen et al. (2016) propose a new strategy to estimate the price of co-skewness and co-kurtosis risk from option prices that avoids problems inherent to the use of two-stage cross-sectional regressions. They show that the price of co-skewness risk corresponds to the spread between the physical and risk-neutral second moment, which is the market variance risk premium. In addition, the price of co-kurtosis risk is given by the spread between the physical and risk-neutral third moment, which is the market skewness risk premium. The information needed to pin down the price of risk comes from option prices, which are naturally time-varying and forward-looking. In simple terms, when moving into a volatile phase, investors are subject to higher uncertainty, and therefore a forward-looking risk premium should become higher. Then immediate price changes could be observed in the option market, while a historically-based estimation cannot be expected to reflect the changing market conditions.

Although it is appealing to consider co-skewness and co-kurtosis risk in cross-sectional asset pricing, there is no widespread consensus on their empirical relevance. This has been shown even in the simplest case when only covariance risk is considered : the CAPM has been heavily tested over the years; however, it has often been rejected. For instance, studies by Lakonishok and Sharpiro (1986), and Fama and French (1992) find no relation between market beta and average returns during the 1963-1990 period; further, recently, Baker, Bradley, and Wurgler (2011) show that high-beta stocks significantly underperform low-beta stocks.

Existing techniques for beta estimation use historical returns. These methods thus assume that the future will be sufficiently similar to the past, justifying simple extrapolation of current or lagged betas. However, no matter how sophisticated the modeling of time-variation in the betas, they are not able to capture sudden changes in the market. In this paper, I use option-implied information to estimate the exposure to co-skewness and co-kurtosis risk (namely, co-skewness beta and co-kurtosis beta, respectively). Option prices are inherently forward-looking and therefore contain valuable information of the future betas as opposed to the lagged ones. These measures can be computed using option data on a single day and, therefore, it is potentially possible to reflect sudden changes in the structure of underlying companies.

There is a growing amount of literature about extracting information from the option market to forecast asset returns and volatilities. Bakshi and Madan (2000) propose a model-free method to estimate the risk-neutral moments of underlying from option prices. For instance, Conrad, Dittmar, and Ghysels (2013) use option prices to estimate ex ante higher moments of risk-neutral returns distribution of underlying individual securities. They find that securities' risk-neutral volatility, skewness, and kurtosis are strongly related to the future returns. Many studies have demonstrated that option-implied volatility is a strong predictor of future volatility in equity markets. Classic contributions in this field include the ones from Christensen and Prabhala (1998), as well as Blair, Poon, and Taylor (2001). The predictive power of option-implied equity volatility has been confirmed recently by Busch, Christensen, and Nielsen (2008), who compare option-implied forecasts with state-of-the-art realized volatility forecasts.

To provide a completely forwarding-looking forecast of equity expected returns, we also need to estimate the corresponding risk premium. The most common way to do this is to calculate the average value of historical realized returns for a given period. However, it is very likely that the risk premium, reflecting a level of risk that is related to a different state of the world, will not occur later in the sample. Elton (1999) points out that future work in asset pricing should consider alternative ways to measure expected returns other than relying on the ex-post realized returns.

Chalamandaris and Rompolis (2016) propose another method to solve this issue. They extend the theoretical model of Duan and Zhang (2014) to a general system of equations that connects the cumulants of the physical distribution of any order to those of the risk-neutral cumulants through the projected relative risk aversion coefficient (PRRAC). Clearly, investors require a higher compensation to hold the market portfolio if a) they are more risk averse; b) they expect that future returns would be more volatile, more negatively skewed, and have a higher level of excess kurtosis. Another implication of their method is that it restricts the shape discrepancy between the physical and risk-neutral distributions by means of the PRRAC.

In this paper, I derive a new formula that expresses the measures of the co-variance, co-skewness, and co-kurtosis risk in terms of the risk-neutral moments of the market return and the higher order risk-neutral co-moments between the market and individual stock returns. I show that, compared to traditional regression-based methods, option-implied parameters perform well empirically. One source of the inputs to the formula—risk-neutral moments are computed directly from option prices; this is forward-looking and time varying. Another source of the inputs—risk-neutral co-moments are estimated by utilizing both information from stock return time series and the option market. Consequently, my approach has several distinctive features that separate it from conventional approaches.

First, because it is based on the current market prices instead of, for instance, accounting information, it can be implemented in real time. In principle, with the data available, we can update the parameters daily.

Second, my approach generates conditional forecasts at individual stock level. Rather than providing a vague unconditional average expected return on a portfolio of large value stocks, the new method is able to answer, for example, "what is the expected return on Microsoft today?".

Third, the parameters have specific, quantitative expressions that can be calculated using current market information. This is contrasted with factor models, in which both factor loadings and factors are estimated from time series data. The classical CAPM model requires forward-looking betas, which, in reality, are estimated based on historical data. My method provides a new perspective to solve this issue.

I empirically investigate the performance of my approach for the returns of individual stock and portfolio. Based on the monthly data for the period 2005-2014, I find that my option-implied forward-looking betas outperform regression-based estimates for both individual and portfolio returns. My results also indicate that the higher moment (co-skewness and co-kurtosis) risk premiums are priced in the cross-sectional asset returns. I further show that, with option-implied estimates for the higher moment risk premium, the predicted returns of portfolio are highly correlated with future realized ones.

This paper proceeds as follows. Section 2 gives a review of the method that Christoffersen et al. (2016) used to estimate the price of co-skewness and co-kurtosis risk. In Section 3, I derive the measures of co-skewness and co-kurtosis risk in terms of index returns risk-neutral moments and covariance, co-skewness, and co-kurtosis between individual equity returns and market index returns. Section 4 presents the estimation strategy for risk-neutral covariance, co-skewness, and co-kurtosis between equities and index returns. Section 5 provides the empirical results for the in-sample fit. Section 6 investigates the estimation of option-implied higher moment risk premium. Finally, Section 7 concludes.

2.2 The Price of Co-Skewness and Co-Kurtosis Risk

Absence of arbitrage implies the existence of a stochastic discount factor, m_{t+1}, that prices any asset with risky return, $R_{j,t+1}$, using the following condition:

$$E_t^P\left[(1+R_{j,t+1})m_{t+1}\right] = 1 \tag{2.1}$$

where $E_t^P(.)$ denotes the expectation under the physical measure. Assume that the SDF can be written as a representative investor's marginal rate of substitution between current and future wealth. Under no arbitrage condition, the stochastic discount factor m_{t+1} must be nonnegative.

$$m_{t+1} = \frac{U'(W_{t+1})}{U'(W_t)}, \tag{2.2}$$

where $U'(.)$ is the marginal utility and W is the aggregate wealth. Since the marginal rate of substitution is not directly observable, to obtain testable restrictions from this first order condition, we usually define observable proxies for the marginal rate of substitution. Different proxies for the marginal rate of substitution and mechanisms have been proposed in different asset pricing models during the last four decades. Researchers use either observed returns of financial assets such as equity portfolios or non-market variables such as growth rate in aggregate consumption as the proxies for the marginal rate of substitution. Its form and specification is determined jointly by the assumptions about preferences and distribution of the proxies. As been pointed out by Harvey and Siddique (2000), the specification for the marginal rate of substitution can be viewed as a restriction on the set of trading strategies that the investors can use to maximize their utility.

Arrow (1971) argues that the desirable properties for an investors' utility function are (a) positive marginal utility for wealth, (b) decreasing marginal utility for wealth, and (c) non-increasing absolute risk aversion. The set of utility functions displaying these attributes are logarithmic, power and negative exponential utility functions. Since the exact form of utility function is unknown, they can be expanded as Taylor series:

$$m_{t+1} \approx h_0 + h_1\frac{U''}{U'}R_{m,t+1} + h_2\frac{U'''}{U'}R_{m,t+1}^2 + ..., \tag{2.3}$$

where $R_{m,t+1}$ is the stock market return, which Christoffersen et al. (2016) used as a proxy for the return on the wealth portfolio. From equation (1), we have $E_t^P(m_{t+1}) = \frac{1}{1+R_{f,t}}$ for the risk-free rate, which gives

$$\frac{1}{(1+R_{f,t})} \approx h_0 + h_1 \frac{U''}{U'} E_t^P(R_{m,t+1}) + h_2 \frac{U'''}{U'} E_t^P(R_{m,t+1}^2) + ... \tag{2.4}$$

Combining equation (1) and (3), we can write the following form for the SDF

$$\begin{aligned} m_{t+1} = a_t + b_{1,t}\big(R_{m,t+1} - E_t^P(R_{m,t+1})\big) \\ +b_{2,t}\big(R_{m,t+1}^2 - E_t^P(R_{m,t+1}^2)\big) + b_{3,t}\big(R_{m,t+1}^3 - E_t^P(R_{m,t+1}^3)\big) \end{aligned} \tag{2.5}$$

Equation(3) shows that the stochastic discount factor can be cubic in the market return. The cubic form is consistent with investor's preference for higher order moments, such as skewness and kurtosis. In this case, the expected excess return will be related to co-kurtosis risk, apart from covariance risk and co-skewness risk. Dittmar (2002) explains that kurtosis measure the possibility of extreme values and co-kurtosis captures the sensitivity of asset returns to extreme market returns. if investors are averse to extreme values, they need compensation for holding co-kurtosis risk, in other word, the price of co-kurtosis risk should be positive.

As discussed by Harvey and Siddique (2000), in the traditional CAPM world, there are usually two routes: (a) A two-period world with homogeneous agents, where the representative agent's derived utility function (in wealth) is quadratic or logarithmic which guarantees that the discount factor is linear in the value-weighted portfolio of wealth; (b) Make distributional assumptions on the asset returns, which also keep the discount factor linear in the value-weighted portfolio of wealth. The assumption that the SDF is linear in the market return produces the classic CAPM model.

Christoffersen et al. (2016) propose that in the absence of arbitrage, if the SDF has the form as equation (5), then the cross-sectional restriction on stock returns is

$$E_t^P(R_{j,t+1}) - R_{f,t} = \lambda_t^{MKT} \beta_{j,t}^{MKT} + \lambda_t^{COSK} \gamma_{j,t}^{COSK} + \lambda_t^{COKU} \delta_{j,t}^{COKU} \tag{2.6}$$

The prices of the *covariance* risk, λ_t^{MKT}, *co-skewness* risk λ_t^{COSK} and *co-kurtosis* risk, λ_t^{COKU}, are

$$\lambda_t^{MKT} = E_t^P(R_{m,t+1}) - R_{f,t}, \tag{2.7}$$

$$\lambda_t^{COSK} = E_t^P(R_{m,t+1}^2) - E_t^Q(R_{m,t+1}^2) \tag{2.8}$$

$$\lambda_t^{COKU} = E_t^P(R_{m,t+1}^3) - E_t^Q(R_{m,t+1}^3) \tag{2.9}$$

The model shows that the price of co-skewness risk is corresponding to the spread between the physical and the risk-neutral second moments of the market return. There is quite a number of studies in the literature for modeling the physical volatility of stock returns and estimating variance risk premium (e.g., Carr and Wu 2009). Bakshi and Madan (2006) also relate the volatility spread to risk aversion, Driessen, Maenhout and Vilkov (2009) study the price of correlation risk based on risk-neutral variance of index and its components.

Empirical studies conclude that the physical variance is lower than the risk-neutral variance, indicating a negative price of co-skewness risk (for instance, Bollerslev, Tauchen, and Zhou (2009), Bakshi and Madan (2006), and Jackwerth and Rubinstein (1996)). A negative price of co-skewness risk is intuitive: assets with lower co-skewness decrease the total skewness of the portfolio (more negative), and increase the probability of extreme losses. Thus, assets with lower co-skewness should have higher risk premium by the risk averse investors.

The price of co-kurtosis is equal to the spread between the risk-neutral and physical third moments. Current literature indicate that the risk-neutral distribution of index return is more left skewed than the physical ones, indicating a positive price of co-kurtosis risk. This is again consistent with theory, similar logic of covariance risk premium can apply here.

2.3 The Measures of Co-Skewness and Co-Kurtosis Risk

The usual way to estimate the measures of co-skewness and co-kurtosis risk is in a multivariate regression framework. However, in the following proposition, I show that these measures can be analytically solved by a system of linear equations, in terms of risk-neutral moments and co-moments of index and equity returns.

Proposition 1. *If the cross-sectional pricing restrictions are*

$$E_t^P(R_{j,t+1}) - R_{f,t} = \lambda_t^{MKT}\beta_{j,t}^{MKT} + \lambda_t^{COSK}\beta_{j,t}^{COSK} + \lambda_t^{COKU}\beta_{j,t}^{COKU}, \tag{2.10}$$

and the prices of the covariance *risk,* λ_t^{MKT}*,* co-skewness *risk* λ_t^{COSK} *and* co-kurtosis *risk,* λ_t^{COKU}*, are defined as*

$$\lambda_t^{MKT} = E_t^P(R_{m,t+1}) - R_{f,t}, \tag{2.11}$$

$$\lambda_t^{COSK} = E_t^P(R_{m,t+1}^2) - E_t^Q(R_{m,t+1}^2) \tag{2.12}$$

$$\lambda_t^{COKU} = E_t^P(R_{m,t+1}^3) - E_t^Q(R_{m,t+1}^3) \tag{2.13}$$

Then measures of the corresponding risk are solved by the following system linear equations

$$\begin{bmatrix} E^Q[(\tilde{R}_j - \bar{R}_j)(\tilde{R}_m - \bar{R}_m)] \\ E^Q[(\tilde{R}_j - \bar{R}_j)(\tilde{R}_m - \bar{R}_m)^2] \\ E^Q[(\tilde{R}_j - \bar{R}_j)(\tilde{R}_m - \bar{R}_m)^3] \end{bmatrix} = \begin{bmatrix} k_{m,2} & k_{m,3} & k_{m,4} \\ k_{m,3} & k_{m,4} - k_{m,2}^2 & k_{m,5} - k_{m,3}\cdot k_{m,2} \\ k_{m,4} & k_{m,5} - k_{m,2}\cdot k_{m,3} & k_{m,6} - k_{m,3}^2 \end{bmatrix} \cdot \begin{bmatrix} \beta_j^{MKT} \\ \gamma_j^{COSK} \\ \delta_j^{COKU} \end{bmatrix}$$

where $k_{m,n} = E^Q[(\tilde{R}_m - \bar{R}_m)^n]$.

The exact solution for betas can be found in Appendix A.

Proof. We could express equation (10) as (drop time subscript):

$$\begin{aligned} \tilde{R}_j - R_f = c_j + \beta_j^{MKT}\left(\tilde{R}_m - R_f\right) + \beta_j^{COSK}\left(\tilde{R}_m - \bar{R}_m\right)^2 \\ + \beta_j^{COSK}\left(\tilde{R}_m - \bar{R}_m\right)^3 + \tilde{e}_j \end{aligned} \tag{2.14}$$

where $\bar{R}_i$ denotes the $E^Q[R_i]$ for $i \in \{j, m\}$, and the zero-mean error term, $\tilde{e}_j$ is assumed to be independent of $\tilde{R}_m - R_f$, $(\tilde{R}_m - \bar{R}_m)^2$ and $(\tilde{R}_m - \bar{R}_m)^3$; Indeed, if we take expectation under physical measure of equation (14), write $[c_j]$ in terms of physical and risk-neutral moments of R_m, we could find the link (equivalence) between equation (10) and (14).

Take the expectation of equation (14) under risk-neutral measure, subtracted from itself:

$$\begin{gathered}(10) - E^Q[(10)] \Rightarrow \tilde{R}_j - \bar{R}_j = \beta_j^{MKT}(\tilde{R}_m - \bar{R}_m) + \\ \gamma_j^{COSK}\{(\tilde{R}_m - \bar{R}_m)^2 - k_{m,2}\} + \delta_j^{COKU}\{(\tilde{R}_m - \bar{R}_m)^3 - k_{m,3}\}\end{gathered} \tag{2.15}$$

where $k_{m,n} = E^Q[(\tilde{R}_m - \bar{R}_m)^n]$, the n_{th}-moment under risk-neutral measure.
Multiply equation (15) both sides by $(\tilde{R}_m - \bar{R}_m)$, and take the expectation under $\mathbb{Q}$:

$$E^Q[(\tilde{R}_j - \bar{R}_j)(\tilde{R}_m - \bar{R}_m)] = \beta_j^{MKT} k_{m,2} + \gamma_j^{COSK} k_{m,3} + \delta_j^{COKU} k_{m,4} \tag{2.16}$$

on the left side of equation (16), it is just the RN covariance between individual stock return R_j and market index return R_m. Repeat previous procedure, multiply equation (15) both sides by $(\tilde{R}_m - \bar{R}_m)^2$ and $(\tilde{R}_m - \bar{R}_m)^3$, respectively, then take the expectation, so that we have equation (17) and (18):

$$\begin{gathered}E^Q[(\tilde{R}_j - \bar{R}_j)(\tilde{R}_m - \bar{R}_m)^2] = \beta_j^{MKT} k_{m,3} + \\ \gamma_j^{COSK}(k_{m,4} - k_{m,2}^2) + \delta_j^{COKU}(k_{m,5} - k_{m,3} \cdot k_{m,2})\end{gathered} \tag{2.17}$$

on the left side of the equation (17), it is the risk-neutral co-skewness.

$$\begin{gathered}E^Q[(\tilde{R}_j - \bar{R}_j)(\tilde{R}_m - \bar{R}_m)^3] = \beta_j^{MKT} k_{m,4} + \\ \gamma_j^{COSK}(k_{m,5} - k_{m,2} \cdot k_{m,3}) + \delta_j^{COKU}(k_{m,6} - k_{m,3}^2)\end{gathered} \tag{2.18}$$

on the left side of the equation (18), it is the risk-neutral co-kurtosis. Combine equation (16), (17) and (18), we could express β_j^{MKT}, β_j^{COSK} and β_j^{COKU} in terms of risk-neutral moments (which can be approximated by OTM option prices, see Appendix) and covariance, co-skewness and co-kurtosis. □

First to note that, as the pricing restriction (equation 10) is a natural extension to CAPM model, so is the measurement of corresponding risk. To see this, considering in a world when skewness and kurtosis risk is not priced, drop the high-moment related item in the matrix, we could have the expression for market Beta, $\beta_j^{MKT} = \frac{cov(R_j, R_m)}{var(R_m)}$. It is exactly the same as CAPM model.

The parameters $\beta_{i,t}^{MKT}$, $\beta_{i,t}^{COSK}$ and $\beta_{i,t}^{COKU}$, are functions of the market higher order moments (variance, skewness, kurtosis, etc), covariance, co-skewness and co-kurtosis, illustrate the relationship between my model and the Kraus and Litzenberger (1976) three-moment CAPM as well as Harvey and Siddique (2000) conditional skewness measure. It provides a testable restriction imposed on the cross section of asset expected returns from the asset pricing model incorporating skewness and kurtosis risk.

Empirical studies conclude that no one model solves the asset pricing puzzle and different combinations of factors work for different settings. Thus, in this paper, I proposed an asset pricing model that is a combination of the multifactor model with nonlinear components derived from asset return (co)skewness and (co)kurtosis. This is also consistent with Ghysels (1998) that nonlinear multi factor models behave better than linear beta model in the empirical studies.

2.3.1 General Case

Although there is relatively little research about the sign of terms in the SDF higher than third order, Christoffersen et al. (2016) give a more general nonlinearities form in the SDF, and its corresponding pricing model. If the stochastic discount factor (SDF) has the following form:

$$m_{t+1} = a_t + \sum_n b_{k,t}\big(G_n(R_{m,t+1}) - E_t^P[G_n(R_{m,t+1})]\big) + \sum_l c_{l,t}\big(f_{l,t+1} - E_t^P(f_{l,t+1})\big), \quad (2.19)$$

then the cross sectional pricing restrictions are

$$E_t^P(R_{j,t+1}) - R_f = \sum_n \lambda_t^n \beta_{j,t}^n + \sum_l \gamma_t^l \beta_{j,t}^l \quad (2.20)$$

and

$$E_t^P(R_{i,t+1}) - R_f = \sum_n \lambda_t^n \beta_{i,t}^n + \sum_l \gamma_t^l \beta_{i,t}^l, \quad (2.21)$$

where the β_t^n and β_t^l are from the projection of asset returns on $G_n(R_{m,t+1})$ and $f_{l,t+1}$ respectively, and γ^l is the price of risk associated with the factor f_l. The price of corresponding risk associated with market return λ_t^n, is

$$\lambda_t^n = E_t^P\big(G_n(R_{m,t+1})\big) - E_t^Q\big(G_n(R_{m,t+1})\big), \quad (2.22)$$

where $E_t^P(.)$ and $E_t^Q(.)$ denote the expectation under the physical and risk-neutral probability measure.

Proposition 2. *If the pricing restriction is in the form of equation (19), then the measures of the corresponding risk β can be calculated from following equation set, $k_{m,n} = E^Q[(\tilde{R}_m - \bar{R}_m)^n]$:*

$$\begin{bmatrix} E^Q[(\tilde{R}_j - \bar{R}_j)(\tilde{R}_m - \bar{R}_m)] \\ E^Q[(\tilde{R}_j - \bar{R}_j)(\tilde{R}_m - \bar{R}_m)^2] \\ ... \\ E^Q[(\tilde{R}_j - \bar{R}_j)(\tilde{R}_m - \bar{R}_m)^n] \end{bmatrix} = \begin{bmatrix} k_{m,2} & k_{m,3} & k_{m,4} \\ k_{m,3} & k_{m,4} - k_{m,2}^2 & k_{m,5} - k_{m,3} \cdot k_{m,2} \\ ... & ... & ... \\ k_{m,n} & k_{m,n+2} - k_{m,2} \cdot k_{m,n} & k_{m,k+3} - k_{m,3} \cdot k_{m,n} \end{bmatrix} \cdot \begin{bmatrix} \beta_j^{MKT} \\ \beta_j^{COSK} \\ ... \\ \beta_j^{COn_{th}} \end{bmatrix}$$

Proof. The structure of the proof is similar to the proof of Proposition 1. The only extension is to multiply equation (15) both sides by $(\tilde{R}_m - \bar{R}_m)^n$, then apply the same procedure described in the proof of Proposition 1. □

2.4 Implied Moments and Correlation

In order to estimate the betas in equation (10), we need the conditional moments-co-moments (e.g., variance-covariance matrix) of the factors and vector of the conditional co-moments between the factors and the stock returns. The usual way to estimate these parameters is using historical stock returns. Nevertheless, what we really need the is the conditional moments-co-moments matrix, the use of historical data implies that the future is sufficiently similar to the past.

Instead of using solely the historical time series of the stock returns, we can estimate these moments from option prices, and use these information to construct the beta-parameters. There is a long history of using option implied information to forecast expected/future realized moments. For instance, the implied volatility of Black and Scholes (1973), or model free option implied moments of Britten-Jones and Neuberger (2000) and Bakshi, Kapadia, and Madan (2003).

The reason for using option implied information in the estimation is that the option prices subsume the current market expectations about future stock dynamics (e.g., Vanden (2008)), which is natural forward looking and time-varying. As shown by Blair, Poon and Taylor (2001), the implied volatility has better predictability for realized volatility in terms of R^2. Thus, using option-implied information in the parameters construction can potential increase their predictability.

Risk-neutral moments for stock and index can be easily computed from the observed option data. However, risk neutral co-moments pose a challenge, regarding to modeling and estimation. Buss and Vilkov (2012) propose a parametric way to estimate implied correlations such that the correlation matrix (a) meets all the necessary requirements of a correlation matrix (positive definite with absolute pairwise correlation smaller than one), (b) satisfies the identifying restriction that the weighted sum of index constituents implied variance is equal to the implied variance of the index. They use historical rolling window correlations (computed from daily and monthly returns) as input to the identification procedure of the implied correlations. In Appendix C, I illustrate the extended version of Buss and Vilkov (2012) method, to construct the estimates for the co-moments.

2.5 Data Description

This study is based on the major U.S. market proxy, the S&P100 index, and its constituents for period from January 4, 2005 to November 31, 2014. In section 5.1, I briefly describe the stock and option data. In section 5.2, I introduce the estimation method for realized and option-implied measures.

2.5.1 Stock and Option Data

The daily stock data consist of prices (close price) and number of shares outstanding, plus S&P 100 index prices from *OptionMetrics* database. Sorted by cusip, there are total 203 names in the data, as the index constituents are changing through the time. The index weights are calculated using the closing market capitalization of all current index components on the previous day.

The data for equity and index options are also obtained from OptionMetrics Volatility Surface that provides Black-Scholes implied volatilities for options with standard maturities and moneyness level. As we are interested in most liquid options and also considering underlying stock investment horizon, I use options with approximately one month to maturity. I select out-of-the-money (OTM) as well as at-the-money (ATM) call and put options with this maturity. As the purpose of using options is not as instruments for trading, but as an information source only. Even if the OTM options are not so liquid some time, it is not be a big issue here.

2.5.2 Variance, Skewness and Kurtosis Estimation

I estimate the realized (co)variance, (co)skewness and (co)kurtosis as central moments from daily returns using the rolling window methodology with six months and one year window length.

For the risk-neutral moments, the model-free methodology is very welcome in the literature. Recent studies (e.g., Bakshi and Madan (2006), Carr and Lee (2009), Carr and Wu (2009)) show that the risk-neutral expected variance is best approximated by the model-free implied variance (MFIV).

Give the fact the MFIV method extracts information from all existing options expiring on one date and does not rely on any parametric model (while there is minor assumption on the stock process), I use this method to estimate all the risk-neutral moments needed in this paper (See Appendix for more detailed information). As pointed out by Carr and Wu (2006), the MFIV method is also chosen by practitioner to trade CBOE VIX.

2.5.3 Parameters Estimation

The empirical part of my study is to show that option-implied betas deliver a better result regarding asset pricing (measured by squared error or R-square). The reason behind this is that we managed to extract information about future dynamics from the option market as well as the information contained in the history of stock return time series.

From **Proposition 1**, we can have analytical closed form solution for betas as long as we have the following components: risk-neutral moments and risk-neutral co-moments. RN moments are estimated based on the Bakshi and Madan (2000) MFIM (model-free option-implied moments), more details can be found in Appendix. Estimating RN co-moments are divided into two parts: RN moments and RN correlations (include higher order correlations), again, RN moments get be extracted from option prices; while for the correlations, we need to build the bridge between the physical measure and the risk-neutral measure. Buss and Vilkov (2012) propose a semi-parametric way to model option-implied correlations, I extend their methods to higher order correlations.

[**Insert Table 2 here**]

Table 2 provides the summary statistics for market beta, co-skewness beta and co-kurtosis beta. The sample period spans from August 2005 to November 2014. For each month I compute three betas for all stock in the S&P100 index. The table reports the time-series median of these statistic. Additionally, since option-implied method is able to capture the sudden change in the market, the ex-ante betas are more volatile then the ones from regression-based method, their distribution consequently is more skewed. Thus, I pool all the betas across time and stocks, and compute the 5%, 20%, 40%, 60%, 80% and 95% observation. Even though these betas are calculated using option-implied information rather than estimated from regression based method, still the value range of β^{MKT}s is quite reasonable (For instance, the median of the market risk beta during the whole sample period is 1.08). The value for co-skewness betas and co-kurtosis betas are much larger, it is intuitive given the fact that the corresponding premium is much smaller in scale.

2.5.4 In-Sample Fitting

Once we have these forward looking parameters, if we are interested in the expected stock return, we still need corresponding risk premium. These risk premium, defined as the difference between the physical and the risk-neutral measure of the first three expected return moments. It is relatively straightforward to estimate the risk-neutral cumulants, using the model-free method proposed by Bakshi and Madan (2000). However, for the physical moments estimation, despite many models/methods have been proposed to forecast the market premium and variance/volatility, there is no such consensus about which method to use. I will discuss this issue in the next section. At this section, we are only interested to see whether option-implied information is able to improve asset pricing, thus I run cross sectional regression to estimate the risk premium for each of the month. The sum of the squared residual is calculated and compared with the classical asset pricing model/method in the literature.

[Insert Table 3 here]

Table 3 shows the comparison between my method OiCCC (Option implied Measures of covariance-co-skewness-co-kurtosis risk) and other classical asset pricing models, $CAPM_{3Y}$ and $CAPM_{5Y}$ stand for CAPM model with three-year and five-year monthly moving window, respectively. $regCCC_{3Y}$ and $regCCC_{5Y}$ stand for regression based covariance-co-skewness-co-kurtosis with three-year and five-year monthly moving window, respectively. The CAPM and regCCC model are estimated using Fama-MacBeth two stage regression, I first use moving-window method to estimate the beta(s) for each stock, then a cross-sectional regression is run to determine the risk premium for each period: $R_{j,T} - r_f = c_j + \beta_j^{MKT}\lambda_T^{MKT} + \beta_j^{COSK}\lambda_T^{COSK} + \beta_j^{COKU}\lambda_T^{COKU}$. The sample period range from 2005.08 to 2014.11, with 112 monthly observation, cross-sectionally, we have 100 (S&P100 constituents, updated for each of the month) stocks for each month. After the OLSregression estimation, we have a 112*100 matrix of squared errorforeach method. Then for each month and each benchmark method, I compare the aggregate squared error (distribution) with the one from OiCCC, to see whether these benchmark methods significantly (at 10% level, t≥1.65) perform better/worse (measured by whether the MSE is higher or lower), and count the frequency. Panel A reports the average mean square error (MSE) for monthly returns, average cross-sectional R-square/adj. R-square and the frequencies that OiCCC outperforms the benchmark methods. Panel B reports the frequency that all the three premium (covariance-co-skewness-co-kurtosis) estimates are significant in the cross-sectional fit. Table 3 shows that the benchmark asset pricing methods very rarely outperform OiCCC while they are much frequently worse than OiCCC. More than quintile of the time, OiCCC performs significantly better than CAPM model, while the advantage shrinks when being compared to regCCC model. Panel B shows the frequency that all three premium estimates are significantly different from zero. Empirically, there is no such consensus about the significance of higher moment risk premium. Indeed, that is the case when we use regression based method to estimate the betas and then run cross-sectional regression to get the premium estimates. Panel B shows that there is only approximately 5% to 7% of the times when all three premium estimates are significant. Given this results, it is very natural to empirically reject the pricing models with higher moment risk premium. However, the frequencies becomes much higher when using option-implied estimation method. Though they may not be high enough (for example, over 90% of the time), still with a frequency around 41% (with $p = 0.1$), it can be the supporting evidence that higher moment risk premium is priced in the cross-sectional asset returns.

2.5.5 Portfolio Analysis

Individual stock returns are relatively difficult to model, empirically, people are sometimes more interested in the portfolio return analysis/forecast. However, in my case, due to the technical difficulty, I only study 100 stock returns for each time period, which means, in order to have a sufficient number of observations for cross-sectional regression practice, the portfolio should be consisting of 4 or 5 stocks.

In order to reflect the implied exposure of the corresponding risk for each stocks, I construct the following four portfolios: 1. portfolio sorted by market beta; 2. portfolio sort by co-skewness beta; 3. portfolio sort by co-kurtosis beta; 4. portfolio sort by aggregate risk exposure. Take 1^{st} portfolio (sort by market beta) as an example, for each month, I sort the stocks based on their option-implied market beta, to form 20 portfolios. The first portfolio thereby contains the stocks with the lowest market betas, and the last portfolio contains the stocks with the highest market betas. The 4^{th} portfolio (sort by aggregate measures) is constructed differently. First, for each month, I calculate the percentile of ranking for each beta, aggregate risk measure (ARM) is then defined as follows:

$$ARM = percentile(\beta^{MKT}) - percentile(\beta^{COSK}) + percentile(\beta^{COKU}) \tag{2.23}$$

Note that a negative sign on the co-skewness beta is also consistent with thoery, and with existing empirical studies that document a negative price of co-skewness risk (for instance, KL (1976) and HS (2000)). This portfolio sorting method is similar to the studies of Jensen, Black, and Scholes (1972), Baker, Bradley, and Wurgler (2011) and Buss and Vilkov (2012). For each portfolio, each month, and each methodology, I compute the equal-weighted expected portfolio risk measures (market beta, co-skewness beta and co-kurtosis beta) and realized portfolio return over the next month. Then I use cross-sectional regression to access the portfolio fit, measured by R-square.

[Insert Table 4 here]

Like the individual stock return case, I use three pricing models as the benchmark, the only difference here is that since we are evaluating the the model performance by the cross-sectional R-square, all of the benchmark models are estimated by two-stage Fama-Macbeth style regression (first betas, then premiums). Table 4 reports the mean R-square for each of the cross-sectional regression, for each of the methodology. As the table shows, for all portfolio construction methods, the OiCCC outperforms the other methods in a large scale. Overall, multi-factors models behave better, even after adjusting the degree of freedom. OiCCC could relatively improve the adj.R-square by 100% comparing to other multi-factor models, and its improvement is consistent across different portfolios.

2.6 Estimating Physical Cumulants

In order to provide a completely forward-looking forecast of equity expected returns, we also need to estimate corresponding market premium. The most common way to do that is to calculate the average value for historical realized returns for a given period. Simple as it be, it has several problems: a) the average realized return is unconditional estimate. Given the fact that varies studies document premium (expected returns) being time-varying and persistent, indicating that the conditional ER is not very likely to be the same as the ex-post unconditional one. b) The long-run average does not take into account short-term changes in the market condition. It is very probable that the risk premium, reflects the level of risk which is related to the different state of the world will not occur in the sample later. Elton (1999) points out that future work in asset pricing should consider alternative ways to measure expected returns rather than relying on the ex-post realized returns.

There are different routes in the literature to solve this issue, for instance, use survey on academics or investors to get their view on the ER. Another literature use information from stock and option markets, together with a parametric option pricing model (Santa-Clara and Yan(2010)) or semi-parametric procedure (Duan and Zhang (2014)). Chalamandaris and Rompolis (2016) extend the theoretical model of Duan and Zhang (2014) to a general system of equations that connects the cumulants of the physical distribution of any order to those of the risk-neutral cumulants through the projected relative risk aversion coefficient (PRRAC).

The expected return of the market portfolio return is connected to higher-order physical cumulants and PRRAC. It is quite intuitive, investors require a higher compensation to hold the market portfolio if a) they are more risk averse; b) they expect the future returns would be more volatile, more negative skewed and higher level of excess kurtosis. Another implication of their method is that it restricts the shape discrepancy between the physical and risk-neutral distributions by means of the PRRAC.

2.6.1 Relationship Between Physical and RN Moments

Chalamandaris and Rompolis (2016) propose a semi-parametric relationship between physical and risk-neutral cumulants. Let $k_{t,n}^{\mathbb{Q}}$ and $k_{t,n}^{P}$ be the n^{th}-order cumulants of the τ period log-return $r_{t,T}$ distribution conditional on the current market information under the physical P and risk-neutral Q measure, respectively. Then, the relationship between physical cumulants and risk-neutral ones:

$$k_{t,n}^{P} = \sum_{m=0}^{\infty} k_{t,n+m}^{Q} \frac{\gamma^m}{m!}, \tag{2.24}$$

or in another form:

$$k_{t,n}^{Q} = \sum_{m=0}^{\infty} k_{t,n+m}^{P} \frac{(-\gamma)^m}{m!} \tag{2.25}$$

This is a general case of the well studied estimating physical moments from the risk-neutral ones during previous year. For instance, Bakshi, Kapadia and Madan (2003), Bakshi, Kapadia and Madan (2006) and Duan and Zhuang (2014) give approximate counterparts of equation (24) and equation (25) based on variance, skewness and kurtosis for n=3, n=2 and n=1, respectively. Their proposition provides a framework that express the physical distribution in terms of the risk-neutral ones and PRRAC γ. To see it intuitively, assume the log-return $r_{t,T}$ follows the normal distribution, which implies that $k_{t,n}^{P} = 0$ for $n > 2$, then the market risk premium will be

$$k_{t,1}^{P} - k_{t,1}^{Q} = \gamma k_{t,2}^{P} \tag{2.26}$$

which is a well-known result derived by CAPM and other models. It indicates that knowledge of γ and higher-order risk-neutral moments provides an estimate of the expected physical moments.

We can further express risk premium, which is defined as the difference between physical and risk-neutral measure, in the following way:

$$\lambda_n = k_{t,n}^{\mathbb{P}} - k_{t,n}^{\mathbb{Q}} = \sum_{m=1}^{\infty} k_{t,n+m}^{Q} \frac{\gamma^m}{m!}, \tag{2.27}$$

where λ_1 denotes the market risk premium, λ_2 and λ_3 denote price of the co-skewness and co-kurtosis risk, respectively.

2.6.2 Estimation of PRRAC coefficient

In order to estimate the PRRAC coefficient γ, we need to use the information contained in the equations of (24) and (25). For instance, if we set n=2, formula (24) and (25) lead to two similar expressions for the variance risk premium,

$$k_{t,2}^{P} - k_{t,2}^{Q} = \gamma k_{t,3}^{P} - \frac{\gamma^2}{2!} k_{t,r}^{P} + \frac{\gamma^3}{3!} k_{t,4}^{P} + ... \tag{2.28}$$

and

$$k_{t,2}^{P} - k_{t,2}^{Q} = \gamma k_{t,3}^{Q} - \frac{\gamma^2}{2!} k_{t,r}^{Q} + \frac{\gamma^3}{3!} k_{t,4}^{Q} + ... \tag{2.29}$$

The implication is interesting, it indicates that the variance risk premium can be attributed to higher-order (higher than second) physical/risk-neutral cumulants. As shown by several empirical researches, the negative variance risk premium can be explained by negative skewness. Following Bakshi and Madan (2006), we can implement a GMM method. Denote I_{t-1} as the information set known at time $t-1$. Then the orthogonality condition can be expressed as

$$E\Big[k_{t,N}^{P,Q} + \sum_{m=1}^{M} k_{t,m+2,m+N}^{P} \frac{(-\gamma)^m}{m!} | I_{t-1}\Big] = 0 \tag{2.30}$$

where $k_{t,N}^{P,Q} = \Big(k_{t,2}^{P} - K_{t,2}^{Q}, .., k_{t,N}^{P} - k_{t,N}^{Q}\Big)$ and $k_{t,m+2,m+N}^{P} = \big(k_{t,m+2}, .., k_{t,m+N}^{P}\ \big)$.

2.6.3 Physical Moments and Forward-Looking Premium

Using the estimates of γ along with higher-order physical and RN cumulants, we can calculate forward-looking market risk premium, co-skewness and co-kurtosis risk premium for each month. I implement third-order approximation of formula 24 and 25 to estimate the ex-ante risk premium. Table reports the descriptive statistics for the estimates of the moments and the forward-looking premium. First, the market risk premium is always positive, time-varying and counter-cyclical. It increases during the financial crisis period, the sub-prime mortgage crisis, as expected. On average the co-skewness risk premium is -0.0183 ($RA = 3$) and the co-kurtosis risk premium is 0.0022 ($RA = 3$). These results are consistent with theory and empirical results as well. In addition, it is important to mention that these existing estimates are usually mean of the price of risk over several years. Most of them use a two-stage Fama-MacBeth (1973) setting and report the average estimates of the monthly cross-sectional regression. It is likely that these prices of risk have the opposite sign over shorter time horizon. The advantage of this method is that we can have conditional monthly estimates of the price of risk that have the theoretically expected sign in almost every month.

[**Insert Table 5 here**]

2.6.4 Forecast Portfolio Return

In the previous section, I introduce the method for estimating the betas for the market risk, co-skewness risk and co-kurtosis risk, together with the corresponding risk premium estimated in this section, we can forecast conditional expected return for each stock and portfolio. I use the same method (as the one used in model fit) to sort the stocks according to their aggregate risk exposure. I then compute the equal-weighted expected portfolio betas (market beta, co-skewness beta and co-kurtosis beta) and multiply the price of the corresponding risk to get expected portfolio return for each month. I sort the portfolios into quintiles based on their expected return for each month, and compute the equal-weighted mean realized return for each quintile across all the time. The first quintile therefore contains the portfolios with highest expected returns, and the last one contains the portfolios with lowest expected returns. Additionally, I also use regression based method to estimate betas for CCC (covariance-co-skewness-co-kurtosis model), denote as regCCC, as a benchmark.

[**Insert Table 6 here**]

Table 6 reports the equal-weighted quintile return for different risk averse (RA) coefficient. As we can see, the option-implied method (OiCCC) generates a monotonic relation across different quintiles, and its performance is stable for different RA coefficients. While for regression based method, its relation is more noisy across different quintiles. The return difference between the extreme quintiles for OiCCC is about 10% (RA=4) annually, while it is only 3% for regCCC.

2.7 Conclusion

In this paper, I derive a new formula that expresses the measures of the co-variance, co-skewness and co-kurtosis risk in terms of the risk-neutral moments of the market return and the higher order risk-neutral co-moments between market and individual stock returns. Then I show that comparing to traditional regression based methods, these option implied parameters perform well empirically. I empirically investigate the performance of my approach for the individual stock and portfolio asset pricing. Based on the monthly data for the period 2005-2014, I find that my option-implied forward-looking betas outperform regression-based estimates, for both individual and portfolio returns. My results also indicate that the higher moment (co-skewness and co-kurtosis) risk premium is priced in the cross-sectional asset returns. I further show that, with an option implied estimates for the higher moment risk premium, the forecast portfolio return is highly correlated with future realized returns.

2.8 Appendix

2.8.1 Analytical Solutions for Betas

$$\beta_t^{MKT} = -\frac{((E^{\mathbb{Q}}[(\tilde{R}_j - \bar{R}_j)(\tilde{R}_m - \bar{R}_m)^2] \cdot (k_{m,5} - k_{m,2} \cdot k_{m,3}) \cdot k_{m,4} - (k_{m,4} - k^2_{m,2}) \cdot E^{\mathbb{Q}}[(\tilde{R}_j - \bar{R}_j)(\tilde{R}_m - \bar{R}_m)^3] \cdot k_{m,4} - E^{\mathbb{Q}}[(\tilde{R}_j - \bar{R}_j)(\tilde{R}_m - \bar{R}_m)] \cdot (k_{m,5} - k_{m,2} \cdot k_{m,3})^2}{D}$$
$$+\frac{k_{m,3} \cdot E^{\mathbb{Q}}[(\tilde{R}_j - \bar{R}_j)(\tilde{R}_m - \bar{R}_m)^3](k_{m,5} - k_{m,2} \cdot k_{m,3}) - E^{\mathbb{Q}}[(\tilde{R}_j - \bar{R}_j)(\tilde{R}_m - \bar{R}_m)^2]k_{m,3} \cdot (k_{m,6} - k^2_{m,3}) + E^{\mathbb{Q}}[(\tilde{R}_j - \bar{R}_j)(\tilde{R}_m - \bar{R}_m)](k_{m,4} - k^2_{m,2}) \cdot (k_{m,6} - k^2_{m,3})}{D} \tag{2.31}$$

$$\beta_t^{COSK} = -\frac{(-k_{m,4} \cdot E^{\mathbb{Q}}[(\tilde{R}_j - \bar{R}_j)(\tilde{R}_m - \bar{R}_m)^2] \cdot k_{m,4} \cdot + k_{m,3} \cdot E^{\mathbb{Q}}[(\tilde{R}_j - \bar{R}_j)(\tilde{R}_m - \bar{R}_m)^3] \cdot k_{m,4} + E^{\mathbb{Q}}[(\tilde{R}_j - \bar{R}_j)(\tilde{R}_m - \bar{R}_m)] \cdot k_{m,4} \cdot (k_{m,5} - k_{m,2} \cdot k_{m,3})}{D}$$
$$-\frac{k_{m,2} \cdot E^{\mathbb{Q}}[(\tilde{R}_j - \bar{R}_j)(\tilde{R}_m - \bar{R}_m)^3] \cdot (k_{m,5} - k_{m,2} \cdot k_{m,3}) - E^{\mathbb{Q}}[(\tilde{R}_j - \bar{R}_j)(\tilde{R}_m - \bar{R}_m)] \cdot k_{m,3} \cdot (k_{m,6} - k^2_{m,3}) + k_{m,2} \cdot E^{\mathbb{Q}}[(\tilde{R}_j - \bar{R}_j)(\tilde{R}_m - \bar{R}_m)^2] \cdot (k_{m,6} - k^2_{m,3})}{D} \tag{2.32}$$

$$\beta_t^{COKU} = -\frac{(-k_{m,4} \cdot E^{\mathbb{Q}}[(\tilde{R}_j - \bar{R}_j)(\tilde{R}_m - \bar{R}_m)^2] \cdot k_{m,3} + E^{\mathbb{Q}}[(\tilde{R}_j - \bar{R}_j)(\tilde{R}_m - \bar{R}_m)] \cdot k_{m,4} \cdot (k_{m,4} - k^2_{m,2}) - E^{\mathbb{Q}}[(\tilde{R}_j - \bar{R}_j)(\tilde{R}_m - \bar{R}_m)] \cdot k_{m,3} \cdot (k_{m,5} - k_{m,2} \cdot k_{m,3})}{D}$$
$$+\frac{k_{m,2} \cdot E^{\mathbb{Q}}[(\tilde{R}_j - \bar{R}_j)(\tilde{R}_m - \bar{R}_m)^2](k_{m,5} - k_{m,2} \cdot k_{m,3}) + k_{m,3} \cdot k_{m,3} \cdot E^{\mathbb{Q}}[(\tilde{R}_j - \bar{R}_j)(\tilde{R}_m - \bar{R}_m)^3] - k_{m,2} \cdot (k_{m,4} - k^2_{m,2}) \cdot E^{\mathbb{Q}}[(\tilde{R}_j - \bar{R}_j)(\tilde{R}_m - \bar{R}_m)^3])}{D} \tag{2.33}$$

where

$$\begin{aligned} D = & -k_{m,4} \cdot (k_{m,4} - k^2_{m,2}) \cdot k_{m,4} + k_{m,3} \cdot (k_{m,5} - k_{m,2} \cdot k_{m,3}) \cdot k_{m,4} + k_{m,4} \cdot k_{m,3} \cdot (k_{m,5} - k_{m,2} \cdot k_{m,3}) \\ & -k_{m,2} \cdot (k_{m,5} - k_{m,2} \cdot k_{m,3})^2 - k_{m,3} \cdot k_{m,3} \cdot (k_{m,6} - k^2_{m,3}) + k_{m,2} \cdot (k_{m,4} - k^2_{m,2}) \cdot (k_{m,6} - k^2_{m,3}) \end{aligned} \tag{2.34}$$

and $k_{m,n} = E^Q[(\tilde{R}_m - \bar{R}_m)^n]$.

0

2.8.2 Appendix: Model Free Option-Implied Moments

Bakshi, Carr and Madan (2000) show that any twice continuously differentiable fnct, $H(S_T)$, of terminal price S_T, can be replicated by a unique position in the risk-free, stocks and European options.

$$H[S] = H[\bar{S}] + (S - \bar{S})H_s[\bar{S}] + \int_{\bar{S}}^{\infty} H_{ss}[K](S-K)^+ dK + \int_0^{\bar{s}} H_{ss}[K](K-S)^+ dK \quad (2.35)$$

The prices of these contracts are

$$\begin{aligned} E_t^Q\{e^{-r\tau}H[S]\} = (H[\bar{S}] - \bar{S}H_s[\bar{S}])e^{-r\tau} + H_s[\bar{s}]S(t) + \\ \int_{\bar{s}}^{\infty} H_{ss}[K]C(t,\tau;K)dK + \int_0^{\bar{S}} H_{ss}[K]P(t,\tau;K)dK. \end{aligned} \quad (2.36)$$

where $C_t(\tau, K)$ and $P_t(\tau, K)$ are prices of the European call and put options with time to maturity τ and strike price K. As a result, we can calculate the prices of derivatives given the price of the risk free zero coupon bond r, the spot price of the underlying, $\bar{S}$, and a series of OTM calls and puts. Since our main interest would be underlying return distribution, consider the function $H[S]$):

$$H[S_{t+\tau}] = R_{t+\tau}^2 = (lnS_{t+\tau} - lnS_t)^2 \quad (2.37)$$

Then the risk-neutral variance, skewness and kurtosis of equity returns could be computed based on the following expressions.

$$E_0^Q\Big[e^{-rT}(\frac{S_T - S_0}{S_0})^2\Big] = \frac{2}{s_0^2}\Big[\int_0^{S_0} P_0(T,X)dX + \int_{S_0}^{\infty} C_0(T,X)dX\Big] \quad (2.38)$$

$$E_0^Q\Big[e^{-rT}(\frac{S_T - S_0}{S_0})^3\Big] = \frac{6}{S_0^2}\Big[\int_0^{S_0} (\frac{X - S_0}{S_0})P_0(T,X)dX + \int_{S_0}^{\infty} (\frac{X - S_0}{S_0})C_0(T,X)dX\Big] \quad (2.39)$$

$$E_0^Q\Big[e^{-rT}(\frac{S_T - S_0}{S_0})^4\Big] = \frac{12}{S_0^2}\Big[\int_0^{S_0}(\frac{X - S_0}{S_0})^2 P_0(T,X)dX + \int_{S_0}^{\infty}(\frac{X - S_0}{S_0})^2 C_0(T,X)dX\Big] \tag{2.40}$$

Since there is no continuity of strike prices, we can approximate the integrals using cubic spline. For a given maturity, I interpolate implied volatilities across different moneyness level (K/S) to obtain a continuum of implied volatilities. Furthermore, the implied volatility of the highest or lowest available strike price is used when moneyness below and above the available moneyness level in the market. More precisely, for moneyness level smaller than 1 $(K/S < 1)$, the corresponding implied volatilities are used to generate put option prices, while for moneyness level larger than 1 $(K/S < 1)$, the corresponding implied volatilities are used to generate call option prices.

2.8.3 Proof of Equation (11) (12) and (13)

Christoffersen et. al (2016) shows that the linear factor models, in which the stochastic discount factor (SDF) is $m_{t+1} = a_t + \mathbf{b}_t'\big(\tilde{f}_{t+1} - E_t^P(\tilde{f}_{t+1})\big) = a_t + \mathbf{b}_t'\mathbf{f}_{t+1}$, are equivalent to beta-representation models with the vector of risk factors $\mathbf{f}$

$$E_t^P(R_{j,t+1}) - R_{f,t} = \lambda_t' \beta_{j,t}, \tag{2.41}$$

where $\lambda_t' = -\frac{b_t' E_t^P(f_{t+1}f_{t+1}')}{a_t}$, $(1 + R_{f,t}) = \frac{1}{a_t} = \frac{1}{E_t^P(m_t+1)}$, $\beta_{j,t} = [E_t^P(f_{t+1}f_{t+1}')]^{-1}E_t^P(f_{t+1}f_{t+1}')$. Since the pricing kernel prices all the assets including contingent claims, the above equation also holds for any claim i whose price is contingent on the stock j and has a payoff function $\Phi(R_{j,t+1})$, for any function $\Phi(.)$. It follows

$$E_t^P\big(\frac{\Phi(R_{j,t+1}) - P_{i,t}}{P_{i,t}}\big) - R_{f,t} = \lambda_t \beta_{i,t}, \tag{2.42}$$

where $P_{i,t}$ is the price of the contingent claim i. Based on the definition of $\beta_{i,t}$, we have

$$E_t^P\big(\frac{\Phi(R_{j,t+1}) - P_{i,t}}{P_{i,t}}\big) - R_{f,t} = \lambda_t'[E_t^P(f_{t+1}f_{t+1'})]^{-1}E_t^P\big(f_{t+1}\frac{\Phi(R_{j,t+1}) - P_{i,t}}{P_{i,t}}\big). \tag{2.43}$$

Rearranging and using $E_t^P(f_{t+1}) = 0$, it follows

$$E_t^P[\Phi(R_{j,t+1})] - P_{i,t}(1 + R_{f,t}) = \lambda_t'[E_t^P(f_{t+1}f_{t+1}')]^{-1}E_t^P[f_{t+1}\Phi(R_{j,t+1})] \tag{2.44}$$

The no-arbitrage condition ensures the existence of at least one risk-neutral measure Q such that $P_{i,t} = \frac{1}{(1+R_{f,t})}E_t^P[\Phi(R_{j,t+1})]$. Therefore, we obtain

$$E_t^P[\Phi(R_{j,t+1})] - E_t^Q[\Phi(R_{j,t+1})] = \lambda_t' \beta_{\Phi,t} \tag{2.45}$$

where $\beta_{\Phi,t}$ is from the projection of $\Phi(R_{j,t+1})$ on f_{t+1}. For $m_{t+1} = a_t + b_{1,t}\big(R_{m,t+1} - E_t^P(R_{m,t+1})\big) + b_{2,t}\big(R_{m,t+1}^2 - E_t^P(R_{m,t+1}^2)\big) + b_{3,t}\big(R_{m,t+1}^3 - E_t^P(R_{m,t+1}^3)\big)$ and $\Phi = R_m$, we have equation (11). Similarly, for $\Phi = R_m^2$ and $\Phi = R_m^3$, we recover equation (12) and equation (13), respectively.

2.8.4 Option Implied co-skewness and co-kurtosis

Start from covariance, which is indeed correlation (since moments are "known"). Buss and Vilkov (2012 RFS) proposed an option-based measure for $\rho_{ij,t}^Q$

1. Consider market index I, with N components, for the identification of the implied correlations $\rho^Q_{ij,t}$, we have only one identifying restriction that links the observed implied variance of the market index, $(\sigma^Q_{M,t})^2$ with the implied variance of a portfolio of all market index components:
$$(\sigma^Q_{I,t})^2 = \sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j \sigma^Q_{i,t}\sigma^Q_{j,t}\rho^Q_{ij,t} \tag{2.46}$$
where $\sigma^Q_{i,t}$ denotes the implied volatility of stock i and w_i is the stock weight in the index.

2. In order to identify $N \cdot (N-1)/2$ correlations that satisfy restriction, they propose the following parametric form for implied correlations $\rho^Q_{ij,t}$:
$$\rho^Q_{ij,t} = \rho^P_{ij,t} - \alpha_t(1 - \rho^P_{ij,t}) \tag{2.47}$$
where $\rho^P_{ij,t}$ is the expected correlation under the objective measure, and α_t denotes the parameter to be identified.

3. Estimate $\rho^P_{ij,t}$ from historical rolling windows, then compute α_t, and identify $\rho^Q_{ij,t}$. Substitute the implied correlation (47) into restriction (46), one can compute α_t in closed form:
$$\alpha_t = -\frac{(\sigma^Q_{M,t})^2 - \sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j \sigma^Q_{i,t}\sigma^Q_{j,t}\rho^P_{ij,t}}{\sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j \sigma^Q_{i,t}\sigma^Q_{j,t}(1 - \rho^P_{ij,t})} \tag{2.48}$$

4. In the end, we have the RN covariance:
$$Cov^Q_t(R_j, R_m) = \sigma^Q_{j,t}\sum_{i=1}^{N} w_i \sigma^Q_{i,t}\rho^Q_{ij,t} \tag{2.49}$$

Similarly, I propose the following strategy to estimate the co-skewness:

1. Define the central co-skewness (third moment):
$$\phi_{ijk} = E[(r_i - \bar{r}_i)(r_j - \bar{r}_j)(r_k - \bar{r}_k)], \tag{2.50}$$
and skewness is a special case of co-skewness when $i = j = k$. also, the central co-kurtosis (fourth moment) can be written in the following way:
$$\psi_{ijkl} = E[(r_i - \bar{r}_i)(r_j - \bar{r}_j)(r_k - \bar{r}_k)(r_l - \bar{r}_l)], \tag{2.51}$$

2. The skewness of the market index:

$$\phi^Q_{mmm} = \sum_i \sum_j \sum_k w_i w_j w_k \phi^Q_{ijk} \tag{2.52}$$

3. Define *Skewness Correlation*

$$K_{ijk} = \frac{S_{ijk}}{\sqrt[2]{\phi_{ii}}\sqrt[4]{\phi_{jjjj}}\sqrt[4]{\phi_{kkkk}}} \tag{2.53}$$

4. Rewrite Equation (19):

$$\phi^Q_{mmm} = \sum_i \sum_j \sum_k w_i w_j w_k K^Q_{ijk} \sqrt[2]{\phi^Q_{ii}} \sqrt[4]{\phi^Q_{jjjj}\phi^Q_{kkkk}} \tag{2.54}$$

5. Impose parametric relationship between K^P_{ijk} and K^Q_{ijk}:

$$K^Q_{ijk,t} = K^P_{ijk,t} + \alpha_t(1 + K^P_{ijk,t}). \tag{2.55}$$

6. Estimate K^P_{ijk} from historical rolling data, identify the relationship parameter α_t from index skewness (substitute equation (23) into restriction (22)) and then compute K^Q_{ijk} accordingly. The expression for α_t

$$\alpha_t = \frac{\phi^Q_{mmm,t} - \sum_{i=1}^N \sum_{j=1}^N \sum_{k=1}^N w_i w_j w_k \sqrt[2]{\phi^Q_{ii,t}} \sqrt[4]{\phi^Q_{jjjj,t}\phi^Q_{kkkk,t}} K^P_{ijk,t}}{\sum_{i=1}^N \sum_{j=1}^N \sum_{k=1}^N w_i w_j w_k \sqrt[2]{\phi^Q_{ii,t}} \sqrt[4]{\phi^Q_{jjjj,t}\phi^Q_{kkkk,t}}(1 - K^P_{ijk,t})} \tag{2.56}$$

7. Recall that on the left side of equation (13), we have the co-skewness between stock j and index m: $S^Q_{jmm} = E[(\tilde{R}_j - \bar{R}_j)(\tilde{R}_m - \bar{R}_m)^2]$, now becomes:

$$S^Q_{jmm} = \sqrt[2]{S^Q_{jj}} \sum_i \sum_k w_i w_k K^Q_{ijk} \sqrt[4]{S^Q_{iiii} S^Q_{kkkk}}, \tag{2.57}$$

with each of the components on the right side of equation (24) estimated, the co-skewness S^Q_{jmm} is acquired.

The procedure for estimating co-kurtosis is analogous, just extend one more dimension.

1. Define the central co-kurtosis (fourth moment):

$$\psi_{ijkl} = E[(r_i - \bar{r}_i)(r_j - \bar{r}_j)(r_k - \bar{r}_k)(r_l - \bar{r}_l)], \tag{2.58}$$

and kurtosis is a special case of co-skewness when $i = j = k = l$

2. The kurtosis of the market index:

$$\psi^Q_{mmmm} = \sum_i \sum_j \sum_k \sum_l w_i w_j w_k w_l \psi^Q_{ijkl} \tag{2.59}$$

3. Define *Kurtosis Correlation*

$$K_{ijkl} = \frac{\psi_{ijkl}}{\sqrt[4]{\psi_{iiii}}\sqrt[4]{\psi_{jjjj}}\sqrt[4]{\psi_{kkkk}}\sqrt[4]{\psi_{llll}}} \tag{2.60}$$

4. Rewrite Equation (26):

$$\psi^Q_{mmmm} = \sum_i \sum_j \sum_k \sum_l w_i w_j w_k w_l K^Q_{ijkl} \sqrt[4]{\psi^Q_{iiii}\psi^Q_{jjjj}\psi^Q_{kkkk}\psi^Q_{llll}} \tag{2.61}$$

5. Impose parametric relationship between K^P_{ijkl} and K^Q_{ijkl}:

$$K^Q_{ijk,t} = K^P_{ijk,t} - \alpha_t(1 - K^P_{ijk,t}). \tag{2.62}$$

6. Estimate K^P_{ijkl} from historical rolling data, identify the relationship parameter α_t from index kurtosis (substitute equation (23) into restriction (22)) and then compute K^Q_{ijkl} accordingly:

$$\alpha_t = \frac{\psi^Q_{mmmm,t} - \sum_{i=1}^N \sum_{j=1}^N \sum_{k=1}^N \sum_{l=1}^N w_i w_j w_k w_l \sqrt[4]{\psi^Q_{iiii,t}\psi^Q_{jjjj,t}\psi^Q_{kkkk,t}\psi^Q_{llll,t}} K^P_{ijkl,t}}{\sum_{i=1}^N \sum_{j=1}^N \sum_{k=1}^N \sum_{l=1}^N w_i w_j w_k w_l \sqrt[4]{\psi^Q_{iiii,t}\psi^Q_{jjjj,t}\psi^Q_{kkkk,t}\psi^Q_{llll,t}}(1 - K^P_{ijkl,t})} \tag{2.63}$$

7. Recall that on the left side of equation (13), we have the co-kurtosis between stock j and index m: $S^Q_{jmmm} = E[(\tilde{R}_j - \bar{R}_j)(\tilde{R}_m - \bar{R}_m)^3]$, now becomes:

$$\psi^Q_{jmmm} = \sqrt[4]{\phi^Q_{jjjj}} \sum_i \sum_k \sum_l w_i w_k w_l K^Q_{ijkl} \sqrt[4]{\phi^Q_{iiii}\phi^Q_{kkkk}\phi^Q_{llll}}, \tag{2.64}$$

with each of the components on the right side of equation (24) estimated, the co-kurtosis S^Q_{jmmm} is acquired.

2.8.5 General Framework of Co-Moments

Suppose we have N assets and wish to determine the corresponding first three co-moments of the asset returns, i.e. the covariance of asset i and j:

$$\sigma_{i,j} = E\big[(R_i - \mu_i)(R_j - \mu_j)\big], \tag{2.65}$$

the products of three returns, i.e. the co-skewness of asset i,j and k:

$$\phi_{i,j,k} = E\big[(R_i - \mu_i)(R_j - \mu_j)(R_k - \mu_k)\big], \tag{2.66}$$

and similarly, we have the products of four returns, i.e. the co-kurtosis of asset i, j, k and l:

$$\psi_{i,j,k,l} = E\big[(R_i - \mu_i)(R_j - \mu_j)(R_k - \mu_k)(R_l - \mu_l)\big]. \tag{2.67}$$

We can also rewrite previous expression into matrix style: $N \times N$ covariance matrix Σ, $N \times N^2$ co-skewness matrix Φ and $N \times N^3$ co-kurtosis matrix Ψ of the corresponding return vector R with mean μ_R:

$$\Sigma = E\big[(R - \mu_R)(R - \mu_R)'\big] \tag{2.68}$$

$$\Phi = E\big[(R-\mu_R)(R-\mu_R)' \otimes (R-\mu_R)'\big] \Psi = E\big[(R-\mu_R)(R-\mu_R)' \otimes (R-\mu_R)' \otimes (R-\mu_R)'\big], \tag{2.69}$$

where $\otimes$ denotes the Kronecker product. For instance, if A is a $M \times N$ matrix and B is a $p \times q$ matrix, then the Kronecker product of A and B is the $mp \times nq$ block matrix:

$$\begin{bmatrix} A_{11}B & A_{12}B & ... & A_{1n}B \\ A_{21}B & A_{22}B & ... & A_{2n}B \\ . & . & . & . \\ . & . & . & . \\ A_{m1}B & A_{m2}B & ... & A_{mn}B \end{bmatrix}$$

Furthermore, as mentioned by Peterson and Boudt (2008), we have the property that allow us to easily calculate the n-th portfolio/index return moment:

$$m_2(W) = E\big[\big(W'(R - \mu_R)\big)^2\big] = w'\Sigma w \tag{2.70}$$

Table 2.1: Number of Parameters

	Number of Elements
sigma	N(N+1)/2
phi	N(N+1)(N+2)/6
psi	N(N+1)(N+2)(N+3)/24
Total	N(N+1)/2+N(N+1)(N+2)/6 +N(N+1)(N+2)(N+3)/24
Total	
N=30	46,345
N=100	4,598,025
N=500	2,656,615,125,000,000

$$m_3(w) = E\left[\left(W'(R - \mu_R)\right)^3\right] = w'\Phi(w \otimes w) \tag{2.71}$$

$$m_4(w) = E\left[\left(W'(R - \mu_R)\right)^4\right] = w'\Psi(w \otimes w \otimes w) \tag{2.72}$$

As been pointed out by previous studies, the challenge of this method is to estimate these co-moments/correlation. AS shown in Table 1, if we consider DJIA as market proxy, and study its components, we need to estimate $46,345$ parameters for each time period, which is quite acceptable. When we increase the number of components to 100, for instance, S&P 100 (which is my case), we would face $4,598,025$ parameters for each time period, under current hardware computing power, this is still acceptable. However, it might be a problem if we want to apply similar procedure to S&P 500 and its constituents.

2.8.6 Regression Methods for Benchmark Comparison

I use those classical asset pricing models in the literature as benchmarks to compare the in-sample fitting performance, measured by squared error (SE) and R-square/adj. R-square. The models are CAPM and regression based covariance-co-skewness-co-kurtosis model (denote as regCCC, proposed by Christoffersen et. al (2016)) and Fama-French three factor model.

2.8.7 CAPM Model

I use Fama-MacBeth two stage regression to estimate the CAPM model. It has better explaining performance in the cross-sectional asset returns comparing to one-stage regression model. At first stage, for each stock j among 100 index components, I estimate the $\beta_{j,t}$ using moving window method based on following regression. The rolling windows I use is three-year and five-year monthly.

$$R_{j,t} - R_f = \alpha_j + \beta_{j,t} \times (R_{mkt,t} - R_f) + \epsilon_{j,t}. \tag{2.73}$$

Then regress all asset returns for each month (T) against the betas to determine the risk premium ($R_{mkt,T}$). For $j = 1$ to 100

$$R_{j,T} - r_f = \alpha_j + \beta_j \times (R_{mkt,T} - r_f) + \epsilon_{j,t}. \tag{2.74}$$

2.8.8 regCCC Model

Christoffersen et. al (2016) introduce this pricing model incorporating co-skewness and co-kurtosis risk apart from covariance risk. I also use Fama-MacBeth two stage regression to estimate this model. First, for each stock j, I estimate the market beta, co-skewness beta and co-kurtosis beta using moving window method based on following regression. The rolling windows I use is three-year and five-year monthly.

$$\tilde{R}_j - R_f = c_j + \beta_j^{MKT} \lambda_t^{MKT} + \beta_j^{COSK} \lambda_t^{COSK} + \beta_j^{COKU} \lambda_t^{COKU} \tag{2.75}$$

Where

$$\lambda_t^{MKT} = E_t^P(R_{m,t+1}) - R_{f,t}, \tag{2.76}$$

$$\lambda_t^{COSK} = E_t^P(R_{m,t+1}^2) - E_t^Q(R_{m,t+1}^2) \tag{2.77}$$

$$\lambda_t^{COKU} = E_t^P(R_{m,t+1}^3) - E_t^Q(R_{m,t+1}^3) \tag{2.78}$$

Then regress all asset returns for each month (T) against the betas to determine the risk premium. For $j = 1$ to 100

$$R_{j,T} - r_f = c_j + \beta_j^{MKT}\lambda_T^{MKT} + \beta_j^{COSK}\lambda_T^{COSK} + \beta_j^{COKU}\lambda_T^{COKU} \tag{2.79}$$

The expected physical moments are approximated by realized ones, while the risk-neutral moments are calculated based on model-free implied moments.

2.8.9 OiCCC model

OiCCC represents Option-implied covariance-co-skewness-co-kurtosis model. Instead of the estimating the betas based on historical moving window method. I use option-implied information to construct these risk measures. Thus, they are naturally forward-looking and time-varying. For a model in-sample fit testing, I use the same method as the Fama-macBeth second stage regression to determine the risk premiums:

$$R_{j,T} - r_f = c_j + \beta_j^{MKT}\lambda_T^{MKT} + \beta_j^{COSK}\lambda_T^{COSK} + \beta_j^{COKU}\lambda_T^{COKU}, \tag{2.80}$$

where the risk premium (λ) is defined as before.

2.8.10 Proof of Formula (24) and (25)

Chalamandaris and Rompolis (2016) shows that if we take the $n^{th}-$order derivatives of $m_t^Q(u) = ln\beta + r_f\tau + m_t^P(u-\gamma)$ (which is the moment-generating function of market portfolio log-return distribution under Q measure, for more details, please refer to Chalamandaris and Rompolis (2016)), with respect to u evaluated at $u = 0$ which follows:

$$\Big[\frac{d^n m_t^Q(u)}{du^n}\Big]_{u=0} = \Big[\frac{d^n m_t^P(u-\gamma)}{du^n}\Big]_{u=0}, \forall n \in \mathbb{N}. \tag{2.81}$$

By definition the n^{th}-order cumulant of $r_{t,T}$ under the risk-neutral measure is equal to $k_{t,n}^Q = \big[\frac{d^n m_t^Q(u)}{du^n}\big]_{u=0}$. Also express $m_t^P(u-\gamma)$ in a power series expansion:

$$m_t^P(u-\gamma) = \sum_{m=1}^{\infty} k_{t,m}^P \frac{(u-\gamma)^m}{m!} \tag{2.82}$$

and calculate the $n^{th}-$order derivative yields

$$\Big[\frac{d^n m_t^P(u-\gamma)}{du^n}\Big]_{u=0} = \sum_{m=0}^{\infty} k_{t,n+m}^P \frac{(u-\gamma)^m}{m!} \tag{2.83}$$

Combine formula (80) (81) and (82), we have

$$k_{t,n}^Q = \sum_{m=0}^{\infty} k_{t,n+m}^P \frac{(-\gamma)^m}{m!}, \forall n \in \mathbb{N}, \tag{2.84}$$

Similar procedure can be applied for formula (24), see Rompolis and Tzavalis (2010).

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Table 2.2: Descriptive Statistic for Betas

Table 2 provides the summary statistics for market beta, co-skewness beta and co-kurtosis beta. The sample period spans from August 2005 to November 2014. For each month I compute three betas for all stock in the S&P100 index. The table reports the time-series median of these statistic. Additional, I pool all the betas across time and stocks, and compute the 5%, 20%, 40%, 60%, 80% and 95% observation.

	Market Risk Beta	co-Skewness Beta	co-Kurtosis Beta
median	1.08	0.02	-17.25
5%	-3.65	-105.43	-6003.13
20%	-0.68	-24.69	-1185.98
40%	0.63	-4.85	-179.87
60%	1.56	4.38	127.77
80%	3.22	21.64	966.19
95%	8.54	64.38	3518.31

Table 2.3: Comparison with Benchmark Methods

Table 3 shows the comparison between my method OiCCC (Option implied Measures of covariance-co-skewness-co-kurtosis risk) and other classical asset pricing models, $CAPM_{3Y}$ and $CAPM_{5Y}$ stand for CAPM model with three-year and five-year monthly moving window, respectively. $regCCC_{3Y}$ and $regCCC_{5Y}$ stand for regression based covariance-co-skewness-co-kurtosis with three-year and five-year monthly moving window, respectively. The CAPM and regCCC model are estimated using Fama-MacBeth two stage regression, I first use moving-window method to estimate the beta(s) for each stock, then a cross-sectional regression is run to determine the risk premium for each period: $R_{j,T} - r_f = c_j + \beta_j^{MKT}\lambda_T^{MKT} + \beta_j^{COSK}\lambda_T^{COSK} + \beta_j^{COKU}\lambda_T^{COKU}$. The sample period range from 2005.08 to 2014.11, with 112 month observation, cross-sectionally, we have 100 (S&P100 constituents, updated for each of the month) stocks for each month. After the OLSregression estimation, we have a 112*100 matrix of squared error-foreach method. Then for each month and each benchmark method, I compare the aggregate squared error (distribution) with the one from OiCCC, to see whether these benchmark methods significantly (at 10% level, t≥1.65) perform better/worse (measured by whether the MSE is higher or lower), and count the frequency. Panel A reports the average mean square error (MSE) for monthly returns, average cross-sectional R-square/adj. R-square and the frequencies that OiCCC outperforms the benchmark methods. Panel B reports the frequency that all the three premium (covariance-co-skewness-co-kurtosis) estimates are significant in the cross-sectional fit.

Panel A. Cross-Sectional Fit of Individual Stock Returns

	OiCCC	$CAPM_{3Y}$	$CAPM_{5Y}$	$regCCC_{3Y}$	$regCCC_{5Y}$
MSE	0.0049	0.0054	0.0054	0.0053	0.0053
R-square	0.1284	0.0454	0.0449	0.0815	0.0781
adj. R-square	0.1012	0.0354	0.0352	0.0528	0.0493
f. better than OiCCC		0	0	1/112	1/112
f. worse than OiCCC		23/112	24/112	12/112	17/112

Panel B. Summary of the Regression Statistics

	OiCCC	$regCCC_{3Y}$	$regCCC_{5Y}$
f. all three premiums are significant (10%)	41.1%	7.1%	7.1%
f. all three premiums are significant (5%)	34.8%	5.4%	6.3%

Figure 2.1: Difference between Mean Squared Error

Figure 1 plots the comparison between my method OiCCC (Option implied Measures of covariance-co-skewness-co-kurtosis risk) and other classical asset pricing models, $CAPM_{3Y}$ and $CAPM_{5Y}$ stand for CAPM model with three-year and five-year monthly moving window, respectively. The CAPM model is estimated using Fama-MacBeth two stage regression, I first use moving-window method to estimate the beta for each stock, then a cross-sectional regression is run to determine the risk premium for each period: $R_{j,T} - r_f = c_j + \beta_j^{MKT} \lambda_T^{MKT}$. The sample period range from 2005.08 to 2014.11, with 112 month observation, cross-sectionally, we have 100 (S&P100 constituents, updated for each of the month) stocks for each month. After the OLS regression estimation, we have a 112*100 matrix of squared errors (SEs) for each method. Then for each month and each benchmark method, I compare the aggregate squared error (distribution) with the one from OiCCC, to see whether these benchmark methods significantly (at 10% level, t≥1.65) perform better/worse (measured by whether the SE is higher or lower).

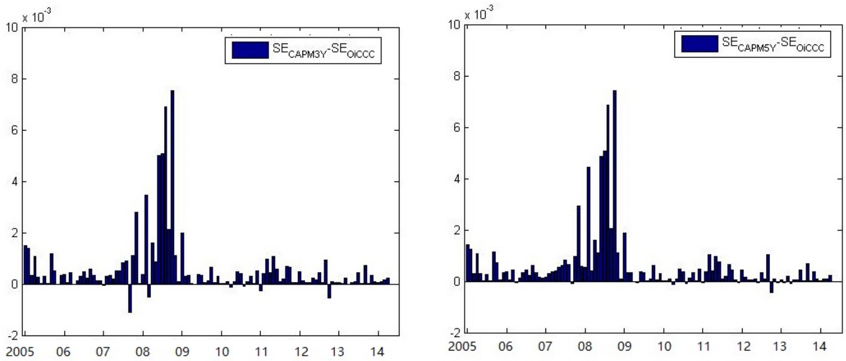


Figure 2.2: Difference between Mean Squared Error

Figure 2 plots the comparison between my method OiCCC (Option implied Measures of covariance-co-skewness-co-kurtosis risk) and other classical asset pricing models, $regCCC_{3Y}$ and $regCCC_{5Y}$ stand for regression based covariance-co-skewness-co-kurtosis with three-year and five-year monthly moving window, respectively. The regCCC model is estimated using Fama-MacBeth two stage regression, I first use moving-window method to estimate the betas for each stock, then a cross-sectional regression is run to determine the risk premium for each period: $R_{j,T} - r_f = c_j + \beta_j^{MKT}\lambda_T^{MKT} + \beta_j^{COSK}\lambda_T^{COSK} + \beta_j^{COKU}\lambda_T^{COKU}$. The sample period range from 2005.08 to 2014.11, with 112 month observation, cross-sectionally, we have 100 (S&P100 constituents, updated for each of the month) stocks for each month. After the OLS regression estimation, we have a 112*100 matrix of squared errors (SEs) for each method. Then for each month and each benchmark method, I compare the aggregate squared error (distribution) with the one from OiCCC, to see whether these benchmark methods significantly (at 10% level, t≥1.65) perform better/worse (measured by whether the SE is higher or lower).

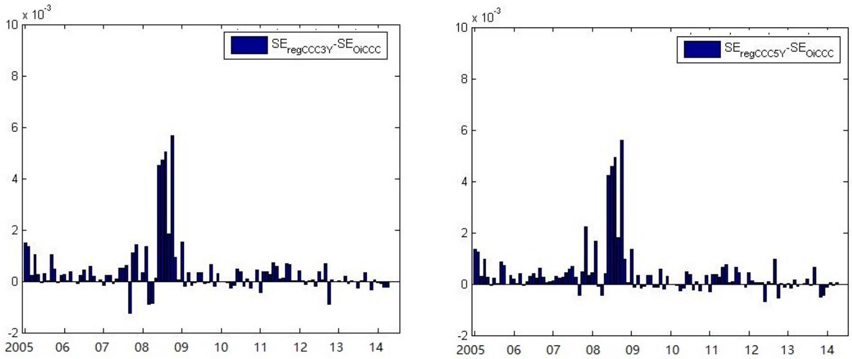


Table 2.4: Portfolio Fit Summary (1/2)

I construct the following four portfolios: 1. portfolio sort by market beta; 2. portfolio sort by co-skewness beta; 3. portfolio sort by co-kurtosis beta; 4. portfolio sort by aggregate risk exposure. Take 1^{st} portfolio (sort by market beta) as an example, for each month, I sort the stocks based on their option-implied market beta, to form 20 portfolios. The first portfolio thereby contains the stocks with the lowest market betas, and the last portfolio contains the stocks with the highest market betas. For each portfolio, each month, and each methodology, I compute the equal-weighted expected portfolio risk measures (market beta, co-skewness beta and co-kurtosis beta) and realized portfolio return over the next month. Then I use cross-sectional regression to access the portfolio fit, measured by R-square (adj. R-square). The 4^{th} portfolio (sort by aggregate measures) is constructed differently. For each month, calculate the percentile of ranking for each beta, aggregate risk measure (ARM) is then defined as follows:

$$ARM = percentile(\beta^{MKT}) - percentile(\beta^{COSK}) + percentile(\beta^{COKU}) \quad (2.85)$$

Portfolio Fitting: Sort by β^{MKT}

	$CAPM_{3Y}$	$CAPM_{5Y}$	$regCCC_{3Y}$	$regCCC_{5Y}$	OiCCC
obervation	2800	2800	2800	2800	2800
MSE (%)	0.13	0.13	0.12	0.12	0.10
R-square (%)	8.50	8.72	19.43	18.49	25.94
adj. R-square (%)	4.53	4.75	7.92	6.85	15.36

Portfolio Fitting: Sort by β^{COSK}

	$CAPM_{3Y}$	$CAPM_{5Y}$	$regCCC_{3Y}$	$regCCC_{5Y}$	OiCCC
obervation	2800	2800	2800	2800	2800
MSE	0.15	0.15	0.13	0.13	0.10
R-square (%)	8.2	8.09	17.94	17.59	27.81
adj. R-square (%)	4.2	4.09	6.22	5.81	17.50

Table 2.5: Portfolio Fit Summary (2/2)

I construct the following four portfolios: 1. portfolio sort by market beta; 2. portfolio sort by co-skewness beta; 3. portfolio sort by co-kurtosis beta; 4. portfolio sort by aggregate risk exposure. Take 1^{st} portfolio (sort by market beta) as an example, for each month, I sort the stocks based on their option-implied market beta, to form 20 portfolios. The first portfolio thereby contains the stocks with the lowest market betas, and the last portfolio contains the stocks with the highest market betas. For each portfolio, each month, and each methodology, I compute the equal-weighted expected portfolio risk measures (market beta, co-skewness beta and co-kurtosis beta) and realized portfolio return over the next month. Then I use cross-sectional regression to access the portfolio fit, measured by R-square (adj. R-square). The 4^{th} portfolio (sort by aggregate measures) is constructed differently. For each month, calculate the percentile of ranking for each beta, aggregate risk measure (ARM) is then defined as follows:

$$ARM = percentile(\beta^{MKT}) - percentile(\beta^{COSK}) + percentile(\beta^{COKU}) \tag{2.86}$$

Portfolio Fitting: Sort by β^{COKU}

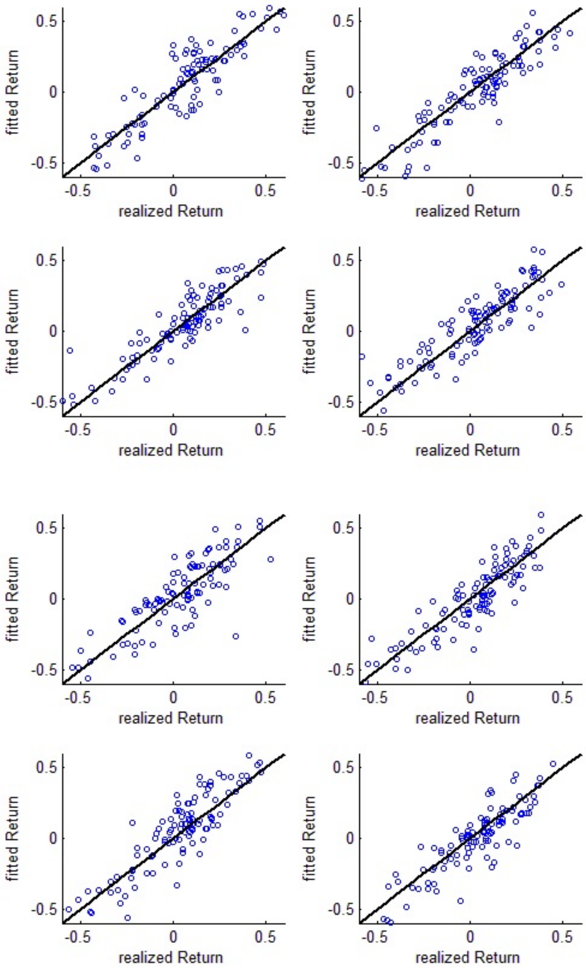
	$CAPM_{3Y}$	$CAPM_{5Y}$	$regCCC_{3Y}$	$regCCC_{5Y}$	OiCCC
observation	2800	2800	2800	2800	2800
MSE	0.14	0.14	0.12	0.12	0.10
R-square	9.25	9.51	18.89	20.04	26.92
adj. R-square	5.30	5.57	7.30	8.61	16.48

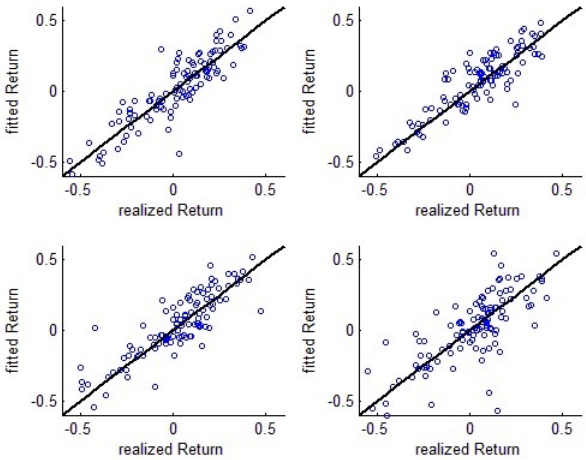
Portfolio Fitting: Sort by ARM

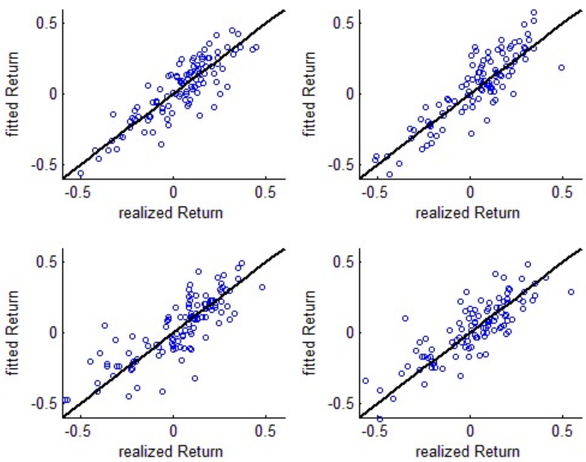
	$CAPM_{3Y}$	$CAPM_{5Y}$	$regCCC_{3Y}$	$regCCC_{5Y}$	OiCCC
observation	2800	2800	2800	2800	2800
MSE	0.14	0.14	0.12	0.13	0.10
R-square	8.65	9.10	19.62	18.11	27.47
adj. R-square	4.68	5.14	8.14	6.41	17.11

Figure 2.3: Scatter Plot between realized return and fitted-return

Figure 2 shows the scatter plot between realized portfolio returns and fitted-returns. For each portfolio, each month, I compute the equal-weighted expected portfolio risk measures (market beta, co-skewness beta and co-kurtosis beta) and realized portfolio return over the next month. Then I use cross-sectional regression to access the portfolio fit. The sample period is from August 2005 to November 2014.




fitted Return
realized Return
0.5
0
-0.5


fitted Return
realized Return
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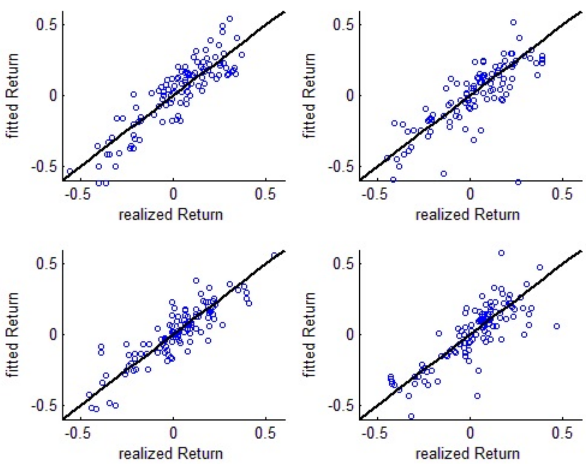
fitted Return
realized Return
0.5
0
-0.5

Table 2.6: Price of Market Risk, Co-Skewness and Co-Kurtosis Risk

The table provides descriptive statistics for the market risk premium, the price of co-skewness and co-kurtosis risk, with different risk-aversion (RA) coefficient. The data are monthly and the mean and the standard deviation are reported in percentages. The risk premiums are estimated using the ex-ante option implied method of Chalamandaris and Rompolis (2016). The risk-neutral moments are estimated using the model-free approach in Bakshi and Madan (2000). The sample period is from August 2005 to November 2014.

RA=3	Market Risk	co-skewness Risk	co-Kurtosis Risk
mean	0.8192	-0.0183	0.0022
std	1.0783	0.0382	0.0031
skew	3.6101	-5.1791	1.9146
kurt	18.0276	33.0156	6.7769

RA=4	Market Risk	co-skewness Risk	co-Kurtosis Risk
mean	1.0819	-0.0233	0.0029
std	1.4164	0.0522	0.0039
skew	3.5824	-5.3883	1.9487
kurt	17.7942	34.9828	6.7542

RA=5	Market Risk	co-skewness Risk	co-Kurtosis Risk
mean	1.3408	-0.0279	0.0036
std	1.7449	0.0679	0.0048
skew	3.5505	-5.5785	2.0111
kurt	17.5307	36.8091	6.7983

Figure 2.4: Market Risk Premium

The Figure 3, Figure 4 and Figure 5 plot the time series for the conditional market risk premium, price of the co-skewness and co-kurtosis risk, with different risk-aversion (RA) coefficient. The data are monthly and the mean and the standard deviation are reported in percentages. The risk premiums are estimated using the ex-ante option implied method of Chalamandaris and Rompolis (2016). The risk-neutral moments are estimated using the model-free approach in Bakshi and Madan (2000). The sample period is from August 2005 to November 2014.

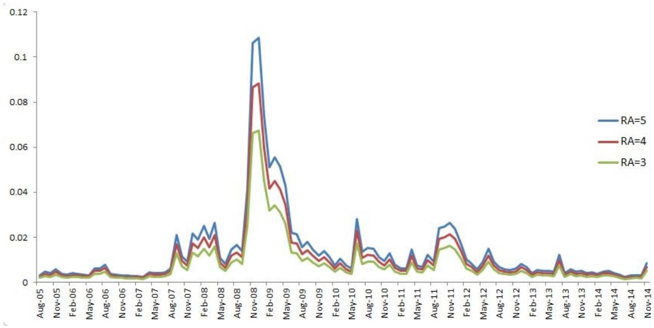

Figure 2.5: Market Risk Premium

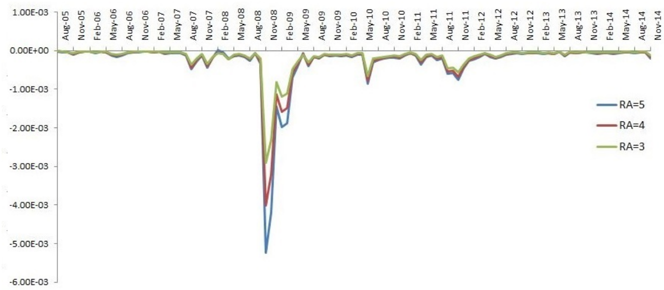

Figure 2.6: Co-Skewness Risk Premium

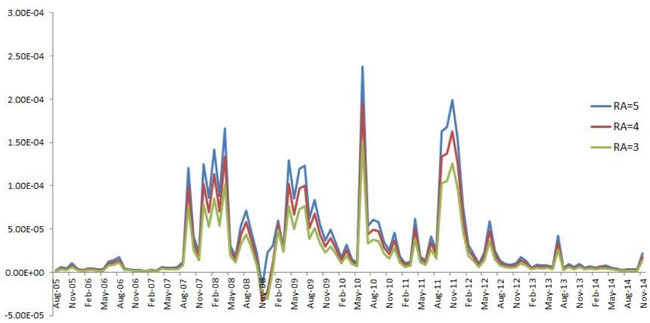

Figure 2.7: Co-Kurtosis Risk Premium

Table 2.7: Out-of-Sample Portfolio return forecast

The betas are estimated by the option implied method introduced in this paper. The risk premiums are estimated based on the option-implied method of Chalamandaris and Rompolis (2016). I sort the stocks according to their aggregate risk exposure. I then compute the equal-weighted expected portfolio betas (market beta, co-skewness beta and co-kurtosis beta) and multiply the price of the corresponding risk to get expected portfolio return for each month. I sort the portfolios into quintiles based on their expected return for each month, and compute the equal-weighted mean realized return for each quintile across all the time. The first quintile therefore contains the portfolios with highest expected returns, and the last one contains the portfolios with lowest expected returns. Additionally, I also use regression based method to estimate betas for CCC (covariance-co-skewness-co-kurtosis model), denote as regCCC, as a benchmark. The sample period is from August 2005 to November 2014.

RA=3

Quintile	OiCCC	regCCC
1	0.1054	0.0618
2	0.0674	0.0593
3	0.0553	0.0525
4	0.0481	0.0770
5	0.0108	0.0346
1-5	0.0945	0.0273
p-value	0.0064	0.4608

RA=4

Quintile	OiCCC	regCCC
1	0.1063	0.0618
2	0.0707	0.0687
3	0.0592	0.0441
4	0.0442	0.0760
5	0.0066	0.0346
1-5	0.0996	0.0273
p-value	0.0043	0.4536

RA=5

Quintile	OiCCC	regCCC
1	0.1130	0.0657
2	0.0841	0.0595
3	0.0510	0.0475
4	0.0358	0.0765
5	0.0031	0.0359
1-5	0.1099	0.0298
p-value	0.0017	0.4112

Chapter 3

Option-Based Range Estimator and Application

Abstract

In Chapter 3, I propose a percentile range estimator (PRE) for SPY weekly returns based on the intra-day minute data. It is a direct measure of return confidence intervals. Then, I use *Monday VIX open price* as a variable to forecast the future realized ranges, the correlation between VIX price and realized range is 0.73 for the whole sample period (2010-2017). The out-of-the-sample forecast R-square is 0.51, indicating a very strong predictive power. Panel regressions also show that the option-implied method dominates the historical-based ones. The results should appeal to financial practitioners, for whom it is very useful to have a reliable forecast of the return distribution, such as VaR (Value-at-Risk) analysis, expected shortfall (ES) for risk management and break-even point for option traders.

3.1 Introduction

Volatility is a core concept in modern finance theory, including in asset pricing, portfolio allocation or risk management. Literature has evolved from a constant volatility (Merton, 1969; Black and Scholes, 1973) to a time-varying fashion (e.g, Andersen and Bollerslev, 1997), and stochastic volatility models are also widely used. Stochastic volatility models are heavily used in both academia and industry (e.g., Hull and Whilte, 1987; Heston, 1993; Bates, 1996; Ghysels, Harvey, and Renault, 1996; Jarrow, 1998; Duffie, Pan, and Singleton, 2000). However, it is not easy to estimate the stochastic volatility model. For instance, the Gaussian quasi-maximum likelihood estimation (QMLE) approach of Harvey, Ruiz, and Shephard (1994), seems appealing due to its simplicity. Nevertheless, the problem embedded in these line of estimation is that standard volatility proxies such as squared returns are contaminated by highly non-Gaussian error.

Among financial practitioner, it is very useful to know the future distribution in advance, such as VaR (Value-at-Risk) analysis, expected shortfall (ES) for risk management, break-even point for option traders. The classical way to estimate the return distribution or interval rely on the estimation of volatility, then transfer to the confidence interval with an assumption on the distribution (normality, for example). Nevertheless, returns are not normally distributed, forecast return distribution confidence interval through volatility is biased.

In the last decades, a number of range based estimators have been proposed. Price range, which is defined as the difference in th high and low price observed during a time interval, is more efficient than return based estimators (e.g., Rogers and Satchell, 1991; Alizadeh, Brandt and Diebold, 2002 and Bali and Weinbaum 2005). However, range based estimators are proposed with assumption that the price dynamic is continuous and lognormally distributed. These estimators may be more efficient and unbiased than return-based ones, however, it can be the case in reality that the prices are not observed continuously.

There is a growing literature about extracting information from option market to forecast asset returns and volatilities. Bakshi and Madan (2000) propose a model-free method to estimate the risk-neutral moments of underlying from option prices. For instance, Conrad, Dittmar and Ghysels (2013) use option prices to estimate ex ante higher moments of the underlying individual securities risk-neutral returns distribution. They find that securities risk-neutral volatility, skewness, and kurtosis are strongly related to future returns. Many studies have demonstrated that option-implied volatility is a strong predictor of the future volatility in equity markets. Classic contributions include Christensen and Prabhala (1998), and Blair, Poon, and Taylor (2001). The predictive power of option-implied equity volatility has been confirmed recently by Busch, Christensen and Nielsen (2008), who compare option-implied forecasts with state-of-the-art realized volatility forecasts.

I propose a percentile range estimator (PRE) for SPY weekly return based on intra-day minute data. It is a direct measure of return distribution interval. Then, I use *Monday VIX open price* as a variable to forecast the future realized ranges, the correlation between VIX price and realized range is 0.73 for the whole sample period (2010-2017). The out-of-the-sample forecast R-square is 0.51, indicating a very strong predictive power. Panel regressions also show that the option-implied method dominates the historical-based ones. The results should appeal to financial practitioners, for whom it is very useful to have a reliable forecast of the future distribution, such as VaR (Value-at-Risk) analysis, expected shortfall (ES) for risk management and break-even point for option traders.

3.2 Range Estimator

It is very useful for financial practitioners, if they can know the future distribution in advance, such as VaR (Value-at-Risk) analysis, expected shortfall (ES) for risk management, break-even point for option traders. For lots of option trading strategies, underlying price confidence interval is critical, for instance, choosing different call and put to short a strangle requires the bet/believe on the break-even point, in another word, the confidence interval. This knowledge might not be so interesting for a stock trader, however, for derivative traders (who are dealing with more sophisticated instruments), they are profitable. In addition, people in risk management area are also interested in price/return distribution, in their word, tail-risk. The philosophy of risk management lies in the Black-Swan, historically, those low probability events crash the industry, and cause significant loss. Thus, with the knowledge of future distribution, risk management team could produce more reliable numbers. The classical way to estimate the return distribution or interval rely on the estimation of volatility, then transfer to the confidence interval with an assumption on the distribution (Normality, for example). Nevertheless, returns are not normally distributed, forecast return confidence interval through volatility is thus biased.

3.2.1 Range Estimator Construction

Denote P_0 the price of the security at time t_0, P_i the price of the security at time t_i, where $i = 1, 2, ...N$. For a given period (can be a day, a week or a month depending on the data frequency), denote the range of the price dynamics with respect to the initial price P_0:

$$C_i = \frac{P_i}{P_0}, i = 1, 2, ...N \tag{3.1}$$

Sort C_i from low to high, for a given confidence level α, calculate the confidence interval $C^{HL} = C^{sorted}_{1-\alpha} - C^{sorted}_{\alpha}$, for instance, if we want to calculate the 90% confidence interval for the future distribution, we can calculate the difference between 95 percentile sorted value and 5 percentile sorted value, $C^{HL}_{90\%}=C^{sorted}_{95\%} - C^{sorted}_{5\%}$. Compare to the classic range estimator, which usually defined as the difference of lowest and highest value, a confidence interval could be less affected by the extreme values—- the sudden change in the market (might recover very soon), indicates a more fruitful information for traders.

3.2.2 Data and Empirical Results

The data are minutes data of SPY (S&P500 ETF) price, range from 2010.06 to 2017.05. For each trading day, we have 405 observations. The data is from CBOE.

[**Insert Figure 1 here**]

[**Insert Table 1 here**]

Figure 1 plots the $80\%, 90\%, 95\%$ and 99% percentile PREs for one week distribution. As the figure shows, it hits high level during second half of 2011, when there are fears of contagion of the European sovereign debt crisis to Spain and Italy, as well as concerns over France's current AAA rating, concerns over the slow economic growth of the United States and its credit rating being downgraded. Among different percentile PREs, the pattern is quite similar, correlations between different PREs are almost 1, indicating its consistence. Table 1 summarize the statistics for the PREs with different percentiles. The means and medians are increasing monotonically as expected. Results of skewness and kurtosis showing the distribution of PREs are highly non-normal. For instance, the mean of PRE with 99 percentile is 0.0234, it indicates on average, the 99% of the one week SPY minute price is deviated within 2.34% of the SPY Monday open price (defined as the S_0).

3.2.3 Correlation with VIX

There is a growing literature about extracting information from option market to forecast asset returns and volatilities. Bakshi and Madan (2000) propose a model-free method to estimate the risk-neutral moments of underlying from option prices. Classic contributions include Christensen and Prabhala (1998), and Blair, Poon, and Taylor (2001). The predictive power of option-implied equity volatility has been confirmed recently by Busch, Christensen and Nielsen

(2008), who compare option-implied forecasts with state-of-the-art realized volatility forecasts. Option, especially OTM (out-of-the-money option) contains unique information about the market expectation of the underlying future movements. When investors/institution traders perceive (or know in advance due to asymmetric information) that certain event is going to happen, we could observe money flows into OTM options, due to the high leverage effect. These money flows are then push the IV (implied volatility) to a higher level, in the end, affect the VIX. One could take the advantage of observing VIX to get aggregate market "emotion". Consequently, it is interesting to see whether/how these information can forecast future.

[**Insert table 3 here**]

Figure 3 plots the Monday VIX open price with the subsequent week price distribution, with different percentile. The correlations of VIX price with different percentile distribution remains high for the whole period, range from 0.65 to 0.75, indicate that VIX is indeed a forward-looking measurement, at least for this percentile range estimator. Its correlation is actually increase as the percentile increase, showing the ability to capture/forecast the tail risk.

3.3 Forecast Percentile Range

There is no doubt that future distribution information is crucial in the finance world, both for academia and industry. The percentile range estimator (PRE) could be applied in both trading and risk management, thus a decent forecast method is appealing. The classical method is to first estimate the volatility, then transfer to the confidence interval based on the assumption of the distribution (normality, for example). However due to highly non-Gaussian feature, this kind of transformation causes biased estimator. In this section, I propose a new simple regression based model to fit and forecast FPE, other benchmark methods are also used for comparison.

3.3.1 VIX as the variable to estimate PRE

In the previous section, I show that the realized percentile range is highly correlated with VIX price, to be more precise, the one week realized percentile range is over 70% correlated with the Monday open price of VIX. Naturally, given this level of correlation, we can use a regression model to estimate the relationship, and forecast accordingly. I use OLS to estimate the following regression,

$$C^{HL}_{\alpha,t,T} = C + b1 \cdot VIX_t + b2 \cdot VIX_{t-1} \tag{3.2}$$

Table 2 summarize the statistic results and R-square for the fitted values. The fitted PREs also show high non-normality, with skewness up to 2 and kurtosis is around 8. Both R squares and adjusted R squares are at high level, more than 50% for all percentiles, indicate very good model fit. Interest to notice, the R-square increases as the chosen percentile increases, showing the potential to capture the edge cases.

3.3.2 Forecast PRE using VIX

In the previous section, the results showing that with VIX variable (and lagged one), the linear model could provide very good fitness. In order to forecast PRE, we need ex-ante coefficients, I use moving window estimation strategy, as its popularity among the literature. The coefficients are estimated using backward 256 weeks observation (approximately 5 years to generate stable estimates):

$$C^{HL,forecast}_{\alpha,t,T} = C_{t-256,t} + b1_{t-256,t} \cdot VIX_t + b2_{t-256,t} \cdot VIX_{t-1}. \tag{3.3}$$

3.3.3 Alternative Methods

The alternative method is to estimate the volatility and then transfer to distribution interval with an assumption on the distribution itself (usually normality). As a benchmark method, I use rolling window to estimate historical volatility, then apply the normality assumption, a 90% distribution interval is equal to $\mu \pm 1.5\sigma$. We can further assume that the weekly mean of deviation from the original price (S_0) is 0, then the 90 percentile range estimator is equal to 3σ. Figure 6 plots the comparison between VIX based forecast PREs, volatility transferred based PREs and realized PREs. As the figure shows, the volatility transferred based method generates biased estimates, they are over estimated through the whole time period. Another method can be using realized volatility (RV) and/or PRE_{t-1} as variables. Table 4 summarizes the panel regressions for forecasting PRE_ts. PRE_t is the realized PRE (Percentile Range Estimator) for the week (Monday to Friday), VIX_t stands for the Monday open price of VIX index. VIX_{t-1} is the last Monday (t-1) open price of VIX index. RV_{t-1} is the last week realized volatility calculating using minute data. PFE_{t-1} stands for the realized PRE of last week. It shows that the both historical and option-implied methods work well alone, although the forecast performance of using VIX_t and VIX_{t-1} as variables is much better than the historical ones.

The $adj.R^2$ of Regression (1) is more than doubled comparing to Regression (2) and (3). This is intuitive, since we believe the VIX index price on Monday should not only reflect the performance of S&P500 during last week, but also absorb the information during weekends, to form a rational expectation of the market for the following week. Interestingly, if we put historical metrics RV_{t-1} and PFE_{t-1} together with the option-implied ones (VIX_t, VIX_{t-1}) in Regression (5) and (6), the $adj.R^2$s do not increase, RV_{t-1} and PFE_{t-1} are now insignificant, showing not much value is added by incorporating historical metrics.

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3.4 Appendix

3.4.1 Appendix: Model Free Option-Implied Moments

Bakshi, Carr and Madan (2000) show that any twice continuously differentiable fnct, $H(S_T)$, of terminal price S_T, can be replicated by a unique position in the risk-free, stocks and European options.

$$H[S] = H[\bar{S}] + (S - \bar{S})H_s[\bar{S}] + \int_{\bar{S}}^{\infty} H_{ss}[K](S-K)^+ dK + \int_0^{\bar{s}} H_{ss}[K](K-S)^+ dK \tag{3.4}$$

The prices of these contracts are

$$\begin{aligned} E_t^Q\{e^{-r\tau}H[S]\} = (H[\bar{S}] - \bar{S}H_s[\bar{S}])e^{-r\tau} + H_s[\bar{s}]S(t) + \\ \int_{\bar{s}}^{\infty} H_{ss}[K]C(t,\tau;K)dK + \int_0^{\bar{S}} H_{ss}[K]P(t,\tau;K)dK. \end{aligned} \tag{3.5}$$

where $C_t(\tau, K)$ and $P_t(\tau, K)$ are prices of the European call and put options with time to maturity τ and strike price K. As a result, we can calculate the prices of derivatives given the price of the risk free zero coupon bond r, the spot price of the underlying, $\bar{S}$, and a series of OTM calls and puts. Since our main interest would be underlying return distribution, consider the function $H[S]$):

$$H[S_{t+\tau}] = R_{t+\tau}^2 = (lnS_{t+\tau} - lnS_t)^2 \tag{3.6}$$

Then the risk-neutral variance, skewness and kurtosis of equity returns could be computed based on the following expressions.

$$E_0^Q\Big[e^{-rT}(\frac{S_T - S_0}{S_0})^2\Big] = \frac{2}{s_0^2}\Big[\int_0^{S_0} P_0(T,X)dX + \int_{S_0}^{\infty} C_0(T,X)dX\Big] \tag{3.7}$$

$$E_0^Q\Big[e^{-rT}(\frac{S_T - S_0}{S_0})^3\Big] = \frac{6}{S_0^2}\Big[\int_0^{S_0} (\frac{X - S_0}{S_0})P_0(T,X)dX + \int_{S_0}^{\infty} (\frac{X - S_0}{S_0})C_0(T,X)dX\Big] \tag{3.8}$$

$$E_0^Q\Big[e^{-rT}(\frac{S_T - S_0}{S_0})^4\Big] = \frac{12}{S_0^2}\Big[\int_0^{S_0} (\frac{X - S_0}{S_0})^2 P_0(T,X)dX + \int_{S_0}^{\infty} (\frac{X - S_0}{S_0})^2 C_0(T,X)dX\Big] \tag{3.9}$$

Since there is no continuity of strike prices, we can approximate the integrals using cubic spline. For a given maturity, I interpolate implied volatilities across different moneyness level (K/S) to obtain a continuum of implied volatilities. Furthermore, the implied volatility of the highest or lowest available strike price is used when moneyness below and above the available moneyness level in the market. More precisely, for moneyness level smaller than 1 $(K/S < 1)$, the corresponding implied volatilities are used to generate put option prices, while for moneyness level larger than 1 $(K/S < 1)$, the corresponding implied volatilities are used to generate call option prices.

Figure 3.1: Percentile Range Estimator

Figure 1 plots the 80% and 90% percentile range estimators for one week SPY distribution. Denote P_0 the price of the security at time t_0, P_i the price of the security at time t_i, where $i = 1, 2, ...N$. For a given period (can be a day, a week or a month depending on the data frequency), denote the range of the price dynamics with respect to the initial price P_0: $C_i = \frac{P_i}{P_0}, i = 1, 2, ...N$. Sort C_i from low to high, for a given confidence level α, calculate the confidence interval $C^{HL} = C^{sorted}_{1-\alpha} - C^{sorted}_{\alpha}$, for instance, if we want to calculate the 90% confidence interval for the future distribution, we can calculate the difference between 95 percentile sorted value and 5 percentile sorted value, $C^{HL}_{90\%} = C^{sorted}_{95\%} - C^{sorted}_{5\%}$. The data are minutes data of SPY (S&P500 ETF) price, range from 2010.06 to 2017.05. For each trading day, we have 405 observations. The data is from CBOE.

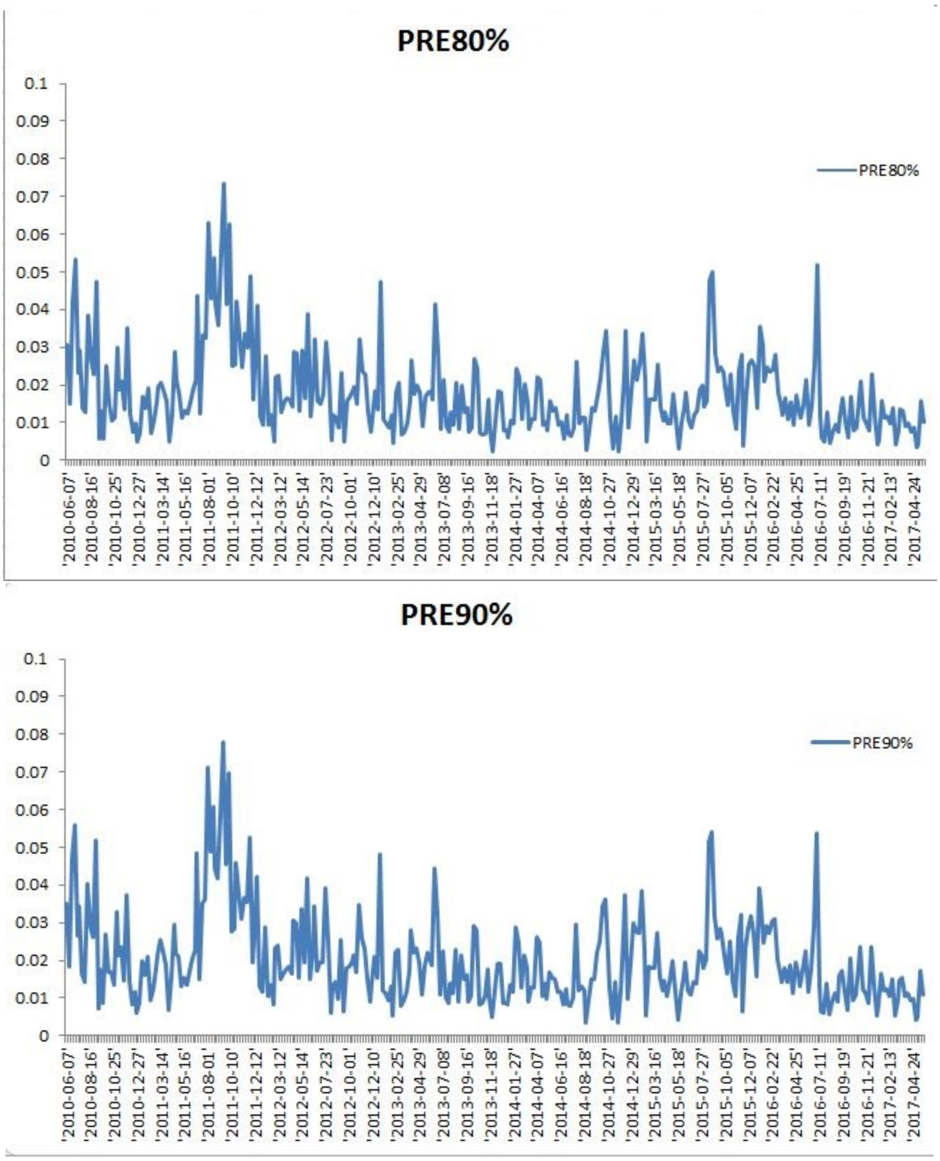


Table 3.1: Summary Statistics for PRE

Table 1 shows the different percentile range estimators for one week SPY distribution. Denote P_0 the price of the security at time t_0, P_i the price of the security at time t_i, where $i = 1, 2, ...N$. For a given period (can be a day, a week or a month depending on the data frequency), denote the range of the price dynamics with respect to the initial price P_0: $C_i = \frac{P_i}{P_0}, i = 1, 2, ...N$. Sort C_i from low to high, for a given confidence level α, calculate the confidence interval $C^{HL} = C^{sorted}_{1-\alpha} - C^{sorted}_{\alpha}$, for instance, if we want to calculate the 90% confidence interval for the future distribution, we can calculate the difference between 95 percentile sorted value and 5 percentile sorted value, $C^{HL}_{90\%}$=$C^{sorted}_{95\%} - C^{sorted}_{5\%}$. The data are minutes data of SPY (S&P500 ETF) price, range from 2010.06 to 2017.05. For each trading day, we have 405 observations. The data is from CBOE.

	PRE(80%)	PRE(90%)	PRE(95%)	PRE(99%)
mean	0.0178	0.0201	0.0218	0.0234
median	0.0149	0.0173	0.0188	0.0201
volatility	0.0113	0.0121	0.0128	0.0135
skewness	1.6364	1.6065	1.6145	1.6917
kurtosis	6.4365	6.3578	6.3623	6.8715

Table 3.2: Summary Statistics for fitted PRE

Table 2 reports the summary statistics for fitted percentile range estimator (PRE) for one week SPY distribution with different percentiles. PRE is defined in the following way: Denote P_0 the price of the security at time t_0, P_i the price of the security at time t_i, where $i = 1, 2, ...N$. For a given period (can be a day, a week or a month depending on the data frequency), denote the range of the price dynamics with respect to the initial price P_0: $C_i = \frac{P_i}{P_0}, i = 1, 2, ...N$. Sort C_i from low to high, for a given confidence level α, calculate the confidence interval $C^{HL} = C^{sorted}_{1-\alpha} - C^{sorted}_{\alpha}$. The fitted PREs are estimated through regression $C^{HL}_{\alpha,t,T} = C + b1 \cdot VIX_t + b2 \cdot VIX_{t-1}$. The data are minutes data of SPY (S&P500 ETF) price, range from 2010.06 to 2017.05. For each trading day, we have 405 observations, which constitute 2025 observations per week. The data is from CBOE.

	PRE(80%)	PRE(90%)	PRE(95%)	PRE(99%)
mean	0.0181	0.0202	0.0218	0.0233
median	0.0154	0.0174	0.0188	0.0201
volatility	0.0089	0.0094	0.01	0.0104
skewness	2.1785	2.1639	2.1496	2.1406
kurtosis	8.3723	8.347	8.3232	8.3185
R square	0.503	0.5296	0.5453	0.5566
adj. Rˆ2	0.5131	0.5281	0.544	0.5553

Table 3.3: Summary Statistics

Table 3 reports the summary statistics for fitted percentile range estimator (PRE) for one week SPY distribution with different percentiles. PRE is defined in the following way: Denote P_0 the price of the security at time t_0, P_i the price of the security at time t_i, where $i = 1, 2, ...N$. For a given period (can be a day, a week or a month depending on the data frequency), denote the range of the price dynamics with respect to the initial price P_0: $C_i = \frac{P_i}{P_0}, i = 1, 2, ...N$. Sort C_i from low to high, for a given confidence level α, calculate the confidence interval $C^{HL} = C^{sorted}_{1-\alpha} - C^{sorted}_{\alpha}$. The fitted PREs are estimated through regression $C^{HL}_{\alpha,t,T} = C + b1 \cdot VIX_t + b2 \cdot VIX_{t-1}$. The data are minutes data of SPY (S&P500 ETF) price, range from 2010.06 to 2017.05. For each trading day, we have 405 observations, which constitute 2025 observations per week. The data is from CBOE.

	PRE(80%)	PRE(90%)	PRE(95%)	PRE(99%)
mean	0.0179	0.0201	0.0216	0.023
median	0.0159	0.0179	0.0192	0.0205
volatility	0.0075	0.0082	0.0086	0.009
skewness	1.9272	1.9187	1.8908	1.8808
kurtosis	7.1501	7.0968	6.9574	6.9656
MSE	0.0205	0.0224	0.0253	0.0295

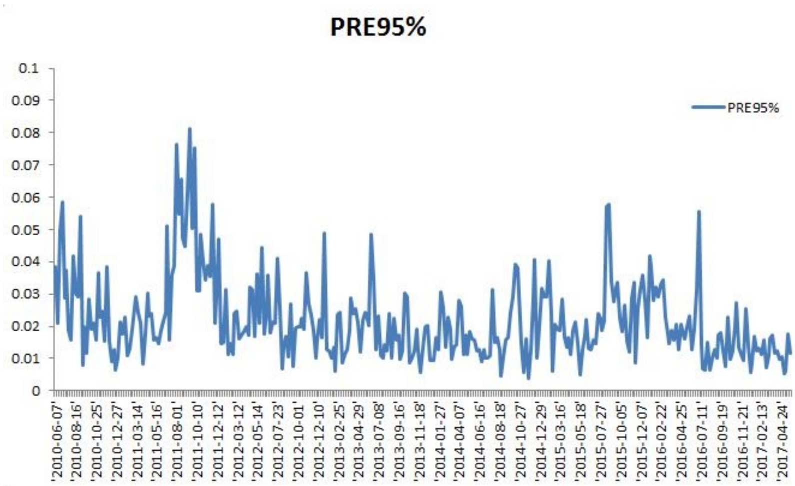
Table 3.4: Panel Regressions for Different Variables

Table 4 summarizes the panel regressions for forecasting PRE_t. PRE_t is the realized PRE (Percentile Range Estimator) for the week (Monday to Friday), VIX_t stands for the Monday open price of VIX index. VIX_{t-1} is the last Monday (t-1) open price of VIX index. RV_{t-1} is the last week realized volatility calculating using minute data. PFE_{t-1} stands for the realized PRE of last week. PRE is defined in the following way: Denote P_0 the price of the security at time t_0, P_i the price of the security at time t_i, where $i = 1, 2, ...N$. For a given period (can be a day, a week or a month depending on the data frequency), denote the range of the price dynamics with respect to the initial price P_0: $C_i = \frac{P_i}{P_0}, i = 1, 2, ...N$. The data are minutes data of SPY (S&P500 ETF) price, range from 2010.06 to 2017.05. For each trading day, we have 405 observations, which constitute 2025 observations per week. The data is from CBOE.

	$PRE_t(80\%)$					
	(1)	(2)	(3)	(4)	(5)	(6)
VIX_t	0.180***				0.158***	0.180***
	(11.36)				(12.56)	(11.07)
VIX_{t-1}	-0.039***					-0.045***
	(-2.56)					(-2.41)
RV_{t-1}		0.532***		0.304***	0.084	-0.004
		(8.84)		(4.38)	(-1.29)	(-0.055)
PRE_{t-1}			0.475***	0.334***	0.016	0.054
			(9.77)	(5.81)	(0.29)	(0.98)
observation	328	328	328	328	328	328
adj. Rˆ2	51.3%	19.1%	22.4%	26.5%	50.4%	51.2%

Figure 3.2: Percentile Range Estimator

Figure 2 plots the 95% and 99% percentile range estimator for one week SPY distribution. Denote P_0 the price of the security at time t_0, P_i the price of the security at time t_i, where $i = 1, 2, ...N$. For a given period (can be a day, a week or a month depending on the data frequency), denote the range of the price dynamics with respect to the initial price P_0: $C_i = \frac{P_i}{P_0}, i = 1, 2, ...N$. Sort C_i from low to high, for a given confidence level α, calculate the confidence interval $C^{HL} = C^{sorted}_{1-\alpha} - C^{sorted}_{\alpha}$, for instance, if we want to calculate the 90% confidence interval for the future distribution, we can calculate the difference between 95 percentile sorted value and 5 percentile sorted value, $C^{HL}_{90\%}=C^{sorted}_{95\%} - C^{sorted}_{5\%}$. The data are minutes data of SPY (S&P500 ETF) price, range from 2010.06 to 2017.05. For each trading day, we have 405 observations. The data is from CBOE.



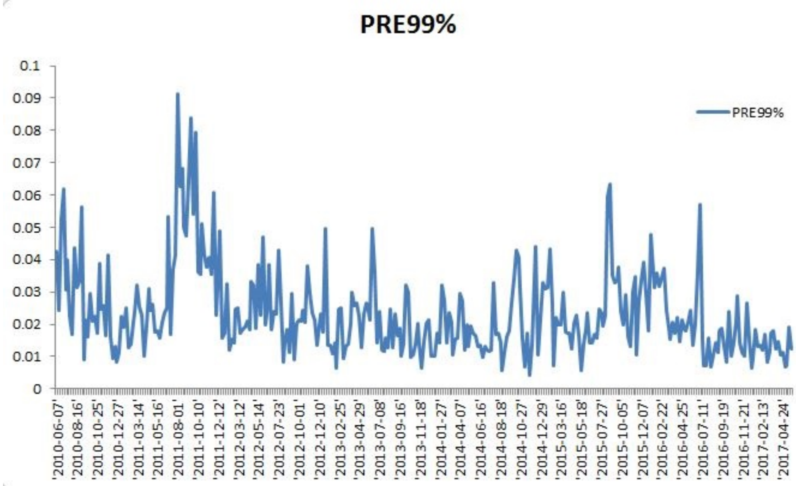


Figure 3.3: Correlation between PRE and VIX

Figure 3 plots the 80% and 90% percentile range estimator for one week SPY distribution and Monday VIX open price. The correlation between VIX price and subsequent realized PRE is also given. The data are minutes data of SPY (S&P500 ETF) price, range from 2010.06 to 2017.05. For each trading day, we have 405 observations. The data is from CBOE.

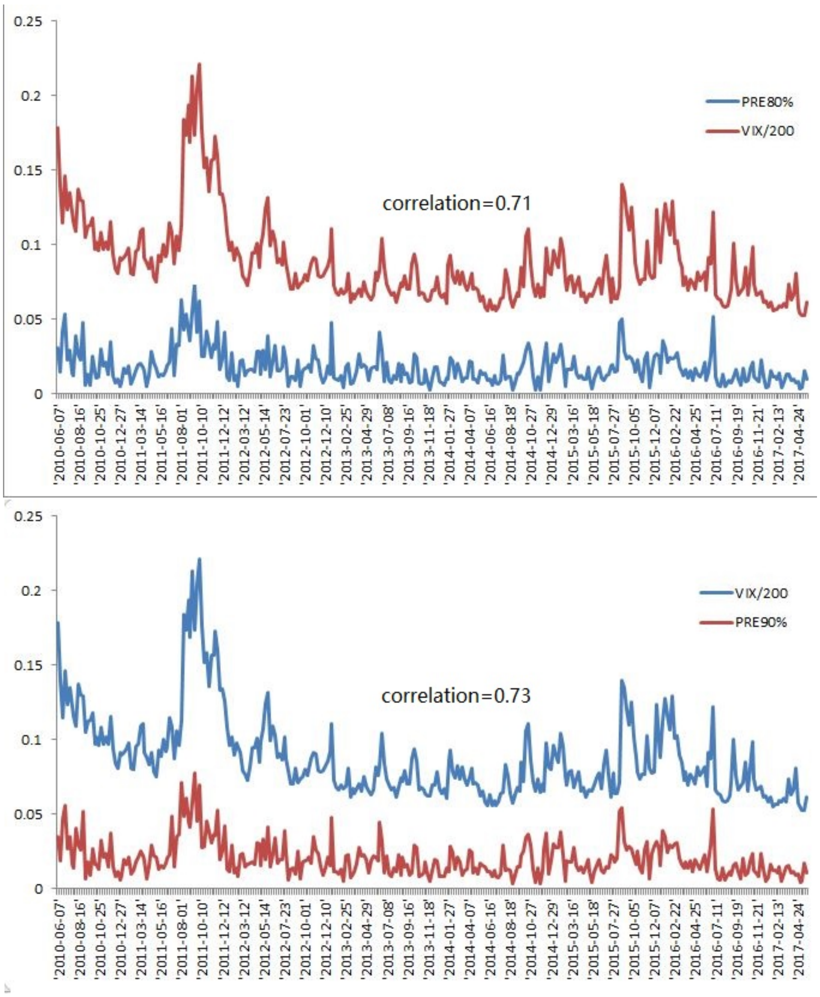


Figure 3.4: Correlation between PRE and VIX

Figure 4 plots the 95% and 99% percentile range estimator for one week SPY distribution and Monday VIX open price. The correlation between VIX price and subsequent realized PRE is also given. The data are minutes data of SPY (S&P500 ETF) price, range from 2010.06 to 2017.05. For each trading day, we have 405 observations. The data is from CBOE.

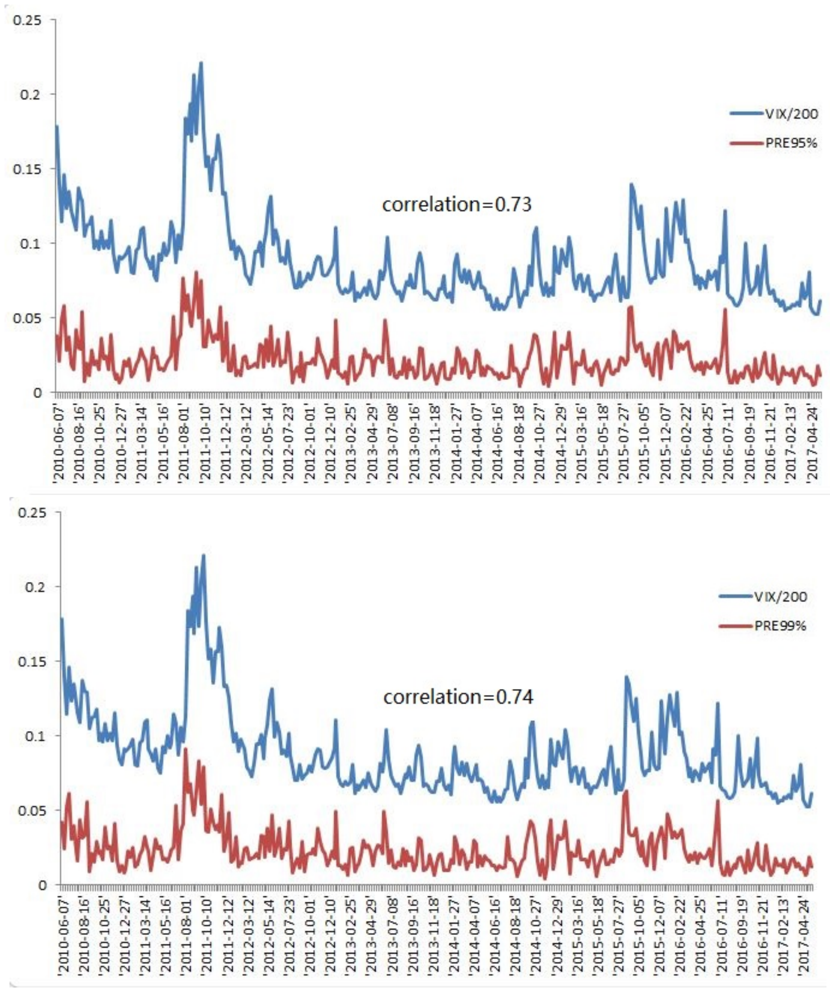


Figure 3.5: Fitted PRE

Figure 5 plots the fitted percentile range estimator (PRE) for one week SPY distribution and realized PRE. The fitted PREs are estimated through regression $C^{HL}_{\alpha,t,T} = C + b1 \cdot VIX_t + b2 \cdot VIX_{t-1}$ with 80 and 90 percentile during the whole sample period. The data are minutes data of SPY (S&P500 ETF) price, range from 2010.06 to 2017.05. For each trading day, we have 405 observations, which constitute 2025 observations per week. The data is from CBOE.

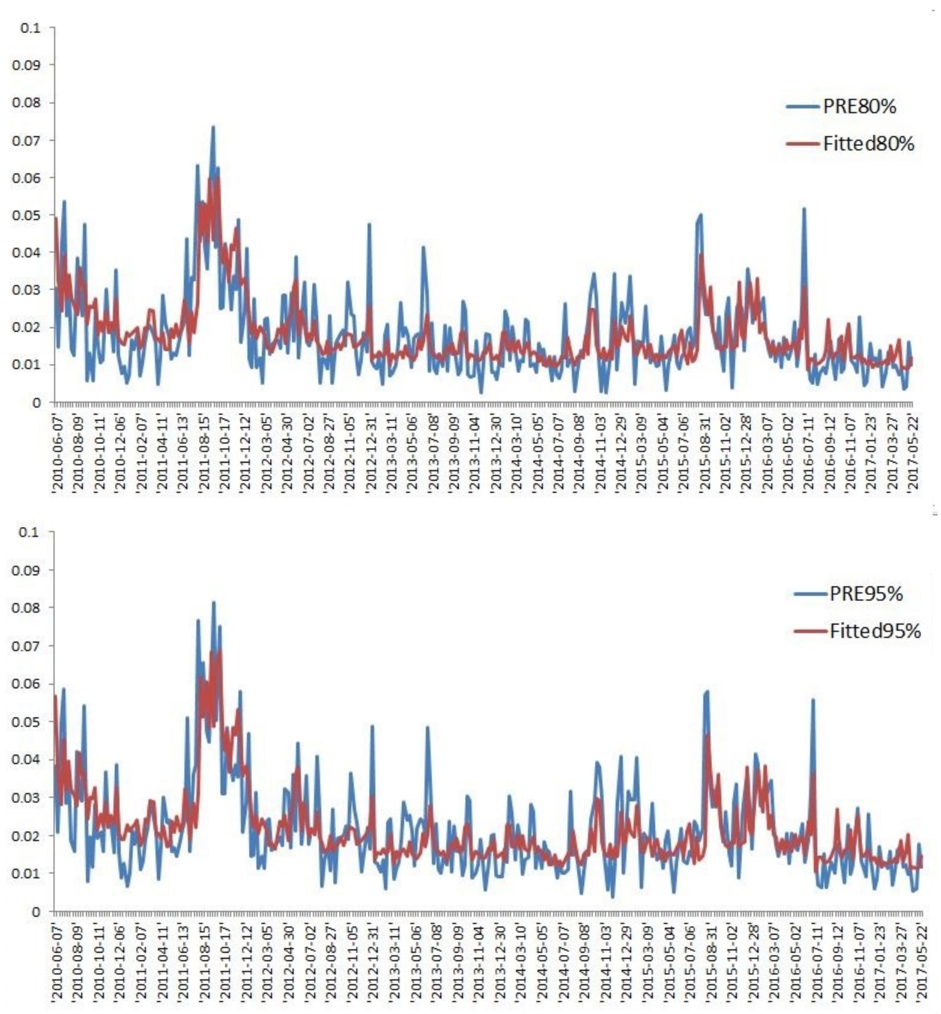


Figure 3.6: Predicted PRE

Figure 6 plots the predicted percentile range estimator (PRE) for one week SPY distribution and realized PRE. The predicted PREs are estimated using moving window estimation strategy, the coefficients are estimated using backward 256 weeks observation (approximately 5 years to generate stable estimates):

$$C_{\alpha,t,T}^{HL,forecast} = C_{t-256,t} + b1_{t-256,t} \cdot VIX_t + b2_{t-256,t} \cdot VIX_{t-1}.$$

As a benchmark method, I use rolling window to estimate historical volatility, then apply the normality assumption, a 90% distribution interval is equal to 3σ (zero-mean assumption).