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Quattro anni fa ero un po' preoccupato di come sarebbe potuta andare. Le mie motivazioni erano di imparare cose nuove, tecniche diverse, e possibilmente cambiare punti di vista su problemi giá visti. Posso dire ora di essere soddisfatto dei risultati, seppur parziali, raggiunti fin'ora e che, senza dubbio, mi piacerebbe continuare su questa strada nei prossimi anni.

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Abstract

This thesis is a collection of several contributions to theoretical statistics and game theory, using tecniques from topology and functional analysis.

In the first Chapter, we present a characterization of the spaces of random variables in the context of vector lattices. This provides the necessary and sufficient conditions for which an abstract mathematical object admits a concrete representation. Specifically, given a probability measure space (X, Σ, μ), it is well known that the Riesz space $L^0(\mu)$ of equivalence classes of measurable functions $f : \Omega \to \mathbf{R}$ is universally complete and the constant function $\mathbf{1}$ is a weak order unit. In addition, the linear functional $L^\infty(\mu) \to \mathbf{R}$ defined by $f \mapsto \int |f| \, \mathrm{d}\mu$ is strictly positive and order continuous. Then we show that the converse holds, that is, any universally complete Riesz space E with a weak order unit $e > 0$ which admits a strictly positive order continuous linear functional on the principal ideal generated by e is lattice isomorphic onto some $L^0(\mu)$.

In the second Chapter, we study a kind of (non-topological) convergence of sequences (x_n) taking values in topological spaces, which is strictly related to Cesàro summability and the Cech–Stone compactification of $\mathbf{N}$. We study several characterizations and relationships between two different generalizations of the set of accumulation points of (x_n). Then, we quantify the largeness of the set of subsequences (of a given sequence) which preserve these two sets, providing an example of non-analogue between measure and category. Finally, we prove that the set of bounded sequences statistically convergent to 0 is not isomorphic to ℓ_∞.

In the last Chapter, we say that two extensive game forms are behaviorally equivalent if they share the same reduced normal form with respect to terminal paths. Then we prove that this is the case if and only if it can be transformed one into the other through a composition of two elementary transformations. This characterization allows us to study the invariance of known solution concepts with respect to these transformations.

"The ideas which are here expressed so laboriously are extremely simple and should be obvious."

John Maynard Keynes.

Introduction

The underlying theme of this thesis is to present characterization results in the contexts of vector lattices, ideal convergence, and games structures with imperfect information. Accordingly, the presentation is divided into three Chapters.

The aim of first Chapter is provide a concrete representation of the spaces $L^0(\mu)$ of all random variables on a given probability space. The main result is that an order complete vector lattice E with a weark order unit $e > 0$ is lattice isomorphic onto some $L^0(\mu)$ if and only if E is laterally complete (that is, every disjoint subset of positive vectors admits a supremum) and admits a strictly positive order continuous linear functional φ on the principal ideal $\bigcup_n [-ne, ne]$. In turn, lateral completeness can be substituted, equivalently, with the completeness of the metric d defined by $d(x, y) = \varphi(|x - y| \wedge e)$ for all $x, y \in E$. It is remarkable that many spaces of stochastic processes have this form, thus allowing to extend important static results to their dynamic counterpart without any extra effort.

In the second Chapter, we start studying two analogues of the accumulation points of sequences x in the context of ideal convergence in topological spaces. To this aim, fix an ideal $\mathcal{I} \subseteq \mathcal{P}(\mathbf{N})$ representing the collection of subsets which are "small" in a suitable sense. Then we study several characterizations and relationships between the sets of $\mathcal{I}$-cluster points of x (that is, the set of all ℓ such that $\{n : x_n \in U\} \notin \mathcal{I}$ for all neighborhood U of ℓ) and $\mathcal{I}$-limit points of x (that is, the set of all ℓ such that $x_{n_k} \to \ell$ for some subsequence (x_{n_k}) such that $\{n_k : k \in \mathbf{N}\} \notin \mathcal{I}$).

We show, among others, that:

(i) $\mathcal{I}$-cluster points are the classical "cluster points of the filter" generated by the sequence x;

(ii) the set of $\mathcal{I}$-cluster points is the smallest closed set containing "almost all" the sequence x;

(iii) the set of $\mathcal{I}$-limit points is an F_σ-set, provided that $\mathcal{I}$ is an analytic P-ideal (identifying $\mathcal{P}(\mathbf{N})$ with the Cantor space $\{0,1\}^{\mathbf{N}}$);

(iv) the set of $\mathcal{I}$-limit points is closed for all sequences x if and only if $\mathcal{I}$-cluster points and $\mathcal{I}$-limit points coincide for all x if and only if $\mathcal{I}$ is an F_σ-ideal, provided that $\mathcal{I}$ is an analytic P-ideal;

(v) $\mathcal{I}$ is maximal if and only if each real sequence x admits at most one $\mathcal{I}$-limit point.

At this point, consider the bijection between $(0,1]$ and the set of subsequences of x. Then, using techniques from probability, we prove that the set of $\mathcal{I}$-cluster points of "almost every" subsequence of x coincides with the set of $\mathcal{I}$-cluster points of the original sequence, in the sense of Lebesgue measure. From a topological point of view, we show that the same set is not meager if and only if the set of $\mathcal{I}$-cluster points of x coincides with the set of accumulation points of x; moreover, in this case, it is comeager. This provides another example of the duality between measure and category.

Finally, in the last Section we prove that, if $\mathcal{I}$ is a meager ideal, then there are no continuous projections from ℓ_∞ onto the set of bounded sequences which are $\mathcal{I}$-convergent to 0. In particular, it follows that the set of bounded sequences statistically convergent to 0 is not isomorphic to ℓ_∞.

The aim of the last Chapter is to provide a characterization of an equivalence relation defined on the class of rooted trees, interpreted as extensive game structures with imperfect information (this is the reason of the word "Applications" in the title). In this regard, the main result can be stated as follows: Two extensive game structures are said to be behaviorally equivalent if they have the same normal representation with respect to terminal paths. Then we show that this is the case if and only if it can be transformed one into the other through a composition of (natural generalizations of) two elementary transformations, commonly known in literature as Interchanging of Simultaneous Moves and Coalescing Moves / Sequential Agent Splitting. Lastly, this characterization allows us to study the invariance of known solution concepts, such as Sequential Equilibrium and Extensive Form Rationalizability, with respect to behavioral equivalence.

The main text contains all the main results in the following works:

1. Cerreia–Vioglio, S., Leonetti, P., and Maccheroni, F. (2018). A characterization of the vector lattice of measurable functions. Submitted.
2. Leonetti, P. and Maccheroni, F. (2018). Ideal cluster points in topological spaces. Submitted.
3. Balcerzak, M. and Leonetti, P. (2019). On the relationship between ideal cluster points and ideal limit points. *Topology Appl.*, 252:178-190.
4. Leonetti, P. (2018). Thinnable ideals and invariance of cluster points. *Rocky Mountain J. Math.*, 48(6):1951-1961.
5. Leonetti, P. (2019). Invariance of ideal limit points. *Topology Appl.*, 252:169-177.
6. Leonetti, P. (2018). Limit points of subsequences. Submitted.
7. Leonetti, P., Miller, H., and Miller-Van Wieren, L. (2019). Duality between measure and category of almost all subsequences of a given sequence. *Period. Math. Hungar.*, to appear.
8. Leonetti, P. (2018). Continuous projections onto ideal convergent sequences. *Results Math.*, to appear.
9. Battigalli, P., Leonetti, P., and Maccheroni, F. (2018). Behavioral equivalence of extensive game structures. Manuscript.

Chapter 1

A Functional Analytic Approach to Random Variables

1.1 Preliminaries on Banach lattices

1.1.1 Motivation

Representation theorems provide us with more concrete description of abstract mathematical objects like operators, functionals, or spaces equipped with algebraic, topological or order structure. They play an important role in many areas of mathematics. Indeed they help us to get a better understanding of the objects they deal with, highlighting the necessary and sufficient conditions for which an abstract object (e.g., an ordered group or a topological vector space) need to have a concrete representation.

Accordingly, we are going to characterize those Riesz algebras that are isomorphic to the $L^0(\mu)$ space of all random variables, that is, measurable functions on some probability space. Such a result is useful because, in particular, it shows that many spaces of stochastic processes (e.g., the spaces of predicatable stochastic processes) have this form, thus allowing to extend important static results to their dynamic counterpart without any extra effort. As a consequence, the main result is a useful tool for getting new and reproving old results with possible easier proofs.

It is worth to remark that there is a recent growing literature on $L^0(\mu)$-modules in mathematical finance for which our repesentation may have some relevance, see Cerreia-Vioglio et al. (2016, 2017) and references therein.

1.1.2 Related results

A classical result of Kakutani states that any AL-space, that is, any Banach lattice with the norm additive on pairs of positive disjoint vectors, has to be a space $L^1(\mu) = L^1(X, \Sigma, \mu)$ of equivalence classes of μ-integrable functions $f : \Omega \to \mathbf{R}$, where Σ is a σ-algebra of subsets of some set X and $\mu : \Sigma \to [0, \infty]$ is a completely additive measure. In addition, if there exists a weak order unit, then μ can be chosen finite; see (Kakutani, 1941, Theorem 7). This can be seen as a characterization of the class of integrable functions by properties of the norm and order.

Relying on this result, Masterson proved a classification for the set of (equivalence classes of) real-valued measurable functions, see (Masterson, 1969, Corollary 2):

Theorem 1.1.1. *Let E be an Archimedean Riesz space. Then there exists an onto lattice isomorphism $E \to L^0(\mu)$, for some completely additive σ-finite measure space (X, Σ, μ), if and only if E is universally complete, has the countable sup property, and the extended order continuous dual of E is separating on E.*

Note that Theorem 1.1.1 involves only order properties. Here, the extended order continuous dual of E, usually denoted by $\Gamma(E)$, is the set of equivalence classes of order continuous linear functionals defined on order dense ideals of E, where two functionals are identified whenever they agree on an order dense ideal of E, cf. (Luxemburg and Masterson, 1967, Section 1). It is well known that $\Gamma(E)$ is separating on E if and only if there exists an order dense ideal I of E such that the order continuous dual of I is separating on I, and in that case there exists an order dense ideal which admits a strictly positive order continuous linear functional, see (Luxemburg and Masterson, 1967, Theorem 2.5). Other equivalent conditions are provided in (Filter, 1994, Theorem 3.4); in particular, if $\Gamma(E)$ is separating on E, then there exists a measure space (X, Σ, μ) for which E can be embedded order densely into $L^0(\mu)$.

Related results concerning representations of Archimedean Riesz spaces as spaces of measurable functions can be found, e.g., in Pinsker (1947), Fremlin (1967), and Labuda (1987); see also (Filter, 1994, Section 3) for a survey.

The aim of this chapter is to obtain an analogous characterization for the space of (equivalence classes of) measurable real-valued functions $L^0(X, \Sigma, \mu)$, where $\mu : \Sigma \to \mathbf{R}$ is a σ-additive probability measure. The result aims at providing a concrete meaning to the property of the extended order dual being separating on the space itself. In turn, the result furnishes a concrete representation for f-algebras of $\mathcal{L}^0$ type considered in Cerreia-Vioglio et al. (2017), which was the original motivation for this work; see Section 1.2 below.

1.1.3 Notation

We refer to Aliprantis and Burkinshaw (2003) for basic aspects of Riesz spaces. Let E be a Riesz space. Then, we denote the positive cone of a Riesz subspace F by $F^+ := \{x \in F : x \geq 0\}$. The principal ideal generated by a vector $x \in E$, that is, the smallest solid Riesz subspace containing x, is denoted by E_x. E is said to be *laterally complete* [respectively, *laterally σ-complete*] if the supremum of every disjoint subset [resp., sequence] of E^+ exists in E. If E is also Dedekind complete, then we say that E is *universally complete*. E has the *countable sup property* if for every subset S of E whose supremum exists in E, there exists an at most countable subset of S having the same supremum as S in E.

Hereafter, a measure space (X, Σ, μ) is a non-empty set X, together with a σ-algebra Σ of subsets of X, and a σ-additive measure $\mu : \Sigma \to \mathbf{R}$. If, in addition, $\mu(X) = 1$ then (X, Σ, μ) is a probability measure space. Also, if μ is completely additive, then (X, Σ, μ) is a completely additive measure space. Moreover, $\mathbf{1}$ stands for the multiplicative unit of $L^0(\mu)$, whenever the underlying measure space is understood. Lastly, given an integrable function $f \in L^1(\mu)$, we shorten $\int f \, \mathrm{d}\mu$ with $\mu(f)$.

1.2 Characterization of $L^0(\mu)$

A preliminary observation is in order, whose easy proof is deferred to Section 1.2.1.

Lemma 1.2.1. *Let E be a Riesz space with weak order unit $e > 0$ and let $\varphi : E_e \to \mathbf{R}$ be a strictly positive linear functional. Then*

$$d_\varphi : E \times E \to \mathbf{R} : (x, y) \mapsto \varphi(|x - y| \wedge e) \tag{1.1}$$

is an invariant metric and the topology τ_φ generated by d_φ is Hausdorff locally solid.

Our main result follows.

Theorem 1.2.2. *Let E be a Dedekind complete Riesz space with weak order unit $e > 0$. Then the following are equivalent:*

(i) *There exist a completely additive probability measure space (X, Σ, μ) and an onto lattice isomorphism $T : E \to L^0(\mu)$ such that $T(e) = \mathbf{1}$.*

(ii) *There exist a (σ-additive) measure space (X, Σ, μ) and an onto lattice isomorphism $T : E \to L^0(\mu)$ such that $T(e) = \mathbf{1}$.*

(iii) *There exists a strictly positive order continuous linear functional $\varphi : E_e \to \mathbf{R}$ such that the metric d_φ is complete.*

(iv) *There exists a strictly positive order continuous linear functional $\psi : E_e \to \mathbf{R}$ and E is laterally complete.*

Moreover, in such case, E_e is lattice isomorphic onto $L^\infty(\mu)$, the metrics d_φ and d_ψ are topologically equivalent, i.e., $\tau_\varphi = \tau_\psi$, and E has the countable sup property.

As an immediate consequence, we obtain a result in the same spirit of Theorem 1.1.1.

Corollary 1.2.3. *Let E be an Archimedean Riesz space. Then E is lattice isomorphic onto $L^0(\mu)$, for some probability measure space (X, Σ, μ), if and only if E is universally complete (hence, with weak order unit $e > 0$) and admits a strictly positive order continuous linear functional on E_e.*

Lastly, we obtain a charaterization of f-algebras of $\mathcal{L}^0$ type, cf. (Cerreia-Vioglio et al., 2017, Definition 6). In this regard, we recall that an f-algebra is a Riesz algebra E for which $(a \cdot c) \wedge b = (c \cdot a) \wedge b = 0$ for all $a, b, c \geq 0$ such that $a \wedge b = 0$. If, in addition, E is Dedekind complete and admits a non-zero multiplicative unit e, then it is said to be a Stonean algebra, cf. (Cerreia-Vioglio et al., 2016, Definition 2). In such case, the following facts are well known and readily provable:

(i) The multiplication is commutative, i.e., $a \cdot b = b \cdot a$ for all $a, b \in E$,

(ii) $x^2 := x \cdot x \geq 0$ for all $x \in E$; in particular, $e > 0$, and

(iii) e is a weak order unit.

Accordingly, a Stonean algebra E is said to be *f-algebra of $\mathcal{L}^0$ type* whenever the principal ideal E_e is an Arens algebra, i.e., a real commutative Banach algebra such that $\|e\| = 1$ and

$$\|a\|^2 \leq \|a^2 + b^2\|$$

for all $a, b \in E_e$, and there exists a strictly positive order continuous linear functional φ on E_e such that the metric d_φ defined in (1.1) is complete.

In this respect, Theorem 1.2.2 implies that f-algebras of $\mathcal{L}^0$ type are (equivalence classes of) spaces of measurable functions.

Corollary 1.2.4. *Let E be an Archimedean f-algebra with non-zero multiplicative unit. Then E is an f-algebra of $\mathcal{L}^0$ type if and only if E is lattice and algebra isomorphic onto $L^0(\mu)$, for some probability measure space (X, Σ, μ).*

Lastly, it is worth noting that the topological equivalence of d_φ and d_ψ at the end of Theorem 1.2.2 cannot be strengthened to strongly equivalence, as it is shown in the following example.

Example 1.2.5. Let μ be the function $\mathcal{P}(\mathbf{N}) \to \mathbf{R} : X \mapsto \sum_{x \in X} 2^{-x}$, where $\mathbf{N}$ is the set of positive integers and $\mathcal{P}(\mathbf{N})$ its powerset.

Then $(\mathbf{N}, \mathcal{P}(\mathbf{N}), \mu)$ is a probability measure space, $L^0(\mu)$ is the space of real-valued sequences (indexed by $\mathbf{N}$), and $L^\infty(\mu)$ is the ideal generated by $e = (1, 1, \ldots)$, i.e., the subspace of bounded sequences ℓ^∞. Accordingly, define the strictly positive order continuous linear functionals $\varphi : \ell^\infty \to \mathbf{R}$ and $\psi : \ell^\infty \to \mathbf{R}$ mapping each $x = (x_1, x_2, \dots)$ into $\sum_{n\geq 1} x_n 2^{-n}$ and $\sum_{n\geq 1} x_n 3^{-n}$, respectively.

At this point, let us suppose for the sake of contradiction that there exists a positive constant c such that $d_\varphi(x, y) \leq c d_\psi(x, y)$ for all $x, y \in L^0(\mu)$. Moreover, for each $n \in \mathbf{N}$, define $e_n = (0, \dots, 0, 1, 1, \dots)$, where 0 is repeated exactly n times. Then, it would follow

$$\begin{aligned}\sum_{k\geq n} 2^{-k} &= \varphi(e_n) = d_\varphi(e_n, 0) \\ &\leq c d_\psi(e_n, 0) = c\psi(e_n) = c \sum_{k\geq n} 3^{-k},\end{aligned}$$

which is false whenever n is sufficiently large.

Proofs of Theorem 1.2.2 and Corollaries 1.2.3-1.2.4 follow in Section 1.2.2.

1.2.1 Preliminary Lemmas

Proof of Lemma 1.2.1. Note that d_φ is well defined since E_e is solid and $0 \leq |x-y| \wedge e \leq e \in E_e$ for all $x, y \in E$. Since e is a weak order unit, $|x-y| \wedge e = 0$ if and only if $x = y$. Then, the strict positivity of φ implies that $d_\varphi(x, y) = d_\varphi(y, x) \geq 0$ for all $x, y \in E$, with equality if and only if $x = y$. Lastly, for each $x, y, z \in E$, we have $|x - z| \leq |x - y| + |y - z|$, so that, thanks to (Aliprantis and Burkinshaw, 2003, Theorem 1.7.(4)),

$$|x - z| \wedge e \leq (|x - y| + |y - z|) \wedge e \leq |x - y| \wedge e + |y - z| \wedge e.$$

Since φ is a positive operator, we obtain $d_\varphi(x, z) \leq d_\varphi(x, y) + d_\varphi(y, z)$. Clearly, d_φ is invariant and (E, τ_φ) is Hausdorff.

Lastly, the local solidness follows by the fact each open ball B centered in 0 and with radius $r > 0$ is solid. Indeed, given $x, y \in E$ with $|x| \leq |y|$ and $y \in B$, then by the positivity of φ we get $\varphi(|x| \wedge e) \leq \varphi(|y| \wedge e)$, that is, $x \in B$. $\square$

Lemma 1.2.6. *Let E, F be Riesz spaces and let $T : E \to F$ be an onto lattice isomorphism. Then T is order continuous.*

Proof. Fix a net $(x_\alpha)_{\alpha\in A}$ of vectors in E such that $x_\alpha \downarrow 0$. Then it will be enough to show thar $T(x_\alpha) \downarrow 0$ in F. Note that, since T is an onto lattice isomorphism, then clearly T and its inverse T^{-1} are both positive operators.

On the one hand, the positivity of T implies that $T(x_\alpha) \geq 0$ for all $\alpha \in A$. On the other hand, if ℓ is a lower bound of $\{T(x_\alpha) : \alpha \in A\}$, then the positivity of T^{-1} and the fact that T is one-to-one and onto imply that

$$x_\alpha = T^{-1}(T(x_\alpha)) \geq T^{-1}(\ell)$$

for all $\alpha \in A$, i.e., $T^{-1}(\ell)$ is a lower bound of $\{x_\alpha : \alpha \in A\}$. Since $\inf\{x_\alpha : \alpha \in A\} = 0$ by hypothesis, then $T^{-1}(\ell) \leq 0$, with the consequence that $\ell \leq 0$. □

In the proof of the main result, we will use the following characterization of $L^\infty(\mu)$, cf. also Abramovich, Aliprantis, and Zame (Abramovich et al., 1995, Corollary 2.2).

Lemma 1.2.7. *Let E be a Dedekind complete Riesz space with strong order unit $e > 0$ which admits a strictly positive order continuous linear functional φ. Then there exist a completely additive probability space (X, Σ, μ) and an onto lattice isomorphism $T : E \to L^\infty(\mu)$ such that $T(e) = \mathbf{1}$ and $\varphi(x) = \mu(T(x))$ for all $x \in E^+$.*

Proof. Since φ is strictly positive, then $\varphi(e) > 0$. Hence, dividing by $\varphi(e)$, we can suppose without loss of generality that $\varphi(e) = 1$. It follows that

$$\|\cdot\| : E \to \mathbf{R} : x \mapsto \varphi(|x|)$$

is an order continuous L-norm. Let $\widehat{E}$ be the topological completion of E. Then, $\widehat{E}$ is an AL-space and, according to (Abramovich et al., 1995, Footnote 6), E is an (order dense) ideal of $\widehat{E}$. It follows by Kakutani's representation theorem (Kakutani, 1941, Theorem 7) that there exists an onto lattice and isometric $\widehat{T}$: $\widehat{E} \to L^1(\mu)$, for some completely additive probability space (X, Σ, μ), such that $\widehat{T}(e) = 1$. In particular,

$$\varphi(x) = \mu(\widehat{T}(x))$$

for all $x \in E^+$. In addition, since e is unit of E and E is an ideal of $\widehat{E}$, then $E = E_e = \widehat{E}_e$. The claim follows by letting T equal to the restriction of $\widehat{T}$ from E to its direct image. □

1.2.2 Proofs

Proof of Theorem 1.2.2. We are going to show the following chain of equivalences:

$$\text{(i)} \implies \text{(ii)} \implies \text{(iii)} \implies \text{(iv)} \implies \text{(iii)} \implies \text{(i)}.$$

(i) $\implies$ (ii). This is obvious.

(ii) $\implies$ (iii). Let us assume that there exist a (σ-additive) measure space (X, Σ, μ) and an onto lattice isomorphism $T : E \to L^0(\mu)$ such that $T(e) = \mathbf{1}$. In particular, T is a positive operator. It follows that $T([-\lambda e, \lambda e]) = [-\lambda T(e), \lambda T(e)]$, hence

$$T(E_e) = T\left(\bigcup_{\lambda>0}[-\lambda e, \lambda e]\right) = \bigcup_{\lambda>0}[-\lambda \mathbf{1}, \lambda \mathbf{1}] = L^\infty(\mu). \tag{1.2}$$

Therefore, the restriction of T on E_e, hereafter denoted by T_e, is a lattice isomorphism onto $L^\infty(\mu)$. Note that, thanks to Lemma 1.2.6, T_e is order continuous.

At this point, define the linear functional

$$\varphi : E_e \to \mathbf{R} : x \mapsto \mu(T(x)).$$

It is routine to check that φ is strictly positive. Moreover, φ is order continuous. To this aim, since φ is a positive operator, it is enough to show that $\varphi(x_\alpha) \downarrow 0$ for every net $(x_\alpha) \downarrow 0$ in E_e. Since $\mathbf{R}$ is an Archimedean Riesz space with the countable sup property and $\varphi : E_e \to \mathbf{R}$ is strictly positive, it follows by (Aliprantis and Burkinshaw, 2003, Theorem 1.45) that E_e has the countable sup property as well. In particular, it is enough to show that $\varphi(x_n) \downarrow 0$ for every sequence $(x_n) \downarrow 0$ in E_e. Since T_e is order continuous, $T_e(x_n) \downarrow 0$ in $L^\infty(\mu)$. Finally $\varphi(x_n) = \mu(T_e(x_n)) \downarrow 0$ by Lebesgue's dominated convergence theorem.

Lastly, we miss to prove that the metric space (E, d_φ) is (topologically) complete. Let d be the metric of convergence in measure on $L^0(\mu)$, that is,

$$d : L^0(\mu) \times L^0(\mu) \to \mathbf{R} : (f, g) \mapsto \mu(|f - g| \wedge \mathbf{1}).$$

Hence, for all $x, y \in E$, we obtain

$$\begin{aligned} d_\varphi(x,y) &= \varphi(|x-y| \wedge e) = \mu(T(|x-y| \wedge e)) \\ &= \mu(T(|x-y|) \wedge T(e)) = \mu(|T(x-y)| \wedge \mathbf{1}) \\ &= \mu(|T(x) - T(y)| \wedge \mathbf{1}) = d(T(x), T(y)). \end{aligned} \tag{1.3}$$

Then, fix a Cauchy sequence (x_n) of vectors in E, i.e., for each $\varepsilon > 0$ there exists $n_0 = n_0(\varepsilon)$ such that $d_\varphi(x_n, x_m) \le \varepsilon$ whenever $n, m \ge n_0$. It follows from (2.2) that $(T(x_n))$ is a Cauchy sequence in $(L^0(\mu), d)$. Since the metric space $(L^0(\mu), d)$ is complete, there exists $f \in L^0(\mu)$ such that $d(T(x_n), f) \to 0$ as $n \to +\infty$. Moreover, T is a bijection, hence there exists $x \in E$ such that $T(x) = f$. Therefore, thanks to (2.2), we obtain $d_\varphi(x_n, x) \to 0$ as $n \to +\infty$.

(iii) $\implies$ (iv). Suppose that there exists a strictly positive order continuous linear functional $\varphi : E_e \to \mathbf{R}$ for which the metric space (E, d_φ) is complete, and set $\varphi = \psi$.

Since e is a weak order unit and E is Dedekind complete, it follows by (Aliprantis and Burkinshaw, 2003, Theorem 7.39) that it is enough to show that E is laterally σ-complete. To this aim, let (x_n) be a sequence of disjoint vectors in E^+ and define the sequences (y_n) by $y_n := x_n \wedge e$ for each $n \ge 1$. Note that (y_n) is a disjoint sequence of vectors in the order interval $[0, e]$. Moreover, for each positive integer n, define

$$a_n := x_1 + \cdots + x_n \quad \text{and} \quad b_n := y_1 + \cdots + y_n.$$

Since E is Dedekind complete and $b_n = y_1 \vee \cdots \vee y_n \le e$ for each $n \ge 1$, then the supremum of the sequence (b_n) exists in $[0, e]$, and we denote it by b. Hence $0 \le b - b_n \downarrow 0$, which implies $0 \le (b - b_n) \wedge e \downarrow 0$. Since φ is order continuous, then

$$\lim_{n \to \infty} d_\varphi(b_n, b) = 0.$$

In particular, (b_n) is a Cauchy sequence in (E, d_φ). In addition, for all positive integers n, m with $n > m$, it holds

$$\begin{aligned} d_\varphi(a_n, a_m) &= \varphi((a_n - a_m) \wedge e) = \varphi((x_{m+1} \vee \cdots \vee x_n) \wedge e) \\ &= \varphi(y_{m+1} \vee \cdots \vee y_n) = \varphi((y_{m+1} + \cdots + y_n) \wedge e) \\ &= \varphi((b_n - b_m) \wedge e) = d_\varphi(b_n, b_m). \end{aligned}$$

It follows that also (a_n) is a Cauchy sequence in (E, d_φ). Since (E, d_φ) is complete by hypothesis, there exists $a \in E$ such that

$$\lim_{n\to\infty} d_\varphi(a_n, a) = 0. \tag{1.4}$$

Thanks to Lemma 1.2.1, (E, τ_φ) is a locally solid Hausdorff Riesz space. Therefore, according to (Aliprantis and Burkinshaw, 2003, Theorem 2.21.(c)) and (1.4), it follows that $x_1 \vee \cdots \vee x_n = a_n \uparrow a$. By the previous argument, this implies that E is laterally complete.

(iv) $\implies$ (iii). Set $\varphi = \psi$ and note that $\tau := \tau_\psi$ is a Fatou topology on E, i.e., it has a neighborhood base at 0 consisting of solid and order closed sets. Let $(\widehat{E}, \hat{\tau})$ be the topological completion of (E, τ).

According to a classical result of Nakano, see e.g. (Aliprantis and Burkinshaw, 2003, Theorem 4.28), since (E, τ) is a Dedekind complete locally solid Riesz space with the Fatou property, then the order intervals of E are τ-complete. Fix $x \in E$ and $\hat{x} \in \widehat{E}$ such that $0 \le \hat{x} \le x$ in $\widehat{E}$ and let (y_α) be a net of positive vectors in E such that $y_\alpha \xrightarrow{\tau} \hat{x}$. This implies that $x_\alpha \xrightarrow{\tau} \hat{x}$, where $x_\alpha := y_\alpha \wedge x$ for each index α. Since $x_\alpha \in [0, x]$ for each α and the order intervals are τ-complete, then $\hat{x} \in E$. Hence E is an ideal of $\widehat{E}$. (An alternative proof of this fact can be found also in (Aliprantis, 1974, Theorem 2.2).)

Moreover, given $0 \le \hat{x} \in \widehat{E}$ and a net (x_α) of positive vectors in E such that $x_\alpha \xrightarrow{\tau} \hat{x}$, then $x_\alpha \wedge \hat{x}$ belongs to E^+ (since E is an ideal). Hence, considering the finite suprema of the net $(x_\alpha \wedge \hat{x})$, we obtain a net (y_β) of vectors in E^+ such that $y_\beta \xrightarrow{\tau} \hat{x}$ and $y_\beta \uparrow \hat{x}$. This means that E is an order dense ideal of $\widehat{E}$.

Therefore, since E is a universally complete order dense Riesz subspace of the Archimedean Riesz space $\widehat{E}$, then $E = \widehat{E}$ by the uniqueness of the universal completion, see e.g. (Aliprantis and Burkinshaw, 2003, Theorem 7.15.(ii)).

(iii) $\implies$ (i). Suppose that an increasing net $(x_\alpha)_{\alpha\in A}$ of positive vectors in E_e is upper bounded by some $y \in E_e$. Then $x := \sup\{x_\alpha : \alpha \in A\}$ exists in E and belongs to the order interval $[0, y]$. Since E_e is solid, then $x \in E_e$. Therefore, thanks to (Aliprantis and Burkinshaw, 2003, Lemma 1.39), E_e is a

Dedekind complete Riesz subspace with strong order unit $e > 0$. It follows by Lemma 1.2.7 that there exist a completely additive probability space (X, Σ, μ) and an onto lattice isomorphism

$$T_{\mathrm{e}} : E_e \to L^\infty(\mu)$$

such that $T_{\mathrm{e}}(e) = \mathbf{1}$ and $\varphi(x) = \mu(T_{\mathrm{e}}(x))$ for all $0 \le x \in E_e$. Then, for all $x, y \in E_e$ we obtain

$$\begin{aligned} d(T_{\mathrm{e}}(x), T_{\mathrm{e}}(y)) &= \mu(|T_{\mathrm{e}}(x) - T_{\mathrm{e}}(y)| \wedge \mathbf{1}) \\ &= \mu(T_{\mathrm{e}}(|x-y|) \wedge \mathbf{1}) = \mu(T_{\mathrm{e}}(|x-y| \wedge e)) \\ &= \varphi(|x-y| \wedge e) = d_\varphi(x, y). \end{aligned} \tag{1.5}$$

Claim 1. *E is the topological closure of E_e in (E, τ_φ).*

Proof. Given $x \in E^+$, then $(x_n) \uparrow x$, where $x_n := x \wedge ne$, by the fact that e is a weak order unit. This implies that $(|x - x_n| \wedge e) \downarrow 0$. Since φ is order continuous, then

$$d_\varphi(x_n, x) = \varphi(|x - x_n| \wedge e) \downarrow 0,$$

i.e., $x_n \to x$ in (E, τ_φ). The claim follows by the fact that $x = x^+ - x^-$ for each $x \in E$ and the topological limits are linear. $\square$

Claim 2. *There exists a positive operator $T : E \to L^0(\mu)$ extending T_{e} for which (1.5) holds for all $x, y \in E$.*

Proof. Define the operator $T : E \to L^0(\mu)$ as the unique extension of

$$E^+ \to L^0(\mu) : x \mapsto \lim_{n\to\infty} T_{\mathrm{e}}(x_n), \tag{1.6}$$

where $(x_n)_{n\ge 1}$ is any sequence in E_e^+ such that $x_n \to x$ in (E, τ_φ). The limit in (1.6) is understood to be in $(L^0(\mu), d)$.

At first, we show that T is well defined. To prove the existence of the limit, fix a sequence (x_n) of vectors in E_e such that $x_n \to x$ (note that such sequence exists by Claim 1). Then (x_n) is a Cauchy sequence. It follows by (1.5) that $(T_{\mathrm{e}}(x_n))$ is a Cauchy sequence in $(L^0(\mu), d)$. Then, by the completeness of the latter space, there exists (a unique) $f \in L^0(\mu)$ such that $\lim_{n\to\infty} T_{\mathrm{e}}(x_n) = f$.

Then, we show that the limit in (1.6) is independent from the choice of the sequence (x_n). Indeed, let us suppose that (x'_n) is another sequence of vectors such that $x'_n \to x$ in (E, τ_φ). This implies that $x_n - x'_n \to 0$, i.e.,

$$\lim_{n\to\infty} \varphi(|x_n - x'_n| \wedge e) = 0.$$

Since $x_n - x'_n \in E_e$ for each n and E_e is Dedekind complete, there exists $\ell \in E_e$ such that $\ell = \inf\{|x_n - x'_n| : n \geq 1\}$. In particular, there exists a real $\lambda > 0$ such that $\ell \leq \lambda e$. Clearly, $\ell \geq 0$ and, by the strict positivity of φ, it follows that

$$\varphi(|x_n - x'_n| \wedge e) \geq \varphi(\ell \wedge e)$$

for all n, proving that $\ell \wedge e = 0$. Hence $\ell = \ell \wedge \lambda e = 0$. By the same argument, it is easy to see that there does not exist any $y > 0$ in E_e such that $|x_n - x'_n| \geq y$ for infinitely many n. In particular, choosing $y = {}^1\!/_k\, e$, we obtain that $x_n - x'_n$ belongs to the order interval $[-{}^1\!/_k\, e, {}^1\!/_k\, e]$ whenever n is sufficiently large. This implies that $x_n - x'_n$ converges to 0 with respect to the order, i.e.,

$$x_n - x'_n \xrightarrow{o} 0. \tag{1.7}$$

Since T_e is a lattice isomorphism onto $L^\infty(\mu)$, then it is also order continuous, thanks to Lemma 1.2.6. Hence $T_\mathrm{e}(x_n - x'_n) \xrightarrow{o} 0$ in $L^\infty(\mu)$, which is equivalent to

$$\lim_{n\to\infty} T_\mathrm{e}(x_n - x'_n)(\omega) = 0$$

for each $\omega \in X$. Since it is well known that puntual convergence implies convergence in measure, then

$$\lim_{n\to\infty} d(T_\mathrm{e}(x_n - x'_n), 0) = 0,$$

which is what we wanted to show.

In addition, it is routine to check that T is a positive operator.

Lastly, for each $x, y \in E$, there exist by Claim 1 two sequences of vectors (x_n) and (y_n) in E_e such that $x_n \to x$ and $y_n \to y$ in (E, τ_φ). Thanks to Claim 1 and (1.5), we get

$$\begin{aligned} d_\varphi(x, y) &= \lim_{n\to\infty} d_\varphi(x_n, y_n) = \lim_{n\to\infty} d(T_\mathrm{e}(x_n), T_\mathrm{e}(y_n)) \\ &= d(\lim_{n\to\infty} T_\mathrm{e}(x_n), \lim_{n\to\infty} T_\mathrm{e}(y_n)) = d(T(x), T(y)). \end{aligned}$$

for all $x, y \in E^+$, hence also for all $x, y \in E$. $\square$

Claim 3. T is an onto lattice isomorphism.

Proof. Fix $0 \le f \in L^0(\mu)$. Since the constant function $\mathbf{1}$ is a weak order unit, then $f_n \uparrow f$, where $f_n := f \wedge n\mathbf{1}$ for each positive integer n. Again, since puntual convergence implies convergence in measure, we get $f_n \to f$ in $(L^0(\mu), d)$. In particular, (f_n) is a Cauchy sequence.

At this point, define $x_n := T_{\mathrm{e}}^{-1}(f_n)$ for each n. Note that (x_n) is a sequence of positive vectors in E_e and, thanks to (1.5), is a Cauchy sequence in (E, τ_φ). Since the metric d_φ is complete by hypothesis, there exists $x \in E^+$ for which $x_n \to x$. According to (1.6), we conclude that

$$T(x) = \lim_{n\to\infty} T_{\mathrm{e}}(x_n) = \lim_{n\to\infty} f_n = f,$$

i.e., $f \in T(E)$. Then, by the arbitrariness of f, T is onto, i.e., $T(E) = L^0(\mu)$.

To sum up, $T : E \to L^0(\mu)$ is a one-to-one and onto linear operator such that T and T^{-1} are both positive operators. Therefore, thanks to (Aliprantis and Burkinshaw, 2003, Exercise 16), T is an onto lattice isomorphism. $\square$

At this point, note that, if one of the equivalent conditions (i)-(iv) hold, then E_e is lattice isomorphic onto $L^\infty(\mu)$, thanks to (1.2).

Also, the metrics d_φ and d_ψ are topologically equivalent: indeed, a laterally complete Riesz space admits at most one Hausdorff Fatou topology, which must be necessarily a Lebesgue topology (i.e., $x_\alpha \xrightarrow{\tau} 0$ whenever $x_\alpha \downarrow 0$), see e.g. (Aliprantis and Burkinshaw, 2003, Theorem 7.53).

Lastly, suppose that $0 \le x_\alpha \uparrow x$ in E, hence by the Lebesgue property $x - x_\alpha \xrightarrow{\tau} 0$, i.e., $\varphi((x - x_\alpha) \wedge e) \to 0$. Then, there exists a subsequence (x_{α_n}) of the net (x_α) such that $\varphi((x - x_{\alpha_n}) \wedge e) \to 0$ as $n \to \infty$, i.e., $x - x_{\alpha_n} \xrightarrow{\tau} 0$. Considering that $x - x_{\alpha_n}$ is a decreasing sequence, we conclude that $x - x_{\alpha_n} \downarrow 0$ by (Aliprantis and Burkinshaw, 2003, Theorem 2.21.(c)), that is, $x_{\alpha_n} \uparrow x$. This means that E has the countable sup property. $\square$

Let us conclude with the proofs of the last two corollaries.

Proof of Corollary 1.2.3. Let us suppose that E is an Archimedean Riesz space and there exists an onto lattice isomorphism $T : E \to L^0(\mu)$, for some probability measure space (X, Σ, μ). Since $L^0(\mu)$ is universally complete (Aliprantis and Burkinshaw, 2003, Theorem 7.73) and has a weak order unit $\mathbf{1}$, then also E is universally complete with a weak order unit $e := T^{-1}(\mathbf{1}) > 0$. In addition, note that the existence of T is implied by (i) and, in turn, implies (ii). The claim follows by Theorem 1.2.2. □

Proof of Corollary 1.2.4. If E is lattice and algebra isomorphic onto $L^0(\mu)$, for some probability measure space (X, Σ, μ), then it is easy to check that E is an f-algebra of $\mathcal{L}^0$ type (we omit details).

Conversely, let us suppose that E is an f-algebra of $\mathcal{L}^0$ type. Then, in particular, E is a Dedekind complete Riesz space with weak order unit $e > 0$ and admits a strictly positive order continuous linear functional $\varphi : E_e \to \mathbf{R}$ such that the metric d_φ defined in (1.1) is complete. It follows by Theorem 1.2.2 that there exists a lattice isomorphism $T : E \to L^0(\mu)$, for some probability measure space (X, Σ, μ).

Then, we have to prove that T is also an algebra isomorphism. Note that the multiplication $\cdot$ defined by

$$x \cdot y := T^{-1}(T(x)T(y))$$

for all $x, y \in E$ makes E an Archimedean f-algebra with multiplicative unit $e > 0$. The claim follows by the fact that there exists at most one algebra multiplication on an Archimedean Riesz space L that makes L an Archimedean f-algebra with given unit, see e.g. (Aliprantis and Burkinshaw, 2006, Theorem 2.58). □

Chapter 2

Accumulation Points and Projections in Ideal Convergence

A real sequence (x_n) is said to be *statistically convergent* to ℓ if

$$\{n \in \mathbf{N} : |x_n - \ell| > \varepsilon\}$$

has zero (upper) asymptotic density for each $\varepsilon > 0$ or, equivalently, if there exists a subsequence (x_{n_k}) converging, in the classical sense, to ℓ and $\mathbf{N} \setminus \{n_k : k \in \mathbf{N}\}$ has zero (upper) asymptotic density, see Section 2.1.1 for definitions.

While statistical convergence has become an active area of research quite recently, it has appeared in the literature in a variety of guises since the beginning of the century. For instance, it is a classical result that a real bounded sequence (x_n) convergences statistically to ℓ if and only if it converges strongly–Cesàro to ℓ, that is,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} |x_i - \ell| = 0,$$

cf. e.g. Connor (1988).

It turns out that statistical convergence is related to a number of other fields of mathematics including number theory, trigonometric series, summability theory, combinatorics, ergodic theory, the Cech–Stone compactification of $\mathbf{N}$ (hence, the collection of its free ultrafilters), convergence in probability, densities on $\mathbf{N}$, descriptive set theory, and locally convex spaces, see Connor et al. (2000) and references therein.

Quite informally, every result in functional analysis identifies a class of ideals on $\mathbf{N}$ (cf. Section 2.1) for which its analogue holds; we refer to Hrušák (2011) for a recent review on the subject, its connections with logic and topology, together with a list of open problems.

Following the concept of statistical convergence as a generalization of the ordinary convergence, Fridy introduced the statistical limit points and statistical cluster points as generalizations of accumulation points; see Fridy (1993).

A real number ℓ is said to be a *statistical limit point* of (x_n) if there exists a subsequence (x_{n_k}) such that $\lim_{k\to\infty} x_{n_k} = \ell$ and the set of indices $\{n_k : k \in \mathbf{N}\}$ has positive upper asymptotic density. Also, ℓ is called *statistical cluster point* provided that $\{n : |x_n - \ell| < \varepsilon\}$ has positive upper asymptotic density for every $\varepsilon > 0$. He proved, among others, that these concepts are not equivalent.

These notions have been studied in a number of recent papers, see e.g. Connor and Kline (1996); Fridy and Orhan (1997); Miller (1995); Nuray and Ruckle (2000). Extensions of statistical convergence to more general spaces can be found in Albayrak and Pehlivan (2012); Di Maio and Kočinac (2008); Maddox (1988); Mamedov and Pehlivan (2001), and to ideal convergence, see e.g. Barbarski et al. (2011); Filipów et al. (2007); Kostyrko et al. (2001).

The aim of this Chapter is to study the analogues of the sets of accumulation points (of a given sequence) and complementability in the context of statistical convergence and, more generally, ideal convergence.

2.1 A non-topological convergence

Let Fin be the collection of finite subsets of $\mathbf{N}$. The upper asymptotic density of a set $S \subseteq \mathbf{N}$ is defined by

$$\mathrm{d}^\star(S) := \limsup_{n\to\infty} \frac{|S \cap [1,n]|}{n}$$

and we denote by $\mathcal{I}_0$ the collection of all S such that $\mathrm{d}^\star(S) = 0$. In particular, $\ell \in \mathbf{R}$ is a statistical cluster point of a real sequence (x_n) if and only if $\{n : |x_n - \ell| < \varepsilon\}$ does not belong to $\mathcal{I}_0$ for every $\varepsilon > 0$.

An ideal $\mathcal{I}$ on $\mathbf{N}$ is a family of subsets of positive integers closed under taking finite unions and subsets of its elements. It is also assumed that $\mathcal{I}$ is different from the power set of $\mathbf{N}$ and contains all the singletons. It is clear that Fin and $\mathcal{I}_0$ are ideals. Many other examples can be found, e.g., in (Farah, 2000, Chapter 1) and (Kwela and Tryba, 2017, Section 2). Intuitively, an ideal represents the collection of subsets of $\mathbf{N}$ which are "small" in a suitable sense.

Given an ideal $\mathcal{I}$ on the positive integers $\mathbf{N}$, we investigate various properties of $\mathcal{I}$-cluster points and $\mathcal{I}$-limit points of sequences taking values in topological spaces (X, τ). In particular, in this section we will show:

(i) a new characterization of $\mathcal{I}$-convergence: informally, a sequence (x_n) is $\mathcal{I}$-convergent if and only if there exists a "big" $\mathcal{I}$-convergent subsequence (see Theorem 2.1.4.(iv) and Corollary 2.1.5);

(ii) the topology generated by the pair $(\tau, \mathcal{I})$ corresponds to the underlying topology τ (see Theorem 2.1.12);

(iii) a characterization of $\mathcal{I}$-cluster points as classical "cluster points of the filter" generated by the sequence (see Theorem 2.1.14);

(iv) a characterization of the set of $\mathcal{I}$-cluster points as the smallest closed set containing "almost all" the sequence (see Theorem 2.1.15).

2.1.1 Preliminaries

We denote by $\mathcal{I}^\star := \{A \subseteq \mathbf{N} : A^c \in \mathcal{I}\}$ the *filter dual* of $\mathcal{I}$ and by $\mathcal{I}^+$ the collection of *$\mathcal{I}$-positive sets*, that is, the collection of all sets which do not belong to $\mathcal{I}$.

Definition 2.1.1. Let X be a topological space. Then, a sequence $x = (x_n)$ is said to be *$\mathcal{I}$-convergent* to ℓ, shortened with $x_n \to_{\mathcal{I}} \ell$, whenever $\{n : x_n \in U\} \in \mathcal{I}^\star$ for all neighborhoods U of ℓ. Moreover, let $\Gamma_x(\mathcal{I})$ denote the set of *$\mathcal{I}$-cluster points* of

x, that is, the set of all $\ell \in X$ such that $\{n : x_n \in U\} \in \mathcal{I}^+$ for all neighborhoods U of ℓ.

Ordinary convergence corresponds to Fin-convergence (thus, we shorten $x_n \to_{\mathrm{Fin}} \ell$ with $x_n \to \ell$) and statistical convergence to $\mathcal{I}_0$-convergence. Now, one may worder whether $\mathcal{I}$-convergence corresponds to ordinary convergence with respect to another topology on the same base set. Essentially, it never happens.

Example 2.1.2. Let us assume that $\mathcal{I} \neq \mathrm{Fin}$ and X is a topological space with at least two distinct points such that its topology τ is not the trivial topology τ_0. Hence, there exists a set $I \in \mathcal{I} \setminus \mathrm{Fin}$; in particular, I is infinite. Fix distinct $a, b \in X$ and define the sequence (x_n) by $x_n = a$ whenever $n \notin I$ and $x_n = b$ otherwise.

It follows by construction that $x_n \to_{\mathcal{I}} a$ in (X, τ). Let us assume, for the sake of contradiction, there exists a topology τ' such that $x_n \to a$ in (X, τ'). If there is a τ'-neighborhood U of a such that $b \notin U$, then $\{n : x_n \notin U\} = I$. This is impossible, since I is not finite. Hence $b \in U$ whenever $a \in U$. By the arbitrariness of a and b, we conclude that $\tau' = \tau_0$. The converse is false: given $U \in \tau \setminus \tau_0$ and $u \in U$, then the constant sequence (u) is not $\mathcal{I}$-convergent to ℓ provided that $\ell \notin U$.

Other notions of convergence have been defined in literature, considering properties of subsequences of x with sufficiently many elements.

Definition 2.1.3. Let X be a topological space. Then, a sequence $x = (x_n)$ is said to be $\mathcal{I}^\star$-*convergent* to ℓ, shortened with $x_n \to_{\mathcal{I}^\star} \ell$, whenever there exists a subsequence (x_{n_k}) such that $x_{n_k} \to \ell$ and $\{n_k : k \in \mathbf{N}\} \in \mathcal{I}^\star$. Moreover, let $\Lambda_x(\mathcal{I})$ denote the set of $\mathcal{I}$-*limit points* of x, that is, the set of all $\ell \in X$ such that there exists a subsequence (x_{n_k}) for which $x_{n_k} \to \ell$ and $\{n_k : k \in \mathbf{N}\} \in \mathcal{I}^+$.

At this point, recall that an ideal $\mathcal{I}$ is a *P-ideal* if it is σ-directed modulo finite sets, i.e., for every sequence (A_n) of sets in $\mathcal{I}$ there exists $A \in \mathcal{I}$ such that $A_n \setminus A$ is finite for all n; equivalent definitions were given, e.g., in (Balcerzak et al., 2007, Proposition 1).

Moreover, given infinite sets $A, B \subseteq \mathbf{N}$ such that A has canonical enumeration $\{a_n : n \in \mathbf{N}\}$, we say that $\mathcal{I}$ a *G-ideal* if

$$A_B := \{a_b : b \in B\} \in \mathcal{I}^\star \quad \text{if and only if} \quad B \in \mathcal{I}^\star$$

provided that $A \in \mathcal{I}^\star$. This condition is strictly related to the so-called "property (G)" considered in Balcerzak et al. (2016) and to the definition of invariant and thinnable ideals considered later in Section 2.3. Note that the class of G-ideals contains the ideals generated by α-densities with $\alpha \geq -1$ (in particular, $\mathcal{I}_0$ and the collection of logarithmic density zero sets), several summable ideals, and the *Pólya ideal*, i.e.,

$$\mathcal{I}_{\mathfrak{p}} := \left\{ S \subseteq \mathbf{N} : \mathfrak{p}^\star(S) := \lim_{s \to 1^-} \limsup_{n \to \infty} \frac{|S \cap [ns, n]|}{(1-s)n} = 0 \right\}.$$

Among other things, the upper Pólya density $\mathfrak{p}^\star$ has found a number of remarkable applications in analysis and economic theory, see e.g. Pólya (1929), Levinson (1940) and Marinacci (1998).

In this regard, we have the following basic result: points (i)-(ii) can be shown by routine arguments, cf. (Albayrak and Pehlivan, 2012, Theorem 3.1) and (Di Maio and Kočinac, 2008, Section 2) (we omit details); although not explicit in the literature, point (iii) can be considered folklore, see (Kostyrko et al., 0001, Theorem 3.2) for the case X being a metric space (we include the proof here for the sake of completeness); lastly, point (iv) provides a new characterization of $\mathcal{I}$-convergence (a related result can be found in (Balcerzak et al., 2016, Theorem 3.4)).

Theorem 2.1.4. *Let X be a topological space and $\mathcal{I}$ be an ideal. Then:*

(i) *$\mathcal{I}$-limits and $\mathcal{I}^\star$-limits are unique, provided X is Hausdorff;*

(ii) *$\mathcal{I}^\star$-convergence implies $\mathcal{I}$-convergence;*

(iii) *$\mathcal{I}$-convergence implies $\mathcal{I}^\star$-convergence, provided X is first countable and $\mathcal{I}$ is a P-ideal;*

(iv) *A sequence $(x_n) \in X^{\mathbf{N}}$ is $\mathcal{I}$-convergent if and only if there exists an $\mathcal{I}$-convergent subsequence (x_{n_k}) such that $\{n_k : k \in \mathbf{N}\} \in \mathcal{I}^\star$, provided $\mathcal{I}$ is a G-ideal.*

Proof. (iii) Let (x_n) be a sequence taking values in X which is $\mathcal{I}$-convergent to some $\ell \in X$. Then, let (U_j) be a countable decreasing local base at ℓ and, for each j, define $A_j := \{n : x_n \notin U_j\}$. Hence, $A_j \in \mathcal{I}$ for each j, (A_j) is increasing, and, since $\mathcal{I}$ is a P-ideal, there exists $A \in \mathcal{I}$ such that $A_j \setminus A$ is finite for all j. Denoting by (n_k) the increasing sequence of integers in A^c (which belongs to $\mathcal{I}^\star$), it follows that $x_{n_k} \to \ell$. Indeed, letting V be a neighborhood of ℓ and $j \in \mathbf{N}$ such that $U_j \subseteq V$, then the finiteness of $\{k : x_{n_k} \notin V\}$ follows by the fact that it has the same cardinality of $\{n_k : x_{n_k} \notin V\}$ and $\{n_k : x_{n_k} \notin V\} \subseteq \{n_k : x_{n_k} \notin U_j\} \subseteq \{n \in A^c : x_n \notin U_j\} = A_j \setminus A$.

(iv) Let us suppose that (x_n) is $\mathcal{I}$-convergent to $\ell \in X$. Fix also $I \in \mathcal{I}$ and let (n_k) be the increasing enumeration of I^c. Then, it is claimed that the subsequence (x_{n_k}) is $\mathcal{I}$-convergent to ℓ. Indeed, for each neighborhood U of ℓ, we have $\{n : x_n \notin U\} \in \mathcal{I}$ by hypothesis, hence $\{n_k : x_{n_k} \in U\} = \{n : x_n \in U\} \setminus I = \mathbf{N} \setminus (\{n : x_n \notin U\} \cup I) \in \mathcal{I}^\star$. It follows by the fact that $\mathcal{I}$ is a G-ideal that $\{k : x_{n_k} \in U\} \in \mathcal{I}^\star$, that is, $x_{n_k} \to_{\mathcal{I}} \ell$. The converse can be shown similarly. □

It is well known that $\mathcal{I}_0$ is a P-ideal (see e.g. (Freedman and Sember, 1981, Proposition 3.2)) and, as recalled before, it is also a G-ideal. Hence:

Corollary 2.1.5. *Let (x_n) be a sequence taking values in a topological space X. Then the following are equivalent:*

(i) *(x_n) is statistically convergent;*

(ii) *There exists a statistically convergent subsequence (x_{n_k}) with $\{n_k : k \in \mathbf{N}\} \in \mathcal{I}_0^\star$.*

If, in addition, X is first countable, then they are also equivalent to:

(iii) *There exists a convergent subsequence (x_{n_k}) with $\{n_k : k \in \mathbf{N}\} \in \mathcal{I}_0^\star$;*

It is worth noting that the equivalence between (i) and (iii) can be already found in (Di Maio and Kočinac, 2008, Theorem 2.2), cf. also (Fridy, 1985, Theorem 1) and (Miller, 1995, Theorem 1).

We obtain also an abstract version of (Connor, 1988, Theorem 2.3), see also (Barbarski et al., 2011, Proposition 1) and (Nabiev et al., 2007, Theorem 1); the proof goes verbatim, hence we omit it.

Corollary 2.1.6. *Let $\mathcal{I}$ be a P-ideal and (x_n) be a sequence taking values in a metrizable group (written additively, with identity 0) such that $x_n \to_{\mathcal{I}} \ell$. Then, there exist sequences (y_n) and (z_n) such that: $x_n = y_n + z_n$ for all n, $y_n \to \ell$, and $\{n : z_n \neq 0\} \in \mathcal{I}$.*

2.1.2 Ideal Cluster points

Given sequences x and y taking values in a topological space X, we say that they are $\mathcal{I}$-*equivalent*, shortened with $x \equiv_{\mathcal{I}} y$, if $\{n : x_n \neq y_n\} \in \mathcal{I}$ (it is easy to see that $\equiv_{\mathcal{I}}$ is an equivalence relation). The following lemmas, which collect and extend several results contained in Di Maio and Kočinac (2008); Fridy (1993); Kostyrko et al. (0001), show some standard properties of $\mathcal{I}$-cluster and $\mathcal{I}$-limit points.

Lemma 2.1.7. *Let x and y be sequences taking values in a topological space X and fix ideals $\mathcal{I} \subseteq \mathcal{J}$. Then:*

(i) $\Lambda_x(\mathcal{J}) \subseteq \Lambda_x(\mathcal{I})$ *and* $\Gamma_x(\mathcal{J}) \subseteq \Gamma_x(\mathcal{I})$*;*

(ii) $\Lambda_x(\mathrm{Fin}) = \Gamma_x(\mathrm{Fin})$*, provided X is first countable;*

(iii) $\Lambda_x(\mathcal{I}) \subseteq \Gamma_x(\mathcal{I})$*;*

(iv) $\Gamma_x(\mathcal{I})$ *is closed;*

(v) $\Lambda_x(\mathcal{I}) = \Lambda_y(\mathcal{I})$ *and* $\Gamma_x(\mathcal{I}) = \Gamma_y(\mathcal{I})$ *provided $x \equiv_{\mathcal{I}} y$;*

(vi) $\Gamma_x(\mathcal{I}) \cap K \neq \emptyset$*, provided $K \subseteq X$ is compact and $\{n : x_n \in K\} \in \mathcal{I}^+$;*

(vii) $\Lambda_x(\mathcal{I}) = \Gamma_x(\mathcal{I}) = \{\ell\}$ *provided $x_n \to_{\mathcal{I}^\star} \ell$ and X is Hausdorff.*

Proof. (i) and (ii) easily follow from the definitions. In addition, (iii) is obvious if $\Lambda_x(\mathcal{I}) = \emptyset$. Otherwise, fix $\ell \in \Lambda_x(\mathcal{I})$ and a neighborhood U of ℓ. Then, there exists an increasing subsequence (n_k) with $\{n_k\} \in \mathcal{I}^+$ such that $x_{n_k} \to \ell$, so that

$S := \{n_k : x_{n_k} \notin U\}$ is finite. This implies that $\{n_k\} \setminus S \subseteq \{n : x_n \in U\}$. To conclude, it is sufficient to note that $\{n_k\} \setminus S \notin \mathcal{I}$, therefore $\{n : x_n \in U\} \in \mathcal{I}^+$.

Similarly, (iv) is clear if $\Gamma_x(\mathcal{I}) = \emptyset$. In the opposite, let y be an accumulation point of $\Gamma_x(\mathcal{I})$ and U a neighborhood of y. Then, there exists $z \in \Gamma_x(\mathcal{I}) \cap U$. Let V be a neighborhood of z contained in U. Considering that $\{n : x_n \in V\} \subseteq \{n : x_n \in U\}$ and $\{n : x_n \in V\} \in \mathcal{I}^+$, we conclude that $y \in \Gamma_x(\mathcal{I})$.

To prove (v), fix $\ell \in \Lambda_x(\mathcal{I})$, so that there exists a subsequence (x_{n_k}) such that $\{n_k\} \in \mathcal{I}^+$ and $x_{n_k} \to \ell$. Since $\{n : x_n \neq y_n\} \in \mathcal{I}$ and $\{n_k : x_{n_k} \neq y_{n_k}\} \subseteq \{n : x_n \neq y_n\}$, then $S := \{n_k : x_{n_k} = y_{n_k}\} \in \mathcal{I}^+$. Denoting by (s_n) the canonical enumeration of S, we obtain $y_{s_n} \to \ell$, hence $\ell \in \Lambda_y(\mathcal{I})$. By the arbitrariness of ℓ, we have $\Lambda_x(\mathcal{I}) \subseteq \Lambda_y(\mathcal{I})$ therefore, by symmetry, $\Lambda_x(\mathcal{I}) = \Lambda_y(\mathcal{I})$. The other claim can be shown similarly.

The proof of (vi) can be found in (Das, 2012, Theorem 6), cf. also (Di Maio and Kočinac, 2008, Theorem 2.14) for the case $\mathcal{I} = \mathcal{I}_0$.

Lastly, suppose that $x_n \to_{\mathcal{I}^\star} \ell$ so that $x_n \to_{\mathcal{I}} \ell$ by Theorem 2.1.4.(ii) and, in particular, $\ell \in \Lambda_x(\mathcal{I})$. Also, thanks to (iii), we have $\{\ell\} \subseteq \Lambda_x(\mathcal{I}) \subseteq \Gamma_x(\mathcal{I})$. To conclude, let us suppose for the sake of contradition that there exists an $\mathcal{I}$-cluster point ℓ' of x different from ℓ. Fix disjoint neighborhoods U and U' of ℓ and ℓ', respectively. On the one hand, since ℓ' is a $\mathcal{I}$-cluster point, then $\{n : x_n \in U'\} \in \mathcal{I}^+$. On the other hand, this is impossible since $\{n : x_n \in U'\} \subseteq \{n : x_n \notin U\} \in \mathcal{I}$. This proves (vii). □

It follows at once from Theorem 2.1.4.(iii) and Lemma 2.1.7.(vii) that:

Corollary 2.1.8. *Let $\mathcal{I}$ be a P-ideal and (x_n) be a sequence taking values in a first countable Hausdorff space such that $x_n \to_{\mathcal{I}} \ell$. Then $\Lambda_x(\mathcal{I}) = \Gamma_x(\mathcal{I}) = \{\ell\}$.*

The converse of Corollary 2.1.8 does not hold: the real sequence x defined by $x_n = n$ if n is even and $x_n = 0$ otherwise satisfies $\Lambda_x(\mathcal{I}_0) = \Gamma_x(\mathcal{I}_0) = \{0\}$ while $x_n \not\to_{\mathcal{I}_0} 0$.

Moreover, Lemma 2.1.7.(v) can be strenghtened if X is a topological group:

Lemma 2.1.9. *Let x and y be sequences taking values in a topological group X (written additively, with identity 0) and fix an ideal $\mathcal{I}$. Then:*

(i) $\Gamma_x(\mathcal{I}) = \Gamma_y(\mathcal{I})$ *provided* $x_n - y_n \to_{\mathcal{I}} 0$;

(ii) $\Lambda_x(\mathcal{I}) = \Lambda_y(\mathcal{I})$ *provided* $x_n - y_n \to_{\mathcal{I}^\star} 0$.

Proof. Let z be the sequence defined by $z_n = x_n - y_n$.

(i) It follows by hypothesis $z_n \to_{\mathcal{I}} 0$ and $-z_n \to_{\mathcal{I}} 0$. Fix $\ell \in \Gamma_x(\mathcal{I})$ and let U be a neighborhood of ℓ. By the continuity of the operation of the group, there exist neighborhoods V and W of ℓ and 0, respectively, such that $V + W \subseteq U$. Considering that $\{n : x_n \in V\} \in \mathcal{I}^+$ and $\{n : -z_n \in W\} \in \mathcal{I}^\star$, it follows that

$$\{n : y_n \in U\} = \{n : x_n - z_n \in U\} \supseteq \{n : x_n \in V\} \cap \{n : -z_n \in W\} \in \mathcal{I}^+.$$

Since ℓ and U were arbitrarily chosen, then $\Gamma_x(\mathcal{I}) \subseteq \Gamma_y(\mathcal{I})$. The opposite inclusion can be shown similarly.

(ii) By hypothesis $z_n \to_{\mathcal{I}^\star} 0$ and $-z_n \to_{\mathcal{I}^\star} 0$. Fix $\ell \in \Lambda_x(\mathcal{I})$, hence there exist $A, B \in \mathcal{I}^\star$ such that $\lim_{a \in A} x_a = \ell$ and $\lim_{b \in B} -z_b = 0$. Setting $C := A \cap B \in \mathcal{I}^\star$, it follows that $\lim_{c \in C} y_c = \lim_{c \in C} x_c - z_c = \ell$, therefore $\Lambda_x(\mathcal{I}) \subseteq \Lambda_y(\mathcal{I})$. The opposite inclusion can be shown similarly. □

We recall that, under suitable assumptions on X and $\mathcal{I}$, the collection of $\mathcal{I}$-cluster and $\mathcal{I}$-limit point sets can be characterized as the closed sets and F_σ sets, respectively; see (Di Maio and Kočinac, 2008, Section 2), (Kostyrko et al., 2001, Theorem 1.1), and (Kostyrko et al., 0001, Section 4). Moreover, the continuity of the map $x \mapsto \Gamma_x(\mathcal{I})$ has been investigated in Kostyrko et al. (2001).

The next result establishes a connection between sets of cluster points with respect to different ideals (the proof is based on (Fridy, 1993, Theorem 2) which focuses on the case $X = \mathbf{R}$, $\mathcal{I} = \mathcal{I}_0$, and $\mathcal{J} = \mathrm{Fin}$).

Lemma 2.1.10. *Let x be a sequence taking values in a strongly Lindelöf space X and fix ideals $\mathcal{J} \subseteq \mathcal{I}$ such that $\mathcal{I}$ is a P-ideal. Then, there exists an $\mathcal{I}$-equivalent sequence y such that $\Gamma_x(\mathcal{I}) = \Gamma_y(\mathcal{J})$ and $\{y_n : n \in \mathbf{N}\} \subseteq \{x_n : n \in \mathbf{N}\}$.*

Proof. The claim is obvious if $\Gamma_x(\mathcal{I}) = \Gamma_x(\mathcal{J})$. Hence, let us suppose that $\Delta := \Gamma_x(\mathcal{J}) \setminus \Gamma_x(\mathcal{I}) \neq \emptyset$ and, for each $z \in \Delta$, let U_z be a neighborhood of z such that $\{n : x_n \in U_z\} \in \mathcal{I}$. Then $\bigcup U_z$ is an open cover of Δ. Since X is strongly

Lindelöf, there exists a countable subset $\{z_k : k \in \mathbf{N}\} \subseteq \Delta$ such that $\bigcup U_{z_k}$ is an open subcover of Δ. Moreover, since $\mathcal{I}$ is a P-ideal, there exists $I \in \mathcal{I}$ such that $\{n : x_n \in U_{z_k}\} \setminus I$ is finite for all k. At this point, let (i_n) be the canonical enumeration of $\mathbf{N} \setminus I$ and define the sequence y by $y_n = x_{i_n}$ if $n \in I$ and $y_n = x_n$ otherwise. Since $\{n : x_n \neq y_n\} \subseteq I \in \mathcal{I}$, then $x \equiv_{\mathcal{I}} y$, hence we obtain by Lemma 2.1.7.(v) that $\Gamma_x(\mathcal{I}) = \Gamma_y(\mathcal{I})$. The claim follows by the fact that every $\mathcal{J}$-cluster point of y is also an $\mathcal{I}$-cluster point, therefore $\Gamma_y(\mathcal{I}) = \Gamma_y(\mathcal{J})$. □

Lastly, given a topological space (X, τ) and an ideal $\mathcal{I}$, define the family

$$\tau(\mathcal{I}) := \left\{ F^c \subseteq X : F = \bigcup_{x \in F^{\mathbf{N}}} \Gamma_x(\mathcal{I}) \right\},$$

that is, F is $\tau(\mathcal{I})$-*closed* if and only if it is the union of $\mathcal{I}$-cluster points of F-valued sequences. In particular, it is immediate that $\tau = \tau(\mathrm{Fin})$.

Lemma 2.1.11. *$\tau \subseteq \tau(\mathcal{I})$.*

Proof. Let F be a τ-closed set. Thanks to Lemma 2.1.7.(i), we have

$$F \subseteq \textstyle\bigcup_{x \in F^{\mathbf{N}}} \Gamma_x(\mathcal{I}) \subseteq \bigcup_{x \in F^{\mathbf{N}}} \Gamma_x(\mathrm{Fin}) = F,$$

where the first inclusion is obtained by choosing the constant sequence (f), for each fixed $f \in F$. Therefore, $F^c \in \tau(\mathcal{I})$. □

The converse holds under some additional assumptions:

Theorem 2.1.12. *Assume that one of the following conditions holds:*

(i) *X is sequentially strongly Lindelöf and $\mathcal{I}$ is a P-ideal;*

(ii) *X is first countable.*

Then $\tau = \tau(\mathcal{I})$.

Proof. Thanks to Lemma 2.1.11, it is sufficient to show that $\tau(\mathcal{I}) \subseteq \tau$. Let F be a $\tau(\mathcal{I})$-closed set. Then, it is enough to show that if $\ell \in F$ is an $\mathcal{I}$-cluster point of some F-valued sequence x, it is also an ordinary limit point of some F-valued sequence y.

(i) This follows directly by Lemma 2.1.10, setting $\mathcal{J} = \mathrm{Fin}$.

(ii) Let (U_k) be a decreasing local base at ℓ. Then, there exists a subsequence (x_{n_k}) converging to ℓ: to this aim, set $S_k := \{n : x_n \in U_k\}$ for each k, fix $n_1 \in S_1$ arbitrarily and, for each $k \in \mathbf{N}$, define $n_{k+1} := \min S_{k+1} \setminus \{1, \dots, n_k\}$ (note that this is possible since each S_k is infinite). □

2.1.3 Characterizations of $\mathcal{I}$-cluster points

Given an ideal $\mathcal{I}$ and a sequence x taking values in a topological space X, we define the *$\mathcal{I}$-filter generated by x* as

$$\mathscr{F}_x(\mathcal{I}) := \{Y \subseteq X : \{n : x_n \notin Y\} \in \mathcal{I}\}.$$

It is immediate that $\mathscr{F}_x(\mathcal{I})$ is a filter on X with filter base

$$\mathcal{B}_x(\mathcal{I}) := \{\{x_n : n \notin I\} : I \in \mathcal{I}\}.$$

In addition, if $\mathcal{I} = \mathrm{Fin}$, then $\mathscr{F}_x(\mathcal{I})$ coincides with the standard filter generated by x, cf. (Bourbaki, 1998, Definition 7, p.64).

With this notation, we are going to show that ℓ is an $\mathcal{I}$-cluster point of x if and only if it is a cluster point of the filter $\mathscr{F}_x(\mathcal{I})$, that is, ℓ lies in the closure of all sets in the filter base $\mathcal{B}_x(\mathcal{I})$, c.f. (Bourbaki, 1998, Definition 2, p.69).

Lemma 2.1.13. $\bigcap_{B \in \mathcal{B}_x(\mathcal{I})} \overline{B} \subseteq \Gamma_x(\mathcal{I})$.

Proof. Let us suppose that $\ell \in \bigcap_{I \in \mathcal{I}} \overline{\{x_n : n \notin I\}}$, that is, for each $I \in \mathcal{I}$ there exists a subsequence (x_{n_k}) converging to ℓ such that $\{n_k : k \in \mathbf{N}\} \cap I = \emptyset$. Suppose for the sake of contradiction that ℓ is not an $\mathcal{I}$-cluster point, i.e., there exists an open neighborhood U of ℓ such that $J := \{n : x_n \in U\}$ belongs to $\mathcal{I}$. Then, it follows that $\{x_n : n \notin J\} \in \mathcal{B}_x(\mathcal{I})$ hence

$$\ell \in \bigcap_{B \in \mathcal{B}_x(\mathcal{I})} \overline{B} \subseteq \overline{\{x_n : n \notin J\}} = \overline{\{x_n : x_n \notin U\}} \subseteq U^c,$$

which is impossible since $\ell \in U$. □

However, if X is first countable, then also the converse holds.

Theorem 2.1.14. *Let $\mathcal{I}$ be an ideal and x be a sequence taking values in a first countable space X. Then $\Gamma_x(\mathcal{I}) = \bigcap_{B \in \mathcal{B}_x(\mathcal{I})} \overline{B}$.*

Proof. Thanks to Lemma 2.1.13, it is sufficient to show that $\Gamma_x(\mathcal{I}) \subseteq \bigcap_{B \in \mathcal{B}_x(\mathcal{I})} \overline{B}$. Let us suppose that ℓ is an $\mathcal{I}$-cluster point of x and fix a decreasing local base (U_k) at ℓ, so that $S_k := \{n : x_n \in U_k\} \in \mathcal{I}^+$ for all k. Fix also $I \in \mathcal{I}$ and note that $T_k := S_k \setminus I \in \mathcal{I}^+$ for all k (in particular, each T_k is infinite). Then, we have to prove that $\ell \in \overline{\{x_n : n \notin I\}}$, i.e., there exists a subsequence (x_{n_k}) converging to ℓ such that $n_k \notin I$ for all k. To this aim, it is enough to fix $n_1 \in T_1$ arbitrarily and $n_{k+1} := \min T_{k+1} \setminus \{1, \ldots, n_k\}$ for all $k \in \mathbf{N}$. It follows by construction that $\lim_{k \to \infty} x_{n_k} = \ell$ and $n_k \notin I$ for all k. $\square$

As a corollary, we obtain another proof of Lemma 2.1.7.(iv), provided X is first countable.

We conclude this section with another characterization of the set of $\mathcal{I}$-cluster points, which subsumes the results contained in Güncan et al. (2004).

Theorem 2.1.15. *Let x be a sequence taking values in a regular Hausdorff space X such that $\{n : x_n \notin K\} \in \mathcal{I}$ for some compact set K. Then $\Gamma_x(\mathcal{I})$ is the smallest closed set C such that $\{n : x_n \notin U\} \in \mathcal{I}$ for all sets U containing C.*

Proof. Fix $\kappa \in K$ and define the sequence y by $y_n = \kappa$ if $x_n \notin K$ and $y_n = x_n$ otherwise. It follows by Lemma 2.1.7.(vi)-(v) that $\emptyset \neq \Gamma_x(\mathcal{I}) = \Gamma_y(\mathcal{I}) \subseteq K$. Let also $\mathscr{C}$ be the family of closed sets C such that $\{n : x_n \notin U\} \in \mathcal{I}$ for all open subsets $U \supseteq C$ (note that $\{n : x_n \notin U\} \in \mathcal{I}$ if and only if $\{n : y_n \notin U\} \in \mathcal{I}$).

First, we show that $\Gamma_x(\mathcal{I}) \in \mathscr{C}$. Indeed, $\Gamma_x(\mathcal{I})$ is closed by Lemma 2.1.7.(iv); moreover, let us suppose for the sake of contradiction that there exists an open set U containing $\Gamma_x(\mathcal{I})$ such that $\{n : x_n \notin U\} \in \mathcal{I}^+$, that is, $\{n : y_n \notin U\} = \{n : y_n \in K \setminus U\} \in \mathcal{I}^+$. Considering that $K \setminus U$ is compact, we obtain by Lemma 2.1.7.(vi) that there exists an $\mathcal{I}$-cluster point of y in $K \setminus U$. This contradicts the fact that $\Gamma_y(\mathcal{I}) = \Gamma_x(\mathcal{I}) \subseteq U$.

Lastly, fix $C \in \mathscr{C}$ and let us suppose that $\Gamma_x(\mathcal{I}) \setminus C \neq \emptyset$. Fix $\ell \in \Gamma_x(\mathcal{I}) \setminus C$ and, by the regularity of X, there exist disjoint open sets U and V containing the closed sets $\{\ell\}$ and $K \cap C$, respectively. This is impossible, indeed the set $\{n : x_n \in V\}$ belongs to $\mathcal{I}$ since $C \in \mathscr{C}$, and, on the other hand, it contains $\{n : x_n \in U\} \in \mathcal{I}^+$ since ℓ is an $\mathcal{I}$-cluster point. $\square$

2.2 $\mathcal{I}$-cluster points Vs $\mathcal{I}$-limit points

In what follows, we shorten $\Lambda_x(\mathcal{I}_0)$, $\Gamma_x(\mathcal{I}_0)$, and $\Lambda_x(\mathrm{Fin})$ with Λ_x, Γ_x, and L_x, respectively (note that the latter coincides with $\Gamma_x(\mathrm{Fin})$ if X is first countable.

By identifying sets of integers with their characteristic function, we equip the power set $\mathcal{P}(\mathbf{N})$ with the Cantor-space topology and therefore we can assign the topological complexity to the ideals on $\mathbf{N}$. In particular, an ideal $\mathcal{I}$ is *analytic* if it is a continuous image of a Borel subset of a Polish space.

In this section, we establish some relationship between the set of ideal cluster points and the set of ideal limit points of a given sequence. In particular, it is shown that:

(i) $\Lambda_x(\mathcal{I})$ is an F_σ-set, provided that $\mathcal{I}$ is an analytic P-ideal (Theorem 2.2.2);

(ii) $\Lambda_x(\mathcal{I})$ is closed, provided that $\mathcal{I}$ is an F_σ-ideal (Theorem 2.2.3);

(iii) $\Lambda_x(\mathcal{I})$ is closed for all x if and only if $\Lambda_x(\mathcal{I}) = \Gamma_x(\mathcal{I})$ for all x if and only if $\mathcal{I}$ is an F_σ-ideal, provided that $\mathcal{I}$ is an analytic P-ideal (Theorem 2.2.5);

(iv) For every F_σ-set A, there exists a sequence x such that $\Lambda_x(\mathcal{I}) = A$, provided that $\mathcal{I}$ is an analytic P-ideal which is not F_σ (Theorem 2.2.6);

(v) Each isolated $\mathcal{I}$-cluster point is also an $\mathcal{I}$-limit point (Theorem 2.2.7).

In addition, we provide in Section 2.2.2 some joint converse results:

(vi) Given $A \subseteq B \subseteq C \subseteq \mathbf{R}$ where A is an F_σ-set, B is non-empty regular closed, and C is closed, then there exists a real sequence x such that $\Lambda_x = A$, $\Gamma_x = B$, and $\mathrm{L}_x = C$ (Theorem 2.2.9 and Corollary 2.2.11);

(vii) Given non-empty closed sets $B \subseteq C \subseteq \mathbf{R}$, there exists a real sequence x such that $\Lambda_x(\mathcal{I}) = \Gamma_x(\mathcal{I}) = B$ and $\mathrm{L}_x = C$, provided $\mathcal{I}$ is an F_σ-ideal different from Fin (Theorem 2.2.12).

Lastly, it is shown in Section 2.2.3 that:

(viii) $\Lambda_x(\mathcal{I})$ is analytic, provided that $\mathcal{I}$ is a co-analytic ideal (Proposition 2.2.13);

(ix) An ideal $\mathcal{I}$ is maximal if and only if each real sequence x admits at most one $\mathcal{I}$-limit point (Proposition 2.2.14 and Corollary 2.2.15).

We conclude by showing that there exists an ideal $\mathcal{I}$ and a real sequence x such that $\Lambda_x(\mathcal{I})$ is not an F_σ-set (Example 2.2.16).

2.2.1 Topological structure of $\mathcal{I}$-limit points

A map $\varphi : \mathcal{P}(\mathbf{N}) \to [0, \infty]$ is a *lower semicontinuous submeasure* provided that:

(i) $\varphi(\emptyset) = 0$;

(ii) $\varphi(A) \le \varphi(B)$ whenever $A \subseteq B$;

(iii) $\varphi(A \cup B) \le \varphi(A) + \varphi(B)$ for all A, B; and

(iv) $\varphi(A) = \lim_n \varphi(A \cap \{1, \ldots, n\})$ for all A.

By a classical result of Solecki, an ideal $\mathcal{I}$ is an analytic P-ideal if and only if there exists a lower semicontinuous submeasure φ such that

$$\mathcal{I} = \mathcal{I}_\varphi := \{A \subseteq \mathbf{N} : \|A\|_\varphi = 0\} \tag{2.1}$$

and $\varphi(\mathbf{N}) < \infty$, where $\|A\|_\varphi := \lim_n \varphi(A \setminus \{1, \ldots, n\})$ for all $A \subseteq \mathbf{N}$, see (Solecki, 1999, Theorem 3.1). Note, in particular, that for every $n \in \mathbf{N}$ it holds

$$\|A\|_\varphi = \|A \setminus \{1, \ldots, n\}\|_\varphi. \tag{2.2}$$

Hereafter, unless otherwise stated, an analytic P-ideal will be always denoted by $\mathcal{I}_\varphi$, where φ stands for the associated lower semicontinuous submeasure as in (2.1).

Given a sequence $x = (x_n)$ taking values in a first countable space X and an analytic P-ideal $\mathcal{I}_\varphi$, define

$$\mathfrak{u}(\ell) := \lim_{k\to\infty} \|\{n : x_n \in U_k\}\|_\varphi \tag{2.3}$$

for each $\ell \in X$, where (U_k) is a decreasing local base of neighborhoods at ℓ. It is easy to see that the limit in (2.3) exists and its value is independent of the choice of (U_k).

Lemma 2.2.1. *The map $\mathfrak{u}$ is upper semi-continuous. In particular, the set*

$$\Lambda_x(\mathcal{I}_\varphi, q) := \{\ell \in X : \mathfrak{u}(\ell) \geq q\}.$$

is closed for every $q > 0$.

Proof. We need to prove that $\mathscr{U}_y := \{\ell \in X : \mathfrak{u}(\ell) < y\}$ is open for all $y \in \mathbf{R}$ (hence $\mathscr{U}_\infty$ is open too). Clearly, $\mathscr{U}_y = \emptyset$ if $y \leq 0$. Hence, let us suppose hereafter $y > 0$ and $\mathscr{U}_y \neq \emptyset$. Fix $\ell \in \mathscr{U}_y$ and let (U_k) be a decreasing local base of neighborhoods at ℓ. Then there exists $k_0 \in \mathbf{N}$ such that $\|\{n : x_n \in U_k\}\|_\varphi < y$ for every $k \geq k_0$. Fix $\ell' \in U_{k_0}$ and let (V_k) be a decreasing local base of neighborhoods at ℓ'. Fix also $k_1 \in \mathbf{N}$ such that $V_{k_1} \subseteq U_{k_0}$. It follows by the monotonicity of φ that

$$\|\{n : x_n \in V_k\}\|_\varphi \leq \|\{n : x_n \in U_{k_0}\}\|_\varphi < y$$

for every $k \geq k_1$. In particular, $\mathfrak{u}(\ell') < y$ and, by the arbitrariness of ℓ', $U_{k_0} \subseteq \mathscr{U}_y$. □

At this point, we provide a useful characterization of the set $\Lambda_x(\mathcal{I}_\varphi)$ (without using limits of subsequences) and we obtain, as a by-product, that it is an F_σ-set.

Theorem 2.2.2. *Let x be a sequence taking values in a first countable space X and $\mathcal{I}_\varphi$ be an analytic P-ideal. Then*

$$\Lambda_x(\mathcal{I}_\varphi) = \{\ell \in X : \mathfrak{u}(\ell) > 0\}. \tag{2.4}$$

In particular, $\Lambda_x(\mathcal{I}_\varphi)$ is an F_σ-set.

Proof. Let us suppose that there exists $\ell \in \Lambda_x(\mathcal{I}_\varphi)$ and let (U_k) be a decreasing local base of neighborhoods at ℓ. Then there exists $A \subseteq \mathbf{N}$ such that $\lim_{n\to\infty, n\in A} x_n = \ell$ and $\|A\|_\varphi > 0$. At this point, note that, for each $k \in \mathbf{N}$, the set $\{n \in A : x_n \notin U_k\}$ is finite, hence it follows by (2.2) that $\mathfrak{u}(\ell) \ge \|A\|_\varphi > 0$.

On the other hand, suppose that there exists $\ell \in X$ such that $\mathfrak{u}(\ell) > 0$. Let (U_k) be a decreasing local base of neighborhoods at ℓ and define $\mathcal{A}_k := \{n : x_n \in U_k\}$ for each $k \in \mathbf{N}$; note that $\mathcal{A}_k$ is infinite since $\|\mathcal{A}_k\|_\varphi \downarrow \mathfrak{u}(\ell) > 0$ implies $\mathcal{A}_k \notin \mathcal{I}_\varphi$ for all k. Set for convenience $\theta_0 := 0$ and define recursively the increasing sequence of integers (θ_k) so that θ_k is the smallest integer greater than both θ_{k-1} and $\min \mathcal{A}_{k+1}$ such that

$$\varphi(\mathcal{A}_k \cap (\theta_{k-1}, \theta_k]) \ge \mathfrak{u}(\ell)\,(1 - 1/k)\,.$$

Finally, define $\mathcal{A} := \bigcup_k (\mathcal{A}_k \cap (\theta_{k-1}, \theta_k])$. Since $\theta_k \ge k$ for all k, we obtain

$$\varphi(\mathcal{A} \setminus \{1, \dots, n\}) \ge \varphi(\mathcal{A}_{n+1} \cap (\theta_n, \theta_{n+1}]) > \mathfrak{u}(\ell)\,(1 - 1/n)$$

for all n, hence $\|\mathcal{A}\|_\varphi \ge \mathfrak{u}(\ell) > 0$. In addition, we have by construction $\lim_{n\to\infty, n\in\mathcal{A}} x_n = \ell$. Therefore ℓ is an $\mathcal{I}_\varphi$-limit point of x. To sum up, this proves (2.4).

Lastly, rewriting (2.4) as $\Lambda_x(\mathcal{I}_\varphi) = \bigcup_n \Lambda_x(\mathcal{I}_\varphi, 1/n)$ and considering that each $\Lambda_x(\mathcal{I}_\varphi, 1/n)$ is closed by Lemma 2.2.1, we conclude that $\Lambda_x(\mathcal{I}_\varphi)$ is an F_σ-set. □

The fact that $\Lambda_x(\mathcal{I}_\varphi)$ is an F_σ-set already appeared in (Das, 2012, Theorem 2), although with a different argument. The first result of this type was given in (Kostyrko et al., 2001, Theorem 1.1) for the case $\mathcal{I}_\varphi = \mathcal{I}_0$ and $X = \mathbf{R}$. Later, it was extended in (Di Maio and Kočinac, 2008, Theorem 2.6) for first countable spaces. However, in the proofs contained in Das (2012); Di Maio and Kočinac (2008) it is unclear why the constructed subsequence $(x_n : n \in \mathcal{A})$ converges to ℓ.

A stronger result holds in the case that the ideal is F_σ. We recall that, by a classical result of Mazur, an ideal $\mathcal{I}$ is F_σ if and only if there exists a lower semicontinuous submeasure φ such that

$$\mathcal{I} = \{A \subseteq \mathbf{N} : \varphi(A) < \infty\}, \tag{2.5}$$

with $\varphi(\mathbf{N}) = \infty$, see (Mazur, 1991, Lemma 1.2).

Theorem 2.2.3. *Let $x = (x_n)$ be a sequence taking values in a first countable space X and let $\mathcal{I}$ be an F_σ-ideal. Then $\Lambda_x(\mathcal{I}) = \Gamma_x(\mathcal{I})$. In particular, $\Lambda_x(\mathcal{I})$ is closed.*

Proof. Since it is known that $\Lambda_x(\mathcal{I}) \subseteq \Gamma_x(\mathcal{I})$, the claim is clear if $\Gamma_x(\mathcal{I}) = \emptyset$. Hence, let us suppose hereafter that $\Gamma_x(\mathcal{I})$ is non-empty. Fix $\ell \in \Gamma_x(\mathcal{I})$ and let (U_k) be a decreasing local base of neighborhoods at ℓ. Letting φ be a lower semicontinuous submeasure associated with $\mathcal{I}$ as in (2.5) and considering that ℓ is an $\mathcal{I}$-cluster point, we have $\varphi(A_k) = \infty$ for all $k \in \mathbf{N}$, where $A_k := \{n : x_n \in U_k\}$.

Then set $a_0 := 0$ and define an increasing sequence of integers (a_k) which satisfies

$$\varphi(A_k \cap (a_{k-1}, a_k]) \geq k$$

for all k (note that this is possible since $\varphi(A_k \setminus S) = \infty$ whenever S is finite). At this point, set $A := \bigcup_k A_k \cap (a_{k-1}, a_k]$. It follows by the monotonocity of φ that $\varphi(A) = \infty$, hence $A \notin \mathcal{I}$. Moreover, for each $k \in \mathbf{N}$, we have that $\{n \in A : x_n \notin U_k\}$ is finite: indeed, if $n \in A_j \cap (a_{j-1}, a_j]$ for some $j \geq k$, then by construction $x_n \in U_j$, which is contained in U_k. Therefore $\lim_{n \to \infty,\, n \in A} x_n = \ell$, that is, $\ell \in \Lambda_x(\mathcal{I})$. □

Since summable ideals are F_σ P-ideals, see e.g. (Farah, 2000, Example 1.2.3), we obtain the following consequence:

Corollary 2.2.4. *Let x be a real sequence and let $\mathcal{I}$ be a summable ideal. Then $\Lambda_x(\mathcal{I})$ is closed.*

It turns out that, within the class of analytic P-ideals, the property that the set of $\mathcal{I}$-limit points is always closed characterizes the subclass of F_σ-ideals:

Theorem 2.2.5. *Let X be a first countable space which admits a non-trivial convergent sequence. Let also $\mathcal{I}_\varphi$ be an analytic P-ideal. Then the following are equivalent:*

(i) $\mathcal{I}_\varphi$ is also an F_σ-ideal;

(ii) $\Lambda_x(\mathcal{I}_\varphi) = \Gamma_x(\mathcal{I}_\varphi)$ for all sequences x;

(*iii*) $\Lambda_x(\mathcal{I}_\varphi)$ *is closed for all sequences* x*;*

(*iv*) *there does not exist a partition* $\{A_n : n \in \mathbf{N}\}$ *of* $\mathbf{N}$ *such that* $\|A_n\|_\varphi > 0$ *for all* n *and* $\lim_n \|\bigcup_{k>n} A_k\|_\varphi = 0$.

Proof. (i) $\implies$ (ii) follows by Theorem 2.2.3 and (ii) $\implies$ (iii) is clear.

(iii) $\implies$ (iv) By hypothesis, there exists a sequence (ℓ_n) converging to $\ell \in X$ such that $\ell_n \neq \ell$ for all n. Let us suppose that there exists a partition $\{A_n : n \in \mathbf{N}\}$ of $\mathbf{N}$ such that $\|A_n\|_\varphi > 0$ for all n and $\lim_k \|\bigcup_{n\ge k} A_n\|_\varphi = 0$. Define the sequence $x = (x_n)$ by $x_n = \ell_i$ for all $n \in A_i$. Then we have that $\{\ell_n : n \in \mathbf{N}\} \subseteq \Lambda_x(\mathcal{I}_\varphi)$. On the other hand, since X is first countable Hausdorff, it follows that for all $k \in \mathbf{N}$ there exists a neighborhood U_k of ℓ such that

$$\{n : x_n \in U_k\} \subseteq \{n : x_n = \ell_i \text{ for some } i \ge k\} = \textstyle\bigcup_{n\ge k} A_n.$$

Hence, by the monotonicity of φ, we obtain $0 < \|\{n : x_n \in U_k\}\|_\varphi \downarrow 0$, i.e., $\mathfrak{u}(\ell) = 0$, which implies, thanks to Theorem 2.2.2, that $\ell \notin \Lambda_x(\mathcal{I}_\varphi)$. In particular, $\mathcal{I}_\varphi$ is not closed.

(iv) $\implies$ (i) Lastly, assume that the ideal $\mathcal{I}_\varphi$ is not an F_σ-ideal. According to the proof of (Solecki, 1999, Theorem 3.4), cf. also (Solecki, 1996, pp. 342–343), this is equivalent to the existence, for each given $\varepsilon > 0$, of some set $M \subseteq \mathbf{N}$ such that $0 < \|M\|_\varphi \le \varphi(M) < \varepsilon$. This allows us to define recursively a sequence of sets (M_n) such that

$$\|M_n\|_\varphi > \sum_{k\ge n+1} \varphi(M_k) > 0. \tag{2.6}$$

for all n and, in addition, $\sum_k \varphi(M_k) < \varphi(\mathbf{N})$. Then, it is claimed that there exists a partition $\{A_n : n \in \mathbf{N}\}$ of $\mathbf{N}$ such that $\|A_n\|_\varphi > 0$ for all n and $\lim_n \|\bigcup_{k>n} A_k\|_\varphi = 0$. To this aim, set $M_0 := \mathbf{N}$ and define $A_n := M_{n-1} \setminus \bigcup_{k\ge n} M_k$ for all $n \in \mathbf{N}$. It follows by the subadditivity and monotonicity of φ that

$$\varphi(M_{n-1} \setminus \{1,\ldots,k\}) \le \varphi(A_n \setminus \{1,\ldots,k\}) + \varphi\left(\textstyle\bigcup_{k\ge n} M_k\right)$$

for all $n, k \in \mathbf{N}$; hence, by the lower semicontinuity of φ and (2.6),

$$\|A_n\|_\varphi \ge \|M_{n-1}\|_\varphi - \varphi\left(\textstyle\bigcup_{k\ge n} M_k\right) \ge \|M_{n-1}\|_\varphi - \textstyle\sum_{k\ge n} \varphi(M_k) > 0$$

for all $n \in \mathbf{N}$. Finally, again by the lower semicontinuity of φ, we get

$$\| \bigcup_{k>n} A_k \|_\varphi = \| \bigcup_{k\geq n} M_k \|_\varphi \leq \varphi \left(\bigcup_{k\geq n} M_k \right) \leq \sum_{k\geq n} \varphi(M_k)$$

which goes to 0 as $n \to \infty$. This concludes the proof. □

At this point, thanks to Theorem 2.2.2 and Theorem 2.2.5, observe that, if X is a first countable space which admits a non-trivial convergent sequence and $\mathcal{I}_\varphi$ is an analytic P-ideal which is not F_σ, then there exists a sequence x such that $\Lambda_x(\mathcal{I}_\varphi)$ is a non-closed F_σ-set. In this case, indeed, all the F_σ-sets can be obtained:

Theorem 2.2.6. *Let X be a first countable space where all closed sets are separable and assume that there exists a non-trivial convergent sequence. Fix also an analytic P-ideal $\mathcal{I}_\varphi$ which is not F_σ and let $B \subseteq X$ be a non-empty F_σ-set. Then there exists a sequence x such that $\Lambda_x(\mathcal{I}_\varphi) = B$.*

Proof. Let (B_k) be a sequence of non-empty closed sets such that $\bigcup_k B_k = B$. Let also $\{b_{k,n} : n \in \mathbf{N}\}$ be a countable dense subset of B_k. Thanks to Theorem 2.2.5, there exists a partition $\{A_n : n \in \mathbf{N}\}$ of $\mathbf{N}$ such that $\|A_n\|_\varphi > 0$ for all n and $\lim_n \| \bigcup_{k>n} A_k \|_\varphi = 0$. Moreover, for each $k \in \mathbf{N}$, set $\theta_{k,0} := 0$ and it is easily seen that there exists an increasing sequence of positive integers $(\theta_{k,n})$ such that

$$\varphi(A_k \cap (\theta_{k,n-1}, \theta_{k,n}]) \geq \frac{1}{2} \|A_k \setminus \{1, \dots, \theta_{k,n-1}\}\|_\varphi = \frac{1}{2} \|A_k\|_\varphi$$

for all n. Hence, setting $A_{k,n} := A_k \cap \bigcup_{m \in A_n} (\theta_{k,m-1}, \theta_{k,m}]$, we obtain that $\{A_{k,n} : n \in \mathbf{N}\}$ is a partition of A_k such that $\frac{1}{2}\|A_k\|_\varphi \leq \|A_{k,n}\|_\varphi \leq \|A_k\|_\varphi$ for all n, k.

At this point, let $x = (x_n)$ be defined by $x_n = b_{k,m}$ whenever $n \in A_{k,m}$. Fix $\ell \in B$, then there exists $k \in \mathbf{N}$ such that $\ell \in B_k$. Let (b_{k,r_m}) be a sequence in B_k converging to ℓ. Thus, set $\tau_0 := 0$ and let (τ_m) be an increasing sequence of positive integers such that $\varphi(A_{k,r_m} \cap (\tau_{m-1}, \tau_m]) \geq \frac{1}{2}\|A_{k,r_m}\|_\varphi$ for each m. Setting $A := \bigcup_m A_{k,r_m} \cap (\tau_{m-1}, \tau_m]$, it follows by construction that $\lim_{n\to\infty, n\in A} x_n = \ell$ and $\|A\|_\varphi \geq \frac{1}{4}\|A_k\|_\varphi > 0$. This shows that $B \subseteq \Lambda_x(\mathcal{I}_\varphi)$.

To complete the proof, fix $\ell \notin B$ and let us suppose for the sake of contradiction that there exists $A \subseteq \mathbf{N}$ such that $\lim_{n\to\infty, n\in A} x_n = \ell$ and $\|A\|_\varphi > 0$. For each $m \in \mathbf{N}$, let U_m be an open neighborhood of ℓ which is disjoint from the closed set

$B_1 \cup \cdots \cup B_m$. It follows by the subadditivity and the monotonicity of φ that there exists a finite set Y such that

$$\|A\|_\varphi \le \|Y\|_\varphi + \|\{n \in A : x_n \notin B_1 \cup \cdots \cup B_m\}\|_\varphi \le \|\bigcup_{k>m} A_k\|_\varphi.$$

The claim follows by the arbitrariness of m and the fact that $\lim_m \|\bigcup_{k>m} A_k\|_\varphi = 0$. □

Note that every analytic P-ideal without the Bolzano-Weierstrass property cannot be F_σ, see (Filipów et al., 2007, Theorem 4.2). Hence Theorem 2.2.6 applies to this class of ideals.

It was shown in (Di Maio and Kočinac, 2008, Theorem 2.8 and Theorem 2.10) that if X is a topological space where all closed sets are separable, then for each F_σ-set A and closed set B there exist sequences $a = (a_n)$ and $b = (b_n)$ with values in X such that $\Lambda_a = A$ and $\Gamma_b = B$.

As an application of Theorem 2.2.2, we prove that, in general, its stronger version with $a = b$ fails (e.g., there are no real sequences x such that $\Lambda_x = \{0\}$ and $\Gamma_x = \{0, 1\}$).

Here, a topological space X is said to be *locally compact* if for every $x \in X$ there exists a neighborhood U of x such that its closure $\overline{U}$ is compact, cf. (Engelking, 1989, Section 3.3).

Theorem 2.2.7. *Let $x = (x_n)$ be a sequence taking values in a locally compact first countable space and fix an analytic P-ideal $\mathcal{I}_\varphi$. Then each isolated $\mathcal{I}_\varphi$-cluster point is also an $\mathcal{I}_\varphi$-limit point.*

Proof. Let us suppose for the sake of contradiction that there exists an isolated $\mathcal{I}_\varphi$-cluster point, let us say ℓ, which is not an $\mathcal{I}_\varphi$-limit point. Let (U_k) be a decreasing local base of open neighborhoods at ℓ such that $\overline{U}_1$ is compact. Let also m be a sufficiently large integer such that $U_m \cap \Gamma_x(\mathcal{I}_\varphi) = \{\ell\}$. Thanks to (Engelking, 1989, Theorem 3.3.1) the underlying space is, in particular, regular, hence there exists an integer $r > m$ such that $\overline{U}_r$ is a compact contained in U_m. In addition, since ℓ is an $\mathcal{I}_\varphi$-cluster point and it is not an $\mathcal{I}_\varphi$-limit point, it follows by Theorem 2.2.2 that

$$0 < \|\{n : x_n \in U_k\}\|_\varphi \downarrow \mathfrak{u}(\ell) = 0.$$

In particular, there exists $s \in \mathbf{N}$ such that $0 < \|\{n : x_n \in U_s\}\|_\varphi < \|\{n : x_n \in U_r\}\|_\varphi$.

Observe that $K := \overline{U}_r \setminus U_s$ is a closed set contained in $\overline{U}_1$, hence it is compact. By construction we have that $K \cap \Gamma_x(\mathcal{I}_\varphi) = \emptyset$. Hence, for each $z \in K$, there exists an open neighborhood V_z of z such that $V_z \subseteq U_m$ and $\{n : x_n \in V_z\} \in \mathcal{I}_\varphi$, i.e., $\|\{n : x_n \in V_z\}\|_\varphi = 0$. It follows that $\bigcup_{z \in K} V_z$ is an open cover of K which is contained in U_m. Since K is compact, there exists a finite set $\{z_1, \ldots, z_t\} \subseteq K$ for which

$$K \subseteq V_{z_1} \cup \cdots \cup V_{z_t} \subseteq U_m. \tag{2.7}$$

At this point, by the subadditivity of φ, it easily follows that $\|A \cup B\|_\varphi \leq \|A\|_\varphi + \|B\|_\varphi$ for all $A, B \subseteq \mathbf{N}$. Hence we have

$$\begin{aligned} \|\{n : x_n \in K\}\|_\varphi &\geq \|\{n : x_n \in \overline{U}_r\}\|_\varphi - \|\{n : x_n \in U_s\}\|_\varphi \\ &\geq \|\{n : x_n \in U_r\}\|_\varphi - \|\{n : x_n \in U_s\}\|_\varphi > 0. \end{aligned}$$

On the other hand, it follows by (2.7) that

$$\|\{n : x_n \in K\}\|_\varphi \leq \|\{n : x_n \in \textstyle\bigcup_{i=1}^t V_{z_i}\}\|_\varphi \leq \sum_{i=1}^t \|\{n : x_n \in V_{z_i}\}\|_\varphi = 0.$$

This contradiction concludes the proof. $\square$

The following corollary is immediate (we omit details):

Corollary 2.2.8. *Let x be a real sequence for which Γ_x is a discrete set. Then $\Lambda_x = \Gamma_x$.*

2.2.2 Joint Converse results

We provide now a kind of converse of Theorem 2.2.7, specializing to the case of the ideal $\mathcal{I}_0$: informally, if B is a sufficiently smooth closed set and A is an F_σ-set containing the isolated points of B, then there exists a sequence x such that $\Lambda_x = A$ and $\Gamma_x = B$.

To this aim, we need some additional notation: let $\mathrm{d}^\star$, $\mathrm{d}_\star$, and d be the upper asymptotic density, lower asymptotic density, and asymptotic density on $\mathbf{N}$, resp.; in particular, $\mathcal{I}_0 = \{S \subseteq \mathbf{N} : \mathrm{d}^\star(S) = 0\}$.

Given a topological space X, the interior and the closure of a subset $S \subseteq X$ are denoted by S° and $\overline{S}$, respectively; S is said to be *regular closed* if $S = \overline{S^\circ}$. We let the Borel σ-algebra on X be $\mathcal{B}(X)$. A Borel probability measure $\mu : \mathcal{B}(X) \to [0,1]$ is said to be *strictly positive* whenever $\mu(U) > 0$ for all non-empty open sets U. Moreover, μ is *atomless* if, for each measurable set A with $\mu(A) > 0$, there exists a measurable subset $B \subseteq A$ such that $0 < \mu(B) < \mu(A)$. Then, a sequence (x_n) taking values in X is said to be *μ-uniformly distributed* whenever

$$\mu(F) \ge \mathrm{d}^\star(\{n : x_n \in F\}) \tag{2.8}$$

for all closed sets F, cf. (Fremlin, 2006, Section 491B).

Theorem 2.2.9. *Let X be a separable metric space and $\mu : \mathcal{B}(X) \to [0,1]$ be an atomless strictly positive Borel probability measure. Fix also sets $A \subseteq B \subseteq C \subseteq X$ such that A is an F_σ-set, and B, C are closed sets such that:*

(i) $\mu(B) > 0$,

(ii) the set S of isolated points of B is contained in A, and

(iii) $B \setminus S$ is regular closed.

Then there exists a sequence x taking values in X such that

$$\Lambda_x = A, \;\; \Gamma_x = B, \;\; \text{and } \mathrm{L}_x = C. \tag{2.9}$$

Proof. Set $F := B \setminus S$ and note that, by the separability of X, S is at most countable. In particular, $\mu(S) = 0$, hence $\mu(F) = \mu(B) > 0$.

Let us assume for now that A is non-empty. Since A is an F_σ-set, there exists a sequence (A_k) of non-empty closed sets such that $\bigcup_k A_k = A$. Considering that X is (hereditarily) second countable, then every closed set is separable. Hence, for each $k \in \mathbf{N}$, there exists a countable set $\{a_{k,n} : n \in \mathbf{N}\} \subseteq A_k$ with closure A_k. Considering that F is a separable metric space on its own right and that the (normalized) restriction μ_F of μ on F, that is,

$$\mu_F : \mathcal{B}(F) \to [0,1] : Y \mapsto \frac{1}{\mu(F)}\mu(Y) \tag{2.10}$$

is a Borel probability measure, it follows by (Fremlin, 2006, Exercise 491Xw) that there exists a μ_F-uniformly distributed sequence (b_n) which takes values in F and satisfies (2.8). Lastly, let $\{c_n : n \in \mathbf{N}\}$ be a countable dense subset of C.

At this point, let $\mathscr{C}$ be the set of non-zero integer squares and note that $\mathrm{d}(\mathscr{C}) = 0$. For each $k \in \mathbf{N}$ define $\mathscr{A}_k := \{2^k n : n \in \mathbf{N} \setminus 2\mathbf{N}\} \setminus \mathscr{C}$ and $\mathscr{B} := \mathbf{N} \setminus (2\mathbf{N} \cup \mathscr{C})$. It follows by construction that $\{\mathscr{A}_k : k \in \mathbf{N}\} \cup \{\mathscr{B}, \mathscr{C}\}$ is a partition of $\mathbf{N}$. Moreover, each $\mathscr{A}_k$ admits asymptotic density and

$$\lim_{n \to \infty} \mathrm{d}\left(\bigcup_{k \ge n} \mathscr{A}_k\right) = 0. \tag{2.11}$$

Finally, for each positive integer k, let $\{\mathscr{A}_{k,m} : m \in \mathbf{N}\}$ be the partition of $\mathscr{A}_k$ defined by $\mathscr{A}_{k,1} := \mathscr{A}_k \cap \bigcup_{n \in \mathscr{A}_1 \cup \mathscr{B} \cup \mathscr{C}} [n!, (n+1)!)$ and $\mathscr{A}_{k,m} := \mathscr{A}_k \cap \bigcup_{n \in \mathscr{A}_m} [n!, (n+1)!)$ for all integers $m \ge 2$. Then, it is easy to check that

$$\mathrm{d}^\star(\mathscr{A}_{k,1}) = \mathrm{d}^\star(\mathscr{A}_{k,2}) = \cdots = \mathrm{d}(\mathscr{A}_k) = 2^{-k-1}.$$

Hence define the sequence $x = (x_n)$ by

$$x_n = \begin{cases} a_{k,m} & \text{if } n \in \mathscr{A}_{k,m}, \\ b_m & \text{if } n \text{ is the } m\text{-th term of } \mathscr{B}, \\ c_m & \text{if } n \text{ is the } m\text{-th term of } \mathscr{C}. \end{cases} \tag{2.12}$$

To complete the proof, let us verify that (2.9) holds true:

Claim (i): $\mathrm{L}_x = C$. Note that $x_n \in C$ for all $n \in \mathbf{N}$. Since C is closed by hypothesis, then $\mathrm{L}_x \subseteq C$. On the other hand, if $\ell \in C$, then there exists a sequence (c_n) taking values in C converging (in the ordinary sense) to ℓ. It follows by the definition of (x_n) that there exists a subsequence (x_{n_k}) converging to ℓ, i.e., $C \subseteq \mathrm{L}_x$.

Claim (ii): $\Gamma_x = B$. Fix $\ell \notin B$ and let U be an open neighborhood of ℓ disjoint from B (this is possible since, in the opposite, ℓ would belong to $\overline{B} = B$). Then, $\{n : x_n \in U\} \subseteq \mathscr{C}$, which implies that $\Gamma_x \subseteq B$.

Note that the Borel probability measure μ_F defined in (2.10) is clearly atomless. Moreover, given an open set $U \subseteq X$ with non-empty intersection with F, then

$U \cap F^\circ \neq \emptyset$: indeed, in the opposite, we would have $F^\circ \subseteq U^c$, which is closed, hence $F = \overline{F^\circ} \subseteq U^c$, contradicting our hypothesis. This proves that every non-empty open set V (relative to F) contains a non-empty open set of X. Therefore μ_F is also strictly positive. With these premises, fix $\ell \in F$ and let V be a open neighborhood of ℓ (relative to F). Since (b_n) is μ_F-uniformly distributed and μ_F is strictly positive, it follows by (2.8) that

$$\begin{aligned}0 < \mu_F(V) = 1 - \mu_F(V^c) &\leq 1 - \mathrm{d}^\star(\{n : b_n \in V^c\}) \\ &= \mathrm{d}_\star(\{n : b_n \in V\}) \leq \mathrm{d}^\star(\{n : b_n \in V\}).\end{aligned}$$

Since $\mathrm{d}(\mathscr{B}) = 1/2$, we obtain by standard properties of $\mathrm{d}^\star$ that

$$\mathrm{d}^\star(\{n : x_n \in V\}) \geq \mathrm{d}^\star(\{n \in \mathscr{B} : x_n \in V\}) = \frac{1}{2}\mathrm{d}^\star(\{n : b_n \in V\}) > 0.$$

We conclude by the arbitrariness of V and ℓ that $F \subseteq \Gamma_x$.

Hence we miss only to show that $S \subseteq \Gamma_x$. To this aim, fix $\ell \in S$, thus ℓ is also an isolated point of A. Hence there exist $k, m \in \mathbf{N}$ such that $a_{k,m} = \ell$. We conclude that $\mathrm{d}^\star(\{n : x_n \in U\}) \geq \mathrm{d}^\star(\{n : x_n = \ell\}) \geq \mathrm{d}(\mathscr{A}_k) > 0$ for each neighborhood U of ℓ. Therefore $B = F \cup S \subseteq \Gamma_x$.

Claim (iii): $\Lambda_x = A$. Fix $\ell \in A$, hence there exists $k \in \mathbf{N}$ for which ℓ belongs to the (non-empty) closed set A_k. Since $\{a_{k,n} : n \in \mathbf{N}\}$ is dense in A_k, there exists a sequence $(a_{k,r_m} : m \in \mathbf{N})$ converging to ℓ. Recall that $x_n = a_{k,r_m}$ whenever $n \in \mathscr{A}_{k,r_m}$ for each $m \in \mathbf{N}$. Set by convenience $\theta_0 := 0$ and define recursively an increasing sequence of positive integers (θ_m) such that θ_m is an integer greater than θ_{m-1} for which

$$\mathrm{d}^\star\left(\mathscr{A}_{k,r_m} \cap (\theta_{m-1}, \theta_m]\right) \geq \frac{\mathrm{d}(\mathscr{A}_k)}{2} = 2^{-k-2}.$$

Then, setting $\mathcal{A} := \bigcup_m \mathscr{A}_{k,r_m} \cap (\theta_{m-1}, \theta_m]$, we obtain that the subsequence $(x_n : n \in \mathcal{A})$ converges to ℓ and $\mathrm{d}^\star(\mathcal{A}) > 0$. In particular, $A \subseteq \Lambda_x$.

On the other hand, it is known that $\Lambda_x \subseteq \Gamma_x$, see e.g. Fridy (1993). If $A = B$, it follows by Claim (ii) that $\Lambda_x \subseteq A$ and we are done. Otherwise, fix $\ell \in B \setminus A = F \setminus A$ and let us suppose for the sake of contradiction that there exists a subsequence (x_{n_k}) such that $\lim_k x_{n_k} = \ell$ and $\mathrm{d}^\star(\{n_k : k \in \mathbf{N}\}) > 0$. Fix a real $\varepsilon > 0$. Then, thanks to (2.11), there exists a sufficiently large integer n_0 such that $\mathrm{d}\left(\bigcup_{k>n_0} \mathscr{A}_k\right) \le \varepsilon$. In addition, since F is a metric space and μ_F is atomless and strictly positive (see Claim (II)), we have

$$\lim_{n\to\infty} \mu_F(V_n) = \mu_F(\{\ell\}) = 0,$$

where V_n is the open ball (relative to F) with center ℓ and radius $1/n$. Hence, there exists a sufficiently large integer m' such that $0 < \mu_F(V_{m'}) \le \varepsilon$. In addition, there exists an integer m'' such that $V_{m''}$ is disjoint from the closed set $A_1 \cup \cdots \cup A_{n_0}$. Then set $V := V_m$ where m is an integer greater than $\max(m', m'')$ such that $\mu_F(V) < \mu_F(V_{\max(m',m'')})$. In particular, by the monotonicity of μ_F, we have

$$0 < \mu_F(V) \le \mu_F(\overline{V}) \le \mu_F(V_{m'}) \le \varepsilon. \tag{2.13}$$

At this point, observe there exists a finite set Y such that

$$\begin{aligned}\{n_k : k \in \mathbf{N}\} &= \{n_k : x_{n_k} \in V\} \cup Y \\ &\subseteq \left(\textstyle\bigcup_{k>n_0} \mathscr{A}_k\right) \cup \{n \in \mathscr{B} : x_n \in V\} \cup \mathscr{C} \cup Y.\end{aligned}$$

Therefore, by the subadditivity of $\mathrm{d}^\star$, (2.8), and (2.13), we obtain

$$\begin{aligned}\mathrm{d}^\star(\{n_k : k \in \mathbf{N}\}) &\le \varepsilon + \mathrm{d}^\star(\{n \in \mathscr{B} : x_n \in V\}) \le \varepsilon + \mathrm{d}^\star(\{n \in \mathscr{B} : b_n \in V\}) \\ &\le \varepsilon + \mathrm{d}^\star(\{n \in \mathscr{B} : b_n \in \overline{V}\}) \le \varepsilon + \mu_F(\overline{V}) \le 2\varepsilon.\end{aligned}$$

It follows by the arbitrariness of ε that $\mathrm{d}(\{n_k : k \in \mathbf{N}\}) = 0$, i.e., $\Lambda_x \subseteq A$.

To complete the proof, assume now that $A = \emptyset$. In this case, note that necessarily $S = \emptyset$, and it is enough to replace (2.12) with

$$x_n = \begin{cases} b_{n-\lfloor\sqrt{n}\rfloor} & \text{if } n \notin \mathscr{C}, \\ c_{\sqrt{n}} & \text{if } n \in \mathscr{C}. \end{cases}$$

Then, it can be shown with a similar argument that $\Lambda_x = \emptyset$, $\Gamma_x = B$, and $\mathrm{L}_x = C$. $\square$

It is worth noting that Theorem 2.2.9 cannot be extended to the whole class of analytic P-ideals. Indeed, it follows by Theorem 2.2.3 that if $\mathcal{I}$ is an F_σ ideal on $\mathbf{N}$ then the set of $\mathcal{I}$-limit points is closed set, cf. also Theorem 2.2.12 below.

In addition, the regular closedness of $B \setminus S$ is essential in the proof of Theorem 2.2.9. On the other hand, there exist real sequences x such that Γ_x is the Cantor set $\mathcal{C}$ (which is a perfect set but not regular closed):

Example 2.2.10. Given a real $r \in [0, 1)$ and an integer $b \geq 2$, we write r in base b as $\sum_n a_n/b^n$, where each a_n belongs to $\{0, 1, \ldots, b-1\}$ and $a_n = \zeta$ for all sufficiently large n only if $\zeta = 0$. This representation is unique.

Let $x = (x_n)$ be the sequence $(0, 0, 1, 0, \frac{1}{2}, 1, 0, \frac{1}{3}, \frac{2}{3}, 1, \ldots)$. This sequence is unifomly distributed in $[0, 1]$, i.e., $\mathrm{d}(\{n : x_n \in [a, b]\}) = b - a$ for all $0 \leq a < b \leq 1$, and $\Gamma_x = [0, 1]$, see e.g. (Fridy, 1993, Example 4). Let also $T : [0, 1] \to \mathcal{C}$ be the injection defined by $r \mapsto T(r)$, where if $r = \sum_n a_n/2^n \in [0, 1)$ in base 2 then $T(r) = \sum_n 2a_n/3^n$ in base 3, and $1 \mapsto 1$. Observe that $\mathcal{C} \setminus T([0, 1))$ is the set of points of the type $2(1/3^{n_1} + \cdots + 1/3^{n_{k-1}}) + 1/3^{n_k}$, for some non-negative integers $n_1 < \cdots < n_k$; in particular, $\overline{T([0, 1))} = \mathcal{C}$.

Since the sequence $T(x) := (T(x_n))$ takes values in the closed set $\mathcal{C}$, it is clear that $\Gamma_{T(x)} \subseteq \mathcal{C}$. On the other hand, fix $\ell \in T([0, 1))$ with representation $\sum_n 2a_n/3^n$ in base 3, where $a_n \in \{0, 1\}$ for all n. For each k, let U_k be the open ball with center ℓ and radius $1/3^k$. It follows that

$$\left\{n : T(x_n) \in \left[\frac{2a_1}{3} + \cdots + \frac{2a_k}{3^k}, \frac{2a_1}{3} + \cdots + \frac{2a_k}{3^k} + \frac{1}{3^k}\right)\right\},$$

that is,

$$\left\{n : x_n \in \left[\frac{a_1}{2} + \cdots + \frac{a_k}{2^k}, \frac{a_1}{2} + \cdots + \frac{a_k}{2^k} + \frac{1}{2^k}\right)\right\}.$$

Since (x_n) is equidistributed, then $\mathrm{d}^\star(\{n : T(x_n) \in U_k\}) \geq 1/2^k$ for all k. In particular, $\Gamma_{T(x)}$ is a closed set containing $T([0, 1))$, therefore $\Gamma_{T(x)} = \mathcal{C}$.

Finally, we provide a sufficient condition for the existence of an atomless strictly positive Borel probability measure:

Corollary 2.2.11. *Let X be a Polish space without isolated points and fix sets $A \subseteq B \subseteq C \subseteq X$ such that A is an F_σ-set, $B \neq \emptyset$ is regular closed, and C is closed. Then, there exists a sequence x taking values in X which satisfies* (2.9).

Proof. First, observe that the restriction $\tilde{\lambda}$ of the Lebesgue measure λ on the set $\mathscr{I} := (0,1) \setminus \mathbf{Q}$ is an atomless strictly positive Borel probability measure. Thanks to (Engelking, 1989, Exercise 6.2.A(e)), X contains a dense subspace D which is homeomorphic to $\mathbf{R} \setminus \mathbf{Q}$, which is turn is homeomorphic to $\mathscr{I}$, let us say through $\eta : D \to \mathscr{I}$. This embedding can be used to transfer the measure $\tilde{\lambda}$ to the target space by setting

$$\mu : \mathcal{B}(X) \to [0,1] : Y \mapsto \tilde{\lambda}(\eta(Y \cap D)). \tag{2.14}$$

Lastly, since B is non-empty closed regular, then it has no isolated points and contains an open set U of X. In particular, considering that η is an open map, we get by (2.14) that $\mu(B) \geq \mu(U) = \tilde{\lambda}(\eta(U \cap D)) > 0$. The claim follows by Theorem 2.2.9. □

Note that, in general, the condition $B \neq \emptyset$ cannot be dropped: indeed, it follows by (Di Maio and Kočinac, 2008, Theorem 2.14) that, if X is compact, then every sequence (x_n) admits at least one statistical cluster point.

We conclude with another converse result related to ideals $\mathcal{I}$ of the type F_σ (recall that, thanks to Theorem 2.2.3, every $\mathcal{I}$-limit point is also an $\mathcal{I}$-cluster point):

Theorem 2.2.12. *Let X be a first countable space where all closed sets are separable and let $\mathcal{I} \neq \mathrm{Fin}$ be an F_σ-ideal. Fix also closed sets $B, C \subseteq X$ such that $\emptyset \neq B \subseteq C$. Then there exists a sequence x such that $\Lambda_x(\mathcal{I}) = \Gamma_x(\mathcal{I}) = B$ and $\mathrm{L}_x = C$.*

Proof. By hypothesis, there exists an infinite set $I \in \mathcal{I}$. Let φ be a lower semicontinuous submeasures associated to $\mathcal{I}$ as in (2.5). Let $\{b_n : n \in \mathbf{N}\}$ and $\{c_n : n \in \mathbf{N}\}$ be countable dense subsets of B and C, respectively. In addition, set $m_0 := 0$ and let (m_k) be an increasing sequence of positive integers such that $\varphi((\mathbf{N} \setminus I) \cap (m_{k-1}, m_k]) \geq k$ for all k (note that this is possible since $\varphi(\mathbf{N} \setminus I) = \infty$

and φ is a lower semicontinuous submeasure). At this point, given a partition $\{H_n : n \in \mathbf{N}\}$ of $\mathbf{N} \setminus I$, where each H_n is infinite, we set

$$M_k := (\mathbf{N} \setminus I) \cap \bigcup_{n \in H_k} (m_{n-1}, m_n]$$

for all $k \in \mathbf{N}$. Then it is easily checked that $\{M_k : k \in \mathbf{N}\}$ is a partition of $\mathbf{N} \setminus I$ with $M_k \notin \mathcal{I}$ for all k, and that the sequence (x_n) defined by

$$x_n = \begin{cases} b_k & \text{if } n \in M_k, \\ c_k & \text{if } n \text{ is the } k\text{-th term of } I. \end{cases}$$

satisfies the claimed conditions. □

In particular, Theorem 2.2.6 and Theorem 2.2.12 fix a gap in a result of Das (Das, 2012, Theorem 3) and provide its correct version.

2.2.3 Additional remarks

In this last part of the section, we are interested in the topological nature of the set of $\mathcal{I}$-limit points when $\mathcal{I}$ is neither F_σ- nor analytic P-ideal.

Let $\mathcal{N}$ be the set of strictly increasing sequences of positive integers. Then $\mathcal{N}$ is a Polish space, since it is a closed subspace of the Polish space $\mathbf{N}^{\mathbf{N}}$ (equipped with the product topology of the discrete topology on $\mathbf{N}$). Let also $x = (x_n)$ be a sequence taking values in a first countable regular space X and fix an arbitrary ideal $\mathcal{I}$ on $\mathbf{N}$. For each $\ell \in X$, let $(U_{\ell,m})$ be a decreasing local base of open neighborhoods at ℓ. Then, ℓ is an $\mathcal{I}$-limit point of x if and only if there exists a sequence $(n_k) \in \mathcal{N}$ such that

$$\{n_k : k \in \mathbf{N}\} \notin \mathcal{I} \;\text{ and }\; \{k : x_{n_k} \notin U_{\ell,m}\} \in \text{Fin} \;\text{ for all } m. \tag{2.15}$$

Set $\mathcal{I}^c := \mathcal{P}(\mathbf{N}) \setminus \mathcal{I}$ and define the continuous function $\psi : \mathcal{N} \to \{0,1\}^{\mathbf{N}} : (n_k) \mapsto \chi_{\{n_k : k \in \mathbf{N}\}}$, where χ_S is the characteristic function of a set $S \subseteq \mathbf{N}$. Moreover, define

$$\zeta_m : \mathcal{N} \times X \to \{0,1\}^{\mathbf{N}} : (n_k) \times \ell \mapsto \chi_{\{k : x_{n_k} \notin U_{\ell,m}\}}$$

for each m. Hence it easily follows by (2.15) that

$$\Lambda_x(\mathcal{I}) = \pi_X\left(\bigcap_m \left(\psi^{-1}(\mathcal{I}^c) \times X\right) \cap \zeta_m^{-1}(\mathrm{Fin})\right),$$

where $\pi_X : \mathcal{N} \times X \to X$ stands for the projection on X.

Proposition 2.2.13. *Let $x = (x_n)$ be a sequence taking values in a first countable regular space X and let $\mathcal{I}$ be a co-analytic ideal. Then $\Lambda_x(\mathcal{I})$ is analytic.*

Proof. For each $(n_k) \in \mathcal{N}$ and $\ell \in X$, the sections $\zeta_m((n_k), \cdot)$ and $\zeta_m(\cdot, \ell)$ are continuous. Hence, thanks to (Srivastava, 1998, Theorem 3.1.30), each function ζ_m is Borel measurable. Since Fin is an F_σ-set, we obtain that each $\zeta_m^{-1}(\mathrm{Fin})$ is Borel. Moreover, since $\mathcal{I}$ is a co-analytic ideal and ψ is continuous, it follows that $\psi^{-1}(\mathcal{I}^c) \times X$ is an analytic subset of $\mathcal{N} \times X$. Therefore $\Lambda_x(\mathcal{I})$ is the projection on X of the analytic set $\bigcap_m (\psi^{-1}(I^c) \times X \cap \zeta_m^{-1}(\mathrm{Fin}))$, which proves the claim. $\square$

The situation is much different for *maximal ideals*, i.e., ideals which are maximal with respect to inclusion. In this regard, we recall if $\mathcal{I}$ is a maximal ideal then every bounded real sequence x is $\mathcal{I}$-convergent, i.e., there exists $\ell \in \mathbf{R}$ such that $\{n : |x_n - \ell| \geq \varepsilon\} \in \mathcal{I}$ for every $\varepsilon > 0$, cf. (Kostyrko et al., 2005, Theorem 2.2).

Let $B(a, r)$ the open ball with center a and radius r in a given metric space (X, d), and denote by $\mathrm{diam}\, S$ the diameter of a non-empty set $S \subseteq X$, namely, $\sup_{a,b \in S} d(a, b)$. Then, the metric space is said to be *smooth* if

$$\lim_{k\to\infty} \sup_{a \in X} \mathrm{diam}\, \overline{B(a, 1/k)} = 0. \tag{2.16}$$

Note that (2.16) holds if, e.g., the closure of each open ball $B(a, r)$ coincides with the corresponding closed ball $\{b \in X : d(a, b) \leq r\}$.

Proposition 2.2.14. *Let x be a sequence taking values in a smooth compact metric space X and let $\mathcal{I}$ be a maximal ideal. Then x has exactly one $\mathcal{I}$-cluster point. In particular, $\Lambda_x(\mathcal{I})$ is closed.*

Proof. Since X is a compact metric space, then X is totally bounded, i.e., for each $\varepsilon > 0$ there exist finitely many open balls with radius ε covering X. Moreover, it is well known that an ideal $\mathcal{I}$ is maximal if and only if either $A \in \mathcal{I}$ or $A^c \in \mathcal{I}$ for every $A \subseteq \mathbf{N}$.

Hence, fix $k \in \mathbf{N}$, let $\{B_{k,1}, \ldots, B_{k,m_k}\}$ be a cover of X of open balls with radius $1/k$, and define $\mathscr{C}_{k,i} := \{n : x_n \in C_{k,i}\}$ for each $i \le m_k$, where $C_{k,i} := B_{k,i} \setminus (B_{k,1} \cup \cdots \cup B_{k,i-1})$ and $B_{k,0} := \emptyset$. Considering that $\{\mathscr{C}_{k,1}, \ldots, \mathscr{C}_{k,m_k}\}$ is a partition of $\mathbf{N}$, it follows by the above observations that there exists a unique $i_k \in \{1, \ldots, m_k\}$ for which $\mathscr{C}_{k,i_k} \notin \mathcal{I}$.

At this point, let (G_k) be the decreasing sequence of closed sets defined by

$$G_k := \overline{C_{1,i_1} \cap \cdots \cap C_{k,i_k}}$$

for all k. Note that each G_k is non-empty, the diameter of G_k (which is contained in $\overline{B_{k,i_k}}$) goes to 0 as $k \to \infty$, and $\{n : x_n \in G_k\} \notin \mathcal{I}$ for all k. Since X is a compact metric space, then $\bigcap_k G_k$ is a singleton $\{\ell\}$. Considering that every open ball with center ℓ contains some G_k with k sufficiently large, it easily follows that $\Gamma_x(\mathcal{I}) = \{\ell\}$. In particular, since each $\mathcal{I}$-limit point is also an $\mathcal{I}$-cluster point, we conclude that $\Lambda_x(\mathcal{I})$ is either empty or the singleton $\{\ell\}$. $\square$

Corollary 2.2.15. *An ideal $\mathcal{I}$ is maximal if and only if every real sequence x has at most one $\mathcal{I}$-limit point.*

Proof. First, let us assume that $\mathcal{I}$ is a maximal ideal. Let us suppose that there exists $k > 0$ such that $A_k := \{n : |x_n| > k\} \in \mathcal{I}$ and define a sequence $y = (y_n)$ by $y_n = k$ if $n \in A_k$ and $y_n = x_n$ otherwise. Then, it follows by (Das, 2012, Theorem 4) and Proposition 2.2.14 that there exists $\ell \in \mathbf{R}$ such that $\Lambda_x(\mathcal{I}) = \Lambda_y(\mathcal{I}) \subseteq \Gamma_y(\mathcal{I}) = \{\ell\}$.

Now, assume that $A_k^c \in \mathcal{I}$ for all $k \in \mathbf{N}$. Hence, letting $z = (z_n)$ be the sequence defined by $z_n = x_n$ if $n \in A_k$ and $z_n = k$ otherwise, we obtain

$$\Lambda_x(\mathcal{I}) = \Lambda_z(\mathcal{I}) \subseteq \mathrm{L}_z \subseteq \mathbf{R} \setminus (-k, k).$$

Therefore, it follows by the arbitrariness of k that $\Lambda_x(\mathcal{I}) = \emptyset$.

Conversely, let us assume that $\mathcal{I}$ is not a maximal ideal. Then there exists $A \subseteq \mathbf{N}$ such that $A \notin \mathcal{I}$ and $A^c \notin \mathcal{I}$. Then the sequence (x_n) defined by $x_n = \chi_A(n)$ for each n has two $\mathcal{I}$-limit points. $\square$

We conclude by showing that there exist an ideal $\mathcal{I}$ and a real sequence x such that $\Lambda_x(\mathcal{I})$ is not an F_σ-set.

Example 2.2.16. Fix a partition $\{P_m : m \in \mathbf{N}\}$ of $\mathbf{N}$ such that each P_m is infinite. Then, define the ideal

$$\mathcal{I} := \{A \subseteq \mathbf{N} : \{m : A \cap P_m \notin \mathrm{Fin}\} \in \mathrm{Fin}\},$$

which corresponds to the Fubini product $\mathrm{Fin} \times \mathrm{Fin}$ on $\mathbf{N}^2$ (it is known that $\mathcal{I}$ is a $F_{\sigma\delta\sigma}$-ideal and it is not a P-ideal). Given a real sequence $x = (x_n)$, let us denote by $x \restriction P_m$ the subsequence $(x_n : n \in P_m)$. Hence, a real ℓ is an $\mathcal{I}$-limit point of x if and only if there exists a subsequence (x_{n_k}) converging to ℓ such that $\{n_k : k \in \mathbf{N}\} \cap P_m$ is infinite for infinitely many m. Moreover, for each m of this type, the subsequence $(x_{n_k}) \restriction P_m$ converges to ℓ. It easily follows that

$$\Lambda_x(\mathcal{I}) = \bigcap_k \bigcup_{m \ge k} \mathrm{L}_{x \restriction P_m}. \tag{2.17}$$

(In particular, since each $\mathrm{L}_{x \restriction P_m}$ is closed, then $\Lambda_x(\mathcal{I})$ is an $F_{\sigma\delta}$-set.)

At this point, let $(q_t : t \in \mathbf{N})$ be the sequence

$$({}^0\!/_1, {}^1\!/_1, {}^0\!/_2, {}^1\!/_2, {}^2\!/_2, {}^0\!/_3, {}^1\!/_3, {}^2\!/_3, {}^3\!/_3, \ldots),$$

where $q_t := a_t/b_t$ for each t, and note that $\{q_t : t \in \mathbf{N}\} = \mathbf{Q} \cap [0,1]$. It follows by construction that $a_t \le b_t$ for all t and $b_t = \sqrt{2t}(1+o(1))$ as $t \to \infty$. In particular, if m is a sufficiently large integer, then

$$\min_{i \le m:\, q_i \neq q_m} |q_i - q_m| \ge \left(\frac{1}{\sqrt{2m}(1+o(1))}\right)^2 > \frac{1}{3m}. \tag{2.18}$$

Lastly, for each $m \in \mathbf{N}$, define the closed set

$$C_m := [0,1] \cap \bigcap_{t \le m} \left(q_t - \frac{1}{2^m}, q_t + \frac{1}{2^m}\right)^c.$$

We obtain by (2.18) that, if m is sufficiently large, let us say $\ge k_0$, then

$$C_m \cup C_{m+1} = [0,1] \cap \bigcap_{t \le m} \left(q_t - \frac{1}{2^{m+1}}, q_t + \frac{1}{2^{m+1}}\right)^c.$$

It follows by induction that

$$C_m \cup C_{m+1} \cup \cdots \cup C_{m+n} = [0,1] \cap \bigcap_{t \le m} \left(q_t - \frac{1}{2^{m+n}}, q_t + \frac{1}{2^{m+n}}\right)^c.$$

for all $n \in \mathbf{N}$. In particular, $\bigcup_{m \ge k} C_m = [0,1] \setminus \{q_1, \ldots, q_k\}$ whenever $k \ge k_0$.

Let x be a real sequence such that each $\{x_n : n \in P_m\}$ is a dense subset of C_m. Therefore, it follows by (2.17) that

$$\begin{aligned}\Lambda_x(\mathcal{I}) = \bigcap_k \bigcup_{m \ge k} C_m &\subseteq \bigcap_{k \ge k_0} \bigcup_{m \ge k} C_m \\ &= \bigcap_{k \ge k_0} [0,1] \setminus \{q_1, \ldots, q_k\} = [0,1] \setminus \mathbf{Q}.\end{aligned}$$

On the other hand, if a rational q_t belongs to $\Lambda_x(\mathcal{I})$, then $q_t \in \bigcup_{m \ge k} C_m$ for all $k \in \mathbf{N}$, which is impossible whenever $k \ge t$. This proves that $\Lambda_x(\mathcal{I}) = [0,1] \setminus \mathbf{Q}$, which is not an F_σ-set.

We leave as an open question to determine whether there exists a real sequence x and an ideal $\mathcal{I}$ such that $\Lambda_x(\mathcal{I})$ is not Borel measurable.

2.3 Limit points of subsequences

It is well known that the set of ordinary limit points of "almost every" subsequence of a real sequence (x_n) coincides with the set of ordinary limit points of the original sequence, in the sense of Lebesgue measure, see Buck (1944). In the same direction, we prove its analogue for ideal cluster points and ideal limit points.

The main question addressed in this Section is to find suitable conditions on X and $\mathcal{I}$ such that the set of $\mathcal{I}$-cluster points of a sequence (x_n) is equal to the set of $\mathcal{I}$-cluster points of "almost all" its subsequences. Finally, we obtain a characterization of ideal convergence. Related results were obtained in Agnew (1944); Dawson (1973); Miller (1995); Miller and Miller-Wan Wieren (2019); Miller and Orhan (2001).

2.3.1 Thinnability

Given $k \in \mathbf{N}$ and *infinite* sets $A, B \subseteq \mathbf{N}$ with canonical enumeration $\{a_n : n \in \mathbf{N}\}$ and $\{b_n : n \in \mathbf{N}\}$, respectively, we write $A \le B$ if $a_n \le b_n$ for all $n \in \mathbf{N}$ and define $A_B := \{a_b : b \in B\}$ and $kA := \{ka : a \in A\}$.

Definition 2.3.1. An ideal $\mathcal{I}$ is said to be *weakly thinnable* if $A_B \notin \mathcal{I}$ whenever $A \subseteq \mathbf{N}$ admits non-zero asymptotic density and $B \notin \mathcal{I}$.

If, in addition, also $B_A \notin \mathcal{I}$ and $X \notin \mathcal{I}$ whenever $X \le Y$ and $Y \notin \mathcal{I}$, then $\mathcal{I}$ is said to be *thinnable*.

Definition 2.3.2. An ideal $\mathcal{I}$ is said to be *strechable* if $kA \notin \mathcal{I}$ for all $k \in \mathbf{N}$ and $A \notin \mathcal{I}$.

The terminology has been suggested from the related properties of finitely additive measures on $\mathbf{N}$ studied in van Douwen (1992). In this regard, Fin is thinnable and strechable.

This is the case of several other ideals:

Proposition 2.3.3. *Let $f : \mathbf{N} \to (0, \infty)$ be a definitively non-increasing function such that $\sum_{n \ge 1} f(n) = \infty$. Define the summable ideal*

$$\mathcal{I}_f := \left\{ S \subseteq \mathbf{N} : \sum_{n \in S} f(n) < \infty \right\}.$$

Then $\mathcal{I}_f$ is thinnable provided $\mathcal{I}_f$ is strechable.

In addition, suppose that

$$\liminf_{n \to \infty} \frac{\sum_{i \in [1,n]} f(i)}{\sum_{i \in [1,kn]} f(i)} \neq 0 \quad \text{for all } k \in \mathbf{N} \tag{2.19}$$

and define the Erdős–Ulam ideal

$$\mathscr{E}_f := \left\{ S \subseteq \mathbf{N} : \lim_{n \to \infty} \frac{\sum_{i \in S \cap [1,n]} f(i)}{\sum_{i \in [1,n]} f(i)} = 0 \right\}.$$

Then $\mathscr{E}_f$ is thinnable provided $\mathscr{E}_f$ is strechable.

Proof. Let us suppose that $A = \{a_n : n \in \mathbf{N}\}$ admits asymptotic density $c > 0$ and $B = \{b_n : n \in \mathbf{N}\} \notin \mathcal{I}_f$, that is, $\sum_{n\ge 1} f(b_n) = \infty$. Define the integer $k := \lfloor 1/c \rfloor + 1 \ge 2$ and note that $\sum_{n \ge 1} f(kb_n) = \infty$ by the fact that $\mathcal{I}_f$ is strechable. Then $a_n = \frac{1}{c}n(1+o(1))$ as $n \to \infty$, which implies

$$\textstyle\sum_{n\ge 1} f(a_{b_n}) \ge O(1) + \sum_{n \ge 1} f(kb_n) = \infty, \tag{2.20}$$

i.e., $A_B \notin \mathcal{I}_f$, hence $\mathcal{I}_f$ is weakly thinnable. Moreover, observe that

$$\begin{aligned} \sum_{n \equiv 1 \bmod k} f(b_n) &\ge \sum_{n \equiv 2 \bmod k} f(b_n) \ge \cdots \\ &\ge \sum_{n \equiv 0 \bmod k} f(b_n) \ge \sum_{\substack{n \equiv 1 \bmod k,\\ n \neq 1}} f(b_n) \end{aligned} \tag{2.21}$$

and note that the first sum is finite if and only if the last sum is finite. Since $I \notin \mathcal{I}_f$, then all the above sums are infinite, which implies that

$$\sum_{n \ge 1} f(b_{a_n}) \ge O(1) + \sum_{n\ge 1} f(b_{kn}) = \infty,$$

i.e., $B_A \notin \mathcal{I}_f$. Lastly, given infinite sets $X, Y \subseteq \mathbf{N}$ with $X \le Y$ and $X \in \mathcal{I}_f$, we have $\sum_{y \in Y} f(y) \le \sum_{x \in X} f(x) < \infty$. Therefore $\mathcal{I}_f$ is thinnable.

The proof of the second part is similar, where (2.20) is replaced by

$$\textstyle\sum_{a_{b_n} \le x} f(a_{b_n}) \ge O(1) + \sum_{b_n \le x/k} f(kb_n).$$

Moreover, $B \notin \mathscr{E}_f$ implies $kB \notin \mathscr{E}_f$ by the hypothesis of strechability, i.e.,

$$\textstyle\sum_{b_n \le x/k} f(kb_n) \neq o\left(\sum_{i \le x/k} f(i)\right);$$

thanks to (2.26), we conclude that

$$\textstyle\sum_{b_n \le x/k} f(kb_n) \neq o\left(\sum_{i \le x} f(i)\right),$$

hence $A_B \notin \mathscr{E}_f$, which shows that $\mathscr{E}_f$ is weakly thinnable. In addition, we get

$$\frac{f(b_{a_1}) + \cdots + f(b_{a_n})}{f(1) + \cdots + f(b_{a_n})} \ge \frac{O(1) + f(b_k) + \cdots + f(b_{kn})}{f(1) + \cdots + f(b_{kn})} \not\to 0,$$

so that $B_A \notin \mathscr{E}_f$, where the last $\not\to$ comes from a reasoning similar to (2.21). Finally, given infinite subsets $X, Y \subseteq \mathbf{N}$ with canonical enumeration $\{x_n : n \in \mathbf{N}\}$ and $\{y_n : n \in \mathbf{N}\}$, respectively, such that $X \le Y$ and $X \in \mathscr{E}_f$, it holds

$$\frac{f(x_1) + \cdots + f(x_n)}{f(1) + \cdots + f(x_n)} \ge \frac{f(y_1) + \cdots + f(y_n)}{f(1) + \cdots + f(y_n)}$$

for all $n \in \mathbf{N}$ therefore $Y \in \mathscr{E}_f$. $\square$

Given a real $\alpha \ge -1$, let $\mathcal{I}_\alpha$ be the collection of subsets with zero α-density, that is,

$$\mathcal{I}_\alpha := \{S : \mathrm{d}^\star_\alpha(S) = 0\}, \text{ where } \mathrm{d}^\star_\alpha(S) = \limsup_{n\to\infty} \frac{\sum_{i \in S \cap [1,n]} i^\alpha}{\sum_{i \in [1,n]} i^\alpha}. \tag{2.22}$$

Proposition 2.3.4. *All ideals $\mathcal{I}_\alpha$ are thinnable.*

Proof. If $\alpha \in [-1, 0]$, the claim follows by Proposition 2.3.3 (we omit details). Hence, let us suppose hereafter than $\alpha > 0$. Fix infinite sets $X, Y \subseteq \mathbf{N}$ with canonical enumerations $\{x_n : n \in \mathbf{N}\}$ and $\{y_n : n \in \mathbf{N}\}$, respectively, such that $Y \notin \mathcal{I}_\alpha$. Then, there exist an infinite set S such that $|Y \cap [1, y_n]| \ge \lambda y_n$ for all $n \in S$, where $\lambda := 1 - \left(1 - \frac{1}{2}\mathrm{d}^\star_\alpha(Y)\right)^{\frac{1}{\alpha+1}} > 0$. Indeed, in the opposite case, we would have that

$$\begin{aligned} \frac{\alpha+1}{y_n^{\alpha+1}} \sum_{i \le n} y_i^\alpha &\le \frac{\alpha+1}{y_n^{\alpha+1}} \sum_{i \in ((1-\lambda)y_n, y_n]} i^\alpha \\ &\le \left(1 - (1-\lambda)^{\alpha+1}\right)(1 + o(1)) < \frac{2}{3}\,\mathrm{d}^\star_\alpha(Y) \end{aligned}$$

for all sufficiently large n. Since $|Y \cap [1, n]| \le |X \cap [1, n]|$ for all n, we conclude that

$$\frac{1}{x_n^{\alpha+1}} \sum_{i \le n} x_i^\alpha \ge \frac{1}{x_n^{\alpha+1}} \sum_{i \le \lambda y_n} i^\alpha \ge \frac{1}{x_n^{\alpha+1}} \sum_{i \le \lambda x_n} i^\alpha \ge \frac{\lambda^{\alpha+1}}{2}$$

for all large $n \in S$, so that $X \notin \mathcal{I}_\alpha$.

At this point, fix sets $A, B \subseteq \mathbf{N}$ with canonical enumerations $\{a_n : n \in \mathbf{N}\}$ and $\{b_n : n \in \mathbf{N}\}$, respectively, such that A admits asymptotic density $c > 0$ and

$B \notin \mathcal{I}_\alpha$. Fix also $\varepsilon > 0$ sufficiently small and note that there exists $n_0 = n_0(\varepsilon) \in \mathbf{N}$ such that $(1/c - \varepsilon)n \le a_n \le (1/c + \varepsilon)n$ for all $n \ge n_0$. In particular, it follows that

$$\frac{1}{a_{b_n}^{\alpha+1}} \sum_{k \le n} (a_{b_k})^\alpha \ge \frac{1}{\left(\frac{1}{c} + \varepsilon\right)^{\alpha+1} {b_n}^{\alpha+1}} \left(O(1) + \sum_{n_0 \le k \le n} \left(\frac{1}{c} - \varepsilon\right)^\alpha {b_k}^\alpha \right).$$

Therefore, setting $\kappa := \min\left\{ \left(\frac{1}{c} + \varepsilon\right)^{-\alpha-1}, \left(\frac{1}{c} - \varepsilon\right)^\alpha \right\} > 0$, we obtain

$$\begin{aligned} \frac{\mathrm{d}^\star_\alpha(A_B)}{\alpha+1} &= \limsup_{n \to \infty} \frac{1}{a_{b_n}^{\alpha+1}} \sum_{k \le n} (a_{b_k})^\alpha \\ &\ge \limsup_{n \to \infty} \frac{\kappa}{{b_n}^{\alpha+1}} \left(O(1) + \sum_{n_0 \le k \le n} \kappa {b_k}^\alpha \right) \\ &= \kappa^2 \limsup_{n \to \infty} \frac{1}{{b_n}^{\alpha+1}} \sum_{n_0 \le k \le n} {b_k}^\alpha = \kappa^2 \frac{\mathrm{d}^\star_\alpha(B)}{\alpha+1} > 0. \end{aligned}$$

This proves that $A_B \notin \mathcal{I}_\alpha$. Finally, let k be an integer greater than $1/c$ and note that $B_A \le B_{k\mathbf{N}} \setminus S$, for some finite set S. By the previous observation, it is sufficient to show that $B_{k\mathbf{N}} \notin \mathcal{I}_\alpha$ and this is straightforward by an analogous argument of (2.21). $\square$

To mention another example, let $\mathcal{I}_\mathfrak{p}$ be the *Pólya ideal*, that is,

$$\mathcal{I}_\mathfrak{p} := \{ S \subseteq \mathbf{N} : \mathfrak{p}^\star(S) = 0 \},$$

where

$$\mathfrak{p}^\star(S) = \lim_{s \to 1^-} \limsup_{n \to \infty} \frac{|S \cap [ns, n]|}{(1-s)n}.$$

Among other things, the upper Pólya density $\mathfrak{p}^*$ has found a number of remarkable applications in analysis and economic theory, see e.g. Pólya (1929), Levinson (1940) and Marinacci (1998).

Corollary 2.3.5. *The Pólya ideal $\mathcal{I}_\mathfrak{p}$ is thinnable.*

Proof. The upper Pólya density $\mathfrak{p}^*$ is the pointwise limit of the real net of the upper α-densities $\mathrm{d}^\star_\alpha$ defined in (2.22), see (Letavaj et al., 2015, Theorem 4.3).

Fix infinite sets $X, Y \subseteq \mathbf{N}$ with canonical enumerations $\{x_n : n \in \mathbf{N}\}$ and $\{y_n : n \in \mathbf{N}\}$, respectively, such that $Y \notin \mathcal{I}_\mathfrak{p}$. Then, there exists $\alpha > 0$ such that $\mathrm{d}^\star_\alpha(Y) > 0$ and, thanks to Proposition 2.3.4, we get $\mathrm{d}^\star_\alpha(X) > 0$ as well. This implies that $X \notin \mathcal{I}_\mathfrak{p}$. Other properties can be shown similarly. □

Lastly, it is worth noting that there exist summable ideals which are not weakly thinnable: for instance, let $\mathcal{I}_f$ be the ideal defined by $f(2n) = 1$ and $f(2n-1) = 0$ for all $n \in \mathbf{N}$, so that

$$\mathcal{I}_f = \{I \subseteq \mathbf{N} : I \cap 2\mathbf{N} \in \mathrm{Fin}\}.$$

Set $A := \mathbf{N} \setminus \{1\}$ and $B := 2\mathbf{N}$. Then, A has asymptotic density 1, $B \notin \mathcal{I}_f$, and $A_B = 2\mathbf{N} + 1 \in \mathcal{I}_f$. Therefore $\mathcal{I}_f$ is not weakly thinnable.

2.3.2 Invariance of $\mathcal{I}$-cluster points

Consider the natural bijection between the collection of all subsequences (x_{n_k}) of (x_n) and real numbers $\omega \in (0, 1]$ with non-terminating dyadic expansion

$$\sum_{i \geq 1} d_i(\omega) 2^{-i},$$

where $d_i(\omega) = 1$ if $i = n_k$, for some integer k, and $d_i(\omega) = 0$ otherwise, cf. (Billingsley, 1995, Appendix A31) and Miller (1995). Accordingly, for each $\omega \in (0, 1]$, denote by $x \restriction \omega$ the subsequence of (x_n) obtained by omitting x_i if and only if $d_i(\omega) = 0$.

Moreover, let $\lambda : \mathscr{M} \to \mathbf{R}$ denote the Lebesgue measure, where $\mathscr{M}$ stands for the completion of the Borel σ-algebra on $(0, 1]$. Our main result, contained in Leonetti (2018b), follows:

Theorem 2.3.6. *Let $\mathcal{I}$ be a thinnable ideal and (x_n) be a sequence taking values in a first countable space X where all closed sets are separable. Then*

$$\lambda\left(\{\omega \in (0, 1] : \Gamma_x(\mathcal{I}) = \Gamma_{x \restriction \omega}(\mathcal{I})\}\right) = 1.$$

Proof. Let Ω be the set of normal numbers, that is,

$$\Omega := \left\{\omega \in (0, 1] : \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} d_i(\omega) = \frac{1}{2}\right\}. \tag{2.23}$$

It follows by Borel's normal number theorem (Billingsley, 1995, Theorem 1.2) that $\Omega \in \mathscr{M}$ and $\lambda(\Omega) = 1$. Then, it is claimed that

$$\Gamma_{x \restriction \omega}(\mathcal{I}) \subseteq \Gamma_x(\mathcal{I}) \quad \text{for all } \omega \in \Omega. \tag{2.24}$$

To this aim, fix $\omega \in \Omega$ and denote by (x_{n_k}) the subsequence $x \restriction \omega$. Let us suppose for the sake of contradiction that $\Gamma_{x \restriction \omega}(\mathcal{I}) \setminus \Gamma_x(\mathcal{I}) \neq \emptyset$ and fix a point ℓ therein. Then, the set of indexes $\{n_k : k \in \mathbf{N}\}$ has asymptotic density $1/2$ and, for each neighborhood U of ℓ, it holds $\{k : x_{n_k} \in U\} \notin \mathcal{I}$. This implies

$$\{n : x_n \in U\} \supseteq \{n_k : x_{n_k} \in U\} \notin \mathcal{I}$$

by the hypothesis that $\mathcal{I}$ is, in particular, weakly thinnable. Therefore $\{n : x_n \in U\} \notin \mathcal{I}$, which is a contradiction since ℓ would be also a $\mathcal{I}$-cluster point of x. This proves (2.24).

To complete the proof, it is sufficient to show that

$$\lambda\left(\{\omega \in (0,1] : \Gamma_x(\mathcal{I}) \subseteq \Gamma_{x \restriction \omega}(\mathcal{I})\}\right) = 1. \tag{2.25}$$

This is clear if $\Gamma_x(\mathcal{I})$ is empty. Otherwise, note that $\Gamma_x(\mathcal{I})$ is closed, hence there exists a non-empty countable dense subset L. Fix $\ell \in L$ and let (U_m) be a decreasing local base of neighborhoods at ℓ. Fix also $m \in \mathbf{N}$ and define $I := \{n : x_n \in U_m\}$ which does not belong to $\mathcal{I}$; in particular, I is infinite and we let $\{i_n : n \in \mathbf{N}\}$ be its enumeration. Again by Borel's normal number theorem,

$$\Theta(\ell, U_m) := \left\{\omega \in (0,1] : \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} d_{i_j}(\omega) = \frac{1}{2}\right\}$$

belongs to $\mathscr{M}$ and has Lebesgue measure 1. Fix ω in the above set and denote by (x_{n_k}) the subsequence $x \restriction \omega$. Hence, the set $J := \{n : i_n \in \{n_k : k \in \mathbf{N}\}\}$ admits asymptotic density $1/2$ and, by the thinnability of $\mathcal{I}$, we get $I_J \notin \mathcal{I}$. Lastly, note that

$$\{k : x_{n_k} \in U_m\} = \{k : n_k \in I\} \leq \{n_k : n_k \in I\} = I_J.$$

Therefore $\{k : x_{n_k} \in U_m\} \notin \mathcal{I}$. In addition, $\Theta(\ell) := \bigcap_{m \geq 1} \Theta(\ell, U_m)$ belongs to $\mathscr{M}$ and has Lebesgue measure 1. This implies that

$$\lambda\left(\{\omega \in (0,1] : \ell \in \Gamma_{x \restriction \omega}(\mathcal{I})\}\right) = 1.$$

(See also (Rao et al., 1977, Theorem 1) for the case $\mathcal{I} = \mathrm{Fin}$.) At this point, since L is countable, we get $\lambda(\{\omega \in (0,1] : L \subseteq \Gamma_{x\restriction\omega}(\mathcal{I})\}) = 1$. Claim (2.25) follows by the fact that also $\Gamma_{x\restriction\omega}(\mathcal{I})$ is closed, so that each of these $\Gamma_{x\restriction\omega}(\mathcal{I})$ contains the closure of L, i.e., $\Gamma_x(\mathcal{I})$. $\square$

Remark 2.3.7. Separable metric spaces X satisfy the hypotheses of Theorem 2.3.6. Indeed, X is first countable and every closed subset F of X is separable. To prove the latter, let A be a countable dense subset of X and note that

$$\mathscr{F} := \{B(a,r) \cap F : a \in A, 0 < r \in \mathbf{Q}\} \setminus \{\emptyset\}$$

is a base for F, where $B(a,r)$ is the open ball with center a and radious r. Then, a set which picks one point for every set in $\mathscr{F}$ is a countable dense subset of F.

As a consequence of Proposition 2.3.4, Theorem 2.3.6, and Remark 2.3.7, we obtain:

Corollary 2.3.8. *Let x be a sequence taking values in a separable metric space. Then the set of statistical cluster points of x is equal to the set of statistical cluster points of almost all its subsequences (in the sense of Lebesgue measure).*

Similarly, setting $\mathcal{I} = \mathrm{Fin}$, we recover Buck's result Buck (1944):

Corollary 2.3.9. *Let x be a sequence taking values in a separable metric space. Then the set of ordinary limit points of x is equal to the set of ordinary limit points of almost all its subsequences (in the sense of Lebesgue measure).*

Lastly, we recall that a sequence $x = (x_n)$ taking values in topological space X converges (with respect to an ideal $\mathcal{I}$) to $\ell \in X$, shortened as $x \to_{\mathcal{I}} \ell$, if

$$\{n : x_n \notin U\} \in \mathcal{I}$$

for all neighborhoods U of ℓ. In this regard, Miller (Miller, 1995, Theorem 3) proved that a real sequence x converges statistically to ℓ, i.e., $x \to_{\mathcal{I}_0} \ell$, if and only if almost all its sequences converge statistically to ℓ.

This is extended in the following result. Here, we say that an ideal $\mathcal{I}$ is *invariant* if, for each $A \subseteq \mathbf{N}$ with positive asymptotic density, it holds $A_B \notin \mathcal{I}$ if and only if $B \notin \mathcal{I}$ (in particular, $\mathcal{I}$ is weakly thinnable). This condition is strictly related with the so-called "property (G)" defined in Balcerzak et al. (2016).

Theorem 2.3.10. *Let $\mathcal{I}$ be an invariant ideal and x be a sequence taking values in a topological space. Then $x \to_{\mathcal{I}} \ell$ if and only if*

$$\lambda\left(\{\,\omega \in (0,1] : x \upharpoonright \omega \to_{\mathcal{I}} \ell\,\}\right) = 1.$$

Proof. First, let us suppose that $x \to_{\mathcal{I}} \ell$ and let U be a neighborhood of ℓ. Let Ω be set of normal numbers defined in (2.23), fix $\omega \in \Omega$, and denote by (x_{n_k}) the subsequence $x \upharpoonright \omega$. Then $I := \{n : x_n \notin U\} \in \mathcal{I}$ and $A := \{n_k : k \in \mathbf{N}\}$ has asymptotic density $1/2$. Define $B := \{k : x_{n_k} \notin U\} = \{k : n_k \in I\}$. Since $\mathcal{I}$ is, in particular, weakly thinnable and $A_B = \{n_k : x_{n_k} \notin U\} \in \mathcal{I}$, it follows that $B \in \mathcal{I}$, i.e., $x \upharpoonright \omega \to_{\mathcal{I}} \ell$.

Conversely, note that $\lambda(\Omega \cap (1-\Omega)) = 1$. Hence, there exists $\omega \in \Omega$ such that $x \upharpoonright \omega \to_{\mathcal{I}} \ell$ and $x \upharpoonright (1-\omega) \to_{\mathcal{I}} \ell$. It easily follows that $x \to_{\mathcal{I}} \ell$. Indeed, denoting by (x_{n_k}) and (x_{m_r}) the subsequences $x \upharpoonright \omega$ and $x \upharpoonright (1-\omega)$, respectively, we have that, for each neighborhood U of ℓ, it holds $\{k : x_{n_k} \notin U\} \in \mathcal{I}$ and $\{r : x_{m_r} \notin U\} \in \mathcal{I}$. Since $\{n_k : k \in \mathbf{N}\}$ and $\{m_r : r \in \mathbf{N}\}$ form a partition of $\mathbf{N}$, then

$$\{n : x_n \notin U\} = \{n_k : x_{n_k} \notin U\} \cup \{m_r : x_{m_r} \notin U\}.$$

The claim follows by the hypothesis that $\mathcal{I}$ is invariant. $\square$

Finally, note that that it is not possible to extend Theorem 2.3.10 on the class of all ideals: indeed, it has been shown in (Balcerzak et al., 2016, Example 2) that there exists an ideal $\mathcal{I}$ and a real sequence x such that $x \to_{\mathcal{I}} \ell$ and, on the other hand, $\lambda\left(\{\,\omega \in (0,1] : x \upharpoonright \omega \to_{\mathcal{I}} \ell\,\}\right) = 0$.

2.3.3 Strong thinnability

To extend Theorem 2.3.6 to the case of $\mathcal{I}$-limit points, we need to define also the notion of strong thinnability:

Definition 2.3.11. An analytic P-ideal $\mathcal{I}_\varphi$ is said to be *strongly thinnable* if:

(i) $\mathcal{I}_\varphi$ is weakly thinnable;

(ii) given $q>0$ and a set $A \subseteq \mathbf{N}$ with asymptotic density $a>0$, there exists $c=c(q,a)>0$ such that $\|B_A\|_\varphi \ge cq$ whenever $\|B\|_\varphi \ge q$;

(iii) there exists $c>0$ such that $\|X\|_\varphi \ge c\|Y\|_\varphi$ whenever $X \le Y$.

A moment thought reveals that strongly thinnability is just a refinement of thinnability, considering that $\|\cdot\|_\varphi$ allows us to quantify the "largeness" of subsets of $\mathbf{N}$.

Proposition 2.3.12. *Let $f:\mathbf{N}\to(0,\infty)$ be a definitively non-increasing function such that $\sum_{n\ge 1} f(n)=\infty$. In addition, suppose that*

$$\liminf_{n\to\infty} \frac{\sum_{i\in[1,n]} f(i)}{\sum_{i\in[1,kn]} f(i)} \neq 0 \quad \textit{for all } k \in \mathbf{N} \tag{2.26}$$

and define the Erdős–Ulam ideal

$$\mathscr{E}_f := \left\{ S\subseteq \mathbf{N} : \lim_{n\to\infty} \frac{\sum_{i\in S\cap[1,n]} f(i)}{\sum_{i\in[1,n]} f(i)} = 0 \right\}.$$

Then, $\mathscr{E}_f$ is a strongly thinnable analytic P-ideal provided that $\mathscr{E}_f$ is strechable, i.e., $kA \notin \mathscr{E}_f$ for all $k\in\mathbf{N}$ and $A\notin\mathscr{E}_f$.

Proof. First, note that $\mathscr{E}_f$ is a Erdős–Ulam ideal, indeed $f(n)=o(f(1)+\cdots+f(n))$ as $n\to\infty$ since f is non-increasing, cf. (Farah, 2000, Section 1.13). Moreover, the weak thinnability of $\mathscr{E}_f$, i.e., property (i), has been shown in Proposition 2.3.3.

Let φ be a lower semicontinuous submeasure associated with $\mathscr{E}_f$. Then, it follows from the proof of (Farah, 2000, Theorem 1.13.3) that there exists a strictly increasing sequence of positive integers (z_n) such that

$$\lim_{n\to\infty} \frac{\sum_{s\in(z_n,z_{n+1}]} f(s)}{\sum_{s\in[1,z_n]} f(s)} = 1 \tag{2.27}$$

and $\|S\|_\varphi = \lim_{n\to\infty} g_n(S)$ for all $S\subseteq\mathbf{N}$, where

$$g_n(S) := \sup_{k\in\mathbf{N}} \frac{\sum_{s\in S\cap(z_k,z_{k+1}]\setminus\{1,\ldots,n\}} f(s)}{\sum_{s\in[1,z_k]} f(s)}.$$

Considering that $g_n(S) \downarrow \|S\|_\varphi$, then also $g_{z_n}(S) \downarrow \|S\|_\varphi$. Hence

$$\|S\|_\varphi = \inf_{n\to\infty} g_{z_n}(S) = \limsup_{n\to\infty} \frac{\sum_{s\in S\cap(z_n,z_{n+1}]} f(s)}{\sum_{s\in[1,z_n]} f(s)}. \tag{2.28}$$

Replacing φ with $\frac{1}{2}\varphi$ (which is possible since $\mathcal{I}_\varphi = \mathcal{I}_{\frac{1}{2}\varphi}$), we obtain by (2.27) and (2.28) that

$$\|S\|_\varphi = \limsup_{n\to\infty} \frac{\sum_{s\in S\cap(z_n,z_{n+1}]} f(s)}{\sum_{s\in[1,z_{n+1}]} f(s)}. \tag{2.29}$$

At this point, fix $q > 0$ and let B be a set of integers such that $\|B\|_\varphi \ge q$. Given $a \in (0,1]$, fix also a set A with canonical enumeration $\{a_n : n \in \mathbf{N}\}$ such that A admits asymptotic density a and set $r := \lfloor 1/a \rfloor + 1$. Then, it follows by (2.29) that

$$\begin{aligned}
\|B_A\|_\varphi &= \limsup_{n\to\infty} \frac{\sum_{z_n<s\le z_{n+1}, s=b_{a_k} \text{ for some } k} f(b_{a_k})}{\sum_{s\in[1,z_{n+1}]} f(s)} \\
&\ge \limsup_{n\to\infty} \frac{O(1) + \sum_{z_n<s\le z_{n+1}, s=b_{a_k} \text{ for some } k} f(b_{rk})}{\sum_{s\in[1,z_{n+1}]} f(s)}
\end{aligned}$$

Hence, considering that by (2.27) it holds $z_{n+1} - z_n \ge z_n \ge n \to \infty$ and

$$\begin{aligned}
\sum_{\substack{s\in S\cap(z_n,z_{n+1}],\\ s\equiv 0 \bmod r}} f(s) \ge O(1) + \sum_{\substack{s\in S\cap(z_n,z_{n+1}],\\ s\equiv 1 \bmod r}} f(s) &\ge \cdots \\
\ge O(1) + \sum_{\substack{s\in S\cap(z_n,z_{n+1}],\\ s\equiv r-1 \bmod r}} f(s)& \\
\ge O(1) + \sum_{\substack{s\in S\cap(z_n,z_{n+1}],\\ s\equiv 0 \bmod r}} f(s)&
\end{aligned}$$

for every $S \subseteq \mathbf{N}$, we obtain that

$$\|B_A\|_\varphi \ge \limsup_{n\to\infty} \frac{\sum_{z_n<s\le z_{n+1}, s=b_{a_k} \text{ for some } k} f(b_{rk})}{\sum_{s\in[1,z_{n+1}]} f(s)} \ge \frac{\|B\|_\varphi}{r} \ge \frac{q}{r},$$

which proves property (ii).

Finally, fix sets $X, Y \subseteq \mathbf{N}$ with $X \le Y$ and define

$$h_n(X) := \frac{\sum_{s\in X\cap(z_n,z_{n+1}]} f(s)}{\sum_{s\in[1,z_{n+1}]} f(s)}$$

and

$$h_n(Y) := \frac{\sum_{s \in Y \cap (z_n, z_{n+1}]} f(s)}{\sum_{s \in [1, z_{n+1}]} f(s)}$$

for each $n \in \mathbf{N}$. It follows by (2.29) that there exists an infinite set $\mathcal{N}$ such that $h_n(Y) \geq \frac{1}{2}\|Y\|_\varphi$ for all $n \in \mathcal{N}$. Set also $\mu_n := \sum_{s \in [1, z_{n+1}]} f(s)$ for each n. Then, thanks to (2.27) and the hypothesis $X \leq Y$, we obtain

$$\begin{aligned} h_n(Y)\mu_n = \sum_{s \in Y \cap (z_n, z_{n+1}]} f(s) \leq \sum_{s \in X \cap [1, z_{n+1}]} f(s) \\ = O(1) + \sum_{i=1}^{n} h_i(X)\mu_i \\ \leq O(1) + \mu_n \sum_{i=1}^{n} \left(\frac{2}{3}\right)^{n-i} h_i(X) \end{aligned}$$

for each $n \in \mathcal{N}$. Since $\mu_n \to \infty$ by hypothesis, it follows that

$$h_n(Y) \leq \sum_{i=1}^{n} \left(\frac{2}{3}\right)^{n-i} h_i(X)$$

whenever $n \in \mathcal{N}$ is sufficiently large. Then $\|X\|_\varphi = \limsup_n h_n(X) \geq \frac{1}{6}\|Y\|_\varphi$: indeed, in the opposite, we would get

$$\frac{1}{2}\|Y\|_\varphi \leq h_n(Y) \leq \frac{1}{6}\|Y\|_\varphi \sum_{i=1}^{n} \left(\frac{2}{3}\right)^{n-i} < \frac{1}{2}\|Y\|_\varphi$$

for each sufficiently large $n \in \mathcal{N}$. This proves property (iii), concluding the proof. □

Note that, for each real parameter $\alpha \geq -1$, the ideal $\mathcal{I}_\alpha$ is an Erdős–Ulam ideal.

Recalling that every Erdős–Ulam ideal is a density ideal (hence, in particular, an analytic P-ideal), see e.g. (Farah, 2000, Theorem 1.13.3), the following is immediate by Proposition 2.3.12 (we omit details):

Corollary 2.3.13. *$\mathcal{I}_\alpha$ is a strongly thinnable analytic P-ideal whenever $\alpha \in [-1, 0]$.*

2.3.4 Invariance of $\mathcal{I}$-limit points

As in the proof of Theorem 2.3.6, let Ω be the set of normal numbers, i.e.,

$$\Omega := \left\{\omega \in (0,1] : \lim_{n\to\infty} \frac{d_1(\omega)+\cdots+d_n(\omega)}{n} = \frac{1}{2}\right\}.$$

Lemma 2.3.14. *Let $\mathcal{I}$ be a weakly thinnable ideal and let $x=(x_n)$ be a sequence taking values in a topological space. Then*

$$\lambda\left(\{\omega \in (0,1] : \Lambda_{x\restriction\omega}(\mathcal{I}) \subseteq \Lambda_x(\mathcal{I})\}\right) = 1.$$

Proof. It follows by Borel's normal number theorem (Billingsley, 1995, Theorem 1.2) that $\Omega \in \mathscr{M}$ and $\lambda(\Omega)=1$. Fix $\omega \in \Omega$ and denote by (x_{n_k}) the subsequence $x \restriction \omega$. Let us suppose that $\Lambda_{x\restriction\omega}(\mathcal{I}) \setminus \Lambda_x(\mathcal{I}) \neq \emptyset$ and fix a point ℓ therein. Then, the set of indexes $\{n_k : k \in \mathbf{N}\}$ has asymptotic density $1/2$ and, by hypothesis, there exists a subsequence $\left(x_{n_{k_m}}\right)$ of (x_{n_k}) such that $\{k_m : m \in \mathbf{N}\} \notin \mathcal{I}$ and $\lim_m x_{n_{k_m}} = \ell$. On the other hand, since $\mathcal{I}$ is weakly thinnable, the set $\{n_{k_m} : m \in \mathbf{N}\}$ does not belong to $\mathcal{I}$. Considering that $\left(x_{n_{k_m}}\right)$ is clearly a subsequence of (x_n), it follows that $\ell \in \Lambda_x(\mathcal{I})$, which contradicts our assumption. This proves that $\Lambda_{x\restriction\omega}(\mathcal{I}) \subseteq \Lambda_x(\mathcal{I})$ for all $\omega \in \Omega$. □

Finally, we can state the analogue of Theorem 2.3.6 for $\mathcal{I}$-limit points:

Theorem 2.3.15. *Let $\mathcal{I}_\varphi$ be a strongly thinnable analytic P-ideal and let $x=(x_n)$ be a sequence taking values in a separable metric space. Then*

$$\lambda\left(\{\omega \in (0,1] : \Lambda_x(\mathcal{I}_\varphi) = \Lambda_{x\restriction\omega}(\mathcal{I}_\varphi)\}\right) = 1.$$

Proof. Thanks to Lemma 2.3.14, it is sufficient to show that

$$\lambda\left(\{\omega \in (0,1] : \Lambda_x(\mathcal{I}_\varphi) \subseteq \Lambda_{x\restriction\omega}(\mathcal{I}_\varphi)\}\right) = 1. \tag{2.30}$$

This is clear if $\Lambda_x(\mathcal{I}_\varphi)$ is empty. Otherwise, let us suppose hereafter that $\Lambda_x(\mathcal{I}_\varphi) \neq \emptyset$. Note that, by the σ-subadditivity of λ, Claim (2.30) would follow from

$$\lambda\left(\{\omega \in (0,1] : \Lambda_x(\mathcal{I}_\varphi, q) \subseteq \Lambda_{x\restriction\omega}(\mathcal{I}_\varphi)\}\right) = 1 \tag{2.31}$$

for each (rational) $q > 0$. At this point, recall that each

$$\Lambda_x(\mathcal{I}_\varphi, q) := \left\{\ell \in X : \lim_{n\to\infty,\, n\in A} x_n = \ell \text{ for some } A \subseteq \mathbf{N} \text{ with } \|A\|_\varphi \ge q\right\}$$

is closed and observe that, since X is a separable metric space, every closed set is separable. Hence, fix a sufficiently small $q > 0$ such that $\Lambda_x(\mathcal{I}_\varphi, q) \neq \emptyset$ and let L be a (non-empty) countable subset with closure $\Lambda_x(\mathcal{I}_\varphi, q)$.

Fix $\ell \in L$. By hypothesis there exists a subsequence (x_{n_k}) such that $\lim_k x_{n_k} = \ell$ and $\|A\|_\varphi \ge q$, where $A := \{n_k : k \in \mathbf{N}\}$. Define the set

$$\Theta_\ell := \left\{\omega \in (0,1] : \lim_{k\to\infty} \frac{d_{n_1}(\omega) + \cdots + d_{n_k}(\omega)}{k} = \frac{1}{2}\right\}.$$

It follows again by Borel's normal number theorem that $\Theta_\ell \in \mathscr{M}$ and $\lambda(\Theta_\ell) = 1$. Fix also $\omega \in \Theta_\ell$ and denote by (x_{m_k}) the subsequence $x \upharpoonright \omega$. Then, letting $B := \{m_k : k \in \mathbf{N}\}$, we obtain that $A \cap B$ admits asymptotic density $1/2$ relative to A, i.e., the set $K := \{k : n_k \in B\}$ admits asymptotic density $1/2$. Since $\mathcal{I}_\varphi$ is strongly thinnable, there exists a positive constant $\kappa = \kappa(q)$ such that

$$\|A_K\|_\varphi = \|A \cap B\|_\varphi \ge \kappa q.$$

In addition, since $C := \{k : m_k \in A_K\} \le A_K$, we get by the strongly thinnability of $\mathcal{I}_\varphi$ that $\|C\|_\varphi \ge cq$, for some $c > 0$. It follows by construction that the subsequence $(x_{m_k} : k \in C)$ of $(x_{m_k} : k \in \mathbf{N})$ converges to ℓ, hence $\ell \in \Lambda_{x\upharpoonright\omega}(\mathcal{I}_\varphi, cq)$ for all $\omega \in \Theta_\ell$.

Thus, define $\Theta := \bigcap_{\ell \in L} \Theta_\ell$ and note that $\Theta \in \mathscr{M}$ and $\lambda(\Theta) = 1$. Therefore $\lambda\left(\{\omega \in \Theta : L \subseteq \Lambda_{x\upharpoonright\omega}(\mathcal{I}_\varphi, cq)\}\right) = 1$. On the other hand, each $\Lambda_{x\upharpoonright\omega}(\mathcal{I}_\varphi, cq)$ is closed, hence it contains the closure of L, that is,

$$\lambda\left(\{\omega \in \Theta : \Lambda_x(\mathcal{I}_\varphi, q) \subseteq \Lambda_{x\upharpoonright\omega}(\mathcal{I}_\varphi, cq)\}\right) = 1.$$

This implies (2.31) since $\Lambda_{x\upharpoonright\omega}(\mathcal{I}_\varphi, cq) \subseteq \Lambda_{x\upharpoonright\omega}(\mathcal{I}_\varphi)$, completing the proof. □

As a consequence of Corollary 2.3.13 and Theorem 2.3.15, we obtain:

Corollary 2.3.16. *Let x be a sequence taking values in a separable metric space. Then the set of statistical limit point of x is equal to the set of statistical limit points of almost all its subsequences (in the sense of Lebesgue measure).*

With a similar argument, the following can be shown (we omit details):

Theorem 2.3.17. *Let $\mathcal{I}_f$ be a thinnable summable ideal and (x_n) be a sequence taking values in a separable metric space X. Then*

$$\lambda\left(\{\omega \in (0,1] : \Lambda_x(\mathcal{I}_f) = \Lambda_{x\restriction\omega}(\mathcal{I}_f)\}\right) = 1.$$

We conclude with the relationship between ideal limit points and ideal cluster points of subsequences of a given sequence. Given an ideal $\mathcal{I}$ and a sequence $x = (x_n)$ taking values in a topological space, recall that ℓ is a $\mathcal{I}$-cluster point of (x_n) provided that $\{n : x_n \in U\} \notin \mathcal{I}$ for all neighborhoods U of ℓ. Denoting by $\Gamma_x(\mathcal{I})$ the set of $\mathcal{I}$-cluster points of (x_n), we obtain:

Corollary 2.3.18. *Let x be a sequence taking values in a separable metric space and let $\mathcal{I}$ be a thinnable summable ideal or a strongly thinnable analytic P-ideal. Then*

$$\lambda\left(\{\omega \in (0,1] : \Lambda_{x\restriction\omega}(\mathcal{I}) = \Gamma_{x\restriction\omega}(\mathcal{I})\}\right)$$

is either 0 *or* 1*. In addition, it is* 1 *if and only if* $\Lambda_x(\mathcal{I}) = \Gamma_x(\mathcal{I})$*.*

Proof. Thanks to Theorem 2.3.15, Theorem 2.3.17, and Theorem 2.3.6, it holds

$$\lambda\left(\{\omega \in (0,1] : \Lambda_{x\restriction\omega}(\mathcal{I}) = \Lambda_x(\mathcal{I}) \text{ and } \Gamma_{x\restriction\omega}(\mathcal{I}) = \Gamma_x(\mathcal{I})\}\right) = 1,$$

which is sufficient to prove the claim. □

2.4 Category analogues

In this Section we provide the topological analogues of Theorem 2.3.6 and Theorem 2.3.15, obtaining another example of the "duality" between measure and category, cf. e.g. Oxtoby (1980). In particular, our main results (Theorems 2.4.2 and 2.4.4) imply that the set of subsequences considered by Buck Buck (1944) is always comeager. In addition, they show that the set of subsequences of x which preserve the statistical cluster points [resp., statistical limit points] is meager if and only if there exists an ordinary limit point of x which is not a statistical cluster point of x [resp., statistical limit point].

We recall that, as a consequence of Theorem 2.3.6 and Theorem 2.3.15, the following holds:

Corollary 2.4.1. *Fix $\alpha \in [-1,0]$ and let x be a sequence taking values in a first countable space where all closed sets are separable. Then*

$$\lambda\left(\{\omega \in (0,1] : \Gamma_x(\mathcal{I}_\alpha) = \Gamma_{x\restriction\omega}(\mathcal{I}_\alpha)\}\right) = 1$$

and

$$\lambda\left(\{\omega \in (0,1] : \Lambda_x(\mathcal{I}_\alpha) = \Lambda_{x\restriction\omega}(\mathcal{I}_\alpha)\}\right) = 1.$$

(Here, $\mathcal{I}_\alpha = \{S \subseteq \mathbf{N} : \mathrm{d}^\star_\alpha(S) = 0\}$.) Note that the key observation in the proof of the above result is that the set of normal numbers

$$\Omega := \left\{\omega \in (0,1] : \frac{1}{n}\sum_{i\le n} d_i(\omega) \to \frac{1}{2} \text{ as } n \to \infty\right\}$$

has full Lebesgue measure. On the other hand, it is well known that Ω is a meager set, that is, a set of first category. This suggests that the category analogue of Corollary 2.4.1 does not hold in general. Our main results show that this is indeed the case.

2.4.1 Main Results

Let (I_n) be a partition of $\mathbf{N}$ in non-empty finite sets and $\mu = (\mu_n)$ be a sequence of submeasures such that each μ_n concentrates on I_n and $\limsup_n \mu_n(I_n) \neq 0$. Then, the *generalized density ideal*

$$\mathcal{Z}_\mu := \{A \subseteq \mathbf{N} : \lim_{n\to\infty} \mu_n(A \cap I_n) = 0\} \tag{2.32}$$

is an analytic P-ideal: indeed, it is easy to check that $\mathcal{Z}_\mu = \mathrm{Exh}(\varphi_\mu)$, where $\varphi_\mu := \sup_k \mu_k$ and $\mathrm{Exh}(\varphi)$ is the exaustive ideal $\mathcal{I}_\varphi = \{A : \lim_{n\to\infty} \varphi(A \setminus [1,n]) = 0\}$. The class of generalized density ideals has been introduced by Farah in (Farah, 2002, Section 2.10), see also Farah (2006). In particular, each $\mathcal{Z}_\mu$ is an $F_{\sigma\delta}$-ideal and it is tall if and only if $\lim_n \sup_k \mu_n(\{k\}) = 0$ (an ideal $\mathcal{I}$ is *tall* if each infinite $A \subseteq \mathbf{N}$ has a subset in $\mathcal{I}$).

It is worth noting that generalized density ideals have been used in different contexts, see e.g. Borodulin-Nadzieja et al. (2015); Filipów et al. (2007), and it is a very rich class. Indeed, if each μ_n is a measure then $\mathcal{Z}_\mu$ is a density ideal, as defined in (Farah, 2000, Section 1.13). In particular, it includes $\emptyset \times \mathrm{Fin}$ and also the Erdős–Ulam ideals introduced in Just and Krawczyk (1984), i.e., ideals of the type $\mathrm{Exh}(\varphi_f)$ where $f : \mathbf{N} \to (0, \infty)$ is a function such that $\sum_{n \in \mathbf{N}} f(n) = \infty$ and $f(n) = o\left(\sum_{i \le n} f(i)\right)$ as $n \to \infty$ and $\varphi_f : \mathcal{P}(\mathbf{N}) \to (0, \infty)$ is the submeasure defined by

$$\varphi_f(A) = \sup_{n \in \mathbf{N}} \frac{\sum_{i \le n,\, i \in A} f(i)}{\sum_{i \le n} f(i)},$$

see (Farah, 2000, pp. 42–43). In addition, it contains the ideals associated with suitable modifications of the natural density, the so-called simple density ideals, see Balcerzak et al. (2015). Lastly, a large class of generalized density ideals has been defined by Louveau and Veličković in Louveau and Veličković (1994), cf. also (Farah, 2002, Section 2.11).

Note that also the class of F_σ-ideals is quite large: it contains, among others, all the summable ideals (i.e., P-ideals of the form $\{A : \sum_{n \in A} f(n) < \infty\}$, where $f : \mathbf{N} \to [0, \infty)$ is a function such that $\sum_{n \in \mathbf{N}} f(n) < \infty$, see (Farah, 2000, Section 1.12)), finitely generated ideals $\{A : A \setminus B \in \mathrm{Fin}\}$ for some infinite set B as in (Balcerzak et al., 2016, Example 2), Tsirelson ideals defined in Farah (1999); Veličković (1999), and other non-summable F_σ P-ideals, see (Farah, 2000, Section 1.11).

Our main result follows:

Theorem 2.4.2. *Let x be a sequence taking values in a first countable space X where all closed sets are separable and let $\mathcal{I}$ be a generalized density ideal or an F_σ-ideal. Then*

$$\{\omega \in (0,1] : \Gamma_{x \restriction \omega}(\mathcal{I}) = \Gamma_x(\mathcal{I})\} \tag{2.33}$$

is not meager if and only if $\Gamma_x(\mathcal{I}) = \mathrm{L}_x$. Moreover, in this case, it is comeager.

Since the ideal of finite sets Fin is countable (hence, an F_σ-ideal), we obtain the topological analogue of Buck's result Buck (1944):

Corollary 2.4.3. *Let x be a sequence as in Theorem 2.4.2. Then the set of subsequences which preserve the ordinary limit points of x is comeager.*

We have also the analogue of Theorem 2.4.2 for $\mathcal{I}$-limit points:

Theorem 2.4.4. *Let x be a sequence taking values in a first countable space X where all closed sets are separable and let $\mathcal{I}$ be a generalized density ideal or an F_σ-ideal. Then*

$$\{\omega \in (0,1] : \Lambda_{x\upharpoonright\omega}(\mathcal{I}) = \Lambda_x(\mathcal{I})\} \tag{2.34}$$

is not meager if and only if $\Lambda_x(\mathcal{I}) = \mathrm{L}_x$. Moreover, in this case, it is comeager.

Recalling that Erdős–Ulam ideals are density ideals (hence, in particular, generalized density ideals), the following corollaries are immediate (we omit details):

Corollary 2.4.5. *Let x be a sequence taking values in a separable metric space X and let $\mathcal{I}$ be an Erdős–Ulam ideal. Then the set (2.33) [respectively, the set (2.34)] is not meager if and only if $\Gamma_x(\mathcal{I}) = \mathrm{L}_x$ [resp., $\Lambda_x(\mathcal{I}) = \mathrm{L}_x$].*

In this regard, for each $\alpha \ge -1$, the ideal $\mathcal{I}_\alpha$ is an Erdős–Ulam ideal. In particular, setting $\alpha = 0$ and $X = \mathbf{R}$, we obtain the main result given in Leonetti et al. (2019):

Corollary 2.4.6. *Let x be a real sequence. Then the set of its subsequences which preserve the statistical cluster points [resp., statistical limit points] of x is comeager if and only if it is not meager if and only if every ordinary limit point of x is also a statistical cluster point [resp., statistical limit point] of x.*

2.4.2 Proofs of Theorems 2.4.2 and 2.4.4

We start an easy preliminary observation:

Lemma 2.4.7. *Let x be a sequence taking values in a first countable space and let $\mathcal{I}$ be an ideal. Then $\Lambda_{x\upharpoonright\omega}(\mathcal{I}) \subseteq \mathrm{L}_x$ and $\Gamma_{x\upharpoonright\omega}(\mathcal{I}) \subseteq \mathrm{L}_x$ for each $\omega \in (0,1]$.*

Proof. It follows by $\Lambda_{x\upharpoonright\omega}(\mathcal{I}) \subseteq \Gamma_{x\upharpoonright\omega}(\mathcal{I}) \subseteq \mathrm{L}_{x\upharpoonright\omega} \subseteq \mathrm{L}_x$ for all $\omega \in (0,1]$. □

We proceed with some technical lemmas (for the case of generalized density ideals):

Lemma 2.4.8. *Let x be a sequence taking values in a first countable space X, let $\mathcal{I}$ be a generalized density ideal such that $\mathcal{I} = \mathcal{Z}_\mu$ as in (2.32), and fix $q \in (0, \limsup_{n\to\infty} \mu_n(I_n))$. Fix also $\ell \in X$ and a decreasing local base (U_m) at ℓ. Then, the set*

$$\mathscr{V}_\ell = \mathscr{V}_\ell(x;q) := \left\{\omega \in (0,1] : \limsup_{n\to\infty} \mu_n(A_{\omega,m} \cap I_n) \ge q \text{ for all } m\right\},$$

where $A_{\omega,m} := \{k : x_{n_k} \in U_m\}$ and $(x_{n_k}) := x \upharpoonright \omega$, is either comeager or empty.

Proof. Let us suppose $\mathscr{V}_\ell$ is non-empty, so that, in particular, $\ell \in \mathrm{L}_x$. Then, it is claimed that $\mathscr{V}_\ell^c$ is meager. For each $m, n \in \mathbf{N}$ and $\omega \in (0,1]$ set also $\nu_{\omega,m}(n) := \mu_n(A_{\omega,m} \cap I_n)$ to ease the notation. It follows that

$$\begin{aligned} \mathscr{V}_\ell^c &= \left(\bigcap_{m\ge1}\bigcap_{j\ge1} \{\omega : \nu_{\omega,m}(n) \ge q\,(1-2^{-j}) \text{ for infinitely many } n\}\right)^c \\ &= \bigcup_{m\ge1}\bigcup_{j\ge1} \{\omega : \nu_{\omega,m}(n) < q\,(1-2^{-j}) \text{ for all large } n\} \\ &= \bigcup_{m\ge1}\bigcup_{j\ge1}\bigcup_{t\ge1}\bigcap_{s\ge t} \{\omega : \nu_{\omega,m}(s) < q\,(1-2^{-j})\}\,. \end{aligned}$$

Hence, it is sufficient to show that, for every $q \in (0, \limsup_{n\to\infty} \mu_n(I_n))$, each set $B_{m,t} := \bigcap_{s\ge t} \{\omega : \nu_{\omega,m}(s) < q\}$ is nowhere dense: indeed, this would imply that $\mathscr{V}_\ell$ is comeager.

Equivalently, let us prove that, for each fixed $m, t \in \mathbf{N}$, every non-empty open interval $(a,b) \subseteq (0,1)$ contains a non-empty interval disjoint to $B_{m,t}$. Fix $\omega_0 \in (a,b)$ with finite dyadic representation $\sum_{i=1}^r 2^{-h_i}$ such that $\omega_0 + 2^{-h_r} < b$. Moreover, since $\ell \in \mathrm{L}_x$, there exists $\omega_1 \in (0,1]$ such that $x \upharpoonright \omega_1 \to \ell$, hence

$$\limsup_{s\to\infty} \nu_{\omega_1,m}(s) = \limsup_{n\to\infty} \mu_n(I_n) > 0.$$

It follows that there exists an integer $s_\star > \max(t, h_r)$ such that $d_{s_\star}(\omega_1) = 1$ and $\nu_{\omega_\star,m}(s_\star) \ge q$, where $\omega_\star := \omega_0 + \sum_{h_r < i \le s_\star} d_i(\omega_1)/2^i$. Therefore, each $\omega \in (\omega_\star, \omega_\star + 2^{-s_\star})$ starts with the same binary representation of $\omega_\star$, so that $\nu_{\omega,m}(s_\star) \ge q$ and, in particular, does not belong to $B_{m,t}$. This concludes the proof since $(\omega_\star, \omega_\star + 2^{-s_\star}) \subseteq \left(\omega_0, \omega_0 + 2^{-h_r}\right)$ which, in turn, is contained in (a,b). $\square$

Lemma 2.4.9. *With the same notation of Lemma 2.4.8, it holds*

$$\{\omega \in (0,1] : \ell \in \Lambda_{x\restriction\omega}(\mathcal{I})\} = \bigcup_{q>0} \mathscr{V}_\ell(x;q).$$

Proof. Let us fix $\omega \in (0,1]$ such that $\ell \in \Lambda_{x\restriction\omega}(\mathcal{I})$, i.e., there exist $\eta \in (0,1]$ and $q \geq 0$ such that the subsequence $(x \restriction \omega) \restriction \eta \to \ell$ and $\limsup_j \mu_j(\{k_t : t \in \mathbf{N}\} \cap I_j) \geq q$, where we denote by (x_{n_k}) and $(x_{n_{k_t}})$ the subsequences $x \restriction \omega$ and $(x \restriction \omega) \restriction \eta$, respectively. Then, for each $m \in \mathbf{N}$ there is a finite set $F \in \mathrm{Fin}$ such that

$$\begin{aligned}
\limsup_{j\to\infty} \mu_j(\{k : x_{n_k} \in U_m\} \cap I_j) &\geq \limsup_{j\to\infty} \mu_j(\{k_t : x_{n_{k_t}} \in U_m\} \cap I_j) \\
&= \limsup_{j\to\infty} \mu_j((\{k_t : t \in \mathbf{N}\} \setminus F) \cap I_j) \\
&= \limsup_{j\to\infty} \mu_j(\{k_t : t \in \mathbf{N}\} \cap I_j) \geq q.
\end{aligned}$$

This implies that $\omega \in \mathscr{V}_\ell(x;q)$.

Conversely, let us fix $\omega \in \mathscr{V}_\ell(x;q)$ for some $q > 0$, that is, $\limsup_j \mu_j(\{k : x_{n_k} \in U_m\} \cap I_j) \geq q$ for all m. Hence, there exists an increasing sequence (j_r) of positive integers such that

$$\mu_{j_r}(\{k : x_{n_k} \in U_m\} \cap I_{j_r}) \geq q\left(1 - \tfrac{1}{2^r}\right)$$

for all r. Then, define the subsequence $(x_{n_{k_t}})$ of (x_{n_k}) by picking the index k if and only if there exists $r \in \mathbf{N}$ for which $x_{n_k} \in U_{j_r}$ and $k \in I_{j_r}$. It follows by construction that $\ell \in \Lambda_{x\restriction\omega}(\mathcal{I})$. $\square$

Corollary 2.4.10. *Let x be a sequence taking values in a first countable space X, let $\mathcal{I}$ be a generalized density ideal, and assume that $\mathrm{L}_x \neq \emptyset$. Then*

$$\{\omega \in (0,1] : \ell \in \Lambda_{x\restriction\omega}(\mathcal{I})\}$$

is comeager for every $\ell \in \mathrm{L}_x$.

Proof. Fix $\ell \in \mathrm{L}_x$. Then, there exists $\omega_0 \in (0,1]$ such that $x \restriction \omega_0 \to \ell$. Hence, given $q_0 \in (0, \limsup_{n\to\infty} \mu_n(I_n))$, the set $\mathscr{V}_\ell(x;q_0)$ contains ω_0; in particular, it is non-empty and, thanks to Lemma 2.4.8, it is comeager. Lastly, the claim follows by the fact that, thanks to Lemma 2.4.9, the inclusion $\mathscr{V}_\ell(x;q_0) \subseteq \{\omega : \ell \in \Lambda_{x\restriction\omega}(\mathcal{I})\}$ holds. $\square$

Corollary 2.4.11. *Fix x and $\mathcal{I}$ as in Corollary 2.4.10. Then the set*

$$\{\omega \in (0,1] : \ell \in \Gamma_{x\upharpoonright\omega}(\mathcal{I})\}$$

is comeager for every $\ell \in \mathrm{L}_x$.

Proof. An ordinary limit point $\ell \in \mathrm{L}_x$ is an $\mathcal{I}$-cluster point of $x \upharpoonright \omega$ whenever it is an $\mathcal{I}$-limit point of $x \upharpoonright \omega$. Hence, the claim follows by Corollary 2.4.10. □

The analogue of Corollary 2.4.11 holds also for F_σ ideals:

Lemma 2.4.12. *Let x be a sequence taking values in a first countable space X, let $\mathcal{I}$ be an F_σ-ideal, and assume that $\mathrm{L}_x \neq \emptyset$. Then the set $\{\omega \in (0,1] : \ell \in \Gamma_{x\upharpoonright\omega}(\mathcal{I})\}$ is comeager for every $\ell \in \mathrm{L}_x$.*

Proof. Let φ be a lower semicontinuous submeasure such that $\mathcal{I} = \mathrm{Fin}(\varphi)$ as in (2.5). Fix $\ell \in \mathrm{L}_x$ and let (U_m) be a decreasing local base at ℓ. Then, we need to show that $S := \{\omega : \ell \notin \Gamma_{x\upharpoonright\omega}(\mathcal{I})\}$ is meager. Note that $S = \bigcup_{m\ge 1}\bigcup_{k\ge 1} S_{m,k}$, where

$$S_{m,k} := \{\omega \in (0,1] : \varphi\left(\{n : (x \upharpoonright \omega)_n \in U_m\}\right) \le k\}$$

for all $m,k \in \mathbf{N}$. At this point, it is sufficient to show that each $S_{m,k}$ is nowhere dense.

Fix $\omega_0 \in S^c_{m,k} := (0,1] \setminus S_{m,k}$ (since $\ell \in \mathrm{L}_x$ then $S^c_{m,k} \neq \emptyset$). By the lower semicontinuity of φ, there exists $n_0 \in \mathbf{N}$ such that

$$\varphi(\{n \le n_0 : (x \upharpoonright \omega)_n \in U_m\}) > k.$$

It follows that $\omega \in S^c_{m,k}$ whenever $d_n(\omega) = d_n(\omega_0)$ for all $n \le n_0$. Hence $S_{m,k}$ is closed. Finally, we need to show that $S_{m,k}$ contains no nonempty open sets. Fix $\omega_1 \in (0,1]$ such that the subsequence $x \upharpoonright \omega_1$ converges to ℓ and let us suppose for the sake of contradiction that there exists $e_1,\ldots,e_{n_1} \in \{0,1\}$ such that $\omega \in S_{m,k}$ whenever $d_n(\omega) = e_n$ for all $n \le n_1$. Define $\omega^\star := \sum_{n\le n_1} e_n/2^n + \sum_{n>n_1} d_n(\omega_1)/2^n$. Then $\omega^\star \in S_{m,k}$ and, on the other hand, the subsequence $x \upharpoonright \omega^\star$ converges to ℓ, which gives the desired contradiction. □

Corollary 2.4.13. *Fix x and $\mathcal{I}$ as in Lemma 2.4.12. Then the set*

$$\{\omega \in (0,1] : \ell \in \Lambda_{x\restriction\omega}(\mathcal{I})\}$$

is comeager for every $\ell \in \mathrm{L}_x$.

Proof. Since $\mathcal{I}$ an F_σ-ideal and X is first countable, it follows by Theorem 2.2.3 that $\Lambda_{x\restriction\omega}(\mathcal{I}) = \Gamma_{x\restriction\omega}(\mathcal{I})$ for all $\omega \in (0,1]$. Hence, the claim is obtained by Lemma 2.4.12. □

Lastly, we show that a certain subset of $\mathcal{I}$-limit points $\ell \in X$ is closed.

Lemma 2.4.14. *With the same notation of Lemma 2.4.8, the set*

$$\Lambda_x(\mathcal{I}; q) := \left\{\ell \in X : \limsup_{j\to\infty} \mu_j\left(\{n : x_n \in U_m\}\right) \ge q \text{ for all } m\right\}$$

is closed for each $q \in (0, \limsup_{n\to\infty} \mu_n(I_n))$.

Proof. Equivalently, we have to prove that the set

$$G := \left\{\ell \in X : \limsup_{j\to\infty} \mu_j\left(\{n : x_n \in U_m\}\right) < q \text{ for some } m\right\}$$

is open for each q. This is obvious if G is empty. Otherwise, let us fix $\ell \in G$ and let (U_m) be a decreasing local base at ℓ. Then, there exists $m_0 \in \mathbf{N}$ such that $\limsup_j \mu_j(\{n : x_n \in U_m\} \cap I_j) < q$ for all $m \ge m_0$. Fix $\ell' \in U_{m_0}$ and let (V_m) a decreasing local base at ℓ'. Fix also $m_1 \in \mathbf{N}$ such that $V_{m_1} \subseteq U_{m_0}$. It follows by monotonicity that

$$\limsup_{j\to\infty} \mu_j(\{n : x_n \in V_m\} \cap I_j) \le \limsup_{j\to\infty} \mu_j(\{n : x_n \in U_{m_0}\} \cap I_j) < q$$

for every $m \ge m_1$. In particular, since ℓ' has been arbitrarily fixed, $U_{m_0} \subseteq G$. □

Let us finally prove our main results.

Proof of Theorem 2.4.2. If Part. Let us suppose that $\Gamma_x(\mathcal{I}) = \mathrm{L}_x$. Hence, it is claimed that the set $\{\omega \in (0,1] : \Gamma_{x\restriction\omega}(\mathcal{I}) = \mathrm{L}_x\}$ is comeager.

If $\mathrm{L}_x = \emptyset$, then the claim follows by Lemma 2.4.7. Hence, let us suppose hereafter that L_x is non-empty. Since L_x is closed, there exists a non-empty countable set $\mathscr{L}$ whose closure is L_x. Moreover, since the collection of meager sets is a σ-ideal, we get by Corollary 2.4.11 and Lemma 2.4.12 that

$$\mathcal{M} := \{\omega \in (0,1] : \ell \notin \Gamma_{x \restriction \omega}(\mathcal{I}) \text{ for some } \ell \in \mathscr{L}\}$$

is meager. Hence $\mathscr{L} \subseteq \Gamma_{x\restriction\omega}(\mathcal{I})$ for each $\omega \in \mathcal{M}^c := (0,1] \setminus \mathcal{M}$. At this point, fix $\omega \in \mathcal{M}^c$. It follows that $\Gamma_{x\restriction\omega}(\mathcal{I})$ contains also the closure of $\mathscr{L}$, i.e., L_x. On the other hand, $\Gamma_{x\restriction\omega}(\mathcal{I}) \subseteq \mathrm{L}_x$ by Lemma 2.4.7. Therefore $\Gamma_{x\restriction\omega}(\mathcal{I}) = \mathrm{L}_x$ for each $\omega \in \mathcal{M}^c$.

Only If Part. Let us suppose that $\Gamma_x(\mathcal{I}) \neq \mathrm{L}_x$ so that there exists a point $\ell \in \mathrm{L}_x \setminus \Gamma_x(\mathcal{I})$. Therefore, the set of all $\omega \in (0,1]$ such that $\Gamma_{x\restriction\omega}(\mathcal{I}) = \Gamma_x(\mathcal{I})$ is contained in $\{\omega \in (0,1] : \ell \notin \Gamma_{x\restriction\omega}(\mathcal{I})\}$ which, thanks to Corollary 2.4.11 and Lemma 2.4.12, is a meager set. □

Proof of Theorem 2.4.4. If $\mathcal{I}$ is an F_σ-ideal, then the claim follows by Theorem 2.4.2. Indeed, thanks to Theorem 2.2.3, we have $\Lambda_{x\restriction\omega}(\mathcal{I}) = \Gamma_{x\restriction\omega}(\mathcal{I})$ for all $\omega \in (0,1]$. Hence, let us assume hereafter that $\mathcal{I}$ is a generalized density ideal.

If Part. With the same notation of the above proof, suppose that $\Lambda_x(\mathcal{I}) = \mathrm{L}_x$ and, similarly, assume that $\mathrm{L}_x \neq \emptyset$. For each $\ell \in \mathrm{L}_x$, there exists $\omega_\ell \in (0,1]$ such that $x \restriction \omega_\ell \to \ell$ and, in particular, $\omega_\ell \in \Lambda_{x\restriction\omega_\ell}(\mathcal{I})$. Hence, for each fixed $q \in (0, \limsup_{n\to\infty} \mu_n(I_n))$, the set $\{\omega : \ell \in \Lambda_{x\restriction\omega}(\mathcal{I};q)\}$ is non-empty. Moreover, note that

$$\mathscr{V}_\ell(x;q) = \{\omega \in (0,1] : \ell \in \Lambda_{x\restriction\omega}(\mathcal{I};q)\}.$$

Thus, it follows by Lemma 2.4.8 that $\{\omega : \ell \notin \Lambda_{x\restriction\omega}(\mathcal{I};q)\}$ is meager. Therefore also $\mathcal{N} := \{\omega : \ell \notin \Lambda_{x\restriction\omega}(\mathcal{I};q) \text{ for some } \ell \in \mathscr{L}\}$ is meager, that is,

$$\mathcal{N}^c = \{\omega \in (0,1] : \mathscr{L} \subseteq \Lambda_{x\restriction\omega}(\mathcal{I};q)\}$$

is comeager. At this point, for each $\omega \in \mathcal{N}^c$, it follows by Lemma 2.4.14 that $\Lambda_{x\restriction\omega}(\mathcal{I};q)$ contains also the closure of $\mathscr{L}$, i.e., L_x. On the other hand, $\Lambda_{x\restriction\omega}(\mathcal{I};q) \subseteq \Lambda_{x\restriction\omega}(\mathcal{I}) \subseteq \mathrm{L}_x$ by Lemma 2.4.7. Therefore $\Lambda_{x\restriction\omega}(\mathcal{I}) = \mathrm{L}_x$ for each $\omega \in \mathcal{N}^c$.

Only If Part. This part goes verbatim as in the *only if* part of the proof of Theorem 2.4.2 (using Corollary 2.4.10). □

Note that it follows by Corollary 2.4.1 that, for each $\alpha \in [-1, 0]$, the set

$$\{\omega \in (0,1] : \Lambda_{x \upharpoonright \omega}(\mathcal{I}_\alpha) = \Gamma_{x \upharpoonright \omega}(\mathcal{I}_\alpha)\}$$

has full Lebesgue measure if and only if $\Lambda_x(\mathcal{I}_\alpha) = \Gamma_x(\mathcal{I}_\alpha)$.

On the other hand, its topological analogue is quite different. Indeed, we conclude with the following corollary, which follows from the proofs of the main results:

Corollary 2.4.15. *With the same notations of Theorem 2.4.4, the sets*

$$\{\omega \in (0,1] : \Gamma_{x \upharpoonright \omega}(\mathcal{I}) = \mathrm{L}_x\} \quad \text{and} \quad \{\omega \in (0,1] : \Lambda_{x \upharpoonright \omega}(\mathcal{I}) = \mathrm{L}_x\}$$

are comeager. In particular, the set $\{\omega \in (0,1] : \Gamma_{x \upharpoonright \omega}(\mathcal{I}) = \Lambda_{x \upharpoonright \omega}(\mathcal{I})\}$ *is comeager.*

We leave as an open question to check whether Theorems 2.4.2 and 2.4.4 may be extended to the whole class of $F_{\sigma\delta}$-ideals (hence, in particular, analytic P-ideals) and, moreover, whether there exists an ideal $\mathcal{I}$ for which these results fail.

2.5 Ideal continuous projections onto ℓ_∞

A closed subspace X of a Banach space B is said to be complemented in B if there exists a continuous projection from B onto X. It is known that c_0, the space of real sequences convergent to 0, is not complemented in ℓ_∞, cf. Sobczyk (1941); Whitley (1966). The aim of this last Section is to show the ideal analogue of this result.

We denote by $c(\mathcal{I})$ [resp. $c_0(\mathcal{I})$] the space of real sequences which are $\mathcal{I}$-convergent [resp. $\mathcal{I}$-convergent to 0]. The set of bounded real $\mathcal{I}$-convergent sequences has been studied, e.g., in Bartoszewicz et al. (2011); Filipów and Tryba (2019); Kostyrko et al. (2005).

By an easy modification of (Kostyrko et al., 2005, Theorem 2.3), $c_0(\mathcal{I}) \cap \ell_\infty$ is a closed linear subspace of ℓ_∞ (with the sup norm).

The question addressed here, posed at the open problem session of the 45th Winter School in Abstract Analysis (Czech Republic, 2017), follows:

Question 1. Is $c_0(\mathcal{I}) \cap \ell_\infty$ complemented in ℓ_∞?

Before proving our main result, we recall the following:

Lemma 2.5.1. *An infinite dimensional subspace X of ℓ_∞ is complemented in ℓ_∞ if and only if it is isomorphic to ℓ_∞.*

Proof. If X is complemented in ℓ_∞, then X is isomorphic to ℓ_∞, see (Albiac and Kalton, 2006, Definition 2.2.5 and Theorem 5.6.5). For the opposite direction, see (Albiac and Kalton, 2006, Proposition 2.5.2). □

Hence, Question 1 can be reformulated as:

Question 2. Is $c_0(\mathcal{I}) \cap \ell_\infty$ isomorphic to ℓ_∞?

We will prove that the answer is negative for a large class of ideals. To state our result, we recall that a family $\mathscr{A} \subseteq \mathcal{I}^+$ is said to be $\mathcal{I}$*-maximal-almost-disjoint* (in short, $\mathcal{I}$-mad) if $\mathscr{A}$ is a maximal family (with respect to inclusion) such that $A \cap B \in \mathcal{I}$ for all distinct $A, B \in \mathscr{A}$, so that for each $X \in \mathcal{I}^+$ there exists $A \in \mathscr{A}$ such that $X \cap A \in \mathcal{I}^+$. (The minimal cardinality $\mathfrak{a}(\mathcal{I})$ of an $\mathcal{I}$-mad has been studied in the literature: e.g., it is known that, if $\mathcal{I}$ is an analytic P-ideal, $\mathfrak{a}(\mathcal{I}) > \mathbf{N}$ if and only if $\mathcal{I}$ is F_σ, cf. Baumgartner (1983); Farkas and Soukup (2009).)

Our main result, contained in Leonetti (2018a), follows:

Theorem 2.5.2. *Let $\mathcal{I}$ be an ideal for which there exists an uncountable $\mathcal{I}$-*mad *family. Then $c_0(\mathcal{I}) \cap \ell_\infty$ is not complemented in ℓ_∞.*

It can be shown that, if $\mathcal{I}$ is a meager ideal, there is an $\mathcal{I}$-mad family of cardinality $\mathfrak{c}$, see Lemma 2.5.8 below. In particular

Corollary 2.5.3. *$c_0(\mathcal{I}) \cap \ell_\infty$ is not complemented in (and not isomorphic to) ℓ_∞ whenever $\mathcal{I}$ is meager.*

As an important example, the family of asymptotic density zero sets $\mathcal{I}_0$ is an analytic P-ideal, hence meager. Therefore:

Corollary 2.5.4. *The set of bounded real sequences statistically convergent to 0 (i.e., $c_0(\mathcal{I}_0)$) is not is isomorphic to ℓ_∞.*

Lastly, we obtain an analogue of the main result in Lindenstrauss (1963) (for summability matrices):

Corollary 2.5.5. *c is complemented in $c(\mathcal{I}) \cap \ell_\infty$ if and only if $\mathcal{I} = \mathrm{Fin}$.*

It is worth noting that Theorem 2.5.2 *cannot* be extended to all ideals $\mathcal{I}$. Indeed, if $\mathcal{I}$ is maximal, then the set of bounded $\mathcal{I}$-convergent sequences, which is isomorphic to $c_0(\mathcal{I}) \cap \ell_\infty$, is exactly ℓ_∞.

2.5.1 Proofs

Thanks to Lemma 2.5.1, a negative question to Question 1 would follow if $c_0(\mathcal{I}) \cap \ell_\infty$ was separable (indeed ℓ_∞ is nonseparable, hence they cannot be isomorphic). However, this works only if $\mathcal{I} = \mathrm{Fin}$:

Lemma 2.5.6. *$c_0(\mathcal{I})$ is separable if and only if $\mathcal{I} = \mathrm{Fin}$.*

Proof. First, suppose that $\mathcal{I} = \mathrm{Fin}$. Then the set of eventually zero rational-valued sequences $x \in \ell_\infty$ is a countable dense subset of $c_0(\mathcal{I})$, which is therefore separable.

Conversely, let us suppose that there exists $A \in \mathcal{I} \cap [\mathbf{N}]^{\mathbf{N}}$. For each $X \subseteq \mathbf{N}$ and $\varepsilon > 0$, let $B(\mathbf{1}_X, \varepsilon)$ be the open ball with center $\mathbf{1}_X$ and radious ε. The collection $\mathscr{B} := \left\{B(\mathbf{1}_X, 1/2) : X \in [A]^{\mathbf{N}}\right\}$ is an uncountable family of nonempty open sets which are pairwise disjoint, hence $c_0(\mathcal{I})$ is not separable. $\square$

This implies that $c_0 = c_0(\mathrm{Fin})$ is not complemented in ℓ_∞, which is known. At this point, recall the following characterization, see Talagrand (1980) and (Bartoszyński and Judah, 1995, Theorem 4.1.2):

Lemma 2.5.7. *$\mathcal{I}$ is a meager ideal if and only if there exists a finite-to-one function $f : \mathbf{N} \to \mathbf{N}$ such that $f^{-1}(A) \in \mathcal{I}$ if and only if A is finite.*

In other words, the second condition is $\mathrm{Fin} \leq_{\mathrm{RB}} \mathcal{I}$, where $\leq_{\mathrm{RB}}$ is the Rudin–Blass ordering. This is sufficient to prove the existence of an uncountable $\mathcal{I}$-mad family:

Lemma 2.5.8. *There exists an $\mathcal{I}$-*mad *family of cardinality $\mathfrak{c}$, provided $\mathcal{I}$ is meager.*

Proof. It is known that there is a Fin-mad family $\mathscr{A}$ of cardinality $\mathfrak{c}$, cf. Whitley (1966). Then, thanks to Lemma 2.5.7, there exists a finite-to-one function $f : \mathbf{N} \to \mathbf{N}$ such that $f^{-1}(A) \in \mathcal{I}$ if and only if A is finite, hence $\{f^{-1}(A) : A \in \mathscr{A}\}$ is the claimed $\mathcal{I}$-mad family. $\square$

Let us prove our main result:

Proof of Theorem 2.5.2. Let us suppose for the sake of contradiction that $c_0(\mathcal{I}) \cap \ell_\infty$ is complemented in ℓ_∞ and denote by

$$\pi : \ell_\infty \to c_0(\mathcal{I}) \cap \ell_\infty$$

the canonical projection. Define $T := I - \pi$, hence T is bounded linear operator such that $T(x) = 0$ for each $x \in c_0(\mathcal{I}) \cap \ell_\infty$. Note also that, if $B \notin \mathcal{I}$, then $\mathbf{1}_B$ is a bounded sequence which is not $\mathcal{I}$-convergent to 0, hence $\pi(\mathbf{1}_B) \neq \mathbf{1}_B$ and $T(\mathbf{1}_B) \neq 0$.

At this point, let $(A_j : j \in J)$ be an uncountable $\mathcal{I}$-mad family, which exists by hypothesis. We are going to show that there exists $j \in J$ such that $T(\mathbf{1}_{A_j}) = 0$, which is impossible since $A_j \in \mathcal{I}^+$. Indeed, let us suppose that, for each $j \in J$, there exists $x_j = (x_{j,n}) \in \ell_\infty$ supported on A_j with $T(x_j) \neq 0$ and, without loss of generality, $\|x_j\|_\infty = 1$. It follows that there exists $m, k \in \mathbf{N}$ such that $\tilde{J} := \{j \in J : |x_{j,m}| \geq 2^{-k}\}$ is uncountable. Also, by possibly replacing x_j with $-x_j$, let us suppose without loss of generality that $x_{j,m} > 0$ for all $j \in \tilde{J}$.

For each nonempty finite set $F \subseteq \tilde{J}$, define $s_F = (s_{F,n}) := \sum_{j \in F} x_j$. In particular,

$$\|T(s_F)\|_\infty \geq s_{F,m} \geq |F| 2^{-k}. \tag{2.35}$$

Note also that $I := \bigcup (A_i \cap A_j)$, where the sum is extended over all distinct $i, j \in F$, belongs to $\mathcal{I}$. This implies that the sequence $s_F \upharpoonright I$ is $\mathcal{I}$-convergent to 0, hence $T(s_F) = T(s_F \upharpoonright I^c)$. Therefore

$$\|T(s_F)\|_\infty = \|T(s_F \upharpoonright I^c)\|_\infty \leq \|T\| \cdot \|s_F \upharpoonright I^c\|_\infty \leq \|T\|,$$

which, together with (2.35), implies $|F| \leq 2^k \|T\|$. This contradicts the fact the $\tilde{J}$ is infinite. $\square$

Proof of Corollary 2.5.5. There is nothing to prove if $\mathcal{I} = \mathrm{Fin}$. Conversely, fix $I \in \mathcal{I} \setminus \mathrm{Fin}$ and define $X := \{x \in \ell_\infty : x_i \neq 0 \text{ only if } i \in I\}$ and $Y := X \cap c_0$. It is clear that

$$c \subseteq Y \subseteq X \subseteq c(\mathcal{I}) \cap \ell_\infty$$

and that X and Y are isometric to ℓ_∞ and c_0, respectively. Hence, it is known that c can be projected continuously onto Y, let us say through T, see Sobczyk (1941). To conclude the proof, let us suppose that there exists a continuous projection $H : c(\mathcal{I}) \cap \ell_\infty \to c$. Then the restriction $T \circ H \upharpoonright X$ is a continuous projection $\ell_\infty \to c_0$. This contradicts Theorem 2.5.2 (in the case $\mathcal{I} = \mathrm{Fin}$). $\square$

Chapter 3

A Characterization of Behavioral Equivalence

In the context of non-cooperative game theory, two different representations have been widely used to describe interactions in dynamic games: the "extensive form" and the "normal form." The main difference is that the former provides a richer structure, specifying the information available to each active player and making explicit the order of moves. On the other hand, this does not happen in the normal form representation, where it *seems* that each player chooses instantaneously his strategy.

There are situations where the richer information structure of the extensive form can be considered redundant, as in the case of dynamic games where players have to choose their moves, one by one, and ignoring all previous actions chosen by their opponents. Even if the order of playing may not matter, e.g., from some solution-mode perspective, the extensive form structure requires its specification. Actually, there is no representation which is better than the other.

The main goal of this Chapter is to provide a characterization of *behavioral equivalence* between extensive game forms with imperfect information, highlighting the connection between these two representations. Everything will be precise in a few. However, it is worth noting that the focus is not related to strategic considerations, but to the descriptive component of the theory of non-cooperative games: the "rules of the game."

Let us consider the following two examples of behaviorally equivalent games:

Example 3.0.1. Player *1* is allowed to play extensive game forms G and G' represented in Figure 3.1. Neither the consequence function nor the preference relations over the set of terminal nodes $\{z_1, z_2, z_3\}$ are specified. In particular, the normal form representations will be defined with respect to terminal paths. Since player *1* knows ex-ante the extensive forms G and G', he realizes that these games represent "essentially" the same situation. In the former he needs to choose two consecutive actions to reach the terminal nodes z_2 or z_3.

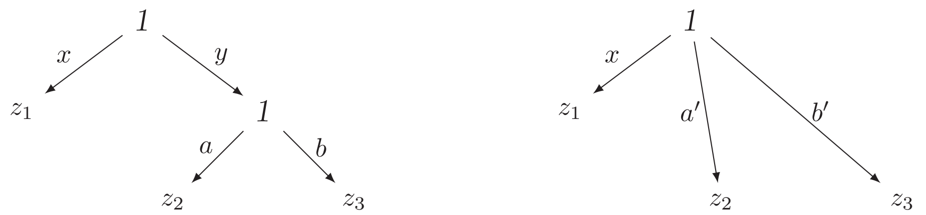

Figure 3.1: Example of Coalescing Moves / Sequential Agent Splitting.

Example 3.0.2. It may be the case that there are players which are not always informed about previous moves of their opponents, according to the rules of the game, i.e., there is imperfect information. For instance, players *1* and *2* play games represented in their extensive forms in Figure 3.2.

Figure 3.2: Example of Interchanging of Simultaneous Moves.

As usual, a dashed line between some nonterminal nodes where the same player is active means that he is not able to realize on which node he is actually playing. To be clear, in the extensive form game represented in the left hand side of Figure

3.2, player *2* knows ex-ante the structure of the extensive form representation; nevertheless, he cannot observe the action chosen by *1*, according to the rules of the game, hence he is not able to infer at which node he is playing. These game structures are "essentially" equivalent for both players.

It will be shown that two game structures are behaviorally equivalent (roughly, this is the case if they share the same normal form with respect to terminal paths, cf. Definition 3.1.3) if and only if one can be transformed into the other through some (natural generalizations of the) transformations used in Examples 3.0.1 and 3.0.2, which are known in the literature as Coalescing Moves / Sequential Agent Splitting and Interchanging of Simultaneous Moves, respectively.

3.0.1 Literature

There is little work on the topic of game equivalence. Being on a theoretical side, part of the problem is the tendency to search for the correct notion of equivalence, as opposed to looking for many of them and what is kept invariant in each case. The starting point of this line of research comes back to Thompson Thompson (1952), where he defines four transformations which preserve the strategic features of the game structures, meaning that the reduced normal forms are essentially kept invariant. These basic transformations are commonly known as "Interchanging of Simultaneous Moves," "Coalescing Moves / Sequential Agent Splitting," "Addition of a Superfluous Move," and "Inflation / Deflation." Relying on the simplification of Krentel, McKinsey, and Quine Krentel et al. (1951) and the extensive model proposed by Kuhn Kuhn (1950), he shows that, up to relabelings, two (finite) game structures share the same reduced normal form if and only if each extensive form representation can be transformed into the other through a finite number of applications of these transformations.

Contributions and extensions can be found sporadically throughout the literature. In particular, Kohlberg and Mertens Kohlberg and Mertens (1986) extend the above result to games with chance moves, proposing two additional transformations which are, essentially, modified versions of Coalescing Moves and Addition of Superfluous Moves for the chance player. They argue that all the "strategic features" are unchanged through the application of these transformations. Some years later, Elmes and Reny Elmes and Reny (1994) point out that, if it is really the case, then the analysis of the strategic interactions should be restricted only to the normal form representations. However, they notice that one of these transformations, Inflation / Deflation, does not preserve the perfect recall property. Hence, proposing a modified version of the Addition of a Superfluous Move transformation, they show that two extensive form games with the same reduced normal forms can be transformed into each other without appealing to the unwanted transformation, preserving the perfect recall property. Other notions of game equivalence have been studied in the literature: see, for example, Hoshi and Isaac Hoshi and Isaac (2010), Dalkey Dalkey (1953), and Bonanno Bonanno (1992).

3.1 Preliminaries on Game structures

3.1.1 Preliminary Notation

Let $(X, \leq)$ be a partially ordered finite set, i.e., a nonempty finite set X with a binary relation $\leq$ contained in $X \times X$ which is transitive, reflexive, and antisymmetric. As usual, given $x, y \in X$, we write $x < y$ as a shorthand of $x \leq y$ and $x \neq y$. We let $[x, y]$ represent the order interval $\{z \in X : x \leq z \leq y\}$ (hence $[x, y] \neq \emptyset$ if and only if $x \leq y$) and denote the immediate predecessor relation by $\ll$, that is, $x \ll y$ with $x \neq y$ if and only if $[x, y] = \{x, y\}$.

The partially ordered finite set $(X, \leq)$ is said to be a tree whenever there exists a (necessarily unique) minimum element e, which is called the root of the tree, and for each $x, y \in X$ with $x < y$ the set $\{z \in X : x < z \leq y\}$ admits a minimum m (note that $x \ll m$). We assume that a finite tree is a meet-semilattice, i.e., for each $x, y \in X$ the set $\{z \in X : z \leq x, z \leq y\}$ admits a maximum, denoted by $x \wedge y$: indeed the order intervals $[e, x]$ and $[e, y]$ are finite so that their intersection is a totally ordered set.

Given a nonempty set S, we denote by 2^S the power set of S and by $(\mathscr{P}\mathrm{art}(S), \leq$ $)$ the collection of (possibly infinite) partitions of S partially ordered by refinement, that is, $\mathscr{P} \le \mathscr{P}'$ for some partition $\mathscr{P}, \mathscr{P}'$ of S whenever each $P' \in \mathscr{P}'$ is contained in some $P \in \mathscr{P}$. In addition, we represent the set of finite sequences of elements from S by

$$S^{<\mathbb{N}_0} := \bigcup_{n\in\mathbb{N}_0} S^n,$$

where $S^0 := \{\varnothing\}$ is the singleton containing the empty sequence (here $\mathbb{N}_0$ and $\mathbb{N}$ stand, respectively, for the set of nonnegative integers and positive integers; in particular $0 \in \mathbb{N}_0$). Accordingly, we define a partial order $\preceq$ on $S^{<\mathbb{N}_0}$ such that $x \preceq y$ if and only if x is a prefix of y, that is, if and only if $x = \varnothing$ or there exists $n \in \mathbb{N}$ such that $x = (x_1, \ldots, x_n) \in S^n$ and $x_i = y_i$ for all positive integers $i \le n$. Then $(S^{<\mathbb{N}_0}, \preceq)$ is a tree: indeed, $\varnothing \preceq x$ for all $x \in S^{<\mathbb{N}_0}$ and every nonempty order interval $[x, y]$ is finite, hence totally ordered and the set $\{z : x \prec z \preceq y\}$ has a minimum provided that $x \neq y$; cf. (Alós-Ferrer and Ritzberger, 2016, pp. 144-145). Here, the immediate predecessor relation is denoted by $\prec\!\!\prec$.

Lastly, given nonempty sets X and Y, a correspondence $f : X \rightrightarrows Y$ is meant to be a mapping from each $x \in X$ to some *possibly empty* subset of Y.

3.1.2 Extensive game forms

Let $\mathcal{G}$ be the collection of all extensive game forms G with imperfect information represented by tuples

$$\langle I, \bar{H}, (A_i, F_i, \boldsymbol{H}_i)_{i\in I}\rangle, \tag{3.1}$$

where each primitive component will be described below (this is inspired by Osborne and Rubinstein (1994)).

We let I stand for a nonempty finite set of players. For each $i \in I$, A_i denotes a nonempty finite set of potentially feasible actions which can be chosen by player i. Then, we set

$$A := \bigcup_{\emptyset\neq J\subseteq I} \left(\prod_{i\in J} A_i\right).$$

The symbol $\bar{H}$ represents the set of histories, i.e., a sub-tree of $(A^{<\mathbb{N}_0}, \preceq)$ such that $\varnothing \in \bar{H}$ and it is closed under the immediate predecessor relation, that is, if $x \prec\!\!\prec y$ in $(A^{<\mathbb{N}_0}, \preceq)$ and $y \in \bar{H}$, then $x \in \bar{H}$. (Hence, each node in $\bar{H}$ can be represented by a chain of elements, i.e., profile of actions, in A.) In this regard, $\bar{H}$ can be partitioned in the sets of terminal histories Z and non-terminal histories $H := \bar{H} \setminus Z$, where

$$Z := \{h \in \bar{H} : h' \in \bar{H}, h \preceq h' \implies h = h'\}.$$

Moreover, for each player $i \in I$, we let $F_i : H \rightrightarrows A_i$ be his feasibility correspondence, that is, a correspondence such that

- ◇ For each non-terminal history $h \in H$, there is at least one player with nonempty set of feasible actions $F_i(h)$, i.e.,

$$I_h := \{i \in I : F_i(h) \neq \emptyset\} \neq \emptyset;$$

- ◇ For each player $i \in I$, there is at least one non-terminal history $h \in H$ with nonempty $F_i(h)$, i.e.,

$$H_i := \{h \in H : F_i(h) \neq \emptyset\} \neq \emptyset;$$

- ◇ For each player $i \in I$ and non-terminal history $h \in H$, the immediate successors of h are the histories (h, a) for which $a = (a_i : i \in I_h)$ and all a_i are feasible actions, i.e.,

$$h \lessdot (h, a) \quad \text{if and only if} \quad a \in \prod_{i \in I_h} F_i(h).$$

(Here, we write $(h, a) := (a_1, \ldots, a_n, a)$ whenever $h = (a_1, \ldots, a_n) \in H$; in particular, $(h, a) = a$ if $h = \varnothing$.) These feasibility correspondences serve us to model symultaneous moves games at each non-terminal history.

At this point, given $h \in H$, we say that a player $i \in I$ is:

- *active* if he has at least two feasible actions, i.e., $|F_i(h)| \geq 2$;
- *observing* if he has exactly one feasible action, i.e., $|F_i(h)| = 1$;
- *inactive* if he has no feasible actions, i.e., $F_i(h) = \emptyset$.

Accordingly, it is assumed that:

(A) *There are no observing players.*

Remark 3.1.1. Assumption (A) is the unique real constraint that we are imposing on game structures $G \in \mathcal{G}$. Even if we are aware of the importance of observing players from the psychological and game-theoretical point of view, we are making this hypothesis only to simplify the presentation of our main result and to avoid several technicalities in its proof.

Indeed, it will be clear that it is sufficient to add in Theorem 3.3.2 a "trivial" transformation which eliminates observing players whenever possible (which, clearly, preserves the Z-reduced normal form defined in Section 3.1.4, cf. Definition 3.1.3).

Lastly, for each player $i \in I$, we let $\boldsymbol{H}_i$ be a partition of H_i which is finer than

$$\tilde{\boldsymbol{H}}_i := \{F_i^{-1}(X) : X \subseteq A_i, X \neq \emptyset\}.$$

Hence, $\boldsymbol{H}_i$ and $\tilde{\boldsymbol{H}}_i$ are partitions of H_i such that $\tilde{\boldsymbol{H}}_i \leq \boldsymbol{H}_i$. Each element $\boldsymbol{h}_i \in \boldsymbol{H}_i$ is interpreted as an information set of player i, i.e., a subset of non-terminal histories where player i is not able to realize where he is actually playing. In particular, for each information set $\boldsymbol{h}_i \in \boldsymbol{H}_i$ and $h, h' \in \boldsymbol{h}_i$, we have $F_i(h) = F_i(h')$. Note that, in each information set, player i could not know who his active opponents are and their feasible actions.

3.1.3 Perfect recall

Game structures $G \in \mathcal{G}$ have to satisfy perfect recall, see (Alós-Ferrer and Ritzberger, 2016, Definition 6.5). This means, among others, that each player knows and remembers everything he did in prior moves. Indeed, the violation of perfect recall, which depends on the intrinsic features of the players, would have nothing to do with the rules of the game.

In particular, perfect recall implies that, in each information set, two distinct histories are not comparable, i.e.,

$$\forall i \in I, \forall \boldsymbol{h}_i \in \boldsymbol{H}_i, \forall h, h' \in \boldsymbol{h}_i, \quad h \preceq h' \implies h = h'. \tag{3.2}$$

The violation of (3.2) is commonly known as “absent-mindedness.”

3.1.4 Z-reduced normal form

For each player $i \in I$, let S_i be his set of strategies. In other words,

$$S_i := \prod_{\boldsymbol{h}_i \in \boldsymbol{H}_i} F_i(\boldsymbol{h}_i).$$

Similarly, $S := \prod_{i \in I} S_i$ denotes the set of strategy profiles. At this point, in a finite game each strategy profile $s \in S$ determines a unique terminal history $z \in Z$.[1] We denote this path function by

$$\zeta : S \to Z.$$

Accordingly, for each game structure $G = \langle I, \bar{H}, (A_i, F_i, \boldsymbol{H}_i)_{i \in I} \rangle \in \mathcal{G}$, we define its Z-*normal form* by

$$\mathrm{n}_Z(G) := \langle I, (S_i)_{i \in I}, Z, \zeta \rangle.$$

Note that the Z-normal form is not graphical per se, but rather represents the game G by means of a (possibly infinite dimensional) matrix.

Definition 3.1.2. Fix $G \in \mathcal{G}$ as in (3.1) and a player $i \in I$. Then two strategies $s_i, s_i' \in S_i$ are said to be *behaviorally equivalent*, shortened with $s_i \overset{i}{\sim} s_i'$, if

$$\forall s_{-i} \in S_{-i}, \quad \zeta(s_i, s_{-i}) = \zeta(s_i', s_{-i}),$$

where $S_{-i} := \prod_{j \in I \setminus \{i\}} S_j$.

Since $\overset{i}{\sim}$ is an equivalence relation on S_i, we can define the quotient space

$$\mathcal{S}_i := S_i / \overset{i}{\sim} .$$

Similarly, we set $\mathcal{S} := \prod_{i \in I} \mathcal{S}_i$ and denote the representative of each $s \in S$ by $s^\bullet \in \mathcal{S}$.

[1] Players who are active at $\varnothing$ determine a unique history $h_1 \in \overline{H}$ of lenght 1; if $h_1 \in Z$ we are done, otherwise players who are active at h_1 determine a unique history $h_2 \in \overline{H}$ of lenght 2; this is repeated only a finite number of times, hence the algorithm terminates.

In this regard, for each $G \in \mathcal{G}$, we let its *Z-reduced normal form* be

$$\mathrm{rn}_Z(G) := \langle I, (\mathcal{S}_i)_{i \in I}, Z, \tilde{\zeta} \rangle,$$

where $\tilde{\zeta} : \mathcal{S} \to Z$ is the map defined by $s^\star \mapsto \arg \left\{ \bigcup_{s \in S,\, s^\bullet = s^\star} \{\zeta(s)\} \right\}$, where $\arg X := x$ if $X = \{x\}$; note that this is also well defined in ZF.

At this point, we have all the ingredients to define the notion of behavioral equivalence.

Definition 3.1.3. Two extensive game structures $G, G' \in \mathcal{G}$ are said to be *behaviorally equivalent* (shortened with $G \equiv G'$) if they share the same Z-reduced normal form up to isomorphism[2] (i.e., $\mathrm{rn}_Z(G) \simeq \mathrm{rn}_Z(G')$).

3.2 Invariant Transformations

To state our main characterization of behavioral equivalence, we need to define the notion of invariant transformation.

Definition 3.2.1. A function $T : \mathrm{dom}(T) \subseteq \mathcal{G} \to \mathcal{G}$ is said to be an *invariant transformation* if $\mathrm{dom}(T) \neq \emptyset$ and $G \equiv T(G)$ for all $G \in \mathrm{dom}(T)$.

The family of invariant transformations will be denoted by $\mathcal{T}$. Hence, the main question addressed in this work can be stated as follows:

Question 3. Characterize the family $\mathcal{T}$ of all invariant transformations T.

More explicitly, we have to describe the family $\mathcal{T}$ so that, given $G, G' \in \mathcal{G}$, then $G \equiv G'$ if and only if $T(G)$ is isomorphic to $T'(G')$ for some $T, T' \in \mathcal{T}$.

At this point, we introduce two basic invariant transformations. To this aim, we need first to introduce the notions of "controlling" and "dominating" sets. Intuitively, these sets are collections of histories in the same information set of a fixed active player i such that, in a sense, control all (or part of) the paths which are successors of a given set of histories $h \in H$.

[2]More explicitly, there exist a bijection $f_i : \mathcal{S}_i \to \mathcal{S}'_i$ for each $i \in I$ and a bijection $g : Z \to Z'$ such that $g(\tilde{\zeta}(s)) = \tilde{\zeta}'(f(s))$ for all $s \in \mathcal{S}$; here $f(s)$ is the reduced strategy $(f_i(s_i) : i \in I) \in \mathcal{S}'$.

3.2.1 Controlling and dominating sets

Fix a game structure $G \in \mathcal{G}$ as defined in (3.1).

Definition 3.2.2. Given a player $i \in I$ and two information sets $\boldsymbol{h}_i, \boldsymbol{h}_i' \in \boldsymbol{H}_i$, we say that $\boldsymbol{h}_i'$ is a *controlling set* of $\boldsymbol{h}_i$, shortened with

$$\boldsymbol{h}_i \ll_i \boldsymbol{h}_i',$$

whenever there exists a feasible action $a_i^\star \in F_i(\boldsymbol{h}_i)$ such that for all histories $h \in \boldsymbol{h}_i$, profile of actions of his opponents $a_{-i} \in F_{-i}(h)$,[3] and terminal histories $z \in Z$, then

$$[(h, (a_i^\star, a_{-i})), z] \neq \emptyset \quad \text{if and only if} \quad [(h, (a_i^\star, a_{-i})), z] \cap \boldsymbol{h}_i' \neq \emptyset. \tag{3.3}$$

In other words, if $z \in Z$ is a terminal history such that $(h, (a_i^\star, a_{-i})) \preceq z$ for some $h \in \boldsymbol{h}_i$ and $a_{-i} \in F_{-i}(h)$, then there exists $h' \in \boldsymbol{h}_i'$ such that $(h, (a_i^\star, a_{-i})) \preceq h' \preceq z$. Lastly, note that there exists at most one such $a_i^\star \in F_i(\boldsymbol{h}_i)$, otherwise perfect recall of G would be violated.

To make an example, the information set in blue of player *2* in the left hand side of Figure 3.3 below is controlling the information set in red.

Remark 3.2.3. Fix $i \in I$. Given an information set $\boldsymbol{h}_i \in \boldsymbol{H}_i$ and a feasible action $a_i^\star \in F_i(\boldsymbol{h}_i)$, it is easily seen, thanks to perfect recall, that there is at most one $\boldsymbol{h}_i' \in \boldsymbol{H}_i$ such that $\boldsymbol{h}_i \ll_i \boldsymbol{h}_i'$ which passes through $a_i^\star$.

Definition 3.2.4. Given a player $i \in I$, a non-terminal history $h \in H$ with $i \notin I_h$, and a non-empty subset $\boldsymbol{d}_i$ of some information set $\boldsymbol{h}_i \in \boldsymbol{H}_i$ which does not contain h, we say that $\boldsymbol{d}_i$ is *dominating set* of h, shortened with

$$h \lessdot_i \boldsymbol{d}_i,$$

whenever

$$\forall z \in Z, \quad [h, z] \neq \emptyset \quad \text{if and only if} \quad [h, z] \cap \boldsymbol{d}_i \neq \emptyset.$$

[3]That is, $(a_j)_{j \in I \setminus \{i\}} \in \prod_{j \in I_h \setminus \{i\}} F_i(h)$.

On a similar note, if z is a terminal history compatible with h (that is, $h \prec z$), then there exists h' in some given subset $\boldsymbol{d}_i$ of an information set $\boldsymbol{h}_i$ for which $h \prec h' \prec z$.

To make an example, the last two nodes on the right of the information set in blue of player *3* in the left hand side of Figure 3.4 is dominaning (with respect to player *3*) the history which consist of the singleton with the rightmost action of player *1*.

Remark 3.2.5. In the same spirit of Remark 3.2.3, fix $h \in H$ and $i \in I$ such that i is inactive at h. Then it is easily seen that there is at most one non-empty set $\boldsymbol{d}_i \subseteq \boldsymbol{h}_i \in \boldsymbol{H}_i$ such that $h \lessdot_i \boldsymbol{d}_i$.

With these notions at hand, we can define two invariant transformations, which are the natural generalizations of the ones provided in Examples 3.0.1 and 3.0.2, respectively.

3.2.2 Coalescing / Sequential Agent Splitting

We represent the so-called "Coalescing" transformation by

$$\gamma : \mathrm{dom}(\gamma) \subseteq \mathcal{G} \to \mathcal{G},$$

and denote its inverse correspondence, commonly known in the literature as "Sequential Agent Splitting," by $\sigma : \gamma(\mathcal{G}) \rightrightarrows \mathcal{G}$. The domain $\mathrm{dom}(\gamma)$ is the collection of all extensive form games G defined by (3.1) for which there exist $i \in I$ and information sets $\boldsymbol{h}_i, \boldsymbol{h}'_i \in \boldsymbol{H}_i$ such that $\boldsymbol{h}'_i$ is a controlling set of $\boldsymbol{h}_i$, that is, $\boldsymbol{h}_i \ll_i \boldsymbol{h}'_i$.

With a slight abuse of notation, we replace $\gamma(G)$ with

$$\gamma(G; \boldsymbol{h}_i, \boldsymbol{h}'_i).$$

In this regard, calling $a_i^\star \in F_i(\boldsymbol{h}_i)$ the (unique feasible) action of the player $i \in I$ for which (3.3) holds, the transformed game

$$\gamma(G; \boldsymbol{h}_i, \boldsymbol{h}'_i) := \langle \tilde{I}, \tilde{\tilde{H}}, (\tilde{A}_i, \tilde{F}_i, \tilde{\boldsymbol{H}}_i)_{i \in \tilde{I}} \rangle$$

is defined by:

- $\tilde{I} = I$;
- $\tilde{\bar{H}}$ coincides with $\bar{H}$ for all histories h such that at least one of the following is satisfied:

  (i) There exists $h' \in \boldsymbol{h}_i$ such that $h \preceq h'$;

  (ii) h is not comparable with every $h' \in \boldsymbol{h}_i$;

  (iii) There exists $h' \in \boldsymbol{h}_i$ and $a_i \in F_i(h) \setminus \{a_i^\star\}$ such that $(h', (a_i, a_{-i})) \preceq h$.

  In the remaining cases, each history $h' \in \bar{H}$ such that $(h, (a_i^\star, a_{-i})) \preceq h'$ for some $a_{-i} \in F_i(h)$ has to be replaced in $\tilde{\bar{H}}$ with $\tilde{h}'$ where the actions chosen by player i at the histories in the information set $\boldsymbol{h}_i'$ are shifted back replacing $a_i^\star$.
- $\tilde{A}_j = A_j$ for all $j \in I$;
- Denoting with $\tilde{h}$ the corresponding history in $\gamma(G)$ of $h \in H$, we have[4]

$$\tilde{F}_i(\tilde{h}) = F_i(\boldsymbol{h}_i) \cup F_i(\boldsymbol{h}_i') \setminus \{a_i'\}$$

  for all $h \in \boldsymbol{h}_i$; otherwise $\tilde{F}_j(\tilde{h}) = F_j(h)$.
- The new information sets $(\tilde{\boldsymbol{H}}_j)_{j \in I}$ are modified accordingly: in particular, $\boldsymbol{h}_i$ and $\boldsymbol{h}_i'$ are "merged" and "new" information sets of players $j \in I \setminus \{i\}$ may be added in the sub-trees with roots $h \in \boldsymbol{h}_i$, see e.g. the information sets of player *3* in Figure 3.3 below.

Loosely speaking, the Coalescing transformation shifts all the actions in a information set $\boldsymbol{h}_i'$ of a given player i upwards, to another information set $\boldsymbol{h}_i$ of i controlled by the first one. Note that the histories in $\tilde{\boldsymbol{h}}'_i$ corresponding to $\boldsymbol{h}_i'$ may "disappear" if player i was the only active player in such histories. Finally, $\boldsymbol{h}_i'$ cannot be a proper subset of some information set of i because, after the transformation, the number of available moves at the new histories in $\boldsymbol{h}_i$ are greater than the remaining ones in $\boldsymbol{h}_i' \setminus \boldsymbol{h}_i$, allowing the player to realize where he actually is (or better, not is). This would contradict the perfect recall of the transformed game.

[4] It can be assumed, up to relabeling of the feasible actions, that $F_i(\boldsymbol{h}_i) \cap F_i(\boldsymbol{h}_i') = \emptyset$.

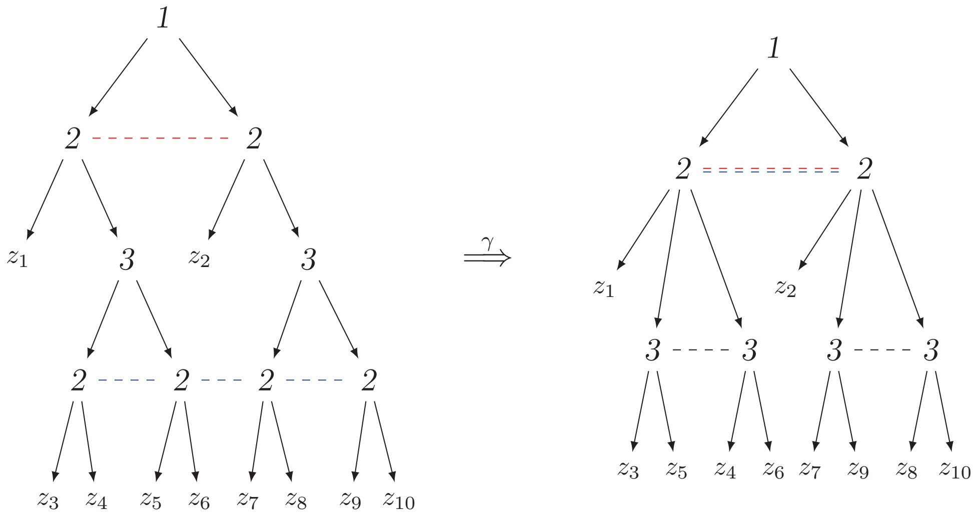

Figure 3.3: An example of the transformation "Coalescing" γ.

Lemma 3.2.6. *Coalescing transformation is an invariant transformation, that is,* $\gamma \in \mathcal{T}$*.*

Proof. Note that the reduced set of strategies are kept fixed for all player $j \in I \setminus \{i\}$, that is, $\tilde{\mathcal{S}}_j = \mathcal{S}_j$ for all $j \in I \setminus \{i\}$. Moreover, replacing $\boldsymbol{h}_i$ and $\boldsymbol{h}'_i$ with the corresponding $\tilde{\boldsymbol{h}}_i$ it is easy to see that $\tilde{\mathcal{S}}_i \simeq \mathcal{S}_i$. This implies that $\mathrm{rn}_Z(G) \simeq \mathrm{rn}_Z(\gamma(G))$, i.e., G and $\gamma(G)$ are behaviorally equivalent. □

3.2.3 Interchanging of Simultaneous Moves

We denote the "Interchanging of Simultaneous Moves" transformation by

$$\iota : \mathrm{dom}(\iota) \subseteq \mathcal{G} \to \mathcal{G},$$

where the domain $\mathrm{dom}(\iota)$ is the collection of all extensive game forms G defined as in (3.1) for which there exist an history $h \in H$, a player $i \in I$, and an information set $\boldsymbol{h}_i \in \boldsymbol{H}_i$ with a non-empty subset $\boldsymbol{d}_i$ such that $h \lessdot_i \boldsymbol{d}_i$, that is, $\boldsymbol{d}_i$ is a dominating set of h.

Similarly, with a slight abuse of notation, we replace $\iota(G)$ with $\iota(G; h, \boldsymbol{d}_i)$. In this respect, the transformed game

$$\iota(G; h, \boldsymbol{d}_i) := \langle \tilde{I}, \tilde{\tilde{H}}, (\tilde{A}_i, \tilde{F}_i, \tilde{\boldsymbol{H}}_i)_{i \in \tilde{I}} \rangle$$

is defined by:

- $\tilde{I} = I$;
- $\tilde{\bar{H}}$ coincides with $\bar{H}$ for all histories h' such that either $h' \preceq h$ or h' is not comparable with h. In the remaining cases, each history $h' \in \bar{H}$ such that $h \prec h'$ has to be replaced in $\tilde{\bar{H}}$ with $\tilde{h}'$ where the actions chosen by player i at the histories in $\boldsymbol{d}_i$ are shifted back at the last coordinate of h (this is possible since, as it been already observed after Definition 3.2.4, player i has to be inactive at h);
- $\tilde{A}_j = A_j$ for all $j \in I$;
- Denoting with $\tilde{h}'$ the corresponding history in $\gamma(G)$ of $h' \in H$, we have
$$\tilde{F}_i(\tilde{h}) = F_i(\boldsymbol{d}_i),$$
and $\tilde{F}_j(\tilde{h}') = F_j(h')$ in all remaining cases.
- The new information sets $(\tilde{\boldsymbol{H}}_j)_{j\in I}$ are modified accordingly: the position of the subset $\boldsymbol{d}_i \subseteq \boldsymbol{h}_i$ is shifted back at the coordinate of h, all the others are kept fixed; see e.g. the information set of player *3* in Figure 3.4 below.

In rough words, the Interchanging of Simultaneous Moves transformation shifts all the actions in a subset $\boldsymbol{d}_i$ of an information set $\boldsymbol{h}_i$ of a player i upwards, to another history h dominated by $\boldsymbol{d}_i$ (where he is not active). In addition, as it shown in Figure 3.4, $\boldsymbol{d}_i$ can be a proper subset of $\boldsymbol{h}_i$.

Lemma 3.2.7. *Interchanging of Simultaneous Moves is an invariant transformation, that is,* $\iota \in \mathcal{T}$.

Proof. It follows by construction that $\tilde{\mathcal{S}}_j \simeq \tilde{\mathcal{S}}_j$ for all $j \in I$. In particular, $\mathrm{rn}_Z(G) \simeq \mathrm{rn}_Z(\iota(G))$, i.e., G and $\iota(G)$ are behaviorally equivalent. □

Remark 3.2.8. It is immediate to see that the composition of invariant transformations is invariant. In particular, by Lemma 3.2.6 and Lemma 3.2.7, the compositions of transformations γ and ι are invariant.

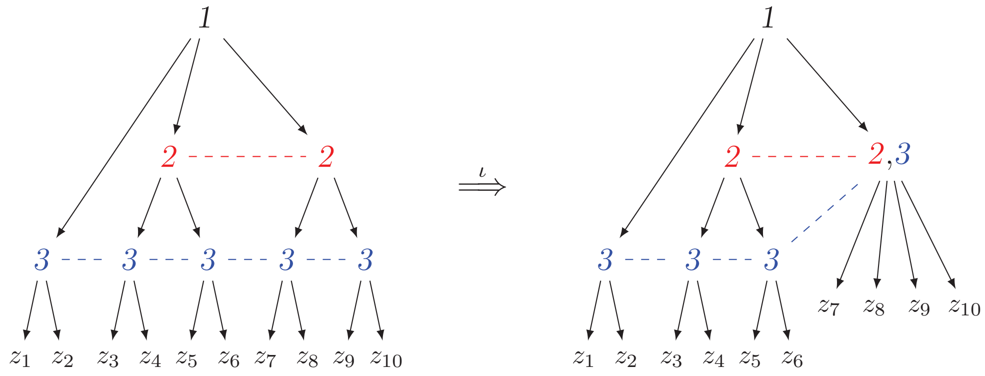

Figure 3.4: An example of the transformation "Interchanging of Simultaneous Moves" ι.

3.3 Characterization of Behavioral Equivalence

We can thus provide a complete answer to Question 3:

Theorem 3.3.1. *$\mathcal{T}$ is the family of compositions of transformations γ and ι.*

As a consequence, we get a characterization of the notion of behavioral equivalence:

Theorem 3.3.2. *Two extensive game forms are behaviorally equivalent if and only if one can be transformed into the other through a (possibly empty) net of transformations γ and ι and their inverses.*

Before we prove our main results, note that the transformations γ and ι are *not* independent meaning that, if $G \in \mathrm{dom}(\gamma) \cap \mathrm{dom}(\iota)$, it may be possible that $\gamma(G) \notin \mathrm{dom}(\iota)$. Indeed, consider the game structure G in left hand side of Figure 3.5: transformation ι may be applied in the subgame with root (t); however, considering that the information set in red of player *1* is controlling the history (t), we may apply also the γ transformation, which yields the game structure $\gamma(G)$ in the right hand side of Figure 3.5. On the other hand, ι cannot be applied now to $\gamma(G)$, that is, $\gamma(G) \notin \mathrm{dom}(\iota)$.

However, we are going to show that for each $G \in \mathcal{G}$ there exists a "minimal" behaviorally equivalent $\widehat{G} \in \mathcal{G}$ obtained by applying, in some order, the transfor-

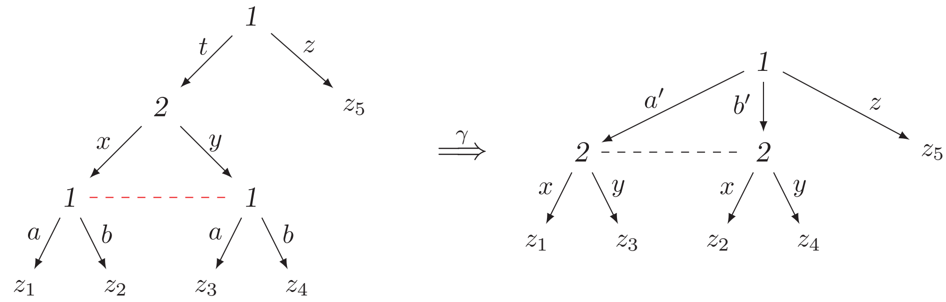

Figure 3.5: γ and ι are not independent.

mations γ and ι. To this aim, let $\widehat{\mathcal{G}}$ be the sub-family of game structures $G \in \mathcal{G}$ for which transformations γ and ι cannot be applied, that is,

$$\widehat{\mathcal{G}} := \mathcal{G} \setminus (\mathrm{dom}(\iota) \cup \mathrm{dom}(\gamma)).$$

Then, for each $G \in \mathcal{G}$, there exists a unique behaviorally equivalent game $\widehat{G} \in \widehat{\mathcal{G}}$. Equivalently:

Lemma 3.3.3. *$\widehat{\mathcal{G}}$ and the quotient space $(\mathcal{G}/\equiv)$ are isomorphic.*

Proof. Fix $G \in \mathcal{G}$. For each player $i \in I$, apply whenever possible the transformations γ and ι at the root $\varnothing$ (note that at histories of the same lenght they can be applied independently). Denote by $G(0) \in \mathcal{G}$ the obtained game.

At this point, suppose that the game $G(n) \in \mathcal{G}$ has been defined recursively. Then, for each player $i \in I$ and for each history h in $G(n)$ of lenght $n+1$, apply whenever possible:

- the γ transformation at the information set $\boldsymbol{h}_i$ containing h with respect to the controlling set $\boldsymbol{h}_i'$ and passing through a given action $a_i^\star \in F_i(h)$ (recall that there is at most one such $\boldsymbol{h}_i'$ by Remark 3.2.3);
- the ι transformations at the history h with respect to the dominating sets $\boldsymbol{d}_i$ (recall that there is at most one such $\boldsymbol{d}_i$ by Remark 3.2.5).

Denote by $G(n+1)$ the new game and note that each game structure in the sequence $(G(n) : n \in \mathbb{N}_0)$ is by construction behaviorally equivalent to G (indeed, each $G(n)$ has been obtained from G through a finite composition of invariant transformations γ and ι).

Let $\widehat{G}$ be the "pointwise limit" of the sequence $(G(n) : n \in \mathbb{N}_0)$, which can be constructed explicitly in the following way: for each $n \in \mathbb{N}_0$, the histories of lenght n in $\widehat{G}$ correspond to the histories of lenght n in $G(n)$. Note that $\widehat{G}$ belongs to $\widehat{\mathcal{G}}$ (since $\widehat{G} \notin \mathrm{dom}(\gamma) \cup \mathrm{dom}(\iota)$) and it is behaviorally equivalent to G (since it can be seen as a transformation of G through an "ordered" net of invariant transformations). Lastly, all possible transformations γ and ι have been applied recursively at the histories of lowest lenght. Since each transformation ι or γ on the original game G can be "included" in the ordered net defined above,[5] it follows that $\widehat{G}$ is the unique behaviorally equivalent game of G in $\widehat{\mathcal{G}}$. □

We recall that a bounded complete lattice is a partially ordered set which admits a greatest and a least element and such that all subsets have both a supremum and an infimum (we omit details).

Lemma 3.3.4. *Fix a non-empty set finite X. Then the partially ordered set $(\mathscr{P}\mathrm{art}(X), \leq)$ is a bounded complete lattice. In particular, for each non-empty collection $\{\mathscr{P}_j : j \in J\}$ of partitions of X, then* $\sup\{\mathscr{P}_j : j \in J\}$ *exists in $\mathscr{P}\mathrm{art}(X)$.*

Lemma 3.3.5. *$\widehat{\mathcal{G}}$ and $\mathrm{rn}_Z(\mathcal{G})$ are isomorphic.*

Proof. We have to prove that there exists a unique $\widehat{G} \in \widehat{\mathcal{G}}$, up to isomorphism, which has a given Z-reduced normal form $\langle I, (\mathcal{S}_i)_{i \in I}, Z, \tilde{\zeta}\rangle$. Thanks to Lemma 3.3.4, for each $i \in I$, there exists an unique partition $\mathscr{P}_i^\star$ of $\mathcal{S}_i$ defined by

$$\sup\left\{\mathscr{P}_i \in \mathscr{P}\mathrm{art}(\mathcal{S}_i) : \{\tilde{\zeta}(P_i, \mathcal{S}_{-i}) : P_i \in \mathscr{P}_i\} \in \mathscr{P}\mathrm{art}(Z)\right\}.$$

Note that $\mathscr{P}_i^\star$ is well defined because the above collection contains $\{\mathcal{S}_i\}$, hence it is non-empty.

[5] The real reason behind the uniqueness of the minimal game $\widehat{G}$ is that the unique substantial difference of the transformations γ and ι is that the former applies to histories with the same active players, while the latter not. However, in both cases, they aim at "regroup" and "minimize" the game structures shifting upwards players' actions whenever possible.

At this point, if $\mathscr{P}_i^\star = \{\mathcal{S}_i\}$ then player i is inactive at the root $\varnothing$. Otherwise i is active and his set of feasible actions $F_i(\varnothing)$ can labeled as $\mathscr{P}_i^\star$; in particular, $|F_i(\varnothing)| = |\mathscr{P}_i^\star|$, i.e., the number of available actions for player i at the root of the game is equal to number of elements of the partition $\mathscr{P}_i^\star$.

Then, for each $i \in I$ fix some $P_i^\star \in \mathscr{P}_i^\star$ and consider the sub-game where the set of available strategies is restricted to $P_i^\star$. With the same argument above, for each $i \in I$, there exists an unique partition $\mathscr{P}_i^{\star\star}$ of $P_i^\star$ defined by

$$\sup\left\{\mathscr{P}_i \in \mathscr{P}\mathrm{art}(P_i^\star) : \{\tilde{\zeta}\left(P_i, \textstyle\prod_{j\in I\setminus\{i\}} P_j^\star\right) : P_i \in \mathscr{P}_i\} \in \mathscr{P}\mathrm{art}(Z^\star)\right\},$$

where $Z^\star$ is the set of terminal paths compatible with $\prod_i P_i^\star$. Similarly, player $j \in I$ is active at the history $h := (P_i^\star : i \in I)$ if and only if $\mathscr{P}_j^{\star\star} \neq \{P_j^\star\}$ and his set of feasible actions $F_j(h)$ can be labeled as $\mathscr{P}_j^{\star\star}$.

This algorithm can be continued by induction on the lenght of the histories.

Lastly, two histories h, h' are on a same information set of a player i if and only if the above algorithm identifies the same partition on the same set at the corresponding histories h, h', that is, $F_i(h) = F_i(h')$. This allows to construct the whole game structure $\widehat{G} \in \widehat{\mathcal{G}}$. $\square$

Example 3.3.6. Consider the game $\widehat{G} \in \widehat{\mathcal{G}}$ in the left hand side of Figure 3.5. Its Z-reduced normal form is given in Figure 3.6.

$1\backslash 2$	x	y
a'	z_1	z_3
b'	z_2	z_4
z	z_5	z_5

Figure 3.6: $\mathrm{rn}_Z(\widehat{G})$ of the game in Figure 3.5.

It follows from the algorithm described in Lemma 3.3.5 that player *1* is active at the root $\varnothing$ with three feasible actions and, on the other hand, player *2* is active because the unique partition of $\{x, y\}$ which divides the terminal paths in disjoint sets is $\{x, y\}$ itself. At this point, for the sub-games starting at the histories (a')

and (b'), player *2* is active and the finest partition dividing the terminal paths in disjoint sets is $\{\{x\},\{y\}\}$. It follows that player *2* is active at such histories, they are on the same information set, and he has two available actions. Finally, player *2* is inactive at the history (z). In other words, we construced the game $\widehat{G}$ in the left hand side of Figure 3.5.

Finally, we are ready to prove our main results.

Proof of Theorem 3.3.1. Thanks to Lemma 3.2.6, Lemma 3.2.7, and Remark 3.2.8, $\mathcal{T}$ contains the family of compositions of γ and ι. Conversely, fix $T \in \mathcal{T}$ and $G \in \mathrm{dom}(T)$, which is possible since $\mathrm{dom}(T)$ is non-empty, so that $\mathrm{rn}_Z(G) \simeq \mathrm{rn}_Z(T(G))$. Since $\mathrm{rn}_Z(\mathcal{G})$ and $\widehat{G}$ are isomorphic thanks to Lemma 3.3.5, we conclude that $\widehat{G} \simeq \widehat{T(G)}$. By construction $\widehat{G}$ and $\widehat{T(G)}$ are the minimal game structures of G and $T(G)$, respectively, through transformations γ and ι. Hence T can be expressed as compositions of γ and ι. $\square$

Proof of Theorem 3.3.2. The IF part follows by Remark 3.2.8. Conversely, fix two behaviorally equivalent game structures $G, G' \in \mathcal{G}$ so that

$$\mathrm{rn}_Z(G) \simeq \mathrm{rn}_Z(G'). \tag{3.4}$$

Thanks to Lemma 3.3.3 there exist unique minimal game structures $\widehat{G}, \widehat{G'} \in \widehat{\mathcal{G}}$ corresponding to G and G', respectively. In particular, by Lemma 3.2.6 and Lemma 3.2.7,

$$\mathrm{rn}_Z(G) \simeq \mathrm{rn}_Z(\widehat{G}) \quad \text{and} \quad \mathrm{rn}_Z(G') \simeq \mathrm{rn}_Z(\widehat{G'}). \tag{3.5}$$

Putting together (3.4) and (3.5), we obtain that $\widehat{G}$ and $\widehat{G'}$ are behaviorally equivalent. Lastly, since $\mathrm{rn}_Z(\mathcal{G})$ and $\widehat{G}$ are isomorphic by Lemma 3.3.5, we conclude that $\widehat{G} \simeq \widehat{G'}$. $\square$

3.4 Invariance of Solution Concepts

In this section, we are going to investigate the invariance of known solution concepts in behaviorally equivalent games. Thanks to Theorem 3.3.2, the question can be restated in easier terms whether solution concepts are invariant with respect to the transformations γ and ι. Variants of this question have been already posed by some authors, cf. Bonanno (1992); Hoshi and Isaac (2010).

To this aim, we associate a consequence to each terminal node and, for each player $i \in I$, a utility function representing his preference over consequences. Hereafter, we assume for simplicity that I and each S_i are finite sets, and that a von Neumann–Morgenstern utility $u_i : Z \to \mathbf{R}$ represents preferences of player i. In addition, given a finite non-empty set A, we denote by $\Delta(A)$ the set of probability measures $2^A \to \mathbf{R}$. We let δ_a be the Dirac measure at $a \in A$ and, for each player $i \in I$, we set

$$U_i : \Delta(S) \to \mathbf{R} : \mu \mapsto \sum_{s \in S} u_i(s)\, \mu(\{s\})$$

Lastly, we let $\mathrm{n}(G)$ and $\mathrm{rn}(G)$ be the classical normal form and reduced normal form, respectively, of a game $G \in \mathcal{G}$. The new isomorphism (which preserves the classical $\mathrm{rn}(G)$) is denoted by $\cong$.

3.4.1 Nash equilibrium.

Without doubt, the most famous equilibrium concept was defined by Nash in Nash (1951): a profile of strategies $s^\star \in S$ is said to be a *Nash equilibrium* whenever $u_i(s_i^\star, s_{-i}^\star) \geq u_i(s_i, s_{-i}^\star)$ for all $i \in I$ and $s_i \in S_i$.

Lemma 3.4.1. *Nash equilibria are preserved in behaviorally equivalent games.*

Proof. By Theorem 3.3.2, we have equivalently to show that Nash equilibria are preserved through γ and ι. Since γ and ι are invariant transformations, they preserve the Z-reduced normal form, i.e., $G \simeq \mathrm{rn}_Z(G)$ for all $G \in \mathrm{dom}(\gamma)$, and similarly for ι. In particular, it implies that $G \cong \mathrm{rn}(G)$. The claim follows by the fact that Nash equilibria are defined on $\mathrm{n}(G)$ for each $G \in \mathcal{G}$. □

3.4.2 Proper equilibrium.

It is implicitly assumed, in the definition of Nash equilibrium, that each player has correct beliefs about the profile of strategies of their opponents and, in addition, that there is no incentive to deviate from their own equilibrium strategy. Moreover, it does not take into account the possibility of events where players fail to act rationally. There is no surprise that, after its introduction, it was attacked from different sides against its too permissive character.

Hence, Myerson proposed in Myerson (1978) the notion of *proper equilibrium*. To this aim, given $\varepsilon > 0$, a totally mixed strategy profile $\alpha \in \otimes_{i\in I}\Delta(S_i)$ (i.e., the support of α_i is S_i for all $i \in I$) is said to be ε-proper if for all $i \in I$ and $s_i, s_i' \in S_i$,

$$U_i\left(\delta_{s_i} \otimes \alpha_{-i}\right) < U_i\left(\delta_{s_i'} \otimes \alpha_{-i}\right) \implies \alpha_i^\star\left(\{s_i\}\right) \le \varepsilon\alpha_i^\star\left(\{s_i'\}\right).$$

At this point, a strategy profile $\alpha^\star \in \otimes_{i\in I}\Delta(S_i)$ is said to be a proper equilibrium if it is a weak limit of ε-proper strategy profiles, as $\varepsilon \to 0$.

Note that proper equilibrium is a refinement of Nash equilibrium (and in turn further refines Selten's notion of trembling hand perfect equilibrium Selten (1975)).

Lemma 3.4.2. *Proper equilibria are preserved in behaviorally equivalent games.*

Proof. Note that ε-proper equilibria are defined on the normal form $\mathrm{n}(G)$ of a game G. Then, the corresponding fully mixed strategy profile on the reduced normal form $\mathrm{rn}(G)$ (assigning the sum of probabilities of equivalent strategies to its representative element) is a ε'-proper equilibria, where $\varepsilon' := \varepsilon \max\{|S_i| : i \in I\}$. The converse is obvious. It means that proper equilibria can be defined directly on $\mathrm{rn}(G)$. The claim follows by the same argument used in Lemma 3.4.1. □

3.4.3 Subgame Perfect Equilibrium

Consider the extensive form game represented in Figure 3.7.

There are two Nash equilibria: (y, a) and (x, b). In particular, the latter one highlights the ineptitude of this concept to represent situations of erring players. Indeed, suppose that the equilibrium of the game prescribes (x, b) and player *1* deviates for some reason. The equilibrium (x, b) still requires player *2* to play b, which is clearly not his optimal choice. The problem relies on the fact a strategy is thought to be a kind of prescription to a player on what to do at all possible

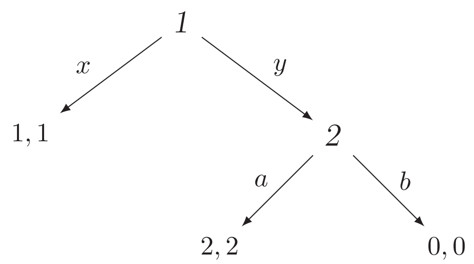

Figure 3.7: An unreasonable Nash equilibrium.

information sets: "Since in a non-cooperative game binding agreements are not possible, the solution of such a game has to be self-enforcing" (van Damme, 1983, Preface). Indeed, Nash and proper equilibria are purely defined on the normal form representations.

A profile of strategies $s^\star$ is a *subgame perfect equilibrium* if it is a Nash equilibrium in every subgame, see Selten (1965).

In particular, it refines the notion of Nash equilibrium. Indeed, in the case of the game represented in Figure 3.7 there is a unique subgame perfect equilibrium, i.e., (y, a).

Lemma 3.4.3. *Subgame perfect equilibria are preserved through ι, but not through γ. Hence, subgame perfect equilibria are not preserved in behaviorally equivalent games.*

Proof. Fix $G \in \mathrm{dom}(\iota)$ and note that the transformation ι does not break any information sets in each subgame of G. Hence, with the same argument of Lemma 3.4.1, subgame perfect equilibria are preserved through ι.

On the other hand, consider the game in Figure 3.8.

Since $\gamma(G)$ has no proper subgames, Nash equilibria and subgame perfect equilibria in $\gamma(G)$ coincide with the set $\{(x, a), (z, b)\}$. On the other hand, the pair (z, b) is not a subgame perfect equilibrium in G (to be precise, neither $(z.x, b)$ nor $(z.y, b)$). $\square$

At this point, one may argue that all types of equilibrium are invariant to the

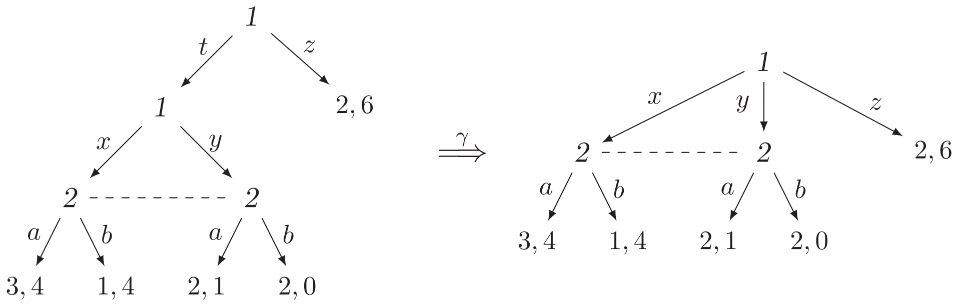

Figure 3.8: Games G and $\gamma(G)$.

transformation ι. This is wrong, as we will see in a moment.

3.4.4 Weak Perfect Bayesian Equilibrium

There exist situations where also subgame perfect equilibria may be unreasonable. Consider for example the game represented in Figure 3.9.

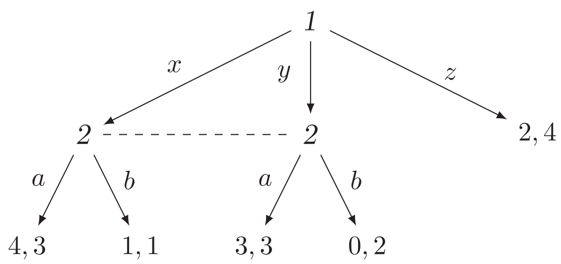

Figure 3.9: An unreasonable subgame perfect equilibrium.

It can be easily seen that there are two Nash equilibria, (x, a) and (z, b), and both of them are subgame perfect equilibria. One may conclude that the notion of subgame is too strict and a different equilibrium concept is needed. The most natural way is to assume that each player has to "guess" what is the correct path, intended as ordered sequence of simultaneous moves games, which brings him to be active. Accordingly, we will define some probabilities measures on the set of his available actions. This is the underlying idea of the weak perfect Bayesian equilibrium, cf. (Mas-Colell et al., 1995, p.285).

A pair (σ, μ) given by a profile of strategies σ and a system of beliefs μ is a *weak perfect Bayesian equilibrium* if:

- σ is sequentially rational given the belief system μ;
- μ is derived from the strategy profile σ through Bayes' rule whenever possibile, i.e.,

$$\forall i \in I, \forall h \in \boldsymbol{h}_i, \ \ \Pr(\boldsymbol{h}_i|\sigma) > 0 \implies \mu(x) = \frac{\Pr(h|\sigma)}{\Pr(\boldsymbol{h}_i|\sigma)}.$$

Lemma 3.4.4. *Weak perfect Bayesian equilibria are not preserved through ι.*

Proof. Consider the two behaviorally equivalent games represented in Figure 3.10 and Figure 3.11. If player *2* chooses the action o the game ends (indeed, there are two terminal nodes associated with this action, z_0 and z_{13}, depending on the choice of player *1*). Otherwise player *3* will have to choose between a, b and c, under the assumption that he can observe the initial choice of player *1*, but not the one of player *2* (or better, he can only infer that player *2* did not choose the action o).

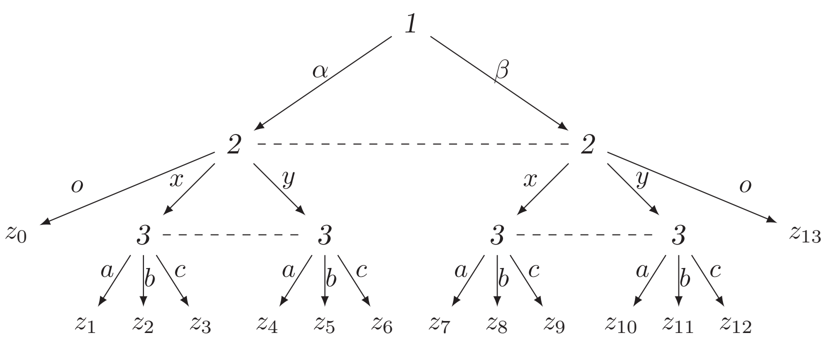

Figure 3.10: ι does not preserve weak perfect Bayesian equilibria.

The utilities associated with terminal nodes are $u(z_0) = (1, 1, 0)$ and $u(z_{13}) = (0, 0, 0)$ if *2* chooses o. Otherwise, for all $j \in \{1, \ldots, 6\}$ let $u(z_j) = u(z_{j+6})$, and

j	1	2	3	4	5	6
$u(z_j)$	$(0, 0, 3)$	$(0, 0, 0)$	$(0, 0, 2)$	$(0, 0, 0)$	$(0, 0, 3)$	$(0, 0, 2)$

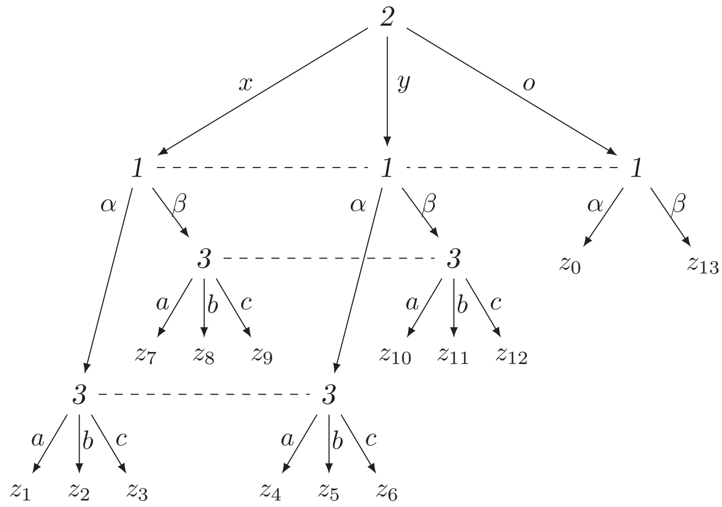

Figure 3.11: ι transformation of the game in Figure 3.10.

On the one hand, it can be checked that the assessment (σ, μ) where $\sigma = (\alpha, o, (a \text{ if } \alpha; b \text{ if } \beta))$ is a weak perfect Bayesian equilibrium whenever the probability assigned by player *3* at the histories (α, x) and (β, y) are both not smaller than $2/3$. In this way, the condition of sequentially rational is verified for player *3* (there is nothing else to check, since *1* and *2* play only once). On the other hand, such pair is not a weak perfect Bayesian equilibrium in the transformed game of Figure 3.11. □

3.4.5 Rationalizability in extensive form games

Since the extensive form representation is richer than its associated normal form in terms of information, and the active players are allowed to update their subjective beliefs as the game unfolds, the study of rationalizability becomes more complex than the one in simultaneous moves games. "Extensive form rationalizability," introduced by Pearce (1984), is a solution concept which relies on the forward induction reasoning: it tries to grab the implications of rationality and common certainty in rationality.

It is defined, as in the case of simultaneous moves games, as an algorithmic procedure of elimination of strategies. The underlying idea is that each active player can make an argument about how he can best rationalize their opponents' behavior, and what can be deduced from this line of reasoning about what their opponent will do. The assumption is that when a player knows that his opponents behaved rationally, then (he believes) they will continue to act rationally in the future. Moreover, if he observes that their opponents' behavior can be rationalized assuming that each one of them believed their opponents to behave rationally, he assumes that they will continue to do also in the future. And so on.

Predicted outcomes are characterized epistemically in Battigalli and Siniscalchi (1999, 2002) as those attained under rationality and common strong belief in rationality: along each path of the game, at the first occasion a player has the occasion to play, he believes that his opponents will behave rationally, and that their opponents believe *their* opponents to behave rationally, and so forth. This notion incorporates a notion of "best rationalization principle," in the sense that, in case of unexpected situations, players will be so optimistic to assign to highest possible degree of strategic sophistication to the other ones.

Let us make an example and consider the extensive form game with perfect information represented in Figure 3.12.

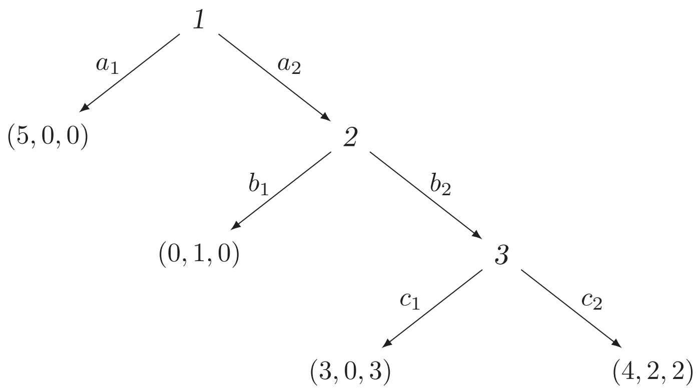

Figure 3.12: How can player *2* be justified to choose b_2?

Here the action a_1 is strictly dominant for player *1*, and action c_1 is strictly dominant for player *3*. At the beginning of the game, player *2* assumes that his opponents are sequentially rational, or better:

$$\forall i \in I, (\forall j \in I \setminus \{i\}, \ \ i \text{ believes that } j \text{ is sequentially rational}) \, . \tag{3.6}$$

Hence, he is expecting that the game will not unfold to his decision node. What happens in the opposite case? He can only deduce that his assumption was not correct, meaning that it is not true that every opponent (viewed as a *group*) is sequentially rational. In particular, he infers that player *3* can choose his strictly dominated action c_2. In this sense, player *2* might be justified to choose action b_2. This is the line of reasoning underlying the "Correlated Extensive Form Rationalizability."

The situation will be clearly different if the assumption (3.6) is replaced with

$$\forall i \in I, \forall j \in I \setminus \{i\}, \ \ (i \text{ believes that } j \text{ is sequentially rational}) \, .$$

In other words, stochastic independence among players' beliefs allows to narrow down the set of possible outcomes. In particular, in the case player *2* has to choose, the assumption about the rationality of player *3* is *not* violated. Under this constraint, the number of possible outcomes shrinks to two. This example highlights the difference between the notion of "Weak Extensive Form Rationalizability" and the Correlated Extensive Form Rationalizability; cf. also Battigalli (1996, 1997); Ben-Porath (1997). The insight is that, as pointed out in Stalnaker (1998), causal independence does not entail epistemic independence.

Let us characterize these different notions of Extensive Form Rationalizability, with the convention that a player "strongly believes" an event E if he assigns probability 1 to E on any information set which is not inconsistent with E itself. The **Correlated Extensive Form Rationalizability** is based on the following axioms, for all players $i \in I$:

- $E_0(i)$: "$\forall j \in I, j$ is (weakly) sequentially rational."
- $\forall n \in \mathbb{N}_0$, $E_{n+1}(i)$: "$\forall j \in I, j$ strongly believes $E_0(i) \,\&\, \cdots \,\&\, E_n(i)$."

It is then assumed that each player assigns the highest possible degree of strategic sophistication to the other ones viewed as *the group* of opponents. Then, it is implicit that each player entertains the hypothesis that his opponents might be coordinating their strategies. As the game unfolds, a player i may realize that the event $E_n(i)$ is not true; in such case he will assign probability 1 to $E_0(i) \,\&\, \cdots \,\&\, E_{n-1}(i)$.

The notion of **Weak Extensive Form Rationalizability** (sometimes called "Initial Extensive Form Rationalizability") relies on the following ones, for all $i \in I$:

- $W_0(i)$: "$\forall j \in I, j$ is (weakly) sequentially rational and has independent beliefs."

- $\forall n \in \mathbb{N}_0$, $W_{n+1}(i)$: "$\forall j \in I, j$ is certain of $W_n(i)$ at the beginning of the game."

In particular, i assigns to each of their opponents the smallest degree of strategic sophistication among them: their opponents are viewed as a group.

The assumption of epistemic independence can be then complemented with an additional restriction on belief revision, so that each player will assign the maximal degree of sophistication to *each one* of his opponents. Hence, a further refinement has been proposed, called **Strong Extensive Form Rationalizability**. This last notion is based on the following axioms, for all players $i \in I$:

- $\forall j \in I$, $S_{0,j}(i)$: "j is (weakly) sequentially rational and has independent beliefs."

- $\forall j \in I, \forall n \in \mathbb{N}_0$, $S_{n+1}(i)$:
  "$\forall k \in I \setminus \{j\}, j$ strongly believes $S_{0,k}(j) \,\&\, \cdots \,\&\, S_{n,k}(j)$."

At this point, one can simply realize that the set of profile of strategies which survive the algorithmic procedure of eliminations is preserved through Interchanging of Simultaneous Moves and Coalescing Moves transformations, with respect to each version of Extensive Form Rationalizability. For all games $G \in \mathcal{G}$ define $\boldsymbol{E}_{\mathrm{Corr}}(G)$ the set of profile of strategies which are selected from the Correlated Extensive Form Rationalizability, and similarly $\boldsymbol{E}_{\mathrm{Weak}}(G)$ and $\boldsymbol{E}_{\mathrm{Strong}}(G)$, so that

$$\boldsymbol{E}_{\mathrm{Strong}}(G) \subseteq \boldsymbol{E}_{\mathrm{Weak}}(G) \subseteq \boldsymbol{E}_{\mathrm{Corr}}(G).$$

Lemma 3.4.5. *Strong, weak, and correlated extensive form rationalizable profiles are preserved in behaviorally equivalent games.*

Proof. The transformation ι essentially preserves the information sets partitions $(\boldsymbol{H}_i)_{i \in I}$. It implies that, as the transformed game unfolds, each player does *not* have a better information structure than in the original game: the validity of axioms is preserved across this transformation. In particular, a profile of strategy will be eliminated from the transformed game $\iota(G)$ if and only if it will on the game G as well.

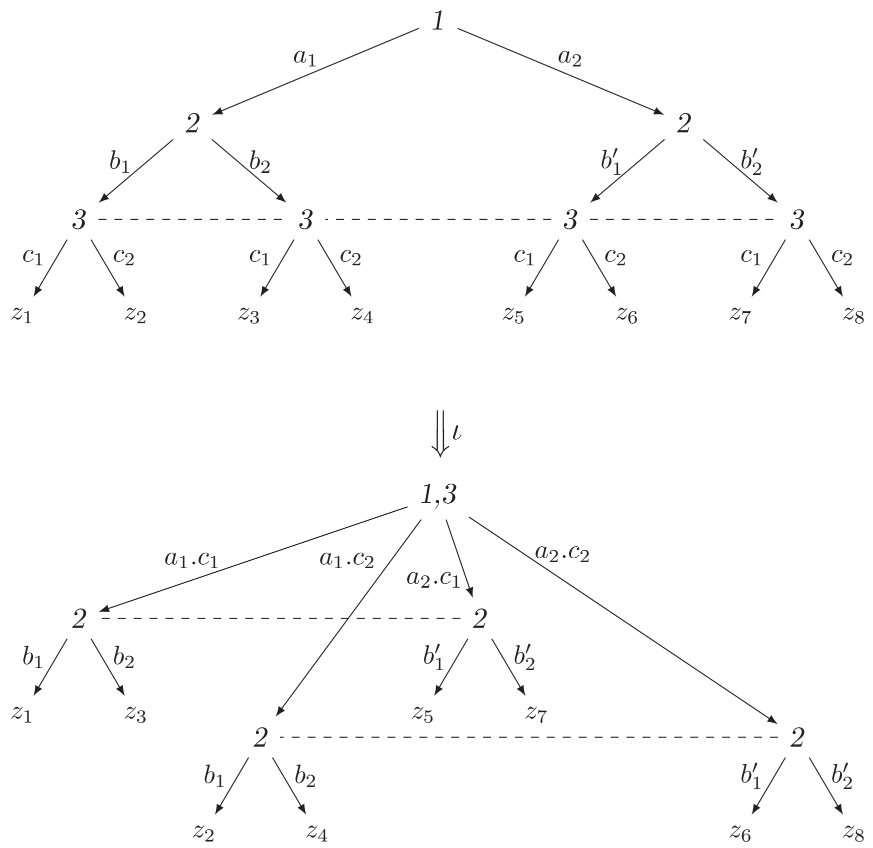

Figure 3.13: EFR profiles are preserved through ι.

Take as example the transformation represented in Figure 3.13. Player *3* is not able to observe the action chosen by his opponents. Also, player *2* can observe the choice of *1*. After the application of transformation ι player *3* is playing at the root of the tree, together with *1*. It is clear by construction that player *2* has always the same amount of available information as the game unfolds: he can still observe only the action chosen by player *1*.

The same argument cannot be used with the transformation γ: the structure of information sets partitions $(\boldsymbol{H}_i)_{i\in I}$ is *not* preserved across the transformation. Nevertheless, also in this case the information received by each player is essentially the same. Indeed, according to the construction of this transformation, the information sets partitions are modified in a way that, if some actions of a player i are *moved up* in the tree, his opponents (in particular, the ones active in the middle part of the tree) can infer nothing more and nothing less after this transformation.

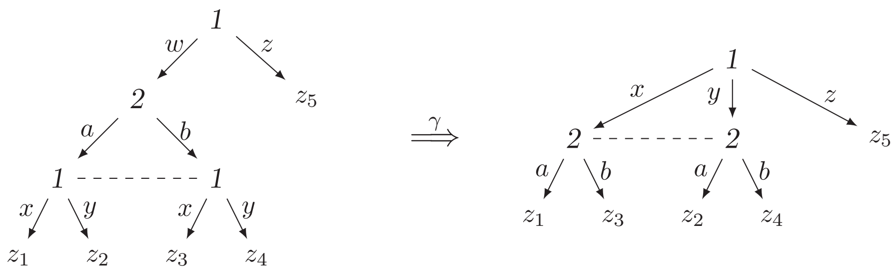

Figure 3.14: EFR profiles are preserved through γ.

Take as example the transformation represented in Figure 3.14. In the transformed game, player *1* has a bigger set of available actions. Clearly, for player *1* both games represent essentially the same situation, as far in both case he will observe nothing about his opponent. Moreover, in the original game player *2* is able to distinguish only between z and w (or better, *if* the game unfolds up to his decision node *then* he can infer that action z has not been chosen by player *1*). In the transformed game, if the game unfolds up to one of his decision nodes, he will deduce only that z has not been chosen. As it was expected, the information structure is essentially preserved through the transformation. □

3.4.6 Concluding remarks

In game theory, there is place for many notions of equivalence. From this perspective, the four transformations provided by Thompson in Thompson (1952) can be considered the starting point of the development of many other equivalence relations. Here, although the focus is still related to the descriptive component of the rules of the game, the theoretical framework does not need the specification of the consequence function. It has been shown that (the compositions of) two simple invariant transformations, i.e., Interchanging of Simultaneous Moves and Coalescing Moves / Sequential Agent Splitting, are sufficient to characterize the notion of behavioral equivalence. It is remarkable that these transformations preserve perfect recall.

Then, several solution concepts have been surveyed to check which ones are preserved under these transformations, and which ones not. In this regard, it is seen in Section 3.4 that many known solution concepts are not invariant under these transformations. Hence, one may argue that this “invariance criterion” can be used to provide a sort of goodness for different notions of solution concepts.

It would be interesting to check whether Theorem 3.3.2 can be extended to more general structures than rooted trees.

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