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Application of Reinforced Urn Processes to Survival Analysis

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The thesis "**Application of Reinforced Urn Processes to Survival Analysis**" by **Paolo Bulla** is recommended for acceptance by the members of the delegated committee, as stated by the enclosed reports, in partial fulfillment of the requirements for the degree of **Dottore di Ricerca**.

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Alla mia famiglia

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Abstract

In order to stress the large applicability of reinforced processes derived from urn schemes, two different problems of survival analysis have been presented: the inference for Markov renewal processes and the estimation of a bivariate survival function.
In the first, the so called *product Dirichlet times beta-Stacy* prior is characterized as the mixing measure generated by a special case of Markov reinforced renewal process. Such prior is shown to be conjugate when the data are collected as a trajectory of a Markov renewal process; the Bayes estimators are computed, too.
To solve the second problem, a bivariate reinforced process is defined starting from the generalized Pólya urn schemes. So, a prior on the bivariate distributions on $\mathbb{N}_0$ is obtained. Nevertheless this prior has a structure making overly complicated to compute the posterior, the Bayesian estimator is computed as the predictive law of the bivariate process and a Gibbs sampler procedure is developed. The support of the prior is studied and a result on the weak consistency is achieved.

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Introduction

Difficulties in the definition of the reinforced random processes arise from the absence of a theory at general level. Nevertheless, from an informal point of view, we can think at these processes as a weakening of markovian property in which the future depends not only on the present state, but, in a particular way, on the overall past history.
In other words, the processes collected in this category are characterized by the existence of a mechanism such that the transition probabilities at time n depend on the empirical measure of the process up to time n, i.e. on a tally of past observations, but not on the order in which they appear.
Pemantle (2001) offers a survey of recent works in the field of processes with reinforcement and provides an enumeration including:

1. *Urn Schemes*: a very rich class of models representing the probability of certain events in terms of sampling and replacing balls in an urn. We shall say more later;

2. *Reinforced Random Walks*: introduced by Coppersmith and Diaconis (1986), they describe the attitude of a random walker in some space (possibly represented as a graph) reaching states already visited;

3. *Stochastic Approximation*: first introduced by Robbins and Monro (1951), it can

be considered an analytical tool related to previous class in order to generalize Newton's method for root finding when the values of the function have noise added;

4. *Continuous Time Negative Reinforced Processes* related to physical model of self avoiding random walks.

Pemantle also asserts that these reinforced models constitute a reasonable and useful tool to describe several phenomena as spatial exploration, learning behavior, evolution and natural selection, determination of location in industry, cooperative interaction and so on.

To be honest most of the works in this field privilege the probabilistic, though applied, aspects of the problems and deal with statistics marginally.

On the other hand, urn models (and, less, reinforced random walks) have found fruitful applications in Bayesian statistics, hence it will be worth spending in the next pages some words about them.

Our interest lies in emphasizing the possibilities of employment of the reinforced urn processes and of models derived from them in statistics, particularly in the Bayesian nonparametric approach. For this purpose, in the present work, models derived from this methodology are applied to tackle two different problems in survival analysis: the inference for Markov renewal processes and the estimation of bivariate survival function.

This line of research in Bayesian nonparametric statistics has been opened by the paper of Blackwell and MacQueen (1973) in that it introduces, by the first time, the idea of characterizing by a sequence of predictive distributions, made concrete in an urn scheme, a measure on the space of distribution function (a Bayesian nonparametric prior) like the Dirichlet process. Given the complexity of a such mathematical object, it is easily understood the advantage of a concrete representation in terms both of intuition and of formal research of the relevant features. Such idea has also found fruitful application in the techniques of computation and simulation of these priors expressed as processes.

This work is prosecuted by Muliere et al. (2000) whose Reinforced Urn Processes, a generalization of the above cited urn scheme, are able to characterize other nonparametric priors as beta-Stacy, Pólya trees and, more generally, neutral to the right and tailfree.

Chapter 1 includes a brief review of the fundamental concepts about all this matter: the Pólya Urn, the Blackwell and Mac Queen's extension and the Reinforced Urn Processes. As special example of these last processes, the scheme generating a beta-Stacy process[1] for discrete distribution functions is proposed with a short digression about the construction, the definition and the properties of this nonparametric prior. Indeed, being conjugate with respect to right-censored observations, it enjoyed a great success in survival analysis; furthermore, it constitutes also a starting point for both the models presented here.

The first problem consists in the definition of a Bayesian nonparametric prior to make inference about Markov renewal processes.

Roughly speaking, this class of models describes the evolution of a system between a countable number of states. The system moves randomly from a state to another like a Markov chain, but, unlike this, it stays in each state a random length of time

[1] This scheme is quite similar to the so called *generalized Pólya urn* scheme of Walker and Muliere (1997).

whose distribution functions may depend both on starting and arrival state. These processes found several applications in very different fields; particularly, in survival analysis, they are used to study multilevel diseases.

So, in chapter 2 the basic theory of renewal, Markov renewal and semi-Markov processes is sketched: definitions, properties and relations between the three classes are clarified. Some papers, classical and Bayesian, on the problem of the estimation of the transition distributions of a Markov renewal processes are summarized, too. Finally, a result about the characterization of the mixtures of semi-Markov processes, useful for the remainder of the work, is recalled.

The following chapter illustrates the construction of two categories of processes with reinforcement in continuous time, the reinforced renewal processes and the reinforced Markov renewal processes (see Muliere et al. (2003)).

Some properties of these processes are considered. In particular the interarrival times of a reinforced renewal processes are studied and, exploiting the product integral representation (see Gill and Johansen (1990)), an explicit computation of the predictive distributions is achieved.

By consequence of backward equation for product integral and continuous time Markov processes, the conditional density function of the interarrival time T_{n+1} given $T_1, \ldots, T_n$ is obtained as well the density of the vector $(T_1, \ldots, T_n)$. These computations are useful to give a new proof of the exchangeability of the sequence of reinforced renewal processes interarrival times and to identify their mixing (de Finetti's) measure.

Muliere et al. (2003), analyzing the array of sojourn times and successor states, showed that a reinforced Markov renewal process is a mixture of semi-Markov processes, but they did not specify the de Finetti's measure.

Nevertheless, assuming the sojourn times and the successor states to be independent and the transitions from a given state to be determined always by the same Pólya urn, the mixing measure consists of a stochastic matrix having on each row a Dirichlet distribution for transition probabilities and beta-Stacy processes for each distribution of the sojourn time in a given state.

Taking this measure as prior, a Bayesian nonparametric inference procedure for Markov renewal process can be developed. It has also been shown that this prior is conjugate and some results of estimation have been given.

Chapter 4 deals with a different problem, the estimation of a bivariate survival function.

Starting from generalized Pólya urn schemes (or the equivalent Reinforced Urn Processes for beta-Stacy process) a bivariate reinforced process is built in such a way as to model couples of dependent lifetimes. More precisely, considered one of these couples, each of the two elements is supposed to be the sum of a specific component and of another one common to both the lifetimes. Each of the three components needed to build the couples is generated by a generalized Pólya urn. So this construction creates a particular form of the dependence between the lifetimes as well the reinforcement rule, specific of the generalized Pólya urn, introduces Bayesian learning from the past observations.

The bivariate reinforced process defines, via de Finetti's representation theorem, a prior on the space of bivariate distributions on $\mathbb{N}_0^2$. Although this prior has a structure which makes overly complicated to compute the posterior, an estimator for the bivariate survival function (the bivariate predictive distribution) can be easily computed by the Gibbs sampler exploiting the predictive distributions of the components. For this reason, this approach may be considered as Bayesian predictive.

Indeed, something more about the structure of the prior can be said: the bivariate distribution function sampled from it has a special form

$$P_{XY}(x,y) = \sum_{a=0}^{x\wedge y} P_A(a)\, P_B(x-a)\, P_C(y-a) \quad \forall (x,y) \in \mathbb{N}_0^2$$

and its marginals can be seen as convolutions of two distributions from beta-Stacy processes.

These remarks are useful to investigate some features of the prior like the width of the support and the consistency. Dealing with these problems, the strategy consists in, by the first, facing the case of the beta-Stacy process and, then, trying to find some extension holding for our prior.

In such a way, the result of proposition 3 of Ferguson (1973) about the width of the support with respect to the weak topology is adapted to the subset of the bivariate distributions having the structure above described. A more detailed knowledge is attained analysing the behaviour of the prior in the Kullback-Leibler neighborhoods. Specifically, some conditions on the parameters of the generalized Pólya urn scheme are provided in order the beta-Stacy prior satisfy the Kullback-Leibler condition. As before, such conditions can be translated quite easily to the case of our prior on the bivariate distributions with the usual structure. By consequence of a well known result of Schwartz (1965), the Kullback-Leibler condition is sufficient to weak consistency, so that our prior enjoys this properties.

So, the parameters of our prior can be chosen to obtain the consistency of the estimated bivariate function. On the other hand, when the concern is not to get the property of consistency, but just to center the prior in according to some researcher's guess, there is an alternative way to fix the parameters. It does not need any complicated idea about the structure of the bivariate distribution, but only some opinions as regards the dependence between the variables and the marginal distributions. Finally, to illustrate how this method works an example is proposed. Two cases are presented: the first replicates the classical Kaplan-Meier estimators when the lifetimes are supposed to be independent, the second, instead, is a "proper" situation in which the bivariate prior expresses some degree of dependence.

Chapter 1

Some examples of Reinforced Urn Processes

The very large family of probabilistic models collected under the name of urn processes (or equivalently urn schemes) represents the probability of certain events in terms of sampling and replacing balls in a urn.

Their most attractive property consists of making intuitive and concrete the probabilistic ideas characterizing the model, yet keeping the level of reasoning abstract enough to find general results.

The simplest urn process is the celebrated Pólya urn, originally employed to study the diffusion of infectious diseases.

Starting from this prototype, a great number of extensions and variations has been suggested.

The structure of the chapter is the following. Section 1.1 presents the basic facts about Pólya urn; the work of Blackwell and MacQueen (1973) describing a generalization of great moment in Bayesian nonparametric statistics is summarized in section 1.2.

Section 1.3 prosecutes this approach introducing Reinforced Urn Processes for the construction of a different type of nonparametric priors.
A short digression around the beta-Stacy process is in section 1.4 in order to clarify the specific example of Reinforced Urn Process generating such nonparametric prior distribution (section 1.5).

1.1 Pólya urn

Bayesian statistics devoted considerable attention to Pólya urn both for its original character and as a scheme implementing the classical statistical model for infinite sequences of Bernoulli random variables, conditionally independent and identically distributed given the probability of success assigned by a Beta distribution.
Consider an urn with W_0 balls of color 0 and B_0 of color 1. At each successive time $n \geq 0$, a ball is randomly sampled from the urn and, after observing the color, replaced in it with m balls of the same color.
By this way, a random sequence $X = \{X_n, n \geq 1\}$ describing the results of the successive drafts is defined; let W_n and B_n indicate the number of 0 and 1 balls and $Z_n = \frac{B_n}{W_n+B_n}$ be the proportion of 1 balls in the urn after the n-th draft.
This scheme is said reinforced because the rule, stating the introduction of new balls according with the color of the sampled ones, increases the probability of extracting a ball of the same color at next stage. Formally, we have

$$P\left[X_{n+1} = 1 | X_1, \ldots, X_n\right] = Z_n = \frac{B_0 + m\sum_{i=1}^{n} X_i}{B_0 + W_0 + mn}. \tag{1.1.1}$$

The theorem below summarizes the foremost results about Pólya urn.

Theorem 1.1.1. The sequence $X = \{X_n, n \geq 1\}$ generated by a Pólya urn is exchangeable and its de Finetti's measure is a Beta distribution with parameters $(\frac{B_0}{m}, \frac{W_0}{m})$.

As n grows to infinity, Z_n converges almost surely to a random limit. Moreover, the distribution of the limit is a Beta with parameters $(\frac{B_0}{m}, \frac{W_0}{m})$.

It is straightforward that a Pólya urn with balls of n different colors generates a n dimensional Dirichlet distribution instead of a Beta.

A number of urn schemes are generalizations of the above described, for example among the others Friedman (1949), Freedman (1965), Hill et al. (1987); Kotz and Johnson (1977) is a thorough reference book on this topic.

The prominence of Beta and Dirichlet distributions in the Bayesian framework can help us to understand why urn processes, as already said, have found an interesting application in Bayesian nonparametric statistics.

1.2 Pólya sequences and Dirichlet process

A seminal paper of Blackwell and MacQueen (1973) on Dirichlet process (Ferguson (1973)) has introduced, by means of a generalization of the Pólya urn, a novel perspective in the construction of nonparametric prior distributions.

They extend the classical Pólya urn scheme considering a Polish space $\mathbb{X}$ endowed with its Borel σ-field $\mathcal{X}$ and a finite measure α on it.

Once again, $\mathbb{X}$ can be seen as a space of colors labeling balls of different type in an urn with unitary reinforcement.

For $n \geq 1$, let $X_n = x$, if x is the color sampled at n-th stage; the sequence $X =$

$\{X_n,\, n \geq 1\}$ is called *Pólya sequence* when we have the initial distribution given by

$$P[X_1 \in B] = \frac{\alpha(B)}{\alpha(\mathbb{X})} = m_0(B), \qquad \forall B \in \mathcal{X} \tag{1.2.1}$$

and, for $n \geq 1$, the predictive of X_{n+1} conditioned on $X_1, \ldots, X_n$ is

$$P[X_{n+1} \in B | X_1, \ldots, X_n] = \frac{\alpha(B) + \sum_{i=1}^{n} \delta_{X_i}(B)}{\alpha(\mathbb{X}) + n} = \frac{\alpha_n(B)}{\alpha_n(\mathbb{X})} = m_n(B). \tag{1.2.2}$$

Therefore, it is possible to state the following.

Theorem 1.2.1. [Blackwell and MacQueen (1973)]

Let $X = \{X_n,\, n \geq 1\}$ be a Pólya sequence on $\mathbb{X}$ with measure α. Then:

1. as n grows to infinity, almost surely the random probability distribution m_n weakly converges to a discrete random probability distribution α^*;

2. the distribution of α^* is $\mathcal{D}_\alpha$, a Dirichlet process with parameter α;

3. the Pólya sequence is exchangeable and its de Finetti's measure is $\mathcal{D}_\alpha$.

The importance of this result lies in the fact that it gives a simple and concrete procedure for constructing an infinite exchangeable sequence of random variables with Dirichlet process as de Finetti's measure.

Moreover, by this way, some of the properties of Dirichlet process, a rather complicate mathematical object, are made more intuitive, for instance conjugacy property or predictive distributions.

1.3 Reinforced Urn Processes

The paper of Muliere et al. (2000) blends the two ideas of urn schemes and reinforced random walk. Indeed, they define the Reinforced Urn Processes as a reinforced random walk on a state space of urns and, through an analysis of their fundamental properties, illustrate how the notions of reinforcement and partial exchangeability are decisive for constructing prior distributions. Also, Cifarelli et al. (2000) provides a survey of several schemes based on Reinforced Urn Processes for the construction of general classes of priors commonly used in Bayesian nonparametric statistics like Pólya tree, beta-Stacy, neutral to the right and tailfree.

Each Reinforced Urn Process is defined by four elements.

Definition 1.3.1. Let

1. S be a countable state space;
2. $E = \{c_1, \ldots, c_k\}$ a finite set of colors of cardinality $k \geq 1$;
3. $U(x) = (n_x(c_1), \ldots, n_x(c_k))$ an urn composition function which maps S into the set of k-tuples of nonnegative numbers whose sum is a strictly positive number;
4. $q : S \times E \to S$ a law of motion such that for every $x, y \in S$, there is at most one color $c(x,y) \in E$ such that $q(x, c(x,y)) = y$.

Fixed $X_0 = x_0 \in S$, if for $n \geq 1$ $X_{n-1} = x$, a ball is sampled from the urn associated with x and if c is its color we set

$$X_n = q(x, c).$$

Finally the ball is replaced in the urn along with one of the same color.

The sequence $X = \{X_n,\, n \geq 0\}$ is said to be a Reinforced Urn Process with initial state x_0 and parameters S, E, U, q.

Obviously, the traditional Pólya's scheme represents an oversimplified example of Reinforced Urn Processes.

For all $x \in S$, let

$$R_x = \{y \in S : n_x(c(x,y)) > 0\} \text{ and } R^{(0)} = \{x_0\}$$

be the set of all states attainable from the state x in one step.

Then define for $n \geq 1$

$$R^{(n)} = \bigcup_{x \in R^{(n-1)}} R_x$$

and

$$R = \bigcup_{n=0}^{\infty} R^{(n)}.$$

the set of all states visited by the Reinforced Urn Process.

Recall that, according to Diaconis and Freedman (1980), two finite sequences σ and τ of elements of S are equivalent if they begin with the same element and, for every $x, y \in S$, the number of transitions from x to y is the same in both sequences.

In addition, a process $Y = \{Y_n,\, n \geq 1\}$ on S is partially exchangeable if, for all $n \geq 0$ and all equivalent sequences $\sigma = (s_0, \ldots, s_n)$ and $\tau = (t_0, \ldots, t_n)$

$$P\left[Y_0 = s_0, \ldots, Y_n = s_n\right] = P\left[Y_0 = t_0, \ldots, Y_n = t_n\right]. \qquad (1.3.1)$$

It possible to show that a Reinforced Urn Process $X = \{X_n,\, n \geq 0\}$ is partially exchangeable. Moreover, if X is also recurrent, a representation theorem of Diaconis and Freedman (1980) states that it is a mixture of Markov chains, that is there exists a probability measure μ on the set $\mathcal{P}$ of stochastic matrix on $R \times R$ such that for all $n \geq 1$ and $(x_1, \cdots, x_n)$,

$$P[X_0 = x_0, \cdots, X_n = x_n] = \int_{\mathcal{P}} \prod_{j=0}^{n-1} \pi(x_j, x_{j+1}) \mu(d\pi) \tag{1.3.2}$$

Now, given a random matrix Π of $\mathcal{P}$ with distribution μ, let, for all $x \in R$, $\Pi(x)$ be the x-th row of Π and $\alpha(x)$ the measure on R which assigns mass $n_x(c)$ to $q(x, c)$ for each $c \in E$ such that $n_x(c) > 0$ and mass 0 to all other elements or R.
Hence, it is possible to show that the following theorem holds.

Theorem 1.3.1. [Muliere et al. (2000)]
If the Reinforced Urn Process $X = \{X_n, n \geq 1\}$ is recurrent, the rows of Π are mutually independent random probability distributions on R and, for all $x \in R$, the law $\Pi(x)$ is that of a Dirichlet process with parameter $\alpha(x)$.

Again, following Diaconis and Freedman (1980), for a process Y on S, a y_0-block is defined to be a finite sequence of states which begins by y_0 and contains no further y_0. Let S^* be the countable space of all finite sequences of elements of S endowed with the discrete topology.
When $Y = \{Y_n, n \geq 0\}$ is recurrent, let $B_1 \in S^*, B_2 \in S^*, \ldots$ be the sequence of the successive y_0-blocks in Y.
It is quite straightforward that if Y is recurrent and partially exchangeable with initial state y_0, the sequence $\{B_n, n \geq 1\}$ of its y_0-blocks is exchangeable.
It follows that the sequence of x_0-blocks of a recurrent Reinforced Urn Process X is exchangeable. This implies that if φ is a measurable function from S^* to another

space, the sequence $\{\varphi(B_n),\, n \geq 1\}$ is exchangeable as well.

Often the de Finetti's measure of this sequence is simply characterized by the properties of the Reinforced Urn Process. This is the case, for instance, of the beta-Stacy and Pólya tree priors. By this way, these prior are constructed very simply and many of their characteristics become intuitively clear.

At this point, in order to understand how the construction of a non-parametric prior works, we present the specific Reinforced Urn Process for the beta-Stacy. Indeed, besides being a clear example, it will be useful also for the following.

However, for the same reason, it is convenient, before, a short digression recalling the definition and the main properties of this prior.

1.4 The beta-Stacy process

The beta-Stacy process, introduced by Walker and Muliere (1997), provides a prior for the cumulative distribution function with the attractive feature of being conjugate with respect to eventually right-censored observations and, therefore, it finds its application in nonparametric survival studies.

Let us start describing the simpler case of discrete time beta-Stacy process. Suppose a partition of the time axis $[0, \infty)$ is given by $0 = t_0 < t_1 < t_2 < \ldots$, a countable sequence of time points indexed by $k = 1, 2, \ldots$

By means of this dicretization, the survival model assumes that if an object is censored in the interval $[t_k, t_{k+1})$ and then dies before t_{k+1}, the information will becomes available just after t_{k+1}.

Let consider a random sequence $Y = \{Y_k, k \geq 1\}$ defined as follows:

$$\begin{aligned} Y_1 &\sim \mathcal{C}(\alpha_1, \beta_1, 1) \\ Y_2 | Y_1 &\sim \mathcal{C}(\alpha_2, \beta_2, 1 - Y_1) \\ &\dots \\ Y_k | Y_1, \dots, Y_{k-1} &\sim \mathcal{C}(\alpha_k, \beta_k, 1 - F_{k-1}) \end{aligned} \tag{1.4.1}$$

with each α_k, β_k positive, $\mathcal{C}$ denoting the beta-Stacy distribution (see Connor and Mosimann (1969)) and $F_{k-1} = \sum_{j=1}^{k-1} Y_j$.

If $F_0 = 0$ and

$$\prod_{k=1}^{\infty} \left(1 - \frac{\alpha_k}{\alpha_k + \beta_k}\right) = 0 \tag{1.4.2}$$

$F = \{F_k, k \in \mathbb{N}_0\}$ is a random discrete cumulative distribution function and it is called discrete beta-Stacy process with parameters $\{\alpha_k, \beta_k, k \in \mathbb{N}_0\}$.

Moreover, it follows that, for $m > 1$,

$$\mathcal{L}(Y_1, \dots, Y_m) = \mathcal{G}(\alpha_1, \beta_1, \dots, \alpha_m, \beta_m) \tag{1.4.3}$$

where $\mathcal{G}$ is a generalized Dirichlet distribution (see Walker and Muliere (1997)).

This distribution is neutral to the right; indeed, considering the random variables

$$V_k = \frac{Y_k}{1 - F_{k-1}}$$

such that $1 - V_k$ represents the random probability of survive in $[t_k, t_{k+1})$ conditioned on the fact of being alive at the beginning of the interval, for any $m > 1$, $V_1, \dots, V_m$ are independent. Moreover, being a beta-Stacy process, each $V_k \sim \mathcal{B}e(\alpha_k, \beta_k)$ marginally. The general definition, holding also for continuous random cumulative distributions, based on Lévy processes, is given below.

Definition 1.4.1. Let $c(\cdot)$ be a positive function, let $G \in \mathcal{F}$ be right continuous and let $\{t_k,\ k \geq 1\}$ be the countable set of points of discontinuity of G, that is, $G(\{t_k\}) = G(t_k) - G(t_k^-) > 0$.
Furthermore, put $G_c(t) = G(t) - \sum_{t_k \leq t} G(\{t_k\})$, the continuous part of G.
We say that F is a beta-Stacy process on $([0,\infty), \mathcal{A})$ with parameters $c(\cdot)$ and G, written $F \sim \mathcal{S}(c(\cdot), G)$, if, for $t \geq 0$

$$F(t) = 1 - \exp(-Z(t))$$

where Z is a Lévy process with Lévy measure for $Z(t)$ given, for $v \geq 0$, by

$$dN_t(v) = \frac{dv}{1 - e^{-v}} \int_0^t e^{-vc(s)G(s,\infty)} dG_c(s) \tag{1.4.4}$$

and moment generating function given by

$$\log Ee^{-\phi Z(t)} = \sum_{t_k \leq t} \log Ee^{-\phi S_k} + \int_0^\infty \left(e^{-\phi v} - 1\right) dN_t(v) \tag{1.4.5}$$

where $1 - e^{-S_k} \sim \mathcal{B}e(c(t_k)G(\{t_k\}), c(t_k)G[t_k, \infty))$.

This definition corresponds to the assumption that, infinitesimally,

$$dF(t)|F(t) \sim \mathcal{C}(c(t)dG(t), c(t)G[t,\infty), 1 - F(t)) \tag{1.4.6}$$

and, for a point of positive mass,

$$F(\{t\})\,|F(t^-) \sim \mathcal{C}\left(c(t)G(\{t\}), c(t)G[t,\infty), 1 - F(t^-)\right). \tag{1.4.7}$$

The beta-Stacy process results conjugate with respect to right censored observations. The following theorem indicates how to update the parameters.

Theorem 1.4.1. [Walker and Muliere (1997)]

Let $T_1, \ldots, T_n$ be an independent and identically distributed sample, possibly with right censoring, from an unknown cumulative distribution function F on $[0, \infty)$ and $F \sim \mathcal{S}(c, G)$.

Then, given the data, F is a beta-Stacy process with parameters c^* and G^*, where

$$G^*(t) = 1 - \prod_{[0,t]} \left[1 - \frac{c(s)dG(s) + dN(s)}{c(s)G[s,\infty) + M(s)}\right]$$
$$c^*(t) = \frac{c(t)G[t,\infty) + M(t) - N\{t\}}{G^*[t,\infty)} \tag{1.4.8}$$

and $N(\cdot)$ is the counting process for uncensored observations and $M(t) = \sum_{i=1}^{n} \mathbb{1}_{\{T_i \geq t\}}$.

By consequence, if F is a beta-Stacy process with parameters c and G, given an independent and identically distributed sample with possible right censoring, the Bayes estimator of $F(t)$, with quadratic loss function, is the predictive distribution of T_{n+1} given the data

$$\hat{F}(t) = E\left[F(t)|\, data\right] = 1 - \prod_{[0,t]} \left[1 - \frac{c(s)dG(s) + dN(s)}{c(s)G[s,\infty) + M(s)}\right]. \tag{1.4.9}$$

This is the same nonparametric estimator of F as that obtained from the beta process (see Hjort (1990)).

Note finally that when $c(t) \to 0\, \forall t$, the Kaplan-Meier estimate is obtained.

1.5 A Reinforced Urn Process for discrete beta-Stacy process

To define a Reinforced Urn Process enerating a discrete beta-Stacy process means to identify its four characterizing elements.

Let $S = \mathbb{N}_0$ and $E = \{w, b\}$, where w is for white and b for black. Assume as initial state $x_0 = 0$ and $n_0(w) = 0$, whereas $n_j(w) > 0$, $\forall j \geq 1$.

So, the Reinforced Urn Process X starts at 0 and, given that, at stage $n \geq 0$, it is at state $x \in S$, it moves to $x + 1$ if the ball sampled from $U(x)$ is black, otherwise it goes back to 0.

Hence, the law of motion is

$$q(x, c) = \begin{cases} x+1 & c = b \\ 0 & c = w \end{cases} \tag{1.5.1}$$

Urn compositions are updated according the usual Pólya rule.

An alternative, but quite similar urn scheme, named generalized Pólya urn, always generating a beta-Stacy process is presented in Walker and Muliere (1997).

If $n_j(b) = 0$ for some $j \geq 1$, let $N = \min\{j \geq 1 : n_j(b) = 0\}$; the process will visit just the states $\{0, \ldots, N\}$.

For an admissible finite sequence $(x_0, \ldots, x_n)$ of elements of S, the probability of the trajectory is

$$P[X_0 = x_0, \cdots, X_n = x_n] = \prod_{j=1}^{\infty} \frac{B(n_j(w) + t(j, 0), n_j(b) + t(j, j+1))}{B(n_j(w), n_j(b))} \tag{1.5.2}$$

where $B(a, b)$ is the usual beta function of parameters a and b, while $t(i, j)$ is the number of transitions from i to j in the sequence $(x_0, \ldots, x_n)$.

For all $n \geq 1$, let $T_n = X_{\tau_n - 1}$ where the sequence of stopping times $\{\tau_n, n \geq 1\}$ is defined by $(\tau_0 = 0)$, for $n \geq 1$,

$$\tau_n = \inf\{j > \tau_{n-1} : X_j = 0\}.$$

When X is recurrent, $T_n = \xi(B_n)$ is the last state of the block B_n or equivalently its length (in both cases a measurable function from S^* to nonnegative integers).

A necessary and sufficient condition for the recurrence of X is

$$\lim_{n\to\infty}\prod_{i=0}^{n}\frac{n_i(b)}{n_i(b)+n_i(w)}=0. \tag{1.5.3}$$

As the sequence $\{T_n,\ n\geq 1\}$ is exchangeable, for the de Finetti's representation theorem, there exist a random distribution function F such that, conditioned on it, the random variables T_n are independent and identically distributed according to F. Finally, we can state

Theorem 1.5.1. [Walker and Muliere (1997)]
The random distribution function F is distributed according to a beta-Stacy process on $\mathbb{N}_0$ with parameters $\{n_j(w), n_j(b),\ j\in\mathbb{N}_0\}$.

The Reinforced Urn Process above described is suitable only to characterize a random cumulative distribution sampled from a discrete beta-Stacy.
The well-known definition of this prior also for continuous distribution functions implies the natural question of searching a reinforced urn scheme for the general version of the process with a non-null continuous part.
In the following, we shall present the reinforced renewal process (Muliere et al. (2003)) solving this problem and we shall show how to use it in the problem of inference for Markov renewal and semi-Markov processes.

Chapter 2

Markov renewal and semi-Markov processes

Markov renewal processes and the equivalent family of semi-Markov processes represent the fusion between renewals and Markov chains theories; hence, they provide a suitable tool to model a system moving from one state to another and sojourning in each of these a random amount of time with a distribution that may depend both on starting and arrival states.

The importance of Markov renewal processes lies in its large domain of applicability rather than its inner richness.

Introduced by Lévy (1954a), Lévy (1954b) and Smith (1955), they were deeply studied in Pyke (1961a), Pyke (1961b) and Pyke and Schaufele (1964) and enjoyed a great success because of many applications in different fields. (A review can be found in Çinlar (1975)).

For inference procedure for such processes see Jain (1990); we cite only some classical works: Moore and Pyke (1968) develops maximum likelihood estimation of transition distributions under certain restrictions; Lagakos et al. (1978), Gill (1980), Voelkel and Crowley (1984) and Phelan (1990b) discuss the problem of right-censoring and asymptotic properties of estimators; also Greenwood and Wefelmeyer (1996) gives asymptotic results in classical framework; while Brock (1973), Phelan (1990a) and Duncan (1990) are Bayesian works.

The first section of the current chapter offers a survey of the definitions of renewal, Markov renewal and semi-Markov processes and their main properties; the relations between them are illustrated as well.

In section 2.2 the problem of the estimation of the transition distributions of a Markov renewal process is faced recalling some works among the above cited, both classical (Moore and Pyke (1968) and Gill (1980)) and Bayesian (Brock (1973) and Phelan (1990a)).

Finally, the last section introduces some results about the characterization of mixtures of semi-Markov processes whose interest by a Bayesian point of view will be clear in the next chapter.

2.1 Renewal and Markov renewal processes

Renewal processes investigate the properties of the successive occurrences of a fixed phenomenon repeating itself in time.

Let $\{T_n,\, n \geq 1\}$ be the sequence of times between the occurrences and define

$$S_0 = T_0 = 0 \quad \text{and} \quad S_{n+1} = S_n + T_{n+1} = \sum_{i=1}^{n+1} T_i, \quad n \geq 0. \tag{2.1.1}$$

Definition 2.1.1. Let $F : [0, \infty) \to [0, 1]$ be a distribution function and $\{T_n,\, n \geq 1\}$

an infinite sequence of non-negative independent random variables on a complete probability space $(\Omega, \mathcal{F}, P)$, distributed according to F.

The sequence $\{S_n;\ n \in \mathbb{N}\}$ is called an *ordinary renewal process*; the times T_n are the *interarrival times*, while the S_n the *renewal* (or *waiting*) *times.*

The name renewal process is often referred to the counting process

$$N(t) = \sup\left\{n \geq 0 : \sum_{i=1}^{n} T_i \leq t\right\} \quad t \geq 0. \tag{2.1.2}$$

The fundamental characteristic of renewal processes is the independence and identical distribution of interarrival times, so that an event taking place in a given time does not influence the successive occurrences.

The simplest renewal process is the homogeneous Poisson process for which the distribution function F is $\exp(\lambda)$ and $N(t)$ has a Poisson distribution, $\mathrm{Po}(\lambda t)$.

Markov renewal and semi-Markov processes constitute a generalization of renewal processes describing the repeated occurrences of different types. In order to clarify, it is possible to think at a system moving between different states and staying in each of these a random length of time (the distribution function of which depends both on present and next states).

Definition 2.1.2. Let E be an arbitrary set with $\mathcal{E}$ a countably generated σ-field. Let $(J,T) = \{(J_n, T_n),\ n \geq 0\}$ be a Markov chain with values in $E \times [0,\infty)$, with transition distributions $Q(i, dj, dt)$ and initial distribution $a(di)\varepsilon_0(dt)$. This process is called Markov renewal process.

In particular, set $T_0 = 0$, $S_n = \sum_{k=0}^{n} T_k$, $n \geq 0$ and $N(t) = \max\{i \geq 0 : T_i \leq t\}$, $t \geq 0$.

Then

$$Y_t = J_{N(t)} \qquad t \geq 0$$

is called a semi-Markov process.

Consider, now, the marginal distributions

$$Q_1(i, dj) = Q(i, dj \times [0, \infty)) \tag{2.1.3}$$

$$Q_2(i, dt) = Q(i, E \times dt). \tag{2.1.4}$$

If the state space E is finite (or countable), let

$$p_{ij} = Q_1(i, \{j\}) \tag{2.1.5}$$

be the transition probabilities for some Markov chain with state space E. It follows that

$$P\left[J_{n+1} = j \mid J_0, \ldots, J_n; T_0, \ldots, T_n\right] = p_{J_n j}. \tag{2.1.6}$$

Let indicate $Q_{ij}(t) = Q(i, \{j\} \times [0, t])$; if $p_{ij} = 0$ for some pair (i, j), then $Q_{ij}(t) = 0$, $\forall\, t \geq 0$ and we set the ratio $Q_{ij}(t)/p_{ij}$ to be the unity.

With this convention we define

$$F_{ij}(t) = \frac{Q_{ij}(t)}{p_{ij}} \tag{2.1.7}$$

that is the conditional distribution of the sojourn time in the state i given the next equal to j.

Finally, we have

$$H_i(t) = Q_2(i, [0, t]) = \sum_{j \in E} p_{ij} F_{ij}(t) \tag{2.1.8}$$

is the distribution function of the sojourn time in state i.

So, in this special case, we can adopt the definition of Pyke (1961b).

Definition 2.1.3. Let $E = \{0, 1, \dots, m\}$, $m \geq 1$, $\mathbf{Q}(t) = \{Q_{ij}(t) : i, j \in E, t \geq 0\}$ be the matrix of transition distributions such that

(i) $Q_{ij}(t) = 0 \quad t \leq 0$,

(ii) $\sum_{j=0}^{m} Q_{ij}(+\infty) = 1$,

and $\mathbf{A} = (a_0, \dots, a_m)$ be the vector of initial probabilities such that

(iii) $a_j \geq 0$,

(iv) $\sum_{j=0}^{m} a_j = 1$.

The process $(J, T) = \{(J_n, T_n); n \geq 1\}$, defined on a complete probability space $(\Omega, \mathcal{F}, P)$, satisfying $T_0 = 0$ *a.s.*,

$$P[J_0 = k] = a_k \quad a.s. \tag{2.1.9}$$

and $\forall n \geq 0$

$$\begin{aligned} & P[J_{n+1} = j, T_{n+1} \leq t | J_0, J_1, T_1, \dots, J_n, T_n] = \\ & = P[J_{n+1} = j, T_{n+1} \leq t | J_n] = \\ & = Q_{J_n j}(t) \quad a.s., \end{aligned} \tag{2.1.10}$$

is called Markov renewal process determined by $(m, \mathbf{A}, \mathbf{Q})$.

We can consider the counting processes $\{N(t); t \geq 0\}$ and $\{N_j(t); t \geq 0\}$ defined by

$$\begin{aligned} N(t) &= \sup\{n \geq 0 : S_n \leq t\} \\ N_j(t) &= \sharp\{k = 1, \dots, N(t) : J_k = j\}. \end{aligned}$$

Sometimes also $\mathbf{N}(t) = (N_1(t), \dots, N_m(t))$ is called Markov renewal process, while $N(t)$ is the counting Markov renewal process.

The following propositions are quite natural and can help us to better understand the structure of the processes.

Proposition 2.1.1. If E consists of a single point, then $\{T_n,\, n \geq 1\}$ is a renewal process.

Proposition 2.1.2. For any integer $n \geq 1$ and $u_1, \ldots, u_n \in \mathbb{R}_+$,

$$P[T_1 \leq u_1, \ldots, T_n \leq u_n | J_0, J_1, \ldots, J_n] = \prod_{k=1}^{n} H_{J_{k-1}J_k}(u_k) \tag{2.1.11}$$

i.e. the sojourn times $T_1, T_2, \ldots$ are conditionally independent given the Markov chain $J_0, J_1, \ldots$

A usual hypothesis made in all the models derived from Markov renewal processes is the conditional independence between the sojourn time and the next state given the present one. Some authors claim that such assumption does not imply a loss of generality and, even, constitutes a nice feature since, isolating the role of waiting time distributions and transition probabilities, it reproduces the attitude of the analysts in many applications (Duncan (1990)).

Hence, under this assumption, $F_{ij}(t)$ does not depend on j and we have

$$F_{ij}(t) = H_i(t) \tag{2.1.12}$$

$$Q_{ij}(t) = p_{ij} H_i(t). \tag{2.1.13}$$

The following proposition in Çinlar (1975) explains the reasons of the name of such processes.

Proposition 2.1.3. Let $j \in E$ be fixed and define $S_0^j, S_1^j, \ldots$ to be the successive S_n for which $J_n = j$.

Then, $S^j = \{S_n^j;\ n \in \mathbb{N}\}$ is a (possibly delayed) renewal process.

So a Markov renewal process can be viewed as a superposition of these renewal processes; the renewal process which contributed the point S_n is the jth if and only if $J_n = j$.

A realization of a semi-Markov process may be obtained in two different ways.

By the first, suppose a matrix of transition probabilities p_{ij} is given and the initial state is i.

The next state is sampled from the distribution $p_{i\cdot}$; if the outcome is j, then the sojourn time u is sampled from the distribution $F_{ij}(\cdot)$. So the process is set to be i for $t < u$ and jumps to j in u.

The second step starts sampling from $p_{j\cdot}$; if the outcome is k, then a sojourn time is sampled from $F_{jk}(\cdot)$, say v; the process is set to be equal to j in $[u, u+v)$ and a jump to k is given in $u+v$.

The following steps are analogous. The process turns out to have a kernel $Q_{ij}(t) = p_{ij}F_{ij}(t)$.

An alternative approach requires, for each $i \in E$, a distribution function $H_i(t)$ on $\mathbb{R}_+$ and, for each $i \in E$ and $t \in \mathbb{R}_+$, probabilities $K_{ij}(t)$ such that $\sum_j K_{ij}(t) = 1$.

Suppose the process starts from the state i; a sojourn time is sampled from $H_i(\cdot)$. If the outcome is u, the process is set to be equal to i for all $t < u$. The next state is chosen by sampling from $K_{i\cdot}(u)$. If it is j, the successive sojourn time is from $H_j(\cdot)$ and so on.

In this case the kernel is given by $Q_{ij}(t) = \int_0^t H_i(du)K_{ij}(u)$.

If a semi-Markov kernel Q have the form

$$Q_{ij}(t) = p_{ij}\left(1 - e^{-\lambda(i)t}\right), \tag{2.1.14}$$

one can show that

$$P\left[Y_{t+s} = j \mid Y_u,\ u \leq t\right] = P\left[Y_{t+s} = j \mid Y_t\right] \tag{2.1.15}$$

for all $t, s \in \mathbb{R}_+$ and $j \in E$; furthermore, $P\left[Y_{t+s} = j \mid Y_t = i\right]$ in independent of t.

The semi-Markov process becomes an homogeneous Markov process.

Thus, with respect to Markov processes, the novel feature of semi-Markov process is the freedom allowed in the choice of the distributions of sojourn times; this freedom is achieved at the expenses of the Markov property that holds only for the jump times.

2.2 Estimation of the transition distributions of a Markov renewal process

Moore and Pyke (1968) is concerned with the estimation of the transition distributions of a Markov renewal process with a finite number of state; a natural estimator is shown to be consistent and its limiting distribution is derived.

It is assumed the Markov renewal process is irreducible, recurrent and that (2.1.12) holds.

Estimators are defined on sample function of the Markov renewal process over a time interval $[0, t]$ or equivalently on $\left(J_0, J_1, \ldots, J_{N(t)}, T_1, T_2, \ldots, T_{N(t)}\right)$.

Remember

$$N(t) = \sup\left\{n \geq 0 : \sum_{i=0}^{n} T_i \leq t\right\} \tag{2.2.1}$$

$$N_j(t) = \sharp\{k : J_k = j,\ 0 \leq k \leq N(t)\} \tag{2.2.2}$$

$$N_{ij}(t) = \sharp\{k : J_k = i, J_{k-1} = j,\ 0 \leq k < N(t)\} \tag{2.2.3}$$

and let T_{ij} denote the sojourn time of the j-th visit to state i, so that we can consider $\{T_{ij}; 1 \leq i \leq m, 1 \leq j \leq N_i(t)\}$ as a simple relabeling of $\{T_i; 1 \leq i \leq N(t)\}$.

Consider the estimator defined by

$$\hat{Q}_{ij}(x;t) = \hat{p}_{ij}(t)\hat{H}_i(x;t) \tag{2.2.4}$$

where $t, x > 0$

$$\hat{p}_{ij}(t) = \frac{N_{ij}(t)}{N_i(t)} \tag{2.2.5}$$

$$\hat{H}_i(x;t) = \frac{1}{N_i(t)} \sum_{k=1}^{N_i(t)} \mathbb{1}_{[T_{ik} \leq x]}. \tag{2.2.6}$$

Hence, $\hat{H}_i(x;t)$ is the ordinary empirical distribution function determined from the sample, of random size $N_i(t)$, of the sojourn times in state i. Set $\hat{Q}_{ij}(x;t) = 0$, if $N_i(t) = 0$.

Theorem 2.2.1. [Moore and Pyke (1968)]

The estimator in (2.2.4) is uniformly strongly consistent as $t \to \infty$ in the sense that

$$\max_{i,j} \sup_{x} \left|\hat{Q}_{ij}(x;t) - Q_{ij}(x)\right| \to 0 \tag{2.2.7}$$

with probability one.

The limiting distributions of the estimators are obtained exploiting the central limit theorem for Makov renewal process.

Theorem 2.2.2. [Moore and Pyke (1968)]

For fixed i, j and x,

$$\begin{pmatrix} t^{1/2}\left[\hat{p}_{ij}(t) - p_{ij}\right] \\ t^{1/2}\left[\hat{H}_i(x;t) - H_i(x)\right] \end{pmatrix} \xrightarrow{L} \mathcal{N}_2\left(\mathbf{0}, \boldsymbol{\Sigma}\right), \quad t \to \infty \tag{2.2.8}$$

where the covariance matrix is given by

$$\begin{aligned} \Sigma_{11} &= \mu_{ii} p_{ij}(1 - p_{ij}) \\ \Sigma_{22} &= \mu_{ii} H_i(1 - H_i) \\ \Sigma_{12} &= \Sigma_{21} = 0. \end{aligned} \tag{2.2.9}$$

where μ_{ij} denote the mean first passage times from state i to j in the Markov renewal process.

The null correlation between $\hat{p}_{ij}(t)$ and $\hat{H}(x;t)$ implies that they are asymptotically independent.

Corollary 2.2.3. [Moore and Pyke (1968)]

For fixed i, j and x, $\hat{p}_{ij}(t)$ and $\hat{H}(x;t)$ are asymptotically independent.

The asymptotic normality can be used to obtain the limiting distribution of $\hat{Q}(x;t)$.

Corollary 2.2.4. [Moore and Pyke (1968)]

For fixed i, j and x,

$$t^{1/2}\left[\hat{Q}_{ij}(x;t) - Q_{ij}(x)\right] \xrightarrow{L} \mathcal{N}\left(0, \sigma_{ij}^2\right), \quad t \to \infty \tag{2.2.10}$$

where $\sigma_{ij}^2 = \mu_{ii} H_i(x) p_{ij} [H_i(x) - 2H_i(x)p_{ij} + p_{ij}]$.

Let $W_{ijk}(t) = \hat{Q}_{ij}(x_k;t) - Q_{ij}(x_k)$ for $1 \le i, j \le m$ and $1 \le k \le s$.

Theorem 2.2.5. [Moore and Pyke (1968)]

For fixed s, the distribution of $\left\{t^{1/2} W_{ijk}(t);\ 1 \le i, j \le m,\ 1 \le k \le s\right\}$ converges in law as $t \to \infty$ to a m^2s-dimensional normal random variable with zero mean and covariance matrix $\mathbf{\Sigma}$ whose generic element is

$$\Sigma_{ijk,uvw} = \mu_{ii}\delta_{iu}p_{ij}\left[H_i\left(\min\left[x_k, x_w\right]\right)p_{iv} + H_i(x_k)H_i(x_w)\left(\delta_{jv} - 2p_{iv}\right)\right]. \tag{2.2.11}$$

This result implies that the finite dimensional distributions of the stochastic processes $\left\{t^{1/2}\left(\hat{Q}_{ij}(x;t) - Q_{ij}(x)\right), x \ge 0\right\}$ converge to those of a gaussian process. It is also possible to prove that these processes converge weakly.

In particular, a consequence of the weak convergence of empirical processes with random sample size is that $\left\{N_i(t)^{1/2}\left(\hat{Q}_{ij}(x;t) - Q_{ij}(x)\right), x \ge 0\right\}$ converges weakly to a tied-down Wiener process.

Lagakos et al. (1978), working with right censored observations, proposes to estimate the distribution functions on the basis of some maximum likelihood considerations by maximizing the probability of n realizations of the process observed on fixed finite time intervals over all discrete transition probabilities.

Gill (1980) gives rigorous derivations of consistency and weak convergence properties of these estimators.

Consider the counting processes

$$\tilde{N}_{ij}(t) = \sharp\left\{n \geq 1 : \sum_{k=1}^{n} T_k \leq t,\, J_{n-1} = i,\, J_n = j\right\} \tag{2.2.12}$$

for $i, j \leq m,\ t \in [0, \infty]\,, n \in \mathbb{N}$.

Define also

$$\tilde{N}_{i\cdot}(t) = \sum_j \tilde{N}_{ij}(t) \tag{2.2.13}$$

$$\tilde{N}_{\cdot j}(t) = \sum_i \tilde{N}_{ij}(t) \tag{2.2.14}$$

$$\tilde{N}(t) = \sum_{i,j} \tilde{N}_{ij}(t). \tag{2.2.15}$$

Let suppose the following assumptions hold:

a1. There exist random variables T_n such that *a.s.*

$$S_n \leq T_n \leq S_{n+1}\ \forall n \text{ and } K(t) = \sum_n \mathbb{1}_{(S_n, T_n]} \forall\, t$$

a2. There exists a sub-σ-algebra $\mathcal{A}$ of $\mathcal{F}$ containing all P-null set of $\mathcal{F}$, conditionally independent of $\sigma\left\{\tilde{N}_{ij}(s);\ i, j \leq m,\ s \in [0, \infty)\right\}$ given J_0, and such that, for each n, T_n is a stopping time with respect to the family of σ-algebras $\{\mathcal{F}_t;\ t \in [0, \infty)\}$ defined by

$$\mathcal{F}_t = \mathcal{A} \vee \sigma\left\{J_0, \tilde{N}_{ij}(s);\ i, j \leq m,\ s \in [0, t]\right\}.$$

a3. $E\left[\sharp\left\{T_n > S_n\right\}\right] < \infty$.

a3*. $E\left[\left(\sharp\left\{T_n > S_n\right\}\right)^{7+\varepsilon}\right] < \infty$.

The first assumption restricts the model to consider only right-censorship, meanwhile a2. implies that the process $K(t)$, signaling the censorship, is predictable.

For a3. and a3*. the number of at least partially observed sojourn times is certainly almost surely finite. The first is sufficient to prove consistency, the second is needed for weak convergence.

Moreover define

$$\begin{aligned} Y_i(u) &= \sharp\{n \geq 1 : J_{n-1} = i, T_n \geq u, K(S_{n-1}+u) = 1\} && (2.2.16)\\ Y_i(0) &= Y_i(0^+), && (2.2.17)\\ Y(u) &= \sum_i Y_i(u) && (2.2.18) \end{aligned}$$

note that $Y(0) = \sharp\{n : T_n > S_n\}$.

If n independent identically distributed observations of N_{ij} and Y_i are given, let N_{ij}^n, Y_i^n, N_i^n denote the sums of the n realizations of $N_{ij}, Y_i, \ldots$

The estimators are the following

$$\begin{aligned} \hat{H}_i^n(t) &= 1 - \prod_{s \leq t}\left(1 - \frac{\Delta N_i^n(s)}{Y_i^n(s)}\right) \\ &= \int_0^t \left(1 - \hat{H}_i^n(s^-)\right)\frac{dN_i^n(s)}{Y_i^n(s)} && (2.2.19)\\ \hat{Q}_{ij}^n(t) &= \int_0^t \left(1 - \hat{H}_i^n(s^-)\right)\frac{dN_{ij}^n(s)}{Y_i^n(s)} && (2.2.20) \end{aligned}$$

The properties of the estimators are exposed in the theorem below.

Theorem 2.2.6. [Gill (1980)]

Let $\tau_i = \sup\{t : E[Y_i(t)] > 0\}$. Then, as $n \to \infty$,

$$\sup_{t \in [0,\tau_i]} \left|\hat{H}_i^n(t) - H_i(t)\right| \xrightarrow{P} 0 \qquad (2.2.21)$$

$$\sup_{t \in [0,\tau_i]} \left|\hat{Q}_{ij}^n(t) - Q_{ij}(t)\right| \xrightarrow{P} 0 \qquad (2.2.22)$$

unless $E[Y(\tau_i)] = 0$ and $\Delta H_i(\tau_i)$ or $\Delta Q_{ij}(\tau_i) > 0$, in which case $[0, \tau_i]$ must be replaced by $[0, \tau_i)$ in the corresponding supremum.

A further result is the following.

Theorem 2.2.7. [Gill (1980)]
Suppose condition a3*. holds and choose τ_i, $i \leq m$, such that $E[Y_i(\tau_i)] > 0$.
Then, considered as a random element of $\prod_i (D[0, \tau_i])^{m+1}$,

$$\left\{n^{1/2}\left(\hat{Q}_{ij} - Q_{ij}\right),\ n^{1/2}\left(\hat{H}_i - H_i\right);\ i, j \leq m\right\}$$

is asymptotically distributed as

$$\left\{\int \frac{1 - H_{i-}}{E[Y_i]} dW_{ij} - Q_{ij} \int^{-} \frac{1 - H_{i-}}{1 - H_i} \frac{1}{E[Y_i]} dW_i + \int^{-} Q_{ij} \frac{1 - H_{i-}}{1 - H_i} \frac{1}{E[Y_i]} dW_i,\right.$$
$$\left.(1 - H_i) \int \frac{1 - H_{i-}}{1 - H_i} \frac{1}{E[Y_i]} dW_i;\ i, j \leq m\right\}. \tag{2.2.23}$$

Relatively less work has been done in the Bayesian approach both parametric and nonparametric.
Brock (1973) considers a Markov renewal process with a finite states space $E = \{1, \ldots, m\}$ such that

$$Q_{ij}(t) = p_{ij} F_{ij}(t)$$

with initial distribution $\mathbf{A} = (a_1, \ldots, a_m)$ where $a_k = P[J_0 = k]$ and studies the distribution of $\mathbf{N}(t) = \{N_{ij}(t),\ i, j \in E\}$ for $t > 0$, where N_{ij} is the number of direct transitions from i to j in the interval $(0, t)$.
Define the Laplace-Stiltjes transform of Q_{ij}

$$q_{ij}(s) = \int_0^\infty e^{-st} dQ_{ij}(t), \quad t > 0 \tag{2.2.24}$$

and denote by $\mathbf{P}_0$, $\mathbf{F}(t)$, $\mathbf{Q}(t)$, $\mathbf{f}(t)$ and $\mathbf{q}(s)$ the matrices whose (i,j)-th elements are p_{ij}, $F_{ij}(t)$, $Q_{ij}(t)$, $f_{ij}(t)$ and $q_{ij}(s)$.

Exploiting some earlier results, some distributions interesting for the Bayesian analysis are computed in terms of Laplace-Stieltijes transform and probability generating function.

In particular, the prior posterior and the preposterior analyses are carried out using the matrix beta distribution as prior for the matrix of transition probabilities $\mathbf{P}_0$. This distribution may be characterized by the density function

$$f_{M\beta}^{(K,m)}(\mathbf{P};\mathbf{N}) = \begin{cases} \prod_{i,j=1}^{m}\prod_{k=1}^{K_i} B_m(\nu_{-1}^{k})(p_{ij}^{k})^{\nu_{ij}^{k}-1} & \text{if } \mathbf{P} \text{ is a } (K,m)\text{-stochastic matrix} \\ 0 & \text{otherwise} \end{cases} \tag{2.2.25}$$

$\mathbf{P} = \left[p_{ij}^{k}\right]$ is a matrix of the type of $\mathbf{P}_0$. The parameter $\mathbf{N}$ is a (K,m)-matrix such that $\nu_{ij}^{k} > 0$, $k = 1,\ldots,K_i$, $i,j = 1,\ldots,m$ and ν_{-1}^{k} is the generic row of $\mathbf{N}$. The total number of rows of both $\mathbf{P}$ and $\mathbf{N}$ is $K = \sum_{i=1}^{m} K_i$ and $B_m(\nu_{-1}^{k})$ is the generalized beta function

$$B_m(\nu_{-1}^{k}) = \frac{\Gamma\left(\sum_{j=1}^{m}\nu_{ij}^{k}\right)}{\prod_{j=1}^{m}\Gamma\left(\nu_{ij}^{k}\right)}. \tag{2.2.26}$$

Its moments are

$$E\left[p_{ij}\right] = \frac{\nu_{ij}}{\sum_j \nu_{ij}} \quad i,j = 1,\ldots,m \tag{2.2.27}$$

$$Var\left[p_{ij}\right] = \frac{\nu_{ij}\left[\sum_j(\nu_{ij}) - \nu_{ij}\right]}{\left[\sum_j \nu_{ij}^{2}\right]\left[1+\sum_j \nu_{ij}\right]} \quad i,j = 1,\ldots,m \tag{2.2.28}$$

$$Cov\left[p_{\alpha\beta}p_{\gamma\delta}\right] = \begin{cases} -\frac{\nu_{\alpha\beta}\nu_{\alpha\delta}}{\left[\sum_j \nu_{\alpha j}\right]^2\left[1+\sum_j \nu_{\alpha j}\right]} & \alpha=\gamma=1,\ldots,m;\beta,\delta=1,\ldots,m,\beta\neq\delta \\ 0 & \alpha\neq\gamma \end{cases} \tag{2.2.29}$$

It is possible to compute the distribution of the transition counts, given that $\mathbf{P}_0$ obeys a matrix beta distribution and i as initial state i and

$$P\left[\mathbf{N}(t) = \mathbf{N} \,|\, i; \mathbf{N}\right] = \begin{cases} \int_{S_m} w_i(\mathbf{N}, t) f^{M\beta}_{(m,m)}(\mathbf{P}_0; \mathbf{N}) d\mathbf{P}_0 & \mathbf{N} \in \Phi_m(i) \\ \\ 0 & \text{otherwise} \end{cases} \tag{2.2.30}$$

where S_m is the set of (m, m)-stochastic matrices, $\Phi_m(i)$ the set of all possible transitions counts arising from a Markov renewal process with initial state and $w_i(\Xi, s)$ the i-th row of

$$w\left(\Xi, s\right) = \left[\mathbf{I} - \mathbf{q}(s) \Box \Xi\right]^{-1} \left[\mathbf{I} - \mathbf{q}(s)\right] e \tag{2.2.31}$$

the Laplace-Stiltjes transform of $\mathbf{N}(t)$ conditional on $J_0 = i$ (note that $\Box$ denotes the Hadamard matrix product and e a vector of ones).

As example consider the case of a two-state Markov renewal process having probability matrix

$$\mathbf{P}_0 = \begin{bmatrix} 1-x & x \\ y & 1-y \end{bmatrix}, \quad 0 \leq x, y \leq 1 \tag{2.2.32}$$

distributed according to a matrix beta prior distribution with parameters

$$\mathbf{N} = \begin{bmatrix} \nu_{11} & \nu_{12} \\ \nu_{21} & \nu_{22} \end{bmatrix}. \tag{2.2.33}$$

Then the density of $\mathbf{P}_0$ is

$$f^{(2,2)}_{M\beta}\left(\mathbf{P}_0; \mathbf{N}\right) = \frac{1}{B(\nu_{11}, \nu_{12}) B(\nu_{21}, \nu_{22})} x^{\nu_{12}-1} (1-x)^{\nu_{11}-1} y^{\nu_{21}-1} (1-y)^{\nu_{22}-1}. \tag{2.2.34}$$

The Markov renewal process is observed for a fixed interval of time $(0, t)$ and two things may happen:

- the system may not make any transition at all, staying in the initial state, say i. If ν indicate the sample, we denote this $\nu = i$;

- the system makes some transitions: $\nu = (j_0, j_1, \ldots, j_n, x_1, x_2, \ldots, x_n) = (\mathbf{j}, \mathbf{x})$.

Hence the likelihood of a trajectory is

$$L(\nu|\mathbf{P}_0) = \begin{cases} a_i[1 - F_i(t)] & \text{if } \nu = i \\ \\ a_i[1 - F_j(u_t)] \prod_{k=0}^{n-1} f_{j_k}(x_{k+1}) \\ \quad x^{N_{12}(t)}(1-x)^{N_{11}(t)} y^{N_{21}(t)}(1-y)^{N_{22}(t)} & \text{if } \nu = (\mathbf{j}, \mathbf{x}) \end{cases} \tag{2.2.35}$$

and the marginal distribution of ν is

$$L(\nu) = \begin{cases} a_i[1 - F_i(t)] & \text{if } \nu = i \\ \\ \frac{a_i[1-F_j(u_t)] \prod_{k=0}^{n-1} f_{j_k}(x_{k+1})}{B(\nu_{11},\nu_{12}), B(\nu_{21},\nu_{22})} \\ \quad B_{11,12}[\mathbf{N}(t), \mathbf{N}] B_{21,22}[\mathbf{N}(t), \mathbf{N}] & \text{if } \nu = (\mathbf{j}, \mathbf{x}). \end{cases} \tag{2.2.36}$$

Finally the posterior distribution of $\mathbf{P}_0$ given ν

$$L(\mathbf{P}_0|\nu) = \begin{cases} \frac{1}{B(\nu_{11},\nu_{12})B(\nu_{21},\nu_{22})} x^{\nu_{12}-1}(1-x)^{\nu_{11}-1} y^{\nu_{21}-1}(1-y)^{\nu_{22}-1} & \text{if } \nu = i \\ \\ \frac{1}{B_{11,12}[\mathbf{N}(t),\mathbf{N}] B_{21,22}[\mathbf{N}(t),\mathbf{N}]} x^{N_{12}(t)+\nu_{12}-1}(1-x)^{N_{11}(t)+\nu_{11}-1} \\ \quad y^{N_{21}(t)+\nu_{21}-1}(1-y)^{N_{22}(t)+\nu_{22}-1} & \text{if } \nu = (\mathbf{j}, \mathbf{x}). \end{cases} \tag{2.2.37}$$

A more complete Bayesian study of Markov renewal processes is carried through by Phelan (1990a). This approach can be seen as a considerable generalization of the previous one and it is based on the parametrization of the transition distributions in terms of transition probabilities of a Markov chain and cumulative hazard functions of life distributions.

Following Pyke and Schaufele (1964) it is assumed

$$Q_{ij}(t) = p_{ij} H_i(t), \; t \geq 0 \tag{2.2.38}$$

and representing the distribution function by the product integral (see Gill and Johansen (1990))

$$Q_{ij}(t) = p_{ij} \prod_{[0,t]} [1 - db_i(s)], \; t \geq 0. \tag{2.2.39}$$

Moreover, a trajectory of a Markov renewal process in the interval $[0, t]$ can be written as $(N(t), R(t))$ where $N(t) = \sum_{n\geq 1} \mathbb{1}_{[S_n \leq t]}$ and $R(t) = (J_0, \ldots, J_{N(t)}, T_1, \ldots, T_{N(t)}, t - S_{N(t)})$ so that

$$P[N(t) = n, R(t) \in G] = \int_G dP(n; \pi), \tag{2.2.40}$$

where

$$dP(n;\pi) = p_{j_0} \prod_{[0,u_t]} [1 - db_{j_n}(s)] \prod_{k=0}^{n-1} p_{j_k j_{k+1}} \prod_{(0,t_{k+1})} [1 - db_{j_k}(s)]\, db_{j_k}(t_{k+1}) \tag{2.2.41}$$

A procedure for the inference in for Markov renewal processes can be defined as follows.

Let $\boldsymbol{M}$ be a stochastic Markov transition matrix having each row $\boldsymbol{M}_i$ independent and distributed according to a Dirichlet distribution of parameter $\boldsymbol{\beta}_i$, so that we say $\boldsymbol{M}$ to have a product Dirichlet distribution with the matrix $\boldsymbol{\beta} = (\boldsymbol{\beta}_0, \ldots, \boldsymbol{\beta}_m)'$ as parameter; let also $\boldsymbol{A}$ be a vector of cumulative hazards independent and distributed as a Beta process (c_i, b_i) (a vector Beta distribution).

As the following theorem claims, this prior is conjugate.

Theorem 2.2.8. [Phelan (1990a)]

Define

$$N_{ij} = \sum_{k=1}^{N(t)} \mathbb{1}_{[J_{k-1}=i, J_k=j]}, \quad i,j \in E, \tag{2.2.42}$$

$$N_i(s) = \sum_{k=1}^{N(0t)} \mathbb{1}_{[J_{k-1}=i, T_k \leq s]}, \quad s \geq 0, i \in E, \tag{2.2.43}$$

$$Y_i(s) = \mathbb{1}_{\left[J_{N(t)}=i, t-S_{N(t)} \geq s\right]} + \sum_{k=1}^{N(t)} \mathbb{1}_{[J_{k-1}, T_k \geq s]}, \quad s \geq 0, i \in E. \tag{2.2.44}$$

Let $\boldsymbol{\beta}$, $\boldsymbol{c}$ and $\boldsymbol{b}$ be the parameters of the above described prior.

Finally, for $i \in E$ ans $s \geq 0$

$$\beta^*_{ij} = \beta_{ij} + N_{ij}, \tag{2.2.45}$$

$$c^*_i(s) = c_i(s) + Y_i(s), \tag{2.2.46}$$

$$b^*_i(s) = \int_0^s \frac{c_i db_i + dN_i}{c_i + Y_i}. \tag{2.2.47}$$

Consider the probability model from a Markov renewal process with the parametrization above described and let $(N(t), R(t))$ be an observation of this process in $[0, t]$. Then $\boldsymbol{M}$ and $\boldsymbol{A}$ are conditionally independent given $(N(t), R(t))$ with conditional distributions given by a product Dirichlet with parameter $\boldsymbol{\beta}^*$ and a vector Beta with parameters $\boldsymbol{c}^*$ and $\boldsymbol{b}^*$.

This prior will be important in our work. Indeed, in the next chapter we will present an urn scheme to characterize a prior for inference in Markov renewal processes arisen from a different parametrization focusin not on cumulative hazards, but on distribution functions. By the way, the relation between the two priors can be brought back to the well relation between Beta and beta-Stacy processes.

2.3 Mixtures of semi-Markov processes

Following Epifani et al. (2002), we consider a minimal chain $Y = (Y_t,\ t \geq 0)$ with a countable state space E, i.e. processes whose trajectories are right-continuous step functions that may have infinitely many jumps in a finite time interval and get stuck in an extra state not belonging to E. In this case, after the explosion time, the behavior of the process is not studied.

The law of Y is a mixture of recurrent minimal semi-Markov processes if and only if row-wise partial exchangeability of the matrix of the successor states and the sojourn times holds.

Without loss of generality, it is possible to take $E = \mathbb{N}$; let introduce a point $\partial \notin E$ and put $E^* = E \cup \{\partial\}$.

More formally, let Ω be the set of *generalized right-continuous step functions* from $[0, \infty)$ into E^*: a function $\omega \in \Omega$ is a right continuous step function with values in E^* up to explosion time, with ∂ an absorbing state, and it remains equal to ∂ after an explosion.

Let $0 = S_0(\omega) < S_1(\omega) < S_2(\omega) < \ldots$ be the discontinuities of ω. If $S_n(\omega) < +\infty$, let $J_n(\omega)$ be the value of ω in the interval $[S_n(\omega), S_{n-1}(\omega))$ and $T_n(\omega) = S_{n+1}(\omega) - S_n(\omega)$ be the sojourn times of ω in $J_n(\omega)$.

If ω has just n discontinuities, set $S_k(\omega) = +\infty$ for $k \geq n$ and $T_n(\omega) = +\infty$.

Finally, $\zeta(\omega) = \sum_{n=0}^{\infty} T_n(\omega)$ is the explosion time of ω.

Introduce the canonical process $Y = (Y_t,\ t \geq 0)$ on Ω such that $Y_t(\omega) = \omega_t$ and endow Ω with the smallest σ-field $\mathcal{F}$ with respect to which all Y_t are measurable. Also J_n and T_n are $\mathcal{F}$-measurable for every $n \geq 0$.

Let define the array of successor states and sojourn times of Y.

Definition 2.3.1. For any $i \in E^*$, let ν_{im} be the m-th *visiting time* to i ($\nu_{i0} = -1$, $\inf \emptyset = +\infty$):

$$\nu_{im} = \inf \{n \geq \nu_{im-1} : J_n = i\} \tag{2.3.1}$$

the *successor state* of the m-th visit of Y to i is

$$\sigma_{im} = J_{\nu_{im}+1} \tag{2.3.2}$$

and the *sojourn time* of Y in the state i after the m-th visit is

$$\tau_{im} := T_{\nu_{im}} \tag{2.3.3}$$

with the convention that $J_{+\infty} = \partial$ and $T_{+\infty} = +\infty$. We have also $\sigma_{\partial m} = \partial$ and $\tau_{\partial m} = +\infty$, $\forall\, m \geq 1$.

The elements of the array $[\boldsymbol{\sigma}, \boldsymbol{\tau}] = \{(\sigma_{im}, \tau_{im}),\, i \in E^*, m \geq 1\}$ are said *row-wise P-partially exchangeable* if

$$P\left[\bigcap_{i\in\mathcal{K}} \bigcap_{m=1}^{n} \{\sigma_{im} \in A_{im}, \tau_{im} \in C_{im}\}\right] = P\left[\bigcap_{i\in\mathcal{K}} \bigcap_{m=1}^{n} \left\{\sigma_{im} \in A_{i\pi_i(m)}, \tau_{im} \in C_{i\pi_i(m)}\right\}\right] \tag{2.3.4}$$

for all $\mathcal{K} = \{\partial, 1, \ldots, k\}$, $A_{im} \in \mathcal{P}(E^*)$, $C_{im} \in \mathcal{B}((0, +\infty])$ for each π_i varying on the permutations of $\{1, \ldots, n\}$ with $k, n \geq 1$.

If $\mathbb{H}$ indicates the set of all probability measures on $(E^* \times (0, +\infty], \mathcal{P}(E^*) \otimes \mathcal{B}((0, +\infty])$ with the topology of the weak convergence, then there exists a sequence of random probability measures $\tilde{H} = \left(\tilde{H}_i\right)_{i\in E^*}$ from $(\Omega, \mathcal{F})$ into $\mathbb{H}^\infty$ such that

$$\frac{1}{n} \sum_{m=1}^{n} \delta_{(\sigma_{im}, \tau_{im})} \Rightarrow \tilde{H}_i \quad a.s. - P \tag{2.3.5}$$

for $i \in E^*$ and $n \to +\infty$.

Moreover, if $P_{\tilde{H}}$ denote the conditional probability on $(\Omega, \mathcal{F})$ given $\tilde{H}$, the condition (2.3.4) is equivalent to

$$P_{\tilde{H}}\left[\bigcap_{i\in\mathcal{K}}\bigcap_{m=1}^{n}\{\sigma_{im}\in A_{im}, \tau_{im}\in C_{im}\}\right] = \prod_{i\in\mathcal{K}}\prod_{m=1}^{n}\tilde{H}_i(A_{im}, C_{im}) \quad a.s.-P \tag{2.3.6}$$

See de Finetti's representation theorem for partially exchangeable arrays.

The theorem below provides a characterization of the mixtures of semi-Markov distributions in terms of partial exchangeability by rows of the bidimensional random variables $(\sigma_{in}, \tau_{in}, i \in E^*, n \geq 1)$. The mixing measure is a probability on a class of transition kernels.

Theorem 2.3.1. Let $E^* = \mathcal{N} \cup \{\partial\}$. The elements of the array $(\boldsymbol{\sigma}, \boldsymbol{\tau})$ are P-partially exchangeable by rows if and only if there exists a probability measure μ on $(\mathbb{H}^\infty, \mathcal{B}(\mathbb{H}^\infty))$ such that

1. $\mu(\mathcal{H}_0) = 1$

2. $\mu\{H \in \mathcal{H}_0 : i_0 \text{ is recurrent for } S(i_0, H)\} = 1$

3. for any $i_1, \ldots, i_n$ in I, $C_0, \ldots, C_{n-1}$ in $\mathcal{B}((0, +\infty])$ and $n \geq 1$:

$$\begin{aligned} P\{J_1 = i_1, \ldots, J_n = i_n, T_0 \in C_0, \ldots, T_{n-1} \in C_{n-1}\} \\ = \int_{\mathbb{Q}^\infty} \prod_{s=0}^{n-1} Q(i_s, i_{s+1}, C_s)\mu(dQ). \end{aligned} \tag{2.3.7}$$

Furthermore, the mixing measure μ is uniquely determined.

The last condition of the theorem is equivalently expressed by saying that there exists a random transition kernel $\tilde{Q}$ such that

$$P_{\tilde{Q}}\{J_1 = i_1, \dots, J_n = i_n, T_0 \in C_0, \dots, T_{n-1} \in C_{n-1}\} \\ = \prod_{s=0}^{n-1} \tilde{Q}(i_s, i_{s+1}, C_s) \quad a.s. - P. \tag{2.3.8}$$

This can be rewritten as

$$P_{\tilde{Q}}\{J_1 = i_1, \dots, J_n = i_n, T_0 \in C_0, \dots, T_{n-1} \in C_{n-1}\} \\ = \prod_{s=0}^{n-1} \tilde{p}_{i_s,i_{s+1}} \tilde{F}_{i_s,i_{s+1}}(C_s) \quad a.s. - P. \tag{2.3.9}$$

The proposition states the equivalence between (2.3.4) and a partial exchangeability condition involving the sojourn times in i when the process next makes a jump into the state j, for all i, j.

Let consider the m-th visit of Y to the string (i, j) ($\nu_{ij0} := -1$, $\inf \emptyset = +\infty$):

$$\nu_{ijm} = \inf\{n > \nu_{ijm-1} : J_n = i, J_{n+1} = j\} \quad m \geq 1 \tag{2.3.10}$$

and $\tau = [\tau_{ijm}, \; i, j \in E^*, m \geq 1]$ the array of sojourn time in the state i when the next is j.

The following condition is equivalent to (2.3.4):

$$P\left[\bigcap_{i,j\in\mathcal{K}} \bigcap_{m=1}^{n} \{\sigma_{im} = x_{im}, \tau_{ijm} \in C_{ijm}\}\right] = P\left[\bigcap_{i,j\in\mathcal{K}} \bigcap_{m=1}^{n} \left\{\sigma_{ij} = x_{i\pi_i(m)}, \tau_{ijm} \in C_{ij\rho_{ij}(m)}\right\}\right] \tag{2.3.11}$$

for all $\mathcal{K}$, $x_{im} \in E^*$, $C_{ijm} \in \mathcal{B}((0, +\infty])$ and for each π_i and ρ_{ij} varying permutations of $\{1, \dots, n\}$.

From this equivalence, it turns out that the representation of the law of (J_n, T_{n-1}) holds under (2.3.11) as well.

Moreover, for every $i, j \in E^*$, $a.s. - P$:

$$\frac{1}{n}\sum_{m=1}^{n}\delta_{\sigma_{im}} \Rightarrow \tilde{p}_{i\cdot} \tag{2.3.12}$$

$$\frac{1}{n}\sum_{m=1}^{n}\delta_{\tau_{ijm}} \Rightarrow \tilde{F}_{ij}(\cdot). \tag{2.3.13}$$

The reason of this short digression about the mixture of semi-Markov processes lies in the fact that these results will be used in the next chapter. More precisely, a multi-state reinforced will be defined and the study of the property of partial exchangeability for the matrix of its successor states and sojourn times will assure, on the basis of these considerations, the existence of a de Finetti's measure useful as prior.

Chapter 3

Reinforced Markov renewal processes

At the end of the first chapter, the problem of looking for an urn scheme able to generate the beta-Stacy process in its version also for continuous distribution functions has been left unanswered.

To this purpose Muliere et al. (2003) have introduced a so called reinforced renewal process whose name points out its derivation from reinforced processes presented in the previous chapter.

In the light of the same relation between reinforced processes and Markov renewal processes, the reinforced renewal processes are exploited to build another class of processes with reinforcement in continuous time, the reinforced Markov renewal processes.

As usual the reinforcement mechanism yields the property of exchangeability. In such a way a Bayesian nonparametric estimation procedure has been developed. In particular, via exchangeability of interarrival times and independence between sojourn times and successor states a prior for Markov renewal processes is naturally characterized.

More precisely, it consists of a stochastic matrix having on each row a Dirichlet distribution for transition probabilities and a beta-Stacy process for each distribution of the sojourn time in a fixed state.

Section 3.1 and section 3.2 deal with the definitions of the processes, their properties and the existence and identification of the mixing measure.

In section 3.3 the prior distribution is thoroughly studied; the conjugacy property is proved and the posterior distribution is computed. Also predictives and Bayesian estimators are obtained; the relation of this prior with Phelan (1990a)'s one is cleared up.

3.1 Reinforced renewal processes

In this section a reinforced process in continuous time is defined following Muliere et al. (2003).

Suppose to have a usual renewal process and drop the essential assumption of independence and identical distribution of interarrival time T_n's in such a way as to allow a particular form of dependence between them in order to make possible a kind of Bayesian learning from the past observations.

The formal definition is the following.

Definition 3.1.1. Let α be a positive measure on $\mathcal{B}[0,\infty)$ such that $\alpha(\{0\}) = 0$ and $0 < d_1 < d_2 < \cdots < d_n < \dots$ are the points in which α concentrates positive mass.

Let $D = \{d_i, i \geq 1\}$ and α_c be the continuous (with respect to Lebesgue measure) part of α defined as follows

$$\alpha_c(0,t] = \alpha(0,t] - \sum_{d_i \leq t} \alpha(\{d_i\}) \tag{3.1.1}$$

Let $\beta : [0,\infty) \to (0,\infty)$ be a positive and measurable function.

The function

$$F_{\alpha,\beta}(t) = \begin{cases} 1 - \prod_{d_i \leq t} \left[1 - \frac{\alpha(\{d_i\})}{\alpha(\{d_i\})+\beta(d_i)}\right] e^{-\int_0^t \frac{d\alpha_c(s)}{\beta(s)}} & t > 0 \\ \\ 0 & t \leq 0 \end{cases} \tag{3.1.2}$$

is nondecreasing and right-continuous, and if α and β are such that

$$\prod_{d_i < \infty} \left[1 - \frac{\alpha(\{d_i\})}{\alpha(\{d_i\}) + \beta(d_i)}\right] e^{-\int_0^\infty \frac{d\alpha_c(s)}{\beta(s)}} = 0, \tag{3.1.3}$$

then, $F_{\alpha,\beta}(t)$ is a proper distribution function.

Consider a random variable T_1 on $[0,\infty)$ with distribution function $F_{\alpha,\beta}$ and define recursively a sequence of interarrival times, for $n \geq 1$, such that the conditional distribution of $T_{n+1}| T_1, \ldots, T_n$ is equal to F_{α_n,β_n} with

$$\begin{aligned} \alpha_n(0,t] &= \alpha(0,t] + \sum_{i=1}^{n} \mathbb{1}_{[T_i \leq t]} \\ \beta_n(t) &= \beta(t) + \sum_{i=1}^{n} \mathbb{1}_{[T_i > t]} \end{aligned} \tag{3.1.4}$$

for $t > 0$.

The sequence of times $\{T_n;\, n \in \mathbb{N}\}$ or, equivalently, the point process

$$N(t) = \sup\left\{n \geq 0 : \sum_{i=1}^{n} T_i \leq t\right\}, \quad t \geq 0, \tag{3.1.5}$$

is said a *reinforced renewal process* with parameters (α, β).

Note that if α and β satisfy (3.1.3), then, for $n \geq 1$, α_n and β_n satisfy the same condition *a.s.* and $\{T_n;\ n \in \mathbb{N}\}$ is well defined.

For the following, it is useful to introduce the product integral representation (see Gill and Johansen (1990)). The cumulative hazard function of a random variable distributed according to $F_{\alpha,\beta}$ is equal to [1]

$$\Lambda(t) = \int_0^t \frac{d\alpha_c(s)}{\alpha(\{s\}) + \beta(s)} + \sum_{d_i \leq t} \frac{\alpha(\{d_i\})}{\alpha(\{d_i\}) + \beta(d_i)}. \tag{3.1.6}$$

So, by means of the (3.1.2), the first interarrival time T_1 of a reinforced renewal process with parameters (α, β) has a survival function, using product integral, equal to [2]

$$\begin{aligned} S(t) &= P[T_1 > t] = \\ &= \prod_{d_i \leq t} \left[1 - \frac{\alpha(\{d_i\})}{\alpha(\{d_i\}) + \beta(d_i)}\right] e^{-\int_0^t \frac{d\alpha_c(s)}{\beta(s)}} = \\ &= \prod_0^t \left[1 - \frac{d\alpha(s)}{\alpha(\{s\}) + \beta(s)}\right] = \\ &= \prod_0^t [1 - d\Lambda(s)] \end{aligned} \tag{3.1.7}$$

Denoting $\mathbf{T}^n = (T_1, \ldots, T_n)$, the vector of the n previous interarrival times, from

[1] In the notation we drop the dependence on α and β.

[2] Let

$$\prod_{[0,t]} = \prod_0^t \quad \text{and} \quad \prod_{[0,t)} = \prod_0^{t^-}.$$

(3.1.4), it is possible to compute the cumulative hazard function of T_{n+1} given $\mathbf{T}^n$

$$\begin{aligned}\Lambda_n(t;\mathbf{T}^n) &= \int_0^t \frac{d\alpha_n(s)}{\alpha_n(\{s\})+\beta_n(s)} \\ &= \int_0^t \frac{d\alpha_c(s)+\sum_{i=1}^n \delta_{T_i}(s)}{\alpha(\{s\})+\beta(s)+\sum_{i=1}^n \mathbb{1}_{[T_i\geq s]}} \\ &\quad + \sum_{d_j\leq t} \frac{\alpha(\{d_j\})+\sum_{i=1}^n \delta_{T_i}(d_j)}{\alpha(\{d_j\})+\beta(d_j)+\sum_{i=1}^n \mathbb{1}_{[T_i\geq d_j]}}.\end{aligned} \tag{3.1.8}$$

So the survival function of T_{n+1} given $\mathbf{T}^n$ (the predictive distribution) is, from the product integral representation,

$$\begin{aligned}P\left[T_{n+1}>t|\mathbf{T}^n\right] &= \prod_0^t \left[1-d\Lambda_n(s;\mathbf{T}^n)\right] \\ &= \prod_0^t \left[1-\frac{d\alpha_n(s)}{\alpha_n(\{s\})+\beta_n(s)}\right]\end{aligned} \tag{3.1.9}$$

Muliere et al. (2003) provides an interpretation based on Pólya urns clarifying the meaning of the name of these processes and how the mechanism of learning works.

It is possible to think at T_n's as sequentially observed. Because of reinforcement mechanisms of (3.1.4), the conditional hazard function is constructed in such a way as the probability of surviving in a given infinitesimal interval $[t, t+dt)$ increases if the previous observations have done so.

By mean of this definition, the interarrival times of a reinforced renewal process turn out to be exchangeable.

Exchangeability is a key concept in Bayesian statistics and in this context reveals that the occurrences, nevertheless each of these gives an additional contribution through the reinforcement mechanism, can be considered similar with respect to the state of information.

This idea is mathematically developed via de Finetti's representation theorem.

Now, we show that the interarrival times are exchangeable and their de Finetti's measure is a beta-Stacy process.

First of all, we compute explicitly the value of the product integral in (3.1.9) that is

$$\prod_0^t [1 - d\Lambda_n(s;\mathbf{T}^n)] =$$

$$\begin{cases} \prod_{j=1}^{k-1}\left[\prod_{T_{(j-1)}}^{T_{(j)}}\left[1-\frac{d\alpha(s)}{\alpha(\{s\})+\beta(s)+(n-j+1)}\right]\left[\frac{\alpha(\{T_{(j)}\})+\beta(T_{(j)})+(n-j)}{\alpha(\{T_{(j)}\})+\beta(T_{(j)})+(n-j+1)}\right]\right] \\ \prod_{T_{(k-1)}}^{t}\left[1-\frac{d\alpha(s)}{\alpha(\{s\})+\beta(s)+(n-k+1)}\right] \\ \qquad\qquad\qquad \text{if} \quad T_{(k-1)} \leq t < T_{(k)},\ k=1,2,\ldots,n \\ \prod_{j=1}^{n}\left[\prod_{T_{(j-1)}}^{T_{(j)}}\left[1-\frac{d\alpha(s)}{\alpha(\{s\})+\beta(s)+(n-j+1)}\right]\left[\frac{\alpha(\{T_{(j)}\})+\beta(T_{(j)})+(n-j)}{\alpha(\{T_{(j)}\})+\beta(T_{(j)})+(n-j+1)}\right]\right] \\ \prod_{T_{(n)}}^{t}\left[1-\frac{d\alpha(s)}{\alpha(\{s\})+\beta(s)}\right] \\ \qquad\qquad\qquad \text{if} \quad t \geq T_{(n)} \end{cases}$$

where $(T_{(1)}, T_{(2)}, \ldots, T_{(n)})$ is the vector obtained by the increasingly ordered permutation of $\mathbf{T}^n$, $T_{(0)} = 0$ and $\prod_1^0 \cdot = 1$.

It is possible to see that the distribution of T_{n+1} given $\mathbf{T}^n$ is the same for any permutation of $(T_1, \ldots, T_n)$.

Following Gill (1994) we obtain, as a consequence of the backward equation for product integral and Markov processes

$$F(dt) = \prod_0^{t^-}\left[1 - \frac{d\alpha(s)}{\alpha(\{s\}) + \beta(s)}\right]\frac{d\alpha(t)}{\alpha(\{t\}) + \beta(t)} \tag{3.1.10}$$

and, therefore, the density function of T_1 is

$$f_{T_1}(t) = \left[\frac{\alpha_c'(t)}{\beta(t)}\mathbb{1}_{[t\notin D]} + \frac{\alpha(\{t\})}{\alpha(\{t\})+\beta(t)}\mathbb{1}_{[t\in D]}\right]\prod_0^{t^-}[1-d\Lambda(s)] \tag{3.1.11}$$

where $\alpha_c'(t)$ is the Radon-Nykodim derivative of α_c with respect to Lebesgue measure λ on $[0,\infty)$.

In general, the equation (3.1.11) is the Radon-Nykodim derivative of the distribution of T_1 with respect to the measure

$$\mu(\cdot) = \sum_{d_i\in D}\mathbb{1}_{[d_i\in\cdot]} + \lambda(\cdot). \tag{3.1.12}$$

that is the sum of the counting measure on positive mass point of α and the Lebesgue measure.

If, for simplicity, we set

$$\alpha'(t) := \alpha_c'(t)\mathbb{1}_{[t\notin D]} + \alpha(\{t\})\mathbb{1}_{[t\in D]},$$

we can write

$$f_{T_1}(t) = \frac{\alpha'(t)}{\alpha(\{t\})+\beta(t)}\prod_0^{t^-}[1-d\Lambda(s)]. \tag{3.1.13}$$

Similarly, the conditional density of T_{n+1} given $\mathbf{T}^n$ results

$$\begin{aligned} f_{T_{n+1}|\mathbf{T}^n}(t) &= \left[\frac{\alpha_c'(t)+\sum_{i=1}^n\delta_{T_i}(t)}{\beta(t)+\sum_{i=1}^n\mathbb{1}_{[T_i\geq t]}}\mathbb{1}_{[t\notin D]} + \frac{\alpha(\{t\})+\sum_{i=1}^n\delta_{T_i}(t)}{\alpha(\{t\})+\beta(t)+\sum_{i=1}^n\mathbb{1}_{[T_i\geq t]}}\mathbb{1}_{[t\in D]}\right] \\ &\quad \prod_0^{t}[1-d\Lambda_n(s;T^n)] = \\ &= \frac{\alpha'(t)+\sum_{i=1}^n\delta_{T_i}(t)}{\alpha(\{t\})+\beta(t)+\sum_{i=1}^n\mathbb{1}_{[T_i\geq t]}}\prod_0^{t^-}[1-d\Lambda_n(s;T^n)] \end{aligned} \tag{3.1.14}$$

and the density of $\mathbf{T}^{n+1}$ computed in $\mathbf{t}^{n+1} = (t_1, \ldots, t_{n+1})$ is

$$f_{\mathbf{T}^{n+1}}(\mathbf{t}^{n+1}) = \prod_{j=0}^{n} \left[\frac{\alpha'(t_{j+1}) + \sum_{i=1}^{j} \delta_{t_i}(t_{j+1})}{\alpha(\{t_{j+1}\}) + \beta(t_{j+1}) + \sum_{i=1}^{j} \mathbb{1}_{[t_i \geq t_{j+1}]}} \prod_{0}^{t_{j+1}^-} [1 - d\Lambda_j(s; t^j)] \right] \tag{3.1.15}$$

Now, we can state the following.

Theorem 3.1.1. In a reinforced renewal process, the sequence of the interarrival times $T = \{T_n, n \geq 1\}$ is exchangeable.

Proof. We have to show that $f_{\mathbf{T}^{n+1}}(t_1, \ldots, t_{n+1}) = f_{\mathbf{T}^{n+1}}(\sigma(t_1, \ldots, t_{n+1}))$ for any $\sigma(t_1, \ldots, t_{n+1})$ permutation of $(t_1, \ldots, t_{n+1})$.

As one can obtain a given permutation by a finite number of permutations of contiguous elements, it suffices to show

$$f_{\mathbf{T}^{n+1}}(t_1, \ldots, t_k, t_{k+1}, \ldots t_{n+1}) = f_{\mathbf{T}^{n+1}}(t_1, \ldots, t_{k+1}, t_k, \ldots t_{n+1}). \tag{3.1.16}$$

Note that the conditional densities depend only on the ordered conditioning random variables and so we have to verify the following:

$$f_{T_k|\mathbf{T}^{k-1}}(t_k|\mathbf{t}^{k-1}) f_{T_{k+1}|\mathbf{T}^k}(t_{k+1}|\mathbf{t}^{k-1}, t_k) = f_{T_k|\mathbf{T}^{k-1}}(t_{k+1}|\mathbf{t}^{k-1}) f_{T_{k+1}|\mathbf{T}^k}(t_k|\mathbf{t}^{k-1}, t_{k+1}). \tag{3.1.17}$$

The case $t_k = t_{k+1}$ is obvious.

Now, take the vector $(t_1, \ldots, t_{k-1}, t_k, t_{k+1})$ and its increasing ordered permutation $(t_{(1)}, \ldots, t_{(k-1)}, t_{(k)}, t_{(k+1)})$. Moreover, suppose $t_k = t_{(l)}$ and $t_{k+1} = t_{(m)}$ for $l \neq m$.

Consider the case $t_k < t_{k+1}$ $(l < m)$, we have

$$
\begin{aligned}
&\sum_{i=1}^{k-1} \mathbb{1}_{[t_i \geq t_k]} = \sum_{i=1}^{k-1} \mathbb{1}_{\left[t_i \geq t_{(l)}\right]} = k - l \\
&\sum_{i=1}^{k} \mathbb{1}_{[t_i \geq t_{k+1}]} = \sum_{i=1}^{k} \mathbb{1}_{\left[t_i \geq t_{(m)}\right]} = k - (m-1) \\
&\sum_{i=1}^{k-1} \mathbb{1}_{[t_i \geq t_{k+1}]} = \sum_{i=1}^{k} \mathbb{1}_{[t_i \geq t_{k+1}]} - \mathbb{1}_{[t_k \geq t_{k+1}]} = k - (m-1) \\
&\mathbb{1}_{[t_{k+1} \geq t_k]} = 1
\end{aligned}
$$

We can also compute the product integrals and obtain the following relations

$$
\begin{aligned}
&\prod_0^{t_{k+1}^-} \left[1 - d\Lambda_{k-1}(s; \mathbf{t}^{k-1})\right] = \\
&= \prod_0^{t_{(l)}^-} \left[1 - d\Lambda_{k-1}(s; \mathbf{t}^{k-1})\right] \prod_{t_{(l)}}^{t_{(l+1)}} \left[1 - \frac{d\alpha(s)}{\alpha(\{s\}) + \beta(s) + k - l}\right] \\
&\quad \frac{\alpha(\{t_{(l+1)}\}) + \beta(t_{(l+1)}) + k - l - 1}{\alpha(\{t_{(l+1)}\}) + \beta(t_{(l+1)}) + k - l} \\
&\quad \prod_{j=l+2}^{m-1} \left\{ \prod_{t_{(j-1)}}^{t_{(j)}} \left[1 - \frac{d\alpha(s)}{\alpha(\{s\}) + \beta(s) + k - j + 1}\right] \left[\frac{\alpha(\{t_{(j)}\}) + \beta(t_{(j)}) + k - j}{\alpha(\{t_{(j)}\}) + \beta(t_{(j)}) + k - j + 1}\right] \right\} \\
&\quad \prod_{t_{(m-1)}}^{t_{(m)}^-} \left[1 - \frac{d\alpha(s)}{\alpha(\{s\}) + \beta(s) + k - m + 1}\right] .
\end{aligned}
\tag{3.1.18}
$$

and

$$\prod_{0}^{t_{k+1}^{-}} \left[1 - d\Lambda_k(s; \mathbf{t}^{k-1}, t_k)\right] =$$

$$= \prod_{0}^{t_{(l)}^{-}} \left[1 - d\Lambda_k(s; \mathbf{t}^{k-1}, t_{k+1})\right] \left[\frac{\alpha(\{t_{(l)}\}) + \beta(t_{(l)}) + k - l}{\alpha(\{t_{(l)}\}) + \beta(t_{(l)}) + k - l + 1}\right]$$

$$\prod_{j=l+1}^{m-1} \left\{ \prod_{t_{(j-1)}}^{t_{(j)}} \left[1 - \frac{d\alpha(s)}{\alpha(\{s\}) + \beta(s) + k - j + 1}\right] \left[\frac{\alpha(\{t_{(j)}\}) + \beta(t_{(j)}) + k - j}{\alpha(\{t_{(j)}\}) + \beta(t_{(j)}) + k - j + 1}\right] \right\}$$

$$\prod_{t_{(m-1)}}^{t_{(m)}^{-}} \left[1 - \frac{d\alpha(s)}{\alpha(\{s\}) + \beta(s) + k - m + 1}\right] \tag{3.1.19}$$

Using (3.1.14) and these relations in (3.1.17), after some simplifications, we obtain

$$\prod_{j=l+1}^{m-1} \left\{ \prod_{t_{(j-1)}}^{t_{(j)}} \left[1 - \frac{d\alpha(s)}{\alpha(\{s\})+\beta(s)+k-j+1}\right] \left[\frac{\alpha(\{t_{(j)}\})+\beta(t_{(j)})+k-j}{\alpha(\{t_{(j)}\})+\beta(t_{(j)})+k-j+1}\right] \right\}$$

$$=$$

$$\prod_{t_{(l)}}^{t_{(l+1)}} \left[1 - \frac{d\alpha(s)}{\alpha(\{s\})+\beta(s)+k-l}\right] \frac{\alpha(\{t_{(l+1)}\})+\beta(t_{(l+1)})+k-l-1}{\alpha(\{t_{(l+1)}\})+\beta(t_{(l+1)})+k-l}$$

$$\prod_{j=l+2}^{m-1} \left\{ \prod_{t_{(j-1)}}^{t_{(j)}} \left[1 - \frac{d\alpha(s)}{\alpha(\{s\})+\beta(s)+k-j+1}\right] \left[\frac{\alpha(\{t_{(j)}\})+\beta(t_{(j)})+k-j}{\alpha(\{t_{(j)}\})+\beta(t_{(j)})+k-j+1}\right] \right\}$$

showing that the condition (3.1.17) holds.

Similarly in the case $t_k > t_{k+1}$. □

So, the sequence $\{T_n, n \geq 1\}$ is exchangeable and, from de Finetti's representation theorem, there exists a random distribution F conditionally on which the T_n's are independent and identically distributed from F: that is, there exists a probability (or de Finetti's) measure defined on the space F of probability measures on $[0, \infty)$ such that, $\forall n$, the joint distribution of $T_1, T_2, \ldots, T_n$ can be written as:

$$P[T_1 \in A_1, T_2 \in A_2, \ldots, T_n \in A_n] = \int_{\mathsf{F}} \prod_{i=1}^{n} F(A_i)\mu(dF) \tag{3.1.20}$$

where μ is the de Finetti (or prior) measure. The problem is how to identify the prior, but the following theorem give us the solution.

Theorem 3.1.2. The de Finetti's measure of the infinite exchangeable sequence $\{T_n,\, n \geq 1\}$ is a beta-Stacy process.

Proof. For the first two interarrival time of a reinforced renewal process, we can compute

$$\begin{aligned}
&P[T_1 > t_1, T_2 > t_2] = \\
&= E_{T_1}\left[\mathbb{1}_{[T_1>t_1]} \prod_{0}^{t_2} [1 - d\Lambda_1(s; T_1)]\right] = \\
&= E_{T_1}\left[\mathbb{1}_{[T_1>t_1]} \left\{\mathbb{1}_{[T_1>t_2]} \prod_{0}^{t_2} \left[1 - \frac{d\alpha(s)}{\alpha(\{s\}) + \beta(t) + 1}\right] + \right.\right. \\
&\left.\left. \mathbb{1}_{[T_1 \leq t_2]} \prod_{0}^{T_1} \left[1 - \frac{d\alpha(s)}{\alpha(\{s\}) + \beta(s) + 1}\right] \frac{\alpha(\{T_1\}) + \beta(T_1)}{\alpha(\{T_1\}) + \beta(T_1) + 1} \prod_{T_1}^{t_2} \left[1 - \frac{d\alpha(s)}{\alpha(\{s\}) + \beta(s)}\right]\right\}\right]
\end{aligned}$$

$$= E_{T_1}\left[\mathbb{1}_{[T_1>t_1\vee t_2]}\prod_0^{t_2}\left[1-\frac{d\alpha(s)}{\alpha(\{s\})+\beta(t)+1}\right]\right]+\mathbb{1}_{[t_1<t_2]}E_{T_1}\left[\mathbb{1}_{[t_1<T_1\le t_2]}\right.$$

$$\left.\prod_0^{T_1}\left[1-\frac{d\alpha(s)}{\alpha(\{s\})+\beta(t)+1}\right]\frac{\alpha(\{T_1\})+\beta(T_1)}{\alpha(\{T_1\})+\beta(T_1)+1}\prod_{T_1}^{t_2}\left[1-\frac{d\alpha(s)}{\alpha(\{s\})+\beta(s)}\right]\right]=$$

$$=\prod_0^{t_2}\left[1-\frac{d\alpha(s)}{\alpha(\{s\})+\beta(s)+1}\right]\prod_0^{t_1\vee t_2}\left[1-\frac{d\alpha(s)}{\alpha(\{s\})+\beta(s)}\right]+$$

$$\mathbb{1}_{[t_1<t_2]}\prod_0^{t_2}\left[1-\frac{d\alpha(s)}{\alpha(\{s\})+\beta(s)}\right]\cdot$$

$$\int_{t_1}^{t_2}\prod_0^{t}\left[1-\frac{d\alpha(s)}{\alpha(\{s\})+\beta(s)+1}\right]\frac{d\alpha(t)}{\alpha(\{t\})+\beta(t)+1} \tag{3.1.21}$$

By backward equation for product integral, (3.1.21) is equal to

$$\prod_0^{t_2}\left[1-\frac{d\alpha(s)}{\alpha(\{s\})+\beta(s)+1}\right]\prod_0^{t_1\vee t_2}\left[1-\frac{d\alpha(s)}{\alpha(\{s\})+\beta(s)}\right]+$$

$$+\mathbb{1}_{[t_1<t_2]}\prod_0^{t_2}\left[1-\frac{d\alpha(s)}{\alpha(\{s\})+\beta(s)}\right]\cdot$$

$$\left[\prod_0^{t_1}\left[1-\frac{d\alpha(s)}{\alpha(\{s\})+\beta(s)+1}\right]-\prod_0^{t_2}\left[1-\frac{d\alpha(s)}{\alpha(\{s\})+\beta(s)+1}\right]\right]=$$

$$=\prod_0^{t_1\wedge t_2}\left[1-\frac{d\alpha(s)}{\alpha(\{s\})+\beta(s)+1}\right]\prod_0^{t_1\vee t_2}\left[1-\frac{d\alpha(s)}{\alpha(\{s\})+\beta(s)}\right]. \tag{3.1.22}$$

Now, to prove the theorem we consider a sequence of random variables $\{T'_n,\, n\ge 0\}$ independent and identically distributed having as distribution function $F(t)$, sampled according to a beta-Stacy process of parameters α and β.

We compute for these random variables $P[T'_1 > t_1, T'_2 > t_2]$ and we show that this coincides with what we have found for $\{T_n,\, n\ge 0\}$.

Then, the uniqueness of de Finetti's measure implies the thesis.

So, let $S(t) = 1 - F(t)$ be the survival function and $Z(t)$ the corresponding log-beta process (see Walker and Muliere (1997)).

By Lévy formula for log-beta process, for $t \geq 0$,

$$E\left[S(t)^2\right] = E\left[e^{-2Z(t)}\right] = \prod_0^t \left[1 - \frac{d\alpha(s)}{\alpha(\{s\}) + \beta(s) + 1}\right] \prod_0^t \left[1 - \frac{d\alpha(s)}{\alpha(\{s\}) + \beta(s)}\right] \tag{3.1.23}$$

and, for $t_1 < t_2$,

$$E\left[e^{-(Z(t_2)-Z(t_1))}\right] = \prod_{t_1}^{t_2} \left[1 - \frac{d\alpha(s)}{\alpha(\{s\}) + \beta(s)}\right]. \tag{3.1.24}$$

So that

$$\begin{aligned}
&P\left[T_1' > t_1, T_2' > t_2\right] = \\
&= E\left[P\left[T_1 > t_1|F\right] P\left[T_2 > t_2|F\right]\right] = \\
&= E\left[S(t_1)S(t_2)\right] = \\
&= E\left[S(t_1 \wedge t_2)S(t_1 \vee t_2)\right] = \\
&= E\left[e^{-2Z(t_1\wedge t_2)}\right] E\left[e^{-(Z(t_1\vee t_2)-Z(t_1\wedge t_2))}\right] = \\
&= \prod_0^{t_1\wedge t_2} \left[1 - \frac{d\alpha(s)}{\alpha(\{s\}) + \beta(s) + 1}\right] \prod_0^{t_1\wedge t_2} \left[1 - \frac{d\alpha(s)}{\alpha(\{s\}) + \beta(s)}\right] \prod_{t_1\wedge t_2}^{t_1\vee t_2} \left[1 - \frac{d\alpha(s)}{\alpha(\{s\}) + \beta(s)}\right] = \\
&= \prod_0^{t_1\wedge t_2} \left[1 - \frac{d\alpha(s)}{\alpha(\{s\}) + \beta(s) + 1}\right] \prod_0^{t_1\vee t_2} \left[1 - \frac{d\alpha(s)}{\alpha(\{s\}) + \beta(s)}\right].
\end{aligned} \tag{3.1.25}$$

□

This result seems to provide a natural justification to make nonparametric Bayesian inference on distribution function of the interarrival times, in the case of exchangeability, of a renewal processes setting a beta-Stacy as prior.

3.2 Reinforced Markov renewal processes

In Muliere et al. (2003) a reinforced random process in continuous time with a finite number of states is introduced and its properties are studied.

This process can be considered as a superposition of the basic building blocks represented by the reinforced renewal processes.

So, let consider a finite state space $E = \{0, \dots, m\}$ and associate to each $i \in E$ a reinforced renewal process $\{T_n^i;\ n \in \mathbb{N}\}$ with parameters α^i and β^i.

The transitions between the states are regulated by a class of Pólya urns; more specifically, for each state $i \in E$ and time $t \geq 0$, let $U^i(t)$ be a Pólya urn, with an initial composition $C^i(t) = (c_0^i(t), \dots, c_m^i(t))$ of balls labelled by the states of E. Let consider, for each $i \in E$, also a function $\gamma^i(t) : [0, \infty) \to [0, \infty)$.

For each $i \in E$, conditionally on $\{T_n^i,\ n \geq 1\}$ and provided $\gamma^i(t)$, the sequence of the successors of the state i $\{s_n^i,\ n \geq 1\}$ is generated as follows.

Given T_1^i, a ball is sampled from the urn associated to the state i and the time $\gamma^i(T_1^i)$, $U^i(\gamma^i(T_1^i))$; the first successor of i, s_1^i, is the state labeling this ball. Similarly for $n \geq 1$, given $T_1^i, s_1^i, \dots, T_n^i, s_1^n, T_{n+1}^i$, let s_{n+1}^i is the state marking the ball sampled from the urn $U^i(\gamma^i(T_{n+1}^i))$ updated according to the Pólya rule. Now, it is possible to state the definition below merging the different reinforced renewal processes to constitute processes whose structure recalls Markov renewal and semi-Markov processes.

Definition 3.2.1. For each $i \in E$, consider the reinforced renewal processes $\{T_n^i, 1, n \geq 1\}$ and the sequences $\{s_n^i,\ n \geq 1\}$ of successor states generated according the description above.

Let $L_0 = s_0 \in E$ be the initial state and $S_0 = 0$. For $n \geq 1$, given $L_0, \cdots, L_{n-1} \in E$

and $S_0, \cdots, S_{n-1} \in [0,\infty)$, let $\natural(L_0, \cdots, L_{n-1})$ indicate the number of times the state L_{n-1} appears in $(L_0, \cdots, L_{n-1})$ and set:

$$L_n = s^{L_{n-1}}_{\natural(L_0,\cdots,L_{n-1})} \quad (3.2.1)$$

$$T_n = T^{L_{n-1}}_{\natural(L_0,\cdots,L_{n-1})} \quad (3.2.2)$$

and $S_n = S_{n-1} + T_n$.

For $t > 0$, define

$$N(t) = \sup\{n \geq 0 : S_n \leq t\}, \quad t \geq 0 \quad (3.2.3)$$

and

$$J_t = L_{N(t)}. \quad (3.2.4)$$

The process $(L,T) = \{L_n, T_n, n \geq 0\}$ is called reinforced Markov renewal process, while $J = \{J_t, t \geq 0\}$ is said reinforced semi-Markov process.

For these two new processes the sojourn times in each state are given by the reinforced renewal processes in such a way as we can think at a clock measuring the time spent in that state. At each return, the clock is reset to 0 and the reinforcements are done. We note that it is possible to modify the definition of the processes establishing different resetting and reinforcement rules (for instance, all the clocks are reset only if a given recurrent state is visited), but, in that case, the situation is more complicated and all the following does not hold.

The propositions clarify the structure of the above-defined processes and their relations with reinforced renewal and semi-Markov processes.

Following Muliere et al. (2003), we state

Proposition 3.2.1. [Muliere et al. (2003)]

For every $i \in E$, the sequence $(T^i, s^i) = \{(T^i_n, s^i_n)\}$ is exchangeable.

Moreover, the sequences $(T^0, s^0), (T^1, s^1), \ldots, (T^m, s^m)$ are independent.

Studying the double array $\{a_{in}, i \in E, n \geq 1\}$ with $a_{in} = (T^i_n, s^i_n)$, it is possible to show, by de Finetti's representation theorem for partially exchangeable array or following Epifani et al. (2002), that there exist $Q_1, \ldots, Q_m$ random and independent transition kernels on $[0, \infty) \times E$ such that, for any $i \in E$, the random elements of the sequences (T^i, s^i) are independent with probability distribution Q_i.

Alternatively, stating

$$p_{i\cdot} = Q_i(\cdot, (0, +\infty]) \tag{3.2.5}$$

$$F_{ij}(\cdot) = \frac{Q_i(j, \cdot)}{p_{ij}} \tag{3.2.6}$$

it is possible to claim the existence , for any $i \in E$, of a random probability distribution on E and, for $i, j \in E$, of a random distribution on $[0, \infty)$ such that they generate the kernels Q_i.

So, the sense of the following is clear.

Proposition 3.2.2. [Muliere et al. (2003)]

The process $J = \{J_t, t \geq 0\}$ is a mixture of semi-Markov processes.

An equivalent reformulation of the previous proposition referring to $(L_n, S_n - S_{n-1})$ as a mixture of Markov renewal processes is possible.

Generally, the de Finetti's measure of the mixture of semi-Markov processes is not known, but, in the following special case, we are able to determine it.

Taking

$$\gamma^i(t) = \gamma^i \in [0, \infty), \ \forall\, t \in [0, \infty), \tag{3.2.7}$$

we have that, if the process is in the state i, the transitions to the next state are always generated by the same Pólya urn $U^i(\gamma^i) = U^i$ with initial composition $(c_0^i, \ldots, c_m^i)$ independently of time spent in this state. In other words, the sequences $\{s_n^i,\, n \geq 1\}$ and $\{T_n^i,\, n \geq 1\}$ are independent so that

$$\begin{aligned} P\left[s_{n+1}^i = k \middle|\, s_1^i, T_1^i, \ldots s_n^i, T_n^i, T_{n+1}^i\right] &= P\left[s_{n+1}^i = k \middle|\, s_1^i, \ldots, s_n^i\right] \\ &= \frac{c_k^i + \sum_{j=1}^n \mathbb{1}_{[s_j^i = k]}}{\sum_{j=1}^m c_j^i + n} \end{aligned} \tag{3.2.8}$$

This case corresponds to the assumption about the structure of a Markov renewal process of conditionally independence between sojourn times and successor states.

If the urns U^i are such that each state i is recurrent (for conditions, see Muliere et al. (2000)), then, for $i \in E$, the sequences $\{s_n^i,\, n \geq 1\}$ are infinite and exchangeable. Therefore, the sequence $\{L_n,\, n \geq 1\}$ is a recurrent Reinforced Urn Process and, hence, there exists a random transition matrix $\boldsymbol{M}$ conditionally on which $\{L_n,\, n \geq 1\}$ is a Markov chain.

Furthermore, $\boldsymbol{M}$ is distributed in such a way as the rows $\boldsymbol{M}_i$ are mutually independent probability distributions on E and, for each i, the law of $\boldsymbol{M}_i$ is that of a Dirichlet process with parameter μ_i being the measure that assigns a mass c_j^i to each state $j \in E$.

Note that the law of vector of probability measure $(\boldsymbol{M}_0, \ldots, \boldsymbol{M}_m)$ is a product of Dirichlet processes.

Because of conditionally independence between sojourn times and successor states, proposition 3.2.1 and theorem 3.1.2, the de Finetti's measure for the mixture of

Markov renewal process is structured as the product of:

- beta-Stacy processes for the cumulative distribution function of sojourn times in each state $i \in E$,
- Dirichlet distributions for each row of transition matrix (as interesting events are $\{s_n^i = j\}$, $j \in E$, Dirichlet process reduces to Dirichlet distribution of parameters $(c_0^i, \ldots, c_m^i)$).

The next section will state more precisely these ideas.

3.3 Bayesian estimation

The special kind of reinforced Markov renewal processes at the end of last section provides a justification to make non-parametric Bayesian inference setting as prior beta-Stacy processes for distribution functions and a stochastic matrix, having independent rows with Dirichlet distribution, on transition probabilities.
Indeed, the previous construction characterizes, via exchangeability, our prior. This choice represents an alternative to the family of conjugate priors for Markov renewal processes proposed by Phelan (1990a): Dirichlet on transitions, as above, and beta processes (see Hjort (1990)) for cumulative hazards.
We show that also the prior in exam is conjugate.

3.3.1 The prior

Now, we formalize the structure of the prior distribution.
Let recall the state space $E = \{0, 1, \ldots, m\}$ and define, for $n \geq 0$,

- $\Pi_n = (E \times \mathbb{R}_+)^{n+1}$,

- $\mathbf{\Pi}_n$ the Borel σ-field on Π_n,

- $\mathbb{N}_0 = \{0, 1, \dots\}$,

- $\mathcal{N}$ the class of all subsets of $\mathbb{N}_0$,

- $\Pi = \bigcup_{n=0}^{\infty} \Pi_n$ and $\mathbf{\Pi} = \bigvee_{n=0}^{\infty} \mathbf{\Pi}_n$,

- $N(t) = \sum_{n=1}^{\infty} \mathbb{1}_{[S_n \leq t]}, \quad t \geq 0$, the counting process of all jumps.

Almost all sample functions can be represented by a point in the sample space $(\mathbb{N} \times \Pi, \mathcal{N} \otimes \mathbf{\Pi})$ given by a finite-tuple $(N(t), \boldsymbol{R}(t))$, with

$$\boldsymbol{R}(t) = (J_0, \dots, J_{N(t)}, T_1, \dots, T_{N(t)}, t - S_{N(t)}). \tag{3.3.1}$$

Because of assumption (3.2.7) implying conditionally independence between sojourn times and successor states, we can write the parameter space as

$$\Theta = \mathsf{M}_{m+1} \times \mathsf{F}^{m+1} \tag{3.3.2}$$

where M_{m+1} is the space of random Markov matrices of dimension $m+1$ and F^{m+1} the $m+1$-product of F, the space of random ditribution functions on $[0, +\infty)$.
Let $\mathbf{\Theta} = \mathfrak{M}_{m+1} \otimes \mathfrak{F}^{m+1}$ be the corresponding Borel σ-field.
Now we give the definitions of the distributions of interest: product Dirichlet and vector beta-Stacy.

Definition 3.3.1. Let $\boldsymbol{\beta} = (\beta_{ij})_{i,j \in E}$ denote a matrix of nonnegative quantities.

Let $\boldsymbol{M} = (M_{ij})_{i,j\in E}$ denote a random matrix defined on $(\Omega, \mathcal{F}, \mathbb{P})$. The rows of $\boldsymbol{M}$ are independent random vectors such that, for each $i \in E$,

$$\boldsymbol{M}_i = (M_{i0}, M_{i1}, \ldots, M_{im}) \sim Dir(\beta_{i0}, \beta_{i1}, \ldots, \beta_{im}).$$

We say that $\boldsymbol{M}$ is the product of Dirichlet distributions and write

$$\boldsymbol{M} \sim \Pi Dir(\beta).$$

Definition 3.3.2. For each $i \in E$, let $c_i(\cdot)$ denote a positive function and let $G_i(\cdot) \in \mathsf{F}$ be right continuous with a countable set of points of discontinuity.
Let $\boldsymbol{F} = (F^1, \ldots, F^m)$ denote a vector valued process defined on $(\Omega, \mathcal{F}, \mathbb{P})$.
The F^i's are independent and

$$F^i \sim \mathcal{S}(c_i(\cdot), G_i(\cdot)).$$

$\boldsymbol{F}$ is a vector beta-Stacy process with parameter $(\boldsymbol{c}, \boldsymbol{G}) = ((c_1, \ldots, c_m), (G_1, \ldots, G_m))$ and we write $\boldsymbol{F} \sim \mathcal{S}(\boldsymbol{c}, \boldsymbol{G})$.

Hence, we write the parameters as $\theta = (\boldsymbol{M}, \boldsymbol{F})$.
A mixture of Markov renewal processes with de Finetti's measure ΠDir *times beta-Stacy* can be written as

$$P\left[N(t) = n,\ \boldsymbol{R}(t) \in A\right] = \int_\Theta \left[\int_A dP(n; \pi|\theta)\right] dQ(\theta) \qquad (3.3.3)$$

for $n \geq 0$, $A \in \boldsymbol{\Pi}_n$, with

$$dP(n; \pi|\theta) = p_{j_0}(1 - F^{j_n}(u_t)) \prod_{k=0}^{n-1} M_{j_k j_{k+1}} dF^{j_k}(t_{k+1}) \qquad (3.3.4)$$

where $\pi = (j_0, j_1, \ldots, j_n, t_1, \ldots, t_n, u_t)$, $u_t = t - \sum_1^n t_i$ and $p_{j_0} = P[J_0 = j_0]$ (in general, we assume $P[J_0 = j_0] = 1$) and Q is the prior on (Θ, Θ) defined by definitions 3.3.1 and 3.3.2.

We can also write

$$P[J_{n+1} = j,\, S_{n+1} - S_n \leq t | J_n = i, \boldsymbol{M}, \boldsymbol{F}] = M_{ij} F^i(t) \tag{3.3.5}$$

Remark 3.3.1. This prior has an intimate relation with Phelan (1990a)'s one.

Indeed, the remark 2 in Walker and Muliere (1997) clarifies the connection between beta and beta-Stacy processes: if A is a beta process and $dZ = -\log(1 - dA)$, then $F(t) = -\exp\{-Z(t)\}$ is a beta-Stacy.

Nevertheless, in our case Π*Dir times beta-Stacy* prior is not an arbitrary choice, but it is shown to enjoy the good feature of directly deriving of exchangeability properties of reinforced Markov renewal processes: hence, if the data reveal a structure characterized by exchangeability of interarrival times and successor states featuring the independence between these, this prior appears natural.

Moreover, as pointed out in Walker and Muliere (1997), to elicit a prior guess for a beta-Stacy process, dealing with cumulative distribution function instead of cumulative hazard, is easier than working with beta process.

3.3.2 Some estimation results

We compute the posterior distribution corresponding to prior described in the last subsection. We show that the property of *conjugacy* holds.

Define:

- the observed transition counts

$$N_{ij} = \sum_{k=1}^{N(t)} \mathbb{1}_{[J_{k-1}=i, J_k=j]} \quad i,j \in E;$$

- $N(i) = \{N_i(s);\ s \geq 0\}$ the counting processes defined over the completed sojourn times in state i

$$N_i(s) = \sum_{k=1}^{N(t)} \mathbb{1}_{[J_{k-1}=i, T_k \leq s]} \quad s \geq 0,\ i \in E;$$

- the risk processes $Y(i) = \{Y_i(s);\ s \geq 0\}$

$$Y_i(s) = \mathbb{1}_{\left[J_{N(t)}=i, t-S_{N(t)} \geq s\right]} + \sum_{k=1}^{N(t)} \mathbb{1}_{[J_{k-1}=i, T_k \geq s]} \quad s \geq 0,\ i \in E.$$

The following proposition presents a fundamental result: the conditioned independence between $\left(T_1, \ldots, T_{N(t)}, \boldsymbol{F}\right)$ and M given $\left(J_0, \ldots, J_{N(t)}, N(t)\right)$, so that it will be possible to compute the posterior distributions separately.

Proposition 3.3.1. Let $(N(t), \boldsymbol{R}(t), \boldsymbol{M}, \boldsymbol{F})$ denote the random element in $(\mathbb{N}_0 \times \Pi \times \Theta, \mathcal{N} \otimes \boldsymbol{\Pi} \otimes \boldsymbol{\Theta}, P)$, where P is given, according to (3.3.3), by

$$P\left[N(t) = n,\ \boldsymbol{R}(t) \in A,\ (\boldsymbol{M}, \boldsymbol{F}) \in B\right] = \int_B \left[\int_A dP(n; \pi|\theta)\right] dQ(\theta) \qquad (3.3.6)$$

for $n \geq 0$, $A \in \boldsymbol{\Pi}_n$, $B \in \boldsymbol{\Theta}$.

Then $(T_1, \ldots, T_{N(t)}, \boldsymbol{F})$ and $\boldsymbol{M}$ are conditionally independent given $(J_0, \ldots, J_{N(t)}, N(t))$.

Proof. As $\Theta = \mathsf{M}_{m+1} \times \mathsf{F}^{m+1}$ and $\Pi_n = (E \times \mathbb{R}_+)^{n+1}$, let $\theta = (\theta_0, \theta_1)$ such that

$$\theta_0 = (p_{ij},\ i,j \in E) \in \mathsf{M}_{m+1}$$

$$\theta_1 = \boldsymbol{F} = (F^0, F^1, \ldots, F^{m+1}) \in \mathsf{F}^{m+1}$$

and $\boldsymbol{\pi} = (\boldsymbol{\pi}_0, \boldsymbol{\pi}_1)$

$$\boldsymbol{\pi}_0 = (j_0, j_1, \dots, j_n) \in E^{n+1}$$

$$\boldsymbol{\pi}_1 = (t_1, \dots, t_n, u_t) \in \mathbb{R}_+^{n+1}.$$

Moreover, define

$$dP\left(n; \boldsymbol{\pi}_0 \middle| \theta_0\right) = \prod_{k=0}^{n-1} p_{j_k j_{k+1}} \tag{3.3.7}$$

and $dP\left(n; \boldsymbol{\pi}_1 \middle| \theta_1, \boldsymbol{\pi}_0\right)$ such that

$$dP\left(n; \boldsymbol{\pi} \middle| \theta\right) = dP\left(n; \boldsymbol{\pi}_1 \middle| \theta_1, \boldsymbol{\pi}_0\right) dP\left(n; \boldsymbol{\pi}_0 \middle| \theta_0\right). \tag{3.3.8}$$

As $N(t)$ is *a.s.* finite, it is sufficient to prove the conditional independence on the set $\{N(t) = n\}$, $\forall n \geq 0$.

Fix n and define the $\mathcal{E}^{n+1}$-measurable function

$$f(n, \boldsymbol{\pi}_0) = \int_{F^{m+1} \times \mathbb{R}_+^{n+1}} dP(n; \boldsymbol{\pi}_1 | \, \boldsymbol{\pi}_0, \theta_1) dQ'(\theta_1), \quad \boldsymbol{\pi}_0 \in E^{n+1}$$

where Q' is the distribution of F and $\mathcal{E}^{n+1}$ is the class of all subsets of E^{n+1}.

Observe that

$$P\left[N(t) = n, (J_0, \dots, J_{N(t)}) \in A, \boldsymbol{M} \in B\right] = \int_{B \times A} f(n, \boldsymbol{\pi}_0) dP(n; \boldsymbol{\pi}_1 | \, \boldsymbol{\pi}_0, \theta_1) dQ_0(\theta_0) \tag{3.3.9}$$

for $A \in \mathcal{E}^{n+1}$, $B \in \mathfrak{M}_{m+1}$.

So, on $(E^{n+1} \times \mathsf{M}_{m+1}, \mathcal{E}^{n+1} \times \mathfrak{M}_{m+1})$, we have defined a measure μ, the restriction of distribution of $(N(t), J_0, \dots, J_{N(t)}, \boldsymbol{M})$ to the set $\{N(t) = n\}$.

Moreover, for all $G \in \mathcal{E}^{n+1} \otimes \mathfrak{M}_{m+1}$, we have

$$P\left[N(t) = n, (T_1, \ldots, T_{N(t)}) \in A, \boldsymbol{F} \in K, (J_0, \ldots, J_{N(t)}, \boldsymbol{M}) \in B\right]$$
$$= \int_B \left[\int_{K \times A} f^{\oplus}(n, \boldsymbol{\pi}_0) dP(n; \boldsymbol{\pi}_1 | \boldsymbol{\pi}_0, \theta_1) dQ'(\theta_1)\right] d\mu(\boldsymbol{\pi}_0, \theta_0) \tag{3.3.10}$$

where $A \in \mathcal{B}(\mathbb{R}_+^n)$, $K \in \mathfrak{F}^{m+1}$ and

$$f^{\oplus} = \begin{cases} f^{-1} & \text{if } f > 0 \\ 0 & \text{if } f = 0 \end{cases}.$$

For each n, the term in brackets is a measurable function of π_0 and, for π_0 outside a set of μ-measure zero, defines a probability distribution over $(\mathbb{R}_+^n \times \mathsf{F}^{m+1}, \mathcal{B}(\mathbb{R}_+^n) \otimes \mathfrak{F}^{m+1})$. Hence this term determines a regular conditional distribution of $(T_1, \ldots, T_{N(t)}, \boldsymbol{F})$ given $(N(t), J_0, \ldots, J_{N(t)}, \boldsymbol{M})$ on $\{N(t) = n\}$ and is independent of $\boldsymbol{M}$. □

Consider that the distribution functions $F^i \sim \mathcal{S}(c_i, G_i)$ and define, $\forall i \in E$, Q_i as the probability measure on $(\mathsf{F}, \mathfrak{F})$ induced by the mapping F^i.

For a given realization of a Markov renewal process with $T_1, \ldots, T_{n+1}$ and $J_0, \ldots, J_n$, set, $\forall\, i \in E$,

$$\boldsymbol{T}^i = \{T_k : J_{k-1} = i, k = 1, \ldots, n\}.$$

We know that $F^i \,|\, \boldsymbol{T}^i \sim \mathcal{S}(c_i^*, G_i^*)$ with [3]

$$G_i^*(t) = 1 - \prod_{[0,t]}\left[1 - \frac{c_i(s)dG_i(s) + dN_i(s)}{c_i(s)G_i[s,\infty) + Y_i(s)}\right] \tag{3.3.13}$$

$$c_i^*(t) = \frac{c_i(t)G_i[t,\infty) + Y_i(t) - N_i\{t\}}{G_i^*[t,\infty)}. \tag{3.3.14}$$

Let $\widetilde{Q}_i$ be the probability measure induced on $(\mathsf{F},\mathfrak{F})$ by $F^i \,|\, \boldsymbol{T}^i$ (the posterior). Following Phelan (1990a), now we state some propositions with some interesting results preliminary to the computation of posterior disitributions.

Proposition 3.3.2. Fix $n \geq 0$ and let

$\boldsymbol{T} = (T_0, T_1, \ldots, T_n)$ with $T_0 = 0$,

$\boldsymbol{Y} = (J_0, J_1, \ldots, J_n)$ and

$\boldsymbol{\pi} = (0, j_1, \ldots, j_n) \in E^{n+1}$.

Consider the event $B(\boldsymbol{\pi}) = \{\boldsymbol{Y} = \boldsymbol{\pi}\}$ and define the σ-algebra $\mathcal{A} = \sigma(\boldsymbol{T}, \boldsymbol{Y}) \vee \sigma(B(\boldsymbol{\pi}))$.

Let $r > 0$ and $A = \bigtimes_{1}^{m+1} A_i \in \mathfrak{F}^{m+1}$ and suppose $P[B(\boldsymbol{\pi})] > 0$.

Then on $B(\boldsymbol{\pi})$ we have

$$P\left[T_{n+1} > r, \boldsymbol{F} \in A \middle| \mathcal{A}\right] = \prod_{i \neq j_n} \widetilde{Q}_i(A_i)\widetilde{E}_{j_n}\left[\left(1 - F^{j_n}(r)\right)\mathbb{1}_{[F^{j_n} \in A_{j_n}]}\right] \tag{3.3.15}$$

[3] If we parametrize the beta-Stacy's (and related log-beta processes) using the measures α_i and the nonnegative functions b_i (see Walker and Mulierę (1997)), these parameters are updated as follows

$$d\alpha_i^*(t) = d\alpha_i(t) + dN_i(t) \tag{3.3.11}$$

$$b_i^*(t) = b_i(t) + Y_i(t) - N_i\{t\}. \tag{3.3.12}$$

where $\widetilde{E}_{j_n}$ is the expectation operator defined by the random measure $\widetilde{Q}_{j_n}$. Hence the $\widetilde{E}_i$ give the regular conditional distributions of the F_i given $\mathcal{A}$.

Proof.

$$\begin{aligned} P\left[T_{n+1}>r, \boldsymbol{F} \in A \middle| \mathcal{A}\right] &= E\left[\mathbb{1}_{[T_{n+1}>r]}\mathbb{1}_{[\boldsymbol{F}\in A]} \middle| \mathcal{A}\right] \\ &= E\left[E\left(\mathbb{1}_{[T_{n+1}>r]}\mathbb{1}_{[\boldsymbol{F}\in A]} \middle| \sigma(\boldsymbol{T},\boldsymbol{Y},\boldsymbol{F}) \vee \sigma(B(\boldsymbol{\pi}))\right) \middle| \mathcal{A}\right] \\ &= E\left[\mathbb{1}_{[\boldsymbol{F}\in A]}\left(1-F^{j_n}(r)\right) \middle| \mathcal{A}\right] \end{aligned} \tag{3.3.16}$$

By hypothesis, $T_1, \ldots, T_{n+1}$ are conditionally independent given $B(\boldsymbol{\pi})$ and $\boldsymbol{F}$. Moreover, the sojourn times in state i, belonging to sets $\boldsymbol{T}^i$ for $i \neq j_n$ and $\{T_{n+1}\} \cup \boldsymbol{T}^{j_n}$ have conditional distributions determined by F^i, $i \in E$.

By independence of $\boldsymbol{M}$ and $\boldsymbol{F}$ and independence of the F^i, the random variables $\left(\boldsymbol{T}^i, F^i\right)$, $i \neq j_n$, and $\left(\{T_{n+1}\} \cup \boldsymbol{T}^{j_n}, F^{j_n}\right)$ are conditionally independent given $B(\boldsymbol{\pi})$. Hence, (3.3.16) is equal to

$$\prod_{i \neq j_n} E\left[\mathbb{1}_{[F^i \in A_i]} \middle| \mathcal{A}\right] E\left\{\left[1-F^{j_n}(r)\right]\mathbb{1}_{[F^{j_n}\in A_{j_n}]} \middle| \mathcal{A}\right\}. \tag{3.3.17}$$

The expectations are equivalent to compute the posterior laws of beta-Stacy processes F^i given a sample of size $n^i = \sharp\{k = 0, 1, \ldots, n-1 : j_k = i\}$ and lifetimes in the set $\boldsymbol{T}^i$ (the above defined measures $\widetilde{Q}_i$).

So the (3.3.17) becomes

$$\prod_{i \neq j_n} \widetilde{Q}_i\left(A_i\right) \widetilde{E}_{j_n}\left\{\left[1-F^{j_n}(r)\right]\mathbb{1}_{[F^{j_n}\in A_{j_n}]}\right\} \tag{3.3.18}$$

□

Remark 3.3.2.

$$\begin{aligned} P\left[T_{n+1} > r \middle| \mathcal{A}\right] &= P\left[T_{n+1} > r, \boldsymbol{F} \in \mathsf{F}^{m+1} \middle| \mathcal{A}\right] \\ &= \prod_{i \neq j_n} \widetilde{Q}_i(\mathsf{F})\, \widetilde{E}_{j_n}\left\{\left[1 - F^{j_n}(r)\right] \mathbb{1}_{[F^{j_n} \in \mathsf{F}]}\right\} \\ &= \widetilde{E}_{j_n}\left[1 - F^{j_n}(r)\right]. \end{aligned} \tag{3.3.19}$$

Note that the above expression corresponds to the predictive distribution of sojourn time T_{n+1} of a reinforced Markov renewal process given all its past history.

As F^i's are distributed according to independent beta-Stacy processes and by definition of $\widetilde{Q}_{j_n}$ and $\widetilde{E}_{j_n}$,

$$\begin{aligned} \widetilde{E}_{j_n}\left[1 - F^{j_n}(r)\right] &= \prod_{[0,t]}\left[1 - \frac{c_{j_n}(s) dG_{j_n}(s) + dN_{j_n}(s)}{c_{j_n}(s) G_{j_n}[s, \infty) + Y_{j_n}(s)}\right] \\ &= \prod_{[0,t]}\left[1 - \frac{d\alpha^*_{j_n}(s)}{\alpha^*_{j_n}(\{s\}) + b^*_{j_n}(s)}\right] \end{aligned} \tag{3.3.20}$$

where last relation is obtained taking α_{j_n} and b_{j_n} corresponding to c_{j_n} and G_{j_n} and updating according to (3.3.11) and (3.3.12).

This is coherent with the construction of reinforced Markov renewal process and what exposed in Muliere et al. (2003).

Remark 3.3.3. If F is a random distribution function and $Y \mid F \sim F$, define

$$Q^r_F(A) := P\left[F \in A \middle| Y > r\right].$$

Note that

$$Q^r_F(A) = \frac{E_F\left[(1 - F(r))\, \mathbb{1}_{[F \in A]}\right]}{E_F\left[1 - F(r)\right]}. \tag{3.3.21}$$

Proposition 3.3.3. Fix $n \geq 0$ and let $B(\boldsymbol{\pi})$, $\mathcal{A}$ and $A \in \mathfrak{F}^{m+1}$ be defined as above. Let $r > 0$ and suppose $P\left[B(\boldsymbol{\pi}) \cap \{T_{n+1} > r\}\right] > 0$.

Then, on $B(\boldsymbol{\pi}) \cap \{T_{n+1} > r\}$, we have:

$$P\left[\boldsymbol{F} \in A \,|\, \mathcal{A} \vee \sigma\left(\{T_{n+1}\}\right)\right] = \prod_{i \neq j_n} \widetilde{Q}_i(A_i) \widetilde{Q}^r_{j_n}(A_{j_n}). \tag{3.3.22}$$

Second, define $\Omega_n = \{N(t) = n\}$ and suppose $P\left[B(\boldsymbol{\pi}) \cap \Omega_n\right] > 0$.

Then, on $B(\boldsymbol{\pi}) \cap \Omega_n$, we have:

$$P\left[\boldsymbol{F} \in A \,|\, \mathcal{A} \vee \sigma\left(\Omega_n\right)\right] = \prod_{i \neq j_n} \widetilde{Q}_i(A_i) \widetilde{Q}^{u_t}_{j_n}(A_{j_n}) \tag{3.3.23}$$

where $u_t = t - \sum_1^n t_i \geq 0$ for $T_1(\omega) = t_1, \ldots, T_n(\omega) = t_n$.

Proof. For $A \in \mathfrak{F}^{m+1}$, consider

$$\mu(A) = P\left[\boldsymbol{F} \in A \,|\, T_{n+1} > r\right].$$

Let compute

$$P\left[\boldsymbol{F} \in A \,|\, \mathcal{A} \vee \{T_{n+1} > r\}\right] = \mu(A|\mathcal{A}).$$

Outside a set of μ-measure zero, for $A = \bigtimes_1^{m+1} A_i$, we have

$$\begin{aligned}
\mu(A|\mathcal{A}) &= \frac{P\left[\boldsymbol{F} \in A, T_{n+1} > r \,|\, \mathcal{A}\right]}{P\left[X_{n+1} > r \,|\, \mathcal{A}\right]} = \\
&= \frac{\prod_{i \neq j_n} \widetilde{Q}_i\left(A_i\right) \widetilde{E}_{j_n} \left\{\left[1 - F^{j_n}(r)\right] \mathbb{1}_{[F^{j_n} \in A_{j_n}]}\right\}}{\widetilde{E}_{j_n}\left[1 - F^{j_n}(r)\right]} \\
&= \prod_{i \neq j_n} \widetilde{Q}_i(A_i) \widetilde{Q}^r_{j_n}(A_{j_n})
\end{aligned}$$

where the last equation holds for (3.3.21).

For the second part, take $A \in \mathfrak{F}^{m+1}$ and let

$$\nu(A) = P\left[\boldsymbol{F} \in A \,|\, \Omega_n\right].$$

Outside a set of ν-measure zero, we have

$$P\left[\boldsymbol{F} \in A \middle| \mathcal{A} \vee \sigma(\Omega_n)\right] \mathbb{1}_{[N(t)=n]} = \frac{P\left[\boldsymbol{F} \in A, N(t) = n \middle| \mathcal{A}\right]}{P\left[N(t) = n \middle| \mathcal{A}\right]} \tag{3.3.24}$$

Noting that $\forall \omega \in \Omega,\ T_1(\omega) = t_1, \ldots, T_n(\omega) = t_n, N(t,\omega) = n$ is equivalent to $T_1(\omega) = t_1, \ldots, T_n(\omega) = t_n, T_{n+1}(\omega) > u_t$, for $A = \overset{m+1}{\underset{1}{\times}} A_i$ and on $B(\boldsymbol{\pi})$ (3.3.24) is equal to

$$\frac{P\left[\boldsymbol{F} \in A, T_{n+1} > u_t \middle| \mathcal{A}\right]}{P\left[T_{n+1} > u_t \middle| \mathcal{A}\right]} = \prod_{i \neq j_n} \widetilde{Q}_i(A_i) \widetilde{Q}_{j_n}^{u_t}(A_{j_n}) \tag{3.3.25}$$

□

Finally, we can compute the posterior distributions of $\boldsymbol{M}$ and $\boldsymbol{F}$.

Theorem 3.3.4. For fixed $\boldsymbol{\beta}$, $\boldsymbol{c}$ and $\boldsymbol{G}$, suppose $\boldsymbol{M}$ and $\boldsymbol{F}$ are independent with $\boldsymbol{M} \sim \Pi Dir(\boldsymbol{\beta})$ and $\boldsymbol{F} \sim \mathcal{S}(\boldsymbol{c}, \boldsymbol{G})$.

Consider the probability model above defined.

For $t \geq 0$, let $(N(t), \boldsymbol{R}(t))$ denote an observation on the process (J, S) on $[0, t]$.

Then $\boldsymbol{M}$ and $\boldsymbol{F}$ are conditionally independent given $(N(t), \boldsymbol{R}(t))$ with posterior distributions given respectively by $\Pi Dir(\boldsymbol{\beta}^*)$ and $\mathcal{S}(\boldsymbol{c}^*, \boldsymbol{G}^*)$ where

$$\beta_{ij}^* = \beta_{ij} + N_{ij} \tag{3.3.26}$$

$$G_i^*(t) = 1 - \prod_{[0,t]} \left[1 - \frac{c_i(s) dG_i(s) + dN_i(s)}{c_i(s) G_i[s, \infty) + Y_i(s)}\right] \tag{3.3.27}$$

$$c_i^*(t) = \frac{c_i(t) G_i[t, \infty) + Y_i(t) - N\{t\}}{G_i^*[t, \infty)}. \tag{3.3.28}$$

Proof. We have shown that $\boldsymbol{M}$ and $\boldsymbol{F}$ are conditionally independent given $(N(t), \boldsymbol{R}(t))$. To compute the conditional law of $\boldsymbol{M}$, it suffices to consider $\boldsymbol{M} | (N(t), J_0, \ldots, J_{N(t)})$. Recall $\Omega_n = \{N(t) = n\}$ and since $N(t)$ is *a.s.* finite

$$P\left[\boldsymbol{M} \in A \middle| \sigma\left(N(t), J_0, \ldots, J_{N(t)}\right)\right] = \sum_{n \geq 0} P\left[\boldsymbol{M} \in A \middle| \sigma\left(J_0, \ldots, J_n\right)\right] \mathbb{1}_{\Omega_n}$$

for $A \in \mathfrak{M}$.

By hypothesis $J_0, \ldots, J_n$ are observations from a Markov chain having transition probabilities $\boldsymbol{M} \sim \Pi Dir(\boldsymbol{\beta})$; hence $P[\boldsymbol{M} \in A | J_0, \ldots, J_n] \mathbb{1}_{\Omega_n}$ is computed from $\Pi Dir(\boldsymbol{\beta}^*)$ with

$$\beta^*_{ij} = \beta_{ij} + N_{ij}$$

where $N(t) = n$.

To compute the conditional law of $\boldsymbol{F}|(N(t), \boldsymbol{R}(t))$ define

$$\mathcal{A}_n = \sigma(J_0, \ldots, J_n, T_1, \ldots, T_n) \vee \sigma(\Omega_n),\ n \geq 0$$

then

$$P[\boldsymbol{F} \in A | \sigma(N(t), \boldsymbol{R}(t))] = \sum_{n \geq 0} P[\boldsymbol{F} \in A | \mathcal{A}_n] \mathbb{1}_{\Omega_n} \quad A \in \mathfrak{F}^m.$$

If $A = \bigtimes_1^{m+1} A_i$, from previous proposition

$$P[\boldsymbol{F} \in A | \mathcal{A}_n] \mathbb{1}_{\Omega_n} = \prod_{i \neq j_n} \widetilde{Q}_i(A_i) \widetilde{Q}^{U_t}_{J_n}(A_{J_n})$$

where $U_t = t - \sum_1^n T_i$ and $\widetilde{Q}_i$ is the posterior of beta-Stacy defined as above. □

At last, we provide the Bayes estimators in the case of squared-error loss:

$$E[M_{ij} | N(t), \boldsymbol{R}(t)] = \frac{\beta^*_{ij}}{\sum_{k \in E} \beta^*_{ik}} \tag{3.3.29}$$

$$E[F_i(t) | N(t), \boldsymbol{R}(t)] = G^*_i(t) = 1 - \prod_{[0,t]} \left[1 - \frac{c_i(s) dG_i(s) + dN_i(s)}{c_i(s) G_i[s, \infty) + Y_i(s)}\right] \tag{3.3.30}$$

$$E[M_{ij} F_i(t) | N(t), \boldsymbol{R}(t)] = \frac{\beta^*_{ij}}{\sum_{k \in E} \beta^*_{ik}} \left\{1 - \prod_{[0,t]} \left[1 - \frac{c_i(s) dG_i(s) + dN_i(s)}{c_i(s) G_i[s, \infty) + Y_i(s)}\right]\right\} \tag{3.3.31}$$

3.4 A different perspective and possible developments

As we have already pointed out, a key assumption in the construction of reinforced Markov renewal processes is resetting the measurement of time spent at each change of state. As many examples show, it is possible to think to modify the definition of the process establishing different resetting and reinforcement rules.

An interesting case is that where all the clocks are reset only when a given recurrent state (for instance the state 0) is visited; figure 3.1 makes an illustration to better understand.

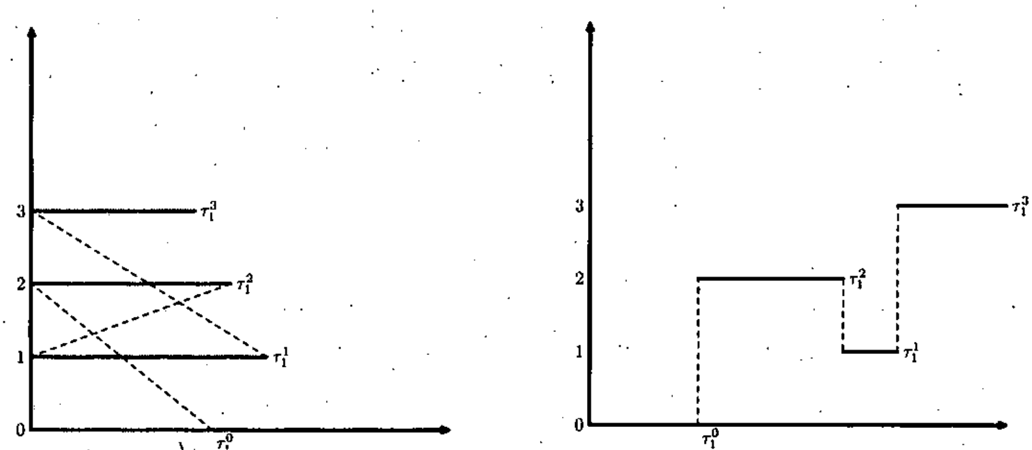

Figure 3.1: Two different resetting rules.

In the left picture, when the process, started from the state 0 at time 0, attains the time τ_1^0, it makes a jump to the state 2 and the clock is reset so that the couple time-state $(0,2)$ is reached. The model has the same behaviour in τ_1^2 and τ_1^1. After τ_1^3, the trajectory goes back to $(0,0)$.

Establishing the second resetting rule, at the moment τ_1^0 when the process passes from 0 to 2, the process does not go back to $(0,2)$, but it reaches $(\tau_1^0, 2)$ and so forth. Once again, only after τ_1^3 it turn to the origin $(0,0)$.

When this last rule is taken, it is possible to model the lifetime of an individual with a multi-state disease, focusing the attention on the successive epochs when the changes of states happen.

Searching for a Bayesian procedure for inference in such models via reinforcement, as hinted in Muliere et al. (2003), could constitute a challenging matter. Here, we sketch some ideas concerning the definition of another reinforced process in continuous time in order to solve this problem.

Recall $E = \{0, \ldots, m\}$ be a finite set of states and consider $\forall i \in E$:

- a positive measure on $\mathcal{B}[0,\infty)$ α_i, finite on bounded sets, with a countable set of points of positive mass D_i and continuous part (w.r.t. the Lebesgue measure) $\alpha_{i,c}((0,t]) = \alpha_i((0,t]) - \sum_{t_j \le t} \alpha_i(\{t_j\})$;

- a measurable function $\beta_i : [0,\infty) \to (0,\infty)$;

- a vector describing an urn composition $c_i(t) = (c_{i1}(t), \ldots, c_{im}(t))$ s.t. $c_{ij}(t) \geq 0 \; \forall i, \forall j, \forall t$ and $c_{ii}(t) = 0 \; \forall i, \forall t$.

Define an intensity measure $\mathbf{\Lambda}$ on the measurable space $([0,\infty), \mathcal{B}[0,\infty))$ as a matrix-valued measure (additive interval function) by

$$d\Lambda_{ij}(t) = \frac{d\alpha_i(t)}{\alpha_i(\{t\}) + \beta_i(t)} \frac{c_{ij}(t)}{\sum_{j=1}^{l} c_{ij}(t)}, \quad i \neq j \tag{3.4.1}$$

$$\begin{aligned} d\Lambda_{ii}(t) &= -\sum_{j\neq i} d\Lambda_{ij} = \\ &= -\frac{d\alpha_i(t)}{\alpha_i(\{t\}) + \beta_i(t)} \end{aligned} \tag{3.4.2}$$

Note that the properties of α_i, β_i and c_i imply that $\mathbf{\Lambda}$ is finite on bounded sets and

$$\Lambda_{ii}(s,t) \leq 0 \quad i \in L,\ s \leq t \tag{3.4.3}$$

$$\Lambda_{ij}(s,t) \geq 0 \quad i \neq j,\ s \leq t \tag{3.4.4}$$

$$\sum_{j=0}^{m} \Lambda_{ij}(s,t) = 0 \quad i \in L,\ s \leq t \tag{3.4.5}$$

$$\Lambda_{ii}(\{t\}) \geq -1 \quad i \in L,\ \forall t. \tag{3.4.6}$$

Furthermore, the measure $\mathbf{\Lambda}$ is dominated by the real measure

$$\mathbf{\Lambda}_o(s,t) = -2\, trace\, \mathbf{\Lambda}(s,t) \tag{3.4.7}$$

bounded on final intervals.

Finally, by mean of the product integral, define

$$\boldsymbol{P}_{\mathbf{\Lambda}}(s,t) = \prod_{(s,t]} [\mathbf{1} + d\mathbf{\Lambda}]. \tag{3.4.8}$$

Theorem 12 of Gill and Johansen (1990) assures that $\boldsymbol{P}_{\mathbf{\Lambda}}(s,t)$ is a stochastic matrix and satisfies

$$\boldsymbol{P}_{\mathbf{\Lambda}}(s,t) = \boldsymbol{P}_{\mathbf{\Lambda}}(s,u)\boldsymbol{P}_{\mathbf{\Lambda}}(u,t) \quad 0 \leq s \leq u \leq t < \infty \tag{3.4.9}$$

$$\boldsymbol{P}_{\mathbf{\Lambda}}(s,s) = \mathbf{1} \quad s \geq 0 \tag{3.4.10}$$

$$\boldsymbol{P}_{\mathbf{\Lambda}}(s,t) \to \mathbf{1} \quad t \downarrow s \tag{3.4.11}$$

So, by this way, a Markov non-homogeneous process in continuous time $X = \{X_t, t \geq 0\}$ with transition matrix $\boldsymbol{P}_\Lambda$ is defined on the finite state space E.

Moreover, this process turns out to have piecewise constant sample paths which are right-continuous and have finitely many jumps on finite intervals.

In this sense the process is well defined, starting from any given state in E at any given time point.

Assume $X_0 = l_0 \in E$, being l_0 a recurrent state for the process X.

Now we define a sequence of Markov non-homogeneous processes in continuous time in the following way. When the process X returns for the first time in the state l_0, it stops and the measurement of time is reset to 0.

Given the realization of the process X, a new process $X^{(2)} = \{X_t^{(2)}, t \geq 0\}$ starts from $X_0^{(2)} = l_0$ and so on for $X^{(n)}$.

These processes are characterized by the intensities measures $\Lambda^{(n)}$ through the following (note $X^{(1)} = X$) for $n \geq 1$:

$$\alpha_i^{(n)}(0,t] = \alpha_i(0,t] + \sum_{k=1}^{n-1} \sharp\{t \in (0,t] : X_{t^-}^{(k)} = i,\ X_t^{(k)} \neq i\} \tag{3.4.12}$$

$$\beta_i^{(n)}(t) = \beta_i(t) + \sum_{k=1}^{n-1} \mathbb{1}_{\left[X_{t^-}^{(k)}=i,\ X_t^{(k)}=i\right]} \tag{3.4.13}$$

$$c_{ij}^{(n)} = c_{ij}(t) + \sum_{k=1}^{n-1} \mathbb{1}_{\left[X_{t^-}^{(k)}=i,\ X_t^{(k)}=j\right]} \tag{3.4.14}$$

so that we have

$$d\Lambda_{ij}^{(n)}(t) = \frac{d\alpha_i(t) + \sum_{k=1}^{n-1} \mathbb{1}_{\left[X_{t-}^{(k)}=i,\, X_t^{(k)}\neq i\right]}}{\alpha_i(\{t\}) + \beta_i(t) + \sum_{k=1}^{n-1} \mathbb{1}_{\left[X_{t-}^{(k)}=i\right]}} \frac{c_{ij}(t) + \sum_{k=1}^{n-1} \mathbb{1}_{\left[X_{t-}^{(k)}=i,\, X_t^{(k)}=j\right]}}{\sum_{j\neq i} c_{ij}(t) + \sum_{k=1}^{n-1} \mathbb{1}_{\left[X_{t-}^{(k)}=i, X_t^{(k)}\neq i\right]}}, \quad i \neq j \tag{3.4.15}$$

$$d\Lambda_{ii}^{(n)}(t) = -\frac{d\alpha_i(t) + \sum_{k=1}^{n-1} \mathbb{1}_{\left[X_{t-}^{(k)}=i,\, X_t^{(k)}\neq i\right]}}{\alpha_i(\{t\}) + \beta_i(t) + \sum_{k=1}^{n-1} \mathbb{1}_{\left[X_{t-}^{(k)}=i\right]}}. \tag{3.4.16}$$

and the matrices of transition probabilities conditioned on the trajectories of the previous processes

$$\boldsymbol{P}_{\boldsymbol{\Lambda}}^{(n)}(s,t) = \prod_{(s,t]} \left[\mathbf{1} + d\boldsymbol{\Lambda}^{(n)}\right]. \tag{3.4.17}$$

Remark 3.4.1. It has already been emphasized that α, β and c with the above properties determine $\boldsymbol{\Lambda}$ such that the continuous time Markov process $X = X^{(1)}$ is well defined.

Since, conditionally on $X^{(1)}, \ldots, X^{(n-1)}$, $\boldsymbol{\Lambda}^{(n)}$ enjoys, by construction, properties like (3.4.3)-(3.4.6) *a.s.*, the process $X^{(n)}$ is well defined *a.s.* So all the processes of the sequence are well defined *a.s.*

More precisely, the transitions probabilities result to be

$$P\left[X_t = j \,|\, X_s = i\right] = \prod_{(s,t)} \left[1 - \frac{d\alpha_i(v)}{\alpha_i(\{v\}) + \beta_i(v)}\right] \frac{d\alpha_i(t)}{\alpha_i(\{t\}) + \beta_i(t)} \frac{c_{ij}(t)}{\sum_{j=1}^{l} c_{ij}(t)} \tag{3.4.18}$$

$$P\left[X_t^{(k)} = j \,\middle|\, X^{(1)}, \ldots, X^{(k-1)}, X_s^{(k)} = i\right] = \prod_{(s,t)} \left[1 - \frac{d\alpha_i^{(k)}(v)}{\alpha_i^{(k)}(\{v\}) + \beta_i^{(k)}(v)}\right] \frac{d\alpha_i^{(k)}(t)}{\alpha_i^{(k)}(\{t\}) + \beta_i^{(k)}(t)} \frac{c_{ij}^{(k)}}{\sum_{j\neq i} c_{ij}^{(k)}}, \quad k > 1 \tag{3.4.19}$$

and if $T_i^{(k)}$ indicates the time the process $X^{(k)}$ is in state i

$$P\left[T_i > t \middle| T_i > s\right] = \prod_{(s,t]}\left[1 - \frac{d\alpha_i(v)}{\alpha_i(\{v\}) + \beta_i(v)}\right] \tag{3.4.20}$$

$$P\left[T_i^{(k)} > t \middle| X^{(1)}, \ldots, X^{(k-1)}, T_i^{(k)} > s\right] = \prod_{(s,t]}\left[1 - \frac{d\alpha_i^{(k)}(v)}{\alpha_i^{(k)}(\{v\}) + \beta_i^{(k)}(v)}\right] \tag{3.4.21}$$

As $\Lambda^{(k)}$ does not depend on the order of $X^{(1)}, \ldots, X^{(k)}$, these predictive distributions are the same for all permutation of $X^{(1)}, \ldots, X^{(k)}$.

Proposition 3.4.1. The sequence of blocks $\left\{X^{(n)}, n \geq 1\right\}$ is exchangeable.

Proof. As any finite permutation can be obtained by a finite sequence of simpler permutations resulting by inversion of contiguous elements, it is sufficient to show:

$$\mathcal{L}\left(X^{(1)}, \ldots, X^{(k)}, X^{(k+1)}, \ldots, X^{(n)}\right) = \mathcal{L}\left(X^{(1)}, \ldots, X^{(k+1)}, X^{(k)}, \ldots, X^{(n)}\right). \tag{3.4.22}$$

Since (3.4.19) does not depend on the order of $X^{(1)}, \ldots, X^{(k-1)}$, (3.4.22) can be reduced to

$$\mathcal{L}\left(X^{(k)}, X^{(k+1)} \middle| X^{(1)}, \ldots, X^{(k)}\right) = \mathcal{L}\left(X^{(k+1)}, X^{(k)} \middle| X^{(1)}, \ldots, X^{(k)}\right). \tag{3.4.23}$$

Now, given $X^{(1)}, \ldots, X^{(k-1)}$, two realizations (i.e. sequences of couples of visited states ad jump times) of $X^{(k)}$ and $X^{(k+1)}$.

Let $\{t_1, t_2, \ldots, t_n\}$ the set of jump times of $X^{(k)}$ and $X^{(k+1)}$ with $0 < t_1 \leq t_2 \leq \cdots \leq t_n$. We can decompose the interval $[0, t_n]$ in the union of $[0, t_1]$, $(t_1, t_2], \ldots$, $(t_{n-1}, t_n]$ and compute the probability of the two realizations as product of the elements on this partition.

These intervals result to be of two different types:

- the ones in which $X^{(k)} \neq X^{(k+1)}$, except in the last point,

- the ones in which $X^{(k)} = X^{(k+1)}$, except in the last point.

In the first kind of sub-intervals, we have $X^{(k)} \neq X^{(k+1)}$, i.e. $X^{(k)}$ and $X^{(k+1)}$ visiting different states, except in the last point.
Hence, given τ_1 and $X^{(1)}, \dots, X^{(k-1)}$, in such intervals $X^{(k)}$ and $X^{(k+1)}$ do not affect each other with reinforcement: so, they are independent (conditionally on τ_1 and $X^{(1)}, \dots, X^{(k-1)}$) and, hence, exchangeable.
In the second kind of intervals, say $(\sigma, \tau]$, we can show by direct calculation the exchangeability. The two trajectories may have different behaviors in the last point: one may jump to another state, while the other may stay in the present state or both may jump.
We consider just the case in which one trajectory jump i.e.

- $X_t^{(k)} = i,\ t \in (\sigma, \tau]$;

- $X_t^{(k+1)} = i,\ t \in (\sigma, \tau),\ X_\tau^{(k+1)} = j$

and viceversa.
Given $\mathbf{X}^{(k-1)} = (X^{(1)}, \dots, X^{(k-1)})$, we compute

$$P\left[X_\tau^{(k)} = i \middle| \mathbf{X}^{(k-1)}, X_\sigma^{(k)} = i\right] = \prod_{(\sigma,\tau]} \left[1 - \frac{d\alpha_i(v) + n(v,i,k-1)}{\alpha_i(\{v\}) + \beta_i(v) + r(v,i,k-1)}\right] \tag{3.4.24}$$

$$P\left[X_\tau^{(k+1)}=j\,\middle|\,\mathbf{X}^{(k-1)};\;X_t^{(k)}=i,\;t\in(\sigma,\tau];\;X_\sigma^{(k+1)}=i\right]=$$
$$=\prod_{(\sigma,\tau)}\left[1-\frac{d\alpha_i(v)+n(v,i,k-1)}{\alpha_i(\{v\})+\beta_i(v)+r(v,i,k-1)+1}\right]$$
$$\frac{d\alpha_i(\tau)+n(\tau,i,k-1)}{\alpha_i(\{\tau\})+\beta_i(\tau)+r(\tau,i,k-1)+1}\frac{c_{ij}(\tau)+s(\tau,i,j,k-1)}{\sum_{j\neq i}(c_{ij}(\tau)+s(\tau,i,j,k-1))} \tag{3.4.25}$$

$$P\left[X_\tau^{(k)}=j\,\middle|\,\mathbf{X}^{(k-1)},X_\sigma^{(k)}=i\right]=\prod_{(\sigma,\tau)}\left[1-\frac{d\alpha_i(v)+n(v,i,k-1)}{\alpha_i(\{v\})+\beta_i(v)+r(v,i,k-1)}\right]$$
$$\frac{d\alpha_i(\tau)+n(\tau,i,k-1)}{\alpha_i(\{\tau\})+\beta_i(\tau)+r(\tau,i,k-1)}\frac{c_{ij}(\tau)+s(\tau,i,j,k-1)}{\sum_{j\neq i}(c_{ij}(\tau)+s(\tau,i,j,k-1))} \tag{3.4.26}$$

$$P\left[X_\tau^{(k+1)}=i\,\middle|\,\mathbf{X}^{(k-1)};\;X_t^{(k)}=i,\;t\in(\sigma,\tau),X_\tau^{(k)}=j;\;X_\sigma^{(k+1)}=i\right]=$$
$$=\prod_{(\sigma,\tau)}\left[1-\frac{d\alpha_i(v)+n(v,i,k-1)}{\alpha_i(\{v\})+\beta_i(v)+r(v,i,k-1)+1}\right]$$
$$\left[1-\frac{d\alpha_i(\tau)+n(\tau,i,k-1)+1}{\alpha_i(\{\tau\})+\beta_i(\tau)+r(\tau,i,k-1)+1}\right] \tag{3.4.27}$$

where

$$n(v,i,k)=\sum_{m=1}^{k}\mathbb{1}_{\left[X_{v-}^{(m)}=i,\;X_v^{(m)}\neq i\right]} \tag{3.4.28}$$
$$r(v,i,k)=\sum_{m=1}^{k}\mathbb{1}_{\left[X_{v-}^{(m)}=i\right]} \tag{3.4.29}$$
$$s(v,i,j,k)=\sum_{m=1}^{k}\mathbb{1}_{\left[X_{v-}^{(m)}=i,\;X_v^{(m)}=j\right]} \tag{3.4.30}$$

Recalling that

$$d\alpha_i(t)=d\alpha_{i,c}(t)\mathbb{1}_{[t\notin D_i]}+\alpha_i(\{t\})\mathbb{1}_{[t\in D_i]}, \tag{3.4.31}$$

it results that (3.4.24)(3.4.25)=(3.4.26)(3.4.27).

The case both the processes have jumps in τ to different is quite similar.

So, $X^{(k)}$ and $X^{(k+1)}$ are exchangeable on all the intervals like $(\sigma, \tau]$.

Considering the product over all the intervals, we obtain the exchangeability of $X^{(k)}$ and $X^{(k+1)}$ on all their definition interval. □

These ideas are a possible development of the last section of Muliere et al. (2003). Indeed, it is possible to consider the special case in which $E = \{0, 1\}$ and $p_{01}(t) = p_{10}(t) = 1 \,\forall\, t$, so that the successor states of 0 and 1 are deterministic and respectively 1 and 0.

Following our construction, starting from α_i and β_i for $i = 0, 1$, after n exchangeable realizations, we obtain

$$d\Lambda_0^{(n)}(t) = \frac{d\alpha_0(t) + \sum_{j=1}^n \mathbb{1}_{[X_{t-}^k=0,\, X_t^k=1]}}{\alpha_0(\{t\}) + \beta_0(t) + \sum_{j=1}^n \mathbb{1}_{[X_{t-}^k=0]}} \tag{3.4.32}$$

$$d\Lambda_1^{(n)}(t) = \frac{d\alpha_1(t) + \sum_{j=1}^n \mathbb{1}_{[X_{t-}^k=1,\, X_t^k=0]}}{\alpha_1(\{t\}) + \beta_1(t) + \sum_{j=1}^n \mathbb{1}_{[X_{t-}^k=1]}} \tag{3.4.33}$$

Note that X^j can be represented as a couple of jump times (T_0^j, T_1^j) and

$$\{X_{t-}^j = 0,\, X_t^j = 1\} = \{T_0^i = t\} \tag{3.4.34}$$

$$\{X_{t-}^j = 0\} = \{T_0^j \geq t\} \tag{3.4.35}$$

$$\{X_{t-}^j = 1,\, X_t^j = 0\} = \{T_1^j = t\} \tag{3.4.36}$$

$$\{X_{t-}^j = 1\} = \{T_1^0 < t \leq T_j^1\} \tag{3.4.37}$$

Therefore,

$$P\left[T_0^{n+1} > t \,\middle|\, X^1, \ldots, X^n\right] = \prod_{[0,t]} \left[1 - \frac{d\alpha_0(v) + \sum_{j=1}^n \delta_{T_0^j}(v)}{\alpha_0(\{v\}) + \beta_0(t) + \sum_{j=1}^n \mathbb{1}_{[T_0^j \geq v]}}\right] \tag{3.4.38}$$

and

$$P\left[T_1^{n+1} > t \middle| X^1, \ldots, X^n, T_0^{n+1}\right] = \prod_{(T_0^{n+1}, t]} \left[1 - \frac{d\alpha_1(v) + \sum_{j=1}^{n} \delta_{T_1^j}(v)}{\alpha_1(\{v\}) + \beta_1(t) + \sum_{j=1}^{n} \mathbb{1}_{\left[T_0^j < v \leq T_1^j\right]}}\right]. \tag{3.4.39}$$

A good understanding of how the new reinforcement and resetting rules work along with the Markovian structure of the trajectories could constitute the objective of future studies. Following this approach, it would be possible to deal with other problems of the survival analysis like interval censored data and truncated observations as in the models of Frydman (1992), Frydman (1994), Frydman (1995) and Lagakos et al. (1988) whose intimate structure is a two or three state Markov or semi-Markov continuous time process.

Chapter 4

Bayesian nonparametric estimation of a bivariate survival function

In this chapter, using a reinforcement scheme, we deal with a difficult problem of survival and reliability analysis arising in studies where the experimental unit consists of couples of components (for example, twins, eyes, kidneys ...) or pairs of lifetimes for the same individual (for example, response times for successive courses of a medical treatment): the estimation of a bivariate survival function for lifetimes subjected to censoring.

We define a bivariate reinforced process, derived from a generalized Pólya urn scheme, to model coupled survival in such a way as the observations are assumed to be exchangeable and, hence, define, by de Finetti's representation theorem, a prior on the space of bivariate distributions. We shall work exclusively with discrete observations and so shall be working with observation space $\mathbb{N}_0^2 = \{0, 1, \ldots\} \times \{0, 1, \ldots\}$.

Although this prior has a structure making the computation of the posterior intractable, a Bayesian nonparametric predictive estimator for the bivariate survival function may be obtained quite easily by mean of the reinforcement rule. The explicit computation can be carried through via an implementation of a Gibbs sampler. Moreover, since the bivariate prior is constructed on a generalized Pólya urn scheme, producing a discrete beta-Stacy prior, it is possible to exploit some good feature of this prior for univariate survival function to obtain some knowledge about the topological support of the bivariate prior.

Let us describe briefly the layout of this chapter. Section 4.1 recalls the definition and some features of generalized Pólya urn scheme. In Section 4.2 the construction of a bivariate reinforced process is done and a prior on the space of distribution functions on $\mathbb{N}_0^2$ is derived and some properties are pointed out, with particular attention to its support and consistency. Section 4.3 shows how to use this prior to make inference about a bivariate survival function with data eventually subjected to censorship. Finally, in Section 4.4, an example is given.

4.1 Preliminaries

The basic building block of the model is the generalized Pólya urn scheme. As we have already pointed out, this scheme is related with both Reinforced Urn Processes and discrete beta-Stacy. Here we recall the definition of Walker and Muliere (1997).

Definition 4.1.1. Consider a sequence of random variables $\{T_n;\, n \geq 1\}$ with values on the non-negative integers $\mathbb{N}_0 = \{0, 1, 2, \dots\}$ and let $\alpha_j, \beta_j \geq 0$ for $j \in \mathbb{N}_0$. The sequence $\{T_n;\, n \geq 1\}$ is said to come from a generalized Pólya urn scheme if

$$P\left[T_1 = t\right] = \frac{\alpha_t}{\alpha_t + \beta_t} \prod_{j=0}^{t-1} \frac{\beta_j}{\alpha_j + \beta_j} \tag{4.1.1}$$

and

$$P\left[T_{n+1}=t \mid \mathbf{T}_n=\mathbf{t}_n\right]=\frac{\alpha_t+m_t\left(\mathbf{t}_n\right)}{\alpha_t+\beta_t+s_t\left(\mathbf{t}_n\right)} \prod_{j=0}^{t-1} \frac{\beta_j+r_j\left(\mathbf{t}_n\right)}{\alpha_j+\beta_j+s_j\left(\mathbf{t}_n\right)} \tag{4.1.2}$$

where

$$m_j\left(\mathbf{z}_n\right) = \sum_{k=1}^{n} \mathbb{1}_{[z_k=j]} \tag{4.1.3}$$

$$r_j\left(\mathbf{z}_n\right) = \sum_{k=1}^{n} \mathbb{1}_{[z_k>j]} \tag{4.1.4}$$

$$s_j\left(\mathbf{z}_n\right) = m_j\left(\mathbf{z}_n\right)+r_j\left(\mathbf{z}_n\right) \tag{4.1.5}$$

(henceforward bold letters will indicate vectors, for example $\mathbf{T}_n = (T_1, \ldots, T_n)$; relations and operations between them will be done componentwise).

The generalized Pólya urn scheme translates the idea of Bayesian learning in follow-up studies where survival times of different subjects are observed.

As we have seen, Walker and Muliere (1997) and Muliere et al. (2000) show that the sequence $\{T_n,\, n \geq 1\}$ is exchangeable and, by de Finetti's representation theorem, there exists a random distribution function F, such that, given F, the random variables T_n are independent and identically distributed with distribution F. Moreover, F is distributed according to a beta-Stacy process on the integers with parameters $\{\alpha_j, \beta_j,\ j \in \mathbb{N}_0\}$. The beta-Stacy process is *neutral to the right* and, so, conjugate to right censored observations.

Let $T_1, \ldots, T_n$ be independent and identically distributed and subjected to right censoring, therefore what is observed is represented by the couples $(T_1^*, \delta_1), \ldots, (T_n^*, \delta_n)$

where

$$T_i^* = t, \delta_i = 0 \quad \Leftrightarrow \quad \text{a censoring took place: } T_i \geq t$$
$$T_i^* = t, \delta_i = 1 \quad \Leftrightarrow \quad \text{a death happened: } T_i = t. \tag{4.1.6}$$

With a quadratic loss function, the predictive distribution of T_{n+1} given $(\mathbf{T}_n, \boldsymbol{\delta}_n)$ is the Bayes estimator for the random distribution function.

So, analogously, for the survival function and under a beta-Stacy prior, we have

$$\begin{aligned} \hat{S}(t) &= P\left[T_{n+1} > t \,|\, \mathbf{T}_n^* = \mathbf{t}_n, \boldsymbol{\delta}_n = \mathbf{d}_n\right] \\ &= \prod_{j=0}^{t} \left[1 - \frac{\alpha_j + m_j^*(\mathbf{t}_n, \mathbf{d}_n)}{\alpha_j + \beta_j + s_j(\mathbf{t}_n)}\right] \end{aligned} \tag{4.1.7}$$

where $\mathbf{d}_n \in \{0,1\}^n$ and $m_j^*(\mathbf{t}_n, \mathbf{d}_n) = \sum_{k=1}^n \mathbb{1}_{[t_k = j, d_k = 1]}$.

Finally, note that:

1. without censoring, the expression in (4.1.7) reduces to

$$\hat{S}(t) = P\left[T_{n+1} > t \,|\, \mathbf{T}_n = \mathbf{t}_n\right] = \prod_{j=0}^{t} \frac{\beta_j + r_j(\mathbf{t}_n)}{\alpha_j + \beta_j + s_j(\mathbf{t}_n)}; \tag{4.1.8}$$

2. if $\alpha_j, \beta_j \to 0$, $\forall j$, $\hat{S}(t)$ reduces to classical Kaplan-Meier estimator.

4.2 A bivariate reinforced random process

4.2.1 Definition

Our aim is to construct a bivariate random process $\{(X_n, Y_n), n \geq 1\}$ providing a model for coupled lifetimes and incorporating the basic principle of reinforcement similar to generalized Pólya urn. The idea underlying this scheme is that of reinforcing the path from the origin to the point $T_n = t$, before generating T_{n+1}, sequentially for $n \geq 1$.

It is relatively easy because the random variables T_n can be represented as a point on the non-negative line. On the other hand, a couple (X_n, Y_n) is a point in the non-negative orthant and unfortunately there is no unique path from $(0,0)$ to (x, y). So, the reinforcement procedure must be introduced in an alternative way.

Let $\{A_n, n \geq 1\}$, $\{B_n, n \geq 1\}$ and $\{C_n, n \geq 1\}$ be independent sequences from generalized Pólya urns with parameters (α_j^A, β_j^B), (α_j^B, β_j^B) and (α_j^C, β_j^C), $j \in \mathbb{N}_0$, respectively.

Now, let us define a bivariate random process $\{(X_n, Y_n), n \geq 1\}$ by

$$\begin{aligned} X_n &= A_n + B_n \\ Y_n &= A_n + C_n, \quad n \geq 1. \end{aligned} \tag{4.2.1}$$

The relations above postulate a particular and very simple form of dependence between X_n and Y_n: for a given couple, each element is supposed to have a common component (A_n) and a specific one (B_n and C_n). In many lifetime studies this assumption appears sensible.

By this construction, it turns out, conditionally on A_n, that X_n and Y_n are independent. Moreover, $\sigma(\mathbf{A}_n, \mathbf{B}_n, \mathbf{C}_n) = \sigma(\mathbf{A}_n, \mathbf{X}_n, \mathbf{Y}_n)$.

The structure of the dependence is described by the following relationships,

$$\mathrm{Cov}(X_1, Y_1) = \mathrm{Var}(A_1) \geq 0 \tag{4.2.2}$$

$$\mathrm{Cov}(X_{n+1}, Y_{n+1} | \mathbf{A}_n, \mathbf{B}_n, \mathbf{C}_n) = \mathrm{Var}(A_{n+1} | \mathbf{A}_n), \quad n \geq 1 \tag{4.2.3}$$

The predictive distribution of this bivariate process can be easily computed in terms of those of the three generalized Pólya urn schemes.

Indeed, $P\left[X_{n+1} > x, Y_{n+1} > y \middle| \mathbf{X}_n = \mathbf{x}_n, \mathbf{Y}_n = \mathbf{y}_n\right]$ is nothing but the ratio

$$P\left[X_{n+1} > x, Y_{n+1} > y \middle| \mathbf{X}_n = \mathbf{x}_n, \mathbf{Y}_n = \mathbf{y}_n\right] = \frac{P\left[X_{n+1} > x, Y_{n+1} > y, \mathbf{X}_n = \mathbf{x}_n, \mathbf{Y}_n = \mathbf{y}_n\right]}{P\left[\mathbf{X}_n = \mathbf{x}_n, \mathbf{Y}_n = \mathbf{y}_n\right]}. \tag{4.2.4}$$

Therefore, it is enough to compute the numerator and the denominator separately.

The first is equal to

$$\begin{aligned} &P\left[X_{n+1} > x, Y_{n+1} > y, \mathbf{X}_n = \mathbf{x}_n, \mathbf{Y}_n = \mathbf{y}_n\right] = \\ &\sum_{a_{n+1}=0}^{x \wedge y} \sum_{a_n=0}^{x_n \wedge y_n} \cdots \sum_{a_1=0}^{x_1 \wedge y_1} \Big[P\left[A_{n+1} = a_{n+1} \middle| \mathbf{A}_n = \mathbf{a}_n\right] \\ &P\left[B_{n+1} > x - a_{n+1} \middle| \mathbf{B}_n = \mathbf{x}_n - \mathbf{a}_n\right] P\left[C_{n+1} > y - a_{n+1} \middle| \mathbf{C}_n = \mathbf{y}_n - \mathbf{a}_n\right] \\ &\prod_{i=1}^{n-1} \left(P\left[A_{i+1} = a_{i+1} \middle| \mathbf{A}_i = \mathbf{a}_i\right] P\left[B_{i+1} = x_{i+1} - a_{i+1} \middle| \mathbf{B}_i = \mathbf{x}_i - \mathbf{a}_i\right]\right. \\ &\left. P\left[C_{i+1} = y_{i+1} - a_{i+1} \middle| \mathbf{C}_i = \mathbf{y}_i - \mathbf{a}_i\right]\right) \\ &P\left[B_1 = x_1 - a_1\right] P\left[C_1 = y_1 - a_1\right] P\left[A_1 = a_1\right] \Big] \end{aligned} \tag{4.2.5}$$

and the second

$$\begin{aligned} &P\left[\mathbf{X}_n = \mathbf{x}_n, \mathbf{Y}_n = \mathbf{y}_n\right] = \\ &\sum_{a_n=0}^{x_n \wedge y_n} \cdots \sum_{a_1=0}^{x_1 \wedge y_1} \Big[\prod_{i=1}^{n-1} \Big(P\left[A_{i+1} = a_{i+1} \middle| \mathbf{A}_i = \mathbf{a}_i\right] \\ &P\left[C_{i+1} = y_{i+1} - a_{i+1} \middle| \mathbf{C}_i = \mathbf{y}_i - \mathbf{a}_i\right] P\left[B_{i+1} = x_{i+1} - a_{i+1} \middle| \mathbf{B}_i = \mathbf{x}_i - \mathbf{a}_i\right] \Big) \\ &P\left[B_1 = x_1 - a_1\right] P\left[C_1 = y_1 - a_1\right] P\left[A_1 = a_1\right] \Big]. \end{aligned} \tag{4.2.6}$$

Substituting the predictive distributions, in the previous, with their own expressions, we obtain for the numerator

$$
\begin{aligned}
&P\left[X_{n+1} > x, Y_{n+1} > y, \mathbf{X}_n = \mathbf{x}_n, \mathbf{Y}_n = \mathbf{y}_n\right] = \\
&\sum_{a_{n+1}=0}^{x \wedge y} \sum_{a_n=0}^{x_n \wedge y_n} \cdots \sum_{a_1=0}^{x_1 \wedge y_1} \left[\frac{\alpha^A_{a_{n+1}} + m_{a_{n+1}}(\mathbf{a}_n)}{\alpha^A_{a_{n+1}} + \beta^A_{a_{n+1}} + s_{a_{n+1}}(\mathbf{a}_n)} \prod_{j=0}^{a_{n+1}-1} \frac{\beta^A_j + r_j(\mathbf{a}_n)}{\alpha^A_j + \beta^A_j + s_j(\mathbf{a}_n)} \right. \\
&\prod_{j=0}^{x-a_{n+1}} \frac{\beta^B_j + r_j(\mathbf{x}_n - \mathbf{a}_n)}{\alpha^B_j + \beta^B_j + s_j(\mathbf{x}_n - \mathbf{a}_n)} \prod_{j=0}^{y-a_{n+1}} \frac{\beta^C_j + r_j(\mathbf{y}_n - \mathbf{a}_n)}{\alpha^C_j + \beta^C_j + s_j(\mathbf{y}_n - \mathbf{a}_n)} \\
&\prod_{i=1}^{n-1} \left(\frac{\alpha^A_{a_{i+1}} + m_{a_{i+1}}(\mathbf{a}_i)}{\alpha^A_{a_{i+1}} + \beta^A_{a_{i+1}} + s_{a_{i+1}}(\mathbf{a}_i)} \prod_{j=0}^{a_{i+1}-1} \frac{\beta^A_j + r_j(\mathbf{a}_i)}{\alpha^A_j + \beta^A_j + s_j(\mathbf{a}_i)} \right. \\
&\frac{\alpha^B_{x_{i+1}-a_{i+1}} + m_{x_{i+1}-a_{i+1}}(\mathbf{x}_i - \mathbf{a}_i)}{\alpha^B_{x_{i+1}-a_{i+1}} + \beta^B_{x_{i+1}-a_{i+1}} + s_{x_{i+1}-a_{i+1}}(\mathbf{x}_i - \mathbf{a}_i)} \prod_{j=0}^{x_{i+1}-a_{i+1}-1} \frac{\beta^B_j + r_j(\mathbf{x}_i - \mathbf{a}_i)}{\alpha^B_j + \beta^B_j + s_j(\mathbf{x}_i - \mathbf{a}_i)} \\
&\left. \frac{\alpha^C_{y_{i+1}-a_{i+1}} + m_{y_{i+1}-a_{i+1}}(\mathbf{y}_i - \mathbf{a}_i)}{\alpha^C_{y_{i+1}-a_{i+1}} + \beta^C_{y_{i+1}-a_{i+1}} + s_{y_{i+1}-a_{i+1}}(\mathbf{y}_i - \mathbf{a}_i)} \prod_{j=0}^{y_{i+1}-a_{i+1}-1} \frac{\beta^C_j + r_j(\mathbf{y}_i - \mathbf{a}_i)}{\alpha^C_j + \beta^C_j + s_j(\mathbf{y}_i - \mathbf{a}_i)} \right) \\
&\frac{\alpha^A_{a_1}}{\alpha^A_{a_1} + \beta^A_{a_1}} \prod_{j=0}^{a_1-1} \frac{\beta^A_j}{\alpha^A_j + \beta^A_j} \frac{\alpha^B_{x_1-a_1}}{\alpha^B_{x_1-a_1} + \beta^B_{x_1-a_1}} \prod_{j=0}^{x_1-a_1-1} \frac{\beta^B_j}{\alpha^B_j + \beta^B_j} \\
&\left. \frac{\alpha^C_{y_1-a_1}}{\alpha^C_{y_1-a_1} + \beta^C_{y_1-a_1}} \prod_{j=0}^{y_1-a_1-1} \frac{\beta^C_j}{\alpha^C_j + \beta^C_j} \right]. \qquad (4.2.7)
\end{aligned}
$$

Similarly, for the denominator

$$
\begin{aligned}
&P\left[\mathbf{X}_n = \mathbf{x}_n, \mathbf{Y}_n = \mathbf{y}_n\right] = \\
&\sum_{a_n=0}^{x_n \wedge y_n} \cdots \sum_{a_1=0}^{x_1 \wedge y_1} \left[\prod_{i=1}^{n-1} \left(\frac{\alpha^A_{a_{i+1}} + m_{a_{i+1}}(\mathbf{a}_i)}{\alpha^A_{a_{i+1}} + \beta^A_{a_{i+1}} + s_{a_{i+1}}(\mathbf{a}_i)} \prod_{j=0}^{a_{i+1}-1} \frac{\beta^A_j + r_j(\mathbf{a}_i)}{\alpha^A_j + \beta^A_j + s_j(\mathbf{a}_i)} \right. \right. \\
&\frac{\alpha^B_{x_{i+1}-a_{i+1}} + m_{x_{i+1}-a_{i+1}}(\mathbf{x}_i - \mathbf{a}_i)}{\alpha^B_{x_{i+1}-a_{i+1}} + \beta^B_{x_{i+1}-a_{i+1}} + s_{x_{i+1}-a_{i+1}}(\mathbf{x}_i - \mathbf{a}_i)} \prod_{j=0}^{x_{i+1}-a_{i+1}-1} \frac{\beta^B_j + r_j(\mathbf{x}_i - \mathbf{a}_i)}{\alpha^B_j + \beta^B_j + s_j(\mathbf{x}_i - \mathbf{a}_i)} \\
&\left. \frac{\alpha^C_{y_{i+1}-a_{i+1}} + m_{y_{i+1}-a_{i+1}}(\mathbf{y}_i - \mathbf{a}_i)}{\alpha^C_{y_{i+1}-a_{i+1}} + \beta^C_{y_{i+1}-a_{i+1}} + s_{y_{i+1}-a_{i+1}}(\mathbf{y}_i - \mathbf{a}_i)} \prod_{j=0}^{y_{i+1}-a_{i+1}-1} \frac{\beta^C_j + r_j(\mathbf{y}_i - \mathbf{a}_i)}{\alpha^C_j + \beta^C_j + s_j(\mathbf{y}_i - \mathbf{a}_i)} \right)
\end{aligned}
$$

$$\frac{\alpha^A_{a_1}}{\alpha^A_{a_1}+\beta^A_{a_1}}\prod_{j=0}^{a_1-1}\frac{\beta^A_j}{\alpha^A_j+\beta^A_j}\frac{\alpha^B_{x_1-a_1}}{\alpha^B_{x_1-a_1}+\beta^B_{x_1-a_1}}\prod_{j=0}^{x_1-a_1-1}\frac{\beta^B_j}{\alpha^B_j+\beta^B_j}$$
$$\left.\frac{\alpha^C_{y_1-a_1}}{\alpha^C_{y_1-a_1}+\beta^C_{y_1-a_1}}\prod_{j=0}^{y_1-a_1-1}\frac{\beta^C_j}{\alpha^C_j+\beta^C_j}\right] \tag{4.2.8}$$

These expressions are relatively simple to obtain theoretically, but they may be rather cumbersome to compute.

From a statistical point of view, the most interesting feature of this bivariate process is illustrated in the following straightforward proposition.

Proposition 4.2.1. The sequence of couples $\{(X_n, Y_n), n \geq 1\}$ is exchangeable.

Proof. Note that (X_n, Y_n) is a measurable function of (A_n, B_n, C_n). This is an exchangeable sequence and hence so is $\{(X_n, Y_n), n \geq 1\}$.

Obviously, then $\{X_n, n \geq 1\}$ and $\{Y_n, n \geq 1\}$ are exchangeable. □

Hence, the de Finetti's Representation Theorem assures the existence of a bivariate random distribution function F_{XY} conditionally on which the couples (X_n, Y_n) are independent and identically distributed and have as distribution F_{XY}.

F_A, F_B and F_C are likewise the random distribution functions corresponding to generalized Pólya urn scheme $\{A_n, n \geq 1\}$, $\{B_n, n \geq 1\}$ and $\{C_n, n \geq 1\}$; moreover, they are distributed according to beta-Stacy processes with parameters (α^A_j, β^A_j), (α^B_j, β^B_j) and (α^C_j, β^C_j), $j \in \mathbb{N}_0$, respectively.

Instead, the de Finetti's measure of $\{(X_n, Y_n), n \geq 1\}$, that is the distribution of the bivariate distribution function F_{XY} is not explicitly known. Nevertheless, it is possible to make some further consideration on it.

First of all, if F_X and F_Y are the two marginal distributions of X_n and Y_n, given F_A, F_B and F_C, then (4.2.1) implies

$$\begin{aligned} F_X &= F_A * F_B \\ F_Y &= F_A * F_C. \end{aligned} \tag{4.2.9}$$

Therefore, each of the two marginals is a convolution of two beta-Stacy processes. In terms of probability functions, we can write

$$P_{XY}(x,y) = \sum_{a=0}^{x \wedge y} P_A(a) P_B(x-a) P_C(y-a) \quad \forall\, (x,y) \in \mathbb{N}_0^2 \tag{4.2.10}$$

where P is the probability function corresponding to the distribution F. For the marginals we get the usual expression for the convolution.

Moreover, given F_{XY}, F_A, F_B and F_C, if $\sigma_A^2 = \mathrm{Var}_{F_A}(A)$, the dependence between X and Y can be described by the covariance

$$\mathrm{Cov}_{F_{XY}}(X,Y) = \sigma_A^2. \tag{4.2.11}$$

Thus, assuming for coupled lifetimes data, a model represented by the bivariate process built on the basis of generalized Pólya urn scheme and the equations (4.2.1) is equivalent, in a Bayesian point of view, to defining a probability measure on the space of the bivariate distribution functions on $\mathbb{N}_0^2$. Let us indicate by Π_2 such a probability measure.

4.2.2 The support of Π_2

Before exploring the properties of the support of Π_2, it is worth to recall explicitly some general definition about the topology on spaces of probability measures.

Given $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$ a measurable space, where $\mathcal{X}$ is a Polish space and $\mathcal{B}(\mathcal{X})$ its Borel σ-algebra, let $\mathcal{M}$ denote the space of all probability measures on $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$ and $\mathcal{B}(\mathcal{M})$ a suitable Borel σ-algebra.

If, for a random element P of $(\mathcal{M}, \mathcal{B}(\mathcal{M}))$, Π is the prior and $\Pi(\cdot\,|\mathbf{X}_n)$ the posterior, given a vector of observations $\mathbf{X}_n = (X_1, \ldots, X_n) \in \mathcal{X}^n$ independent and identically distributed according to P, the definition of a suitable topology on $(\mathcal{M}, \mathcal{B}(\mathcal{M}))$ allows us to deal with some interesting problems.

First of all, we can point out which probability measures belong to the support of Π and, secondly, we can describe the asymptotic behaviour of the posterior, when the number of observations grows. This is the meaning of the following definition, presenting one of the possibilities.

Definition 4.2.1. A subset U of $\mathcal{M}$ is said to be a weak neighborhood of P_0 if it contains a set of the form $\{P : |P(A_i) - P_0(A_i)| < \varepsilon_i,\, i = 1, \ldots, k\}$ where A_i are P_0-continuous sets and $\varepsilon_i > 0$.

The sequence $\{\Pi\left(\cdot\,|\mathbf{X}_n\right), n \geq 1\}$ is said to be weak consistent at P_0, if for every weak neighborhood U of P_0,

$$\Pi\left(U|\mathbf{X}_n\right) \to 1,\ n \to +\infty \quad a.s. - P_0. \tag{4.2.12}$$

So this corresponds intuitively to say that, when the number of observations grows, the posterior approaches a measure degenerate in P_0.

Definition 4.2.2. Let $L(\mu)$ the set of all densities with respect to a σ-finite measure μ.

For $f, f_0 \in L(\mu)$ the Kullback-Leibler divergence of f form f_0 is defined

$$K(f, f_0) = \int f_0 \log \frac{f_0}{f} d\mu. \qquad (4.2.13)$$

Moreover, for $\varepsilon > 0$,

$$\{f : K(f_0, f) < \varepsilon\} \qquad (4.2.14)$$

is called Kullback-Leibler neighborhood of f_0 with radius ε.

The Kullback-Leibler divergence is not a formal distance; nevertheless, the definition of neighborhood makes possible to define Kullback-Leibler consistency, but we shall not deal with it. More important for our aim is a corollary of a classical and well-known result of Schwartz (1965) providing a sufficient condition for the weak consistency.

Proposition 4.2.2. [Schwartz (1965)]
Given $f_0 \in L(\mu)$, if $\forall \varepsilon > 0$,

$$\Pi(f : K(f_0, f) < \varepsilon) > 0, \qquad (4.2.15)$$

then the posterior is weakly consistent at f_0.

Exploiting the properties of the beta-Stacy process, we can obtain some knowledge about the width of the support of Π_2 with respect to the weak topology. We denote $P_{XY}(i,j) = p_{ij}, \forall (i,j) \in \mathbb{N}_0^2$. Moreover, for a distribution (or more generally for a measure) μ, let $\mathcal{S}_\mu$ indicate its support.
Define $\mathcal{P}_{c2}$ the set of the probability distributions on $\mathbb{N}_0^2$ such that there exist three probability distributions on $\mathbb{N}_0$ such that (4.2.10) is satisfied.

The above proposition transfers a well-known result for the Dirichlet process, easily proved also for discrete beta-Stacy, to $\mathcal{P}_{c2}$, subset of bivariate probability distributions.

Proposition 4.2.3. If Π_2 is the prior on the space of the bivariate distribution on $\mathbb{N}_0^2$ determined by the parameters $\alpha_j^A, \beta_j^A, \alpha_j^B, \beta_j^B, \alpha_j^C, \beta_j^C > 0$, $\forall j \in \mathbb{N}_0$, then

$$\mathcal{S}_{\Pi_2} \supseteq \mathcal{P}_{c2}. \tag{4.2.16}$$

Proof. As $\mathbb{N}_0^2$ is discrete we only need to show

$$\Pi_2\left[\left|p_{x_iy_i} - p^o_{x_iy_i}\right| < \varepsilon,\ i = 1, \ldots, k\right] > 0 \tag{4.2.17}$$

for each $p^o \in \mathcal{P}_{c2}$, $\forall \varepsilon > 0$, $k \geq 1$, $\forall (x_i, y_i), \ldots (x_k, y_k) \in \mathbb{N}_0^2$.

Now, $\forall i$ $(m_i = x_i \wedge y_i)$

$$\begin{aligned}
\left|p_{x_iy_i} - p^o_{x_iy_i}\right| &= \left|\sum_{a=0}^{m_i}\left(p_a^A p_{x_i-a}^B p_{y_i-a}^C - p_a^{A,o} p_{x_i-a}^{B,o} p_{y_i-a}^{C,o}\right)\right| &(4.2.18)\\
&\leq \sum_{a=0}^{m_i}\left|p_a^A p_{x_i-a}^B p_{y_i-a}^C - p_a^{A,o} p_{x_i-a}^B p_{y_i-a}^C + p_a^{A,o} p_{x_i-a}^B p_{y_i-a}^C\right.\\
&\qquad \left. - p_a^{A,o} p_{x_i-a}^{B,o} p_{y_i-a}^C + p_a^{A,o} p_{x_i-a}^{B,o} p_{y_i-a}^C - p_a^{A,o} p_{x_i-a}^{B,o} p_{y_i-a}^{C,o}\right| &(4.2.19)\\
&\leq \sum_{a=0}^{m_i}\left|p_a^A - p_a^{A,o}\right| + \sum_{a=0}^{m_i}\left|p_{x_i-a}^B - p_{x_i-a}^{B,o}\right| + \sum_{a=0}^{m_i}\left|p_{y_i-a}^C - p_{y_i-a}^{C,o}\right|\\
&= \sum_{a=0}^{m_i}\left|p_a^A - p_a^{A,o}\right| + \sum_{j=x_i-m_i}^{x_i}\left|p_j^B - p_j^{B,o}\right| + \sum_{j=y_i-m_i}^{y_i}\left|p_j^C - p_j^{C,o}\right| \quad a.s. - \Pi_2 &(4.2.20)
\end{aligned}$$

as $p_i^{A,o}, p_i^{B,o}, p_i^{C,o} \in [0,1]$ and $p_i^A, p_i^B, p_i^C \in [0,1]$ a.s. Π_2, $\forall i$.

So that, since the sequences $\{p_i^A,\ i \in \mathbb{N}_0\}$, $\{p_i^B,\ i \in \mathbb{N}_0\}$ and $\{p_i^C,\ i \in \mathbb{N}_0\}$ are, by construction, independent beta-Stacy processes on $\mathbb{N}_0$

$$\Pi_2\left[\left|p_{x_iy_i}-p^o_{x_iy_i}\right|<\varepsilon,\, i=1,\ldots,k\right]\geq \Pi_2\left[\sum_{j=0}^{m_i}\left|p_j^A-p_j^{A,o}\right|<\frac{\varepsilon}{3},\, i=1,\ldots,k\right]\cdot$$
$$\Pi_2\left[\sum_{j=0}^{x_i}\left|p_j^B-p_j^{B,o}\right|<\frac{\varepsilon}{3},\, i=1,\ldots,k\right]$$
$$\Pi_2\left[\sum_{j=0}^{y_i}\left|p_j^C-p_j^{C,o}\right|<\frac{\varepsilon}{3},\, i=1,\ldots,k\right]. \tag{4.2.21}$$

Consider, for instance, the first element of the product. (But it is analogous for the others).

Let $M=\bigvee_{i=1}^k m_i$, if

$$\left|p_j^A-p_j^{A,o}\right|<\frac{\varepsilon}{3(M+1)}=\varepsilon_M \quad j=0,1,\ldots,M \tag{4.2.22}$$

then

$$\sum_{j=0}^{m_i}\left|p_j^A-p_j^{A,o}\right|<\frac{\varepsilon}{3},\, i=1,\ldots,k \tag{4.2.23}$$

so that

$$\Pi_2\left[\sum_{j=0}^{m_i}\left|p_j^A-p_j^{A,o}\right|<\frac{\varepsilon}{3},\, i=1,\ldots,k\right]\geq\Pi_2\left[\left|p_j^A-p_j^{A,o}\right|<\varepsilon_M, j=0,1,\ldots,M\right]. \tag{4.2.24}$$

Since $(p_0^A,\ldots,p_M^A)$ has generalized Dirichlet distribution with parameters $\alpha_j^A,\beta_j^A>0$, $j=0,1,\ldots,M$ (see Walker and Muliere (1997)), for the full support of this distribution, the result in Proposition 3 of Ferguson (1973) can be generalized to discrete beta-Stacy process, and the last probability in (4.2.24) is strictly positive. □

Remark 4.2.1. As it will be pointed out in the next section, it is possible to center Π_2 with respect to three distributions on $\mathbb{N}_0$, Q_A, Q_B and Q_C; in that way, some of the parameters could be null, and instead of (4.2.16) we have

$$\begin{aligned} \mathcal{S}_{\Pi_2} \supseteq & \{P_{XY} \text{ s.t. } \exists P_A, P_B, P_c \text{ satisfying (4.2.10) and} \\ & \mathcal{S}_{P_A} \subseteq \mathcal{S}_{Q_A}, \mathcal{S}_{P_B} \subseteq \mathcal{S}_{Q_B}, \mathcal{S}_{P_C} \subseteq \mathcal{S}_{Q_C}\} \end{aligned} \tag{4.2.25}$$

where the set on the right-hand side is just a subset of $\mathcal{P}_{c2}$.

A more detailed knowledge of the support of Π_2 is provided analysing the behaviour of the prior in the Kullback-Leibler neighborhoods.

As in the previous case, the best strategy is to extend to $\mathcal{P}_{c2}$ the properties of beta-Stacy process on $\mathbb{N}_0$.

Before we need to prove a lemma stating a useful result for the expected value of the logarithm of a variable with beta distribution.

Lemma 4.2.4. If $X \sim Beta\,(\alpha, \beta)$, $\alpha > 0, \beta > 0$, then

$$E\,[-\log X] = \frac{\beta}{\alpha}\left[\frac{1}{\alpha+\beta} + \alpha \sum_{n=0}^{\infty} \frac{1}{\alpha+n+1} \frac{1}{\alpha+\beta+n+1}\right] < +\infty. \tag{4.2.26}$$

Proof. Let us denote

$$B(x; \alpha, \beta) = \int_0^x t^{\alpha-1}(1-t)^{\beta-1} dt \tag{4.2.27}$$

the incomplete beta function and

$$B(\alpha, \beta) = B(1; \alpha, \beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \tag{4.2.28}$$

$$I(x; \alpha, \beta) = \frac{B(x; \alpha, \beta)}{B(\alpha, \beta)} \tag{4.2.29}$$

Now

$$\begin{aligned} E\,[-\log X] &= \int_0^1 -\log x \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)}\, dx \\ &= |-\log x\, I(x; \alpha, \beta)|_0^1 + \int_0^1 \frac{1}{x} I(x; \alpha, \beta)\, dx. \end{aligned} \tag{4.2.30}$$

The first term is 0 as

$$
\begin{aligned}
\lim_{x\to 0}\log x\, B(x;\alpha,\beta) &= \lim_{x\to 0}\frac{\log x}{B(x;\alpha,\beta)^{-1}}\\
&= \lim_{x\to 0}\frac{\frac{1}{x}}{-B(x;\alpha,\beta)^{-2}x^{\alpha-1}(1-x)^{\beta-1}} \qquad (4.2.31)\\
&= \lim_{x\to 0}-\frac{B(x;\alpha,\beta)^2}{x^{\alpha}}\\
&= \lim_{x\to 0}-\frac{2\,B(x;\alpha,\beta)x^{\alpha-1}(1-x)^{\beta-1}}{\alpha x^{\alpha-1}} = 0 \qquad (4.2.32)
\end{aligned}
$$

(4.2.31) and (4.2.32) by application of De l'Hôpital theorem.

So, considering the following expansion for $I(x;\alpha,\beta)$ (see Abramowitz and Stegun (1964))

$$
I(x;\alpha,\beta) = \frac{x^{\alpha}(1-x)^{\beta}}{\alpha B(\alpha,\beta)}\left[1+\sum_{n=0}^{\infty}\frac{B(\alpha+1,n+1)}{B(\alpha+\beta,n+1)}x^{n+1}\right], \qquad (4.2.33)
$$

the term in (4.2.30) reduces to

$$
\begin{aligned}
&\int_0^1 \frac{x^{\alpha-1}(1-x)^{\beta}}{\alpha B(\alpha,\beta)}\left[1+\sum_{n=0}^{\infty}\frac{B(\alpha+1,n+1)}{B(\alpha+\beta,n+1)}x^{n+1}\right]dx\\
&= \frac{1}{\alpha}\left[\frac{B(\alpha,\beta+1)}{B(\alpha,\beta)}+\sum_{n=0}^{\infty}\frac{B(\alpha+1,n+1)}{B(\alpha+\beta,n+1)B(\alpha,\beta)}\int_0^1 x^{\alpha+n}(1-x)^{\beta}dx\right]\\
&= \frac{1}{\alpha}\left[\frac{\beta}{\alpha+\beta}+\sum_{n=0}^{\infty}\frac{B(\alpha+1,n+1)B(\alpha+n+1,\beta+1)}{B(\alpha+\beta,n+1)B(\alpha,\beta)}\right]\\
&= \frac{1}{\alpha}\left[\frac{\beta}{\alpha+\beta}+\sum_{n=0}^{\infty}\frac{\alpha}{\alpha+n+1}\frac{\beta}{\alpha+\beta+n+1}\right]. \qquad (4.2.34)
\end{aligned}
$$

Since it is straightforward

$$
\sum_{n=0}^{\infty}\frac{1}{\alpha+n+1}\frac{1}{\alpha+\beta+n+1} < +\infty, \qquad (4.2.35)
$$

we get the thesis. □

The previous lemma helps to find constraints on the parameters of beta-Stacy process to assure that the Kullback-Leibler condition holds.

Proposition 4.2.5. Let Π the measure induced by a beta-Stacy process on $\mathbb{N}_0$ with parameters α_j, β_j, $j \in \mathbb{N}_0$, on the space of distribution function on $\mathbb{N}_0$ with $p^o = \{p_j^o, j \in \mathbb{N}_0\}$ being one of them.

If α_j, β_j, $j \in \mathbb{N}_0$ are such that

$$\sum_{j=0}^{+\infty} p_j^o \left[\frac{\beta_j}{\alpha_j(\alpha_j + \beta_j)} + \beta_j T(\alpha_j, \beta_j)\right] < +\infty \tag{4.2.36}$$

$$\sum_{j=0}^{+\infty} \overline{p_j^o} \left[\frac{\alpha_j}{\beta_j(\alpha_j + \beta_j)} + \alpha_j T(\beta_j, \alpha_j)\right] < +\infty \tag{4.2.37}$$

where $\overline{p_j^o} = \sum_{k>j} p_k^o$ and

$$T(\alpha, \beta) = \sum_{n=0}^{+\infty} \frac{1}{\alpha + n + 1} \frac{1}{\alpha + \beta + n + 1} < +\infty \qquad \alpha, \beta > 0, \tag{4.2.38}$$

then, $\forall \varepsilon > 0$,

$$\Pi\left[p : \sum_{j=0}^{+\infty} p_j^o \log \frac{p_j^o}{p_j} < \varepsilon\right] > 0. \tag{4.2.39}$$

Proof. As, under Π, for $j \in \mathbb{N}_0$

$$p_j = u_j \prod_{k=0}^{j-1} [1 - u_k] \tag{4.2.40}$$

where $u_k \sim Beta(\alpha_k, \beta_k)$ and independent, we can compute

$$\begin{aligned}
E\left[-\sum_{j=0}^{+\infty} p_j^o \log p_j\right] &= \sum_{j=0}^{+\infty} p_j^o E\left[-\log p_j\right] \\
&= \sum_{j=0}^{+\infty} p_j^o E\left[-\log u_j\right] + \sum_{j=0}^{+\infty}\sum_{k=0}^{j-1} p_j^o E\left[-\log\left(1 - u_k\right)\right] \\
&= \sum_{j=0}^{+\infty} p_j^o E\left[-\log u_j\right] + \sum_{k=0}^{+\infty} \overline{p_k^o} E\left[-\log\left(1 - u_k\right)\right] \\
&= \sum_{j=0}^{+\infty} p_j^o \left[\frac{\beta_j}{\alpha_j(\alpha_j + \beta_j)} + \beta_j T(\alpha_j, \beta_j)\right] + \sum_{j=0}^{+\infty} \overline{p_j^o} \left[\frac{\alpha_j}{\beta_j(\alpha_j + \beta_j)} + \alpha_j T(\beta_j, \alpha_j)\right]
\end{aligned}$$

by lemma 4.2.4.

The conditions (4.2.36) and (4.2.37) imply the finiteness of the expected value and also

$$-\sum_{j=0}^{+\infty} p_j^o \log p_j < +\infty \quad a.s. - \Pi, \tag{4.2.41}$$

hence $\forall \varepsilon > 0 \; \exists M$ s.t.

$$-\sum_{j>M} p_j^o \log p_j < \frac{\varepsilon}{2} \quad a.s. - \Pi.$$

Considering

$$\sum_{j=0}^{+\infty} p_j^o \log \frac{p_j^o}{p_j} = \sum_{j=0}^{+\infty} p_j^o \log p_j^o - \sum_{j=0}^{+\infty} p_j^o \log p_j < \varepsilon, \tag{4.2.42}$$

it is sufficient to enquire the event

$$\sum_{j=0}^{M} p_j^o \log p_j^o - \sum_{j=0}^{M} p_j^o \log p_j < \frac{\varepsilon}{2}, \tag{4.2.43}$$

given that $(p_0, \ldots, p_M)$, under Π, has a generalized Dirichlet distribution with parameters α_j, β_j, $j = 0, \ldots, M$.

Since this distribution has full support on the $M+1$-dimensional sub-simplex, and the event above is implied by $\left\{p_j > \frac{p_j^o}{\exp\frac{\varepsilon}{2}},\ j = 0, \ldots, M\right\}$, (4.2.43) has strictly positive probability. □

The previous proposition gives sufficient conditions indicating how to choose the parameters to have positive probability in each Kullback-Leibler neighborhood of p^o; straightforwardly, from Schwartz' result, we get the weak consistency at p^o.

An easy condition for consistency when p^o is unknown is, for instance,

$$0 < \underline{\alpha} \leq \alpha_j \leq \overline{\alpha},\ \forall j \in \mathbb{N}_0 \tag{4.2.44}$$

$$0 < \underline{\beta} \leq \beta_j \leq \overline{\beta},\ \forall j \in \mathbb{N}_0. \tag{4.2.45}$$

For the bivariate prior Π_2, the following holds.

Proposition 4.2.6. Let Π_2 indicate the prior on the space of bivariate distribution function on $\mathbb{N}_0^2$ with parameters $\alpha_j^A, \beta_j^A, \alpha_j^B, \beta_j^B, \alpha_j^C, \beta_j^C > 0$, $j \in \mathbb{N}_0$ and $p^o \in \mathcal{P}_{c2}$, with marginals $p_X^o = \{p_{i\cdot}^o, i \in \mathbb{N}_0\}$ and $p_Y^o = \{p_{\cdot j}^o, j \in \mathbb{N}_0\}$ where $p_{i\cdot}^o = \sum_{j=0}^{+\infty} p_{ij}^o$ and $p_{\cdot j}^o = \sum_{i=0}^{+\infty} p_{ij}^o$.

If α_j^B, β_j^B and α_j^C, β_j^C for $j \in \mathbb{N}_0$ satisfy conditions like (4.2.36) and (4.2.37) with respect to p_X^o and p_Y^o, then $\forall \varepsilon > 0$

$$\Pi_2 \left[p : \sum_{i=0}^{+\infty} \sum_{j=0}^{+\infty} p_{ij}^o \log \frac{p_{ij}^o}{\sum_{a=0}^{i \wedge j} p_a^A p_{i-a}^B p_{j-a}^C} < \varepsilon \right] > 0. \tag{4.2.46}$$

Proof. Let compute

$$E\left[\sum_{i=0}^{+\infty}\sum_{j=0}^{+\infty} -p_{ij}^o \log \sum_{a=0}^{i\wedge j} p_a^A p_{i-a}^B p_{j-a}^C\right] = \sum_{i=0}^{+\infty}\sum_{j=0}^{+\infty} p_{ij}^o E\left[-\log \sum_{a=0}^{i\wedge j} p_a^A p_{i-a}^B p_{j-a}^C\right]$$

$$\leq \sum_{i=0}^{+\infty}\sum_{j=0}^{+\infty} p_{ij}^o E\left[-\log p_0^A p_i^B p_j^C\right] \tag{4.2.47}$$

$$= \sum_{i=0}^{+\infty}\sum_{j=0}^{+\infty} p_{ij}^o E\left[-\log p_0^A\right] + \sum_{i=0}^{+\infty}\sum_{j=0}^{+\infty} p_{ij}^o E\left[-\log p_i^B\right] + \sum_{i=0}^{+\infty}\sum_{j=0}^{+\infty} p_{ij}^o E\left[-\log p_j^C\right]$$

$$= E\left[-\log p_0^A\right] + \sum_{i=0}^{+\infty} p_{i\cdot}^o E\left[-\log p_i^B\right] + \sum_{j=0}^{+\infty} p_{\cdot j}^o E\left[-\log p_j^C\right], \tag{4.2.48}$$

hence if α_j^B, β_j^B and α_j^C, β_j^C satisfy the above hypotheses this expected value is finite.

Thus

$$-\sum_{i=0}^{+\infty}\sum_{j=0}^{+\infty} p_{ij}^o \log \sum_{a=0}^{i\wedge j} p_a^A p_{i-a}^B p_{j-a}^C < +\infty \quad a.s. - \Pi_2 \tag{4.2.49}$$

and $\forall \varepsilon > 0\ \exists M$ s.t.

$$-\sum_{i>M}\sum_{j>M} p_{ij}^o \log \sum_{a=0}^{i\wedge j} p_a^A p_{i-a}^B p_{j-a}^C < \frac{\varepsilon}{2}. \tag{4.2.50}$$

So that to prove the thesis we can just show

$$\sum_{i=0}^{M}\sum_{j=0}^{M} p_{ij}^o \log p_{ij}^o - \sum_{i=0}^{M}\sum_{j=0}^{M} p_{ij}^o \log \sum_{a=0}^{i\wedge j} p_a^A p_{i-a}^B p_{j-a}^C < \frac{\varepsilon}{2} = \varepsilon' \tag{4.2.51}$$

has positive probability.

Now, as $p^o_{ij} = \sum_{a=0}^{i\wedge j} p_a^{A,o} p_{i-a}^{B,o} p_{j-a}^{C,o}$, $\forall i,j \in \mathbb{N}_0^2$ the event above is implied by

$$p_i^A > \frac{p_i^{A,o}}{\delta} \quad i = 0, \dots, M \tag{4.2.52}$$

$$p_i^B > \frac{p_i^{B,o}}{\delta} \quad i = 0, \dots, M \tag{4.2.53}$$

$$p_i^C > \frac{p_i^{C,o}}{\delta} \quad i = 0, \dots, M \tag{4.2.54}$$

where $\delta = \exp\left(\frac{1}{3}\varepsilon'\right) > 1$, but, under Π_2, $\left(p_0^A, \dots, p_M^A\right)$, $\left(p_0^B, \dots, p_M^B\right)$ and $\left(p_0^C, \dots, p_M^C\right)$ have independent generalized Dirichlet distribution with parameters α_j^A, β_j^A, α_j^B, β_j^B, $\alpha_j^C, \beta_j^C > 0$ $j = 0, \dots, M$, respectively.

The independent events in (4.2.52), (4.2.53) and (4.2.54) define subsets of the $M+1$-dimensional sub-simplex (the support of the above distribution), so that they have positive probability under Π_2 as well the (4.2.51). □

Therefore, provided (27) holds, the crucial conditions must be satisfied just for α_j^B, β_j^B and α_j^C, β_j^C with respect to the marginals. Putting these parameters as suggested, for instance, in (4.2.44) and (4.2.45), we obtain that $\Pi_2\left(\cdot \,|\mathbf{X}_n, \mathbf{Y}_n\right)$ is consistent at every $p^o \in \mathcal{P}_{c2}$.

The next section shows how to make inference about the bivariate survival function when Π_2 is the prior.

4.3 Estimation of a bivariate survival function

Though the lack of a well-determined form prevents the direct computation of the posterior, it is possible to obtain an estimate of the survival function.

More precisely, let

$$S(x,y) = P\left[X > x, Y > y\right] \qquad (4.3.1)$$

be a bivariate survival function on $\mathbb{N}_0^2$ and $(\mathbf{X}_n, \mathbf{Y}_n)$ be an independent and identically distributed sample from S where each of the two components is subject to independent right censoring, so that the data take form $(\mathbf{X}_n^*, \boldsymbol{\delta}_n, \mathbf{Y}_n^*, \boldsymbol{\xi}_n)$.
Assuming the above described Π_2 as prior and a quadratic loss function, as usual in a Bayesian framework, we have

$$\hat{S}(x,y) = P\left[X_{n+1} > x, Y_{n+1} > y \,|\, \mathbf{X}_n^*, \boldsymbol{\delta}_n, \mathbf{Y}_n^*, \boldsymbol{\xi}_n\right]. \qquad (4.3.2)$$

As the censorings are independent, we assume, referring to (4.2.1), that B_n and C_n can be censored, but not A_n. Therefore, the predictive distributions of B_n and C_n are analogous to (4.1.7), with the respective parameters.
Before any inference, an interpretation of the parameters of the model must be provided. This coincides, in the Bayesian perspective, with specifying the initial distribution centering it in accordance with some prior choice.
Recall that a random distribution function F, distributed according to a discrete beta-Stacy process, can be centered on a given discrete distribution G putting, $\forall j, c_j > 0$ and

$$\begin{aligned} \alpha_j &= c_j G(\{j\}) \\ \beta_j &= c_j \left[1 - \sum_{i=0}^{j} G(\{i\})\right], \end{aligned} \qquad (4.3.3)$$

so that

$$E\left[F(\{j\})\right] = G(\{j\}) \cdot \qquad (4.3.4)$$

(see Walker and Muliere (1997)).

Furthermore, if $c_j = c\ \forall j$, then $\alpha_j + \beta_j = \beta_{j-1}$ and the beta-Stacy reduces to a discrete Dirichlet process.

Hence, to determine the parameters of our model we do not need any complicated idea about the bivariate distribution of the lifetimes, but just some *a priori* guesses on the covariance between X and Y, $\mathrm{Cov}(X,Y)$, and on their marginal distributions. In fact, (4.2.9) and (4.2.11) suggest how to proceed:

1. choose an initial distribution for A, F_A^0 having the variance equal to the desired $\mathrm{Cov}(X,Y)$ and determine α_j^A, β_j^A;

2. given the prior guess F_X^0, F_Y^0 and F_A^0 solve (4.2.9) to obtain F_B^0 and F_C^0 and consequently compute $\alpha_j^B, \beta_j^B, \alpha_j^C, \beta_j^C$.

Note, in absence of further information about F_A, the choice of the distribution for A is free, except for the variance; hence, it could be convenient to choose a distribution making easier the procedure at point 2.

For particular values of the parameters it is possible to abandon the hypothesis of dependence between X and Y.

Indeed, taking

$$\begin{aligned} &\alpha_0^A > 0, \quad \alpha_j^A = 0\ \forall j \geq 1, \\ &\beta_j^A = 0\ \forall j \geq 0 \end{aligned} \tag{4.3.5}$$

we obtain $A_n = 0$, $X_n = B_n$, $Y_n = C_n$, $\forall n, a.s.$ and

$$\begin{aligned} &P\left[X_{n+1} > x, Y_{n+1} > y \,|\, \mathbf{X}_n^* = \mathbf{x}_n, \boldsymbol{\delta}_n = \mathbf{d}_n, \mathbf{Y}_n^* = \mathbf{y}_n, \boldsymbol{\xi}_n = \mathbf{e}_n\right] = \\ &\quad P\left[B_{n+1} > x \,|\, \mathbf{B}_n^* = \mathbf{x}_n, \boldsymbol{\delta}_n = \mathbf{d}_n\right] P\left[C_{n+1} > y \,|\, \mathbf{C}_n^* = \mathbf{y}_n, \boldsymbol{\xi}_n = \mathbf{e}_n\right] = \\ &\quad P\left[X_{n+1} > x \,|\, \mathbf{X}_n^* = \mathbf{x}_n, \boldsymbol{\delta}_n = \mathbf{d}_n\right] P\left[Y_{n+1} > y \,|\, \mathbf{Y}_n^* = \mathbf{y}_n, \boldsymbol{\xi}_n = \mathbf{e}_n\right]. \end{aligned} \tag{4.3.6}$$

This corresponds to assuming as prior for F_A a Dirichlet process centered in a distribution degenerate in 0.

If, in addition, $\alpha_j^A, \beta_j^A, \alpha_j^B, \beta_j^B \to 0 \ \forall j$, the result is the product of Kaplan-Meier estimators for X and Y.

When the parameters are fixed, the right hand side element of (4.3.2) can be directly expressed in terms of the predictives of A_n, B_n and C_n, as in the previous section. The practice computation may be, yet, cumbersome and laborious.

A Markov Chain Monte Carlo estimation procedure can be achieved without any difficulty by the following steps:

1. given the observations $(\mathbf{X}_n^*, \boldsymbol{\delta}_n, \mathbf{Y}_n^*, \boldsymbol{\xi}_n)$, $\mathbf{A}_n$ is generated via a Gibbs sampler.

   So the full conditional of A_n, $P_{A_n|\mathbf{A}_{n-1},\mathbf{X}_n^*,\boldsymbol{\delta}_n,\mathbf{Y}_n^*,\boldsymbol{\xi}_n}$, is

$$\begin{aligned} &P\left[A_n = a_n \,|\mathbf{A}_{n-1} = \mathbf{a}_{n-1}, \mathbf{X}_n^* = \mathbf{x}_n, \boldsymbol{\delta}_n = \mathbf{d}_n, \mathbf{Y}_n^* = \mathbf{y}_n, \boldsymbol{\xi}_n = \mathbf{e}_n\right] \\ &\propto P\left[B_n^* = x_n - a_n, \delta_n = d_n \left|\mathbf{B}_{n-1}^* = \mathbf{x}_{n-1} - \mathbf{a}_{n-1}, \boldsymbol{\delta}_{n-1} = \mathbf{d}_{n-1}\right.\right] \\ &\quad P\left[C_n^* = y_n - a_n, \xi_n = e_n \left|\mathbf{C}_{n-1}^* = \mathbf{y}_{n-1} - \mathbf{a}_{n-1}, \boldsymbol{\xi}_{n-1} = \mathbf{e}_{n-1}\right.\right] \\ &\quad P\left[A_n = a_n \,|\mathbf{A}_{n-1} = \mathbf{a}_{n-1}\right] \end{aligned} \tag{4.3.7}$$

where

$$P\left[B_n^* = b, \delta_n = d \,\middle|\, \mathbf{B}_{n-1}^* = \mathbf{b}_{n-1}, \boldsymbol{\delta}_{n-1} = \mathbf{d}_{n-1}\right] = \begin{cases} P\left[B_n \geq b \,\middle|\, \mathbf{B}_{n-1}^* = \mathbf{b}_{n-1}, \boldsymbol{\delta}_{n-1} = \mathbf{d}_{n-1}\right], & d = 0 \\ \\ P\left[B_n = b \,\middle|\, \mathbf{B}_{n-1}^* = \mathbf{b}_{n-1}, \boldsymbol{\delta}_{n-1} = \mathbf{d}_{n-1}\right], & d = 1 \end{cases} \quad (4.3.8)$$

and similarly for $P\left[C_n^* = y_n - a_n, \xi_n = d_n \,\middle|\, \mathbf{C}_{n-1}^* = \mathbf{c}_{n-1}, \boldsymbol{\xi}_{n-1} = \mathbf{e}_{n-1}\right]$.

On the other hand, for the exchangeability of $\{A_n,\ n \geq 1\}$, the other full conditionals $P_{A_i|\mathbf{A}_{-i},\mathbf{X}_n^*,\delta_n,\mathbf{Y}_n^*,\xi_n}$, $i = 1, \ldots, n-1$ have an analogous form. (Let $\mathbf{A}_{-i} = (A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_n)$).

$\mathbf{B}_n^* = \mathbf{X}_n^* - \mathbf{A}_n$ and $\mathbf{C}_n^* = \mathbf{Y}_n^* - \mathbf{A}_n$ are computed.

2. A_{n+1}, B_{n+1} and C_{n+1} are sampled by the predictive distributions $P_{A_{n+1}|\mathbf{A}_n}$, $P_{B_{n+1}|\mathbf{B}_n^*,\delta_n}$ and $P_{C_{n+1}|\mathbf{C}_n^*,\xi_n}$.

3. Finally, $X_{n+1} = A_{n+1} + B_{n+1}$ and $Y_{n+1} = A_{n+1} + C_{n+1}$.

This will be straightforward to implement in practice.

4.4 An example

This section presents an illustration involving a real data set. The following table presents the data (Woolson and Lachenbruch (1980) and Lin and Ying (1993)) consisting of survival times of two kinds of skin grafts on the same burn patient; more precisely, X is referred to closely matched grafts, while Y to poorly matched ones in relation to HL-A antigen system between donor and recipient.

Patient	1	2	3	4	5	6	7	8	9	10	11
Close match (X)	37	19	57^+	93	16	22	20	18	63	29	60^+
Poor match (Y)	29	13	15	26	11	17	26	21	43	15	40

Table 4.1: Days of survival of skin grafts on burn patients

First of all we try to reproduce the Kaplan-Meier ("empirical") estimator. Using well understood ideas connecting Bayes nonparametric estimators and empirical estimators, we allow F_A to be the distribution degenerate at 0. Hence, we take

$$\begin{aligned} \alpha_0^A &= 1000, \ \alpha_j = 0 \, \forall j \geq 1 \\ \beta_j^A &= 0 \, \forall j \geq 0, \end{aligned} \tag{4.4.1}$$

while $\alpha_j^B, \beta_j^B, \alpha_j^C, \beta_j^C$ are all taken close to 0. The marginal estimators are presented in Figure 4.1 and compare very well with the Kaplan-Meier estimates, which are provided with their standard associated confidence bounds. In Figure 4.1 (and also Figure 4.3) the circle line represents our estimates, the bold line represents the Kaplan-Meier estimate and the dotted line the confidence bounds. Figure 4.2 presents the joint estimator. We know in this case that the estimators are independent; that is we have lost any dependence between the pairs in order to obtain the empirical marginal estimates.

Here we present a "smooth" case when the prior choice embodies some knowledge about the dependence between X and Y. This will carry through, and be updated, into the posterior. We choose to center the priors on Poisson (Po) distributions, to exploit some of their properties of closure under the convolution operation. For illustrative purposes, we center F_A on the Po(10) distribution, which corresponds to $\text{Cov}(X, Y) = 10$ and center F_X and F_Y on Po(40) and Po(25) distributions, respectively. Solving the equations in (4.2.9), we obtain a Po(40) distribution for F_B and for F_C a Po(15) distribution. Finally, we put all the parameters related to the degree of belief c_j^A, c_j^B and c_j^C equal to $1, \forall j \in \mathbb{N}_0$.

The estimates are presented in Figures 4.3 and 4.4. As can be seen the marginal estimators are smoothed versions of the Kaplan-Meier estimators. The joint survival function is now also a smoothed version of Figure 4.2.

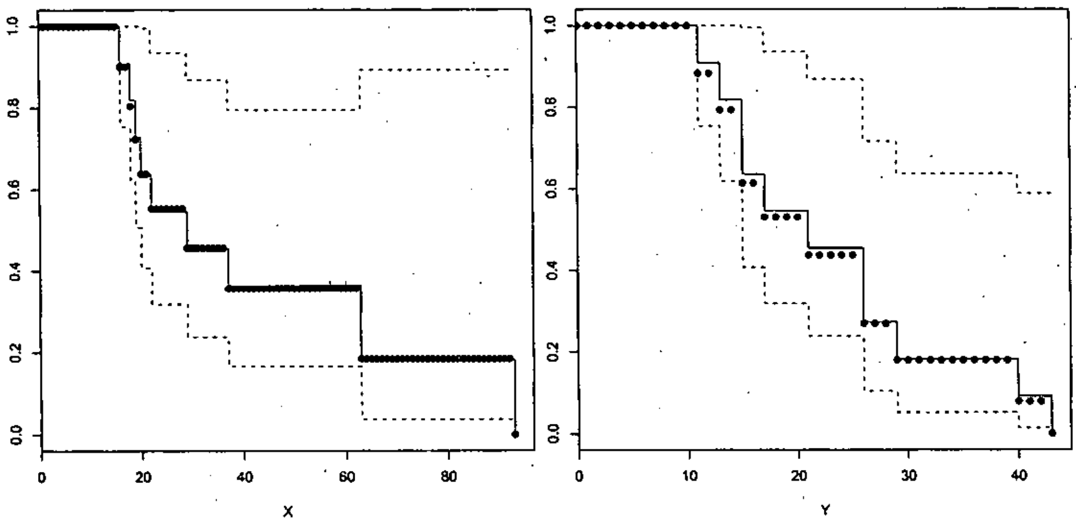

Figure 4.1: Empirical independent case, estimate of marginal survivals of X and Y.

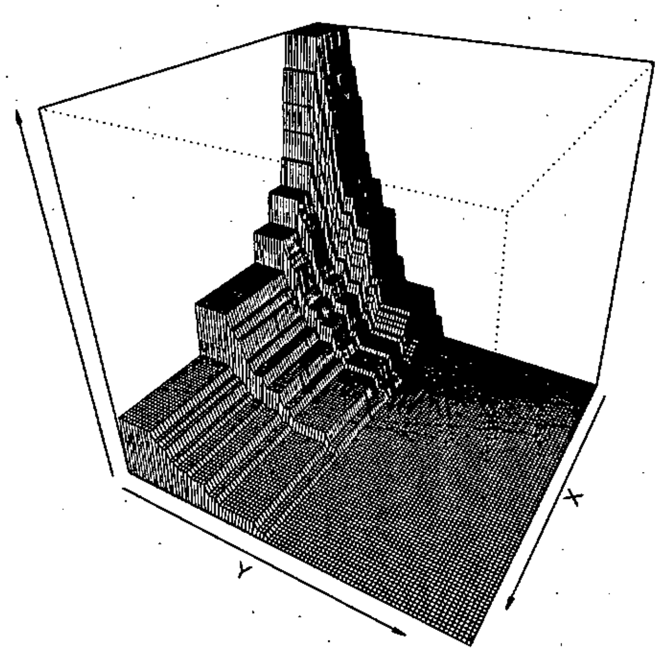

Figure 4.2: Empirical independent case, estimate of the bivariate survival function.

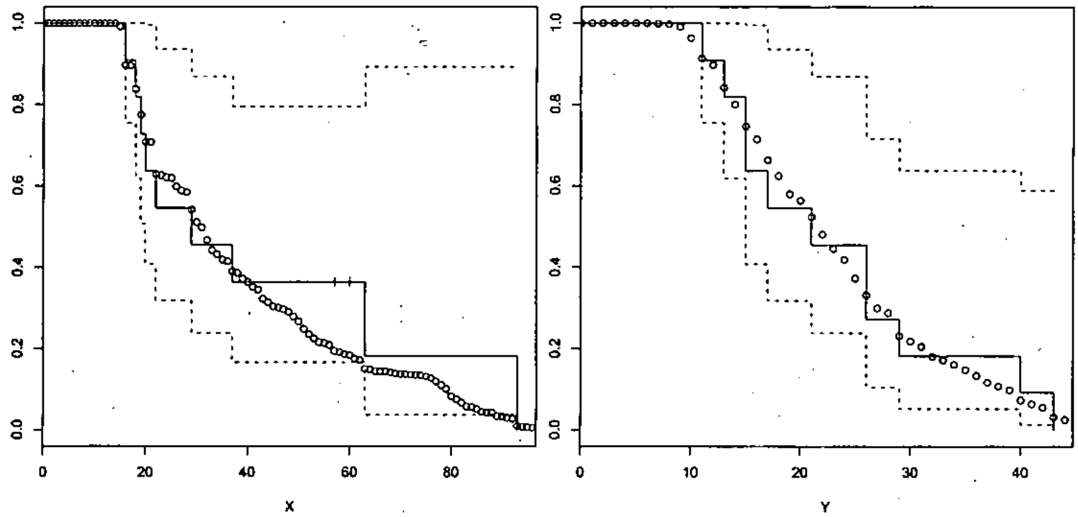

Figure 4.3: Smooth case, estimate of marginal survivals of X and Y

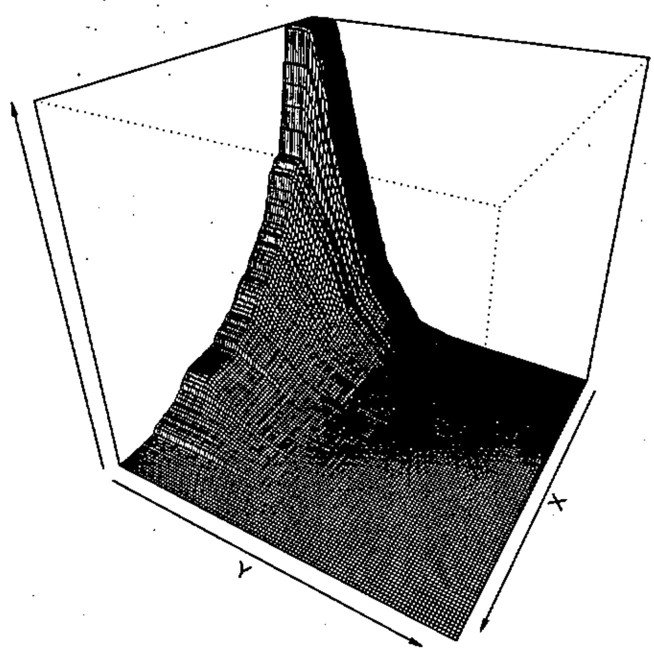

Figure 4.4: Smooth case, estimate of the bivariate survival function.

Conclusion

The interest of Bayesian nonparametric statistics in reinforced processes derived from urn schemes dates back to the work of Blackwell and MacQueen (1973).
Since this starting point a number of contributions has been given, especially in the construction of nonparametric priors (see, for instance, Muliere et al. (2000) and Cifarelli et al. (2000)). Indeed, the translation of a such complex mathematical object in terms of a concrete sequence of sampling and replacement of balls in an urn makes easier the research and the proof of its desirable properties.
In order to stress the large applicability of a such approach, two different problems of survival analysis have been presented: the inference for Markov renewal processes and the estimation of bivariate survival function.
To deal with the first problem we studied two types of reinforced processes in continuous time introduced by Muliere et al. (2003): the reinforced renewal and the Markov reinforced renewal processes.
Determining the conditional density of the $(n+1)$-th sojourn time given the previous n, we gave a new and computational proof of the property of exchangeability of these times and identified their de Finetti's measure with a beta-Stacy process.
The Markov renewal processes, too, enjoy of a property of partial exchangeability of array of the successor states and the sojourn times and, in a special case, we noticed

that its mixing measure consists in a so called *product Dirichlet times beta-Stacy*, that is a stochastic transition matrix, ruling the probability to reach a particular state given the present one, such that the rows are independent Dirichlet distributions and independent beta-Stacy processes for the cumulative distributions of the sojourn times in each state.

So, it is natural to propose this distribution as a prior when the data are supposed to come from exchangeable observations of a Markov renewal process. We showed that this distribution results to be conjugate and we computed the corresponding Bayes estimators.

Because of the well-known relation between Beta and beta-Stacy processes, our prior is strictly related with Phelan (1990a)'s one and the reinforced process in exam characterizes the last, as well.

Future works could be developed adopting an alternative perspective that we briefly sketched at the end of the Chapter 3. The key assumption in the construction of reinforced Markov renewal processes of resetting the measurement of the time spent at each change of state can be substituted supposing that the time is reset to 0 only when a given recurrent state is reached and assuming a consequent reinforcement rule. By this way, we proved the exchangeability of the blocks between successive visits to the same recurrent state, so it would be possible to make Bayesian predictive inference, for instance, for the lifetimes of individuals subjected to multi-state disease. Following this approach it would be worth to try to tackle other problems of survival analysis like interval censored data and truncated observations as in Frydman (1992), Frydman (1994) and Frydman (1995) and Lagakos et al. (1988), models whose deep structure is a two or three state process in continuous time.

Our contribution to the difficult problem of the estimation of a bivariate survival function relied on the definition of a bivariate reinforced process built on the basis of the generalized urn processes in such a way as to model couples of dependent lifetimes: each element is supposed to be the sum of a common component and of a specific one. At the same time, by the reinforcement rule a structure of dependence between the couples has been introduced so that the Bayesian learning becomes possible.

As usual, the reinforced process defined, via exchangeability and de Finetti's representation Theorem, a prior on the space of bivariate distributions on $\mathbb{N}_0$. This prior distribution has a structure not completely known and that makes complicated to compute the posterior, but we made some remarks on the form of the bivariate distributions on which it put positive mass.

Hence, a Bayesian estimator (the predictive distribution) has been obtained and, in order to compute it, a Gibbs sampler procedure has been implemented exploiting the predictive laws of the generalized urn processes.

On the other hand, the considerations on the form of the bivariate distributions made possible to study the support of the prior, profiting by and adapting for our case some properties of beta-Stacy process. A result similar to proposition 3 of Ferguson (1973) has been given; also some constraints on the parameters of the prior in order to satisfy the Kullback-Leibler conditions has been obtained. As consequence, for a well-known result of Schwartz (1965), our prior enjoys the property of weak consistency.

An alternative way to choose the value of the prior parameters based on the researcher guess has been illustrated via an example.

Once again it is worth underlining that, in the future works in the field of multi-state processes as well in the estimation of the bivariate survival function, the reinforced processes do not characterize one of the usual Bayesian nonparametric priors. So, the usefulness of this kind of approach do not reside only in making concrete priors to better understand their properties, but in an alternative method to define them in such a way as to overcome technical difficulties and to make possible inference at least by a predictive point of view. For such reasons, we reckon this framework widely applicable and, particularly, in survival analysis also in consideration of the symmetry between the reinforced processes in the Bayesian approach and the counting processes largely employed in the field of the classical statistics (Andersen et al. (1995)).

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