

# Multinomial Logit Processes and Preference Discovery: Inside and Outside the Black Box

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We provide two characterizations, one axiomatic and the other neuro-computational, of the dependence of choice probabilities on deadlines, within the widely used *softmax* representation

$$p_t(a, A) = \frac{e^{\frac{u(a)}{\lambda(t)} + \alpha(a)}}{\sum_{b \in A} e^{\frac{u(b)}{\lambda(t)} + \alpha(b)}}$$

where  $p_t(a, A)$  is the probability that alternative  $a$  is selected from the set  $A$  of feasible alternatives if  $t$  is the time available to decide,  $\lambda$  is a time-dependent noise parameter measuring the unit cost of information,  $u$  is a time-independent utility function, and  $\alpha$  is an alternative-specific bias that determines the initial choice probabilities (reflecting prior information and memory anchoring).

Our axiomatic analysis provides a behavioural foundation of softmax (also known as Multinomial Logit Model when  $\alpha$  is constant). Our neuro-computational derivation provides a biologically inspired algorithm that may explain the emergence of softmax in choice behaviour. Jointly, the two approaches provide a thorough understanding of softmaximization in terms of internal causes (neuro-physiological mechanisms) and external effects (testable implications).

*Key words:* Discrete choice analysis, Drift Diffusion Model, Heteroscedastic extreme value models, Luce model, Metropolis algorithm, Multinomial Logit Model, Quantal response equilibrium, Rational inattention

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## 1. INTRODUCTION

Human decisions are often made under pressing time limits that substantially affect choice. Think of a trader deciding among alternative investments in fast-moving financial markets, a triage nurse

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*The editor in charge of this paper was Adam Szeidl.*

screening patients in life-threatening conditions, a team player under pressure choosing an action in a split second, a civic official determining whether to evacuate an area because of an impending natural hazard.

### 1.1. *Choice under time constraint*

In all of these examples, the decision-maker is given an exogenously constrained deliberation time during which he can gather and process noisy information about alternatives, whose qualities are *ex ante* only imperfectly known. This binding constraint may prevent the decision-maker from fully learning the values of alternatives and select with certainty the best course of action. The stochastic nature of information acquisition translates into stochastic choice behaviour: when facing the same choice problem in different occasions, the decision-maker might gather different information and choose different alternatives. We study how the choice distribution over available alternatives varies as a function of deliberation time.

As an additional illustration, more amenable to laboratory analysis than the initial field examples, consider the choice of food. In many empirical studies of food decision problems, a specific product  $a$  is identified with the vector  $a = (a_1, a_2, \dots, a_n)$  of its attributes, which may include price, brand, origin, preparation method, nutrition facts, and so on. A few papers show that the relative importance of attributes for consumer choice depends on the time available to decide.<sup>1</sup> In a nutshell, when no time is available to choose, consumers resort to habit or make their decisions based on the immediately available information, say price and brand  $(a_1, a_2)$ , the so-called “package attributes.” Instead when information acquisition is possible, their choice is informed by all of the attributes  $(a_1, a_2, \dots, a_n)$  and thus also less overt “label attributes”—such as origin, preparation method, and nutrition facts—play a role. As to why the resulting choice is stochastic, observe that the sampling of information from the label and from memory is typically random (e.g. [Shadlen and Shohamy, 2016](#)), thus the chosen food items may differ in different choice episodes.

Our analysis will produce a *softmax* stochastic choice in which the probability of choosing alternative  $a$  from a menu  $A$  depends on two key parameters, the slope with respect to the utility of the option and the intercept. More precisely, the representation we will identify is:

$$p_t(a, A) = \frac{e^{\frac{u(a)}{\lambda(t)} + \alpha(a)}}{\sum_{b \in A} e^{\frac{u(b)}{\lambda(t)} + \alpha(b)}}. \quad (1)$$

Here,  $u(a)$  is the true, but *ex ante* unknown to the decision-maker and to the analyst, subjective value of alternative  $a$ ,  $\lambda(t)$  is the cost of processing one unit of information in  $t$  seconds, and  $\alpha(a)$  is the initial bias for alternative  $a$ . Bias captures the information that is immediately available and does not require processing. Such “prior information” is provided by two channels which anchor evaluation: memory and the immediate perception of the alternatives.<sup>2</sup> In our food choice example, attributes such as price and brand are evident, while origin, preparation, and nutrition facts are more difficult to detect and interpret. This suggests that  $\alpha$  might depend only on the immediate attributes that can be effortlessly appreciated,<sup>3</sup> while  $u$  reasonably depends on all of them.

1. See [Silayoi and Speece \(2004\)](#), [Grunert and Wills \(2007\)](#), and the more recent [Huseynov and Palma \(2021\)](#).

2. See [Bordalo, Gennaioli and Shleifer \(2020\)](#) and [Bordalo, Conlon, Gennaioli, Kwon and Shleifer \(2021\)](#) for a recent analysis of the anchoring role of memory.

3. With the attribute “brand” acting as a memory cue, see, e.g., [Warlop, Ratneshwar and van Osselaer \(2005\)](#).

The functional form (1) extends well-known representations. When the initial bias is absent—i.e., when  $\alpha$  is constant—formula (1) reduces to the *Multinomial Logit* specification. In general, under the natural assumption that the unit cost of information processing  $\lambda(t)$  decreases as more time is available to analyse evidence,<sup>4</sup> three scenarios emerge. First, the initial bias determines choice behaviour when there is no deliberation time

$$p_0(a, A) = \lim_{t \rightarrow 0} p_t(a, A) = \frac{e^{\alpha(a)}}{\sum_{b \in A} e^{\alpha(b)}}.$$

Second, at the opposite extreme, with no time pressure, only the best alternatives are selected

$$p_\infty(a, A) = \lim_{t \rightarrow \infty} p_t(a, A) > 0 \iff a \in \arg \max_A u$$

as prescribed by standard microeconomic theory. Third, under constrained but nonzero deliberation time  $t$ , an intermediate stochastic behaviour results. It gives, as  $t$  increases, a higher chance—in the sense of stochastic dominance—of choosing better alternatives: with more time to decide, more observations become available and better estimates of values are possible.

Matejka and McKay (2015) showed that softmax stochastic choice behaviour arises when a decision-maker optimally processes information about the unknown “state of nature”  $u$  under an entropic unit cost of information  $\lambda(t)$ , with  $\alpha$  depending on the decision-maker’s prior. Their study provides an important optimal information acquisition foundation for softmax behaviour. In this article, we study this behaviour from two different, yet complementary, external and internal viewpoints that integrate their analysis. Next, we outline them.

## 1.2. Outside the black box: psychometric axioms

We provide a framework for the external, behavioural, study of an analyst who observes the choices of a decision-maker and interprets them in the “as if” mode of revealed preference analysis. We show that softmax is captured by a few simple behavioural axioms, Theorems 2 and 7, and that its parameters can be elicited from behavioural data, Theorems 3 and 8. In order to achieve the softmax representation, the classic Luce’s axioms are obviously necessary, as it must be the case that

$$p_t(a, A) = \frac{e^{v_t(a)}}{\sum_{b \in A} e^{v_t(b)}}$$

for all deadlines  $t$ , menus  $A$ , and alternatives  $a$  in  $A$ . Yet, these axioms cannot be sufficient because formula (1) imposes the additional separable structure

$$v_t(a) = \frac{u(a)}{\lambda(t)} + \alpha(a) \quad (2)$$

on the Lucean  $v_t$ ’s. We thus need to introduce new axioms. They involve, for each deliberation time  $t$ , a standard *preference order*  $\succ_t$  that ranks alternatives  $a$  and  $b$ , as well as two binary relations  $\succ_t^{\square}$  and  $\succ_t^*$  that account, respectively, for the *intensity of preference* and for the *ease of comparison* across pairs of distinct alternatives  $(a, b)$  and  $(c, d)$ . All these relations are revealed by the choice probabilities under different deadlines  $t$ , and so they can be retrieved from observables. Our axioms connect choice behaviour at different  $t$ ’s by requiring the consistency of the relations

4. With  $\lambda(t) \rightarrow \infty$  as  $t \rightarrow 0$ , and  $\lambda(t) \rightarrow 0$  as  $t \rightarrow \infty$ .

( $\succ_t, \succ_t^{\flat}, \succ_t^*$ ). For instance, consistency of the intensity of preference  $\succ_t^{\flat}$  requires that, for any two deadlines  $t < s$ ,

$$(a, b) \succ_t^{\flat}(c, d) \iff (a, b) \succ_s^{\flat}(c, d)$$

for all distinct pairs  $(a, b)$  and  $(c, d)$ . In words, if deliberation for  $t$  seconds reveals stronger evidence in favour of the hypothesis “ $a$  is preferable to  $b$ ” over the hypothesis “ $c$  is preferable to  $d$ ”, the same happens after a longer deliberation time  $s$ . The consistency requirements for the preference order and for the ease of comparison take similar forms.

While the elicitation of these three relations is somehow standard in the absence of bias (Section 2.1.1), based on the classic analyses of [Debreu \(1958\)](#) and [Davidson and Marschak \(1959\)](#), the exercise has never been performed in the presence of bias. To carry it out, we have to introduce novel elicitation methods to reveal  $(\succ_t, \succ_t^{\flat}, \succ_t^*)$  from choice probabilities (Section 3.1.2).

The next issue that we study axiomatically is whether the overall performance in choice improves (or degrades) with the amount of time available to decide. In our representation (1), the question can be made precise, and amounts to the characterization of decreasing (or increasing) monotonicity of  $\lambda(t)$ . In the study of fast choice over food items, when  $t$  is small and the time constraint is binding, both the findings of [Milosavljevic, Malmaud, Huth, Koch and Rangel \(2010\)](#) and [Reutskaja, Nagel, Camerer and Rangel \(2011\)](#) suggest that the quality of choice (measured by comparing the actual choice with preferences elicited earlier among the items) becomes worse as time pressure increases.

### 1.3. *Inside the black box: neuro-computational process*

Besides the external analysis just described, we also pursue an internal, neuro-computational, approach that provides a causal analysis of the decision-maker choices through a biologically inspired algorithm that may explain softmax emergence in intelligent behaviour. Our algorithm has a sequential structure, motivated by the well-known limits of human working memory and supported by eye-tracking studies—from the seminal [Russo and Rosen \(1975\)](#) to the recent [Reutskaja, Nagel, Camerer and Rangel \(2011\)](#). Our neuro-computational model naturally links multialternative choice with the classic Drift Diffusion Model of binary choice (DDM, hereafter) proposed by [Ratcliff \(1978\)](#). Specifically, we show that a procedure consisting of:

1. Markovian exploration of the menu,
2. sequential comparisons of pairs of alternatives, each based on a DDM,
3. choice of the incumbent at the deadline,

results in a stationary distribution that approximates softmax choice probabilities, Theorems 5 and 13. When binary comparisons are performed according to the symmetric DDM, this distribution is Multinomial Logit with no initial bias (Section 2.2.1). In contrast, a bias appears when general DDMs are considered. Again, the biased case goes beyond the standard analysis (see [Bogacz, Brown, Moehlis, Holmes and Cohen, 2006](#)) and requires a novel, formal, understanding of how past information is encoded, behaviourally and neurally, in the DDM (Section 3.2.1). We also present physiologically calibrated simulations that support the biological plausibility of this neuro-computational foundation (Section 2.3).

### 1.4. *The joint achievement*

The two approaches just outlined provide, along with the optimality analysis of [Matejka and McKay \(2015\)](#), a complete perspective on softmaximization as a model of preference

discovery, in terms of internal causes (neuro-physiological mechanisms) and external effects (testable behavioural implications). In particular, our two approaches show that softmax:

1. is empirically testable and its parameters can be identified by behavioural data,
2. is plausible from a neural viewpoint.

The parameters  $u$ ,  $\lambda$ , and  $\alpha$  of the behavioural softmax process (1) that we axiomatize have neural counterparts in the neural softmax process that our algorithm generates. The integration of the inner and outer analyses allows us to empirically identify and cross-validate these, behavioural and neural, unobservable parameters. Thus, the two analyses complement each other conceptually, through a cause-effect nexus, as well as empirically (Section 2.3).

To ease the exposition, we first introduce in Section 2 the main concepts and ideas in the special case when the initial bias is absent, which corresponds to the Multinomial Logit Model (MNL) and the symmetric DDM. They are widely studied specifications of softmax (in economics) and of the DDM (in neuroscience), and so in this section, we can borrow concepts and techniques from the existing literature and go quicker to the point.

The analysis of the general case is, instead, significantly more demanding since it requires the introduction of new tools that allow us to disentangle the effects of past information and of evidence accumulation on observed choice behaviour. This analysis is developed in Section 3. As anticipated, in this section, new techniques to elicit the relations ( $>_t$ ,  $>_t^h$ ,  $>_t^*$ ) from choice probabilities are introduced and a novel connection between the softmax initial bias and the DDM starting point bias emerges.

We close this introduction by mentioning the relevance of our research for discrete choice analysis and by discussing the related literature.

### 1.5. Discrete choice analysis

A byproduct of our analysis is an axiomatic foundation of the heteroscedastic MNL, the workhorse of discrete choice analysis.<sup>5</sup> Indeed, (1) can be rewritten in terms of random utility (see [Luce and Suppes, 1965](#) and [McFadden, 1973](#)) as

$$p_t(a, A) = \Pr\{u(a) + \lambda(t)\epsilon(a) > u(b) + \lambda(t)\epsilon(b) \quad \text{for all } b \in A \setminus \{a\}\},$$

where  $\{\epsilon(a)\}_{a \in A}$  is a collection of independent errors with type I extreme value distribution, specific mean  $\alpha(a)$ , and common variance  $\pi^2/6$ . Here,  $p_t(a, A)$  describes the stochastic behaviour of a decision-maker who is trying to maximize  $u$  but, because of time pressure, makes mistakes in evaluating the various alternatives. The standard deviation of mistakes is proportional to  $\lambda(t)$  and their bias is captured by  $\alpha$ . In discrete choice analysis,  $t$  may be time or, more in general, an index describing the experimental conditions under which data have been collected (i.e. the different data sets available to the analyst).<sup>6</sup> Heteroscedasticity, i.e., the dependence of  $\lambda$  on  $t$  and the presence of  $\alpha$ , was introduced because, while the decision-makers' utility  $u$  is a stable trait to be learned, disturbances are affected by experimental conditions and alternative-specific biases.

5. See, e.g., the textbooks of [Louviere, Hensher and Swait \(2000\)](#) and [Train \(2009\)](#).

6. For example, the indexes  $t$  represent locations in [Train \(2009, pp. 24–25\)](#), while they distinguish between stated intentions and market choices in [Ben-Akiva and Morikawa \(1990\)](#). In the working paper version of this article, [Cerrea-Vioglio, Maccheroni, Marinacci and Rustichini \(2021a\)](#), we extend our axiomatic analysis to allow for completely general choice and index sets.

The present article permits to test for misspecification of the heteroscedastic MNL and provides simple techniques to directly identify its parameters from data. In return, the discrete choice analysis literature provides a number of methods to estimate the parameters of the softmax specification (1).<sup>7</sup>

### 1.6. *Related theory literature*

This article considers exogenous deliberation times, thus we focus our discussion on the literature dealing with this issue.<sup>8</sup> Our treatment is axiomatic; to the best of our knowledge, there is only one other axiomatic foundation of the softmax model based on choice frequencies, due to [Matejka and McKay \(2015\)](#). The main difference is that they assume that the analyst knows the state that determines the decision-maker's utility,<sup>9</sup> while we consider the general case in which the analyst may possibly ignore this state, or even the state space. Outside the laboratory, presuming such knowledge is a quite strong assumption.

As to the behavioural (external) analysis, in a Random Expected Utility perspective, [Lu \(2016\)](#) axiomatically captures preference learning through increasingly informative priors on the set of probabilistic beliefs of the decision-maker. [Fudenberg and Strzalecki \(2015\)](#) axiomatize a discounted adjusted logit model. Differently from our work, these papers study stochastic choice in a dynamic setting where choices made today can influence the possible choices available tomorrow, and consumption may occur in multiple periods. [Frick, Iijima and Strzalecki \(2017\)](#) characterize the general random utility extension. [Saito \(2017\)](#) obtains several characterizations of the Mixed Logit Model. Finally, [Baldassi, Cerreia-Vioglio, Maccheroni, Marinacci and Pirazzini \(2020\)](#) and [Fudenberg, Newey, Strack and Strzalecki \(2020\)](#) axiomatize the value-based DDM.

As to the algorithmic (internal) analysis, while we consider a sequence of binary comparisons with evidence accumulation in each comparison, the vast majority of the extensions of the DDM to choice tasks with  $|A| > 2$  alternatives considers simultaneous evidence accumulation for all the  $|A|$  alternatives in the menu. In most studies, the choice task is assumed to simultaneously activate  $|A|$  accumulators, each of which is primarily sensitive to one of the alternatives and integrates the evidence relative to that alternative. Choices are then based on absolute or relative evidence levels, with endogenous or exogenous stopping times. See, e.g., [Roe, Busemeyer and Townsend \(2001\)](#), [Anderson, Goeree and Holt \(2004\)](#), [McMillen and Holmes \(2006\)](#), [Bogacz, Usher, Zhang and McClelland \(2007\)](#), [Ditterich \(2010\)](#), and [Krajbich and Rangel \(2011\)](#). [Natenzon \(2019\)](#) also belongs to this family and proposes a Multinomial Bayesian Probit model to jointly accommodate similarity, attraction, and compromise effects in a preference learning perspective. According to Natenzon's model, when facing a menu of alternatives the decision-maker—who has *a priori* i.i.d. standard normally distributed beliefs on the possible utilities of alternatives—receives a random vector of jointly normally distributed signals that represents how much he is able to learn about the ranking of alternatives before making a choice (say within time  $t$ ). The decision-maker updates the prior according to Bayes' rule and chooses the option with the highest posterior mean utility. Natenzon's

7. The econometric study of this model, now textbook material, dates back to [Ben-Akiva and Morikawa \(1991\)](#), [Swait and Louviere \(1993\)](#), [Hensher and Bradley \(1993\)](#), and [Bhat \(1995\)](#).

8. Models where decision time is endogenously—say, optimally—chosen are the subject of active research and we refer readers to [Woodford \(2014\)](#), [Tajima, Drugowitsch and Pouget \(2016\)](#), [Steiner, Stewart and Matejka \(2017\)](#), [Fudenberg, Strack and Strzalecki \(2018\)](#), [Callaway, Rangel and Griffiths \(2019\)](#), [Tajima, Drugowitsch, Patel and Pouget \(2019\)](#), [Webb \(2019\)](#), and [Jang, Sharma and Drugowitsch \(2020\)](#) for updated perspectives.

9. Choice situations of this kind have been studied since [Saltzman and Garner \(1948\)](#) and [Kaufman, Lord, Reese and Volkman \(1949\)](#). More recent contributions are [Gabaix, Laibson, Moloche and Weinberg \(2006\)](#), [Caplin and Dean \(2014\)](#), [Dean and Neligh \(2019\)](#), and [Dewan and Neligh \(2020\)](#).



paper is arguably the closest to ours. Like us, Natenzon builds on a parameterized family of random choice rules and tackles the problem of accounting for prior information and ease of comparison.<sup>10</sup> He also clearly spells out the interpretation of random utility on which, in the unbiased case, our identification between the intensity of preference and intensity of evidence rests upon. At the same time, Natenzon provides testable implications but not a full characterization of his model, while we fully characterize ours. Moreover, he identifies the parameters of the model by assuming that the analyst observes choice from menus with phantom options, while we obtain identification by assuming that the analyst observes choice behaviour at different deadlines.

Reutskaja, Nagel, Camerer and Rangel (2011) propose three two-stage models in which subjects randomly search through the feasible set during an initial search phase and, when this phase is concluded, select the best item encountered during the search (up to some noise). Their approach involves a “quasi-exhaustive” search in that the presence of a deadline may terminate the search phase before all alternatives have been evaluated and introduces an error probability. Although different from these models, and from those of Krajbich and Rangel (2011), our model is consistent with some of their experimental findings about the menu-exploration process and shares the reliance on the classical choice theory approach in which multialternative choice proceeds through binary comparison and elimination.

Finally, to the best of our knowledge, in the economics literature the first to showcase the fact that value-based symmetric DDMs generate (binary) logit probabilities were Clithero (2018) and Webb (2019).

### 1.7. Organization of the article

Section 2 is dedicated to the unbiased case of multinomial logit processes. Section 3 studies the general case of softmax processes. Section 4 concludes. All proofs and additional simulations are relegated to the [Supplementary Appendices](#).

## 2. UNBIASED ANALYSIS: MULTINOMIAL LOGIT PROCESSES

As previously discussed, the MNL and the symmetric DDM both capture the case in which initial bias is absent, as Bogacz *et al.* (2006) and Matejka and McKay (2015) remark with essentially the same words from altogether different perspectives. With these models the psychometric complication of separating the effects of prior information and those of acquired evidence on observed choice is avoided, a simplification that permits a brisker exposition of the main results and ideas of our research and allows us to borrow more from the literature. For this reason, in this section, we focus on the MNL and on the symmetric DDM.

In so doing we neglect, however, the fundamental role of memory and initial bias (and their pitfalls) in decision-making with limited cognitive resources. In Section 3, we extend the analysis to general softmax processes to cope with these issues, and this will require us to tackle some novel, and nontrivial, conceptual and technical challenges.

### 2.1. Outside the black box: behavioural MNL

**2.1.1. Preamble: random choice rules.** Let  $\mathcal{A}$  be the collection of all non-empty finite subsets  $A$  of a universal space  $X$  of possible alternatives, called *menus* (or *choice sets* or *choice*

10. He also takes the similarity of alternatives into account, an important feature that we do not consider here.

problems). We assume throughout that  $X$  is a connected topological space,<sup>11</sup> and we denote by  $\Delta(X)$  the set of all probability measures on  $X$  with finite support.

A *random choice rule* is a function

$$\begin{aligned} p: \mathcal{A} &\rightarrow \Delta(X) \\ A &\mapsto p(\cdot, A) \end{aligned}$$

such that  $p(A, A) = 1$  for all  $A$  in  $\mathcal{A}$ . Given any alternative  $a$  in  $X$ , we interpret  $p(\{a\}, A)$ , simply denoted  $p(a, A)$ , as the probability that a decision-maker chooses  $a$  when the set of available alternatives is  $A$ . More generally, if  $B$  is a subset of  $X$ ,  $p(B, A)$  is the probability  $\sum_{b \in B} p(b, A)$  that the selected element lies in  $B$ . This probability can be viewed as the *frequency* with which an element in  $B$  is chosen. The requirement  $p(A, A) = 1$  simply means that unavailable alternatives cannot be chosen. Indeed, it is equivalent to  $p(x, A) = 0$  for all  $x$  that do not belong to  $A$ .

As usual, given any  $a$  and  $b$  in  $X$ , we indicate by

$$p(a, b) = p(a, \{a, b\}) \quad (3)$$

the probability with which  $a$  is chosen from the doubleton  $\{a, b\}$ . Luce (1959) proposes the most classical random choice model. Its main assumptions on  $p$  are:

**Positivity**  $p(a, b) > 0$  for all  $a, b \in X$ .

**Choice Axiom**  $p(a, A) = p(a, B)p(B, A)$  for all  $B \subseteq A$  in  $\mathcal{A}$  and all  $a \in B$ .

The Choice Axiom says that the probability of choosing an alternative  $a$  from menu  $A$  is that of first selecting  $B$  from  $A$  and then  $a$  from  $B$ . As observed by Luce, this amounts to require that  $\{p(\cdot, A) : A \in \mathcal{A}\}$  is a conditional probability system in the sense of Renyi (1955).

As well known, both axioms can be expressed in terms of odds. In particular, the Choice Axiom is equivalent to the odds independence condition  $p(a, b)/p(b, a) = p(a, A)/p(b, A)$ , for all menus  $A$  in  $\mathcal{A}$  and all alternatives  $a$  and  $b$  in  $A$ , that requires the odds for  $a$  against  $b$  to be independent of the other alternatives available in the menu.<sup>12</sup>

A convenient continuity assumption is often automatically satisfied in applications.

**Continuity** The function  $(a, b) \mapsto p(a, b)$  is continuous on the set of all pairs of distinct alternatives in  $X$ .

Next, we state Luce's classic representation theorem.

**Theorem 1 (Luce)** *The following conditions are equivalent for a random choice rule  $p: \mathcal{A} \rightarrow \Delta(X)$ :*

1.  $p$  satisfies Positivity, the Choice Axiom, and Continuity;
2. there exists a continuous  $v: X \rightarrow \mathbb{R}$  such that

$$p(a, A) = \frac{e^{v(a)}}{\sum_{b \in A} e^{v(b)}} \quad (4)$$

for all  $A \in \mathcal{A}$  and all  $a \in A$ .

Moreover,  $v$  is unique up to location (i.e. up to translation by a constant).

11. With at least three elements, e.g., a non-singleton convex subset of  $\mathbb{R}^n$ . In the working paper version of this article, Cerreia-Vioglio *et al.* (2021a), our analysis is extended to a general set  $X$ , even without a topology.

12. This odds independence condition is often called *independence from irrelevant alternatives*. Cerreia-Vioglio, Lindberg, Maccheroni, Marinacci and Rustichini (2021b) characterizes the random choice rules that satisfy this condition but not necessarily Positivity.



Under Luce’s axioms, the function  $v$  is a *psychometric utility*, that is, it represents the *intensity of preference relation*  $\triangleright$  revealed by  $p$ . Formally, the relation  $\triangleright$  is defined by

$$(a, b) \triangleright (c, d) \iff p(a, b) > p(c, d) \tag{5}$$

and the function  $v$  is such that

$$(a, b) \triangleright (c, d) \iff v(a) - v(b) > v(c) - v(d) \tag{6}$$

for all  $a \neq b$  and all  $c \neq d$  in  $X$ . Discussed by [Debreu \(1958, p. 440\)](#), [Luce \(1959, p. 39\)](#), and especially [Davidson and Marschak \(1959, p. 237\)](#), the interpretation of  $\triangleright$  as the revealed intensity of preference is based on the following classic psychophysical principle, traditionally attributed to Cattell and Fullerton.

**Discrimination principle** *Equally often noticed differences are equal on the sensation scale, unless always or never noticed.*

In a random utility perspective, the Lucean psychometric utility  $v(a)$  of alternative  $a$  is unknown and a noisy signal  $V(a)$  is received.<sup>13</sup> Since  $p(a, b) = \Pr\{V(a) \geq V(b)\}$ , it follows that

$$p(a, b) > p(c, d) \iff \Pr\{V(a) \geq V(b)\} > \Pr\{V(c) \geq V(d)\} \tag{7}$$

that is,  $(a, b) \triangleright (c, d)$  if and only if the probability of receiving a signal in favour of  $a$  over  $b$  is higher than that of receiving a signal in favour of  $c$  over  $d$ . Intensity of preference is thus best understood in terms of intensity of evidence. The notions of this subsection are classic—e.g., the equivalence (7) dates back to [Block and Marschak \(1960, p. 110\)](#) and [Luce and Suppes \(1965, p. 338\)](#). Yet, this relationship between intensity of preference and intensity of evidence is less known and, as it will be seen later in the article, is key for our analysis of preference discovery.

**2.1.2. Axiomatic MNL.** Let  $T \subseteq (0, \infty)$  be a—discrete or continuous—set of points in time.

**Definition 1.** A random choice process is a collection  $\{p_t\}_{t \in T}$  of random choice rules.

For each  $t$ , we interpret  $p_t(a, A)$  as the probability that a decision-maker chooses alternative  $a$  from menu  $A$  if  $t$  is the *deliberation time*, that is, the maximum amount of time he is (exogenously) given to decide.<sup>14</sup> A random choice process thus describes the decision-maker stochastic choice behaviour under different levels of time pressure.

If each  $p_t$  satisfies Positivity, the Choice Axiom, and Continuity, then for each  $t$  in  $T$  there exists a psychometric utility  $v_t : X \rightarrow \mathbb{R}$  such that

$$p_t(a, A) = \frac{e^{v_t(a)}}{\sum_{b \in A} e^{v_t(b)}} \tag{8}$$

13. As observed by [Natenzon \(2019\)](#), this interpretation of random utility in which the value of options is perceived with noise is the one originally proposed by random utility models in psychology.

14. Say, by an experimenter, a script, or a spouse (see [Agranov, Caplin and Tergiman, 2015](#), for a simple protocol that allows to observe these probabilities for human agents).

When this is the case, we say that  $\{p_t\}_{t \in T}$  satisfies the *Psychometric Luce Axioms*. From there, to attain the MNL representation

$$p_t(a, A) = \frac{e^{\frac{u(a)}{\lambda(t)}}}{\sum_{b \in A} e^{\frac{u(b)}{\lambda(t)}}}$$

we need to express all the functions  $v_t$  by means of a time-independent utility function  $u$  and an alternative-independent noise coefficient  $\lambda$  such that

$$v_t(a) = \frac{u(a)}{\lambda(t)}.$$

This is achieved through the next axiom which requires that, over deliberation times, there are no ordinal reversals in the preference intensities.

**Consistency** Given any  $s > t$  in  $T$ ,

$$(a, b) \triangleright_t (c, d) \iff (a, b) \triangleright_s (c, d) \quad (9)$$

for all  $a \neq b$  and  $c \neq d$  in  $X$ .

In words, if the amount of evidence in favour of the hypothesis “ $a$  is preferable to  $b$ ” is, after a given deliberation time, greater than that in favour of the hypothesis “ $c$  is preferable to  $d$ ”, the same happens after a longer deliberation time.

In terms of primitive observables, Consistency reads naturally: given any  $s > t$  in  $T$ ,

$$p_t(a, b) > p_t(c, d) \iff p_s(a, b) > p_s(c, d)$$

for all  $a \neq b$  and  $c \neq d$  in  $X$ . This is coherent with the idea that information depends on an underlying true value that does not change over time and is gradually discovered as time goes by. The next thought experiment illustrates such a concept.<sup>15</sup>

**Example 1.** Subjects are given  $t$  seconds to observe a QR code and take one of the following actions: choose  $b$  and receive a number of dollars equal to the number of black squares, choose  $w$  and receive a number of dollars equal to the number of white squares. Like in [Caplin and Dean \(2014, p. 59\)](#), they are also told that QR codes have been stochastically generated with an *a priori* equal chance of a black majority or a white majority. Two different decision problems are portrayed in [Figure 1](#).

Inspection and experiments—starting with the seminal [Thurstone \(1929\)](#) to more recent random dot motion studies on humans, rats, and pigeons—suggest that, irrespective of  $t$ ,  $p_t(b_1, w_1) > p_t(b_2, w_2)$  because the true, but unknown, frequency of black squares in Problem 1 is bigger than that in Problem 2. It is from this frequency that agents sample for  $t$  seconds before choosing  $b$  or  $w$  in each problem. ▲

15. Notice that Consistency admits a formulation which does not rely on the order structure of  $T$ : Given any  $s$  and  $t$  in  $T$ ,

$$(a, b) \triangleright_t (c, d) \implies (a, b) \triangleright_s (c, d) \quad (10)$$

for all  $a \neq b$  and  $c \neq d$  in  $X$ . This formulation allows to consider alternative interpretations of the index set  $T$ , and to extend our results accordingly.

Problem 1:  $\{b_1, w_1\}$ Problem 2:  $\{b_2, w_2\}$ 

FIGURE 1  
Two decision problems.

We are now ready to state our first representation theorem.

**Theorem 2.** For a random choice process  $\{p_t\}_{t \in T}$ , the following conditions are equivalent:

1.  $\{p_t\}$  satisfies the Psychometric Luce Axioms and Consistency;
2. there exist a continuous  $u: X \rightarrow \mathbb{R}$  and a  $\lambda: T \rightarrow (0, \infty)$  such that

$$p_t(a, A) = \frac{e^{\frac{u(a)}{\lambda(t)}}}{\sum_{b \in A} e^{\frac{u(b)}{\lambda(t)}}} \quad (\text{MNL})$$

for all  $A \in \mathcal{A}$ , all  $a \in A$ , and all  $t \in T$ .

In this case,  $u$  is a cardinally unique psychometric utility for  $p_t$  for all  $t \in T$ , and  $\lambda$  is unique given  $u$ , unless  $\{p_t\}$  is uniform.<sup>16</sup>

These processes are called *Multinomial Logit (MNL) processes*. The next result permits to elicit both the psychometric utility and the noise coefficient. Given any  $a$  and  $b$  in  $X$  and any  $t$  in  $T$ ,

$$r_t(a, b) = \frac{p_t(a, b)}{p_t(b, a)} \quad \text{and} \quad \ell_t(a, b) = \ln r_t(a, b)$$

denote the *odds* for  $a$  against  $b$ —that is, the ratio between the number of episodes in which  $a$  is chosen and the number of episodes in which  $b$  is— and the *log-odds*, respectively. The latter are analytically convenient because they are positive if and only if odds are favourable to  $a$ .

**Theorem 3.** A MNL process  $\{p_t\}$  is not uniform if and only if there exist  $\hat{a}, \hat{b} \in X$  and  $\hat{t} \in T$  such that  $p_{\hat{t}}(\hat{a}, \hat{b}) > p_{\hat{t}}(\hat{b}, \hat{a})$ . In this case, the functions  $\hat{u}: X \rightarrow \mathbb{R}$  and  $\hat{\lambda}: T \rightarrow (0, \infty)$  given by

$$\hat{u}(x) = \frac{\ell_{\hat{t}}(x, \hat{b})}{\ell_{\hat{t}}(\hat{a}, \hat{b})} \quad ; \quad \hat{\lambda}(t) = \frac{1}{\ell_t(\hat{a}, \hat{b})} \quad (11)$$

16. A random choice process  $\{p_t\}$  is *uniform* if  $p_t(a, A) = 1/|A|$  for all  $a \in A \in \mathcal{A}$  and all  $t \in T$ . In this case, formula (MNL) holds for any  $\lambda$  provided  $u$  is constant.

are well defined, with  $\hat{u}$  continuous and

$$p_t(a, A) = \frac{e^{\frac{\hat{u}(a)}{\hat{\lambda}(t)}}}{\sum_{b \in A} e^{\frac{\hat{u}(b)}{\hat{\lambda}(t)}}}$$

for all  $A \in \mathcal{A}$ , all  $a \in A$ , and all  $t \in T$ .

Together the last two results permit to test whether the MNL model is consistent with available choice data and to identify its parameters.

How can Consistency drive these results? The intuition is, in retrospect, simple. The Psychometric Luce Axioms guarantee that each  $v_t$  in (8) is a psychometric utility for the corresponding intensity of preference relation  $\triangleright_t$ . Consistency, in turn, ensures that  $\triangleright_t$  coincides with  $\triangleright_s$ . But, then,  $v_t$  has to be a positive affine transformation of  $v_s$  because of the cardinal uniqueness of psychometric utilities. This observation allows us to transform (8) into (MNL) and also ensures that the functions  $\hat{u}$  and  $\hat{\lambda}$  defined in (11) are “essentially independent” of the choice of  $\hat{a}$ ,  $\hat{b}$ , and  $\hat{t}$ .

This answer, however, rises a few more questions. Why does Consistency imply that alternatives are a priori homogeneous, with no initial bias? How should it be weakened to have  $\alpha$  appear in the general softmax formula (1)? When is  $\lambda$  monotone/continuous? What is the limit behaviour of  $\{p_t\}$  as deliberation time diverges and so loses its binding constraint nature?

We will address all of these questions in the next section on general softmax processes. Here, we close by characterizing the affinity of  $u$ , a case of relevance for both applications and experiments; for instance, when choice under risk is considered and  $X$  is a set of lotteries.<sup>17</sup> In the last lines of this subsection, we assume that  $X$  is a convex subset of a topological vector space.

**Stochastic Betweenness** *There exists  $t \in T$  such that*

$$p_t\left(a, \frac{1}{2}a + \frac{1}{2}b\right) = p_t\left(\frac{1}{2}a + \frac{1}{2}b, b\right)$$

for all  $a \neq b$  in  $X$ .

In words, this axiom says that “equal probability differences are equally often noticed.” It parallels its risk theory counterpart.

**Proposition 4.** *The following conditions are equivalent for a MNL process:*

1.  $\{p_t\}$  satisfies Stochastic Betweenness;
2. the psychometric utility  $u$  is affine.

## 2.2. Inside the black box: neuro-computational MNL

So far, we regarded the components  $p_t$  of a random choice process  $\{p_t\}$  as the output of a black box: our axioms characterize MNL processes but remain silent about what decision procedure may generate the corresponding choice probabilities. Here, we address this issue by combining

17. MNLs with affine utility functions are also used in discrete choice analysis, as well as in quantal response equilibrium (QRE) theory (Section 4.4), and in most multicriteria decision analysis’ applications. Recall that  $u: X \rightarrow \mathbb{R}$  is affine when  $u(\kappa x + (1 - \kappa)y) = \kappa u(x) + (1 - \kappa)u(y)$  for all  $\kappa$  in  $[0, 1]$  and all  $x, y$  in  $X$ .

DDM pairwise comparisons of alternatives and efficient Markovian search of the menu. Both assumptions, motivated by the well-known limits of working memory and by the seminal eye-tracking study of Russo and Rosen (1975), find support in recent theories about memory,<sup>18</sup> as well as in recent eye-tracking experiments on multialternative choice.<sup>19</sup>

Again, we consider a decision-maker who has to choose an optimal alternative from a finite subset  $A$  of  $X$  within an *exogenously controlled* deliberation time  $t$ . For extra clarity,  $t$  is the time given to the agent to think through a single decision episode involving choice problem  $A$ , so it captures the overall level of time pressure.

The procedure we present can be summarized as follows. The decision-maker's search:

1. starts from an arbitrary element  $b$  of the menu, the *incumbent*;
2. selects a candidate alternative  $a$ , the *proposal*;
3. compares them via the DDM, and makes the winner of the DDM comparison the new incumbent;
4. repeats Steps 1–3 until deliberation time  $t$  comes and the current incumbent is chosen.

This is a basic variation over the standard brute force comparison-and-elimination algorithm of classical optimization—termed *standard revision* in marketing. According to standard revision, multiple alternatives are compared in a pairwise fashion and one alternative is permanently eliminated after each binary comparison. Thus, after  $|A| - 1$  of these comparisons the incumbent solution is an optimal choice. The implicit assumption which this brute force procedure rests upon is that pairwise comparisons are instantaneous and exact. In our procedure, instead, the fact that comparisons are time consuming may lead to incomplete exploration of the menu, while the fact that comparisons may be erroneous makes it inadvisable to eliminate permanently an alternative that was judged inferior at a previous stage.

**2.2.1. Binary comparisons—the symmetric DDM.** According to the DDM, when two alternatives  $a$  and  $b$  are compared, an alternative is selected as soon as the net evidence in its favour reaches a posited decision threshold  $\beta$ . Such a threshold represents the amount of evidence that the decision-maker needs to decide and determines the speed-accuracy tradeoff: lower thresholds produce faster but less accurate responding, whereas higher thresholds produce more accurate but slower responses (see Bogacz *et al.*, 2006). In experiments, high time pressure is known to reduce thresholds to speed up decisions.<sup>20</sup> In our setting, this means that the DDM decision thresholds for binary comparisons depend on the (overall) time constraint  $t$  for choice, that is,  $\beta$  is a function  $\beta(t)$  of  $t$ .<sup>21</sup>

Specifically, the comparison of  $a$  and  $b$  is believed to activate two neuronal populations whose activities (firing rates) provide evidence for the two alternatives.<sup>22</sup> The mean activities  $v(a)$  and  $v(b)$  are interpreted as neural indexes of value of the alternatives. For this reason, we call  $v: X \rightarrow \mathbb{R}$  the *neural utility* of the decision-maker. The stochastic nature of firing rates is captured

18. See, e.g., Luck and Vogel (1997), Vogel and Machizawa (2004), and Shadlen and Shohamy (2016).

19. See, e.g., Krajbich and Rangel (2011) and Reutskaja *et al.* (2011).

20. See, e.g., Milosavljevic *et al.* (2010) and Karsilar, Simen, Papadakis and Balci (2014, especially p. 14).

21. A caveat:  $\beta(t)$  depends on the deadline  $t$ , but is constant given  $t$  within each binary comparison. Here we are not considering DDMs with “collapsing thresholds” like in Drugowitsch, Moreno-Bote, Churchland, Shadlen and Pouget (2012) and Fudenberg *et al.* (2018). More on this in the concluding remarks (Section 4.3).

22. See Bogacz *et al.* (2006) and Shadlen and Shohamy (2016) for the neuro-physiological and neuro-psychological analyses of this mechanism; and Roe *et al.* (2001), Krajbich, Armel and Rangel (2010), Milosavljevic *et al.* (2010), and Rangel and Clithero (2014) for applications of the DDM to the choice of consumption goods.

by the assumption that the actual activities of neuronal populations experience instantaneous and independent white noise fluctuations. Evidence accumulation in favour of  $a$  and  $b$  is thus represented by two uncorrelated Brownian motions with drift

$$V_a(\tau) = v(a)\tau + W_a(\tau) \quad \text{and} \quad V_b(\tau) = v(b)\tau + W_b(\tau)$$

Absence of bias corresponds to the fact that the normals  $W_a(\tau)$  and  $W_b(\tau)$  have zero mean (and variance  $\tau$ ). With these assumptions,<sup>23</sup> we have the *symmetric DDM* (*sDDM*), where:

- the *net evidence* in favour of  $a$  against  $b$  is given by the difference

$$Z_{a,b}(\tau) = [v(a) - v(b)]\tau + \sqrt{2}W(\tau) \quad \forall \tau \in (0, \infty) \quad (\text{DDM})$$

where  $W$  is the Wiener process  $(W_a - W_b)/\sqrt{2}$ ;

- comparison ends when  $Z_{a,b}(\tau)$  reaches either the threshold  $\beta(t)$  or  $-\beta(t)$ ; so the *response time* is the random variable

$$\text{RT}_t(a, b) = \min \{ \tau \in (0, \infty) : |Z_{a,b}(\tau)| = \beta(t) \}$$

- at which time, the alternative supported by evidence is selected; so, the *comparison outcome* is the random variable

$$\text{CO}_t(a, b) = \begin{cases} a & \text{if } Z_{a,b}(\text{RT}_t(a, b)) = \beta(t) \\ b & \text{if } Z_{a,b}(\text{RT}_t(a, b)) = -\beta(t) \end{cases}$$

The presence of noise in evidence accumulation is what makes DDM comparisons time consuming and subject to error. In this regard, notice that the sDDM is an optimal Bayesian test of the hypothesis “ $v(a) > v(b)$ ” that alternative  $a$  be preferable to alternative  $b$ .<sup>24</sup>

**2.2.2. Proposal mechanism—efficient Metropolis exploration.** The standard Markovian way to navigate a menu starts with a candidate solution  $b$  drawn from an initial distribution  $\mu$  in  $\Delta(A)$  and then, given an incumbent solution  $b$ , continues by considering an alternative candidate solution  $a \neq b$ , called proposal, with probability  $Q(a|b)$ . This exploration procedure was introduced by [Metropolis, Rosenbluth, Rosenbluth, Teller and Teller \(1953\)](#) to calculate statistical mechanics equilibria,<sup>25</sup> and adopted in the *Simulated Annealing* heuristic of [Kirkpatrick, Gelatt and Vecchi \(1983\)](#) to compute extrema in high-dimensional combinatorial optimization problems.

The only formal requirements of [Metropolis et al. \(1953\)](#) on the probability transition matrix  $Q$ , called *exploration matrix*, are symmetry and irreducibility. At the same time, starting with the seminal [Kirkpatrick et al. \(1983\)](#), the adequate selection of the exploration matrix  $Q$  has been the object of fine tuning for the efficiency of optimization algorithms. The natural exploration matrix for a “travelling salesman” might not be equally compelling to describe our “spouse sent to buy wine 10 minutes before the arrival of the in-laws.” In this case, time pressure compels towards

23. Which will be relaxed in Section 3.2, where more general DDM’s are considered.

24. See [Gold and Shadlen \(2002, 2007\)](#), [Bogacz et al. \(2006\)](#), [Shadlen and Shohamy \(2016\)](#), and especially [Fudenberg et al. \(2018, Section II.B\)](#).

25. See also [Barker \(1965\)](#).

comparisons of category “white or red?” rather than more sophisticated distinctions “Champagne or Prosecco?”.

The sDDM provides a specific suggestion for exploration under time pressure. As observed by Fudenberg *et al.* (2018), the Bayesian decision problem that the sDDM solves features correct expectations on the time that it will take to compare two alternatives, *before* engaging in evidence accumulation to try and find the superior one. Easy choices will produce fast and accurate responses, while difficult ones will be time consuming and poorly efficient, absorbing a lot of time in exchange for a small payoff difference and presenting high-error probabilities. These considerations suggest the adoption of an exploration matrix of the form

$$Q_t(a|b) \propto \frac{1}{\mathbb{E}[\text{RT}_t(a,b)]} \quad \forall a \neq b,$$

where the probability with which  $a$  is proposed if  $b$  is the incumbent is inversely proportional to the expected response time. This choice adapts exploration to time constraints by giving priority to fast and accurate comparisons.<sup>26</sup>

Finally, the absence of initial biases assumed in this section suggests a uniform initial distribution  $\mu$ .<sup>27</sup>

**2.2.3. The Metropolis-DDM algorithm.** We now combine sDDM pairwise comparisons and efficient exploration. The resulting procedure describes a decision-maker who, given time  $t$  to decide, first automatically adjusts his evidence threshold  $\beta(t)$  and then compares alternatives according to the sDDM. His search, starting with a random initial solution, is driven by speed and accuracy concerns, and it continues until time  $t$  is reached, at which point the incumbent solution is chosen.

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### Metropolis-DDM Algorithm

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**Input:** Given  $t > 0$ .

**Start:** Draw  $a_0$  from  $A$  according to  $\mu$  and

- set  $\tau_0 = 0$ ,
- set  $b_0 = a_0$ .

**Repeat:** Draw  $a_{n+1}$  from  $A$  according to  $Q_t(\cdot | b_n)$  and compare it to  $b_n$  via the sDDM, so:

- set  $\tau_{n+1} = \tau_n + \text{RT}_t(a_{n+1}, b_n)$ ,
- set  $b_{n+1} = \text{CO}_t(a_{n+1}, b_n)$ ,

**until**  $\tau_{n+1} > t$ .

**Stop:** Set  $b^* = b_n$ .

**Output:** Choose  $b^*$  from  $A$ .

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26. See Baldassi, Maccheroni, Marinacci and Pirazzini (2022) for a semi-Markov analysis of general exploration matrices.

27. Mathematically, we can use any initial distribution. Yet, we find this specification conceptually compelling.



Notice that  $Q_t$ ,  $RT_t$ , and  $CO_t$  all depend on both the neural utility  $v$  and the threshold level  $\beta(t)$ , which in turn depends on the deadline  $t$ . The Metropolis-DDM algorithm randomly produces a sequence

$$(b_0, a_1, \tau_1, b_1, \dots, b_n, a_{n+1}, \tau_{n+1}, b_{n+1}, \dots)$$

of incumbents  $b_n$ , proposals  $a_{n+1}$ , and elapsed times  $\tau_{n+1}$ , which is truncated by the stopping rule  $\tau_{n+1} > t$  at the chosen alternative  $b^* = b_n$ .

At each iteration of the “repeat-until” loop, the proposal  $a$  is accepted as the new incumbent with probability

$$\mathbb{P}_t(a, b) = \mathbb{P}[CO_t(a, b) = a],$$

while  $a$  is rejected and the old incumbent  $b$  is maintained with the complementary probability  $1 - \mathbb{P}_t(a, b)$ . Therefore, the resulting probability of selecting  $a$  as a new incumbent given old incumbent  $b$  is

$$M_t(a | b) = Q_t(a | b) \mathbb{P}_t(a, b) \quad \forall a \neq b \text{ in } A$$

This transition probability combines the stochasticity of the proposal mechanism and that of the acceptance/rejection rule. Specifically, if the algorithm stops at the  $(n+1)$ -th iteration, the resulting choice probabilities are given by the vector  $M_t^n \mu$  in  $\Delta(A)$ .<sup>28</sup>

**Theorem 5.** *Let  $v: X \rightarrow \mathbb{R}$  and  $\beta: T \rightarrow (0, \infty)$ . Given any  $A \in \mathcal{A}$  and any  $t \in T$ , the exploration matrix  $Q_t = Q_t(A)$  is irreducible and symmetric and the incumbents’ transition matrix  $M_t = M_t(A)$  is aperiodic, irreducible, and reversible, with stationary distribution*

$$m_t(a, A) = \frac{e^{\beta(t)v(a)}}{\sum_{b \in A} e^{\beta(t)v(b)}} \quad \forall a \in A. \quad (12)$$

*In particular,  $\lim_{n \rightarrow \infty} M_t^n(A) \mu = m_t(\cdot, A)$  for all  $A \in \mathcal{A}$  and all  $\mu \in \Delta(A)$ .*

What does this theorem say about the output of the Metropolis-DDM algorithm? Since the average duration of each iteration is bounded above by  $\beta^2(t)/2$ , then the average number of iterations is

$$\bar{n}_t \geq \frac{2t}{\beta^2(t)} - 1 \quad (13)$$

and the “average” output of the algorithm is

$$M_t^{\bar{n}_t} \mu. \quad (14)$$

The distance between this output and the stationary distribution (12) decreases exponentially in  $\bar{n}_t$ .<sup>29</sup> Therefore, if  $\beta^2(t)$  is small relative to  $t$ , then  $\bar{n}_t$  is large and Theorem 5 suggests that the choice probabilities produced by the Metropolis-DDM algorithm are essentially given by (12),<sup>30</sup>

28. Here,  $M_t = [M_t(a | b)]_{a, b \in A}$  is regarded as the probability transition matrix of a Markov chain with initial distribution  $\mu$ , and  $M_t^n$  as its  $n$ th power. Moreover, we write  $M_t(A)$  and  $Q_t(A)$  since these matrices depend on the menu  $A$  under consideration.

29. This follows from Doeblin Theorem. See, e.g., Stroock (2005, p. 28) for an explicit bound on the variation distance between  $M_t^{\bar{n}_t} \mu$  and  $m_t$ .

30. See Baldassi *et al.* (2022) for further elaboration.

which is a MNL distribution under the following identification

Inner		Outer	
Neural utility	$v$	Psychometric utility	$u$
Threshold	$\beta$	Inverse noise	$1/\lambda$

(15)

This important table connects our inside and outside analyses, as we discuss next.

### 2.3. Inside and outside the black box

Remarkably, the neural utility  $v$  and threshold parameter  $\beta$  appearing in the neuro-computational specification (12) of the MNL are the ones governing the sDDM pairwise comparisons. Thus, table (15) allows us to combine Theorems 2, 3, and 5 in order to *identify* and *cross-validate* the unobservable parameters of internal and external processes.

Specifically, if the multialternative choice data available to the analyst do not reject the hypotheses of Theorem 2, she can use Theorem 3 and identify the psychometric parameters  $u$  and  $\lambda$ . At the same time, she can obtain sDDM data from binary comparisons under different levels of time pressure and estimate the neural parameters  $v$  and  $\beta$ .<sup>31</sup> Theorem 5 and, again, Theorem 2 say that, if the Metropolis-DDM algorithm converges, then the neural utility  $v$  must be a cardinal transformation of the psychometric utility  $u$ , and the threshold  $\beta$  must be an inverse transformation of the noise parameter  $\lambda$ . In sum, inside and outside analyses cross-validate each other (see Section 2.3.2 below for the details of the identification and cross-validation procedure).

**2.3.1. Simulations.** The crucial hypothesis behind this cross-validation procedure is the numerical convergence of the Metropolis-DDM algorithm to its stationary distribution in the finite number of iterations allowed by the time limit  $t$ . For a large  $t$ , thanks to inequality (13) numerical convergence is guaranteed whenever  $\beta(t)$  is  $o(\sqrt{t})$ . But, often  $t$  is small. It is therefore important to understand whether numerical convergence takes place, with laboratory calibrated parameters  $v$  and  $\beta(t)$ , when  $t$  is in the order of a few seconds.

In the following simulations, we consider choice from two different menus of eight snacks  $A = \{a_0, a_1, \dots, a_7\}$  and  $B = \{b_0, b_1, \dots, b_7\}$ , under two different deadlines of  $t = 4$  and  $t = 12$  seconds (s) that induce high and low time pressure on binary choices. Our sDDM parameters  $v$ ,  $\beta(4)$ , and  $\beta(12)$  are derived from the estimates of Milosavljevic *et al.* (2010).

Menu  $A$  consists of alternatives that are equally spaced in utility, and no two alternatives are indifferent. Menu  $B$  instead consists of four pairs of indifferent alternatives that have the same utility.

For each menu, in each time pressure condition, we run the Metropolis-DDM algorithm ten thousand times. This procedure simulates 10,000 identical subjects who choose according to the Metropolis-DDM algorithm and delivers the empirical choice probabilities that an analyst would observe. These probabilities are plotted in orange below. Instead, in blue we plotted the stationary distribution of the algorithm. The indistinguishability of the orange and blue plots says that, even if the algorithm only performs a few iterations because of the imposed deadline  $t$ , numerical convergence to the stationary MNL distribution (12) is achieved. The average number of iterations—that is, of binary comparisons—performed in each simulation is also reported. Its small size makes the observed convergence results even more surprising.<sup>32</sup>

31. This has been done, e.g., by Milosavljevic *et al.* (2010).

32. As anticipated, and in line with Krajbich and Rangel (2011) and Reutsckaja *et al.* (2011), we use a uniform initial distribution  $\mu$  in the simulations below; but the observed convergence, a number of alternative simulations, and

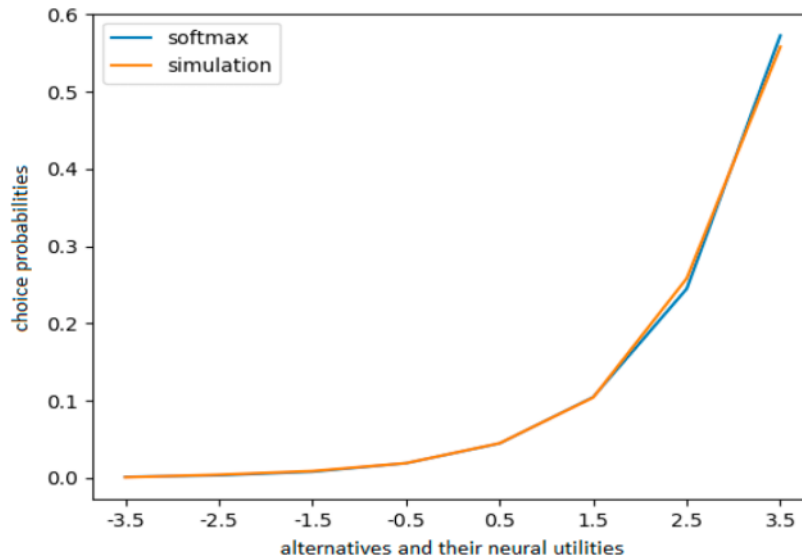


FIGURE 2  
Simulation 1.

#### **Simulation 1 (high time pressure, menu A, no indifferent alternatives)**

Choose in 4 s with  $v(a_i) = i - 3.5$  (for  $i = 0, \dots, 7$ ) and  $\beta(4) = 0.849$ . In Figure 2, the horizontal axis represents the alternatives. Specifically, the eight ticks on the horizontal axis correspond to the alternatives, with the number below each tick representing the neural utility of the corresponding alternative. The choice probability of each alternative is reported on the vertical axis: again, the orange plot describes the empirical choice probabilities obtained by simulating the Metropolis-DDM algorithm, the blue plot describes the theoretical MNL choice probabilities. The average number of binary comparisons in this simulation is 18.

#### **Simulation 2 (high time pressure, menu B, four pairs of indifferent alternatives)**

Choose in 4 s with  $v(b_i) = |i - 3.5|$  (for  $i = 0, \dots, 7$ ) and  $\beta(4) = 0.849$ , see Figure 3. The average number of binary comparisons in this simulation is 12.

#### **Simulation 3 (low time pressure, menu A, no indifferent alternatives)**

Choose in 12 s with  $v(a_i) = i - 3.5$  (for  $i = 0, \dots, 7$ ) and  $\beta(12) = 1.442$ , see Figure 4. The average number of binary comparisons in this simulation is 32.

#### **Simulation 4 (low time pressure, menu B, four pairs of indifferent alternatives)**

Choose in 12 s with  $v(b_i) = |i - 3.5|$  (for  $i = 0, \dots, 7$ ) and  $\beta(12) = 1.442$ , see Figure 5. The average number of binary comparisons in this simulation is 19.

Summing up, the Metropolis-DDM algorithm seems to numerically converge when calibrated with physiological data.<sup>33</sup> This substantiates the identification and cross-validation techniques connecting the axiomatic and neuro-computational approaches that we described at the beginning of this section, and that we further discuss below.

the Doeblin Theorem (on the exponential convergence of Markov chains) tell us that any other  $\mu$  would lead to similar results.

33. See also [Supplementary Appendix D](#) for some stress testing.

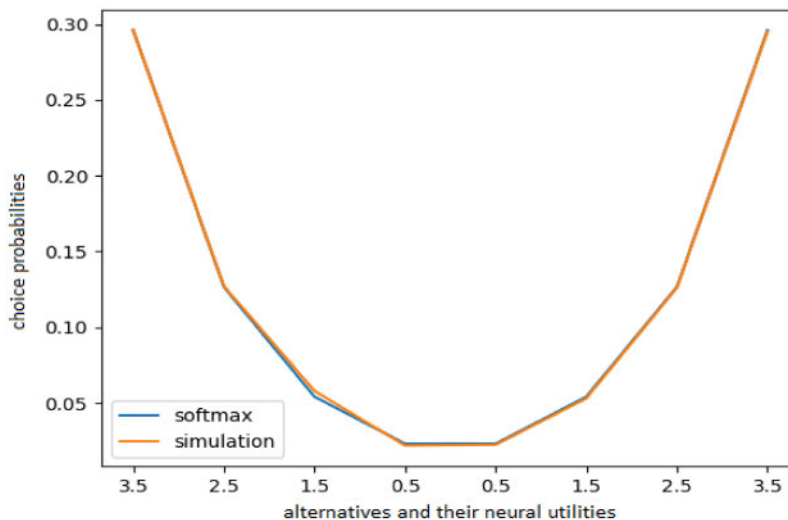


FIGURE 3  
Simulation 2.

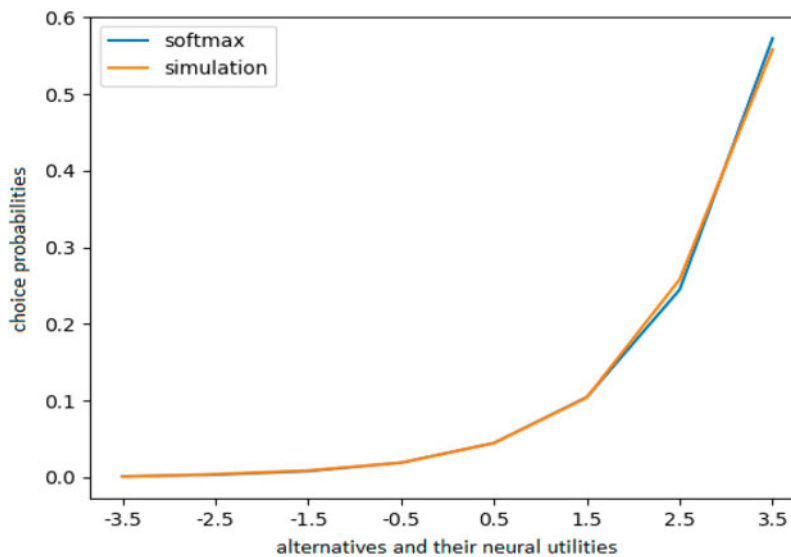


FIGURE 4  
Simulation 3.

**2.3.2. Taking stock.** Our axiomatic characterization of the MNL, Theorem 2, permits to test the Metropolis-DDM algorithm, which in turn, as shown by Theorem 5, approximates Matejka and McKay’s optimal information acquisition strategy in a biologically feasible way. Our identification result, Theorem 3, allows the analyst to discover the “unknown state” that the decision-maker is trying to determine, and reveals that this state simultaneously coincides with the psychometric utility (a behavioural measurement) and the neural utility (a physiological parameter). The same result also relates the unit cost of information  $\lambda(t)$  of the MNL (a behavioural measurement) with the evidence threshold  $\beta(t)$  of the DDM (a physiological parameter).

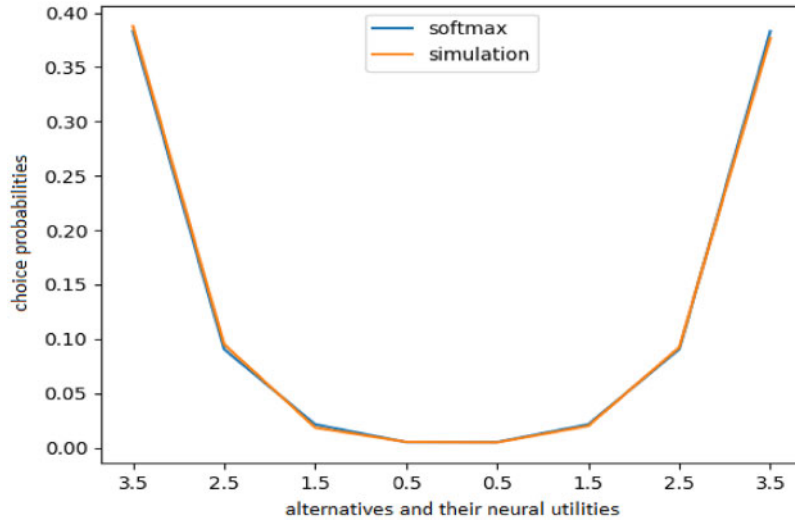


FIGURE 5  
Simulation 4.

Our inside–outside analysis thus provides a procedure enabling the analyst to combine behavioural and physiological measurements in studying human choices. We can summarize it as follows.

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### Identification and cross-validation procedure

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**Neural MNL hypothesis** For all  $A$  in  $\mathcal{A}$  and all  $t$  in  $T$ , the Metropolis-DDM approximates its MNL distribution

$$m_t(a, A) = \frac{e^{\beta(t)v(a)}}{\sum_{b \in A} e^{\beta(t)v(b)}} \quad \forall a \in A$$

with unknown neural components  $v$  and  $\beta$ .

**Behavioural data** The analyst observes a random choice process  $\{p_t\}$ , describing the frequencies of choice.

**Behavioural test** The analyst checks whether  $\{p_t\}$  satisfies the axioms of Theorem 2. If this is the case, the neural MNL hypothesis is not rejected, and she posits that  $m = p$ , that is,

$$\frac{e^{\frac{u(a)}{\lambda(t)}}}{\sum_{b \in A} e^{\frac{u(b)}{\lambda(t)}}} = p_t(a, A) = m_t(a, A) = \frac{e^{\beta(t)v(a)}}{\sum_{b \in A} e^{\beta(t)v(b)}} \quad \forall a \in A$$

for all  $A$  in  $\mathcal{A}$  and all  $t$  in  $T$ , with unknown behavioural components  $u$  and  $\lambda$ .

**Identification** If not uniform, the MNL choice process  $\{p_t\}$  reveals, by Theorem 3, to the analyst the values  $\hat{u}$  and  $\hat{\lambda}$  of the behavioural components of  $\{p_t\}$ . By the uniqueness properties of MNL

representations (Theorem 2), we have

$$v = u = \hat{u} \quad \text{and} \quad \beta = \frac{1}{\lambda} = \frac{1}{\hat{\lambda}}$$

up to cardinal transformations.<sup>34</sup> The neural and behavioural parameters are thus identified.

**Cross-validation** If available physiological data permit to identify the neural components  $v$  and  $\beta$ , the analyst can cross-validate the values previously obtained.

This procedure elucidates the interrelation between the inner and outer perspectives on stochastic choice studied in this article. Far from being disconnected, these two perspectives complement each other conceptually—by providing external (behavioural) verification and internal (causal) explanation of MNL stochastic choice—as well as empirically—by permitting to identify and cross-validate the components of MNL specifications.

### 3. GENERAL ANALYSIS: SOFTMAX PROCESSES

We now present a general analysis of softmax processes able to account for the role of memory and initial biases.

#### 3.1. *Outside the black box: behavioural softmax*

**3.1.1. Preamble: psychometric utilities.** The previous MNL analysis presents the advantages of going beyond the traditional ordinal framework—in which preferences only rank alternatives—by introducing a richer setting where preference intensities and their utility representations play a crucial role.

To extend the analysis, we consider three asymmetric and negatively transitive relations  $\succ$ ,  $\succ^{\natural}$ , and  $\succ^*$  on  $X$ .<sup>35</sup> The first,  $\succ$ , is a standard *preference order* that ranks alternatives, in the sense of Debreu (1954, 1964). The second,  $\succ^{\natural}$ , ranks pairs of alternatives in terms of *intensity of preference*, in the sense of Shapley (1975). The third,  $\succ^*$ , ranks binary choice problems in terms of *ease of comparison*, in the sense of Suppes and Winet (1955).

Formally, the relation  $\succ$  is defined on the set of alternatives  $X$  and ranks them

$$a \succ b$$

In words, “ $a$  is preferred to  $b$ .” The relation  $\succ^{\natural}$  is defined on the set of pairs of distinct alternatives  $X_{\neq}^2 = \{(a, b) : a \neq b \text{ in } X\}$  and ranks them

$$(a, b) \succ^{\natural} (c, d)$$

In words, “the strength of preference for  $a$  over  $b$  is higher than that for  $c$  over  $d$ .” Finally, the relation  $\succ^*$  is defined on the set of binary choice sets  $\mathcal{A}_2 = \{\{a, b\} : a \neq b \text{ in } X\}$  and ranks them

$$\{a, b\} \succ^* \{c, d\}$$

In words, “choosing between  $a$  and  $b$  is easier than choosing between  $c$  and  $d$ .”

34. That is, there exist  $k > 0$  and  $h \in \mathbb{R}$  such that  $v = ku + h$  and  $\beta = 1/k\lambda$ , and the same holds for  $\hat{u}$  and  $\hat{\lambda}$  (with possibly different constants  $\hat{k}$  and  $\hat{h}$ ).

35. Asymmetry and negative transitivity are the standard assumptions for a strict preference (see Definition 2.2 of Kreps, 1988).

Next, we introduce a joint numerical representation of these three binary relations that extends the traditional ordinal representation.

**Definition 2.** A function  $u: X \rightarrow \mathbb{R}$  is a psychometric utility for the triplet  $(\succ, \succ^{\square}, \succ^*)$  if, for each pair of alternatives  $a, b \in X$ ,

$$a \succ b \iff u(a) > u(b) \quad (16)$$

and if, for each quadruple of alternatives  $a \neq b$  and  $c \neq d$  in  $X$ ,

$$(a, b) \succ^{\square} (c, d) \iff u(a) - u(b) > u(c) - u(d) \quad (17)$$

as well as

$$\{a, b\} \succ^* \{c, d\} \iff |u(a) - u(b)| > |u(c) - u(d)| \quad (18)$$

A psychometric utility not only represents the preference order  $\succ$  in the standard fashion but also accounts for the intensity of preferences, quantified via utility differences, as well as for the ease of comparison, quantified via absolute values of utility differences.

Psychometric utilities are cardinal, as the next routine lemma shows.<sup>36</sup>

**Lemma 6.** *Continuous psychometric utilities are cardinally unique.*

The function  $v$  of Theorem 1 is, aptly, a psychometric utility for the triplet  $(\succ, \succ^{\square}, \succ^*)$  defined by

$$\begin{aligned} a \succ b &\iff p(a, b) > p(b, a) \\ (a, b) \succ^{\square} (c, d) &\iff p(a, b) > p(c, d) \\ \{a, b\} \succ^* \{c, d\} &\iff \text{ER}(a, b) < \text{ER}(c, d) \end{aligned} \quad (19)$$

for all quadruples of alternatives  $a \neq b$  and  $c \neq d$  in  $X$ . The error rate

$$\text{ER}(a, b) = \min\{p(a, b), p(b, a)\} = \begin{cases} p(a, b) & \text{if } b \succ a \\ p(b, a) & \text{if } a \succ b \\ 1/2 & \text{otherwise} \end{cases}$$

is the probability of choosing an alternative from  $\{a, b\}$  which is inferior under  $\succ$ . To check that  $v$  is a psychometric utility for this triplet  $(\succ, \succ^{\square}, \succ^*)$ , first observe that

$$(a, b) \succ^{\square} (c, d) \iff (a, b) \triangleright (c, d)$$

i.e.,  $\succ^{\square}$  reveals intensity of preference (and of evidence) as discussed in Section 2. Second,  $\succ$  reveals ordinal preference according to the standard (unbiased) stochastic notion

$$a \succ b \iff a \neq b \text{ and } p(a, b) > 1/2$$

36. Recall that  $X$  is a connected topological space.



which informed economics and psychology since the 1950s.<sup>37</sup> Finally, the idea that error rates measure difficulty of comparison, which dates back to Fechnerian psychophysics,<sup>38</sup> is often spelled out via the following classic psychophysical principle.

**Psychometric Principle** *Easier choice problems are more likely to elicit correct responses than harder ones.*<sup>39</sup>

We conclude that  $v$  is, indeed, a psychometric utility for the triplet  $(\succ, \succ^{\natural}, \succ^*)$  defined in (19).

**3.1.2. Measurement and revelations.** The MNL analysis of Section 2 showed that—in the absence of initial bias—stochastic choice behaviour under time pressure can be faithfully described by means of preference intensities and psychometric utilities. Our general softmax representation theorem, Theorem 7 below, will show that this continues to be the case also when initial bias is present. To take advantage of psychometric techniques, however, we will have to extract a triplet  $(\succ_t, \succ_t^{\natural}, \succ_t^*)$  from each component  $p_t$  of a general random choice process. But, how can an analyst detect and measure the preference order and its intensity, or ease of comparison, if the decision-maker’s behaviour is initially biased?

To address this question, we first augment the set  $T$  of strictly positive deliberation times to  $T_0 = T \cup \{0\}$  and, given any two distinct alternatives  $a$  and  $b$ , we interpret the *initial probability*

$$p_0(a, b)$$

as the frequency with which  $a$  is chosen over  $b$ , when no evidence-based deliberation is possible. With this, we can introduce a key notion.

**Definition 3.** *Alternatives  $a$  and  $b$  are a priori homogeneous if  $p_0(a, b) = 1/2$*

In words, alternatives are a priori homogeneous when there is no initial bias for one over the other. The interpretation of

$$p_t(a, b)$$

for  $t > 0$  is still that discussed in Section 2 as the frequency with which  $a$  is chosen over  $b$  after the decision-maker had the possibility to deliberate for  $t$  seconds, during which information about the two alternatives is gathered and processed. Depending on the obtained evidence, be it from environment or memory (or both),<sup>40</sup> the choice probability  $p_t(a, b)$  at deliberation time  $t$  may well be different from the initial  $p_0(a, b)$ .<sup>41</sup> We interpret this change in light of the following basic principle.

**Measurement Principle** *Prior behaviour gets transformed into posterior behaviour through consideration of evidence, and the transformation itself represents the amount of evidence processed during deliberation.*

37. See, e.g., Georgescu-Roegen (1936, 1958), Mosteller and Nogee (1951), Papandreou (1953) and Papandreou, Sauerlender, Brownlee, Hurwicz and Franklin (1957), Quandt (1956), Debreu (1958), and Davidson and Marschak (1959).

38. See, e.g., Falmagne (1985).

39. This principle is often discussed under the name “Psychometric Function.” See, e.g., Alos-Ferrer, Fehr and Netzer (2021).

40. See, e.g., Bogacz et al. (2006), Gold and Shadlen (2007), Shadlen and Shohamy (2016), and Bordalo et al. (2020).

41. See, again, Silayoi and Speece (2004) and Huseynov and Palma (2021).

This principle is best formalized through a change in odds as:

$$\underbrace{r_t(a,b)}_{\text{posterior odds}} = \underbrace{f}_{\text{strength of evidence}} \times \underbrace{r_0(a,b)}_{\text{prior odds}} \quad (20)$$

The ratio

$$f = f_t(a,b) = \frac{r_t(a,b)}{r_0(a,b)}$$

represents the *strength of evidence*, processed in  $t$  seconds, in favour of the hypothesis “ $a$  is preferable to  $b$ ”.

That said, in both statistics and neuroscience, additive measurements are preferred.<sup>42</sup> This is routinely achieved by taking logarithms on both sides of (20):

$$\underbrace{\ell_t(a,b)}_{\text{posterior log-odds}} = \underbrace{\ln f_t(a,b)}_{\text{weight of evidence}} + \underbrace{\ell_0(a,b)}_{\text{prior log-odds}}$$

The difference

$$w_t(a,b) = \ln f_t(a,b) = \ell_t(a,b) - \ell_0(a,b)$$

is the additive version of  $f_t(a,b)$ , called *weight of evidence*, a convenient logarithmic rescaling of strength of evidence.

Summing up, the strength of evidence is the change in odds for  $a$  against  $b$  induced by evidence accumulation before the deadline. Next, we show how this important notion permits to reveal preference order, intensity of preference, and ease of comparison—in symbols, to reveal the triple  $(\succ_t, \succ_t^\square, \succ_t^*)$  for all  $t$  in  $T$ .

We begin with the traditional ordinal notion. In the following, as usual, the term “revealed” is short for “revealed to an analyst”.

**Definition 4.** After a deliberation time  $t$ , an alternative  $a$  is revealed preferred to  $b$ , written  $a \succ_t b$ , if  $p_t(a,b) > p_0(a,b)$ .

In words,  $a$  is revealed preferred to  $b$  if deliberation favours  $a$  over  $b$ . In particular, when alternatives are a priori homogeneous, i.e.,  $p_0(a,b) = 1/2$ , this definition coincides with the standard unbiased notion of stochastically revealed preference

$$a \succ_t b \iff p_t(a,b) > p_t(b,a)$$

introduced in (19). In general, when alternatives are not necessarily a priori homogeneous, preference for  $a$  over  $b$  is equivalently revealed by an increase in the odds for  $a$  against  $b$  after deliberation. Indeed,

$$a \succ_t b \iff w_t(a,b) > 0 \iff f_t(a,b) > 1$$

Starting from this observation, Luce (1957, pp. 17–19) observes that, while the preference order is determined by the sign of  $w_t(a,b)$ , the intensity of preference is determined by its value. This motivates the next definition.

**Definition 5.** After a deliberation time  $t$ , the preference for  $a$  over  $b$  is revealed to be stronger than that for  $c$  over  $d$ , written  $(a,b) \succ_t^\square (c,d)$ , if  $w_t(a,b) > w_t(c,d)$ .

42. See, again, Gold and Shadlen (2007).

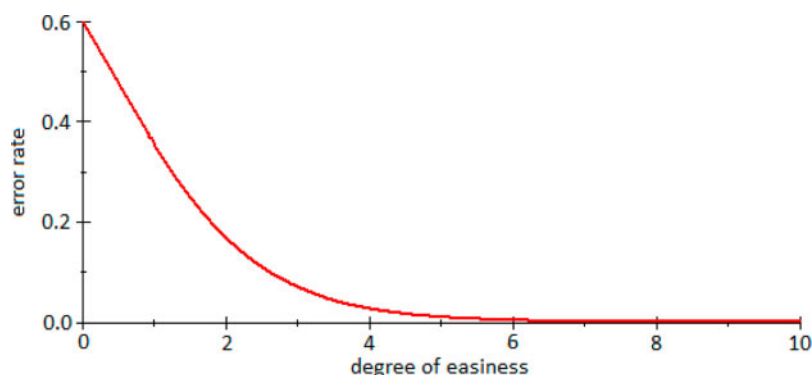


FIGURE 6

Error rate as a function of the degree of easiness, with an initial bias of 10% in favour of the inferior alternative.<sup>44</sup>

In words, the preference for  $a$  over  $b$  is stronger than that for  $c$  over  $d$  if deliberation provides stronger evidence in favour of  $a$  against  $b$  than in favour of  $c$  against  $d$ . Formally,

$$(a, b) \succ_t^b (c, d) \iff w_t(a, b) > w_t(c, d) \iff f_t(a, b) > f_t(c, d).$$

This definition thus extends the identification between intensity of preference and intensity of evidence that we discussed in Section 2. Clearly, in the unbiased case the relation  $\succ_t^b$  reduces to the (unbiased) intensity of preference relation  $\triangleright_t$  adopted there.

In a series of papers,<sup>43</sup> Georg Rasch calls *degree of easiness* of a decision problem  $\{a, b\}$  the quantity

$$|w_t(a, b)|.$$

The reason why this quantity captures the Psychometric Principle is immediately seen by drawing the error rate—the probability of choosing the inferior alternative—in the decision problem  $\{a, b\}$  as a function of the degree of easiness (see Figure 6).

When the degree of easiness is zero, the error rate is maximal and coincides with the initial probability of choosing the inferior alternative. It then decreases exponentially as the degree of easiness increases, and eventually vanishes. Analytically, this is seen by observing that, if  $w_t(a, b) > 0$ , that is,  $a \succ_t b$ , then the error rate is the probability

$$\underbrace{\text{ER}_t(a, b)}_{\text{error rate}} = p_t(b, a) = \frac{1}{1 + \exp\left(\ell_0(a, b) + \underbrace{|w_t(a, b)|}_{\text{degree of easiness}}\right)}$$

of choosing  $b$  from  $\{a, b\}$ , and similar considerations apply if  $w_t(a, b) < 0$ .

Since  $w_t(a, b) = -w_t(b, a)$ , the evidence in favour of  $a$  coincides with that against  $b$ . The degree of easiness  $|w_t(a, b)|$  thus represents the *total amount of evidence* that can be obtained by comparing  $a$  and  $b$  for  $t$  seconds. A decision problem is difficult when this quantity is small, say because sensory evidence or memory do not provide information to the decision-maker about the alternatives. All this leads to the following definition.

43. See Rasch (1960, 1961, 1980).

44. For instance, when  $a \succ_t b$  the inferior alternative is  $b$ .

**Definition 6.** After a deliberation time  $t$ , a decision problem  $\{a, b\}$  is revealed to be easier than a decision problem  $\{c, d\}$ , written  $\{a, b\} \succ_t^* \{c, d\}$ , if  $|w_t(a, b)| > |w_t(c, d)|$ .

At this point, it should not surprise the reader that also this definition reduces to (19) when alternatives are a priori homogeneous.

Summing up, weight of evidence—or, equivalently, strength of evidence—can be elicited from choice data by looking at the variation of choice probabilities *before* and *after* deliberation. It reveals three relations: preference order, preference intensity, and ease of comparison—in symbols  $(\succ_t, \succ_t^{\flat}, \succ_t^*)$ .

**3.1.3. Axiomatic softmax.** The natural extension of Consistency to the general, possibly biased case, is:

**Intensity Consistency** Given any  $s > t$  in  $T$ ,

$$(a, b) \succ_t^{\flat} (c, d) \iff (a, b) \succ_s^{\flat} (c, d)$$

for all  $a \neq b$  and  $c \neq d$  in  $X$ .

This axiom says that if the weight of evidence in favour of the hypothesis “ $a$  is preferable to  $b$ ” is, after a given deliberation time  $t$ , greater than that in favour of the hypothesis “ $c$  is preferable to  $d$ ”, the same happens after a longer deliberation time  $s$ . Formally,

$$w_t(a, b) > w_t(c, d) \iff w_s(a, b) > w_s(c, d)$$

for all  $s > t$ . In the unbiased case,  $\succ_t^{\flat}$  reduces to  $\triangleright_t$  and so Intensity Consistency reduces to Consistency.<sup>45</sup>

The next representation theorem will show that—together with the Psychometric Luce Axioms—Intensity Consistency characterizes general softmax processes. Yet, an alternative characterization is obtained by using non-reversal conditions for the preference order  $\succ_t$  and the ease of comparison relation  $\succ_t^*$ . Interestingly, these conditions have a one-way form, weaker than the two-way form of Intensity Consistency.

**Preference (Order) Consistency** Given any  $s > t$  in  $T$ ,

$$a \succ_t b \implies a \succ_s b$$

for all  $a, b \in X$ .

**Ease (of Comparison) Consistency** Given any  $s > t$  in  $T$ ,

$$\{a, b\} \succ_s^* \{c, d\} \implies \{a, b\} \succ_t^* \{c, d\}$$

for all  $a \neq b$  and  $c \neq d$  in  $X$ .

In terms of primitives, Preference Consistency is equivalent to

$$p_t(a, b) > p_0(a, b) \implies p_s(a, b) > p_0(a, b)$$

for all  $s > t$ . It says that preferences are stable: as time passes, they are not reverted. This is in accord with the idea that, during deliberation, statistically correct (yet noisy) evidence is gathered and analysed by the decision-maker to inform his choice between the two alternatives.

45. Like Consistency, Intensity Consistency admits a formulation which is viable for general index sets  $T$ , see Footnote 15.

Ease Consistency, instead, says that the difficulty of decision problem  $\{a, b\}$  relative to decision problem  $\{c, d\}$  is inherent to the alternatives involved and independent of deliberation times. If the comparison between  $a$  and  $b$  is not easier than that between  $c$  and  $d$ , given deliberation time  $t$ , then the passage of time does not make  $a$  and  $b$  easier to compare than  $c$  and  $d$ . In terms of degree of easiness, Ease Consistency is equivalent to

$$|w_t(a, b)| \leq |w_t(c, d)| \implies |w_s(a, b)| \leq |w_s(c, d)|$$

for all  $s > t$ .

We can now state the general softmax representation theorem. We adopt the convention  $\lambda(0) = \infty$ .

**Theorem 7.** *For a random choice process  $\{p_t\}_{t \in T_0}$ , the following conditions are equivalent:*

1.  $\{p_t\}$  satisfies the Psychometric Luce Axioms and Intensity Consistency;
2.  $\{p_t\}$  satisfies the Psychometric Luce Axioms, Preference Consistency, and Ease Consistency;
3. there exist two continuous functions  $u: X \rightarrow \mathbb{R}$  and  $\alpha: X \rightarrow \mathbb{R}$ , and a  $\lambda: T \rightarrow (0, \infty)$  such that

$$p_t(a, A) = \frac{e^{\frac{u(a)}{\lambda(t)} + \alpha(a)}}{\sum_{b \in A} e^{\frac{u(b)}{\lambda(t)} + \alpha(b)}} \quad (\text{softmax})$$

for all  $A \in \mathcal{A}$ , all  $a \in A$ , and all  $t \in T_0$ .

In this case,  $u$  is a psychometric utility for  $(\succ_t, \succ_t^\dagger, \succ_t^*)$  for all  $t$  in  $T$ ,  $\alpha$  is unique up to location, and  $\lambda$  is unique given  $u$ , unless  $\{p_t\}$  is constant.

Moreover, the process  $\{p_t\}_{t \in T}$  is MNL if and only if all distinct alternatives are a priori homogeneous.

These processes are called *softmax processes*. An analyst, who observes that the decision-maker's behaviour does not contradict the axioms of the theorem, can thus interpret this behaviour in terms of preference discovery, that is, *as if* the decision-maker were trying to learn the value that alternatives have for him and choose the best one.

Like in the unbiased case, our analyst can identify from the probabilistic choices of the decision-maker the softmax parameters  $u$ ,  $\alpha$ , and  $\lambda$ .<sup>46</sup> Specifically,  $u$  is a psychometric utility for each triplet  $(\succ_t, \succ_t^\dagger, \succ_t^*)$ , and as such it is cardinally unique (Lemma 6). Then, if the process is constant,  $u$  must be constant,  $\alpha(a) - \alpha(b) = \ell_0(a, b)$  for all  $a, b \in X$ , and  $\lambda$  is undefined; else there exist at least a pair of alternatives  $\hat{a}$  and  $\hat{b}$  and a deliberation time  $\hat{t}$  such that the preference  $\hat{a} \succ_{\hat{t}} \hat{b}$  is revealed, and the next result provides the explicit expression of the parameters.

**Theorem 8.** *A softmax process  $\{p_t\}$  is not constant if and only if there exist  $\hat{a}, \hat{b} \in X$  and  $\hat{t} \in T$  such that  $p_{\hat{t}}(\hat{a}, \hat{b}) > p_0(\hat{a}, \hat{b})$ . In this case, the functions  $\hat{u}, \hat{\alpha}: X \rightarrow \mathbb{R}$  and  $\hat{\lambda}: T \rightarrow (0, \infty)$  given by*

$$\hat{u}(x) = \frac{w_{\hat{t}}(x, \hat{b})}{w_{\hat{t}}(\hat{a}, \hat{b})} \quad ; \quad \hat{\alpha}(x) = \ell_0(x, \hat{b}) \quad ; \quad \hat{\lambda}(t) = \frac{1}{w_t(\hat{a}, \hat{b})} \quad (21)$$

46. Their estimation is standard, typically carried out by maximum likelihood. See, e.g., Ben-Akiva and Lerman (1985) on the econometric side and McKelvey and Palfrey (1995) on the game-theoretic one.

are well defined, with  $\hat{u}$  and  $\hat{\alpha}$  continuous, and

$$p_t(a, A) = \frac{e^{\frac{\hat{u}(a)}{\lambda(t)} + \hat{\alpha}(a)}}{\sum_{b \in A} e^{\frac{\hat{u}(b)}{\lambda(t)} + \hat{\alpha}(b)}} \quad (22)$$

for all  $A \in \mathcal{A}$ , all  $a \in A$ , and all  $t \in T_0$ .

The last two results extend the “unbiased” Theorems 2 and 3 to the general case. So, they enable the analyst to interpret the stochastic choice behaviour of the decision-maker in terms of softmax preference discovery and to empirically identify the softmax parameters.

**3.1.4. Ordinality and learning.** We conclude this section by characterizing the case in which, as deliberation time increases, the stochastic choice behaviour of a decision-maker improves and becomes less prone to errors.<sup>47</sup>

**Decreasing Error Rates** Given any  $s > t$  in  $T$ ,

$$p_t(a, b) > p_0(a, b) \implies p_s(a, b) \geq p_t(a, b)$$

for all  $a, b \in X$ .

Under Preference Consistency, this axiom requires the frequency of mistakes to decrease over deliberation time. Indeed,  $p_t(a, b) > p_0(a, b)$  is equivalent to the fact that  $a \succ_t b$ , Preference Consistency implies  $a \succ_s b$ , and  $p_s(a, b) \geq p_t(a, b)$  is equivalent to a positive reduction of the error rate. In words, longer deliberation times decrease the chance of selecting an inferior alternative. To appreciate the consequences of this axiom, we need an additional one.

**Stochastic Dominance Improvements** Given any  $s > t$  in  $T$ ,

$$p_s(\{a \in A : a \succ_s b\}, A) \geq p_t(\{a \in A : a \succ_t b\}, A) \quad (23)$$

for all  $A \in \mathcal{A}$  and all  $b \in A$ .

Stochastic Dominance Improvements requires that, for any given benchmark alternative  $b$ , the probability of choosing a superior alternative  $a$  is higher after deliberating for a longer amount of time. This notion thus records, in preferential terms, a probabilistic improvement of the decision maker stochastic choice behaviour as deliberation times increase.

The next proposition shows that Decreasing Error Rates and Stochastic Dominance Improvements are equivalent axioms for softmax processes. Moreover, they characterize the choice behaviour of a decision-maker who takes better and better decisions, according to stochastic dominance in payoffs, as deliberation time increases. In terms of the softmax specification, each of these axioms corresponds to a decreasing noise function  $\lambda$ . In terms of rational inattention, to a time-decreasing unit cost of information processing (e.g. the attention cost of reading and understanding a given paragraph decreases with the time available to do so).

47. For instance, in medical decision-making under severe time pressure, the longer the time a doctor has to process information, the lower the chance of selecting a suboptimal treatment seems to be (see e.g. ALQahtani, Rotgans, Mamede, ALAlwan, MagzoM, Altayeb, Mohamedani and Schmidt, 2016). Yet, other evidence from psychology suggests that overly slack deadlines leave room to procrastination, distractions, and fatigue that may deteriorate choice performance (see e.g. Ariely and Wertenbroch, 2002). Clearly, the study of these time-increasing error rate situations mirrors the one we consider here.

**Proposition 9.** *Let  $\{p_t\}$  be a nonconstant softmax process with utility  $u$ , bias  $\alpha$ , and noise  $\lambda$ . The following conditions are equivalent:*

1.  $\{p_t\}$  satisfies Decreasing Error Rates;
2.  $\{p_t\}$  satisfies Stochastic Dominance Improvements;
3. given any  $s > t$  in  $T$ ,

$$p_s(\{a \in A : u(a) > \bar{u}\}, A) \geq p_t(\{a \in A : u(a) > \bar{u}\}, A)$$

for all  $\bar{u} \in \mathbb{R}$  and all  $A \in \mathcal{A}$ ;

4.  $\lambda$  is decreasing.

In view of this result, it is natural to wonder whether, for longer and longer deliberation times, the decision-maker eventually learns his ranking over alternatives, that is, his preference order over them. In other words, is the preference discovery interpretation of softmax processes true to its name?

To address this question, assume for simplicity that  $T = (0, \infty)$ .<sup>48</sup> By the last result, under Decreasing Error Rates, the noise function  $\lambda$  is decreasing on  $(0, \infty)$ . This permits to define a limit random choice rule  $p_\infty : \mathcal{A} \rightarrow \Delta(X)$  by

$$p_\infty(a, A) = \lim_{t \rightarrow \infty} p_t(a, A)$$

for all  $A$  in  $\mathcal{A}$  and all  $a$  in  $A$ . This random choice rule describes behaviour in the absence of time pressure.

**Asymptotic Tie-breaking** Given any  $a, b \in X$ ,

$$p_\infty(a, b) \neq 0, 1 \implies p_\infty(a, b) = p_0(a, b)$$

In conjunction with the previous ones, this axiom says that the decision-maker is unable to make up his mind between alternatives  $a$  and  $b$ , irrespective of deliberation time, only if it is impossible for him to retrieve new information about them.

**Proposition 10.** *Let  $\{p_t\}$  be a non-constant softmax process with utility  $u$ , bias  $\alpha$ , and noise  $\lambda$ . If  $\{p_t\}$  satisfies Decreasing Error Rates and Asymptotic Tie-breaking, then*

$$p_\infty(a, A) = \delta_a(\operatorname{argmax}_A u) \frac{e^{\alpha(a)}}{\sum_{b \in \operatorname{argmax}_A u} e^{\alpha(b)}}$$

for all  $A \in \mathcal{A}$  and all  $a \in A$ . In particular,

$$u(a) > u(b) \iff p_\infty(a, b) = 1$$

for all  $a \neq b$  in  $X$ .

48. Otherwise, the role of  $\infty$  is played by the supremum of  $T$ .



According to this proposition, the choice rule  $p_\infty$  reveals a preference  $\succ$  on  $X$  defined by

$$a \succ b \iff p_\infty(a, b) = 1$$

and represented by  $u$ . This preference permits to interpret the non-stochastic limit choice behaviour in a traditional ordinal way, as if carried out by a decision-maker who learned his preference – so, his psychometric utility  $u$  up to an ordinal transformation—and accordingly selects the best alternatives.<sup>49</sup>

Standard ordinal analysis thus emerges as the limit version, as deliberation time becomes arbitrarily large, of our cardinal analysis. Alternatively, one can regard standard theory as assuming deliberation time to be virtual; in real time, decision-makers act as if they know their preferences.

### 3.2. Inside the black box: neuro-computational softmax

Symmetry of the DDM presented in Section 2.2.1 permits the description of the net evidence obtained by the decision-maker in differential form as

$$\begin{aligned} dZ_{a,b}(\tau) &= [v(a) - v(b)]d\tau + \sqrt{2}dW(\tau) \\ Z_{a,b}(0) &= 0, \end{aligned} \tag{24}$$

where the initial condition  $Z_{a,b}(0) = 0$  captures the lack of prior information/initial bias in favour or against either  $a$  or  $b$ . As extensively discussed by [Bogacz et al. \(2006\)](#) and [Ratcliff, Smith, Brown and McKoon \(2016\)](#), prior information/initial bias are captured in the DDM by replacing the null initial condition (24) with

$$Z_{a,b}(0) = \zeta_t(a, b),$$

where  $\zeta_t(a, b) > 0$  means that  $a$  is initially favoured, and  $\zeta_t(a, b) < 0$  that  $b$  is. In fact, with starting point position  $\zeta_t(a, b) \neq 0$ , the evidence needed to select  $a$  and  $b$  becomes  $\beta(t) - \zeta_t(a, b)$  and  $\beta(t) + \zeta_t(a, b)$ , respectively.

In this section, we study what happens when a general DDM replaces the symmetric DDM in the Metropolis-DDM algorithm. The additional ingredient, relative to the neural utility  $v: X \rightarrow \mathbb{R}$  and threshold function  $\beta: T \rightarrow (0, \infty)$  of Section 2.2.1, is the *starting point function*

$$\begin{aligned} \zeta: X \times X \times T &\rightarrow \mathbb{R} \\ (a, b, t) &\mapsto \zeta_t(a, b) \end{aligned}$$

such that

$$-\beta(t) < \zeta_t(a, b) = -\zeta_t(b, a) < \beta(t)$$

for all  $(a, b, t)$ . The unbiased case of Section 2.2.1 corresponds to  $\zeta = 0$ . In what follows, we denote by

$$\text{DDM}(v, \beta, \zeta)$$

a DDM with parameters  $v$ ,  $\beta$ , and  $\zeta$ .

49. In contrast, an analyst learns  $u$  up to a cardinal transformation by observing the decision-maker softmax stochastic behaviour (Theorem 8).

**3.2.1. Gibbs transitions.** In this subsection, we relate starting point positions and initial probabilities as equivalent ways to express initial biases. To this end, consider a choice between two alternatives  $a$  and  $b$  to be carried out through a DDM featuring neural utility  $v$ , threshold  $\beta$ , and starting point  $\zeta$ . As we just observed, the values

$$\zeta_t(a, b) = -\zeta_t(b, a)$$

at  $(a, b)$  and  $(b, a)$ , determine whether the DDM comparison favours either  $a$  or  $b$ . However, it should be possible to express the same bias with the values

$$\pi_t(a, b) = 1 - \pi_t(b, a)$$

of an *ex ante* (binary preference) probability  $\pi_t: X_{\neq}^2 \rightarrow \mathbb{R}$  at  $(a, b)$  that describes the chances of choosing either  $a$  or  $b$  before the DDM comparison takes place, that is, before new evidence is gathered and processed.<sup>50</sup> Both  $\zeta_t$  and  $\pi_t$  incorporate the decision-maker past information, in particular his past memories. Both lack—differently from  $dZ$  and  $\beta$ —an obvious physiological counterpart and their values are posited by the analyst to better interpret the model and fit the data.<sup>51</sup>

How do they relate? Which *ex ante* probability  $\pi_t$  corresponds to a starting point function  $\zeta_t$ ? To address these natural questions, observe that  $\zeta$  induces an *ex post* (binary preference) probability

$$\mathbb{P}_t^\zeta(a, b) = \mathbb{P}[\text{CO}_t^\zeta(a, b) = a]$$

of choosing  $a$  over  $b$ , after the DDM that starts at position  $\zeta_t(a, b)$  has accumulated neural evidence  $\beta(t)$ .

**Definition 7.** Given a DDM( $v, \beta, \zeta$ ), the Gibbs transition of  $\zeta$  is

$$\pi_t^\zeta(a, b) = \frac{e^{-\beta(t)v(a)} \mathbb{P}_t^\zeta(a, b)}{e^{-\beta(t)v(a)} \mathbb{P}_t^\zeta(a, b) + e^{-\beta(t)v(b)} \mathbb{P}_t^\zeta(b, a)} \quad (25)$$

for all  $a \neq b$  in  $X$  and all  $t$  in  $T$ .

Formula (25) associates an *ex ante* probability  $\pi_t$  to each section  $\zeta_t$  at  $t$  of the starting point function  $\zeta$ .<sup>52</sup> The Gibbs transition of  $\zeta$  can be equivalently expressed in terms of odds as follows:

$$\underbrace{\frac{\mathbb{P}_t^\zeta(a, b)}{\mathbb{P}_t^\zeta(b, a)}}_{\text{odds post DDM}} = \underbrace{e^{\beta(t)[v(a)-v(b)]}}_{\text{strength of evidence}} \times \underbrace{\frac{\pi_t^\zeta(a, b)}{\pi_t^\zeta(b, a)}}_{\text{odds ante DDM}} \quad (26)$$

Therefore, this formula can be interpreted according to the Measurement Principle of Section 3.1.2. With one caveat: in equation (20) of that section, the analyst observes the *ex ante* and the *ex post* odds, and aims to measure the unknown strength of evidence. Here, in

50. The “binary preference probability” terminology is due to Luce and Suppes (1965). Like  $\zeta_t$ , also  $\pi_t$  may in principle depend on  $t$ .

51. See Bogacz et al. (2006).

52. It is, *mutatis mutandis*, the analogue of the Gibbs prior of Zhang (2006a,b).

contrast, the analyst observes the *ex post* odds and the evidence threshold, and aims to measure the unknown *ex ante* odds that are implied by a posited bias  $\zeta$ . Yet, the underlying measurement principle is the same: the change in odds for  $a$  against  $b$  resulting from the DDM is proportional, via an exponential factor, to the accumulated neural evidence  $\beta(t)$  weighted by the neural utility difference  $v(a) - v(b)$ . The quantity  $\beta(t)[v(a) - v(b)]$  is thus the weight of evidence for  $a$  against  $b$  that makes the neural system move from the *ex ante* to the *ex post* probability of choosing  $a$  over  $b$ .

Observe that, for  $\zeta_t(a, b) = 0$ , formula (26) is easily seen to imply  $\pi_t^\zeta(a, b) = 1/2$ , as the intuition for the unbiased sDDM suggests. In words, a null starting point position corresponds to the a priori homogeneity of alternatives.

The definition of Gibbs transition can also be justified through large deviations arguments.<sup>53</sup> But perhaps more importantly, the relation between starting point positions and *ex ante* probabilities, defined via (25), features some remarkable properties.

**Proposition 11.** *Given a DDM( $v, \beta, \zeta$ ), for any  $a \neq b$  in  $X$  we have*

$$\zeta_t(a, b) \geq 0 \iff \pi_t^\zeta(a, b) \geq \frac{1}{2} \quad (27)$$

and

$$\left| \mathbb{P}_t^\zeta(a, b) - \pi_t^\zeta(a, b) \right| \leq \frac{\beta(t)}{4} |v(a) - v(b)| \quad (28)$$

for all  $t$  in  $T$ .

The monotonicity formula (27) ensures that a positive bias  $\zeta_t(a, b)$  in favour of  $a$  against  $b$  corresponds to a higher *ex ante* probability  $\pi_t^\zeta(a, b)$  of selecting  $a$  over  $b$ . It implies, *inter alia*, that a null  $\zeta$  corresponds to a uniform  $\pi$ , as we previously checked in a direct way.

The monotonicity formula thus substantiates the claim that  $\pi_t^\zeta$  is the *ex ante* probability naturally associated to the initial condition  $\zeta_t$ . Inequality (28) further corroborates this role of  $\pi_t^\zeta$  by showing that it actually governs the DDM's probabilistic choices when the accumulated evidence  $\beta$  is small or alternatives have similar neural utilities, thus initial bias prevails.

This leads to a crucial observation: since it is not affected by evidence accumulation, the *ex ante* choice probability  $\pi_t$  should not depend on the level of time pressure.

**Gibbs Transition Consistency** *Given any  $s > t$  in  $T$ ,*

$$\pi_t^\zeta(a, b) = \pi_s^\zeta(a, b)$$

for all  $a \neq b$  in  $X$ .

If this consistency condition is violated, there is a pair  $(a, b)$  of alternatives whose *ex ante* choice probability differ,  $\pi_t^\zeta(a, b) \neq \pi_s^\zeta(a, b)$ , depending on the level of time pressure,  $t$  or  $s$ . But, this contradicts the intuition that  $\zeta$  and  $\pi^\zeta$  describe *ex ante* information only. In other words, if a starting point function does not satisfy Gibbs Transition Consistency, then it captures something more than (or something different from) memory anchoring.

53. See, e.g., Dupuis and Ellis (1997, p. 27), and the working paper version of this article, Cerreia-Vioglio et al. (2021a).

**3.2.2. Transitive DDMs and stationarity.** When considering a Metropolis-DDM algorithm with general  $\text{DDM}(v, \beta, \zeta)$  binary comparisons, the first observation is that, even if  $\text{DDM}(v, \beta, \zeta)$  satisfies Gibbs Transition Consistency, the transition matrix  $M_t(A)$  of the algorithm may fail to be reversible for some  $A$  in  $\mathcal{A}$ . But reversibility is the Markovian condition which convergence of Markov Chain Monte Carlo (MCMC) algorithms to a distribution that satisfy Luce's Choice Axiom rests upon (see [Baldassi et al., 2022](#); [Valkanova, 2021](#)).

The next proposition will show that reversibility has the following meaningful behavioural counterpart.

**(Stochastic) Transitivity** Given any  $t \in T$ ,

$$\mathbb{P}_t^\zeta(b, a) \mathbb{P}_t^\zeta(c, b) \mathbb{P}_t^\zeta(a, c) = \mathbb{P}_t^\zeta(c, a) \mathbb{P}_t^\zeta(b, c) \mathbb{P}_t^\zeta(a, b) \quad (29)$$

for all distinct alternatives  $a, b, c \in A$ .

In words,  $\text{DDM}(v, \beta, \zeta)$  is transitive when the violations of transitivity that it determines are due only to the presence of noise. Indeed, condition (29) amounts to require that the intransitive cycles

$$a \rightarrow b \rightarrow c \rightarrow a \quad \text{and} \quad a \rightarrow c \rightarrow b \rightarrow a$$

be equally likely. The symmetric (unbiased, value-based) DDMs, considered in Section 2.2, are an important example of transitive DDMs that satisfy Gibbs Transition Consistency. Biased value-based DDMs, instead, might well not be transitive, so may result in choices between alternatives that feature systematic intransitivities, violating a basic rationality tenet. Transitivity ensures that this is not the case.

The next result, which builds upon [Kolmogorov \(1936\)](#) and [Luce and Suppes \(1965\)](#),<sup>54</sup> shows the importance of transitive DDMs in our setting, and a key feature of the corresponding *ex ante* binary probabilities.

**Proposition 12.** *The following conditions are equivalent for a  $\text{DDM}(v, \beta, \zeta)$  which satisfies Gibbs Transition Consistency:*

1.  $\text{DDM}(v, \beta, \zeta)$  is transitive;
2. the incumbents' transition matrix  $M_t(A)$  is reversible for all  $A \in \mathcal{A}$  and all  $t \in T$ ;
3. there exists  $\zeta: X \rightarrow \mathbb{R}$  such that,

$$\pi_t^\zeta(a, b) = \frac{e^{\zeta(a)}}{e^{\zeta(a)} + e^{\zeta(b)}}$$

for all  $(a, b, t) \in X_{\neq}^2 \times T$ .

In this case,  $\zeta$  is uniquely determined by  $\zeta$ , and vice versa (up to location).

Remarkably, this proposition connects properties of altogether different nature:

1. transitivity of the DDM, a behavioural property which ensures that violations of transitivity in the probabilistic choices that it determines are due only to the presence of noise;

54. On reversibility, see, e.g., [Kelly \(2011\)](#).

2. reversibility of  $M$ , an algorithmic property which is an important sufficient condition for the existence of stationary distributions of Markov chains,<sup>55</sup> which is necessary to guarantee that the generated distributions satisfy Renyi conditioning as the set of Markov states changes;<sup>56</sup>
3. existence of a universal *Gibbs density*  $\zeta : X \rightarrow \mathbb{R}$  for  $\zeta$  that, via conditioning, determines all Gibbs transitions.

Because of the one-to-one correspondence between  $\zeta$  and  $\zeta$  (up to location) established by Proposition 12, transitive DDMs that satisfy Gibbs Transition Consistency can be simply denoted by  $\text{DDM}(v, \beta, \zeta)$ .<sup>57</sup> With this, we can establish the general version of Theorem 5.

**Theorem 13.** *Let  $\text{DDM}(v, \beta, \zeta)$  be a transitive DDM which satisfies Gibbs Transition Consistency. Given any  $A \in \mathcal{A}$  and any  $t \in T$ , the exploration matrix  $Q_t = Q_t(A)$  is irreducible and symmetric and the incumbents' transition matrix  $M_t = M_t(A)$  is aperiodic, irreducible, and reversible, with stationary distribution*

$$m_t(a, A) = \frac{e^{\beta(t)v(a) + \zeta(a)}}{\sum_{b \in A} e^{\beta(t)v(b) + \zeta(b)}} \quad \forall a \in A$$

In particular,  $\lim_{n \rightarrow \infty} M_t^n(A) \mu = m_t(\cdot, A)$  for all  $A \in \mathcal{A}$  and all  $\mu \in \Delta(A)$ .

This result concludes our analysis by extending the identification and cross-validation procedure that we described in Section 2.3 to the general, possibly biased, case. Table (15) becomes

Inner		Outer	
Neural utility	$v$	Psychometric utility	$u$
Threshold	$\beta$	Inverse noise	$1/\lambda$
Starting point bias	$\zeta$	Behavioural bias	$\alpha$

where the Gibbs density  $\zeta$  corresponding to the starting point function  $\zeta$  is shown to be the counterpart of initial bias  $\alpha$ . As mentioned, the neuroscience literature and the rational inattention literature interpreted these parameters in the same way, but the formal connection between them has been so far elusive.

Theorem 13 reveals the connection and explains why the Metropolis-DDM algorithm (approximately) generates softmax probabilities thus implementing the optimal information acquisition strategy of [Matejka and McKay \(2015\)](#). In turn, the axiomatic representation Theorem 7 characterizing the latter provides testable implications for the former.<sup>58</sup> For brevity, we omit the, by now, routine details of identification and cross-validation that parallel those spelled out in Section 2.3.

#### 4. CONCLUDING REMARKS

In a few concluding remarks, we explore limitations and possible future extensions of our analysis.

55. As [Geyer \(2011, p. 6\)](#) writes “All known methods for constructing transition probability mechanisms that preserve a specified equilibrium distribution in non-toy problems are ... reversible.”

56. See, again, [Baldassi et al. \(2022\)](#) and [Valkanova \(2021\)](#).

57. In practice, once  $v$ ,  $\beta$ , and  $\zeta$  are estimated in the lab, then checking whether  $\text{DDM}(v, \beta, \zeta)$  satisfies Gibbs Transition Consistency and transitivity is a matter of hypothesis testing. For example, [Milosavljevic et al. \(2010\)](#) cannot reject the nullity of  $\zeta$ , and *a fortiori* these weaker assumptions.

58. If numerical convergence is achieved, see the discussion in Section 2.3.

#### 4.1. Latency

Our main interpretation of  $t$  is as the amount of time available to *gather and process new additional evidence* when facing a decision problem. In Section 3, at the “limit deadline”  $t=0$  only the *immediately available prior information* can then be used by the agent to make a decision.

Going back to the first example in the introduction of a trader deciding among alternative investments,  $t=0$  is an abstraction for the fact that he has to choose “immediately,” that is, within the smallest possible amount of time without looking at his terminal; instead,  $t=10$  means that the trader has 10 s to look at the terminal and decide.

The “immediately” available prior information about the alternative investments resides in the trader’s memory and builds on previous experience. As such, it must be quickly retrieved, a process which however—in reality—takes some time  $\varepsilon > 0$  called *latency*. This  $\varepsilon$  is the smallest possible amount of time after which a decision-maker who has to choose immediately is able to answer. Thus,  $p_0(a, A)$  corresponds, more precisely, to  $p_\varepsilon(a, A)$ , while  $p_t(a, A)$  corresponds to  $p_{\varepsilon+t}(a, A)$ . Moreover,  $\varepsilon$  might depend on  $A$ , with larger sets commanding larger latency. This leads to a model where  $p_t(a, A)$  is replaced by  $p_{\varepsilon(A)+t}(a, A)$ , so that

$$p_{\varepsilon(A)+t}(a, A) = \frac{e^{\frac{u(a)}{\lambda(t)} + \alpha(a)}}{\sum_{b \in A} e^{\frac{u(b)}{\lambda(t)} + \alpha(b)}}$$

and, for  $t=0$ ,

$$p_{\varepsilon(A)}(a, A) = \frac{e^{\alpha(a)}}{\sum_{b \in A} e^{\alpha(b)}}$$

*All of our results* continue to hold in this more sophisticated setting. Thus, our “abuse” in Section 3 consists in writing  $p_t(a, A)$  instead of  $p_{\varepsilon(A)+t}(a, A)$ . This abuse is practically negligible when the minimum element of  $T$  is large relative to  $\varepsilon$  (it is safe to assume that  $\varepsilon$  is smaller than a second). What happens when the experimenter gives the subject deadlines that are smaller than  $\varepsilon$  is an interesting research question that goes beyond the scope of this article.

#### 4.2. Beyond softmax

To put our softmax revealed preference analysis (Section 3) in a wider perspective, we briefly discuss a general specification of a random choice process. In particular, in the next definition we generalize to our deliberation context the definition of utility for random choice rules introduced by [Debreu \(1958\)](#) and [Davidson and Marschak \(1959\)](#).

**Definition 8.** *A psychometric utility function  $u$  and an initial bias  $\alpha$  on  $X$  rationalize a random choice process  $\{p_t\}$  if, at each deliberation time  $t$  and all alternatives  $a \neq b$  and  $c \neq d$  in  $X$ :*

$$u(a) - u(b) \leq u(c) - u(d) \quad \text{and} \quad \alpha(a) - \alpha(b) \leq \alpha(c) - \alpha(d) \implies p_t(a, b) \leq p_t(c, d).$$

It is easy to see that this amounts to requiring the existence, at each deliberation time  $t$ , of a time-dependent function  $\phi_t$ , increasing in both arguments, such that

$$p_t(a, b) = \phi_t(u(a) - u(b), \alpha(a) - \alpha(b))$$

for all distinct alternatives  $a, b \in X$ .<sup>59</sup> Softmax is the special case

$$\phi_t(x, y) = \frac{1}{1 + e^{-\frac{x}{\lambda(t)} - y}}.$$

In fact, in the binary case the softmax formula becomes:

$$p_t(a, b) = \frac{1}{1 + e^{-\frac{u(a) - u(b)}{\lambda(t)} - [\alpha(a) - \alpha(b)]}}.$$

In the presence of bias, a pair  $(u, \alpha)$  is thus needed to understand random choice processes,  $u$  alone is no longer enough—as it was, instead, in the analyses of Debreu, Davidson, and Marschak of random choice rules. The noise  $\lambda$  is peculiar to the softmax case, where it parameterizes the exponential form of  $\phi_t$ .

A natural question, which may be explored in future research, is how the analysis of Debreu (1958) may generalize in this setup, determining which conditions on a random choice process ensure the existence of the pair  $(u, \alpha)$ . A result along these lines would be in relation to Debreu (1958) what our softmax representation theorem (Theorem 7) is to Luce (1959).

#### 4.3. Testing of the Metropolis-DDM algorithm and possible extensions

A novel object such as the Metropolis-DDM algorithm calls for accurate experimental tests, both based on existing datasets and on *ad hoc* experiments focused to trace its specific moving parts. This research is ongoing, it rests on the dialogue between axioms and mechanisms introduced in this article but goes well beyond the scope of this work. More in general, the hypothesis that sequential binary comparisons form the basic structure of multialternative choice remains an open challenge that we plan to explore.

A natural extension of the Metropolis-DDM is obtained by considering either more general DDMs—like the ones with random starting points *à la* Ratcliff *et al.* (2016), or with collapsing barriers *à la* Churchland, Kiani and Shadlen (2008), or with attentional weights *à la* Krajbich *et al.* (2010)—or recent non-DDM models—like the optimal coding models of Rustichini and Padoa-Schioppa (2015) and Rustichini, Conen, Cai and Padoa-Schioppa (2017), or the extrema detection models of Stine, Zylberberg, Ditterich and Shadlen (2020). In all these cases, very weak assumptions guarantee that the resulting algorithm has a stationary distribution which approximates the resulting stochastic choice behaviour; but, since these models do not have closed form binary choice probabilities, the chances of finding a closed form for such a stationary distribution do not seem to be high; but see Baldassi *et al.* (2022).

#### 4.4. Quantal response equilibrium

The softmax functional form can be regarded as a formalization of the Discovered Preference Hypothesis (DPH) outlined in Plott (1996). According to this hypothesis, decision-makers learn how their basic needs are satisfied by the different alternatives in the choice environment through a process of reflection and practice that, in the long run, leads to optimizing behaviour.

Reflection is readily captured in our model by the deliberation time. If one considers applications to repeated choice situations, the DPH points to a different natural interpretation

59. The real-valued function  $\phi_t$  has domain  $(\text{Im}u - \text{Im}u) \times (\text{Im}\alpha - \text{Im}\alpha)$ .



of the set  $T$ . Instead of a deadline, each  $t$  of  $T$  may represent the number of times that the decision-maker has been facing choice problem  $A$ . Under this interpretation, softmax can be seen as capturing preference discovery through practice.

MNL, that is, unbiased softmaximization is the form that preference discovery takes in the QRE theory of McKelvey and Palfrey (1995). In their theory,  $t$  is the number of times an agent played the game, and thus measures his experience level,  $u(a)$  is the expected payoff of action  $a$ , and  $\lambda(t)$  indexes the agent's degree of rationality. From the original data analysis of McKelvey and Palfrey (1995) to the recent Agranov *et al.* (2015), Goeree, Holt and Palfrey (2016), and Ortega and Stocker (2016) evidence seems to suggest that, for sophisticated players, the function  $\lambda$  decreases as time passes and the decision-making environment is better understood.<sup>60</sup>

Our axiomatic and neuro-computational characterizations of MNL can thus be seen as alternative foundations of QRE. The first identifies the discovery outcome, and the second explains the discovery process. QRE is thus the equilibrium concept that corresponds to the decision theory developed in this article. Goeree *et al.* (2016) give a broad perspective of its different applications.<sup>61</sup>

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### Supplementary Data

Supplementary data are available at *Review of Economic Studies* online.

### Data Availability Statement

This article does not involve analysis of external data. The replication packages and the code generating all simulations' plots are available at <https://doi.org/10.5281/zenodo.6599226>.

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60. Interestingly, in Agranov *et al.* (2015) and Ortega and Stocker (2016),  $t$  is not the experience level, but the time the player had to contemplate the alternatives in  $A$  before choosing.

61. For an epistemic characterization of QRE, see the recent Liu and Maccheroni (2021).



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