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Optimal ordering for Product-as-a-Service models with circular economy practices

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ABSTRACT

The Product-as-a-Service (PaaS) model offers the opportunity to implement circular actions such as repairing, reusing, collecting end-of-life products, and recycling. However, adopting circular practices causes more complexities in managing the inventory flow due to repetitive product subscriptions. Accordingly, this paper aims to optimise a PaaS model's order quantity and profit, considering circular economy practices and various quality levels for subscription products. In the proposed model, the subscription firm defines a quality check and repair procedure at the end of each quality period before sending the product to another subscriber. Moreover, the firm recycles end-of-life products and sells the recycled material to the supplier. This study aims to compare a closed-loop PaaS model with the traditional economic order quantity model in terms of operational costs, revenue, and inventory flow. The results show that factors such as the difference between the demand rates of the consecutive periods, the relationship between the recycling capacity and the final collection rate, and the difference between the screening and demand rates have essential roles in alleviating the extra inventory costs related to circular economy actions like reusing, repairing, and recycling.

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Product-as-a-Service (PaaS); subscription-based business model (SBBM); optimal order quantity; circular economy; reusing; recycling

1. Introduction

Nowadays, firms are revising their business models to change their value propositions from selling ownership-based tangible products to offering intangible services, called product-service systems (PSS). This can be a win-win strategy as it increases the firm's profits and mitigates its environmental performance simultaneously (Xing, Wang, and Qian 2013). The subscription-based business model improves servitization and is known as an innovative product distribution approach that offers a range of benefits to both customers and companies. Subscription services have changed how consumers purchase products and services from a pay-per-product model to a membership-based one (Tao and Xu 2018). However, SBBM may cause an increment in operational costs as it requires additional processes to maintain the ongoing service (Wagner, Pinto, and Amorim 2021). This is a crucial issue in SBBM because price and demand typically have a negative relationship, and the subscription firm cannot necessarily compensate for the extra costs by increasing subscription fees (Li et al. 2024).

Moreover, new mindsets, such as the Product-as-a-Service (PaaS) business model, retain ownership of a product and offer it to one or many customers as a service. The PaaS business model has several forms. However, some of the well-known PaaS firms use subscription models in which the customers benefit from the services by paying membership and having recurrent payments based on the type of products (Lacy and Rutqvist 2015). For instance, MUD Jeans[1] offers customers pairs of jeans for a fixed amount per month; Signify[2] provides companies with lights as a subscription service; Bluemovement[3] offers household devices to its customers so they can use the appliance instead of owning it.

Furthermore, although the transition toward business models and supply chains that implement the Circular Economy (CE) concept is challenging (Bressanelli, Perona, and Saccani 2019), PaaS is known as a significant enabler of the life cycle extension of the product, and it increases resource efficiency through reusing goods (Patwa et al. 2021). However, implementing CE initiatives results in a more complicated inventory system due to the necessity of devising a precise inventory plan to deliver, collect, screen, repair, and recycle the products in a reasonable time and cost. Moreover, in subscribing models, the revenue of each period depends on many

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elements, such as the number of customers that subscribed in the previous stages and production quantity decisions of every stage, which complicates the inventory analysis (Huh, Kachani, and Sadighian 2010). In other words, although sustainable manufacturing practices can bring a competitive advantage to the company (Jayaraman, Singh, and Anandnarayan 2012), implementing a circular economy supply chain is challenging from the economic perspective and profitability (Genovese et al. 2017).

To the best of our knowledge, no article discusses the optimised inventory level of products in a PaaS model, considering circular economy practices and different product quality levels. Accordingly, this study investigates the inventory flow of a rental PaaS business model in which customers can rent the goods for a predefined period and pay subscription fees based on the product's quality level. In addition, the core firm remains responsible for screening and repairing products, recycling the end-of-life products, and selling the recycled materials to the supplier. The goal is to maximise the firm's profit by optimising order quantity for the new products concerning different operational costs related to ordering, holding, screening and repairing, and recycling operations. Moreover, this study compares the optimal order quantity of the PaaS model with the traditional Economic Order Quantity (EOQ) model to provide new insights into the impact of adopting CE practices on optimal decisions. This paper discusses various operational costs and conditions to help business owners adopt PaaS models to improve their profit and enhance their sustainability perfromance.

The contribution of this study is related to answering the following research questions: RQ1. What is the optimal order quantity of subscription products in a PaaS model to maximise profit considering adopting CE practices? RQ2. How does the adoption of CE practices in a PaaS model change its optimal order quantity, operational costs, and profit compared to the traditional EOQ model? RQ3. What is the role of the demand rates of different quality levels in optimising the profit of the circular PaaS model? RQ4. What is the role of the screening and recycling rates in optimising the profit of the circular PaaS model?

The results show that compared to the traditional EOQ model, the operational cost increases in the PaaS models. However, the firm has opportunities to increase its revenue by renting the products to a broader range of customers (reusing products) and selling recycled end-of-life products to the supplier. Besides, reducing the difference between the screening and demand rates decreases the holding and ordering costs. Additionally, the lower the difference between consecutive demand rates (demands of consecutive quality periods), the lower the holding and ordering costs. Moreover, improving the recycling rate of the end-of-life products when it is lower than the final collection rate will enhance the holding costs. Finally, the optimal order quantity of the PaaS model is always lower than the EOQ model, and under specific conditions, PaaS's profit outweighs the traditional model.

The remainder of this study is as follows. Section 2 provides the literature review and clarifies the research gap. Section 3 describes the optimisation model and formulates the related revenue and cost functions. Following this, sections 4 and 5 provide the model analysis with numerical examples to discuss the essential factors and conditions under which firms can mitigate their extra operational costs in PaaS models. Besides, section 6 provides further analysis through sensitivity analysis, and section 7 explains the main managerial insights that can be helpful for policymakers and business owners. Finally, section 8 concludes with future research directions.

2. Literature review

In both business-to-business (B2B) and business-to-consumer (B2C) models, companies have become more interested in offering services besides tangible products to improve their value propositions and revenue streams (Durugbo 2020). However, acquiring new business models that include additional services like maintenance and repair options (Barron 2023; Long et al. 2013; Saccani, Johansson, and Perona 2007), warranty and buyback contracts (Chen and Chen 2016; Gong, Lian, and Xiao 2022; Park, Jung, and Park 2020), efficient channel relationship (Mandi et al. 2022), and shortage penalty contracts (Sato, Yagi, and Shimazaki 2018) increases the operations management complexities.

One of the product-service models that is grabbing more attention from business owners is the PaaS model. Although there is a well-developed literature on inventory models, no model of PaaS operational decisions includes inventory analysis with CE practices. In PaaS models, customers do not buy the products, and instead of making a single upfront payment, they subscribe to the product and pay a recurring fee on a perpetual per-outcome basis (Ghobakhloo 2018). PaaS provides new opportunities for companies to extend their market and differentiate their offerings from those of their competitors by providing tailor-made services. Besides, consumers do not face the risks and responsibilities of owning the products and do not bear a noticeable initial payment (Kesavapanikkar, Amit, and Ramu 2023).

In the literature, Access-Based Product-Service Systems (AB-PSS) is a concept close to PaaS and allows consumers to use the products sequentially after each other, like renting shared bicycles or clothing for a fee without purchasing the products. This model, like PaaS, transfers the responsibility for maintaining the products to the provider. Tunn et al. (2021) study AB-PSS adoption barriers that matter to consumers, and the results show that the duration of use, the time a consumer obtains exclusive access to a specific product, product quality, and the type of product are some of the most critical factors that affect AB-PSS adoption. Moreover, as different customers rent the products consecutively, inventory management has a vital role in satisfying the customers in terms of availability and reducing the queue length and the average waiting time for renting or returning products. For example, docked public bicycle-sharing systems require strategies that balance different stations' inventory levels to minimize the rejected demands of bikes and empty lockers (Vishkaei et al. 2020; Vishkaei et al. 2021). However, in AB-PSS, the rental periods are usually shorter than in PaaS, and the customers use them for daily affairs like accessing shared bicycles and driving them to a distance. Therefore, in AB-PSS, firms do not need to screen the products after each use, but in the PaaS model, customers return the products after a longer period. Then, the subscription firm screens and repairs the products before sending them to other users.

The literature mainly discusses the concept, characteristics, opportunities, and challenges of subscription and PaaS models (Bischof, Boettger, and Rudolph 2020; Kesavapanikkar, Amit, and Ramu 2023; Lacy and Rutqvist 2015; Lindström, Maleki Vishkaei, and De Giovanni 20246) rather than analyzing the operational costs, inventory flow, and optimising order quantities. However, some articles study pricing strategies in subscription models in service and manufacturing sectors (Danaher 2002; Jain and Kannan 2002; Kash, Key, and Zoumpoulis 2023; Penmetsa, Gal-Or, and May 2015; Wang, Dada, and Sahin 2019), and some articles discuss pricing for subscription models considering digital products (Alaei, Makhdoumi, and Malekian 2023; Avinadav and Levy 2023). Besides, some studies investigate the impact of subscription models on consumer ordering behaviour and delivery services (Wagner, Pinto, and Amorim 2021), the relationship between quality and price of reusable items in PaaS (Randhawa and Kumar 2008), and optimising the portfolio of subscribing items and subscription boxes (Bernstein and Guo 2023). Nevertheless, no article studies the impact of CE practices on the inventory flow of physical products and operational costs of PaaS models.

Furthermore, calamities like deforestation, global warming, waste disposal, ocean acidification, and ozone depletion are forcing governments, firms, and customers to revise their production and consumption behaviour concerning circular economy strategies (De Giovanni and Folgiero 2023). The shift toward the circular economy requires a mindset change to implement circular business models and create economic value solutions that exploit waste in which the products and materials are recyclable, reusable, or last longer, possibly via offering product-service systems, instead of providing linear business models (Nujen et al. 2023). Moreover, the circular Economy goes beyond sustainable production and acts on responsible consumption. Accordingly, the PaaS business model can help reduce the consumption of raw materials for producing goods as it serves a larger group of individuals with the same resource (Dantas et al. 2021). In extended producer responsibility schemes like PaaS, the producer remains responsible for its product even in the post-consumer stage to implement CE practices (Van Engeland et al. 2020). However, as the core firm will be responsible for maintaining and replacing defective items, implementing an efficient product-service system model requires developing durable products and materials (Szwejczewski, Goffin, and Anagnostopoulos 2015).

In the literature, some articles study optimising CE strategies considering different decision variables. Shokohyar, Mansour, and Karimi (2014) develop an analytical model aligned with a multi-objective genetic algorithm that can help manufacturers analyze their sustainable product service system plans to optimise the environmental and economic impacts of the product during the consumption and end-of-life phase. Agrawal and Bellos (2017) study the financial and environmental potential of servicizing business models considering pure sales, pure servicizing, and a hybrid model with both sales and servicizing options. Their analysis indicates that structural characteristics such as pay-per-use pricing, the degree of pooling, and the type of business model, which can be hybrid or pure servicizing, significantly impact environmental performance.

Moreover, Rabta (2020) develops a circular indicator and defines the demand rate as a function of the proposed index to discuss the optimal order quantity, operational costs, and revenue, considering the circularity level as a decision variable. Reddy and Kumar (2021) develop a hybrid manufacturing-remanufacturing system in the circular economy for a firm that produces both new and re-manufactured products and sells them in primary and secondary markets in which re-manufacturing products does not cannibalise the demand for new products. They discuss inventory and capacity decisions based on the trade-off between penalty costs, holding costs, and disposal costs. Similarly, Cesur et al. (2022) study the remanufacturing of PVC products to maximise profitability by calculating the optimal number of re-manufacturing times considering demand, setup, and variable costs. The output shows that re-manufacturing makes sense if the demand exceeds a specific point. Khan et al. (2023) develop a model for a production system in which customers are socially responsible for their consumption practices. In addition, different operations in the production system, such as the setup and warehouse operations, are responsible for carbon emissions. They calculate the optimal production and circularity level to maximise profit and decrease negative environmental impacts.

The literature provides several studies about closed-loop business models to discuss the production and order quantity considering CE initiatives. Nevertheless, no article investigates the impact of CE practices on PaaS models' operations and inventory flow. Table A1 in Appendix (1) compares the most relevant articles in the literature with this study to clarify the research gap.

3. Model description

To describe the model, consider a firm offering its customers subscription products. For a specific product type, the firm considers n rental periods with predefined lengths, meaning there are n quality levels for that particular product. Customers interested in using products with quality level i, rent them with demand rate D_i, and return them in m_i months for a quality check and possible repair. The firm defines the renting fees as descending based on the level of quality. Moreover, the quality of the products decreases gradually till the end of their lives.

Remark 1: A lower quality level does not mean that the products are defective or in bad shape. It is only an indicator of how long or how many times previous consumers used them.

After each quality period, the firm is responsible for quality checking and repairing products before sending them to other users. The idea is to share products between different classes of consumers to extend the product's usability as a CE practice. Moreover, after n quality periods, the firm recycles end-of-life products and sells the recycled materials to the supplier.

Remark 2: The number of satisfied requests for second-hand items ($i = 2,3,\ldots,m$) depends on the demand for first-hand products because there will be no second-hand products if there is no demand for new products.

Other assumptions of the model are as follows:

Table 1. Notation Summary.

Notation	Definition
i	Index for indicating quality level or quality period
n	Number of quality levels/periods
m_i	Length of quality period i
D_i	Demand for the product with quality level i
r_i	Screening and repairing rate of the products with quality level i
R	Recycling rate
h	Holding cost per unit of product per unit of time
A	Ordering cost per order
p_i	Total subscription fees (renting price) for a product with quality level i
c_R	Recycling cost per unit of product
p_R	Price of recycled materials per unit of product for selling to the supplier
v_i	Screening and repairing cost for a product with quality level i
c	Supplier's unit product price
x_i	A binary parameter that equals one if $D_{i-1} \geq D_i$ ($i = 2, 3, \ldots, n$)
y_i	A binary parameter that equals one if $r_i \leqslant D_{i-1}$ ($i = 2, 3, \ldots, n$)
u_i	A binary parameter that equals one if $r_i \leqslant D_i$ ($i = 2, 3, \ldots, n$)
s	A binary parameter that equals one if $R < D_n$
Z	Total profit (objective function)
GR	Total revenue
TC	Total purchasing cost
TA	Total ordering cost
TH	Total holding cost
TS	Total screening and repair cost
TR	Total recycling cost
Q	Order quantity (decision variable)

- The firm orders new products from the supplier based on the demand for first-hand products. Demand rates for different quality levels may also differ.
- The model aims to satisfy the first-hand product demand and maximise responses to the needs of other quality levels.
- Customers pay subscription fees according to the quality levels and only for the periods they subscribe to a product.
- The firm predefines the length of the renting periods, and the screening rates may differ for each quality level.
- The firm recycles end-of-life products at a recycling rate (capacity) after n quality periods and sells the recycled material to the supplier.
- The model maximises the firm's profit by considering subscription and recycling revenue in addition to purchasing, holding, ordering, screening and repairing, and recycling costs.

Table 1 describes all the parameters, indices, and decision variables required for developing the mathematical formulations. The model requires one index to show the product's quality level or quality period indicated by i.

In the PaaS model, after renting the products with quality level i, the firm starts collecting them after m_i unit of time with the collection rate D_i. Simultaneously, the firm begins the screening procedure with the rate r_{i+1} with the unit screening cost, v_{i+1}. Then, it rents the products during the quality period $i + 1$ with a demand rate

D_{i+1} considering a unit subscription fee of p_{i+1}. Finally, after n quality levels, it recycles end-of-life products with the rate R, and the unit value of the recycled products equals p_R.

Remark 3: To satisfy the demand for first-hand products, the number of orders from the supplier equals $\frac{D_1}{Q}$. Considering n quality levels, the model includes n different quality periods besides one recycling period for each received cargo from the supplier. So, during the planning horizon, the model consists of $\frac{D_1}{Q} \times (n+1)$ periods in which $\frac{D_1}{Q} \times n$ of them belong to renting periods and n periods are related to recycling operation.

Figure 1 provides all possible conditions for quality level i. Accordingly, Figure 1(a) shows the first period when the firm receives Q number of products from the supplier and rents them with the demand rate D_1 during time $\frac{Q}{D_1}$. Figure 1(b) indicates the inventory level when $D_{i-1} \geq D_i$ and $r_i \geq D_i$. In this case, regardless of relationship between r_i and D_{i-1}, the firm collects the rented products with the rate D_{i-1}. Besides, as $r_i \geq D_i$, it starts renting them again with the rate D_i during time $\frac{Q}{D_{i-1}}$ till it collects all the products from the previous customers. So, first, the inventory level increases with the rate $D_{i-1} - D_i$; after that, it decreases with the rate D_i. In this condition, the highest level of the inventory reaches $\frac{Q(D_{i-1}-D_i)}{D_{i-1}}$.

Figure 1(c) depicts the inventory level when $r_i \leqslant D_i$ and $r_i \leqslant D_{i-1}$. So, compared to Figure 1(b), the inventory level increases with the rate $D_{i-1} - r_i$ considering $\frac{Q}{D_{i-1}}$ as the required time for receiving all the products from the previous users. Then, the firm rents the $\frac{Q(D_{i-1}-r_i)}{D_{i-1}}$ remaining items with the rate r_i. Accordingly, the total time for renting the products is $\frac{Q}{D_{i-1}} + \frac{Q(D_{i-1}-r_i)}{D_{i-1}r_i}$, and the maximum inventory level equals $\frac{Q(D_{i-1}-r_i)}{D_{i-1}}$.

Figure 1(d) discusses the inventory condition when $D_{i-1} < D_i$ and $r_i \geq D_{i-1}$. In this case, regardless of the relationship between r_i and D_i, all the products the firm receives from the previous customers will be immediately checked and sent to new customers, which leads to zero inventory. If $r_i \geq D_i$, the length of the required time for renting all the products with quality level i equals $\frac{Q}{D_i}$; otherwise, it equals $\frac{Q}{r_i}$. Finally, Figure 1(e and f) show the inventory level for the recycling period when the recycling rate, R, is greater and lower than the demand rate of the final quality period (D_n), respectively. Accordingly, $\frac{Q}{D_n}$ is the required time for receiving all the end-of-life products from the users, and $\frac{Q}{R}$ is the time needed to recycle Q items. Obviously, when $R < D_n$, it takes more time for the firm to recycle all the products, in which the additional time equals $\frac{Q}{R} - \frac{Q}{D_n}$.

Table 2. The area under the inventory function for conditions in Figure 1.

Condition	Calculations
a	$\frac{Q^2}{2D_1}$
b	$\frac{Q^2(D_{i-1}-D_i)}{2D_{i-1}^2} + \frac{Q^2(D_{i-1}-D_i)^2}{2D_{i-1}^2 D_i}$
c	$\frac{Q^2(D_{i-1}-r_i)}{2D_{i-1}^2} + \frac{Q^2(D_{i-1}-r_i)^2}{2D_{i-1}^2 r_i}$
d	0
e	$\frac{Q^2}{2D_n}$
f	$\frac{Q^2}{2D_n} + Q\left(\frac{Q}{R} - \frac{Q}{D_n}\right)$

Figure A1 in Appendix (2) presents the inventory flow in the closed-loop PaaS model, considering screening, reusing, and recycling procedures. In PaaS model, the total purchasing cost equals the demand for the new products multiplied by the unit cost ($D_1 \times c$). The total ordering cost and recycling costs equal $\frac{D_1 A}{Q}$ and $D_1 \times c_R$, respectively. The screening and repair cost for each quality period is different, and its total cost equals $\frac{D_1}{Q}\sum_{i=2}^{n} Qv_i$, which results in $D_1 \sum_{i=2}^{n} v_i$. Moreover, Equation (1) calculates the total revenue in which the first term determines the revenue from recycling, and the latter indicates the revenue from subscription fees.

$$GR = D_1 p_R + \frac{D_1}{Q}\sum_{i=1}^{n} Qp_i = D_1\left(p_R + \sum_{i=1}^{n} p_i\right) \quad (1)$$

Table 2 summarises the calculations related to the area under the inventory level functions for each condition in Figure 1, which is needed to formulate the total holding cost.

According to Table 2, Equation (2) calculates the total holding cost for $\frac{D_1}{Q}$ iterations. The first part shows the total inventory cost for the first-hand products (Condition 'a'). The second part calculates the total holding cost for other quality levels. When $x_i(1-u_i) = 1$ it means that quality period i follows condition 'b,' and if $u_i y_i = 1$, it follows Condition 'c.' In the third part of Equation (2), the binary parameter s distinguishes Conditions 'e' and 'f' for the recycling periods.

$$TH = \frac{hQ}{2} + \frac{hD_1}{Q}\sum_{i=2}^{n}\Bigg(x_i(1-u_i)\bigg(\frac{Q^2(D_{i-1}-D_i)}{2D_{i-1}^2} + \frac{Q^2(D_{i-1}-D_i)^2}{2D_{i-1}^2 D_i}\bigg) + u_i y_i\bigg(\frac{Q^2(D_{i-1}-r_i)}{2D_{i-1}^2} + \frac{Q^2(D_{i-1}-r_i)^2}{2D_{i-1}^2 r_i}\bigg)\Bigg) + \frac{hD_1}{Q}\left((1-s)\frac{Q^2}{2D_n} + s\left(\frac{Q^2}{2D_n}\right.\right.$$

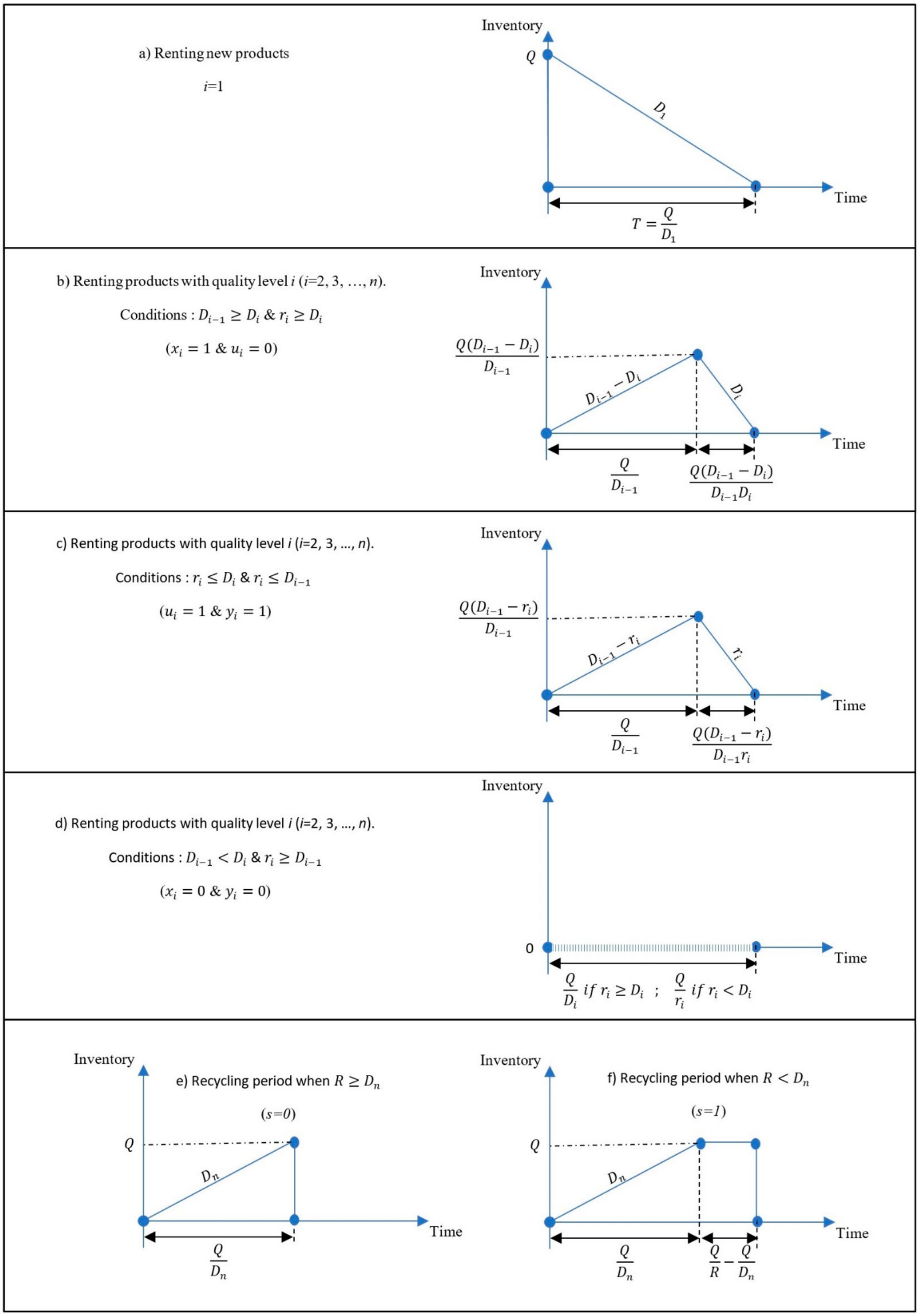

Figure 1. Inventory level for quality period *i* considering all possible conditions.
Source: Author.

$$+\,Q\left(\frac{Q}{R}-\frac{Q}{D_n}\right)\Bigg)\Bigg) \tag{2}$$

Equation (3) shows the final objective function (profit) considering revenue, purchasing costs, ordering costs, screening and repairing costs, holding costs, and recycling costs.

$$Z = GR - TC - TA - TS - TR - TH$$

$$= D_1\left(p_R + \sum_{i=1}^{n} p_i\right) - D_1 c - D_1 \sum_{i=2}^{n} v_i - \frac{D_1 A}{Q} - D_1 c_R - \frac{hQ}{2} - \frac{hD_1}{Q}\sum_{i=2}^{n}\left(x_i(1-u_i)\left(\frac{Q^2(D_{i-1}-D_i)}{2D_{i-1}^2} + \frac{Q^2(D_{i-1}-D_i)^2}{2D_{i-1}^2 D_i}\right) + u_i y_i\left(\frac{Q^2(D_{i-1}-r_i)}{2D_{i-1}^2} + \frac{Q^2(D_{i-1}-r_i)^2}{2D_{i-1}^2 r_i}\right)\right) - \frac{hD_1}{Q} \times \left((1-s)\frac{Q^2}{2D_n} + s\left(\frac{Q^2}{2D_n} + Q\left(\frac{Q}{R} - \frac{Q}{D_n}\right)\right)\right) \quad (3)$$

It is possible to revise Equations (3)–(4) considering the decision variable Q.

$$Z = D_1\left(\sum_{i=1}^{n} p_i - \sum_{i=2}^{n} v_i + p_R - c_R - c\right) - \frac{D_1 A}{Q} - \frac{hQ}{2} - hD_1 Q \sum_{i=2}^{n}\left(x_i(1-u_i) \times \left(\frac{(D_{i-1}-D_i)}{2D_{i-1}^2} + \frac{(D_{i-1}-D_i)^2}{2D_{i-1}^2 D_i}\right) + u_i y_i\left(\frac{(D_{i-1}-r_i)}{2D_{i-1}^2} + \frac{(D_{i-1}-r_i)^2}{2D_{i-1}^2 r_i}\right)\right) - hD_1 Q\left(\frac{(1-s)}{2D_n} + s\left(\frac{1}{R} - \frac{1}{2D_n}\right)\right) \quad (4)$$

Equation (5) indicates the final objective function in which $E = \sum_{i=1}^{n} p_i - \sum_{i=2}^{n} v_i + p_R - c_R - c$ (it does not depend on the order quantity), $F = D_1 \sum_{i=2}^{n} x_i(1-u_i)\left(\frac{(D_{i-1}-D_i)}{2D_{i-1}^2} + \frac{(D_{i-1}-D_i)^2}{2D_{i-1}^2 D_i}\right) + u_i y_i\left(\frac{(D_{i-1}-r_i)}{2D_{i-1}^2} + \frac{(D_{i-1}-r_i)^2}{2D_{i-1}^2 r_i}\right)$, and $G = D_1\left(\frac{(1-s)}{2D_n} + s\left(\frac{1}{R} - \frac{1}{2D_n}\right)\right)$ as fixed coefficients. So, the total profit equals Equation (5), noting that $H = h(1+2F+2G)$.

$$Z = ED_1 - \frac{hQ}{2}(1+2F+2G) - \frac{D_1 A}{Q} = ED_1 - \frac{HQ}{2} - \frac{D_1 A}{Q} \quad (5)$$

Lemma 1: *One necessary condition for a feasible circular PaaS is $c \geq p_R - c_R$.*

Proof: Appendix (3). ■

Lemma (1) emphasises that the recycling cost is lower than the selling price of the recycled materials ($p_R \geq c_R$); otherwise, the model is not economically feasible. Moreover, the supplier only purchases the recycled material if $c \geq p_R$.

Theorem 1: *The first necessary condition but insufficient for having a profitable PaaS model is $E \geq 0$. This means the revenue from the subscription fees and the recycled materials must compensate for the screening and purchasing costs.*

Proof: Appendix (3). ■

According to Theorem (1), there is no feasible solution for the PaaS model if the revenue cannot compensate for the screening and purchasing costs. However, sufficient conditions occur when the revenue from the subscriptions and recycled materials compensates for the total operation costs related to screening, purchasing, holding, and recycling.

Lemma 2: *Both the coefficients F and G always acquire positive values.*

Proof: Appendix (3). ■

According to the model's main assumptions (see Lemma 2 in Appendix 3), coefficients F and G are greater than zero. This lemma is essential in analyzing the optimal order quantity and the operational costs in the following theorems.

Theorem 2: *Considering the order quantity (Q) as the decision variable, maximizing the total profit in the PaaS model is related to minimizing $\frac{hQ}{2}(1+2F+2G) + \frac{D_1 A}{Q}$. Accordingly, the optimal order quantity equals $\sqrt{\frac{2D_1 A}{h(1+2F+2G)}}$.*

Proof: Appendix (3). ■

Equation (6) shows the optimal order quantity for the new products (Q^*) considering Equations (7) and (8). Interpreting H as the unit holding cost of the PaaS model, the optimal order quantity is lower compared to its equivalent value in the traditional EOQ model as $F > 0$ and $G > 0$ (see Lemma 2), which results in $H > h$.

$$Q^* = \sqrt{\frac{2D_1A}{H}} = \sqrt{\frac{2D_1A}{h(1+2F+2G)}} \quad (6)$$

In which:

$$F = D_1\sum_{i=2}^{n} x_i(1-u_i)\left(\frac{(D_{i-1}-D_i)}{2D_{i-1}^2} + \frac{(D_{i-1}-D_i)^2}{2D_{i-1}^2D_i}\right) + u_iy_i\left(\frac{(D_{i-1}-r_i)}{2D_{i-1}^2} + \frac{(D_{i-1}-r_i)^2}{2D_{i-1}^2r_i}\right) \quad (7)$$

$$G = D_1\left(\frac{(1-s)}{2D_n} + s\left(\frac{1}{R} - \frac{1}{2D_n}\right)\right) \quad (8)$$

The next section analyzes the cost and profit functions of the PaaS model compared to the EOQ model.

4. Model analysis

This section discusses coefficients F and G, to analyze their effects on the optimal solutions. As coefficient E does not depend on the decision variable, this study ignores further analysis of it.

Remark 4: In this section, to better explain the analysis, the prime symbol indicates the new parameters after making any changes in their values, and the double prime symbol distinguishes between the parameters of the EOQ model and the PaaS model.

4.1. Analyzing coefficient F

According to Equation (7), coefficient F is related to the holding cost for quality periods $i = 2, 3, \ldots, n$.

Lemma 3: *For the quality period i in which $y_iu_i = 1$ and $i > 1$, the smaller the difference between its screening rate (r_i) and its collection rate (D_{i-1}), which is $D_{i-1} - r_i$, the lower the coefficient F.*

Proof: Appendix (3). ■

Lemma 3 shows that in the PaaS model, when the renting period i satisfies condition 'c' and $D_{i-1} \geq r_i$, using a screening rate that is closer to the demand rate of the previous quality period can decrease coefficient F. This leads to decreasing the unit holding cost (H) and, consequently, based on Equation (6), increasing the optimal order quantity.

Theorem 3: *For the quality period i, if $u_iy_i = 1$ ($r_i \leqslant D_i$ and $r_i \leqslant D_{i-1}$), increasing the value of the screening rate (r_i) with the goal of reducing its difference with the collection rate (D_{i-1}), reduces the total holding and ordering costs in which $TH' = \sqrt{1 - \frac{hD_1}{r_i^2}}TH$ and $TA' = \sqrt{1 - \frac{hD_1}{r_i^2}}TA$ when one unit is added to r_i.*

Proof: Appendix (3). ■

This theorem indicates that subscription firms must improve their screening rates considering the return rates of different quality periods to reduce the storage time of subscription items at the warehouse. In other words, when $u_iy_i = 1$, a significant difference between the collection and screening rates causes an increment in the inventory level, as well as the holding and ordering costs.

Lemma 4: *For the quality period i in which $x_i(1 - u_i) = 1$ and $i > 1$, the lower the difference between its demand rate (D_i) and the demand of the previous period (the collection rate of period i, D_{i-1}), the lower the coefficient F.*

Proof: Appendix (3). ■

This proves that for Condition 'b,' minimising the difference between the demand rates in consecutive inventory periods decreases coefficient F. Then, as $H = h(1 + 2F + 2G)$, it decreases the unit holding cost (H) and, consequently, increases the optimal order quantity based on Equation (6).

Theorem 4: *Considering $S_{dif} = \sum_{i=2}^{n} x_i(1-u_i)\frac{(D_{i-1}-D_i)}{2D_{i-1}^2} + \frac{(D_{i-1}-D_i)^2}{2D_{i-1}^2D_i}$, the more the difference between consecutive demand rates, the greater S_{dif} and the higher the total holding and ordering costs. Accordingly, $TH' = \sqrt{1 + \frac{2hD_1}{H}}TH$ and $TA' = \sqrt{1 + \frac{2hD_1}{H}}TA$ when one unit is added to S_{dif}.*

Proof: Appendix (3). ■

This emphasises that for condition 'b,' having smooth consecutive demand rates for quality periods decreases the holding and ordering costs. This point reveals the importance of defining quality periods with a reasonable balance between the demands of successive subscription periods to mitigate coefficient F and the unit holding cost (H).

4.2. Analyzing coefficient G

According to Equation (8), coefficient G, is related to the holding cost of the end-of-life products.

Lemma 5: *The increment of R when $R > D_n$ does not affect the value of G or H. Moreover, when the recycling rate is lower than the final demand rate ($R < D_n$), the unit holding cost (H) is higher compared to the situation that $R > D_n$. This leads to a lower value for the optimal order quantity, which increases the total ordering cost (TA).*

Proof: Appendix (3). ■

Lemma 6: *When $R < D_n$, the subscription firm can decrease coefficient G by increasing the recycling rate R. By adding one unit to R, G decreases with a value $\frac{D_1}{R^2}$, and consequently, H decreases with a value $\frac{2hD_1}{R^2}$.*

Proof: Appendix (3). ■

Lemmas (5) and (6) show the importance of defining the subscription firms' recycling rate according to the demand rate of the final quality period, which is the same as the collection rate of end-of-life products. The lower the difference between the recycling rate and the collection rate of recyclable products, the lower coefficient G and, consequently, the lower the unit holding cost (H).

Theorem 5: *When the recycling rate is lower than the demand rate of the final quality period ($R < D_n$), increasing the recycling rate (R) decreases both the total holding and ordering costs. Accordingly, the new total holding cost equals $\sqrt{\frac{HR^2-2hD_1}{HR^2}}TH$, and the new total ordering cost equals $\sqrt{\frac{HR^2-2hD_1}{HR^2}}TA$.*

Proof: Appendix (3). ■

According to Theorem (5), when the recycling rate is lower than the collection rate of the end-of-life products, the subscription firm needs more time to recycle them, which increases the total holding and ordering costs and decreases the optimal order quantity.

Theorem 6: *When $R > D_n$, decreasing the demand rate for the final quality period (D_n) increases the optimal order quantity and, consequently, decreases the holding and ordering costs.*

Proof: Appendix (3). ■

If $s = 0$ ($R > D_n$), the firm finishes recycling the end-of-life products when the inventory level reaches its maximum (see Figure 1(e)). So, as D_n has a reverse relationship with coefficient G ($G = \frac{D_1}{2D_n}$), it is crucial to have a good market for the final quality period to decrease this coefficient and consequently, the unit holding cost (H). Moreover, this will result in mitigating the holding and ordering costs.

4.3. Analyzing the optimal solution

Based on Equations (6)–(8), the optimal order quantity for the PaaS model, like the traditional EOQ model, hinges on the trade-offs between the ordering and the holding costs. In contrast, in the PaaS model, the holding cost per unit of product per unit of time depends on two other coefficients related to the screening and repairing periods (F) and the recycling periods (G). Therefore, H is a dummy unit holding cost, which equals $h(1 + 2F + 2G)$.

Theorem 7: *If $F = 0$ and $G = 0$, the optimal order quantity of the PaaS model equals the EOQ model; if $F > 0$ or $G > 0$, the optimal order quantity of the PaaS model is lower than that of the traditional EOQ model.*

Proof: Appendix (3). ■

When $F = 0$ and $G = 0$, it means the firm ignores the implementation of CE practices and there is no difference between the PaaS and EOQ models. However, if $F > 0$ or $G > 0$ (see Lemma 2), the unit holding cost in the PaaS model (H) outweighs the unit holding cost in the EOQ model (h), resulting in a reduction in the optimal order quantity. Figure 2 compares the optimal order quantity of the traditional EOQ model with the PaaS model. This figure shows that the optimal order quantity for PaaS is lower than traditional EOQ, and the total cost of it is greater than the EOQ model.

4.4. Analyzing the objective function

To compare the profit of the PaaS model with the EOQ model, consider Equations (9)–(11) as the average revenue, the average cost, and the average profit per unit of product in the PaaS model, respectively.

$$\alpha = p_R + \sum_{i=1}^{n} p_i \tag{9}$$

$$\beta = c_R + c + \sum_{i=2}^{n} v_i + \frac{\frac{hQ}{2} + \frac{D_1 A}{Q} + hQ(F+G)}{D_1}$$

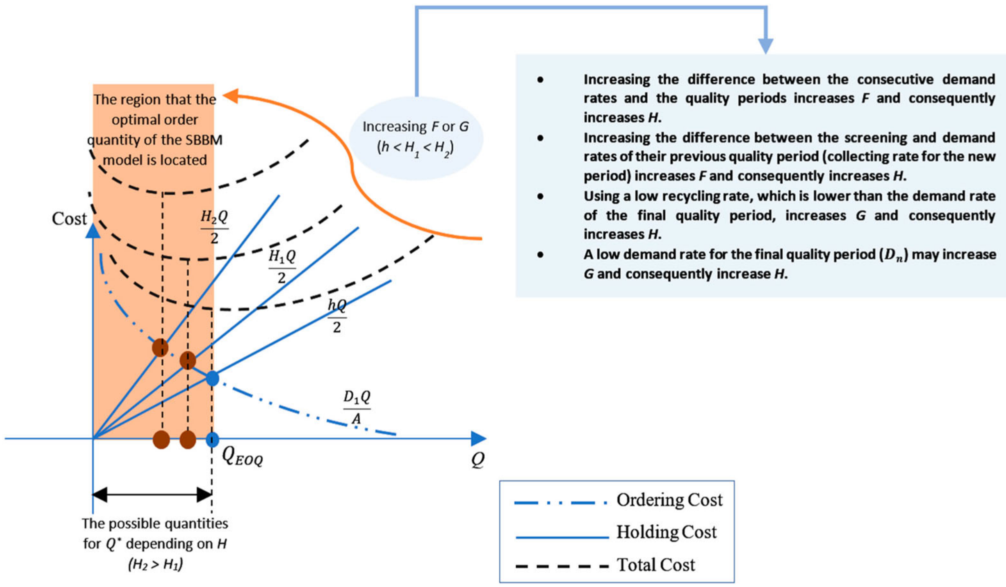

Figure 2. Comparing the optimal order quantity of PaaS with the EOQ model.
Source: Author.

$$= c_R + c + \frac{A}{Q} + \sum_{i=2}^{n} v_i + \frac{hQ(1+2F+2G)}{D_1} \quad (10)$$

$$\gamma = \alpha - \beta = p_R + \sum_{i=1}^{n} p_i - \sum_{i=2}^{n} v_i - c_R - c - \frac{A}{Q} - \frac{hQ(1+2F+2G)}{D_1} \quad (11)$$

Considering D'' as the demand, Q'' as the order quantity, α'' as the unit revenue, and β'' as the unit operational cost in the traditional EOQ model, its profit per unit of product (γ'') equals $\alpha'' - \beta''$ which leads to $\alpha'' - \frac{hQ''}{2D''} - \frac{A}{Q''} - c$. Note that the unit holding cost (h), the ordering cost (A), and the unit purchasing cost (c) are identical in both models. Accordingly, if $\gamma < \gamma''$, the firm can shift from the traditional model to the PaaS model only in the case that the demand increases in the PaaS ($D_1 > D''$) and can compensate for the lower revenue ($D_1\gamma > D'\gamma''$). Otherwise, the firm should increase its unit profit by increasing the subscription fees, reducing screening and recycling costs, and defining more accurate quality levels that smooth the demand rates of consecutive quality periods.

Theorem 8: *If the demand rate of new products is identical for both the traditional and PaaS models ($D_1 = D''$), the additional holding and ordering costs due to defining screening and recycling periods equals $\left(\sqrt{\lambda} - 1\right)(TH'' + TA'')$ considering $H = \lambda h$, $\lambda > 1$.*

Proof: Appendix (3). ■

In the PaaS model, the subscription firm has to deal with more inventory periods due to implementing CE practices, leading to additional holding costs compared to the EOQ model. Moreover, as $H > h$, the optimal order quantity is always lower in the PaaS model, leading to additional ordering costs.

Theorem 9: *The profit of the PaaS model will exceed the traditional model (PaaS will be a more economical model compared to EOQ) if $\left(\sqrt{\theta} - \sqrt{1+2F+2G}\right)\sqrt{2D_1Ah} + D_1(E - \theta(\alpha' - c)) > 0$ considering $D'' = \theta D_1, \theta > 0$.*

Proof: Appendix (3). ■

In terms of profit, the PaaS model outperforms the EOQ model, considering the conditions mentioned in Theorem (9), despite its additional operational costs due to implementing circular economy practices. Equation (12) calculates the optimal profit of PaaS, and Equation (13) calculates the difference between PaaS's profits and the traditional EOQ model.

$$Z^* = ED_1 - \sqrt{2D_1Ah(1+2F+2G)} \quad (12)$$

$$P_{extra} = Z^* - Z''^* = \left(\sqrt{\theta} - \sqrt{1+2F+2G}\right)\sqrt{2D_1Ah} + D_1(E - \theta(\alpha'' - c)) \quad (13)$$

Providing a more precise analysis requires considering the planning horizon to compare EOQ and PaaS models. The planning horizon for PaaS is longer because, after the final period of receiving the new products, the firm continues renting them till all of them are recycled. So, the

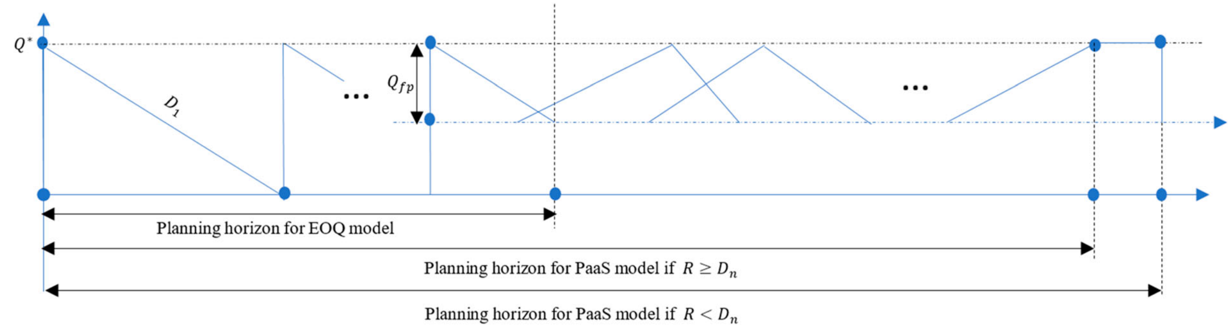

Figure 3. The planning horizon of the PaaS model compared to the EOQ model.
Source: Author.

extra time in PaaS depends on the final renting period of the new products. Figure 3 compares the planning horizon of the PaaS model with the traditional model.

The difference between the planning horizon of the two proposed models is related to the part of the final cargo (Q_{fp}) that is sufficient to cover the remaining demand of the year. Therefore, we consider the end-of-life of the Q_{fp}th item to calculate the planning horizon for PaaS.

Theorem 10: *The planning horizon of the PaaS model is longer than that of the traditional model, considering the same demand rate for the new products. The difference between the planning horizons equals* $\sum_{i=1}^{n} m_i + \sum_{i=2}^{n}\left(x_i(1-u_i)\frac{Q_{fp}(D_{i-1}-D_i)}{D_{i-1}D_i} + u_i y_i \frac{Q_{fp}(D_{i-1}-r_i)}{D_{i-1}r_i}\right) + s\left(\frac{Q_{fp}}{R} - \frac{Q_{fp}}{D_n}\right)$.

Proof: Appendix (3). ■

According to Theorem (10), the planning horizon of the PaaS model is longer than the EOQ model due to extra inventory periods for the subscription items. The additional inventory periods consist of different quality periods for reusing products besides the final period to recycle the end-of-life products. Accordingly, Equation (14) calculates the planning horizon of the PaaS model. Note that, T'_{PH} and T_{PH} are the planning horizons for the EOQ and PaaS models, respectively.

$$T_{PH} = T'_{PH} + \sum_{i=1}^{n} m_i + \sum_{i=2}^{n}\left(x_i(1-u_i)\frac{Q_{fp}(D_{i-1}-D_i)}{D_{i-1}D_i} + u_i y_i \frac{Q_{fp}(D_{i-1}-r_i)}{D_{i-1}r_i}\right) + s\left(\frac{Q_{fp}}{R} - \frac{Q_{fp}}{D_n}\right) \quad (14)$$

This equation shows that by reducing the difference between consecutive demand rates, decreasing the difference between the screening and collection rates, and considering a reasonable recycling rate, the difference between the planning horizon of the proposed models decreases. To be more precise in comparing the profit of the proposed models, firms can consider the profit per unit of time. So, if $\frac{D_1\gamma}{T_{PH}} > \frac{D''\gamma''}{T'_{PH}}$, the PaaS model outperforms the traditional model in terms of profit.

5. Numerical examples

Several cases in the real world can benefit from the results of this study to improve their business models concerning CE practices. For instance, Nuuly[4] provides subscription programmes to its consumers, which allow users to rent six styles every month and then return them to the company for quality check and repair and receive new items. Customers of MUD Jeans[5] can subscribe to pairs of jeans, and the subscription firm will remain responsible for reusing and recycling the garments. Moreover, Cyclon TM[6] provides a subscription programme for running shoes in which the users return the shoes to the company after three or six months to recycle them and subscribe to new items. In most of the case studies, the subscription period is fixed (e.g. three months and six months), which aligns with our assumption for defining a linear demand rate and predefined subscription periods.

This section discusses the optimal ordering model for a company that uses the PaaS model to rent child clothes for children under the age of 2 years old. Based on 'Kid to Kid Franchise,' one of the leading upscale kids' resale franchises, babies typically go up a clothing size around every ten weeks. This means infants might go through around seven clothing sizes in just their first two years of life. With regular wear, most children's clothing can be expected to last around two years, and play clothes might have a shorter lifespan (around one year). So, for the numerical examples, this study considers a garment that

Table 3. Costs and revenue parameters for the numerical examples.

Periods	p_i	v_i	c	c_R	p_R	A	h
Renting period 1	25	–	20	–	–	100	10
Renting period 2	18	0.5	–	–	–	–	10
Renting period 3	10	1	–	–	–	–	10
Renting period 4	5	1.5	–	–	–	–	10
Recycling period	–	–	–	0.5	2	–	10

Table 4. Numerical examples considering various demand rates.

Example	D_1	D_2	D_3	D_4
1	212	636	636	636
2	320	500	600	700
3	700	600	500	320
4	800	220	400	700
5	900	500	200	520
6	1000	100	600	420
7	1199	307	307	307
8	1300	380	300	140
9	1800	90	110	120
10	200	800	300	820
11	850	750	420	100
12	1000	400	500	220
13	630	720	330	440
14	650	470	420	580
15	530	530	530	530
16	530	400	550	640
17	530	850	650	90
18	530	300	450	840
19	530	700	100	790
20	530	220	1200	170

fits newborn babies, defining four quality levels (renting periods), each lasting 2.5 months (0.25 years). The population of Rome in Italy is around three million people, and the birth rate in Italy is anticipated to be 7.036 births per 1000 people, which results in 21180 newborn babies in Rome per year. Considering that the company can gain 10 percent of the market, the average demand rate for the product will be around 2120 per year. Table 3 shows the different costs and revenue parameters used in the numerical examples. Table 4 indicates 20 numerical examples considering different demand rates for the quality levels.

Moreover, the screening and repairing rates for the second, third, and fourth renting periods equal 350, 300, and 250 units per year ($r_2 = 350, r_3 = 300$, and $r_4 = 250$), respectively, and the recycling rate equals 220 units per year ($R = 220$). The screening rates decrease gradually as the products require a more precise check and repair after reusing. Table 5 shows the values of the binary parameters for all the numerical examples based on the relationships between the demand, screening, and recycling rates.

Table 6 shows the outputs of solving the examples indicating the optimal value of different costs, revenue, coefficients, and the final profit. Note that the last two columns show the length of the planning horizon for the PaaS model, which is more than one year (planning horizon for the EOQ model), and the average profit per unit of time, respectively. For instance, in the first numerical example, the planning horizon for the PaaS model is 2.2021 years, which results in a profit of 3226.8 per year. Considering 1000 units per year as the demand for the product for the EOQ model and the selling price of 30, its profit per year will equal 8586. So, in this case, the EOQ model outperforms the PaaS model. In the third example, the profit of the PaaS model per year equals 10351.26, which indicates a better performance for the PaaS model. In this example, although the demand rate for the new products in PaaS is lower than in the EOQ model, the profit is higher.

To better compare the examples, as the number of reused products depends on the number of first-hand products, the demand rates for the new products in the last six examples are identical. Accordingly, among them, example 17 has the highest value for the coefficient G because the demand rate of the final quality period is meagre compared to the other examples, which leads to a longer time for collecting the end-of-life products. After example 17, the highest value of coefficient G belongs to examples 18 and 19, respectively. The main reason is that their recycling rate is lower than the demand of the final period, and they need extra time to recycle all the products after collecting them.

Moreover, example 17 also has the highest value of coefficient F among the last six examples. The reason is the noticeable gap between the demand rates of its two final quality periods. Besides, there is a high difference between the screening and collection rates of its second and third renting periods.

6. Sensitivity analysis

Table 7 summarises the sensitivity analysis of the subscription firm considering the main parameters, coefficients, and different conditions. Accordingly, when the collection rate is greater than the screening rate, increasing the screening rate mitigates both holding and ordering costs besides increasing the optimal order quantity. Moreover, the more significant the difference between consecutive demand rates, the more the optimal holding and ordering cost and the lower the optimal order quantity. Additionally, when the recycling rate is lower than the demand rate of the final quality period, increasing the recycling capacity can reduce the firm's ordering and holding costs. Besides, it increases the optimal order quantity. Nevertheless, when the recycling rate is greater than the final demand rate, improving the market demand for the final period reduces the holding and ordering costs.

Table 5. Binary parameters of the numerical examples.

Example	x_2	x_3	x_4	y_2	y_3	y_4	u_2	u_3	u_4	s
1	0	1	1	0	1	1	1	1	1	1
2	1	1	1	1	1	1	1	1	1	1
3	1	1	1	1	1	1	1	1	1	1
4	1	0	0	1	0	1	0	1	1	1
5	1	1	0	1	1	0	1	0	1	1
6	1	0	1	1	0	1	0	1	1	1
7	1	1	1	1	1	1	0	1	1	1
8	1	1	1	1	1	1	1	1	0	0
9	1	0	0	1	0	0	0	0	0	0
10	0	1	0	0	1	1	1	1	1	1
11	1	1	0	0	0	0	0	0	1	0
12	1	0	1	1	1	1	1	1	0	0
13	0	1	0	1	1	1	1	1	1	1
14	1	1	0	1	1	1	1	1	1	1
15	1	1	1	1	1	1	1	1	1	1
16	1	0	0	1	1	1	1	1	1	1
17	0	1	1	1	1	1	1	1	0	0
18	1	0	0	1	1	1	0	1	1	1
19	0	1	0	1	1	0	1	0	1	1
20	1	0	1	1	0	1	0	1	0	0

Table 6. The optimal values of the PaaS model for different numerical examples.

Example	E	F	G	Q^*	TC	TR	TA	TS	GR	TH	Z^*	T_{PH}	Z/T_{PH}
1	36.50	0.44	0.80	35	4240.00	106.00	608.00	636.00	12720	608	6522.95	2.021	3226.80
2	36.50	0.54	1.23	38	6400.00	160.00	852.00	960.00	19200	852	9975.50	1.883	5298.45
3	36.50	1.78	2.09	40	14000.00	350.00	1749.00	2100.00	42000	1749	22051.40	2.130	10351.26
4	36.50	1.92	3.06	38	16000.00	400.00	2094.00	2400.00	48000	2094	25011.20	1.984	12605.31
5	36.50	2.14	3.23	39	18000.00	450.00	2297.00	2700.00	54000	2297	28256.50	1.993	14180.51
6	36.50	5.67	3.35	32	20000.00	500.00	3086.00	3000.00	60000	3086	30328.60	1.933	15693.89
7	36.50	1.94	3.50	45	23980.00	599.50	2669.00	3597.00	71940	2669	38425.70	1.937	19842.04
8	36.50	4.29	4.64	37	26000.00	350.00	3502.00	3900.00	78000	3502	40446.60	2.007	20156.78
9	36.50	9.50	7.50	32	36000.00	900.00	5612.00	5400.00	108000	5612	54475.00	2.042	26674.40
10	36.50	0.44	0.80	35	4240.00	106.00	608.00	636.00	12720	608	6522.95	2.250	2899.19
11	36.50	0.54	1.23	38	6400.00	160.00	852.00	960.00	19200	852	9975.50	2.245	4444.12
12	36.50	1.78	2.09	40	14000.00	350.00	1749.00	2100.00	42000	1749	22051.40	2.261	9754.11
13	36.50	1.92	3.06	38	16000.00	400.00	2094.00	2400.00	48000	2094	25011.20	2.302	10863.90
14	36.50	2.14	3.23	39	18000.00	450.00	2297.00	2700.00	54000	2297	28256.50	2.289	12345.88
15	36.50	1.20	1.91	38	10600.00	265.00	1383.00	1590.00	31800	1383	16578.70	1.957	8472.05
16	36.50	1.06	2.00	39	10600.00	265.00	1372.00	1590.00	31800	1372	16601.20	1.923	8631.40
17	36.50	3.37	2.94	28	10600.00	265.00	1900.00	1590.00	31800	1900	15545.40	2.000	7772.70
18	36.50	0.85	2.09	39	10600.00	265.00	1352.00	1590.00	31800	1352	16641.30	1.875	8875.41
19	36.50	2.53	2.07	32	10600.00	265.00	1644.00	1590.00	31800	1644	16056.10	2.180	7366.87
20	36.50	2.04	1.56	36	10600.00	265.00	1474.00	1590.00	31800	1474	16396.30	1.931	8492.72

Table 7. Sensitivity analysis for the subscription firm.

Parameter	Condition	Q^*	TH^*	TA^*
$r_i \uparrow$	$r_i \leqslant D_i$ and $r_i \leqslant D_{i-1}$	↑	↓	↓
$D_{i-1} - D_i \uparrow$	$D_{i-1} > D_i$	↓	↑	↑
$R \uparrow$	$R < D_n$	↑	↓	↓
$D_n \uparrow$	$R > D_n$	↑	↓	↓
$h \uparrow$	–	↓	↑	↑
$F \uparrow$	–	↓	↑	↑
$G \uparrow$	–	↓	↑	↑
$A \uparrow$	–	↑	↑	↑

Moreover, expanding the unit holding cost (h) or the ordering cost (A) adversely affects the total holding and ordering costs. However, the first condition decreases the order quantity, and the latter increases it. In other words, increasing h and A has the same impact as the traditional EOQ model.

Furthermore, increasing coefficients G and F have the same impact as h because an increment in any of them will increase H and consequently lead to higher holding and ordering costs, in addition to decreasing the optimal order quantity.

To provide more explanations, we focus on example 1 to discuss the effects of the recycling rate, the screening rates, and the relationships between consecutive demand rates. In the proposed example, the recycling rate is lower than the collection rate of the end-of-life products ($D_4 = 636$ and $R = 220$, so $R < D_4$), which leads to $s = 1$. So, increasing the recycling rate can mitigate the unit holding cost as it decreases the coefficient G. Figure 4 indicates the impact of increasing the recycling rate on different costs and the total profit. The recycling rate (R) is in a range between 220 and 636, as raising it to greater than 636 does not affect the unit holding cost, and the condition shifts from 'f' to 'e' ($s = 0$). According to Figure 4, by increasing R, coefficient G and consequently, the unit holding cost (H) declines, leading to a decrease in the total holding

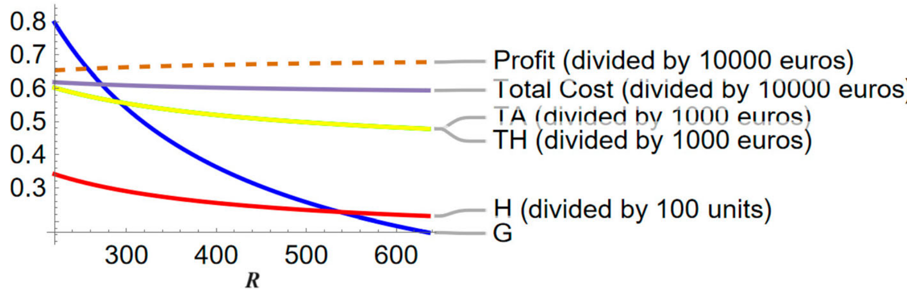

Figure 4. The impact of increasing the recycling rate on different coefficients, costs, and profit.
Source: Author.

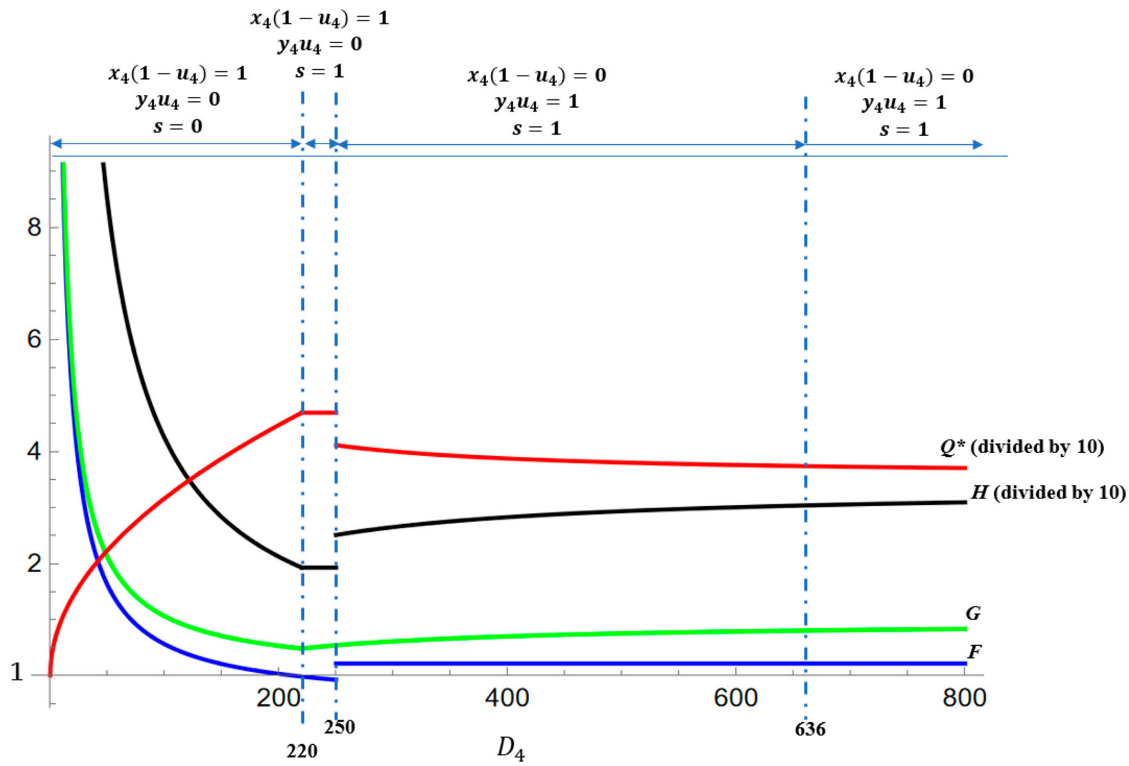

Figure 5. Analyzing the impact of the collection rate on the unit holding cost and the optimal order quantity.
Source: Author.

and ordering costs, and accordingly, the profit of the firm grows.

Figure 5 shows the impact of the collection rate of the end-of-life products (D_4) on coefficients F and G, as well as the unit holding cost (H) and the optimal order quantity (Q^*) in Example 1.

Analyzing the changes of D_4 can provide important information as it is possible to discuss it based on its relationship with the previous demand rate (D_3), with the screening rate (r_4), and with the recycling rate (R). Considering $D_4 < 220$, for this domain $D_3 \geq D_4$ and the higher D_4, the lower the difference between the two final demand rates and the lower G and F. This reduces the value of the unit holding cost and increases the optimal order quantity (Q^*). So, for this domain, the unit holding cost declines rapidly, and based on it there is a sharp growth in the optimal order quantity. For the domain [220, 250], Q^* does not change meaningfully. Besides, although G goes up because the recycling rate is lower than the final collection rate, coefficient F decreases due to the reduction between the consecutive demand rates. Therefore the decrement in F compensates for the increment in G, and Q^* remains steady. For $D_4 > 220$, the inventory flow of the recycling period follows Condition 'f' ($s = 1$) and the recycling rate is lower than the collection rate. So, coefficient G has a gradual growth because of the increment in the difference between the recycling rate (R) and the final collection rate (D_4). Besides, for $D_4 > 250$, nothing changes in the value of coefficient F because it satisfies Condition 'c' ($u_4y_4 = 1$), and the inventory flow depends on D_3 and r_4 but not D_4. Moreover, for this domain, the unit holding cost (H) grows moderately according to the growth rate of coefficient G. In addition, the optimal order quantity has a reverse relationship with the unit holding cost (H), and accordingly, it declines gradually based on the moderate growth of H.

7. Managerial insights

Transferring the business models to PaaS causes new challenges for business owners and policymakers despite PaaS's significant positive impacts in terms of profit and sustainability performance. On the one hand, there are challenges related to new complexities in different operations due to adopting CE practices like reusing, repairing, and recycling. On the other hand, PaaS models can mitigate environmental and social sustainability issues and improve economic sustainability performance through continuous revenue streams. According to the results of this study, the main managerial insights are as follows:

- The optimal order quantity in the PaaS is lower than the traditional model, considering identical demand rates for the first-hand products and the same unit holding and ordering costs. Moreover, the total holding costs in PaaS outweigh EOQ as there are more inventory periods for reusing and recycling subscription items. However, the PaaS model can outperform the traditional model in terms of profit by wisely devising the main parameters of the business model, such as screening, collection, and recycling rates, according to the following notes.
- In PaaS models, it is necessary to tactfully define the quality levels and subscription periods to minimize the difference between the consecutive demand and collection rates. The lower the difference between the demand of successive periods, the lower the total holding and ordering costs.
- Accurate screening capacity is essential in mitigating holding and ordering costs. Therefore, subscription firms must consider the minimum difference between the screening and collection rates. In other words, the lower the difference between the screening and the collection rates, the lower the total holding and ordering costs.
- The subscription firms must consider a reasonable recycling capacity for the final quality period to ensure a minimal difference between the recycling and the demand rates of end-of-life products. The lower the difference between the recycling and final demand rates, the lower the total cost.
- A lack of sufficient demand rate for the final period and products with the lowest quality level can adversely affect the holding and ordering costs. So, subscription firms must pay attention to reaching a good market for products close to their final stage of usage and preparing a sufficient recycling capacity for end-of-life products.

8. Conclusions

This paper studies the inventory flow of a PaaS subscription model in that customers rent the products based on the quality levels of the subscription items and return them after a predefined subscription period for quality check and possible repair. The quality and subscription fees of the products reduce gradually due to repetitive subscription periods. Finally, the firm recycles end-of-life products and sells the recycled materials to the supplier. Analyzing the operational costs and profit of PaaS is essential due to its more complicated inventory flow related to reverse logistics, reusing goods after screening and repairing procedures, and recycling damaged items. The screening periods lead to a higher holding cost, which depends on the difference in the consecutive demand rates of the quality levels as well as the difference between the screening rate of each period and the demand rate of the previous renting period. Moreover, the total total operations cost also depends on the relationship between the recycling rate and the final collection rate of the end-of-life products.

However, the subscription firm can compensate for the extra costs related to CE actions (e.g. screening, repairing, reusing, and recycling) through its two primary sources of revenue: subscription fees and the value of the salvaged materials. Comparing the traditional and PaaS models shows that besides the sustainability advantages through implementing CE actions, the subscription model can outperform the conventional model in terms of profit depending on defining precise values for the main parameters that have a significant impact on mitigating the total cost. So, devising quality periods with the minimum difference between the consecutive demand rates, adjusting the screening rates with the demand rates of the previous periods (collection rates), having sufficient customers for the final quality period (high collection rate), and using a proper recycling rate for the end-of-life products have essential roles in controlling the extra costs in PaaS model.

This study has several limitations and assumptions that will inspire future research based on them. First, this paper considers linear demand rates without shortages. So, developing the model by using nonlinear distributions and the possibility of shortage and backorder can be one of the topics for future research. Second, despite the positive impact of the model on improving CE practices, analyzing the adverse effects of reverse logistics on the environment can provide further insights into PaaS models. Third, this study does not discuss defective items that cannot be repaired and must be recycled before their end-of-life. This requires defining some fines for the customers who do not use the products properly, which can affect the revenue and recycling costs of the model. Fourth, investing in new technologies such as blockchain and AI for monitoring, tracing, and improving trust in PaaS models requires more investigation in the future. Fifth, in some subscription models, consumers can subscribe to a product portfolio rather than one item. Therefore, considering the possibility of renting more than one item can be an exciting research topic. Finally, as the retailer and supplier have a mutual relationship for selling products and recycled materials, analyzing the trade-offs between these two main partners using game theory methods is helpful to improve operations in PaaS models.

Notes

1. https://mudjeans.eu/pages/our-mission-about-us (accessed July, 2024).
2. https://www.signify.com/nl-nl (accessed July, 2024).
3. https://www.bluemovement.com/nl-nl (accessed July, 2024).
4. https://www.nuuly.com/ (accessed July, 2024).
5. https://mudjeans.com/ (accessed July, 2024).
6. https://www.on.com/en-it/collection/cyclon (accessed July, 2024).

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Disclosure statement

No potential conflict of interest was reported by the author(s).

Notes on contributor

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Data availability statement

The authors confirm that the data supporting the findings of this study are available within the article.

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Appendices

Appendix 1. Research gap

Table A1 compares the most relevant articles with this study considering: A. Servicizing, B. Discussing Subscription Concept, C. Subscription or PaaS Model Development and Analysis, D. Optimal Order or Manufacturing Quantity, E. Circular Economy Practices, F. Considering Quality Levels, G. Pricing Strategy, and H. Comparing the Profit and Operational Costs of Subscription Models with Traditional Models.

Appendix 2. Inventory flow of the PaaS model

Figure A1 summarises the main assumptions, parameters, decision variables, operations, costs, revenue flow, and inventory flow in the closed-loop PaaS model, considering screening, reusing, and recycling practices.

Appendix 3. Proofs

Lemma 1: *On the one hand, the firm recycles the materials only if $p_R \geq c_R$, and the supplier buys the recycled materials only if $c \geq p_R$; otherwise, it is not affordable for them. This results in $c > p_R - c_R$, which shows that recycling the products compensates only for a part of the purchasing cost.*

Theorem 1: *Based on Lemma (1), $p_R - c_R - c < 0$, which shows the recycled materials cannot compensate for the purchasing price, so it cannot cover the whole cost as well. Therefore, E can acquire positive or negative values depending on the subscription fees, recycled materials price, screening and repairing price, product price, and recycling cost. If $E < 0$, the revenue is insufficient to reach a positive profit for the PaaS model. So, $E > 0$ is necessary for a profitable model. Still, the firm needs to calculate the objective function Z to check whether there is a feasible (profitable) solution for the firm, depending on the total holding and ordering costs. However, $E > 0$ means $\sum_{i=1}^{n} p_i + p_R > \sum_{i=2}^{n} v_i + c_R + c$, which ensures the unit revenue exceeds the unit screening, purchasing, and recycling costs.*

Lemma 2: *The first term of the coefficient F equals zero if $x_i = 0$ or $u_i = 1$, and it will be a positive value if $x_i = 1$ ($D_{i-1} \geqslant D_i$). The second term will be zero if $u_i = 0$ and otherwise ($u_i = 1$ or $r_i \leqslant D_i$), it will be greater than zero. Therefore, it results in $F \geqslant 0$. It is evident that the first term of G is always positive, and its second term is nonzero if $s = 1$, which signifies $R < D_n$ and accordingly $R < 2D_n$, which results in $\frac{1}{R} > \frac{1}{2D_n}$ and consequently, $G \geq 0$.*

Theorem 2: *As $F \geq 0$ and $G \geq 0$ (Lemma 2) results in $H = h(1 + 2F + 2G) \geq 0$. Moreover, H can be interpreted as holding cost per unit of product per unit of time in the PaaS model. In Equation (5), the first term (ED_1) does not depend on the decision variable Q, so to maximize the revenue, we need to minimize $\frac{hQ}{2}(1 + 2F + 2G) + \frac{D_1 A}{Q}$. Calculating $\frac{\partial Z}{\partial Q} = 0$ results in $Q^* = \sqrt{\frac{2D_1A}{H}} = \sqrt{\frac{2D_1A}{h(1+2F+2G)}}$ in which $F = D_1 \sum_{i=2}^{n} x_i(1 - u_i)\left(\frac{(D_{i-1}-D_i)}{2D_{i-1}^2} + \frac{(D_{i-1}-D_i)^2}{2D_{i-1}^2 D_i}\right) + u_i y_i \left(\frac{(D_{i-1}-r_i)}{2D_{i-1}^2} + \frac{(D_{i-1}-r_i)^2}{2D_{i-1}^2 r_i}\right)$ and $G = D_1\left(\frac{(1-s)}{2D_n} + s\left(\frac{1}{R} - \frac{1}{2D_n}\right)\right)$.*

Lemma 3: *This Lemma discusses Condition 'c' in Figure 1(c). Considering Equation (7) with $x_i(1 - u_i) = 0$ and $u_i y_i = 1$ for period i ($i > 1$), we define two different values for r_i that are B and C in which $B \leqslant C$. So, $D_{i-1} - B \geq D_{i-1} - C$ and as a results $(D_{i-1} - B)/2D_{i-1}^2 \geq (D_{i-1} - C)/2D_{i-1}^2$, and accordingly, $(D_{i-1} - B)^2/2D_{i-1}^2 B \geq (D_{i-1} - C)^2/2D_{i-1}^2 C$.*

Theorem 3: *Based on Lemma (3), for a quality period that follows Condition 'c' in Figure 1(c), decreasing the difference between its screening and collection rates will reduce coefficient F. In other words, $\frac{\partial F}{\partial r_i} = \frac{-D_1}{2r_i^2}$ and $\frac{\partial H}{\partial r_i} = \frac{-hD_1}{r_i^2}$ and accordingly, $H' = H - \frac{hD_1}{r_i^2}$. Moreover, $\frac{Q'}{Q} = \sqrt{\frac{H}{H'}}$, $\frac{TH'}{TH} = \frac{H'Q'}{HQ} = \sqrt{\frac{H'}{H}}$, and $\frac{TA'}{TA} = \frac{Q}{Q'} = \sqrt{\frac{H'}{H}}$. Accordingly, $TH' = \sqrt{1 - \frac{hD_1}{r_i^2}}\,TH$ and $TA' = \sqrt{1 - \frac{hD_1}{r_i^2}}\,TA$. As $1 - \sqrt{\frac{hD_1}{r_i^2}} < 1$, the holding cost and ordering cost decreases when the difference between the screening and collection rates declines. Note that in Condition 'c,' the maximum possible increment in the screening rate equals $D_{i-1} - r_i$; after that, the condition changes.*

Lemma 4: *This Lemma discusses Condition 'b' in Figure 1(b) in which $D_{i-1} > D_i$. When the demand rate of quality period i (D_i,*

Table A1. Comparing this study with the most relevant article.

Article	A	B	C	D	E	F	G	H
Kesavapanikkar, Amit, and Ramu (2023)	✓	✓						
Lacy and Rutqvist (2015)	✓	✓						
Bischof, Boettger, and Rudolph (2020)	✓	✓						
Lindström, Maleki Vishkaei, and De Giovanni (20246)	✓	✓						
Jain and Kannan (2002)			✓				✓	
Penmetsa, Gal-Or, and May (2015)			✓				✓	
Wang, Dada, and Sahin (2019)			✓				✓	
Danaher (2002)			✓				✓	
Kash, Key, and Zoumpoulis (2023)			✓				✓	
Avinadav and Levy (2023)			✓				✓	
Alaei, Makhdoumi, and Malekian (2023)			✓				✓	
Randhawa and Kumar (2008)			✓				✓	
Shokohyar, Mansour, and Karimi (2014)	✓			✓	✓			
Agrawal and Bellos (2017)	✓				✓		✓	
Rabta (2020)				✓	✓			
Reddy and Kumar (2021)	✓			✓	✓			
Cesur et al. (2022)	✓			✓	✓			
Khan et al. (2023)				✓	✓			
This Study		✓	✓	✓	✓	✓		✓

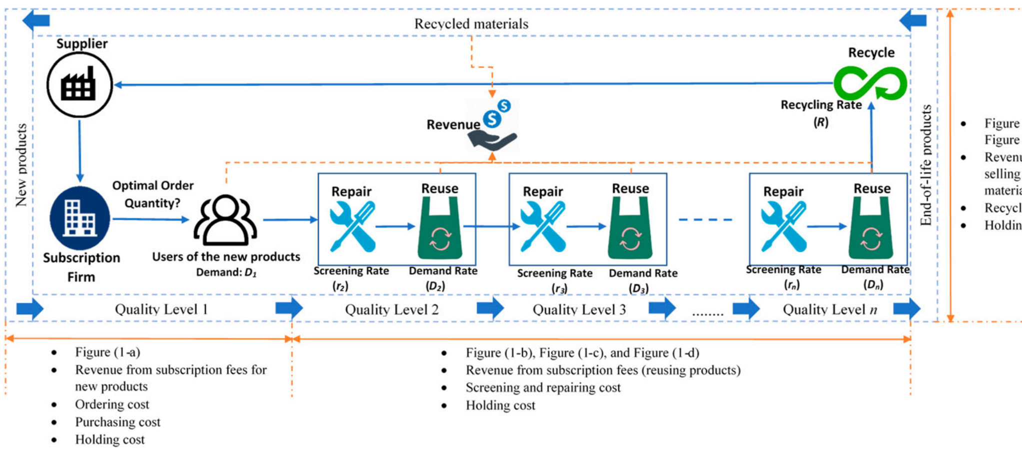

Figure A1. Inventory flow of the PaaS model.
Source: Author.

$i > 1$) is closer to its collection rate (D_{i-1}), coefficient F will have a lower value. To analyze the effects of this case, consider two different values for D_i, which are B and C, in which $B \leqslant C$. Then, substituting these values in Equation (7) considering $x_i(1-u_i) = 1$ and $u_i y_i = 0$, results in $D_{i-1} - B \geq D_{i-1} - C$ and accordingly, $(D_{i-1} - B)/2D_{i-1}^2 \geq (D_{i-1} - C)/2D_{i-1}^2$. Then, the conclusion will be $(D_{i-1} - B)^2/2D_{i-1}^2 B \geq (D_{i-1} - C)^2/2D_{i-1}^2 C$.

Theorem 4: *According to Lemma (4), increasing the difference between two consecutive demand rates in a quality period that satisfies Condition 'b' increases F. To analyze the effect of the difference between consecutive demand rates on the total holding and ordering costs, it is necessary to consider all quality periods simultaneously because changes in a demand rate like D_i must be analyzed for period i as the demand rate and for period $i+1$ as the collection rate. So, if the changes in the demand rates considering all quality periods increase S_{dif}, it will increase F and H as well. Accordingly, calculating the derivative of F with respect to S_{dif}, leads to $\frac{\partial F}{\partial S_{dif}} = D_1$ and $\frac{\partial H}{\partial S_{dif}} = 2hD_1$ which means $H' = H + 2hD_1$. Then, the new total holding and ordering costs will be $TH' = \frac{H'}{H}TH = \sqrt{1 + \frac{2hD_1}{H}}TH$ and $TA' = \frac{H'}{H}TA = \sqrt{1 + \frac{2hD_1}{H}}TA$. So, as $\sqrt{1 + \frac{2hD_1}{H}} > 1$, there is an increment in both of the costs.*

Lemma 5: *The minimum value of G is related to the situation that $R = D_n$, and in that case, $G = \frac{D_1}{2D_n}$. This value remains constant for $\forall R > D_n$ as it will be independent of R (considering $s = 0$). If $R < D_n$, more time is needed to recycle the products, which leads to an increment in coefficient G. In other words, based on Figure 1(e), it requires additional time, which equals $\frac{Q}{R} - \frac{Q}{D_n}$ to recycle the end-of-life products. So, the increment in G decreases the optimal order quantity due to increasing H. Moreover, it increases the total ordering cost as TA has a reverse relationship with the optimal order quantity.*

Lemma 6: *Considering $\frac{\partial G}{\partial R} = \frac{-sD_1}{R^2}$ and $\frac{\partial H}{\partial R} = \frac{-2shD_1}{R^2}$, condition $R < D_n$ ($s = 1$) results in $\frac{\partial G}{\partial R} = \frac{-D_1}{R^2}$ and $\frac{\partial H}{\partial R} = \frac{-2hD_1}{R^2}$. This proves that coefficients G and H will decrease ($G' = G - \frac{D_1}{R^2}$ and $H' = H - \frac{2hD_1}{R^2}$) when there is a unit increment in the recycling rate. Note that the maximum increment in the new recycling rate equals $D_n - R$.*

Theorem 5: *According to Lemmas (5) and (6), $H' = H - \frac{2hD_1}{R^2}$ which results in $\frac{TH'}{TH} = \sqrt{\frac{H'}{H}} = \sqrt{\frac{HR^2 - 2hD_1}{HR^2}}$ and as $H' = H - \frac{2hD_1}{R^2} > 0$ (because changes in R only affects G not the other parameters in H) then $\frac{H - \frac{2hD_1}{R^2}}{H} = \frac{HR^2 - 2hD_1}{HR^2} > 0$ which shows $\frac{TH'}{TH} > 0$. Therefore, as $HR^2 - 2hD_1 < HR^2$, then $0 < \sqrt{\frac{HR^2 - 2hD_1}{HR^2}} < 1$, which indicates a reduction in the total holding cost. Moreover, $\frac{TA'}{TA} = \sqrt{\frac{H'}{H}} = \sqrt{\frac{HR^2 - 2hD_1}{HR^2}} TA$ proves a reduction in the new total ordering cost.*

Theorem 6: *When $R > D_n$, then the recycling period follows Condition 'e,' and $G = \frac{D_1}{2D_n}$. So, according to the reverse relationships between G and D_n, by increasing the demand rate of the final period, G decreases and H decreases as well. Then as $\frac{Q'}{Q} = \sqrt{\frac{H}{H'}}$, $\frac{TH'}{TH} = \frac{H'Q'}{HQ} = \sqrt{\frac{H'}{H}}$, and $\frac{TA'}{TA} = \frac{Q}{Q'} = \sqrt{\frac{H'}{H}}$, it results in increasing the optimal order quantity, in addition to decreasing the holding and ordering costs.*

Theorem 7: *Considering $F = 0$ and $G = 0$ in Equation (6), the optimal order quantity of PaaS will be the same as the traditional model and equals $\sqrt{\frac{2D_1A}{h}}$. If $F > 0$ or $G > 0$, it signifies that there are screening or recycling periods, and in this case, the optimal order quantity of the PaaS will be lower than the traditional optimal order quantity ($\sqrt{\frac{2D_1A}{h(1+2F+2G)}} < \sqrt{\frac{2D_1A}{h}}$) as the unit holding cost increases ($H > h$).*

Theorem 8: *If $F > 0$ or $G > 0$, then $H > h$, which means $H = \lambda h$, $\lambda > 1$. Accordingly, comparing the total holding and ordering costs of the two models indicates that by increasing the unit holding cost, the optimal ordering quantity decreases, but both the holding and ordering costs increase ($TH = \sqrt{\lambda} TH''$ and $TA = \sqrt{\lambda} TA''$). So, $TH + TA - TH'' - TA'' = \left(\sqrt{\lambda} - 1\right)(TH'' + TA'')$ which shows the extra holding and ordering costs of the PaaS model.*

Theorem 9: *Proving this theorem requires comparing the profits of the two proposed models, which means comparing $D_1\gamma$ and $D''\gamma''$. Then, considering $Q = \sqrt{\frac{2D_1A}{h(1+2F+2G)}}$ and $Q'' = \sqrt{\frac{2D''A}{h}}$, the optimal profit for the traditional EOQ model equals $D''(\alpha'' - c) - \sqrt{2D''Ah}$, and by substituting Equation (6) in Equation (4), the optimal profit of PaaS (Z^*) equals $ED_1 - \sqrt{2D_1Ah(1 + 2F + 2G)}$. Subtracting Z''^* from Z^* leads to the difference between the profits (P_{extra}). So, considering $D'' = \theta D_1$, then $P_{extra} = Z^* - Z''^* = \left(\sqrt{\theta} - \sqrt{1 + 2F + 2G}\right)\sqrt{2D_1Ah} + D_1(E - \theta(\alpha'' - c))$. Accordingly, if $P_{extra} > 0$, the PaaS outperforms EOQ in terms of profit.*

Theorem 10: *In the PaaS model, the final new item will be rented at the end of the period T'_{PH} (end of planning horizon for EOQ), and the customer returns it after the renting time m_1. Then, if $x_2 = 1$ and $u_2 = 0$, it takes $\frac{Q_{fp}(D_1 - D_2)}{D_1D_2}$ (see Figure 1(b)) and if $u_2 = 1$ and $y_2 = 1$, it takes $\frac{Q_{fp}(D_1 - r_2)}{D_1r_2}$ (see Figure 1(c)) to rent the proposed product to another customer. Moreover, if $x_2 = 0$ and $y_2 = 0$, it will be rented on the same day the firm receives it from the previous customer (Figure 1(d)). This cycle continues until the product lifetime is over, and it takes time $\frac{Q_{fp}}{R} - \frac{Q_{fp}}{D_n}$ to recycle it if $s = 1$. Otherwise ($s = 0$), the firm recycles the end-of-life product on the same day it receives it from the user. Accordingly, the difference between the planning horizons equals $\sum_{i=1}^{n} m_i + \sum_{i=2}^{n}\left(x_i(1 - u_i)\frac{Q_{fp}(D_{i-1} - D_i)}{D_{i-1}D_i} + u_iy_i\frac{Q_{fp}(D_{i-1} - r_i)}{D_{i-1}r_i}\right) + s\left(\frac{Q_{fp}}{R} - \frac{Q_{fp}}{D_n}\right)$.*