# OPTIMAL HIGH-DIMENSIONAL AND NONPARAMETRIC DISTRIBUTED TESTING UNDER COMMUNICATION CONSTRAINTS 

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#### Abstract

We derive minimax testing errors in a distributed framework where the data is split over multiple machines and their communication to a central machine is limited to $b$ bits. We investigate both the $d$ - and infinite-dimensional signal detection problem under Gaussian white noise. We also derive distributed testing algorithms reaching the theoretical lower bounds.

Our results show that distributed testing is subject to fundamentally different phenomena that are not observed in distributed estimation. Among our findings, we show that testing protocols that have access to shared randomness can perform strictly better in some regimes than those that do not. We also observe that consistent nonparametric distributed testing is always possible, even with as little as 1-bit of communication and the corresponding test outperforms the best local test using only the information available at a single local machine. Furthermore, we also derive adaptive nonparametric distributed testing strategies and the corresponding theoretical lower bounds.


1. Introduction. Distributed methods are concerned with inference in a framework where the data resides at multiple machines. Such settings occur naturally when data is observed and processed locally, at multiple locations, before sent to a central location where they are aggregated to obtain a final result. By working with smaller sample sizes locally distributed methods can substantially speed up the computation compared to centralized, classical methods. Furthermore, they reduce memory requirements and help protecting privacy by not storing all the information at a single location. For these reasons, the study of distributed methods has attracted significant attention in recent years.

In our analysis we first consider the many normal means model, which is often used as a platform to investigate more complex statistical problems. In the classical version of the model one obtains an observation $X$ subject to the dynamics $X=f+n^{-1 / 2} Z$, where $f \in \mathbb{R}^{d}$ is an unknown signal, and $Z$ an unobserved, $d$-dimensional standard normal noise vector. This is equivalent to observing $n$ independent copies of a $N_{d}\left(f, I_{d}\right)$ vector. Our focus is on testing the absence or presence of the signal component $f$ in the model. Rejecting the null hypothesis $H_{0}: f=0$ means declaring that there is a non-zero signal underlying the observation $X$. The difficulty of distinguishing between the two hypotheses depends on signal strength, the noise ratio $n$ and dimension $d$. It is well known that the signal strength in terms of the Euclidean norm of $f$ needs to be at least of the order $d^{1 / 4} / \sqrt{n}$ for the hypotheses to be distinguishable, see e.g. [6].

We study this signal detection problem in a distributed setting. In the distributed version of the above normal-means model, the $n$ observations are divided over $m$ machines (assuming

[^0]without loss of generality that $n$ is a multiple of $m$ ). Equivalently, each local machine $j \in$ $\{1, \ldots, m\}$ observes
\[

$$
\begin{equation*}
X^{j}=f+\sqrt{\frac{m}{n}} Z^{j}, \tag{1}
\end{equation*}
$$

\]

where $f \in \mathbb{R}^{d}$ and the noise vectors $Z^{j}$ are independent $d$-dimensional standard normal random vectors. Each machine $j$ transmits a $b$-bit transcript $Y^{j}$ to a central machine. By aggregating these $m$ local transcripts, the central machine computes a test for the hypothesis $H_{0}: f=0$. We derive, for this distributed setting, the order of the minimal signal strength $\rho$ for which the null hypothesis can be distinguished from the alternative $H_{1}:\|f\|_{2} \geqslant \rho$. In the distributed setting, $\rho$ is considered as a function of the number of machines $m$ and the communication budget $b$, in addition to the dimension $d$ and noise level $n$. We allow all the parameters $b, m$ and $d$ to depend on $n$.

The transcripts generated by the machines may be either deterministic or randomized. When randomizing the transcript, we consider two different possibilities for the source of randomness. In the private coin setup, the machines may only use their own local (independent) source of randomness. In the public coin setup, the machines have access to a shared source of randomness in addition to their own independent source. This is akin to a situation in which the machines have access to the same random seed. We show that depending on the size of the communication budget, having access to a public coin strictly improves the distinguishability of the null- and alternative hypothesis.

Our results indicate that, in the case where $b$ and $m$ are small relative to the dimension $d$ in an appropriate sense, the one-bit protocols have similar properties, in terms of separation rate, as multi-bits protocols, i.e. one can achieve the minimax optimal $b$-bit testing rates with taking the majority vote of appropriately chosen local (one-bit) test outcomes. This is a striking difference with estimation, where for small values of $b$, increases in the communication budget result in (sometimes exponential) improvements in convergence rate. We find that, as $m$ increases, the local testing problems become more difficult as the local sample size deceases, but at a certain threshold, this effect is compensated for by the increase in total communication budget $b m$. This threshold occurs when $b m$ exceeds the dimension. At this point, we find that public coin protocols start to strictly outperform private coin protocols, in the sense that smaller signals can be detected with the same amount of transmitted bits $b$. This is also a dissimilarity with estimation, where having access to public randomness offers no benefit, as we show it in our paper. When the communication budget $b$ per machine exceeds that of the dimension $d$ of the problem, the minimax rates of the classical, non-distributed setting can be attained.

We then extend our results for the $d$-dimensional Gaussian model to the nonparametric signal in white noise setting. This latter model is of interest as it serves as benchmark and starting point to investigate more complicated nonparametric models. Here, the local observations for $j=1, \ldots, m$ constitute $\int_{0}^{j} f(s) d s+\sqrt{\frac{m}{n}} W^{j}$ where the $W^{j}$,s are independent Brownian motions and $f \in L_{2}[0,1]$ the unknown functional parameter of interest. Our results for the infinite dimensional model comes in the form of minimax rates for distributed protocols in terms of the strength of the signal in $L_{2}$-norm, the smoothness $s$ of the signal, the amount of bits $b$ allowed to be communicated by each machine, the signal to noise ratio $n$ and the number of machines $m$. In contrast to nonparametric distributed estimation, we show that consistent distributed testing is always possible, even when $m$ and $b$ are small. Having a shared source of randomness results in better rates in certain regimes in the nonparametric setting, whilst we show that this is never the case for distributed estimation. Finally, we consider the more realistic, adaptive setting where the regularity $s$ is considered to be unknown.

We show that in contrast to the non-distributed setting where the cost for adaptation is a multiplicative $\log \log n$ factor, in the distributed case a more severe $\log n$ penalty is necessary. We also propose a nonparametric distributed testing procedure based on Bonferroni's correction reaching the theoretical limits (up to a $\log \log n$ factor) and observe additional, unexpected phase transitions compared to the non-adaptive setting.
1.1. Related literature. Starting a few decades ago, earlier investigations into similar topics originate in the electrical engineering community, under the names "decentralized decision theory / the CEO problem" e.g. [38, 4, 39, 9, 25, 37] or "inference under multiterminal compression" (see [21] for an overview). Motivated by applications such as surveillance systems and wireless communication, the inference problems are approached from a "ratedistortion" angle in this body of literature. However, these results typically consider fixed, finite sample spaces and a fixed number of machines $m$ and investigate asymptotics only in the sample size $n$.

Understanding the fundamental statistical performance of distributed methods in context of non-discrete, higher-dimensional sample spaces has been considered only recently. Most of the literature focused on estimating the parameter/signal of the model in a distributed framework. Minimax lower and (up to a possible logarithmic factor) matching upper bounds were derived for the minimax risk in terms of communication constraints in context of the many normal means and simple parametric problems, see $[43,19,30,11,41,22,13,12]$. These results were extended to nonparametric models, including Gaussian white noise [44], nonparametric regression [32], density estimation [7] and general, abstract settings [36]. Distributed techniques for adapting to the unknown regularity of the functional parameter of interest were derived in [32, 33, 14].

For distributed testing, much less is known. In [1], the authors consider a setting in which each machine obtains a single observation from a distribution on a finite sample space and derive lower bounds for testing uniformity of this distribution. Similar distributed uniformity testing is considered in [2], where matching upper bounds are exhibited for this setting. In [34], the authors derive matching upper and lower bounds for the distributed version of the classical many normal means model (see (1) above) for the case that only the outcome of local tests can be communicated (e.g. 1-bit of communication). In [3] less stringent communication requirements are considered, in the special case of the model in (1) above with $m=n$. Questions regarding nonparametric models and adaptation in the setting of distributed testing have remained completely open thus far.

To summarize the state of the art, the lower bounds derived in the literature so far are only optimal in case of constant communication budget in the public coin setting, i.e. $b=O(1)$. So far no lower bound results are available in the public coin setting if $b$ can tend to infinity as $n$ increases. Furthermore, there is a lack of any lower bound result in the private coin setup. The traditional methods based on mutual information and Taylor expansion as considered in [34] and [3], respectively, do not extend to the setting of multiple bits or private coin protocols. In this article we fill this gap and derive the first rigorous minimax lower bounds for distributed testing procedures in the normal means model for arbitrary communication budget $b$ both for private and public coin settings. In order to prove the lower bounds, we provide a novel Bayesian testing argument based on a Brascamp-Lieb type inequality with distributed version of testing lower bounding techniques.

The upper bounds derived in [3] are more complete for both the private and public coin settings and go beyond the above described restrictive setting in which the lower bounds were derived, but do not cover all possible cases. For instance, in [3] it is assumed that the separation distance between the null and alternative hypotheses is bounded from above by one, which does not cover the case $\sqrt{d m} \gg n$. Also, only the $m=n$ case was considered in the
preceding paper. Therefore, in certain regimes new testing procedures and proof techniques had to be derived for full treatment of the problem (e.g. our novel test $T_{\text {III }}$ in the high-budget private coin case, see Section 4.3).

The literature on distributed testing has so far solely focused on finite dimensional models. We provide the first results for distributed testing in nonparametric models. Besides deriving lower and matching upper bounds we also derive an adaptive testing procedure, not depending on the typically unknown regularity of the underlying functional parameter of interest.
1.2. Overview of our results and organization. For a quick overview, the main contributions of this article are:

- Sharp minimax upper- and lower bounds for all values of $n, m, d, b$ for the $d$-dimensional distributed-signal-in-white-noise model, for both private and public coin settings (Section 3), with accompanying methods achieving these rates (Theorem 3.1 and Theorem 3.2).
- We extend the $d$-dimensional distributed-signal-in-white-noise model to the nonparametric setting where the signal is a Sobolev regular functional parameter of known regularity and establish the minimax rates within this setting for all values of $n, m, b$ for both the private and public coin settings (Theorem 6.1).
- We consider the nonparametric setting in which the regularity of underlying signal is unknown and derive adaptive private and public coin procedures. Furthermore, we establish private and public coin lower bounds for the adaptive setting that are tight up to a $\log \log n$ factor for all values of $n, m, b$ (Theorem 7.1 and Theorem 7.2).

The remainder of the paper is organized as follows. In Section 2 we describe the distributed-signal-in-white-noise model with $d$-dimensional signal $f \in \mathbb{R}^{d}$ and formalize the distributed testing problem both for private and public coin protocols. In Section 3 we provide the minimax lower and matching upper bounds for both testing protocols. Section 5 gives a sketch of the proof of the lower bound. We exhibit constructive algorithms that achieve matching upper bounds in Section 4. We extend our results to the nonparametric distributed-signal-in-white-noise model with Sobolev regular functional parameter in Section 6. Here, we also compare distributed testing and estimation rates and highlight the similarities and differences between them both in the private and public coin settings. In Section 7, we consider adaptation to the unknown regularity level in the nonparametric setting and present theoretical lower and matching upper bounds. In Section 8, we derive constructive algorithms achieving these upper bounds. The detailed proof of the lower bound for the $d$-dimensional signal is deferred to Section 9 and a key technical lemma described in Section 10. Detailed proofs for this lemma, as well as some of the technical details of the other main results and various auxilliary results, have been deferred to the Supplementary Material [35] to this manuscript. Results, equations and sections in the Supplementary Material are indexed by capital letters as opposed to numerals, as is used in the article.
1.3. Notation. We write $a \wedge b=\min \{a, b\}$ and $a \vee b=\max \{a, b\}$. For two positive sequences $a_{n}, b_{n}$ we use the notation $a_{n} \lesssim b_{n}$ if there exists a universal positive constant $C$ such that $a_{n} \leqslant C b_{n}$. We write $a_{n} \asymp b_{n}$ which holds if $a_{n} \lesssim b_{n}$ and $b_{n} \lesssim a_{n}$ are satisfied simultaneously. We shall use $a_{n} \gg b_{n}$ to denote $b_{n} / a_{n} \rightarrow 0$. The Euclidean norm of a vector $v \in \mathbb{R}^{d}$ is denoted by $\|\cdot\|_{2}$. For absolutely continuous probability measures $P \ll Q$, we denote by $D_{K L}(P \| Q)=\int \log \frac{d P}{d Q} d P$ and $D_{\chi^{2}}(P \| Q)=\int\left(\frac{d P}{d Q}-1\right)^{2} d P$ their Kullback-Leibler and Chi-square divergences, respectively. Throughout the whole text we use for convenience the abbreviation rhs and lhs for right-hand-side and left-hand-side, respectively, and cdf for the cumulative distribution function.
2. Problem formulation and setting. We consider testing in the distributed-signal-in-white-noise model. In this section, we provide the formulation of the distributed setup for data coming from the finite dimensional model. Except for obvious modifications to the sample space, the same setup is considered when the local data is from the infinite dimensional distributed-signal-in-white-noise model, which is formulated in Section 6. For $j=1, \ldots, m$ machines, the local observations constitute $X^{j}$ taking values in $\mathcal{X} \subset \mathbb{R}^{d}$, subject to dynamics (1) under $P_{f}$. Each machine $j$ communicates a $b$-bit transcript $Y^{j}$ to a central machine. That is, the transcript $Y^{j}$ takes values in some space $\mathcal{Y}^{j}$ with $\left|\mathcal{Y}^{j}\right| \leqslant 2^{b}$ for $b \in \mathbb{N}$. Let $Y=\left(Y^{1}, \ldots, Y^{m}\right)$ denote the aggregated data in the central machine. The central machine computes a test $T(Y)$, where $T$ is a map from $\mathcal{Y}:=\bigotimes_{j=1}^{m} \mathcal{Y}^{j}$ to $\{0,1\}$ that has to distinguish between the null hypothesis $f=0$ and the alternative hypothesis. As alternative hypothesis, we consider whether

$$
f \in H_{\rho}:=\left\{f \in \mathbb{R}^{d}:\|f\|_{2} \geqslant \rho\right\}
$$

for some appropriately chosen $\rho=\rho_{m, n, d, b}$.
We distinguish two mechanisms through which the local machines $j=1, \ldots, m$ can generate their transcripts $Y^{j}$. In the first setup, machines can use only their local observation $X^{j}$ when generating $Y^{j}$, possibly in combination with a local source of randomness. In the second setup, we allow the machines to access a common source of randomness $U$, which is independent of the data $X:=\left(X^{1}, \ldots, X^{m}\right)$. In the latter setup, which we call the public coin setting, the machines may use both local randomness, the observed draw of $U$ and their local observation $X^{j}$ when generating their transcript $Y^{j}$. The setup where only local randomness is available shall be referred to as the private coin setting. A formal definition of these two setups is as follows.

- A private coin distributed testing protocol consists of a map $T: \mathcal{Y} \rightarrow\{0,1\}$ and a collection of Markov kernels $K^{j}: 2^{\mathcal{Y}^{j}} \times \mathcal{X}^{j} \rightarrow[0,1], j=1, \ldots, m$, and the transcript satisfies $Y^{j} \mid X^{j} \sim K^{j}\left(\cdot \mid X^{j}\right)$.
- A public coin distributed testing protocol consists of a map $T: \mathcal{Y} \rightarrow\{0,1\}$, a random variable $U$ taking values in a probability space $\left(\mathcal{U}, \mathscr{U}, \mathbb{P}^{U}\right)$ and a collection of Markov kernels $K^{j}: 2^{\mathcal{Y}^{j}} \times \mathcal{X}^{j} \times \mathcal{U} \rightarrow[0,1], j=1, \ldots, m$, such that $Y^{j} \mid\left(X^{j}, U\right) \sim K^{j}\left(\cdot \mid X^{j}, U\right)$.
The choices for the kernels induce the conditional distribution of $Y=\left(Y^{1}, \ldots, Y^{m}\right)$, which we will denote $K:=\bigotimes_{j=1}^{m} K^{j}$. For the joint distribution of $X, Y$ and $U$ we shall write $\mathbb{P}_{f, K} \equiv \mathbb{P}_{f}$, where the $f$ subscript indicates the dynamics underlying $X$ and the subscript $K$ is used to stress that the conditional distribution of $Y$ induced by the choice of kernels. Furthermore, we denote by $\mathbb{P}_{f}^{X}$ the corresponding marginal distribution of X, i.e. $\mathbb{P}_{f}^{X}=P_{f}$. Our distributed architecture in the public coin case then follows the following Markov chain structure at each local machine $j=1, \ldots, m$


Note that any private coin testing protocol can effectively be considered a public coin testing protocol for which $U$ has degenerate distribution, i.e. $U=u \in \mathcal{U}$ almost surely. In our proofs below, for the sake of compactness, we consider without loss of generality that the private coin setting implies $U$ has a degenerate distribution. When no confusion can arise, we will refer to a distributed testing protocol as "distributed test", and we will refer to the tuple $\left(T,\left\{K^{1}, \ldots, K^{m}\right\}, \mathbb{P}^{U}\right)$ by $T$ for ease of notation. We use $\mathcal{T}_{p r i v}(b)$ and $\mathcal{T}_{p u b}(b)$ to denote the classes of all private and public coin distributed tests, respectively, each with communication budget $b$ per machine.

We define the testing risk of a distributed test $T \equiv\left(T, K, \mathbb{P}^{U}\right)$ for the alternative hypothesis $H_{\rho}$ as the sum of the Type I and Type II errors, i.e.

$$
\begin{equation*}
\mathcal{R}\left(H_{\rho}, T\right):=\mathbb{P}_{0}(T(Y)=1)+\sup _{f \in H_{\rho}} \mathbb{P}_{f}(T(Y)=0) \tag{3}
\end{equation*}
$$

3. Minimax upper and lower bounds in the normal means model. Our main results come in the form of two theorems. The first establishes the lower bounds for the detection threshold for both the public- and private coin distributed tests. We provide the proof of this theorem in Section 9. The second theorem establishes the optimality of the lower bound posed in the first theorem by providing distributed tests in both the public and private coin cases which attain the respective rates posed by the lower bounds. These optimal distributed testing procedures are described in Section 4. We note that our results are not asymptotic in nature as they hold for every combination of $b, n, m$ and $d$, hence going beyond the classical parametric framework.

THEOREM 3.1. [Distributed testing lower bound] For each $\alpha \in(0,1)$ there exists a constant $c_{\alpha}>0$ (depending only on $\alpha$ ) such that if

$$
\begin{equation*}
\rho^{2}<c_{\alpha} \frac{\sqrt{d}}{n}\left(\sqrt{\frac{d}{b \wedge d}} \bigwedge \sqrt{m}\right) \tag{4}
\end{equation*}
$$

then in the public coin protocol case

$$
\inf _{T \in \mathcal{T}_{\text {pub }}(b)} \mathcal{R}\left(H_{\rho}, T\right)>\alpha \text { for all } n, m, d, b \in \mathbb{N} .
$$

Similarly, for

$$
\begin{equation*}
\rho^{2}<c_{\alpha} \frac{\sqrt{d}}{n}\left(\frac{d}{b \wedge d} \bigwedge \sqrt{m}\right) \tag{5}
\end{equation*}
$$

we have under the private coin protocol that

$$
\inf _{T \in \mathcal{T}_{\text {priv }}(b)} \mathcal{R}\left(H_{\rho}, T\right)>\alpha \text { for all } n, m, d, b \in \mathbb{N} .
$$

The approach to proving the lower bound theorem can be summarized as follows. We start out by lower bounding the testing risk by a type of Bayes risk, where the parameter $f$ is drawn from an adversarial prior distribution $\pi$. By taking $\pi$ to be Gaussian, we can exploit the conjugacy of the model in order to show that optimal transcripts are either invariant to the prior or "Gaussian" in an appropriate sense. After this, the results follow by data processing arguments that are geometric in nature. We defer a more elaborate sketch of the proof to Section 5 and the detailed proof to Section 9. The techniques used in this work are novel and drastically different than those used in [3,34], which provide tight bounds only in the 1-bit case.

The above theorem implies that if (4) holds, no consistent public coin distributed testing protocol with communication budget $b$ bits per machine exists for the hypotheses $H_{0}: f=0$ versus the alternative $H_{1}:\|f\|_{2} \geqslant \rho$. In other words, no public coin distributed test manages to consistently distinguish all signals from 0 if the signals are smaller than the rhs of (4). When considering only private coin distributed testing protocols, the detection threshold (5) is more stringent than the public coin threshold (4) for certain values of $d, m$ and $b$. Theorem 3.2 below affirms that, in these cases, the best private coin protocol have a strictly worse performance compared to the best public coin protocol.

Theorem 3.2. For each $\alpha \in(0,1)$ there exists a constant $C_{\alpha}>0$ (depending only on a) such that if

$$
\rho^{2} \geqslant C_{\alpha} \frac{\sqrt{d}}{n}\left(\sqrt{\frac{d}{b \wedge d}} \bigwedge \sqrt{m}\right)
$$

there exists $T \in \mathcal{T}_{\text {pub }}(b)$ such that

$$
\mathcal{R}\left(H_{\rho}, T\right) \leqslant \alpha \text { for all } n, m, d, b \in \mathbb{N}
$$

Similarly, for

$$
\rho^{2} \geqslant C_{\alpha} \frac{\sqrt{d}}{n}\left(\frac{d}{b \wedge d} \bigwedge \sqrt{m}\right)
$$

there exists $T \in \mathcal{T}_{\text {priv }}(b)$ such that

$$
\mathcal{R}\left(H_{\rho}, T\right) \leqslant \alpha \text { for all } n, m, d, b \in \mathbb{N}
$$

The achievability of arbitrarily small testing risk is shown using a constructive proof, see Section 4. That is, we derive distributed testing protocols that distinguish the null hypothesis from any $f \in \mathbb{R}^{d}$ in the alternative class.

Theorem 3.1 together with Theorem 3.2 establish the minimax distributed testing rate for public and private coin protocols. As a sanity check, note that when $m=1$, we obtain the nondistributed minimax testing rate $\rho^{2}=\sqrt{d} / n$. Furthermore, when $b \gtrsim d$, enough information about the coefficients can be communicated to obtain the non-distributed minimax rate also, for both the public coin and private coin distributed protocols. When the communication budget is smaller than the dimension $(b=o(d))$, the class of public coin protocols starts to exhibit strictly better performance than the private coin ones in scenarios as long as $d=$ $o(m b)$. That is, as long as the total communication budget $m b$ of the system exceeds the dimension $d$ of the parameter, public coin protocols achieve a strictly better rate than private coin ones. This remarkable phenomenon disappears when the dimension is larger than the total communication budget (i.e. $m b=o(d)$ ), at which point there exists a one-bit private coin protocol achieving the optimal rate of $\rho^{2}=\frac{\sqrt{m d}}{n}$ in both cases. Consistent distributed testing turns out to be possible even for small values of $b$ and $m$, as long as $n$ is large enough compared to $d$. This stands in contrast to estimation in the $d$-dimensional Gaussian mean model, where consistent estimation is not possible when $m b=o(d)$, regardless of sample size $n$ (see e.g. [13]). Furthermore, as long as $m b=o(d)$ in the public coin case or $m b^{2}=o\left(d^{2}\right)$ in the private coin case, an increase in communication budget does not lead to a better rate. This stands in stark contrast to estimation, where for small budgets an increase can lead to an exponential improvement in convergence rate.
4. Distributed testing protocols achieving the lower bound in the many normal means model. In this section, we exhibit three distributed testing procedures achieving the rates posed by the lower bound. The first distributed testing procedure $T_{\mathrm{I}}$ communicates only a single bit per machine and can detect signals with a squared Euclidean norm of larger or equal order than $\frac{\sqrt{d m}}{n}$ and does not need a public coin. As a second procedure, we consider a test satisfying the public coin protocol $T_{\text {II }}$ that achieves the rate $\frac{d}{n \sqrt{b \wedge d}}$. The third procedure satisfies the private coin protocol and achieves the corresponding slower rate $\frac{d}{n(b \wedge d)}$. Note that, depending on the values of $n, m, d$ and $b$, the existence of such distributed testing protocols proves Theorem 3.2 and implies that the lower bounds in Theorem 3.1 are in fact tight.

A common denominator in the construction of the three protocols is that the transcripts $Y^{j}$ are generated as vector of $p_{f}^{j}$-Bernoulli random variables taking values in $\{0,1\}^{b}$ where $p_{f}^{j} \in[0,1]^{b}$ depends on the underlying signal $f$, with $p_{f}^{j}=(1 / 2, \ldots, 1 / 2)$ under the null hypothesis $(f=0)$. The concentration inequality for groups of Bernoulli random variables given in Lemma 4.1 provides a recipe for the construction of a central test for each of the three regimes. The Type I error can be controlled since the distribution under the null hypothesis is known. The Type II error is small whenever the vectors of probabilities $p_{f}^{1}, \ldots, p_{f}^{m}$ are sufficiently separated from $(1 / 2, \ldots, 1 / 2)$ in Euclidean norm.

Lemma 4.1. Consider for $k, l \in \mathbb{N}, l \geqslant 2$, independent random variables $\left\{B_{i}^{j}: i=\right.$ $1, \ldots, k, j=1, \ldots, l\}$ with $B_{i}^{j} \sim \operatorname{Ber}\left(p_{i}\right)$. If $p_{i}=1 / 2$ for $i=1, \ldots, k$, it holds that for all $\alpha \in(0,1)$ there exists $\kappa_{\alpha}>0$ such that

$$
T:=\mathbb{1}\left\{\left|\frac{1}{\sqrt{k} l} \sum_{i=1}^{k}\left(\sum_{j=1}^{l}\left(B_{i}^{j}-\frac{1}{2}\right)\right)^{2}-\sqrt{k} / 4\right| \geqslant \kappa_{\alpha}\right\}
$$

satisfies $\mathbb{E} T \leqslant \alpha / 2$. On the otherhand, if

$$
\begin{equation*}
\eta_{p, l, k}:=\frac{l-1}{2 \sqrt{k}} \sum_{i=1}^{k}\left(p_{i}-\frac{1}{2}\right)^{2} \geqslant \kappa_{\alpha} \tag{6}
\end{equation*}
$$

it holds that

$$
\begin{equation*}
\mathbb{E}(1-T) \leqslant \frac{1 / 2+16 \eta_{p, l, k} / \sqrt{k}}{\eta_{p, l, k}^{2}} \tag{7}
\end{equation*}
$$

The proof of the lemma can be found in Section A. 2 of the Supplementary Material where it is restated as Lemma A.4. We also provide a version of this lemma (Lemma A. 5 in the Supplement) used in the high-budget private coin protocol case.
4.1. Low communication budget: construction of $T_{I}$. The protocol presented here is similar to the one given in [34], with some adjustment allowing the application of Lemma 4.1 for a simpler proof.

We first compute the local test statistic $S_{\mathrm{I}}^{j}=(n / m)\left\|X^{j}\right\|_{2}^{2}$ at every machine $j=1, \ldots, m$. Under the null hypothesis, $S_{\mathrm{I}}^{j}$ follows a chi-square distribution with $d$ degrees of freedom, i.e. $S_{\mathrm{I}}^{j} \sim \chi_{d}^{2}$. Letting $F_{\chi_{d}^{2}}$ denote $\chi_{d}^{2}$-cdf, the quantity $F_{\chi_{d}^{2}}\left(S_{\mathrm{I}}^{j}\right)$ can be seen as the p -value for the local test statistic $S_{\mathrm{I}}^{j}$. Based on these "local p-values", we then generate the randomized transcript $Y_{\mathrm{I}}^{j}$ for every $j$ using Bernoulli random variables:

$$
Y_{\mathrm{I}}^{j} \mid S_{\mathrm{I}}^{j} \sim \operatorname{Ber}\left(F_{\chi_{d}^{2}}\left(S_{\mathrm{I}}^{j}\right)\right)
$$

For a given $\alpha \in(0,1)$, we can construct the test

$$
\begin{equation*}
T_{\mathrm{I}}=\mathbb{1}\left\{\left|\frac{1}{m}\left(\sum_{j=1}^{m}\left(Y_{\mathrm{I}}^{j}-1 / 2\right)\right)^{2}-1 / 4\right| \geqslant \kappa_{\alpha}\right\} \tag{8}
\end{equation*}
$$

at the central machine. In applications, one could set for instance $\kappa_{\alpha}$ such that $\mathbb{P}_{0} T_{\mathrm{I}} \approx \alpha$ by considering that $\sum_{j=1}^{m} Y_{I}^{j}$ is $(m, 1 / 2)$-binomially distributed under the null. Lemma A. 6 in the Supplementary Material yields that for each $\alpha \in(0,1)$, there exist constants
$\kappa_{\alpha}, C_{\alpha}, M_{\alpha}, D_{0}>0$ such that for $m \geqslant M_{\alpha}$ and $d \geqslant D_{0}$ it holds that $\mathcal{R}\left(H_{\rho}, T_{\mathrm{I}}\right) \leqslant \alpha$, whenever $\rho^{2} \geqslant C_{\alpha} \frac{\sqrt{m d}}{n}$.

The case $m \leqslant M_{\alpha}$ corresponds essentially to the non-distributed setting and is treated separately for technical reasons. In practice, one would simply use the test given in (8) also for $m \leqslant M_{\alpha}$. Furthermore, if one allows for a slightly larger amount of bits (e.g. $\log _{2}(n)$ bits), one could opt to transmit an (approximation of) the test statistics $S_{\mathrm{I}}^{j}$ themselves, see e.g. Lemma 2.3 in [33], for which it is easy to prove that the rate of $\frac{\sqrt{m d}}{n}$ is achieved without requiring any assumptions on $m$. For the sake completeness: by considering $\rho^{2} \geqslant C_{\alpha} \sqrt{M_{\alpha}} \frac{\sqrt{d}}{n}$, we see that the optimal rate of $\frac{\sqrt{m d}}{n}$ can be achieved in the $m \leqslant M_{\alpha}$ case by simply taking

$$
\begin{equation*}
T_{\mathrm{I}}^{\prime}:=Y_{\mathrm{I}}^{1}:=\mathbb{1}\left\{\frac{1}{\sqrt{d}}\left(S_{\mathrm{I}}^{1}-d\right) \geqslant \kappa_{\alpha}\right\} \tag{9}
\end{equation*}
$$

for an appropriately large choice of the constant $\kappa_{\alpha}$. Similarly, the requirement that $d$ is larger than some constant $D_{0}$ (which is independent of $\alpha$ ) appears for technical reasons. The case where $d \leqslant D_{0}$ is covered by the private coin protocol $T_{\text {III }}$ in Section 4.3.
4.2. Public coin, high communication budget: construction of $T_{I I}$. We now switch our attention to exhibiting a testing procedure that is optimal when $b m \gtrsim d$ in the public coin case. That a shared source of randomness in distributed settings can be strictly better than private ones in terms of communication complexity, is an idea that goes back to [42]. Essentially, the use of shared randomness allows for the machines coordinate their efforts in "covering" each of the $d$ dimensions of the data even though all communication happens in just one round. See also e.g. Chapter 3 in [29] for an extensive treatment of this phenomenon. We adopt ideas proposed by [3], who consider the setting where $m=n$ with asymptotics in $m$. We exhibit this testing protocol below and provide a full proof covering also the case where $m \neq n$. To that extend, let $U$ be a random rotation, i.e. $U$ is drawn from the Haar measure (see e.g. Theorem F. 13 in [5]) on the set of orthonormal matrices in $\mathbb{R}^{d \times d}$. At each machine, for $b \leqslant d$, we can compute the $b$-bit transcript $Y_{\mathrm{II}}^{j} \in\{0,1\}^{b}$ conditionally on the shared public coin draw $U$, where each of the $1 \leqslant i \leqslant b$ components is defined through

$$
\left(Y_{\mathrm{II}}^{j}\right)_{i} \mid U, X^{j}=\mathbb{1}\left\{\left(\sqrt{n / m} U X^{j}\right)_{i}>0\right\},
$$

where $(v)_{i}$ denotes the projection onto the $i$-th coordinate of the vector $v \in \mathbb{R}^{d}$. The random rotation fulfills a similar purpose as the random reweighting algorithm proposed in [34], but leads to an easier proof in the $d$-dimensional case because of rotational invariance of the Gaussian distribution.

Centrally, after transmitting $\left(Y^{1}, \ldots, Y^{m}\right)$, we compute the aggregated test statistics $S_{\text {II }}=$ $\sum_{j=1}^{m} Y_{\mathrm{II}}^{j}$ and define the corresponding test as

$$
\begin{equation*}
T_{\mathrm{II}}=\mathbb{1}\left\{\left|\frac{1}{\sqrt{b} m} \sum_{i=1}^{b}\left(\left(S_{\mathrm{II}}\right)_{i}-\frac{m}{2}\right)^{2}-\sqrt{b} / 4\right|>\kappa_{\alpha}\right\} . \tag{10}
\end{equation*}
$$

Lemma A. 7 in the Supplementary Material shows that this test achieves the public coin lower bound when $m b \gtrsim d$ and $m \geqslant M_{\alpha}$.
4.3. Private coin, high total communication budget: constructing $T_{I I I}$. Finally, we consider the case of not having access to a public coin, but having a relatively large communication budget ( $b^{2} m \gtrsim d^{2}$ ). Note that we can assume without loss of generality that $m \geqslant M_{\alpha} d^{2} / b^{2}$ for a constant $M_{\alpha}>0$, as otherwise the optimal rate is $\sqrt{m d} / n$, obtained
by the 1-bit private coin test described by (8) (see Section 4.1). This case is the most involved one and we construct a test consisting two sub-tests optimal in different sub-regimes.

The most obvious approach in this case is to divide the communication budget of each machine over the $d$ coordinates as uniformly as possible. That is to say, to partition the coordinates $\{1, \ldots, d\}$ into approximately $d / b$ sets of size $b$ (we assume without loss of generality that $b \leqslant d$, as we can always throw away excess budget and $b=d$ bits suffices for achieving the minimax rate). The machines are then equally divided over each of these partitions and communicate the coefficients corresponding to their partition. More formally, such a strategy entails taking sets $\mathcal{I}_{i} \subset\{1, \ldots, m\}$ such that $\left|\mathcal{I}_{i}\right|=\left\lfloor\frac{m b}{d}\right\rfloor$ and each $j \in\{1, \ldots, m\}$ is in $\mathcal{I}_{i}$ for $b$ different indexes $i \in\{1, \ldots, d\}$. For $i=1, \ldots, d$ and $j \in \mathcal{I}_{i}$, generate the transcripts according to

$$
\begin{equation*}
Y_{i}^{j} \mid X_{i}^{j}=\mathbb{1}\left\{X_{i}^{j}>0\right\} . \tag{11}
\end{equation*}
$$

Centrally, a natural test based on these transcripts is

$$
\begin{equation*}
T_{\mathrm{III}}^{1}:=\mathbb{1}\left\{\left|\frac{1}{\left|\mathcal{I}_{1}\right| \sqrt{d}} \sum_{i=1}^{d}\left(\sum_{j \in \mathcal{I}_{i}}\left(Y_{i}^{j}-1 / 2\right)\right)^{2}-\sqrt{d} / 4\right|>\kappa_{\alpha}\right\} . \tag{12}
\end{equation*}
$$

It turns out that such a test does not cover all regimes where $m \gtrsim d^{2} / b^{2}$, because, there is a certain amount of information loss due to the nonlinearity of the quantization step (11), i.e. the test induces soft thresholding for the signal components which is sub-optimal for (relatively) large signal components. For the exact statement on the testing error of this test, see Lemma A. 9 below.

For detecting signals including large coordinates we propose an adaptation of test $T_{\text {III }}^{1}$. We start by assuming that $b \geqslant 2 \log (d+1)$ otherwise we do not construct the test. Then for $i=1, \ldots, d$ and $j=1, \ldots, m$, let us generate

$$
B_{l i}^{j} \stackrel{i i d}{\sim} \operatorname{Ber}\left(F_{\chi_{1}^{2}}\left(\left(\sqrt{n / m} X_{i}^{j}\right)^{2}\right)\right), \quad l \in\left\{1, \ldots, C_{b, d}=\left\lfloor 2^{b} /(d+1)\right\rfloor\right\} .
$$

Note that $C_{b, d} \geqslant 1$ by assumption. Then machine $j$ communicate the transcripts

$$
\begin{equation*}
N^{j}=\sum_{l=1}^{C_{b, d}} \sum_{i=1}^{d} B_{l i}^{j} \in\left\{0,1, \ldots, C_{b, d} d\right\} \tag{13}
\end{equation*}
$$

which can be done using $\log _{2}\left(C_{b, d} d+1\right) \leqslant b$ bits in total. Based on these transcripts, we compute the test

$$
T_{\mathrm{III}}^{2}=\mathbb{1}\left\{\left|\frac{1}{d m C_{b, d}}\left(\sum_{j=1}^{m}\left(N^{j}-L d / 2\right)\right)^{2}-\frac{1}{4}\right| \geqslant \kappa_{\alpha}\right\}
$$

centrally. The testing risk bound for the above test is given in Lemma A. 10 below.
Finally, we construct our test by combining the above ones. We construct both partial tests $T_{\mathrm{III}}^{1}$ and $T_{\mathrm{III}}^{2}$ if $b \geqslant 2 \log (d+1)$ by transmitting $b^{\prime}=\lfloor b / 2\rfloor$ bits per machine for each, otherwise we just construct $T_{\text {III }}^{1}$. Then we merge them by taking

$$
\begin{equation*}
T_{\mathrm{III}}=T_{\mathrm{III}}^{1} \vee T_{\mathrm{III}}^{2} \mathbb{1}_{\{b \geqslant 2 \log (d+1)\}}, \tag{14}
\end{equation*}
$$

where the indicator should be understood to rule out cases in which the transcripts for $T_{\text {III }}^{2}$ cannot necessarily be communicated. This case, as shown below, is covered by the first test $T_{\text {III }}^{1}$. Lemma A. 8 in the Supplementary Material shows that $T_{\text {III }}$ has sufficiently small testing risk in all cases where $m \geqslant M_{\alpha} d^{2} / b^{2}$.
5. A sketch of proof for the testing lower bound (Theorem 3.1). In this section, we provide a sketch of proof of Theorem 3.1, of which the full details are given in Section 9. The proof starts out the same way for both the private and public coin cases, but bifurcates later on. We consider for the time being a generic collection of $b$-bit distributed testing protocols $\mathcal{T}(b)$.

As a first step, we introduce a prior distribution $\pi$ on $\mathbb{R}^{d}$ and lower bound the testing risk by a type of Bayes risk and the mass of $\pi$ that resides outside of the alternative hypothesis $H_{\rho}$, akin to e.g. [24]. Recall that $\mathbb{P}_{f}$ denotes the joint distribution of $Y, U$ and $X$ where $X^{j}$ follows (1) and $Y \sim \mathbb{E}_{f}^{X, U} K(\cdot \mid X, U)=: \mathbb{P}_{f, K}^{Y}=\mathbb{P}_{f}^{Y}$. For $\pi$ a given a distribution on $\mathbb{R}^{d}$, define the mixture distribution $\mathbb{P}_{\pi}^{X}=P_{\pi}$ on $\mathbb{R}^{m d}$ by $P_{\pi}(A)=\int P_{f}(A) d \pi(f)$, where we recall the notational convention $\mathbb{P}_{f}^{X}=P_{f}$ from Section 2.

Through the Markov chain relation $f \rightarrow X \rightarrow Y$ this defines a distribution $\mathbb{P}_{\pi}^{Y}=\mathbb{P}_{\pi, K}^{Y}$ on $\mathcal{Y}$ and let us denote by $\mathbb{E}_{\pi}^{Y}$ the corresponding expectation. Lemma A. 1 in the Supplementary Material lower bounds the infimum testing risk $\inf _{T \in \mathcal{T}} \mathcal{R}\left(H_{\rho}, T\right)$ using a version of Le Cam's lemma adapted to the distributed setting. The lemma yields that, for any distribution on $U$,

$$
\inf _{T \in \mathcal{T}}\left(\mathbb{E}_{0}^{Y} T(Y)+\sup _{f \in H_{\rho}} \mathbb{E}_{f}^{Y}(1-T(Y))\right) \geqslant \inf _{K} \sup _{\pi}\left(1-\left\|\mathbb{P}_{0, K}^{Y}-\mathbb{P}_{\pi, K}^{Y}\right\|_{T V}-\pi\left(H_{\rho}^{c}\right)\right),
$$

where the infimum on the rhs is over all kernels on $\mathcal{Y}$.
Using that the measure $d \mathbb{P}_{f}^{Y}$ disintegrates as $d \mathbb{P}_{f}^{Y \mid U=u} d \mathbb{P}_{f}^{U}(u)$, and the fact that $U$ is independent of the prior $\pi$, we find by Jensen's inequality that

$$
\left\|\mathbb{P}_{0, K}^{Y}-\mathbb{P}_{\pi, K}^{Y}\right\|_{T V} \leqslant \int\left\|\mathbb{P}_{0, K}^{Y \mid U=u}-\mathbb{P}_{\pi, K}^{Y \mid U=u}\right\|_{T V} d \mathbb{P}^{U}(u)
$$

By Pinsker's second inequality and the fact that $\log (x) \leqslant x-1$, we obtain that

$$
\begin{equation*}
\inf _{T \in \mathcal{T}(b)} \mathcal{R}\left(H_{\rho}, T\right) \geqslant 1-\sup _{K} \inf _{\pi}\left(\int \sqrt{2 D_{\chi^{2}}\left(\mathbb{P}_{0, K}^{Y \mid U=u} ; \mathbb{P}_{\pi, K}^{Y \mid U=u}\right)} d \mathbb{P}^{U}(u)+\pi\left(H_{\rho}^{c}\right)\right), \tag{15}
\end{equation*}
$$

where

$$
\begin{equation*}
D_{\chi^{2}}\left(\mathbb{P}_{0, K}^{Y \mid U=u} ; \mathbb{P}_{\pi, K}^{Y \mid U=u}\right)=\mathbb{E}_{0, K}^{Y \mid U=u}\left(\frac{\mathbb{P}_{\pi, K}^{Y \mid U=u}}{\mathbb{P}_{0, K}^{Y \mid U=u}}(Y)\right)^{2}-1 . \tag{16}
\end{equation*}
$$

From hereon, the proof can be broken down into two steps. We provide the skeleton of the proof here and defer the full details to Section 9.

1. The first term on the rhs of (16) can be expressed in terms of a conditional expectation of the likelihood of $X$ :

$$
\begin{equation*}
\mathbb{E}_{0}^{Y \mid U=u}\left(\mathbb{E}_{0}\left[\left.\int \prod_{j=1}^{m} \frac{d \mathbb{P}_{f}^{X^{j}}}{d \mathbb{P}_{0}^{X^{j}}}\left(X^{j}\right) d \pi(f) \right\rvert\, Y, U=u\right]^{2}\right) \tag{17}
\end{equation*}
$$

which we compare to the quantity

$$
\begin{equation*}
\prod_{j=1}^{m} \mathbb{E}_{0}^{Y^{j} \mid U=u}\left(\mathbb{E}_{0}\left[\left.\int \frac{d \mathbb{P}_{f}^{X^{j}}}{d \mathbb{P}_{0}^{X^{j}}}\left(X^{j}\right) d \pi(f) \right\rvert\, Y^{j}, U=u\right]^{2}\right) \tag{18}
\end{equation*}
$$

which corresponds to the product of the first terms of the local chi-square divergences. In particular, we compare the ratio of the expressions in the above two displays and show that when $\pi$ is taken to be centered Gaussian, this ratio is maximized when the protocol's
kernel $K: L_{2}(\mathcal{Y}) \rightarrow L_{2}(\mathcal{X})$ with Hilbert space adjoint $K^{*}$ satisfies that $K^{*} K: L_{2}(\mathcal{X}) \rightarrow$ $L_{2}(\mathcal{X})$ is Gaussian in an appropriate sense. This is the content of Lemma 10.1, which forms the crux of our proof. This lemma, on which we expound in Section 10, exploits the conjugacy between the prior and the model which enables the use of techniques applied in [26]. Consequently, we obtain that the first term on the rhs of (16) is bounded from above by a multiple of

$$
\begin{equation*}
\prod_{j=1}^{m} \mathbb{E}_{0}\left[\mathscr{L}_{\pi}\left(X^{j}\right)^{2}\right] \int \exp \left(f^{\top} \Xi_{u} g\right) d(\pi \times \pi)(f, g) d \mathbb{P}^{U}(u) \tag{19}
\end{equation*}
$$

where

$$
\begin{equation*}
\Xi_{u}:=\sum_{j=1}^{m} \mathbb{E}_{0}^{Y^{j} \mid U=u} \mathbb{E}_{0}\left[X^{j} \mid Y^{j}, U=u\right] \mathbb{E}_{0}\left[X^{j} \mid Y^{j}, U=u\right]^{\top} . \tag{20}
\end{equation*}
$$

2. The final step combines data processing techniques with what is essentially a geometric argument. The first term in (19) is handled using classical, non-distributed techniques, i.e. decoupling argument of the measure and the moment generating function of the Gaussian chaos, see e.g. [40]. In the second term in (19) the $d \times d$ matrix $\Xi_{u}$ geometrically captures how well $Y$ allows to "reconstruct" the compressed sample $X$. The information lost by compressing a $d$ dimensional observation $X^{j}$ into a $b$-bit transcript $Y^{j}$ is captured in a data processing inequality for the matrix $\Xi_{u}$ and its trace, which comes in the form of Lemma A. 2 and Lemma A.3. From hereon out, the proof of the private and the public coin cases separate. Recalling the order of the supremum, infimum and expectation with respect to the public coin in (15), we see that in the private coin case, $\pi$ can be chosen with knowledge of $\Xi_{u}$, as $U$ is degenerate in this case. To obtain the stricter lower bound in the private coin case, we choose $\pi$ 's covariance in order to exploit the "weakest directions" of the protocol $Y$ and the proof is finished by matrix algebra arguments.
3. Nonparametric testing with known regularity. A natural extension of the above finite dimensional signal in Gaussian noise setting is the infinite dimensional signal in white noise model. Here, the $j=1, \ldots, m$ machines observe iid $X^{j}$ taking values in $\mathcal{X} \subset L_{2}[0,1]$ and subject to the stochastic differential equation

$$
\begin{equation*}
d X_{t}^{j}=f(t) d t+\sqrt{\frac{m}{n}} d W_{t}^{j} \tag{21}
\end{equation*}
$$

under $P_{f}$, with $W^{1}, \ldots, W^{m}$ iid Brownian motions and $f \in L_{2}[0,1]$. Besides the difference in the local observations, the distributed setup considered for this model remains exactly the same. The results derived for the alternatives $H_{\rho}$ in the finite dimensional model translate to testing in the infinite dimensional model against the alternative hypotheses

$$
f \in H_{\rho}^{s, R}:=\left\{f \in \mathcal{H}^{s, R}[0,1]:\|f\|_{L_{2}} \geqslant \rho \text { and }\|f\|_{\mathcal{H}^{s}} \leqslant R\right\} .
$$

Here, $\mathcal{H}^{s, R}=\mathcal{H}^{s, R}([0,1])$ denotes the Sobolev ball of radius $R$ in the space of $s$-smooth Sobolev functions and $\|\cdot\|_{\mathcal{H}^{s}}$ the Sobolev norm, see Section G for recalling the definitions. The smoothness parameter $s>0$ determines the difficulty of the classical (non-distributed, $m=1$ ) nonparametric testing problem as considered in e.g. [24]. The asymptotic minimax rate for the non-distributed case is $\rho^{2}=n^{-\frac{2 s}{2 s+1 / 2}}$ for the $s$-smooth Sobolev alternative class.

We allow for asymptotics in $b$ and $m$ in the sense that they can depend on $n$. Consequently, we consider the separation rate $\rho$ in the nonparametric problem to be a sequence of positive numbers in both $n, m$ and the budget $b$. A distributed test $T$ in the nonparametric setting is called $\alpha$-consistent for $\alpha \in(0,1)$ if $\mathcal{R}\left(H_{\rho}^{s, R}, T\right) \leqslant \alpha$ for all $n$ large enough.

The distributed setting for the nonparametric model remains unchanged in comparison with the finite dimensional model introduced in Section 2, except of course for the sample space in which the observations $X^{j}$ take values. This becomes $L_{2}[0,1]$ instead of $\mathbb{R}^{d}$. The following theorem describes the minimax rate for the nonparametric distributed problem.

THEOREM 6.1 (Nonparametric signal in white noise minimax rate). Take $f \in H^{s, R}$ for some $s, R>0$ and let $b \equiv b_{n}$ and $m \equiv m_{n}$ be sequences of natural numbers and take $\rho \equiv$ $\rho_{n, b, m, s}$ be a sequence of positive numbers satisfying

$$
\rho^{2} \asymp \begin{cases}n^{-\frac{2 s}{2 s+1 / 2},} & \text { if } b \geqslant n^{\frac{1}{2 s+1 / 2}}  \tag{22}\\ (\sqrt{b} n)^{-\frac{2 s}{2 s+1}}, & \text { if } n^{\frac{1}{2 s+1 / 2} / m^{\frac{2 s+1}{2 s+1 / 2}} \leqslant b<n^{\frac{1}{2 s+1 / 2}}} \\ (n / \sqrt{m})^{-\frac{2 s}{2 s+1 / 2},} & \text { if } b<n^{\frac{1}{2 s+1 / 2}} / m^{\frac{2 s+1}{2 s+1 / 2}}\end{cases}
$$

In the public coin protocol case the minimax testing rate is $\rho^{2}$ given in (22), i.e. for all $\alpha \in(0,1)$ there exist constants $C_{\alpha}, c_{\alpha}>0$ depending only on $\alpha$, s and $R$ such that for all $n$ large enough,

$$
\inf _{T \in \mathcal{T}_{\text {pub }}(b)} \mathcal{R}\left(H_{c_{\alpha} \rho}^{s, R}, T\right)>1-\alpha \text { and } \inf _{T \in \mathcal{T}_{p u b}(b)} \mathcal{R}\left(H_{C_{\alpha} \rho}^{s, R}, T\right) \leqslant \alpha
$$

Similarly, in the private coin protocol case $\rho \equiv \rho_{n, b, m}$ given below

$$
\rho^{2} \asymp \begin{cases}n^{-\frac{2 s}{2 s+1 / 2}} & \text { if } b \geqslant n^{\frac{1}{2 s+1 / 2}},  \tag{23}\\ (b n)^{-\frac{2 s}{2 s+3 / 2}} & \text { if } n^{\frac{1}{2 s+1 / 2} / m^{\frac{s+3 / 4}{2 s+1 / 2}} \leqslant b<n^{\frac{1}{2 s+1 / 2}}} \\ (n / \sqrt{m})^{-\frac{2 s}{2 s+1 / 2}} & \text { if } b<n^{\frac{1}{2 s+1 / 2}} / m^{\frac{s+3 / 4}{2 s+1 / 2}}\end{cases}
$$

provides the minimal testing rate, i.e. for all $\alpha \in(0,1)$ there exist constants $C_{\alpha}, c_{\alpha}>0$ depending only on $\alpha$ and $R$ such that for all $n$ large enough,

$$
\inf _{T \in \mathcal{T}_{\text {priv }}(b)} \mathcal{R}\left(H_{c_{\alpha} \rho}^{s, R}, T\right)>1-\alpha \text { and } \inf _{T \in \mathcal{T}_{\text {priv }}(b)} \mathcal{R}\left(H_{C_{\alpha} \rho}^{s, R}, T\right) \leqslant \alpha
$$

The proof of the theorem is given in Section B. The theorem reveals the relationship between the signal-to-noise-ratio $n$, communication budget per machine $b$, the number of machines $m$ and the smoothness of the signal $s$. Before providing the proof we briefly discuss the connection with distributed minimax estimation rates.

The distributed minimax estimation rates under private coin protocol were established in Corollary 2.2 of [32] or Theorem 3.1 in [44]. A slight reformulation of the latter yields that

$$
\inf _{(\hat{f}, \mathcal{L}(Y)) \in \mathcal{E}_{p r i v}(b)} \sup _{f \in \mathcal{H}^{s, R}} \mathbb{E}_{f}^{Y}\|\hat{f}(Y)-f\|_{L_{2}}^{2}= \begin{cases}n^{-\frac{2 s}{2 s+1}}, & \text { if } b \geqslant n^{\frac{1}{2 s+1}}  \tag{24}\\ (b n)^{-\frac{2 s}{2 s+2}}, & \text { if }\left(n / m^{2+2 s}\right)^{\frac{1}{2 s+1}} \leqslant b \leqslant n^{\frac{1}{2 s+1}} \\ (b m)^{-2 s}, & \text { if } b \leqslant\left(n / m^{2+2 s}\right)^{\frac{1}{2 s+1}}\end{cases}
$$

where $\mathcal{E}_{\text {priv }}(b)$ is the class of all distributed estimators based on $b$-bit transcripts $Y=$ $\left(Y^{1}, \ldots, Y^{m}\right)$.

A first observation is that consistent testing is possible in any regime of $b \geqslant 1$ and $m$, whereas this is not the case in estimation. Consider for instance the regime where $m$ and $b$ are fixed. In nonparametric distributed estimation, the $L_{2}$-risk does not improve once the sample size is large enough. In fact, even when allowing for asymptotics in $b$ and $m$ (but assuming that $\left(n / m^{2+2 s}\right)^{\frac{1}{2 s+1}} \geqslant b$ ) one is better off performing the estimation locally using just one of the machines with local signal-to-noise-ratio $n / m$, attaining the locally optimal rate $(n / m)^{-\frac{2 s}{2 s+1}}$.

In the case of nonparametric testing, not only can we consistently test for any fixed $m$ and $b$, the distributed testing rate is bounded from above by $(n / \sqrt{m})^{-2 s /(2 s+1 / 2)}$ (regardless of the communication budget $b$ ), which is significantly smaller (for large $m$ ) than the minimax testing rate based on the local signal-to-noise-ratio $(n / m)^{-2 s /(2 s+1 / 2)}$, which can be achieved by using only a single local machine. One possible explanation for this discrepancy is that in nonparametric estimation, the output of the inference is a high-dimensional object, which requires a large total communication budget to be reconstructed with sufficient granularity. In testing, the output of our inference is binary.

A perhaps less surprising difference is that a larger budget is needed for testing at the nondistributed minimax testing rate compared to estimation. That is, in order to obtain the nondistributed minimax rate of $\rho^{2}=n^{-\frac{2 s}{2 s+1 / 2}}$, the communication budget needs to satisfy $b \gtrsim$ $n^{\frac{1}{2 s+1 / 2}}$. On the other hand, the non-distributed minimax estimation rate $n^{-\frac{2 s}{2 s+1}}$ requires only $b \gtrsim n^{\frac{1}{2 s+1}}$. This follows from the fact that the $L_{2}$ testing rate is faster than the estimation rate and hence to achieve this faster rate one has to collect information about the signal at higher frequency level as well (up to the $O\left(n^{\frac{1}{2 s+1 / 2}}\right)$ coefficients in the spectral decomposition).

Increasing $m$ decreases the local signal-to-noise-ratio. When the total budget $b m$ grows at a similar or faster rate than the "effective dimension" of the model, the rate that can be achieved no longer depends on $m$ in both estimation and testing settings. In this regime, this effect is offset by the total number of bits being received by the central machine. What is different in testing problem, however, is that having access to shared randomness strictly improves the performance (until the local communication budget $b$ reaches the effective dimension $n^{\frac{1}{2 s+1 / 2}}$ as after that both method reaches the minimax non-distributed testing rate $n^{-\frac{2 s}{2 s+1 / 2}}$ ). One might wonder whether having access to a public coin improves the rate in the estimation setting also. It turns out that this is not the case. We show in Theorem C. 1 in the Supplementary Material that under the public coin protocol the distributed minimax estimation rate does not improve compared to the private coin protocol.
7. Adaptation in nonparametrics. In the previous section we have derived minimax lower and matching upper bounds for the nonparametric distributed testing problem in context of the Gaussian white noise model. The proposed tests, however, depend on the regularity hyper-parameter $s$ of the functional parameter of interest $f$. Typically, the regularity of the function is not known in practice and one has to use data driven methods to find the best testing strategies. In this section we derive distributed tests adapting to this unknown regularity. We derive both lower and upper bounds and observe surprising, additional phase transition in the small budget regime which was not present in the non-adaptive setting.

First, we note that even in the non-distributed setting, we have to pay an additional $\log \log n$ factor as a price for adaptation (see e.g. Theorem 2.3 in [31] or Section 7 in [24]). More concretely, if $\rho_{s}=n^{-s /(2 s+1 / 2)}$, it holds that for any $s_{\min }<s_{\max }$,

$$
\sup _{s \in\left[s_{\min }, s_{\max }\right]} \mathcal{R}\left(H_{c_{n} M_{n, s} \rho_{s}}^{s, R}, T\right) \rightarrow 1
$$

for all tests $T, M_{n, s}=(\log \log n)^{\frac{s / 4}{2 s+1 / 2}}$ and any $c_{n}=o(1)$ whilst there exists a test $T$ satisfying

$$
\sup _{s \in\left[s_{\min }, s_{\max }\right]} \mathcal{R}\left(H_{C M_{n, s} \rho_{s}}^{s, R}, T\right) \rightarrow 0 .
$$

for large enough constant $C>0$.
The distributed testing problem is more complicated as we have to consider different regimes based on the number of transmitted bits, see Theorem 6.1. These regimes, however,
depend on the unknown regularity hyper-parameter and require different testing procedures to achieve consistent testing. The transcripts transmitted require a larger communication budget to attain the same performance as in Theorem 6.1. Theorem 7.1 and 7.2 below capture this increased difficulty in terms of lower- and upper bounds on the detection rate (tight up to a $\log -\log$ factor). In the proof of the theorem, we derive such an adaptive distributed testing method which adapts to the smoothness. These methods are in principle based on taking a $1 / \log n$ grid of the regularity interval $\left[s_{\min }, s_{\text {max }}\right]$, constructing optimal tests for each of the grid points and combining them using Bonferroni's correction. This results in loosing a logarithmic factor in the intermediate case as the budget has to be divided over $O(\log n)$ tests, each capturing a different possible level of smoothness.

This additional incurred cost in the distributed setting due to additional communication budget required is fundamental, as our accompanying lower bound shows. This additional difficulty translates to a $\sqrt{\log (n)}$ and $\log (n)$ factor more observations required in the intermediate budget regimes for the public and private coin settings, respectively. In the small budget regime, such a loss is incurred when the local communication budget $b$ is of smaller order than $\log (n)$. When $b \gtrsim \log (n)$ in the small budget regime, the same rate as in Theorem 6.1 can be obtained, up to the $\log \log (n)$ factor incurred by the Bonferroni correction.

The above described results are split over two theorems. The first, Theorem 7.1, concerns the case where $b \gtrsim \log (n)$. In the second, Theorem 7.2, the case where $b \lesssim \log (n)$ (both theorems coincide when $b=\log (n))$. The case where $b=O(1)$ is of special interest, as $b=1$ means each machine's local transcript forms a test itself and the global test can be seen as a "meta-analysis" on the basis of these $m$ tests. The proofs of the upper bounds in both theorems are given in Section 8, while the proofs of the lower bound are deferred to Section D in the supplement.

THEOREM 7.1. Let us consider some $0<s_{\min }<s_{\max }<\infty, R>0$, let $b \equiv b_{n}$ such that $b \gg \log n$ and $m \equiv m_{n}$ be sequences of natural numbers and take a sequence of positive numbers $\rho_{s} \equiv \rho_{n, b, m, s}$ satisfying

$$
\rho_{s}^{2} \leftrightharpoons \begin{cases}n^{-\frac{2 s}{2 s+1 / 2}}, & \text { if } b \geqslant \log (n) n^{\frac{1}{2 s+1 / 2}},  \tag{25}\\ \left(\frac{\sqrt{b} n}{\sqrt{\log (n)}}\right)^{-\frac{2 s}{2 s+1}}, & \text { if } \log (n)\left(\frac{n^{\frac{1}{2 s+1 / 2}}}{m^{2 s+1}} \bigvee 1\right) \leqslant b<\log (n) n^{\frac{1}{2 s+1 / 2}} \\ \left(\frac{n}{\sqrt{m}}\right)^{-\frac{2}{2 s+1 / 2}}, & \text { if } \log (n) \leqslant b<\log (n)\left(\frac{n^{\frac{1}{2 s+1 / 2}}}{m^{2 s+1 / 2}} \bigvee 1\right)\end{cases}
$$

in the public coin case, and

$$
\rho_{s}^{2}= \begin{cases}n^{-\frac{2 s}{2 s+1 / 2}} & \text { if } b \geqslant \log (n) n^{\frac{1}{2 s+1 / 2}}  \tag{26}\\ \left(\frac{b n}{\log (n)}\right)^{-\frac{2 s}{2 s+3 / 2}} & \text { if } \log (n)\left(\frac{n^{\frac{1}{2 s+1 / 2}}}{\left.m^{\frac{s+3 / 4}{2 s+1 / 2}} \bigvee 1\right) \leqslant b<\log (n) n^{\frac{1}{2 s+1 / 2}}}\right. \\ \left(\frac{n}{\sqrt{m}}\right)^{-\frac{2 s}{2 s+1 / 2}} & \text { if } \log (n) \leqslant b<\log (n)\left(\frac{n^{\frac{1}{2 s+1 / 2}}}{m^{\frac{s+3 / 4}{2 s+1 / 2}}} V^{2}\right)\end{cases}
$$

in the case of a private coin. Then, there exits a sequence of distributed testing procedures in the respective setups such that

$$
\sup _{s \in\left[s_{\min }, s_{\max }\right]} \mathcal{R}\left(H_{M_{n} \rho_{s}}^{s, R}, T\right) \rightarrow 0,
$$

for arbitrary $M_{n} \gg(\log \log (n))^{1 / 4}$. Similarly, for all distributed testing procedures in the respective setups, we have that for all $\alpha \in(0,1)$ there exists $c_{\alpha}>0$ such that

$$
\sup _{s \in\left[s_{\min }, s_{\max }\right]} \mathcal{R}\left(H_{c_{\alpha} \rho_{s}}^{s, R}, T\right)>\alpha .
$$

The above theorem recovers (up to log-factors) the three rates corresponding to the three regimes also found in Theorem 6.1, the different regimes corresponding to different testing strategies. Since the true smoothness is unknown, these different distributed testing strategies are to be conducted simultaneously.

We note that for $m \geqslant n^{\frac{1}{2 s_{\min }+1}}$ or $m \geqslant n^{\frac{1}{s_{\min }+3 / 4}}$ in the public and private coin cases, respectively, the small budget regime no longer occurs. The reason for this is that, even though $b$ could be relatively small, the total communication budget $b m$ is large enough to warrant the strategy for the intermediate and high budget regimes. Furthermore, whenever $b>\log (n) n^{\frac{1}{2 s+1 / 2}}$, the budget is large enough to recover the non-distributed regime rate.

For $b \lesssim \log (n)$ the separation rate is different from the non-adaptive low budget regime. Depending on the interplay between $n$ and $m$ either the minimax rate corresponding to the intermediate case applies or an additional $(\log (n) / b)^{\delta}$ factor is present compared to the nonadaptive low budget regime, both in the private and public coin settings. This results in an additional phase transition at $b=\log n$. The reason for this, is that in order to cover approximately $\log (n)$ different levels of smoothness using less than $\log (n)$ bits, each of the machines can no longer send an adequate amount of information on all of the relevant smoothness levels. Instead, an optimal strategy is to divide the different machines over each of the smoothness levels, where each machines foregoes sending information regarding certain smoothness levels all together.

THEOREM 7.2. Assume the conditions of Theorem 7.1 with $b \lesssim \log (n)$ and assume $b m \gg$ $\log (n)$. Let us consider

$$
\rho_{s}^{2}= \begin{cases}\left(\frac{\sqrt{b} n}{\sqrt{\log (n)}}\right)^{-\frac{2 s}{2 s+1}}, & \text { if } m \geqslant n^{\frac{1}{2 s+1}}  \tag{27}\\ \left(\frac{\sqrt{b} n}{\sqrt{m \log (n)}}\right)^{-\frac{2 s}{2 s+1 / 2}}, & \text { if } m<n^{\frac{1}{2 s+1}}\end{cases}
$$

in the public coin case and

$$
\rho_{s}^{2}= \begin{cases}\left(\frac{b n}{\log (n)}\right)^{-\frac{2 s}{2 s+3 / 2}} & \text { if } m \geqslant n^{\frac{2}{2 s+3 / 2}}\left(\frac{b}{\log (n)}\right)^{\frac{s-1 / 4}{2 s+3 / 2}}  \tag{28}\\ \left(\frac{n \sqrt{b}}{\sqrt{m \log (n)}}\right)^{-\frac{2 s}{2 s+1 / 2}} & \text { if } m<n^{\frac{2}{2 s+3 / 2}}\left(\frac{b}{\log (n)}\right)^{\frac{s-1 / 4}{2 s+3 / 2}}\end{cases}
$$

in the private coin case. Then, there exits a sequence of distributed testing procedures in the respective setups such that

$$
\sup _{s \in\left[s_{\min }, s_{\max }\right]} \mathcal{R}\left(H_{M_{n} \rho_{s}}^{s, R}, T\right) \rightarrow 0
$$

for arbitrary $M_{n} \gg(\log \log (n))^{1 / 4}$. Similarly, for all distributed testing procedures in the respective setups, we have that for all $\alpha \in(0,1)$ there exists $c_{\alpha}>0$ such that

$$
\sup _{s \in\left[s_{\min }, s_{\max }\right]} \mathcal{R}\left(H_{c_{\alpha} \rho_{s}}^{s, R}, T\right)>\alpha
$$

REMARK 7.3. Both theorems together cover all cases where $m b \gg \log (n)$. The cases where $m b \lesssim \log (n)$ are excluded for technical reasons, as well as the fact that when $m b \lesssim$ $\log (n)$, the optimal rate in (27)-(28) (up to at most a $\sqrt{\log \log (n)}$ factor) is attained by using a standard non-distributed method using just the data of one machine (see e.g. [31]). Similarly, in order to contain the level of technicality, we have foregone the $(\log \log (n))^{1 / 4}$ additional factor in the lower bound which we esteem also to be present in the distributed setting. We
refer the reader to the argument of Theorem 2.3 in [31] for how to obtain the $(\log \log (n))^{1 / 4}$ factor in the lower bound in addition to the $\sqrt{\log (n)}$ and $\log (n)$ factors in the public and private coin cases respectively.
8. Adaptive tests attaining the adaptation bounds in Theorem 7.1 and 7.2. Let us consider the smooth orthonormal wavelet basis $\left\{\psi_{l i}: l \in \mathbb{N}_{0}, i=0,1, \ldots, 2^{l}-1\right\}$. See Section G for a brief introduction of wavelets and collection of properties used in this proof. For $L=L \in \mathbb{N}$, let $V_{L}=\left\{\psi_{l i}: l \leqslant L, i=0,1, \ldots, 2^{l}-1\right\}$. For $f \in L_{2}[0,1]$, let $f^{L}$ denote the projection of $f$ onto $V_{L}$, i.e.

$$
\begin{equation*}
f^{L}=\sum_{l=0}^{L} \sum_{i=0}^{2^{l}-1} \tilde{f}_{l i} \psi_{l i} \tag{29}
\end{equation*}
$$

with $\tilde{f}_{l i}:=\int f \psi_{l i}$. We denote the wavelet coefficients of $X^{j}$ by $\tilde{X}_{l i}^{j}:=\int_{0}^{1} \psi_{l i} d X_{t}^{j}$. For the coefficients at resolution level $L$, write $\tilde{X}_{L}^{j}=\left(\tilde{X}_{L 0}^{j}, \ldots, \tilde{X}_{L\left(2^{L}-1\right)}^{j}\right) \in \mathbb{R}^{2^{L}}$ and let $\tilde{X}_{L^{\prime}: L}^{j}$ denote the concatenated coefficients from resolution level $L^{\prime}<L$ up to resolution level $L$, i.e. $\tilde{X}_{L^{\prime}: L}^{j}=\left(\tilde{X}_{L^{\prime}}^{j}, \ldots, \tilde{X}_{L}^{j}\right) \in \mathbb{R}^{2^{L+1}-2^{L^{\prime}+1}}$. The vector $\tilde{X}_{0: L}^{j}:=\left(\tilde{X}_{0}^{j}, \tilde{X}_{1}^{j}, \ldots, \tilde{X}_{L}^{j}\right)$ follows the dynamics

$$
\begin{equation*}
\tilde{X}_{0: L}^{j}=\tilde{f}^{L}+\sqrt{\frac{m}{n}} Z^{j} \tag{30}
\end{equation*}
$$

where $Z^{j} \sim^{i i d} N\left(0, I_{2^{L+1}-1}\right), j=1, \ldots, m$, and $\tilde{f}^{L}:=\left(\tilde{f}_{l i}\right)_{l=0, \ldots, L ; i=0, \ldots, 2^{l}-1}$.
Let $\nu_{L}=2^{L+1}-1$ and let us introduce the notations $L_{s}=\left\lfloor s^{-1} \log \left(1 / \rho_{s}\right)\right\rfloor \vee 1$, and for shorthand write $L_{\min }=L_{s_{\max }}$ and $L_{\max }=L_{s_{\min }}$ and note that $L_{s} \in \mathcal{C}:=\left\{L_{\min }, \ldots, L_{\max }\right\}$ for all $s \in\left[s_{\min }, s_{\max }\right]$. Note that $|\mathcal{C}| \leqslant \log n$.

For each regularity hyper-parameter $s$, we distinguish low-budget ( $2^{L_{s}} \gtrsim m b$ in the public coin, and $2^{\frac{3}{2} L_{s}} \gtrsim m b$ in the private coin setting) and high-budget (corresponding to $2^{L_{s}} \lesssim m b$ in the public coin and $2^{\frac{3}{2} L_{s}} \lesssim m b$ in the private coin setting) cases. Since $m$ and $b$ are known for any given regularity $s$ we know which regime it falls and is sufficient to construct that test. For notational convenience, without loss of generality, for each $s$ we construct both the high-budget and the low-budget optimal tests using all the $m$ machines (and do not split them between these two cases).
8.1. Proof of the upper bound in the low-budget regime. First we deal with the lowbudget case (where the total budget is small compared to the effective dimension), which coincides in both setups. For each $L \in \mathcal{C}$ we take a subset of machines $M_{L} \subset\{1, \ldots, m\}$ such that $\left|M_{L}\right|=m^{\prime}:=\frac{m(\log (n) \wedge b)}{\log (n)}$ and each machine appears in at most $b$ such subsets. We note that this is possible since $m^{\prime}|\mathcal{C}| \leqslant m b$. Then for each $j \in M_{L}, L \in \mathcal{C}$ we communicate

$$
\begin{equation*}
Y_{I}^{j}(L) \mid X^{j} \sim \operatorname{Ber}\left(\chi_{\nu_{L}}^{2}\left(\sqrt{n / m}\left\|\tilde{X}_{0: L}^{j}\right\|_{2}^{2}\right)\right) \tag{31}
\end{equation*}
$$

and at the central machine, we can compute

$$
S_{I}(L)=\frac{1}{\sqrt{m^{\prime}}} \sum_{j \in M_{L}}\left(2 Y_{I}^{j}(L)-1\right)
$$

Then we consider the following adaptive test based on Bonferroni's correction

$$
T_{I}^{\text {adapt }}=\mathbb{1}\left\{\max _{L \in \mathcal{C}} S_{I}(L) \geqslant 2 \sqrt{\log \log n}\right\}
$$

Since for $L \in \mathcal{C}$, it holds that $L \asymp \log (n)$, the above $\sqrt{\log \log n}$ blow up suffices to guarantee that the test has asymptotically vanishing Type I error control, i.e. $\mathbb{E}_{0} T_{I}^{\text {adapt }}=o(1)$ by Lemma E. 1 in the Supplementary Material (as the random variables $2 Y_{I}^{j}(L)-1$ are iid Rademacher under $\mathbb{P}_{0}$ ).

For the Type II error note that

$$
\mathbb{E}_{f}\left(1-T_{I}^{\text {adapt }}\right) \leqslant \mathbb{P}_{f}\left(S_{I}\left(L_{s}\right)<2 \sqrt{\log \log n}\right)
$$

and aim to apply Lemma A.4. In view of Lemma A.6, (with $\|f\|_{2}$ replaced by $\left\|\tilde{f}^{L_{s}}\right\|_{2}$ and $d=$ $\nu_{L_{s}}$ ), noting that by triangle inequality $\left\|\tilde{f}^{L_{s}}\right\|_{2}^{2} \geqslant\|f\|_{2}^{2} / 2-2^{-2 L_{s} s} R^{2}$ (see also Section B in the Supplementary Material), we get for $\|f\|_{2}^{2} \geqslant C_{0}^{2} \sqrt{\log \log (n)} \rho_{s}^{2} \geqslant C_{0}^{2} \sqrt{\log \log (n)} \frac{\sqrt{2^{L_{s} m \log (n)}}}{n \sqrt{b \wedge \log (n)}}$, that for $m$ large enough

$$
\eta_{p, m^{\prime}, 1} \gtrsim\left(m^{\prime}-1\right)\left(\frac{n\left\|\tilde{f}^{L_{s}}\right\|_{2}^{2}}{m 2^{L_{s} / 2}} \wedge \frac{1}{2}\right)^{2} \gtrsim m^{\prime}\left(\left(\tilde{C} \frac{\log \log n}{m^{\prime}}\right) \wedge(1 / 4)\right),
$$

with $\tilde{C}=C_{0}^{2} / 2-R^{2}$. By the assumption that $b m \gg \log (n), m^{\prime}$ can be taken larger than arbitrary constant $M_{0}>0$. This means that, in view of Lemma A. 4 with $c_{\alpha, n}=4 \log \log n$ and large enough constant $C_{0}$ (depending on $R$ ), the Type II error is bounded by $\alpha$.
8.2. Proof of the upper bound in the public coin, high budget regime. We use similar arguments as before, applying a Bonferroni-type of correction. First let us consider the public coin setting and take a one-to-one mapping $\xi_{L}$ from $\left\{1, \ldots, \nu_{L}\right\}$ to $\{(l, i): l=0, \ldots, L, i=$ $\left.0,1, \ldots, 2^{l}-1\right\}$. Let us define the test

$$
\begin{equation*}
\left(Y_{\mathrm{II}}^{j}(L)\right)_{i} \mid U_{L}=\mathbb{1}\left\{\left(\sqrt{n / m} U_{L} \tilde{X}_{\xi_{L}(i)}^{j}\right)_{i}>0\right\}, \tag{32}
\end{equation*}
$$

where the random variable $U_{L} \in \mathbb{R}^{\nu_{L} \times \nu_{L}}$ is drawn from the Haar measure on the rotation group on $\mathbb{R}^{\nu_{L}}$. Similarly to before for each $L$ we take a subset of machines $M_{L} \subseteq\{1, \ldots, m\}$ such that $\left|M_{L}\right|=m^{\prime}:=\frac{m(b \wedge \log (n))}{\log (n)}$, and each machine appears at most in $b$ such sets.

Then machine $j \in M_{L}, L \in \mathcal{C}$, transmits the bits $\left(Y_{\mathrm{II}}^{j}(L)\right)_{i}, i=1, \ldots, b^{\prime}:=\frac{m b}{m^{\prime}|\mathcal{C}|} \wedge \nu_{L}$ to the central machine, where these local test statistics are aggregated, similarly to (10), as

$$
\begin{equation*}
S_{\mathrm{II}}(L)=\frac{1}{\sqrt{b^{\prime}} m^{\prime}} \sum_{i=1}^{b^{\prime}}\left[\left(\sum_{j \in M_{L}}\left[\left(Y_{\mathrm{II}}^{j}(L)\right)_{i}-1 / 2\right]\right)^{2}-\frac{m^{\prime}}{4}\right] . \tag{33}
\end{equation*}
$$

In view of Lemma E. 1 the Type I error of the test

$$
T_{\mathrm{II}}^{p u b, \text { adapt }}:=\mathbb{1}\left\{\max _{L \in \mathcal{C}} S_{\mathrm{II}}(L) \geqslant 2 \sqrt{\log \log n}\right\}
$$

is $o(1)$. For the Type II error note that

$$
\mathbb{E}_{f}\left(1-T_{\mathrm{II}}^{\text {pub,adapt }}\right) \leqslant \mathbb{E}_{f} \mathbb{1}\left\{S_{\mathrm{II}}\left(L_{s}\right)<2 \sqrt{\log \log n}\right\}
$$

By Lemma E.2, the above display is $o(1)$ whenever $\rho^{2} \gtrsim M_{n} \frac{2^{L_{s}}}{n \sqrt{\frac{b}{\log (n)} \wedge 2^{L_{s}}}}$, which, for the choice of $L_{s}=\left\lfloor s^{-1} \log \left(1 / \rho_{s}\right)\right\rfloor \vee 1$ yields the rates of Theorem 7.1 and 7.2.
8.3. Proof of the upper bound in the private coin, high-budget regime. We proceed by adapting the test $T_{\text {III }}$ provided in Section 4.3 to the nonparametric setting with unknown regularity using again a Bonferroni type correction to achieve adaptation. For simplicity we again apply the map $\xi_{L}$ introduced previously to move between the single and double index notations of the sequence model.

For all $L \in \mathcal{C}$, similarly to the previous cases we consider a collection of machines $M_{L}$ with $\left|M_{L}\right|=m^{\prime}=\frac{m(b \wedge \log (n))}{\log (n)}$ and similarly to Section 4.3 let us use the notation $\mathcal{I}_{i}(L) \subset M_{L}$ for the collection of machines corresponding the $i$ th coordinate. We note that without loss of generality we can assume that $m^{\prime} \geqslant M_{\alpha} \sqrt{\log \log n} 2^{2 L_{s}} /\left(b^{\prime}\right)^{2}$, for some large enough constant $M_{\alpha}$, otherwise the test $T_{I}^{\text {adapt }}$ above covers the corresponding range. Then we modify the test given in (12) by increasing the threshold with the Bonferroni correction, i.e.

$$
\begin{aligned}
T_{\text {III }}^{\text {priv,adapt }, 1} & =\mathbb{1}\left\{\max _{L \in \mathcal{C}} S^{\text {III, }, 1}(L) \geqslant 2 \sqrt{\log \log n}\right\}, \text { where } \\
S^{\mathrm{III}, 1}(L) & =\left\lvert\, \frac{1}{\left|\frac{\mathcal{I}_{1}(L) \mid 2^{L / 2}}{} \sum_{i=1}^{\nu_{L}}\left(\sum_{j \in \mathcal{I}_{i}(L)}\left(Y_{i}^{j}-1 / 2\right)\right)^{2}-2^{L / 2} / 4\right|, \quad Y_{i}^{j} \mid \tilde{X}_{\xi_{L}(i)}^{j}=\mathbb{1}_{\tilde{X}_{\xi_{L}(i)}^{j}>0} .}\right.
\end{aligned}
$$

To deal with large signal components, similarly to (12) (with $d=\nu_{L}$ and including the Bonferroni correction in the threshold), we propose the test,

$$
\begin{aligned}
T_{\mathrm{III}}^{\text {priv,adapt }, 2} & =\mathbb{1}\left\{\max _{L \in \mathcal{C}, 2 \log (L) \leqslant b} S^{\mathrm{III}, 2}(L) \geqslant \kappa_{\alpha} \sqrt{\log \log n}\right\}, \quad \text { where } \\
S^{\mathrm{II}, 2}(L) & =\left|\frac{1}{d m^{\prime} C_{b, L}}\left(\sum_{j=1}^{m^{\prime}}\left(N^{j}-C_{b, L} 2^{L-1}\right)\right)^{2}-\frac{1}{4}\right|,
\end{aligned}
$$

with $C_{b, L}=2^{b-L}$ and $N^{j}$ given in (13). Finally, we aggregate these tests by taking

$$
T_{\mathrm{III}}^{p r i v, a d a p t}=T_{\mathrm{III}}^{p r i v, a d a p t, 1} \vee T_{\mathrm{III}}^{p r i v, a d a p t, 2}
$$

In view of the law of Lemma E. 1 the Type I error tends to zero for both tests. Therefore it remained to show that the Type II error is bounded by $\alpha$. Similarly to the previous cases, note that
$E_{f}\left(1-T_{\text {III }}^{\text {priv,adapt }}\right) \leqslant \mathbb{E}_{f}\left(\mathbb{1}\left\{S^{\text {III }, 1}\left(L_{s}\right)<2 \sqrt{\log \log n}\right\} \wedge \mathbb{1}\left\{S^{\text {III }, 2}\left(L_{s}\right)<2 \sqrt{\log \log n}\right\}\right)$.
Following the proofs of Lemmas A.8, A. 9 and A. 10 (with $d=\nu_{L_{s}}, f$ taken to be the $\nu_{L_{s}}$ dimensional vector $\tilde{f}^{L_{s}}, b$ replaced by $b^{\prime}$, and $M_{\alpha}$ replaced by $M_{0} \sqrt{\log \log n}$, for some large enough $M_{0}>0$ ), noting that for $C_{0}^{2}>4 R^{2}$

$$
\begin{aligned}
\left\|\tilde{f}^{L_{s}}\right\|_{2}^{2} & \geqslant\|f\|_{2}^{2} / 2-R^{2} 2^{-2 L_{s} s} \gtrsim C_{0} \sqrt{\log \log (n)} \rho_{s}^{2} \\
& =\frac{C_{0} 2^{3 L_{s} / 2} \sqrt{\log \log n}}{2 n\left(\frac{b}{\log (n)} \wedge 2^{L_{s}}\right)} \gtrsim \frac{C_{0} 2^{L_{s}} \sqrt{\log \log n}}{n b^{\prime} \frac{m^{\prime}}{m}},
\end{aligned}
$$

and applying Lemmas A. 11 and A. 4 with $c_{n, \alpha}=2 \sqrt{\log \log n}$, we get that the Type II error of $T_{\text {III }}^{\text {priv,adapt }}$ is bounded from above by $\alpha / 2$.

Finally, we combine the above tests by taking

$$
T^{p r i v, a d a p t}=T_{\mathrm{III}}^{p r i v, a d a p t} \vee T_{\mathrm{I}}^{p r i v, a d a p t} \quad \text { and } \quad T^{p u b, a d a p t}=T_{\mathrm{II}}^{p u b, a d a p t} \vee T_{\mathrm{I}}^{p u b, a d a p t}
$$

Note that both of the above tests still have vanishing Type I error, while the Type II errors are bounded by the prescribed level $\alpha$ in view of taking the union of the above optimal tests.
9. Proof of the testing lower bound. We provide the details for Steps 1 and 2 as outlined in Section 5. We shall write $\mathscr{L}_{\pi}(x)=\int \mathscr{L}_{f}(x) d \pi(f)$ with $\mathscr{L}_{f}(x):=\frac{d P_{f}}{d P_{0}}(x)$ and $P_{f}=\mathbb{P}_{f}^{X}$.

Step 1. In view of the Markov chain structure given in (2), the probability measure $d \mathbb{P}_{\pi}(x, u, y)$ disintegrates as $d \mathbb{P}_{K}^{Y \mid(X, U)=(x, u)} d \mathbb{P}_{f}^{X}(x) d \mathbb{P}^{U}(u) d \pi(f)$. Using the Markov chain structure, the first term on the rhs of (16) can be seen to equal

$$
\begin{equation*}
\sum_{y \in \mathcal{Y}} \mathbb{P}_{0}^{Y \mid U=u}(y)\left(\int \mathscr{L}_{\pi}(x) \frac{K(y \mid x, u)}{\mathbb{P}_{0}^{Y \mid U=u}(y)} d P_{0}(x)\right)^{2}=\mathbb{E}_{0}^{Y \mid U=u} \mathbb{E}_{0}\left[\mathscr{L}_{\pi}(x) \mid Y, U=u\right]^{2} \tag{34}
\end{equation*}
$$

Decoupling the square in $X$ and using Fubini's theorem we can write the above display as

$$
\begin{equation*}
\int \mathscr{L}_{\pi}\left(x_{1}\right) \mathscr{L}_{\pi}\left(x_{2}\right) q_{u}\left(x_{1}, x_{2}\right) d\left(P_{0} \times P_{0}\right)\left(x_{1}, x_{2}\right) \tag{35}
\end{equation*}
$$

where by independence between the transcripts,

$$
q_{u}\left(x_{1}, x_{2}\right):=\sum_{y \in \mathcal{Y}} \frac{K\left(y \mid x_{1}, u\right) K\left(y \mid x_{2}, u\right)}{\mathbb{P}_{0}^{Y \mid U=u}(y)}=\prod_{j=1}^{m}\left(\sum_{y^{j} \in \mathcal{Y}^{j}} \frac{K^{j}\left(y^{j} \mid x_{1}^{j}, u\right) K^{j}\left(y^{j} \mid x_{2}^{j}, u\right)}{\mathbb{P}_{0}^{Y^{j} \mid U=u}\left(y^{j}\right)}\right)
$$

Note that in the above display, $x_{i}^{j}$ and $y^{j}$ denote the projection of $x_{i}$ and $y$ on the coordinates indexed by $\{(j-1) d+1, \ldots, j d\}$, respectively. In addition, let us denote by $\prod_{j=1}^{m} q_{u}^{j}\left(x_{1}^{j}, x_{2}^{j}\right)$ the rhs of the preceding display. Since $K$ is a Markov kernel, the function $q_{u} \in L_{2}\left(\mathbb{R}^{2 d m}, P_{0} \times\right.$ $\left.P_{0}\right)$ is bounded and nonnegative. Furthermore,

$$
\int q_{u}\left(x_{1}, x_{2}\right) d P_{0}\left(x_{1}\right)=\sum_{y \in \mathcal{Y}} \frac{K\left(y \mid x_{2}, u\right)}{\mathbb{P}_{0}^{Y \mid U=u}(y)} \int K\left(y \mid x_{1}, u\right) d P_{0}\left(x_{1}\right)=\sum_{y \in \mathcal{Y}} K\left(y \mid x_{2}, u\right)=1
$$

similarly $\int q_{u}\left(x_{1}, x_{2}\right) d P_{0}\left(x_{2}\right)=1$,

$$
\begin{equation*}
\int x_{i} q_{u}\left(x_{1}, x_{2}\right) d\left(P_{0} \times P_{0}\right)\left(x_{1}, x_{2}\right)=\int x_{i} d P_{0}\left(x_{i}\right)=0 \in \mathbb{R}^{m d} \tag{36}
\end{equation*}
$$

for $i=1,2$, and

$$
\begin{equation*}
\int\binom{x_{1}}{x_{2}}\left(x_{1}^{\top} x_{2}^{\top}\right) q_{u}\left(x_{1}, x_{2}\right) d\left(P_{0} \times P_{0}\right)\left(x_{1}, x_{2}\right)=: \Sigma \in \mathbb{R}^{2 m d \times 2 m d} \tag{37}
\end{equation*}
$$

where the former display can be seen to follow by the law of total expectation, $\Sigma=$ $\operatorname{Diag}\left(\Sigma^{1}, \ldots, \Sigma^{m}\right) \in \mathbb{R}^{2 m d}$ for

$$
\Sigma^{j}:=\left(\begin{array}{cc}
\frac{m}{n} I_{d} & \Xi_{u}^{j} \\
\Xi_{u}^{j} & \frac{m}{n} I_{d}
\end{array}\right)
$$

with

$$
\Xi_{u}^{j}:=\mathbb{E}_{0}^{Y^{j} \mid U=u} \mathbb{E}_{0}\left[X^{j} \mid Y, U=u\right] \mathbb{E}_{0}\left[X^{j} \mid Y^{j}, U=u\right]^{\top}
$$

Writing $\mathscr{L}_{\pi}^{j}:=\int \frac{d \mathbb{P}_{f}^{X^{j}}}{d \mathbb{P}_{0}^{X_{j}^{j}}} d \pi(f),(18)$ can be seen to equal

$$
\begin{equation*}
\prod_{j=1}^{m} \int \mathscr{L}_{\pi}^{j}\left(x_{1}^{j}\right) \mathscr{L}_{\pi}^{j}\left(x_{2}^{j}\right) q_{u}^{j}\left(x_{1}^{j}, x_{2}^{j}\right) d\left(P_{0} \times P_{0}\right)\left(x_{1}^{j}, x_{2}^{j}\right) \tag{38}
\end{equation*}
$$

Lemma 10.1 below applies to the ratio between (35) and (38) whenever $\pi$ is chosen to be centered Gaussian. The lemma yields that the aforementioned ratio is maximized when
$q_{u}\left(x_{1}, x_{2}\right) d\left(P_{0} \times P_{0}\right)\left(x_{1}, x_{2}\right)$ is a Gaussian distribution on $\mathbb{R}^{2 m d}$ with covariance $\Sigma$, where the maximization is among all choices of $q_{u}$ such that $q_{u}$ is nonnegative, bounded and satisfying (36)-(37). Deliberation and proof of the lemma is deferred to Section 10 and the Supplementary Material to the article. For $\pi$ a centered Gaussian distribution on $\mathbb{R}^{d}$, the above lemma applies with $k=2 d, \sigma^{2}=m / n$, we obtain that the ratio between (35) and (38) is bounded above by

$$
\begin{equation*}
\frac{\int \mathscr{L}_{\pi}\left(x_{1}\right) \mathscr{L}_{\pi}\left(x_{2}\right) d N(0, \Sigma)\left(x_{1}, x_{2}\right)}{\prod_{j=1}^{m} \int \mathscr{L}_{\pi}^{j}\left(x_{1}^{j}\right) \mathscr{L}_{\pi}^{j}\left(x_{2}^{j}\right) d N\left(0, \Sigma^{j}\right)\left(x_{1}^{j}, x_{2}^{j}\right)} \tag{39}
\end{equation*}
$$

Combining the result of the lemma with the bound

$$
\begin{equation*}
\prod_{j=1}^{m} \mathbb{E}_{0}^{Y^{j} \mid U=u} \mathbb{E}_{0}\left[\mathscr{L}_{\pi}\left(X^{j}\right) \mid Y^{j}, U=u\right]^{2} \leqslant \prod_{j=1}^{m} \mathbb{E}_{0}^{X^{j} \mid U=u}\left[\mathscr{L}_{\pi}\left(X^{j}\right)^{2}\right] \tag{40}
\end{equation*}
$$

following from Jensen's inequality, we obtain that

$$
\begin{array}{r}
\mathbb{E}_{0}^{Y \mid U=u}\left(\frac{\mathbb{P}_{\pi}^{Y \mid U=u}}{\left.\mathbb{P}_{0}^{Y \mid U=u}(Y)\right)^{2} \leqslant} \frac{\int \mathscr{L}_{\pi}\left(x_{1}\right) \mathscr{L}_{\pi}\left(x_{2}\right) d N(0, \Sigma)\left(x_{1}, x_{2}\right)}{\prod_{j=1}^{m} \int \mathscr{L}_{\pi}^{j}\left(x_{1}^{j}\right) \mathscr{L}_{\pi}^{j}\left(x_{2}^{j}\right) d N(0, \Sigma)\left(x_{1}, x_{2}\right)}\right. \\
\times \prod_{j=1}^{m} \mathbb{E}_{0}^{X^{j} \mid U=u}\left[\mathscr{L}_{\pi}\left(X^{j}\right)^{2}\right] . \tag{41}
\end{array}
$$

By the block diagonal matrix structure of $\Sigma$, the denominator in the first factor of the rhs of (41) satisfies

$$
\begin{aligned}
& \prod_{j=1}^{m} \int \mathscr{L}_{\pi}^{j}\left(x_{1}^{j}\right) \mathscr{L}_{\pi}^{j}\left(x_{2}^{j}\right) d N(0, \Sigma)\left(x_{1}, x_{2}\right)=\prod_{j=1}^{m} \int e^{\frac{n}{2 m}\left(\frac{n}{m}\left\|\sqrt{\Sigma^{j}}(f, g)\right\|_{2}^{2}-\|(f, g)\|_{2}^{2}\right)} d(\pi \times \pi)(f, g) \\
&=\prod_{j=1}^{m} \int e^{\frac{n^{2}}{m^{2}} f^{\top} \Xi_{u}^{j} g} d(\pi \times \pi)(f, g) \\
& \geqslant \prod_{j=1}^{m} e^{\frac{n^{2}}{m^{2}}} f^{\top} f^{\top} \Xi_{u}^{j} g d(\pi \times \pi)(f, g) \\
&
\end{aligned} .
$$

Similarly, the numerator is equal to

$$
\int \mathscr{L}_{\pi}\left(x_{1}\right) \mathscr{L}_{\pi}\left(x_{2}\right) d N(0, \Sigma)\left(x_{1}, x_{2}\right)=\int e^{\frac{n^{2}}{m^{2}} f^{\top} \sum_{j=1}^{m} \Xi_{u}^{j} g} d(\pi \times \pi)(f, g) .
$$

Combining the above displays (i.e. (16) and the last three displays), we obtain that

$$
\begin{equation*}
D_{\chi^{2}}\left(\mathbb{P}_{0, K}^{Y \mid U=u} ; \mathbb{P}_{\pi, K}^{Y \mid U=u}\right) \leqslant \prod_{j=1}^{m} \mathbb{E}_{0}^{X^{j} \mid U=u}\left[\mathscr{L}_{\pi}\left(X^{j}\right)^{2}\right] \cdot \int e^{\frac{n^{2}}{m^{2}} f^{\top} \sum_{j=1}^{m} \Xi_{u}^{j} g} d(\pi \times \pi)(f, g)-1 . \tag{42}
\end{equation*}
$$

Step 2. What is left to show in this step, is that for $\pi=N(0, \Gamma), \Gamma \in \mathbb{R}^{d \times d}$ can be chosen such that the rhs of the previous display is small enough whilst also ensuring that $\pi\left(H_{\rho}^{c}\right)$ is controlled whenever $\rho^{2}$ satisfies (4)-(5) for $c_{\alpha}$ depending only on $\alpha \in(0,1)$.

For a given $c_{\alpha}>0$, set $\epsilon:=\frac{\rho}{c_{\alpha}^{1 / 4} d^{1 / 2}}$ and $\Gamma:=\epsilon^{2} \bar{\Gamma}$ for some $\bar{\Gamma} \in \mathbb{R}^{d \times d}$ to be specified later, separately for the private and public coin protocols. The remaining mass $\pi\left(H_{\rho}\right)$ can now be seen to equal

$$
\pi\left(f:\|f\|_{2}^{2} \leqslant \rho^{2}\right)=\operatorname{Pr}\left(Z^{\top} \bar{\Gamma} Z \leqslant \sqrt{c_{\alpha}} d\right),
$$

where $Z$ is a $d$-dimensional standard normal vector. If $\bar{\Gamma}$ is symmetric, idempotent and has rank (proportional to) $d$, the concentration inequality in Lemma A. 13 yields that the probability on the rhs of the above display can be made arbitrarily small for small enough choice of $c_{\alpha}>0$.

We now proceed to bound the first factor in the product on the rhs of (42), which for a positive semi-definite choice of $\bar{\Gamma}$ equals
$\prod_{j=1}^{m} \int \mathbb{E}_{0}^{X^{j} \mid U=u} \exp \left(\frac{n}{m}(\sqrt{\bar{\Gamma}}(f+g))^{\top} X^{j}-\frac{n}{2 m}\|\sqrt{\bar{\Gamma}} f\|_{2}^{2}-\frac{n}{2 m}\|\sqrt{\bar{\Gamma}} g\|_{2}^{2}\right) d N\left(0, \epsilon^{2} I_{2 d}\right)(f, g)$.
By direct computation, the latter display equals

$$
\prod_{j=1}^{m} \int \exp \left(\frac{n \epsilon^{2}}{m} z^{\top} \bar{\Gamma} z^{\prime}\right) d N\left(0, I_{2 d}\right)\left(z, z^{\prime}\right)
$$

By applying the moment generating function of the Gaussian chaos, e.g. Lemma 6.2.2 in [40] to the above display and using that $\rho^{2}$ satisfies (4) or (5), we obtain that for $\frac{n \epsilon^{2}}{m}\|\bar{\Gamma}\| \lesssim$ $\frac{n \rho^{2}}{c_{\alpha}^{1 / 2} m \sqrt{d}} \leqslant \sqrt{c_{\alpha} / m} \leqslant \sqrt{c_{\alpha}}$ small enough, where $\|\cdot\|$ denotes the spectral norm of a matrix, there exists a constant $C \geqslant\|\bar{\Gamma}\|^{2} / d$ such that

$$
\begin{equation*}
\prod_{j=1}^{m} \mathbb{E}_{0}^{X^{j} \mid U=u}\left[\mathscr{L}_{\pi}\left(X^{j}\right)^{2}\right] \leqslant \exp \left(C c_{\alpha}^{-1} \frac{n^{2} \rho^{4}}{m d}\right) \leqslant \exp \left(C c_{\alpha}\right) . \tag{43}
\end{equation*}
$$

The exponent can be made arbitrarily close to zero per choice of $c_{\alpha}>0$.
We switch our attention now to the second factor in the product on the rhs of (42), which we bound by applying Lemma 6.2.2 in [40] once more,

$$
\begin{equation*}
\int e^{\frac{n^{2}}{m^{2}}(\sqrt{\Gamma} f)^{\top} \sum_{j=1}^{m} \Xi_{u}^{j}(\sqrt{\Gamma} g)} d N\left(0, \epsilon^{2} I_{2 d}\right)(f, g) \leqslant e^{C \frac{n^{4} \epsilon^{4}}{m^{4}} \operatorname{Tr}\left(\left(\sqrt{\Gamma}^{\top} \Xi_{u} \sqrt{\Gamma}\right)^{2}\right)}, \tag{44}
\end{equation*}
$$

whenever

$$
\begin{equation*}
\frac{n^{2} \epsilon^{2}}{m^{2}}\left\|\sqrt{\bar{\Gamma}}^{\top} \Xi_{u} \sqrt{\bar{\Gamma}}\right\| \tag{45}
\end{equation*}
$$

is small enough.
It remains to choose a symmetric, idempotent positive semi-definite $\bar{\Gamma}$ that sufficiently bounds (45) and to combine the above displays providing the stated lower bound for the testing risk. For the exact choice of $\bar{\Gamma}$, we distinguish between the public coin and private coin cases. In both cases, we employ the data processing inequalities of Lemma A. 3 and Lemma A.2, which yield that

$$
\begin{equation*}
\operatorname{Tr}\left(\Xi_{u}\right)=\sum_{j=1}^{m} \operatorname{Tr}\left(\Xi_{u}^{j}\right) \leqslant \min \left\{2 \log 2 \cdot \frac{b}{d}, 1\right\} \frac{m^{2} d}{n} . \tag{46}
\end{equation*}
$$

The public coin case: In this case, it suffices to take $\bar{\Gamma}=I_{d}$, which is trivially symmetric, idempotent and positive semi-definite. By Lemma A.2, $\frac{n}{m} \Xi_{u}^{j} \leqslant I_{d}$, so (45) holds as well for this choice of $\bar{\Gamma}$ :

$$
\frac{n^{2} \epsilon^{2}}{m^{2}}\left\|\Xi_{u}\right\| \leqslant n \epsilon^{2} \leqslant \frac{n \rho^{2}}{\sqrt{c_{\alpha}} d} \leqslant \sqrt{c_{\alpha}},
$$

where the second to last inequality holds for $\rho^{2}$ satisfying (4).
It remains to combine our results and provide a lower bound for the testing risk. Note that

$$
\operatorname{Tr}\left(\Xi_{u}^{2}\right)=\left\|\Xi_{u}\right\| \operatorname{Tr}\left(\Xi_{u}\right) \leqslant \frac{m^{2}}{n} \operatorname{Tr}\left(\Xi_{u}\right) \lesssim \frac{(b \wedge d) m^{4}}{n^{2}},
$$

where the last inequality follows from (46). Combining the above bound with assertions (44), (43), (42), (16), and (15), $\epsilon^{4}=c_{\alpha}^{-1} d^{-2} \rho^{4}$ and the fact that $\pi\left(H_{\rho}\right) \leqslant \alpha / 2$, we obtain that

$$
\begin{aligned}
\inf _{T \in \mathcal{T}_{\text {pub }}(b)} \mathcal{R}\left(H_{\rho}, T\right) & \geqslant 1-\sqrt{2\left(e^{C\left(\frac{n^{2} \rho^{4}}{c_{\alpha} m_{d}}+\frac{n^{2} \rho^{4}(b \alpha d)}{\alpha_{\alpha} d^{2}}\right)}-1\right)}-\pi\left(H_{\rho}^{c}\right) \\
& \geqslant 1-\sqrt{2\left(e^{\left.2 C c_{\alpha}-1\right)}-\alpha / 2>1-\alpha,\right.}
\end{aligned}
$$

whenever $\rho^{2}$ satisfies (4) for $c_{\alpha}>0$ small enough. This finishes the proof for the public coin case.

The private coin case: Since without loss of generality we can assume that $U$ is degenerate in the private coin case, $\Xi_{u}=\Xi$ for $\mathbb{P}^{U}$-almost every $u$. The matrix $\Xi$ is positive definite and symmetric, therefore it possesses a spectral decomposition $V^{\top} \operatorname{Diag}\left(\xi_{1}, \ldots, \xi_{d}\right) V$. Without loss of generality, assume that $\xi_{1} \geqslant \xi_{2} \geqslant \ldots \geqslant \xi_{d}$ with corresponding eigenvectors $V=\left(v_{1} \ldots v_{d}\right)$. Let $\check{V}$ denote the $d \times\lceil d / 2\rceil$ matrix $\left(v_{\lfloor d / 2\rfloor+1} \ldots v_{d}\right)$. The choice of prior may depend on $\Xi$, to see this, note the order of the supremum and infimum in (15) and the fact that $\Xi$ soley depends on the choice of kernel. To that extent, set $\bar{\Gamma}=\check{V}^{\prime} \check{V}^{\top}$. It holds that

$$
\operatorname{Tr}\left(\check{V} \check{V}^{\top}\right)=\sum_{i=1}^{d} \sum_{k=\lfloor d / 2\rfloor+1}^{d}\left(v_{k}\right)_{i}^{2}=\lceil d / 2\rceil
$$

The choice $\Gamma=\epsilon^{2} \bar{\Gamma}$ is thus seen to satisfy the conditions of symmetry and positive definiteness and is idempotent with rank $\lceil d / 2\rceil$.

Since the eigenvalues are decreasingly ordered,

$$
\xi_{\lfloor d / 2\rfloor} \leqslant \frac{2}{d} \sum_{i=1}^{\lfloor d / 2\rfloor} \xi_{i} \leqslant \frac{2}{d} \operatorname{Tr}(\Xi)
$$

By orthogonality of the columns of $V, \check{V}^{\top} \Xi \check{V}=\operatorname{Diag}\left(\xi_{\lfloor d / 2\rfloor+1}, \ldots, \xi_{d}\right)$. Combining this inequality with the last display and assertion (46) we get that for $\rho^{2}$ satisfying (5) the term (45) can be made arbitrarily small for small enough choice of $c_{\alpha}$, i.e.

$$
\begin{aligned}
\frac{n^{2} \epsilon^{2}}{m^{2}}\left\|\sqrt[\bar{\Gamma}]{ }^{\top} \Xi_{u} \sqrt{\bar{\Gamma}}\right\| & \leqslant \frac{n^{2} \epsilon^{2}}{m^{2}} \xi_{\lfloor d / 2\rfloor} \leqslant 2 \frac{n^{2} \rho^{2}}{\sqrt{c_{\alpha}} d^{2} m^{2}} \operatorname{Tr}(\Xi) \\
& \leqslant(4 \log 2) \frac{n \rho^{2}(b \wedge d)}{\sqrt{c_{\alpha}} d^{2}} \leqslant(4 \log 2) \sqrt{c_{\alpha} / d}
\end{aligned}
$$

Finally, a similar argument will be used to bound the right hand side of (44) and finally to provide a lower bound for the testing risk. Note that

$$
\operatorname{Tr}\left(\left(\sqrt{\bar{\Gamma}}^{\top} \Xi_{u} \sqrt{\bar{\Gamma}}\right)^{2}\right)=\operatorname{Tr}\left(\left(\check{V}^{\top} \Xi \check{V}\right)^{2}\right)=\sum_{i=\lfloor d / 2\rfloor+1}^{d} \xi_{i}^{2} \leqslant d \xi_{\lfloor d / 2\rfloor}^{2} \leqslant \frac{4}{d} \operatorname{Tr}(\Xi)^{2}
$$

which implies in turn that

$$
\frac{n^{4} \epsilon^{4}}{m^{4}} \operatorname{Tr}\left(\left(\check{V}^{\top} \Xi \check{V}\right)^{2}\right) \leqslant 4 \frac{n^{4} \rho^{4}}{c_{\alpha} m^{4} d^{3}} \operatorname{Tr}(\Xi)^{2} \leqslant 4 \frac{n^{2} \rho^{4}(b \wedge d)^{2}}{c_{\alpha} d^{3}}
$$

where the last inequality follows from (46). Consequently, we have obtained that

$$
\begin{aligned}
\inf _{T \in \mathcal{T}_{p r i v}(b)} \mathcal{R}\left(H_{\rho}, T\right) & \geqslant 1-\sqrt{2\left(e^{C\left(\frac{n^{2} \rho^{4}}{c_{\alpha} m d}+\frac{n^{2} \rho^{4}(b \wedge d)^{2}}{c_{\alpha} d^{3}}\right)}-1\right)}-\pi\left(H_{\rho}^{c}\right) \\
& \geqslant 1-\sqrt{2\left(e^{2 C c_{\alpha}-1}-\alpha / 2>1-\alpha\right.}-\alpha
\end{aligned}
$$

for $\rho^{2}$ satisfying (5) and $c_{\alpha}>0$ small enough.
10. Lemma 10.1: Gaussian maximization. Before giving the detailed statement of the lemma below, we briefly contemplate on its aim and proof. The lemma bears a close connection to Brascamp-Lieb inequalities [10, 26, 8]. Brascamp-Lieb type inequalities have appeared in context of information theory in the literature before, see e.g. [16, 27], where Gaussian extremality is established for certain information theoretic optimization problems. Instead of the information theoretic entropy based route, we rely on the technique of [26]. The resulting lemma allows us to bound the ratio between (35) and (38), i.e.

$$
\begin{equation*}
\frac{\int \mathscr{L}_{\pi}\left(x_{1}\right) \mathscr{L}_{\pi}\left(x_{2}\right) q_{u}\left(x_{1}, x_{2}\right) d\left(P_{0} \times P_{0}\right)\left(x_{1}, x_{2}\right)}{\prod_{j=1}^{m} \int \mathscr{L}_{\pi}^{j}\left(x_{1}^{j}\right) \mathscr{L}_{\pi}^{j}\left(x_{2}^{j}\right) q_{u}^{j}\left(x_{1}^{j}, x_{2}^{j}\right) d\left(P_{0} \times P_{0}\right)\left(x_{1}^{j}, x_{2}^{j}\right)} \tag{47}
\end{equation*}
$$

by (39), i.e. a Gaussian distribution with matching mean and covariance. Consequently, we obtain a quadratic form in the covariance that we would otherwise obtain via a Taylor expansion. That such a quadratic form does not follow through more standard means such as Taylor expansion is described in [3], Section 4.

The proof of the lemma exploits the conjugacy between likelihood of the observation $X$ and the Gaussian prior on the parameter to obtain that a Gaussian distribution is in fact an extremal case. For reasons of space, we defer the proof to Section F of the Supplementary Material.

LEMMA 10.1. For $x \in \mathbb{R}^{m k}$, let $x^{j} \in \mathbb{R}^{k}, j=1, \ldots, m$, denote the projection of $x$ on the coordinates $\{(j-1) k+1, \ldots, j k\}$. Let $\Lambda \in \mathbb{R}^{k \times k}$ a positive definite symmetric matrix and $\Lambda^{\otimes m}=\operatorname{Diag}(\Lambda, \ldots, \Lambda) \in \mathbb{R}^{m k \times m k}$. For $h \in \mathbb{R}^{k}$, let $p_{h}$ denote the density of a $N(h, \Lambda)$ distribution with respect to the Lebesgue measure on $\mathbb{R}^{k}$ and let $p_{h}^{m}(x):=\prod_{j=1}^{m} p_{h}\left(x^{j}\right)$. Consider for some $M>0, \mathcal{Q} \equiv \mathcal{Q}(M, \Sigma)$ the class of all nonnegative functions $q \in L_{\infty}\left(\mathbb{R}^{m k}\right)$ satisfying $\frac{q(x)}{\int q(x) p_{0}^{m}(x) d x} \leqslant M P_{0}^{m}$-a.e., $\int x q(x) p_{0}^{m}(x) d x=0$ and $\int x x^{\top} q(x) p_{0}^{m}(x) d x=\Sigma$. Furthermore, let $H$ a $N(0, \Upsilon)$-distributed random vector in $\mathbb{R}^{k}$. Then

$$
\sup _{q \in \mathcal{Q}} \frac{\int \mathbb{E}^{H} \Pi_{j=1}^{m} \frac{p_{H}}{p_{0}}\left(x^{j}\right) q(x) p_{0}^{m}(x) d x}{\int \Pi_{j=1}^{m} \mathbb{E}^{H} \frac{p_{H}}{p_{0}}\left(x^{j}\right) q(x) p_{0}^{m}(x) d x} \leqslant \frac{\int \mathbb{E}^{H} \Pi_{j=1}^{m} \frac{p_{H}}{p_{0}}\left(x^{j}\right) d N(0, \Sigma)(x)}{\int \Pi_{j=1}^{m} \mathbb{E}^{H} \frac{p_{H}}{p_{0}}\left(x^{j}\right) d N(0, \Sigma)(x)} .
$$

Acknowledgements: We would like to thank Elliot H. Lieb for a helpful comment regarding the proof of Lemma 10.1. This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No. 101041064).

## SUPPLEMENTARY MATERIAL

Supplementary Material to Optimal high-dimensional and nonparametric distributed testing under communication constraints
In the supplement to this paper [35], we present the detailed proofs for the main theorems in the paper "Optimal high-dimensional and nonparametric distributed testing under communication constraints".

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# SUPPLEMENTARY MATERIAL TO "OPTIMAL HIGH-DIMENSIONAL AND NONPARAMETRIC DISTRIBUTED TESTING UNDER COMMUNICATION CONSTRAINTS" 

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#### Abstract

In this supplement, we present the detailed proofs for the main theorems in the paper "Optimal high-dimensional and nonparametric distributed testing under communication constraints".


## A. Auxilliary lemmas for finite dimensional Gaussian mean testing.

A.1. Lemmas related to the lower bound (Theorem 3.1). Following the notation of Section 2 in the article, let $\mathbb{P}_{f} \equiv \mathbb{P}_{f, K}$ denote the joint distribution of $Y, U$ and $X$ where $X^{j}$ follows $N\left(f, \frac{m}{n} I_{d}\right)$ and $Y \sim \mathbb{E}_{f}^{X, U} K(\cdot \mid X, U)=: \mathbb{P}_{f, K}^{Y}$ for $f \in \mathbb{R}^{d}$. Let $\pi$ be a probability distribution on $\mathbb{R}^{d}$ and define the mixture distribution $P_{\pi}$ by

$$
P_{\pi}(A)=\int P_{f}(A) d \pi(f)
$$

where $P_{f}=\mathbb{P}_{f}^{X}$.
Lemma A.1. [Le Cam bound] For any distribution on $U$, it holds that

$$
\inf _{\varphi, K}\left(\mathbb{E}_{0, K}^{Y} \varphi+\sup _{f \in H_{\rho}} \mathbb{E}_{f, K}^{Y}(1-\varphi)\right) \geqslant \inf _{K}\left(\sup _{\pi}\left(1-\left\|\mathbb{P}_{0, K}^{Y}-\mathbb{P}_{\pi, K}^{Y}\right\|_{T V}\right)-\pi\left(H_{\rho}^{c}\right)\right),
$$

where

- the infimum on the lhs is taken over all Markov kernels $K: 2^{\mathcal{Y}} \times \mathcal{X} \times \mathcal{U} \rightarrow[0,1]$ in a suitable way and maps $\varphi: \mathcal{Y} \rightarrow\{0,1\}$,
- the infimum on the rhs is over the same class od Markov kernels,
- the supremum on the rhs is over all prior distributions $\pi$ on $\mathbb{R}^{d}$.

Proof. It trivially holds that for any $\varphi^{\prime}: \mathcal{Y} \mapsto\{0,1\}$,

$$
\left(\mathbb{E}_{0, K}^{Y} \varphi^{\prime}(Y)+\sup _{f \in H_{\rho}} \mathbb{E}_{f, K}^{Y}\left(1-\varphi^{\prime}(Y)\right)\right) \geqslant \inf _{\varphi}\left(\mathbb{E}_{0, K}^{Y} \varphi(Y)+\sup _{f \in H_{\rho}} \mathbb{E}_{f, K}^{Y}(1-\varphi(Y))\right)
$$

where the infimum is over all $\varphi: \mathcal{Y} \mapsto\{0,1\}$. Furthermore, for any prior distribution $\pi$ on $\mathbb{R}^{d}$ it holds that

$$
\sup _{f \in H_{\rho}} \mathbb{E}_{f, K}^{Y}(1-\varphi(Y)) \geqslant \int_{\left\{f \in H_{\rho}\right\}} \mathbb{E}_{f, K}^{Y}(1-\varphi(Y)) d \pi(f)
$$

$$
\begin{equation*}
\geqslant \int \mathbb{E}_{f, K}^{Y}(1-\varphi(Y)) d \pi(f)-\pi\left(H_{\rho}^{c}\right) . \tag{S.1}
\end{equation*}
$$

Hence the rhs of the second last display is further bounded from below by

$$
\inf _{\varphi}\left(\mathbb{E}_{0, K}^{Y} \varphi(Y)+\mathbb{E}_{\pi, K}^{Y}(1-\varphi(Y))-\pi\left(H_{\rho}^{c}\right)\right)
$$

for all prior distributions $\pi$ on $\mathbb{R}^{d}$. For any $\varphi$, write $A_{\varphi}=\varphi^{-1}(\{0\})$ and note that

$$
\mathbb{P}_{0, K}^{Y} \varphi(Y)+\mathbb{P}_{\pi, K}^{Y}(1-\varphi(Y))=1-\left(\mathbb{P}_{0, K}^{Y}\left(Y \in A_{\varphi}\right)-\mathbb{P}_{\pi, K}^{Y}\left(Y \in A_{\varphi}\right)\right) .
$$

By combining the above two displays we get that

$$
\inf _{\varphi}\left(\mathbb{E}_{0, K}^{Y} \varphi(Y)+\sup _{f \in H_{\rho}} \mathbb{E}_{f, K}^{Y}(1-\varphi(Y))\right) \geqslant 1-\sup _{A}\left|\mathbb{P}_{0, K}^{Y}(A)-\mathbb{P}_{\pi, K}^{Y}(A)\right|-\pi\left(H_{\rho}^{c}\right) .
$$

Since the above is true for any distribution $\pi$ on $\mathbb{R}^{d}$, the statement is true after taking the supremum over $\pi$ also. Since the above holds for an arbitrary Markov kernel $K: 2^{\mathcal{Y}} \times \mathcal{X} \times$ $\mathcal{U} \rightarrow[0,1]$, the proof is concluded.

## Lemma A.2. Let $\Xi_{u}^{j}$ denote the matrix

$$
\Xi_{u}^{j}=\mathbb{E}_{0}^{Y^{j}} \mathbb{E}_{0}^{Y^{j} \mid U=u}\left[X^{j} \mid Y^{j}, U=u\right] \mathbb{E}_{0}^{Y^{j} \mid U=u}\left[X^{j} \mid Y^{j}, U=u\right]^{\top} .
$$

It holds that $\Xi_{u}^{j} \leqslant \frac{m}{n} I_{d}$.
Proof. Let $v \in \mathbb{R}^{d}$, then

$$
\begin{aligned}
v^{\top} \Xi_{u}^{j} v & =\mathbb{E}_{0}^{Y j} \mathbb{E}_{0}^{Y \mid U=u}\left[v^{\top} X^{j} \mid Y^{j}, U=u\right] \mathbb{E}_{0}^{Y \mid U=u}\left[\left(X^{j}\right)^{\top} v \mid Y^{j}, U=u\right] \\
& =\mathbb{E}_{0}^{Y j} \mathbb{E}_{0}^{Y \mid U=u}\left[v^{\top} X^{j} \mid Y^{j}, U=u\right]^{2} .
\end{aligned}
$$

Since the conditional expectation contracts the $L_{2}$-norm, we obtain that the latter is bounded by

$$
\mathbb{E}_{0} v^{\top} X^{j}\left(X^{j}\right)^{\top} v=\frac{m}{n}\|v\|_{2}^{2},
$$

which completes the proof.
The previous lemma is in some sense a data processing inequality: the covariance matrix of $X \mid Y$ is strictly dominated by the covariance of the original process $X$. The following lemma extends this and shows that the trace of the covariance satisfies a different data processing inequality, where the loss of information due to $Y^{j}$ having only $b^{j}$ bits available is captured. When $b^{j}<d$, the latter data processing inequality is stronger than the one implied by Lemma A.2. The lemma below is essentially Theorem 2 of [7] adapted to our setting, for which we provide a different proof that results in a smaller constant.

Lemma A.3. Consider the matrix $\Xi_{u}^{j}$ given in Lemma A.2, then

$$
\begin{equation*}
\operatorname{Tr}\left(\Xi_{u}^{j}\right) \leqslant 2 \log (2) \frac{m}{n}\left(\log _{2}\left|\mathcal{Y}^{j}\right|\right) . \tag{S.2}
\end{equation*}
$$

In particular, for $\log _{2}\left|\mathcal{Y}^{j}\right|=b^{j}$,

$$
\begin{equation*}
\operatorname{Tr}\left(\Xi_{u}^{j}\right) \leqslant\left(2 \log (2) \frac{b^{j}}{d} \bigwedge 1\right) \frac{m d}{n} \tag{S.3}
\end{equation*}
$$

Proof. We start by noting that under $\mathbb{P}_{0}, X^{j}$ follows a $N\left(0, \frac{m}{n} I_{d}\right)$ distribution. For any unit vector $v \in \mathbb{R}^{d}$ and $s \in \mathbb{R}$ this means that

$$
\mathbb{E}_{0} e^{s\left\langle X^{j}, v\right\rangle} \leqslant e^{\frac{s^{2} m}{2 n}}
$$

Furthermore, for arbitary $y \in \mathcal{Y}$,

$$
\begin{aligned}
\sum_{y} \mathbb{P}^{Y^{j} \mid U=u}(y) \mathbb{E}_{0}\left[e^{s\left\langle X^{j}, v\right\rangle} \mid Y^{j}=y, U=u\right] & \geqslant \mathbb{P}^{Y^{j} \mid U=u}(y) \mathbb{E}_{0}\left[e^{s\left\langle X^{j}, v\right\rangle} \mid Y^{j}=y, U=u\right] \\
& \geqslant \mathbb{P}^{Y^{j} \mid U=u}(y) e^{s \mathbb{E}_{0}\left[\left\langle X^{j}, v\right\rangle \mid Y^{j}=y, U=u\right]},
\end{aligned}
$$

where the last line follows by Jensen's inequality. By combining the above displays we obtain that

$$
s \mathbb{E}_{0}\left[\left\langle X^{j}, v\right\rangle \mid Y^{j}=y, U=u\right] \leqslant \frac{s^{2} m}{2 n}-\log \mathbb{P}^{Y^{j} \mid U=u}(y)
$$

for all $s \in \mathbb{R}$. Choosing $s=\frac{n}{m} \mathbb{E}_{0}\left[\left\langle X^{j}, v\right\rangle \mid Y^{j}=y, U=u\right]$, we have for any unit vector $v \in$ $\mathbb{R}^{d}$,

$$
\mathbb{E}_{0}\left[\left\langle X^{j}, v\right\rangle \mid Y^{j}=y, U=u\right]^{2} \leqslant-2 \frac{m}{n} \log \mathbb{P}^{Y^{j} \mid U=u}(y) .
$$

Next define for $y \in \mathcal{Y}^{j}$

$$
\begin{equation*}
w_{1, y}=\frac{1}{\left\|\mathbb{E}_{0}\left(X^{j} \mid Y^{j}=y, U=u\right)\right\|_{2}} \mathbb{E}_{0}\left[X^{j} \mid Y^{j}=y, U=u\right] \tag{S.4}
\end{equation*}
$$

Choose now $w_{2, y}, \ldots, w_{d, y}$ such that together with $w_{1, y}$ the vectors form an orthonormal basis for $\mathbb{R}^{d}$. We then have

$$
\begin{aligned}
\operatorname{Tr}\left(\Xi_{u}^{j}\right) & =\sum_{y \in \mathcal{Y}^{j}} \mathbb{P}^{Y^{j} \mid U=u}(y) \sum_{i=1}^{d} \mathbb{E}_{0}\left[\left\langle w_{i, y}, X^{j}\right\rangle \mid Y^{j}=y, U=u\right]^{2} \\
& =\sum_{y \in \mathcal{Y}^{j}} \mathbb{P}^{Y^{j} \mid U=u}(y) \mathbb{E}_{0}\left[\left\langle w_{1, y}, X^{j}\right\rangle \mid Y^{j}=y, U=u\right]^{2} \\
& \leqslant-2 \frac{m}{n} \sum_{y \in \mathcal{Y}^{j}} \mathbb{P}^{Y^{j} \mid U=u}(y) \log \mathbb{P}^{Y^{j} \mid U=u}(y) \leqslant 2 \frac{m}{n} \log \left|\mathcal{Y}^{j}\right|,
\end{aligned}
$$

where the last inequality follows from the fact that uniform distribution on $\mathcal{Y}^{j}$ maximizes the entropy on the lhs. For the second statement note that by construction $\log \left|\mathcal{Y}^{j}\right| \leqslant b^{j} \log 2$. Furthermore in view of of Lemma A.2, $\log \left|\mathcal{Y}^{j}\right| \leqslant d m / n$. Then the statement follows by combining the above upper bounds for $\log \left|\mathcal{Y}^{j}\right|$ with the preceding display.
A.2. Lemmas for the upper bound theorems in the finite dimensional Gaussian mean model. We state a slightly extended version of Lemma 4.1.

Lemma A.4. Consider for $k, l \in \mathbb{N}, l \geqslant 2$, independent random variables $\left\{B_{i}^{j}: i=\right.$ $1, \ldots, k, j=1, \ldots, l\}$ with $B_{i}^{j} \sim \operatorname{Ber}\left(p_{i}\right)$. If $p_{i}=1 / 2$ for $i=1, \ldots, k$, for each $\alpha \in(0,1)$ there exists $\kappa_{\alpha}>0$ such that

$$
\operatorname{Pr}\left(\left|\frac{1}{\sqrt{k} l} \sum_{i=1}^{k}\left(\sum_{j=1}^{l}\left(B_{i}^{j}-\frac{1}{2}\right)\right)^{2}-\sqrt{k} / 4\right| \geqslant \kappa_{\alpha}\right) \leqslant \alpha .
$$

On the other hand, for arbitrary $c_{\alpha, n}>0$,

$$
\begin{equation*}
\eta_{p, l, k}:=\frac{l-1}{2 \sqrt{k}} \sum_{i=1}^{k}\left(p_{i}-\frac{1}{2}\right)^{2} \geqslant c_{\alpha, n} \tag{S.5}
\end{equation*}
$$

it holds that

$$
\begin{equation*}
\operatorname{Pr}\left(\left|\frac{1}{\sqrt{k} l} \sum_{i=1}^{k}\left(\sum_{j=1}^{l}\left(B_{i}^{j}-\frac{1}{2}\right)\right)^{2}-\sqrt{k} / 4\right| \leqslant c_{\alpha, n}\right) \leqslant \frac{1 / 2+16 \eta_{p, l, k} / \sqrt{k}}{\eta_{p, l, k}^{2}} . \tag{S.6}
\end{equation*}
$$

Proof. The LHS in the event having bounded variance: a straightforward computation (using that for $B_{i}^{j} \sim \operatorname{Bern}\left(p_{i}\right)$, the central fourth moment is $E\left(B_{i}^{j}-p_{i}\right)^{4}=p_{i}\left(1-p_{i}\right)(1-$ $\left.3 p_{i}\left(1-p_{i}\right)\right) \leqslant 1 / 16$ and $\operatorname{Var}(X) \leqslant E X^{2}$ ) yields

$$
\begin{gather*}
\mathbb{E}\left[\frac{1}{\sqrt{k} l} \sum_{i=1}^{k}\left(\sum_{j=1}^{l}\left(B_{i}^{j}-\frac{1}{2}\right)\right)^{2}-\frac{\sqrt{k}}{4}\right]^{2}=\frac{1}{k l^{2}} \sum_{i=1}^{k} \operatorname{Var}\left[\left(\sum_{j=1}^{l}\left(B_{i}^{j}-1 / 2\right)\right)^{2}\right] \\
\leqslant \frac{1}{l^{2}} \sum_{j=1}^{l} \mathbb{E}\left(B_{i}^{j}-1 / 2\right)^{4}+\frac{1}{l^{2}} \sum_{j=1}^{l}\left(\mathbb{E}\left(B_{i}^{j}-1 / 2\right)^{2}\right)^{2} \leqslant 1 / 8 \tag{S.7}
\end{gather*}
$$

after which Chebyshev's inequality yields the first statement.
We turn to the second statement. Adding and subtracting $p_{i}$ and expanding the square, the lhs of the display in the lemma can be written as

$$
\begin{equation*}
\operatorname{Pr}\left(\left|\frac{1}{\sqrt{k} l} \sum_{i=1}^{k}\left(\sum_{j=1}^{l} B_{i}^{j}-l p_{i}\right)^{2}-\mu_{p}+\frac{l-1}{\sqrt{k}} \sum_{i=1}^{k}\left(p_{i}-\frac{1}{2}\right)^{2}+\zeta\right| \leqslant c_{\alpha, n}\right) \tag{S.8}
\end{equation*}
$$

where

$$
\mu_{p}:=\frac{1}{\sqrt{k}} \sum_{i=1}^{k} p_{i}\left(1-p_{i}\right) \text { and } \zeta:=\frac{2}{\sqrt{k}} \sum_{i=1}^{k}\left(p_{i}-\frac{1}{2}\right)\left(\sum_{j=1}^{l} B_{i}^{j}-l p_{i}\right) .
$$

The first term in the event of (S.8) has mean $\mu_{p}$ and variance (by the same computations as in (S.7))

$$
\operatorname{Var}\left[\frac{1}{\sqrt{k} l} \sum_{i=1}^{k}\left(\sum_{j=1}^{l} B_{i}^{j}-l p_{i}\right)^{2}\right]=\frac{1}{k l^{2}} \sum_{i=1}^{k} \operatorname{Var}\left[\left(\sum_{j=1}^{l} B_{i}^{j}-l p_{i}\right)^{2}\right] \leqslant 1 / 8
$$

The term $\zeta$ has mean 0 and

$$
\operatorname{Var}(\zeta)=\frac{4 l}{k} \sum_{i=1}^{k}\left(p_{i}-\frac{1}{2}\right)^{2} p_{i}\left(1-p_{i}\right) \leqslant \frac{l}{k} \sum_{i=1}^{k}\left(p_{i}-\frac{1}{2}\right)^{2} .
$$

Applying the reverse triangle inequality and condition (S.5), the probability in (S.8) is bounded from above by

$$
\begin{aligned}
\operatorname{Pr}\left[\left\lvert\, \frac{1}{\sqrt{k} l} \sum_{i=1}^{k}\right.\right. & \left(\sum_{j=1}^{l} B_{i}^{j}-l p_{i}\right)^{2}-\mu_{p}\left|+|\zeta| \geqslant \frac{l-1}{2 \sqrt{k}} \sum_{i=1}^{k}\left(p_{i}-\frac{1}{2}\right)^{2}\right] \\
& \leqslant \operatorname{Pr}\left[\left|\frac{1}{\sqrt{k}} \sum_{i=1}^{k}\left(\sum_{j=1}^{l} B_{i}^{j}-l p_{i}\right)^{2}-\mu_{p}\right| \geqslant \eta_{p, l, k} / 2\right]+\operatorname{Pr}\left[|\zeta| \geqslant \eta_{p, l, k} / 2\right] \\
& \leqslant \frac{1 / 8}{\left(\eta_{p, l, k} / 2\right)^{2}}+\frac{2 l k^{-1 / 2} \eta_{p, l, k} /(l-1)}{\left(\eta_{p, l, k} / 2\right)^{2}} \leqslant \frac{1 / 2+16 \eta_{p, l, k} / \sqrt{k}}{\eta_{p, l, k}^{2}},
\end{aligned}
$$

where the last line follows by Chebyshev's inequality.

Next we provide another version of the above lemma, with the sum over the index $i$ moved inside of the square.

LEMMA A.5. Consider for $k, l \in \mathbb{N}, l \geqslant 2$, independent random variables $\left\{B_{i}^{j}: i=\right.$ $1, \ldots, k ; j=1, \ldots, l\}$ with $B_{i}^{j} \sim \operatorname{Ber}\left(p_{i}\right)$. If $p_{i}=1 / 2$ for $i=1, \ldots, k$, for each $\alpha \in(0,1)$ there exists $\kappa_{\alpha}>0$ such that

$$
\operatorname{Pr}\left(\left|\frac{1}{l k}\left(\sum_{i=1}^{k} \sum_{j=1}^{l}\left(B_{i}^{j}-\frac{1}{2}\right)\right)^{2}-1 / 4\right| \geqslant \kappa_{\alpha}\right) \leqslant \alpha
$$

On the other hand, if $p_{i} \geqslant 1 / 2$ for all $i=1, \ldots, k$ and for arbitrary $c_{\alpha, n}>0$

$$
\begin{equation*}
\eta_{p, l, k}^{\prime}:=\frac{l-1}{2 k}\left(\sum_{i=1}^{k}\left(p_{i}-\frac{1}{2}\right)\right)^{2} \geqslant c_{\alpha, n} \tag{S.9}
\end{equation*}
$$

it holds that

$$
\operatorname{Pr}\left(\left|\frac{1}{k l}\left(\sum_{i=1}^{k} \sum_{j=1}^{l}\left(B_{i}^{j}-\frac{1}{2}\right)\right)^{2}-1 / 4\right| \leqslant c_{\alpha, n}\right) \leqslant \frac{1 / 2+16 \eta_{p, l, k}^{\prime} / k}{\left(\eta_{p, l, k}^{\prime}\right)^{2}}
$$

PROOF. The lhs in the event having bounded variance: by the same arguments as in (S.7) we have

$$
\mathbb{E}\left[\frac{1}{l k}\left(\sum_{i=1}^{k} \sum_{j=1}^{l}\left(B_{i}^{j}-\frac{1}{2}\right)\right)^{2}-\frac{1}{4}\right]^{2} \leqslant 1 / 8
$$

after which Chebyshev's inequality yields the first statement.
We turn to the second statement. Adding and subtracting $p_{i}$ and expanding the square, the lhs of the display in the lemma can be written as

$$
\begin{equation*}
\operatorname{Pr}\left(\left|\frac{1}{l k}\left(\sum_{i=1}^{k} \sum_{j=1}^{l}\left(B_{i}^{j}-p_{i}\right)\right)^{2}-\mu_{p}^{\prime}+\frac{l-1}{k}\left(\sum_{i=1}^{k}\left(p_{i}-\frac{1}{2}\right)\right)^{2}+\zeta\right| \leqslant c_{\alpha, n}\right) \tag{S.10}
\end{equation*}
$$

where

$$
\mu_{p}^{\prime}:=1 / 4-\frac{1}{k}\left(\sum_{i=1}^{k}\left(p_{i}-\frac{1}{2}\right)\right)^{2} \quad \text { and } \quad \zeta:=\frac{2}{k}\left(\sum_{i=1}^{k}\left(p_{i}-\frac{1}{2}\right)\right)\left(\sum_{i=1}^{k} \sum_{j=1}^{l}\left(B_{i}^{j}-p_{i}\right)\right)
$$

Next we note that in view of the assumption $p_{i} \geqslant 1 / 2$ we have that

$$
\mu_{p}^{\prime} \leqslant 1 / 4-\frac{1}{k} \sum_{i=1}^{k}\left(p_{i}-\frac{1}{2}\right)^{2}=\frac{1}{k} \sum_{i=1}^{k} p_{i}\left(1-p_{i}\right)=: \mu_{p}
$$

The first term in the event of (S.10) has mean $\mu_{p}$ and variance (by the same computations as in (S.7))

$$
\operatorname{Var}\left[\frac{1}{l k}\left(\sum_{i=1}^{k} \sum_{j=1}^{l}\left(B_{i}^{j}-p_{i}\right)\right)^{2}\right]=\frac{1}{l^{2} k^{2}} \operatorname{Var}\left[\left(\sum_{j=1}^{l} \sum_{i=1}^{k}\left(B_{i}^{j}-p_{i}\right)\right)^{2}\right] \leqslant 1 / 8
$$

The term $\zeta$ has mean 0 and

$$
\operatorname{Var}(\zeta)=\frac{4 l}{k^{2}}\left(\sum_{i=1}^{k}\left(p_{i}-\frac{1}{2}\right)\right)^{2} \sum_{i=1}^{k} p_{i}\left(1-p_{i}\right) \leqslant \frac{l}{k^{2}}\left(\sum_{i=1}^{k}\left(p_{i}-\frac{1}{2}\right)\right)^{2} .
$$

Applying the reverse triangle inequality, condition (S.9) and the inequality $\mu_{p} \geqslant \mu_{p}^{\prime}$, the probability in (S.10) is bounded from above by

$$
\begin{aligned}
\operatorname{Pr}\left[\left\lvert\, \frac{1}{k l}\left(\sum_{i=1}^{k}\right.\right.\right. & \left.\sum_{j=1}^{l}\left(B_{i}^{j}-p_{i}\right)\right)^{2}-\mu_{p}\left|+|\zeta| \geqslant \frac{l-1}{2}\left(\sum_{i=1}^{k} p_{i}-\frac{1}{2}\right)^{2}\right] \\
& \leqslant \operatorname{Pr}\left[\left|\frac{1}{k l}\left(\sum_{i=1}^{k} \sum_{j=1}^{l}\left(B_{i}^{j}-p_{i}\right)\right)^{2}-\mu_{p}\right| \geqslant \eta_{p, l, k}^{\prime} / 2\right]+\operatorname{Pr}\left[|\zeta| \geqslant \eta_{p, l, k}^{\prime} / 2\right] \\
& \leqslant \frac{1 / 8}{\left(\eta_{p, l, k}^{\prime} / 2\right)^{2}}+\frac{2 l k^{-1} \eta_{p, l, k}^{\prime} /(l-1)}{\left(\eta_{p, l, k}^{\prime} / 2\right)^{2}} \leqslant \frac{1 / 2+16 \eta_{p, l, k}^{\prime} / k}{\eta_{p, l, k}^{\prime 2}}
\end{aligned}
$$

where the last line follows by Chebyshev's inequality.

Next we provide the lemmas used in Section 4, providing guarantees for the testing procedures $T_{\mathrm{I}}, T_{\text {II }}$ and $T_{\text {III }}$, proposed in subsections 4.1, 4.2 and 4.3, respectively.

LEmmA A.6. For each $\alpha \in(0,1)$, there exist constants $\kappa_{\alpha}, C_{\alpha}, M_{\alpha}, D_{0}>0$ such that for $m \geqslant M_{\alpha}$ and $d \geqslant D_{0}$ it holds that

$$
\mathcal{R}\left(H_{\rho}, T_{I}\right) \leqslant \alpha,
$$

whenever $\rho^{2} \geqslant C_{\alpha} \frac{\sqrt{m d}}{n}$.
Proof of Lemma A.6. Under the null hypothesis the random variables $Y_{\mathrm{I}}^{j} \sim^{i i d} \operatorname{Bern}(1 / 2)$. Next we shall apply Lemma 4.1 with $k=1$, and $l=m$. By the first statement of the lemma, we obtain that there exists $\kappa_{\alpha}>0$ such that $\mathbb{P}_{0} T_{\mathrm{I}} \leqslant \alpha / 2$.

We give an upper bound for the Type II error by using the second statement of the lemma, but before that we show that condition (6) holds. Note that the law of total expectation yields

$$
\mathbb{E}_{f} Y_{I}^{j}=\mathbb{E}_{f} \mathbb{E}_{f}\left[Y_{I}^{j} \mid S_{\mathrm{I}}^{j}\right]=\mathbb{E}_{f} F_{\chi_{d}^{2}}\left(S_{\mathrm{I}}^{j}\right)=\operatorname{Pr}\left(S_{\mathrm{I}}^{j} \geqslant W_{d}\right),
$$

where $S_{\mathrm{I}}^{j}$ is noncentral Chi-square distributed under $\mathbb{P}_{f}$ with $d$-degrees of freedom and noncentrality parameter $\frac{n}{m}\|f\|_{2}^{2}$ and $W_{d}$ is an independent chi-square distributed random variable with $d$-degrees of freedom. Then Lemma 4 in [34] yields that

$$
\begin{equation*}
\eta_{p, m, 1}=\frac{m-1}{2}\left(\mathbb{E}_{f} Y_{I}^{j}-\frac{1}{2}\right)^{2} \geqslant \frac{m-1}{3200}\left(\frac{n\|f\|_{2}^{2}}{m \sqrt{d}} \bigwedge \frac{1}{2}\right)^{2} . \tag{S.11}
\end{equation*}
$$

whenever $d \geqslant D_{0}$ for some universal constant $D_{0}>0$. Consequently, as $\|f\|_{2}^{2} \geqslant \rho^{2} \geqslant$ $C_{\alpha} \frac{\sqrt{m d}}{n}$, we obtain that condition (6) is satisfied whenever $m \geqslant M_{\alpha}$ for some large enough $C_{\alpha}>0$ and $M_{\alpha}>0$. Therefore the Type II error is bounded by the rhs of (7), which is monotone decreasing in $\eta_{p, m, 1}$ hence also in $C_{\alpha}$. Therefore by large enough choice of $C_{\alpha}$ the Type II error is bounded from above by $\alpha / 2$.

LEMMA A.7. For each $\alpha \in(0,1)$, there exist constants $\kappa_{\alpha}, C_{\alpha}, M_{\alpha}>0$ such that for $m \geqslant M_{\alpha}$

$$
\mathcal{R}\left(H_{\rho}, T_{I I}\right) \leqslant \alpha
$$

whenever $\rho^{2} \geqslant C_{\alpha} \frac{d}{n \sqrt{d \wedge b}}$.
Proof of Lemma A.7. First note that it is sufficient to consider the case $b \leqslant d$ as one can simply take $b=b \wedge d$. Then note that under $\mathbb{P}_{f}, \sqrt{n / m} U X^{j} \mid U \sim N_{d}\left(\sqrt{n / m} U f, I_{d}\right)$ by the rotational invariance of the Gaussian distribution. By linearity of the coordinate projection, conditionally on $U$,

$$
\mathbb{1}\left\{\left(\sqrt{n / m} U X^{j}\right)_{i}>0\right\} \stackrel{d}{=} \mathbb{1}\left\{\sqrt{n / m}(U f)_{i}+Z>0\right\}
$$

where $Z \sim N(0,1)$. As a consequence, the vector $S_{\text {II }}$ is conditionally on $U$ coordinate wise independent binomially distributed with parameters $m$ and $p_{f, U} \in[0,1]^{b}$ under $\mathbb{P}_{f}^{Y \mid U}$, where

$$
\left(p_{f, U}\right)_{i}=\Phi\left(\sqrt{n / m}(U f)_{i}\right)
$$

with $\Phi$ the standard normal cdf. Under the null hypothesis, $\left(S_{\mathrm{II}}\right)_{i}$ is $\operatorname{Bin}(m, 1 / 2)$ distributed since $p_{0, U}=(1 / 2, \ldots, 1 / 2) \in[0,1]^{b}$. Next we apply Lemma 4.1 with $k=b$ and $l=m$. By the first statement of the lemma, it follows that for $\kappa_{\alpha}$ large enough, $\mathbb{P}_{0} T_{\mathrm{II}} \leqslant \alpha / 2$.

In order apply the second statement of the lemma, which yields that the Type II error is bounded by $\alpha / 2$, it suffices to show that the event

$$
A=\left\{\frac{m-1}{2 \sqrt{b}} \sum_{i=1}^{b}\left(\left(p_{f, U}\right)_{i}-\frac{1}{2}\right)^{2} \geqslant N_{\alpha}\right\}
$$

where $N_{\alpha}:=\kappa_{\alpha} \vee \frac{16}{\alpha}$, occurs with $\mathbb{P}^{U}$-probability greater than $1-\alpha / 4$. Note that for this choice of $N_{\alpha}$, (6) is satisfied on the event $A$ and the rhs of (7) is smaller than $\alpha / 4$. The Type II error is then bound by $\mathbb{P}_{f} T_{\mathrm{II}} \leqslant \mathbb{P}_{f} T_{\mathrm{II}} \mathbb{1}_{A}+\mathbb{P}_{f} \mathbb{1}_{A^{c}} \leqslant \alpha / 2$.

We proceed to show that $\mathbb{P}_{f} \mathbb{1}_{A^{c}} \leqslant \alpha / 4$. By a standard bound on the Gaussian error function $x \mapsto 2 \Phi(x)-1$ (see Lemma A.11),

$$
\left(\Phi\left(\sqrt{n / m}(U f)_{i}\right)-\frac{1}{2}\right)^{2} \geqslant \frac{1}{12} \min \left\{\frac{n}{m}(U f)_{i}^{2}, 1\right\}
$$

which in turn implies that

$$
\mathbb{P}^{U}\left(\frac{m-1}{2 \sqrt{b}} \sum_{i=1}^{b}\left(\left(p_{f, U}\right)_{i}-\frac{1}{2}\right)^{2} \leqslant N_{\alpha}\right) \leqslant \mathbb{P}^{U}\left(\frac{m-1}{24 \sqrt{b}} \sum_{i=1}^{b} \min \left\{\frac{n}{m}(U f)_{i}^{2}, 1\right\} \leqslant N_{\alpha}\right)
$$

Note that $U f \stackrel{d}{=}\|f\|_{2}\left(Z_{1}, \ldots, Z_{d}\right) /\|Z\|_{2}$, where $Z=\left(Z_{1}, \ldots, Z_{d}\right) \sim N\left(0, I_{d}\right)$ (see e.g. Section 3.4 of [40]). Using that $\|f\|_{2} \geqslant \rho$ and $\rho^{2} \geqslant C_{\alpha} \frac{d}{n \sqrt{b}}$, the previous display is further bounded by

$$
\operatorname{Pr}\left(\frac{m-1}{24 \sqrt{b}} \sum_{i=1}^{b} \min \left\{C_{\alpha} \frac{d Z_{i}^{2}}{m \sqrt{b}\|Z\|_{2}^{2}}, 1\right\} \leqslant N_{\alpha}\right)
$$

Considering the intersection with the event $\left\{\|Z\|_{2}^{2} \leqslant k d\right\}$ for some $k>0$, the above display can be bounded by

$$
\operatorname{Pr}\left(\sum_{i=1}^{b} \min \left\{Z_{i}^{2}, C_{\alpha}^{-1} m \sqrt{b} k\right\} \leqslant \frac{24 b m k}{C_{\alpha}(m-1)} N_{\alpha}\right)+\operatorname{Pr}\left(\|Z\|_{2}^{2} \geqslant k d\right)
$$

For $k$ large enough (independent of $d$ ), the second term is less than $\alpha / 8$. By Lemma A.12,

$$
\operatorname{Pr}\left(\max _{1 \leqslant i \leqslant b} Z_{i}^{2} \geqslant C_{\alpha}^{-1} m \sqrt{b} k\right) \leqslant \frac{2 b}{e^{C_{\alpha}^{-1} m \sqrt{b} k / 4}} .
$$

For large enough $M_{\alpha} \geqslant C_{\alpha}$, the condition $m \geqslant M_{\alpha}$ implies that the right hand side is less than $\alpha / 8$. The first term in the second to last display is consequently bounded by

$$
\begin{gathered}
\operatorname{Pr}\left(\sum_{i=1}^{b} Z_{i}^{2} \leqslant \frac{24 b m k}{C_{\alpha}(m-1)} N_{\alpha}\right)+\operatorname{Pr}\left(\max _{1 \leqslant i \leqslant b} Z_{i}^{2} \geqslant C_{\alpha}^{-1} m \sqrt{b} k\right) \\
\leqslant \operatorname{Pr}\left(\sum_{i=1}^{b} Z_{i}^{2} \leqslant \frac{24 b m k}{C_{\alpha}(m-1)} N_{\alpha}\right)+\alpha / 8
\end{gathered}
$$

For $m \geqslant M_{\alpha} \geqslant 25$ and by choosing $C_{\alpha}$ large enough such that the Chernoff-Hoeffding bound on the left tail of the chi-square distribution (see Lemma A.13) can be applied to the first term of the preceding display we get that

$$
\begin{equation*}
\operatorname{Pr}\left(\sum_{i=1}^{b} Z_{i}^{2} \leqslant \frac{25 k N_{\alpha}}{C_{\alpha}} b\right) \leqslant \exp \left(-b \frac{\frac{25 k N_{\alpha}}{C_{\alpha}}-1-\log \left(\frac{25 k N_{\alpha}}{C_{\alpha}}\right)}{2}\right) \leqslant \alpha / 8 \tag{S.12}
\end{equation*}
$$

finishing the proof of the lemma.
Lemma A.8. For $\alpha \in(0,1)$, there exist constants $M_{\alpha}, C_{\alpha}>0$ such that when $m \geqslant$ $M_{\alpha} d^{2} / b^{2}$, the $b$-bit distributed private testing protocol $T_{\text {III }}$ given in (14) satisfies

$$
\mathcal{R}\left(H_{\rho}, T_{I I I}\right) \leqslant \alpha,
$$

whenever $\rho^{2} \geqslant C_{\alpha} \frac{d \sqrt{d}}{n b}$.
Proof. Fix an arbitrary $f \in H_{\rho}$ and define

$$
\begin{equation*}
\mathcal{J}=\left\{i: 1 \leqslant i \leqslant d, \frac{n}{m} f_{i}^{2} \geqslant 1\right\} . \tag{S.13}
\end{equation*}
$$

By Lemma A. 9 below, the test $T_{\text {III }}^{1}$ given in (12) with $\kappa_{\alpha}, C_{\alpha}, M_{\alpha}>0$ large enough satisfies

$$
\mathbb{E}_{0} T_{\mathrm{III}}^{1} \leqslant \alpha / 6, \quad \text { and } \quad \mathbb{E}_{f}\left(1-T_{\mathrm{III}}^{1}\right) \leqslant \alpha / 6,
$$

whenever

$$
\begin{equation*}
\sum_{i \notin \mathcal{J}} f_{i}^{2} \geqslant \rho^{2} / 2 \text { or } \frac{m b}{d \sqrt{d}}>M_{\alpha} . \tag{S.14}
\end{equation*}
$$

Next we consider the case where (S.14) does not hold. Then $M_{\alpha} \geqslant \frac{m b}{d \sqrt{d}} \geqslant M_{\alpha} \frac{\sqrt{d}}{b}$, where the second inequality follows from the assumption of the lemma. This implies that $b \geqslant \sqrt{d}$. Since $\frac{m b}{d \sqrt{d}} \leqslant M_{\alpha}$ and $m$ can be taken to be larger than arbitrary constant (otherwise we are in the non-distributed regime in which the minimax rate can be achieved locally), we can without loss of generality assume $d$ is larger than an arbitrary constant (depending only on $\alpha$ ), hence $b \geqslant \sqrt{d} \geqslant 2 \log (d+1)$ and the test $T_{\text {III }}^{2}$ and the corresponding transcripts can be constructed. Furthermore, $\sum_{i \in \mathcal{J} c} f_{i}^{2}<\rho^{2} / 2$ implies $\mathcal{J} \neq \varnothing$ in view of $\sum_{i} f_{i}^{2} \geqslant \rho^{2}$. Consequently, the conditions of Lemma A. 10 are satisfied, yielding that there exists a test $T_{\text {III }}^{2}$ such that $\mathbb{E}_{0} T_{\mathrm{III}}^{2} \leqslant \alpha / 6$ and $\mathbb{E}_{f}\left(1-T_{\mathrm{III}}^{2}\right) \leqslant \alpha / 6$. We note that in case $\frac{m b}{d \sqrt{d}}>M_{\alpha}$, the test $T_{\mathrm{III}}^{2}$ cannot necessarily be computed (not enough communication budget), but this is not required as this case is covered by $T_{\text {III }}^{1}$.

We now have that for any $f \in H_{\rho}$, whenever $\frac{m b}{d \sqrt{d}} \leqslant M_{\alpha}$, the test $T_{\text {III }}$ can be computed and using that for nonnegative $x, y \geqslant 0, x \vee y \leqslant x+y$ and $x \vee y \geqslant x$, we obtain that

$$
\begin{aligned}
\mathcal{R}\left(H_{\rho}, T_{\text {III }}\right) \leqslant & \mathbb{E}_{0} T_{\text {III }}^{1}+\mathbb{E}_{0} T_{\text {III }}^{2} \mathbb{1}_{\{b \geqslant 2 \log (d+1)\}} \\
& +\sup _{f \in H_{\rho}} \min \left\{\mathbb{E}_{f}\left(1-T_{\text {III }}^{1}\right), \mathbb{E}_{f}\left(1-T_{\text {III }}^{2} \mathbb{I}_{\{b \geqslant 2 \log (d+1)\}}\right)\right\} \\
\leqslant & 2 \alpha / 6+\alpha / 6=\alpha / 2 .
\end{aligned}
$$

Next we provide the risk bounds for the partial tests $T_{\text {III }}^{1}$ and $T_{\text {III }}^{2}$, used in the previous lemma.

Lemma A.9. For any $\alpha \in(0,1)$ there exist constants $\kappa_{\alpha}, M_{\alpha}, C_{\alpha}>0$ such that $\mathbb{E}_{0} T_{I I I}^{1} \leqslant$ $\alpha / 2$. Furthermore, for $f \in H_{\rho}$ if $\rho^{2} \geqslant C_{\alpha} \frac{d \sqrt{d}}{n(d \wedge b)}$ and either $\frac{m b}{d \sqrt{d}} \geqslant M_{\alpha}$ or

$$
\begin{equation*}
\sum_{i \in \mathcal{J}_{c}} f_{i}^{2} \geqslant \rho^{2} / 2 \tag{S.15}
\end{equation*}
$$

holds, where $\mathcal{J}$ was defined in (S.13), then

$$
\mathbb{E}_{f}\left(1-T_{I I I}^{1}\right) \leqslant \alpha / 2
$$

Proof. Under the null hypothesis, $Y_{i}^{j} \sim^{\text {iid }} \operatorname{Bern}(1 / 2)$. For each $\alpha \in(0,1)$ by applying Lemma 4.1 (with $k=d$ and $l=\left|\mathcal{I}_{1}\right|$ ) we get that $\mathbb{E}_{0} T_{\text {III }}^{1} \leqslant \alpha / 2$ for large enough constant $\kappa_{\alpha}$. For $f \in H_{\rho}$, we have

$$
\mathbb{E}_{f} Y_{i}^{j}=\mathbb{E}_{f} \mathbb{E}_{f}\left[Y_{i}^{j} \mid X_{i}^{j}\right]=\Phi\left(\sqrt{\frac{n}{m}} f_{i}\right)
$$

To bound the Type II error, we use the second statement of Lemma 4.1 (with $k=d$ and $l=\left|\mathcal{I}_{1}\right|$, but before that we show that condition (6) holds. Note that by Lemma A.11,

$$
\begin{equation*}
\frac{\left|\mathcal{I}_{1}\right|-1}{2 \sqrt{d}} \sum_{i=1}^{d}\left(\mathbb{E}_{f} Y_{i}^{j}-\frac{1}{2}\right)^{2} \geqslant \frac{\left|\mathcal{I}_{1}\right|-1}{24 \sqrt{d}} \sum_{i=1}^{d}\left(\frac{n}{m} f_{i}^{2} \bigwedge 1\right) . \tag{S.16}
\end{equation*}
$$

In case (S.15) holds, the preceding display is bounded from below by

$$
\frac{\left|\mathcal{I}_{1}\right|-1}{24 \sqrt{d}} \sum_{i \in \mathcal{J}^{c}} \frac{n}{m} f_{i}^{2} \geqslant \frac{n\left(\left|\mathcal{I}_{1}\right|-1\right) \rho^{2}}{48 m \sqrt{d}}
$$

Note, that for large enough $C_{\alpha}>0, \frac{n\left(\left|\mathcal{I}_{1}\right|-1\right) \rho^{2}}{48 m \sqrt{d}} \geqslant n\left(\frac{m b}{d}\right) C_{\alpha} \frac{d \sqrt{d}}{n b} /(96 m \sqrt{d}) \geqslant \kappa_{\alpha} \vee \frac{16}{\alpha}$. If (S.15) does not hold, then there exists $i^{*} \in\{1, \ldots, d\}$ such that $f_{i *} \geqslant \sqrt{m / n}$, so (S.16) is lower bounded by

$$
\frac{\left|\mathcal{I}_{1}\right|-1}{24 \sqrt{d}} \geqslant \frac{m b}{24 d \sqrt{d}}-\frac{1}{12 \sqrt{d}} \geqslant \frac{M_{\alpha}}{24}-\frac{1}{12} .
$$

Then for large enough $M_{\alpha}>0$, the condition (6) is satisfied. Consequently, the statement of the proof follows by the second statement of Lemma 4.1.

Lemma A.10. For any $\alpha \in(0,1)$ there exists a $\kappa_{\alpha}>0$ large enough such that $\mathbb{E}_{0} T_{I I I}^{2} \leqslant$ $\alpha / 2$. Furthermore, if $\rho^{2} \geqslant C_{\alpha} \frac{d \sqrt{d}}{n(d \wedge b)}, m \geqslant M_{\alpha}$, for some large enough $C_{\alpha}, M_{\alpha}>0$, the set $\mathcal{J}$ defined in $(\mathrm{S} .13)$ is non-empty and $b \geqslant 2 \log (d+1)$, then $\mathbb{E}_{f} T_{I I I}^{2} \leqslant \alpha / 2$.

Proof of Lemma A.10. We apply Lemma A. 5 (with $k=d$ and $l=C_{b, d} m$ ), which is a version of Lemma 4.1, given in the Supplement. Under the null hypothesis, $\left(\sqrt{n / m} X_{i}^{j}\right)^{2}$ follows a chi-square distribution with one degree of freedom. Consequently,

$$
\mathbb{E}_{0} B_{l i}^{j}=\mathbb{E}_{0} F_{\chi_{1}^{2}}\left(\left(\sqrt{\frac{n}{m}} X_{i}^{j}\right)^{2}\right)=1 / 2
$$

and

$$
\begin{equation*}
\sum_{j=1}^{m} N^{j} \sim \operatorname{Bin}\left(1 / 2, m d C_{b, d}\right) \tag{S.17}
\end{equation*}
$$

Then Lemma A. 5 yields that $\mathbb{E}_{0} T_{\text {III }}^{2} \leqslant \alpha / 2$.
Next we deal with the upper bound for the Type II error. Let $p_{i}:=\mathbb{E}_{f} F_{\chi_{1}^{2}}\left(\left(\sqrt{n / m} X_{i}^{j}\right)^{2}\right)$ and note that $p_{i} \geqslant 1 / 2$. We apply again Lemma A. 5 (with $k=d, l=m C_{b, d}$ ). Hence it is sufficient to show that the condition (S.9) of the lemma holds. For this first note that $\left(\sqrt{n / m} X_{i}^{j}\right)^{2}$ is a non-central chi-square distributed random variable with non-centrality parameter $\frac{n}{m} f_{i}^{2}$ and one degree of freedom. Consequently, for all $i \in \mathcal{J} \neq \varnothing$ we have

$$
\begin{equation*}
p_{i}=\mathbb{E}_{f} F_{\chi_{1}^{2}}\left(\left(\sqrt{\frac{n}{m}} X_{i}^{j}\right)^{2}\right)=\operatorname{Pr}(V \geqslant 1)>3 / 5, \tag{S.18}
\end{equation*}
$$

where it is used that $V$ is noncentral F -distributed with noncentrality parameter $\frac{n}{m} f_{i}^{2} \geqslant 1$ and $(1,1)$-degrees of freedom. Then by recalling that $\tilde{p}_{i} \geqslant 1 / 2$ we get that

$$
\begin{aligned}
\frac{m C_{b, d}-1}{2 d}\left(\sum_{i=1}^{d}\left(p_{i}-\frac{1}{2}\right)\right)^{2} & \geqslant \frac{m C_{b, d}-1}{2 d}\left(\sum_{i \in \mathcal{J}}\left(\tilde{p}_{i}-\frac{1}{2}\right)\right)^{2} \\
& \geqslant \frac{m C_{b, d}-1}{2 d}(|\mathcal{J}| / 10)^{2} \geqslant \frac{m 2^{b}}{400 d^{2}} \geqslant M_{\alpha} / 400
\end{aligned}
$$

yielding (S.9) for large enough choice of $M_{\alpha}$ and hence concluding the proof of our statement.

The following three lemmas are standard, technical results, nevertheless we provided them for completeness.

Lemma A.11. Let $\Phi$ denote the cdf of a standard normal random variable. It holds that

$$
\left(\Phi(x)-\frac{1}{2}\right)^{2} \geqslant \frac{1}{12} \min \left\{x^{2}, 1\right\} .
$$

Proof. Since $\Phi(x)=1-\Phi(-x)$, it holds that $\left(\Phi(x)-\frac{1}{2}\right)^{2}=\left(\Phi(-x)-\frac{1}{2}\right)^{2}$ hence one can consider $x \geqslant 0$ without loss of generality. We first show that $\Phi(x)-\frac{1}{2} \geqslant \frac{x^{2}}{8}$ for $0 \leqslant x \leqslant$ $1 / \sqrt{2}$. We have

$$
\begin{equation*}
\Phi(x)-\frac{1}{2}=\frac{1}{\sqrt{2 \pi}} \int_{0}^{x} e^{-\frac{1}{2} z^{2}} d z=\frac{1}{\sqrt{2 \pi}} \int_{0}^{x} \sum_{i=0}^{\infty} \frac{(-1)^{i} z^{2 i}}{2^{i} i!} d z=\frac{x}{\sqrt{2 \pi}}\left(\sum_{i=0}^{\infty} \frac{(-1)^{i}(x / \sqrt{2})^{2 i}}{(2 i+1) i!}\right) \tag{S.19}
\end{equation*}
$$

where the last equation follows by Fubini's theorem. The series in the rhs is decreasing in $x \in[0, \sqrt{2}]$, as for each odd $i$ it holds that

$$
\frac{d}{d \epsilon}\left[\frac{(-1)^{i} \epsilon^{2 i}}{(2 i+1) i!}+\frac{(-1)^{i+1} \epsilon^{2 i+2}}{(2 i+3)(i+1)!}\right]=\frac{\epsilon^{2 i-1} 2 i}{i!(2 i+1)}\left(\frac{2 \epsilon^{2}(2 i+1)(2 i+2)}{(i+1) 2 i(2 i+3)}-1\right)<0
$$

for $0 \leqslant \epsilon \leqslant 1$. Hence, for $0 \leqslant x / \sqrt{2} \leqslant c \leqslant 1$,

$$
\frac{x}{\sqrt{2 \pi}}\left(\sum_{i=0}^{\infty} \frac{(-1)^{i}(x / \sqrt{2})^{2 i}}{(2 i+1) i!}\right) \geqslant \frac{x}{\sqrt{2 \pi}}\left(\sum_{i=0}^{\infty} \frac{(-1)^{i} c^{2 i}}{(2 i+1) i!}\right)=\frac{x}{\sqrt{2} c}\left(\Phi(\sqrt{2} c)-\frac{1}{2}\right),
$$

where the last equality follows by (S.19). For $x>\sqrt{2} c$, it holds that

$$
\Phi(x)-1 / 2 \geqslant \Phi(\sqrt{2} c)-1 / 2
$$

as $x \mapsto \Phi(x)-1 / 2$ is increasing. Taking $c=1$ we obtain

$$
\Phi(x)-1 / 2 \geqslant \min \{x(\Phi(\sqrt{2})-1 / 2) / \sqrt{2}, \Phi(\sqrt{2})-1\}>\min \{x, 1\} / \sqrt{12}
$$

which finishes the proof.
Lemma A.12. Let $Z=\left(Z_{1}, \ldots, Z_{d}\right) \sim N\left(0, I_{d}\right)$. It holds that $\mathbb{E} \max _{1 \leqslant i \leqslant d}\left|Z_{i}\right| \leqslant 3 \sqrt{\log (d) \vee \log (2)}$ and

$$
\operatorname{Pr}\left(\max _{1 \leqslant i \leqslant d} Z_{i}^{2} \geqslant x\right) \leqslant \frac{2 d}{e^{x / 4}},
$$

for all $x>0$.
Proof. The case where $d=1$ follows by standard Gaussian concentration properties. Assume $d \geqslant 2$. For $0 \leqslant t \leqslant 1 / 4$,

$$
\mathbb{E} e^{t \max _{i}\left(Z_{i}\right)^{2}}=e^{t} \mathbb{E} \max _{i} e^{t\left(Z_{i}^{2}-1\right)} \leqslant d e^{2 t^{2}+t}
$$

see e.g. Lemma 12 in [34]. Taking $t=1 / 4$ and applying Markov's inequality yields the second statement of the lemma. Furthermore, in view of Jensen's inequality

$$
\mathbb{E} \max _{i}\left(Z_{i}\right)^{2} \leqslant \frac{\log (d)}{t}+2 t+1,
$$

which in turn yields $\max _{i}\left|Z_{i}\right| \leqslant 3 \sqrt{\log (d)}$.
Lemma A.13. Let $X_{d}$ be Chi-square random variable with d-degrees of freedom. For $0<c<1$ it holds that

$$
\operatorname{Pr}\left(X_{d} \leqslant c d\right) \leqslant e^{-d \frac{c-1-\log (c)}{2}} .
$$

Similarly, for $c>1$ it holds that

$$
\operatorname{Pr}\left(X_{d} \geqslant c d\right) \leqslant e^{-d \frac{c-1-\log (c)}{2}} .
$$

Proof. Let $t<0$. We have

$$
\operatorname{Pr}\left(X_{d} \leqslant c d\right)=\operatorname{Pr}\left(e^{t X_{d}} \geqslant e^{t c d}\right) \leqslant \frac{\mathbb{E} e^{t X_{d}}}{e^{t c d}}
$$

Using that $\mathbb{E} e^{t X_{d}}=(1-2 t)^{-d / 2}$, the latter display equals

$$
\exp \left(-d\left(t c+\frac{1}{2} \log (1-2 t)\right)\right)
$$

The expression $t c+\frac{1}{2} \log (1-2 t)$ is maximized when $t=\frac{1}{2}\left(1-\frac{1}{c}\right)<0$ which leads to the result. The second statement follows by similar steps.
B. Proof of Theorem 6.1. For convenience, we consider a sufficiently smooth orthonormal wavelet basis $\left\{\psi_{l i}: l \in \mathbb{N}_{0}, i=0,1, \ldots, 2^{l}-1\right\}$ for $L_{2}[0,1]$, see Section G for a brief introduction of wavelets and collection of properties used during the proof. Nevertheless we note, that other basis (e.g. Fourier) could be used equivalently. Let $f^{L}, \tilde{X}_{L^{\prime}: L}^{j}$ and $\tilde{f}^{L}$ as defined in (29), (30) and below (30), respectively. Furthermore, let $\Psi_{L}: \mathbb{R}^{2^{L}} \rightarrow L_{2}[0,1]$ be the measurable map defined by

$$
\begin{equation*}
\Psi_{L} \tilde{f}^{L}=\sum_{i=0}^{2^{L}-1} \tilde{f}_{i} \psi_{L i}, \tag{S.20}
\end{equation*}
$$

for $\tilde{f}^{L}=\left(\tilde{f}_{0}, \ldots, \tilde{f}_{2^{L}-1}\right)$.
The existence of $C_{\alpha}>0$ such that $f \in H_{C_{\alpha} \rho}^{s, R}$ can be detected.
In view of Theorem 3.2, there exists a constant $C_{\alpha}^{\prime}>0$ and a $b$-bit public coin distributed testing protocol $T$ with transcripts generated according to $Y^{j} \mid\left(\tilde{X}_{0: L}^{j}, U\right) \sim K^{j}\left(\cdot \mid \tilde{X}_{0: L}^{j}, U\right)$ such that if $\left\|\tilde{f^{L}}\right\|_{2}^{2} \geqslant\left(C_{\alpha}^{\prime}\right)^{2} \frac{\sqrt{2^{L}}}{n}\left(\sqrt{\frac{2^{L}}{b \wedge 2^{L}}} \bigwedge \sqrt{m}\right)$, we have

$$
\mathbb{E}_{0} T+\mathbb{E}_{\tilde{f}^{L}}(1-T) \leqslant \alpha
$$

Similarly, there exists a constant $C_{\alpha}^{\prime}>0$ and a $b$-bit private coin distributed testing protocol $T$ such that the above display holds if $\left\|\tilde{f}^{L}\right\|_{2}^{2} \geqslant\left(C_{\alpha}^{\prime}\right)^{2} \frac{\sqrt{2^{L}}}{n}\left(\frac{2^{L}}{b \wedge 2^{L}} \bigwedge \sqrt{m}\right)$. See Section 4 for the construction of such testing protocols.

Consequently, it suffices to show that for $f \in H_{C_{\alpha} \rho}^{s, R},\left\|\tilde{f} \tilde{f}^{L}\right\|_{2}^{2}$ satisfies the above lower bounds for some $L \in \mathbb{N}$ and $c>0$. In view of $(a+b)^{2} / 2-b^{2} \leqslant a^{2}$,

$$
\left\|f^{L}\right\|_{L_{2}}^{2} \geqslant \frac{\|f\|_{L_{2}}^{2}}{2}-\left\|f-f^{L}\right\|_{L_{2}}^{2}
$$

Furthermore, $f \in H_{C_{\alpha} \rho}^{s, R}$ implies that

$$
\left\|f-f^{L}\right\|_{L_{2}}^{2}=\sum_{l>L} \sum_{i=0}^{2^{l}-1} \tilde{f}_{l i}^{2} \leqslant 2^{-2 L s} \sum_{l>L} \sum_{i=0}^{2^{l}-1} \tilde{f}_{l i}^{2} 2^{2 l s} \leqslant \frac{\|f\|_{\mathcal{H}^{s}}^{2}}{2^{2 L s}} \leqslant \frac{R^{2}}{2^{2 L s}} \quad \text { and } \quad\|f\|_{L_{2}}^{2} \geqslant C_{\alpha}^{2} \rho^{2} .
$$

Consequently, in view of Plancharel's theorem and taking $L=1 \vee\left\lceil-\frac{1}{s} \log \rho\right\rceil$,

$$
\begin{equation*}
\left\|\tilde{f^{L}}\right\|_{2}^{2}=\left\|f^{L}\right\|_{L_{2}}^{2} \geqslant \rho^{2} C_{\alpha}^{2} / 2-R^{2} 2^{-2 L s} \geqslant \rho^{2}\left(C_{\alpha}^{2} / 2-R^{2}\right) \tag{S.21}
\end{equation*}
$$

Consequently, there exists a $b$-bit public coin distributed testing protocol such that

$$
\mathbb{E}_{0} T+\mathbb{E}_{f}(1-T) \leqslant \alpha
$$

whenever

$$
\begin{equation*}
\rho^{2} \gtrsim \frac{\sqrt{2^{L}}}{n}\left(\sqrt{\frac{2^{L}}{b \wedge 2^{L}}} \bigwedge \sqrt{m}\right)=\frac{\sqrt{1 \vee \rho^{-1 / s}}}{n}\left(\sqrt{\frac{1 \vee \rho^{-1 / s}}{b \wedge\left(1 \vee \rho^{-1 / s}\right)}} \bigwedge \sqrt{m}\right) \tag{S.22}
\end{equation*}
$$

since the constant $\left(\frac{C_{\alpha}^{2}}{2}-R^{2}\right)$ can be made arbitrary large by large enough choice of $C_{\alpha}>0$. In the case that $b \geqslant\left(1 \vee \rho^{-1 / s}\right)$, the above display is satisfied whenever $\rho^{2+\frac{1}{2 s}} \gtrsim n^{-1}$, which provides the first case in (22). Similarly, if $b \leqslant \rho^{-1 / s}$, the above display boils down to $\rho^{2+\frac{1}{s}} \gtrsim$ $(\sqrt{b} n)^{-1}$ whenever $b m \geqslant \rho^{-1 / s}$, which leads to the second case in (22). If $b m \leqslant \rho^{-1 / s}$, the inequality (S.22) reduces to $\rho^{2+\frac{1}{2 s}} \gtrsim \sqrt{m} / n$ and consequently provides the third case in (22).

By similar argument as for the public coin protocol above, there exists a $b$-bit private coin distributed testing protocol with testing risk less than $\alpha$ whenever

$$
\rho^{2} \gtrsim \frac{\sqrt{1 \vee \rho^{-1 / s}}}{n}\left(\frac{1 \vee \rho^{-1 / s}}{b \wedge\left(1 \vee \rho^{-1 / s}\right)} \bigwedge \sqrt{m}\right)
$$

and $C_{\alpha}>0$ large enough. Then a similar computation as in the public coin case above leads to the three cases in (23).

The existence of $c_{\alpha}$ for which the risk is lower bounded.
For any distribution $\pi_{L}$ on $\mathbb{R}^{L}, \pi_{L} \circ \Psi_{L}^{-1}$ defines a probability measure on the Borel sigma algebra of $L_{2}[0,1]$. For $\tilde{f}^{L} \in \mathbb{R}^{2^{L}}$, the likelihood ratio $\frac{d P_{f}}{d P_{0}}\left(X^{j}\right)$ with $f=\Psi_{L} \tilde{f}^{L}$ equals

$$
\begin{equation*}
\exp \left(\frac{n}{m} \int_{0}^{1} f d X_{t}^{j}-\frac{n}{2 m}\|f\|_{2}^{2}\right)=\exp \left(\frac{n}{m}\left(\tilde{f}^{L}\right)^{\top} \tilde{X}_{L}^{j}-\frac{n}{2 m}\left\|\tilde{f}^{L}\right\|_{2}^{2}\right)=: \mathscr{L}_{\tilde{f}^{L}}\left(\tilde{X}_{L}^{j}\right) \tag{S.23}
\end{equation*}
$$

where $\tilde{X}_{L}^{j}=\left(\int_{0}^{1} \psi_{L 0}(t) d X_{t}^{j}, \ldots, \int_{0}^{1} \psi_{L\left(2^{L}-1\right)}(t) d X_{t}^{j}\right) \in \mathbb{R}^{2^{L}}$. For an arbitrary $b$-bit distributed testing protocol $T=\left(T, K, \mathbb{P}^{U}\right)$, following the proof of Theorem 3.1 up until equation (15) we obtain that

$$
\begin{equation*}
\mathcal{R}\left(H_{\rho}, T\right) \geqslant 1-\left(\int \sqrt{2 D_{\chi^{2}}\left(\mathbb{P}_{0, K}^{Y \mid U=u} ; \mathbb{P}_{\pi, K}^{Y \mid U=u}\right)} d \mathbb{P}^{U}(u)+\pi\left(\tilde{f}^{L} \in \mathbb{R}^{L}: \Psi \tilde{f}^{L} \notin H_{c_{\alpha} \rho}^{s, R}\right)\right) \tag{S.24}
\end{equation*}
$$

By (S.23),

$$
\begin{equation*}
D_{\chi^{2}}\left(\mathbb{P}_{0, K}^{Y \mid U=u} ; \mathbb{P}_{\pi, K}^{Y \mid U=u}\right)=\mathbb{E}_{0}^{Y \mid U=u}\left(\mathbb{E}_{0}\left[\int \prod_{j=1}^{m} \mathscr{L}_{\tilde{f}^{L}}\left(\tilde{X}_{L}^{j}\right) d \pi\left(\tilde{f}^{L}\right) \mid Y, U=u\right]^{2}\right)-1 \tag{S.25}
\end{equation*}
$$

Under $\mathbb{P}_{0}, \mathscr{L}_{\tilde{f}^{L}}\left(\tilde{X}_{L}^{j}\right)$ is equal in distribution to the likelihood ratio

$$
\frac{d N\left(\tilde{f}^{L}, \frac{m}{n} I_{2^{L}}\right)}{d N\left(0, \frac{m}{n} I_{2^{L}}\right)}
$$

That means that the argument of the proof of Theorem 3.1 for bounding the Chi-square divergence applies to the first term in (S.24). Choosing $\pi_{L}=N(0, \Gamma)$ with $\Gamma=\frac{\sqrt{C_{\alpha}} \rho^{2}}{2^{L}} \bar{\Gamma} \in$ $\mathbb{R}^{2^{L} \times 2^{L}}$ and $\bar{\Gamma}$ as in the proof of Theorem 3.1. In particular, we obtain that for some constant $C>0$ not depending on $\rho, n, m, b, c_{\alpha}$ or $L$,

$$
\int \sqrt{2 D_{\chi^{2}}\left(\mathbb{P}_{0, K}^{Y \mid U=u} ; \mathbb{P}_{\pi, K}^{Y \mid U=u}\right)} d \mathbb{P}^{U}(u) \leqslant \begin{cases}\sqrt{2}\left(e^{C c_{\alpha}^{-1}\left(\frac{n^{2} \rho^{4}}{2 L_{m}}+\frac{n^{2} \rho^{4}\left(b \wedge 2^{L}\right)^{2}}{2^{3 L}}\right)}-1\right), & \text { if } U \text { is degenerate }  \tag{S.26}\\ \sqrt{2}\left(e^{C c_{\alpha}^{-1}\left(\frac{n^{2} \rho^{4}}{L_{m}}+\frac{n^{2} \rho^{4}\left(b \wedge 2^{L}\right)}{2^{2 L}}\right)}-1\right), & \text { otherwise }\end{cases}
$$

Note that for $\rho^{2} \leqslant c \frac{\sqrt{2^{L}}}{n}\left(\sqrt{\frac{2^{L}}{b \wedge 2^{L}}} \bigwedge \sqrt{m}\right)$ in the degenerate $U$ and for $\rho^{2} \leqslant c \frac{\sqrt{2^{L}}}{n}\left(\frac{2^{L}}{b \wedge 2^{L}} \bigwedge \sqrt{m}\right)$ in the not degenerate $U$ case, both terms on the rhs of the preceding display are bounded by $\sqrt{2}\left(e^{2 c_{\alpha} C}-1\right)$, which is further bounded by $2^{5 / 2} C c_{\alpha} \leqslant \alpha$ for small enough choice of $c_{\alpha}$. Taking again $L=2 \vee\left\lceil\log \rho^{-1 / s}\right\rceil$, by similar argument as given below display (S.22) the above upper bounds for $\rho^{2}$ result in (22) and (23).

It remained to bound the prior mass term in (S.24) for $L=2 \vee\left\lceil\log \rho^{-1 / s}\right\rceil$. That is, we will show that

$$
\begin{equation*}
\pi_{L}\left(\tilde{f}^{L} \in \mathbb{R}^{2^{L}}:\left\|\Psi_{L} \tilde{f}^{L}\right\|_{L_{2}}^{2} \geqslant c_{\alpha} \rho^{2},\left\|\Psi_{L} \tilde{f}^{L}\right\|_{\mathcal{H}^{s}}^{2} \leqslant R^{2}\right) \geqslant 1-\alpha / 2 \tag{S.27}
\end{equation*}
$$

for all $n$ large enough. Note that for all $L \in \mathbb{N},\left\|\Psi_{L} \tilde{f}^{L}\right\|_{\mathcal{H}^{s}}^{2} \leqslant 2^{2 L s}\left\|\Psi_{L} \tilde{f}^{L}\right\|_{L_{2}}$. Consequently using Plancharel's theorem, we obtain that the lhs of (S.27) is bounded from below by

$$
\begin{align*}
\pi_{L}\left(\tilde{f}^{L} \in \mathbb{R}^{2^{L}}: c_{\alpha} \rho^{2} \leqslant\left\|\tilde{f}^{L}\right\|_{2}^{2} \leqslant 2^{-2 L s} R^{2}\right) & \geqslant \operatorname{Pr}\left(c_{\alpha} \rho^{2} \leqslant Z^{\top} \Gamma Z \leqslant R^{2} \rho^{2}\right) \\
& =\operatorname{Pr}\left(\sqrt{c_{\alpha}} 2^{L} \leqslant Z^{\top} \bar{\Gamma} Z \leqslant \frac{R^{2}}{\sqrt{c_{\alpha}}} 2^{L}\right), \tag{S.28}
\end{align*}
$$

where $Z$ is a $2^{L}$-dimensional standard normal vector. For both the public and private coin choices of $\bar{\Gamma}$ in the proof of Theorem 3.1, $\bar{\Gamma}$ is symmetric, idempotent and has rank $2^{L}$ and $\left\lceil 2^{L} / 2\right\rceil$ respectively. In the public coin case $Z^{\top} \Gamma Z \sim \chi_{2^{L}}^{2}$, hence Lemma A. 13 yields that the rhs of the above display is bounded from below by

$$
1-\exp \left(-2^{L} \frac{\sqrt{c_{\alpha}}-1-0.5 \log c_{\alpha}}{4}\right)-\exp \left(-2^{L} \frac{R^{2} / \sqrt{c_{\alpha}}-1-0.5 \log \left(R^{4} / c_{\alpha}\right)}{4}\right)
$$

which can be set arbitrarily close to 1 per small enough choice of $c_{\alpha}>0$, verifying the prior mass condition.

In the private coin protocol case $Z^{\top} \Gamma Z \sim \chi_{\left[2^{L} / 2\right]}^{2}$ and by applying again Lemma A. 13 (with $d=\left[2^{L} / 2\right\rceil$ ) we get by similar computations as above that the rhs of (S.28) is arbitrarily close to one for small enough choice of $c_{\alpha}$.
C. Public coin protocols for estimation. Consider the distributed signal-in-Gaussian-white-noise model as described in Section 6, i.e. local $X=\left(X^{1}, \ldots, X^{m}\right)$ observations satisfying the dynamics of (21) and $b$-bit transcripts $Y=\left(Y^{1}, \ldots, Y^{m}\right)$ communicated to a central machine taking values in a space $\mathcal{Y}^{m}$ with $|\mathcal{Y}|=b$. Let $\mathcal{E}_{p u b}(b)$ denote the class of all distributed estimation protocols generating transcripts that may depend on a public coin $U$. That is, $\mathcal{E}_{p u b}(b)$ consists of pairs $(\hat{f}, \mathcal{L}(Y, U \mid X))$ where $\hat{f}: \mathcal{Y} \rightarrow L_{2}[0,1]$ and $\mathcal{L}((Y, U) \mid X)$ is such that

$$
\mathbb{P}_{f}^{Y}(y)=\iint \mathbb{P}^{Y \mid(X, U)=(x, u)}(y) d \mathbb{P}_{f}^{X}(x) d \mathbb{P}^{U}(u),
$$

$X$ is independent of $U$ and $Y^{1}, \ldots, Y^{m}$ are independent given $(X, U)$. Let $\mathcal{E}_{\text {priv }}(b)$ denote the class of all distributed estimation protocols that do not depend on a public coin. This is equivalent to the definition of $\mathcal{E}_{p u b}(b)$ above with $U$ set to a degenerate random variable. Below, we shall write $\hat{f} \equiv(\hat{f}, \mathcal{L}(Y, U \mid X))$ when no confusion can arise.

THEOREM C.1. The distributed minimax estimation rates under communication constraints are the same in the public and private coin protocols, i.e.

$$
\inf _{\hat{f} \in \mathcal{E}_{p u b}(b)} \sup _{f \in \mathcal{H}^{s}, R} \mathbb{E}_{f}^{(Y, U)}\|\hat{f}(Y)-f\|_{L_{2}}^{2}=\inf _{\hat{f} \in \mathcal{E}_{p r i v}(b)} \sup _{f \in \mathcal{H}^{s}, R} \mathbb{E}_{f}^{Y}\|\hat{f}(Y)-f\|_{L_{2}}^{2}
$$

Proof. Since a private coin protocol can be seen as a public coin protocol with a degenerate random variable $U$, it remained to deal with the " $Z$ " inequality. To that extend, it is sufficient to show that the same lower bound as for the private coin case holds.

Following the proof of Theorem 3.1 of [44], there exists a distribution $\pi$ on $\mathcal{H}^{s, R}$ such that

$$
\begin{equation*}
\inf _{\hat{f} \in \mathcal{E}_{p r i v}(b)} \sup _{f \in \mathcal{H}, \mathcal{H}^{s, R}} \mathbb{E}_{f}^{Y}\|\hat{f}(Y)-f\|_{L_{2}}^{2} \geqslant \inf _{\hat{f} \in \mathcal{E}_{\text {priv }}(b)} \int \mathbb{E}_{f}^{Y}\|\hat{f}(Y)-f\|_{L_{2}}^{2} d \pi(f)(1+o(1)), \tag{S.29}
\end{equation*}
$$

where the $o(1)$ term is concerned with asymptotics in $n$ only. The particular choice of $\pi$ considered in [44] does not depend on the law of $Y$ and satisfies

$$
\int \mathbb{E}_{f}^{Y}\|\hat{f}(Y)-f\|_{L_{2}}^{2} d \pi(f) \gtrsim \begin{cases}n^{-\frac{2 s}{2 s+1}} & \text { if } b \gtrsim n^{\frac{1}{2 s+1}}  \tag{S.30}\\ (b n)^{-\frac{2 s}{2 s+2}} & \text { if } b \lesssim n^{\frac{1}{2 s+1}} \text { and } b \gtrsim\left(n / m^{2 s+2}\right)^{\frac{1}{2 s+1}} \\ (b m)^{-2 s} & \text { if } b \lesssim\left(n / m^{2 s+2}\right)^{\frac{1}{2 s+1}}\end{cases}
$$

Furthermore, this lower bound is tight, see Section 4 in [44]. By the Markov chain structure of (2), we have that

$$
\int \mathbb{P}_{f}^{Y}(y) d \pi(f)=\iiint \mathbb{P}_{f}^{Y \mid(X, U)=(x, u)}(y) d \mathbb{P}_{f}^{X}(x) d \mathbb{P}^{U}(u) d \pi(f)
$$

Consequently, for the same choice of prior and any $(\hat{f}, \mathcal{L}(Y, U \mid X)) \in \mathcal{E}_{p u b}(b), f \in \mathcal{H}^{s, R}$,

$$
\begin{aligned}
\mathbb{E}_{f}^{Y}\|\hat{f}(Y)-f\|_{L_{2}}^{2} & \geqslant \iint \mathbb{E}_{f}^{Y \mid U=u}\|\hat{f}(Y)-f\|_{L_{2}}^{2} d \pi(f)(1+o(1)) d \mathbb{P}^{U}(u) \\
& \geqslant \int \inf _{\hat{f} \in \mathcal{E}_{\text {pub }}(b)} \int \mathbb{E}_{f}^{Y \mid U=u}\|\hat{f}(Y)-f\|_{L_{2}}^{2} d \pi(f)(1+o(1)) d \mathbb{P}^{U}(u) \\
& =\int \inf _{\hat{f} \in \mathcal{E}_{\text {priv }}(b)} \int \mathbb{E}_{f}^{Y}\|\hat{f}(Y)-f\|_{L_{2}}^{2} d \pi(f)(1+o(1)) d \mathbb{P}^{U}(u) \\
& =\inf _{\hat{f} \in \mathcal{E}_{\text {priv }}(b)} \int \mathbb{E}_{f}^{Y}\|\hat{f}(Y)-f\|_{L_{2}}^{2} d \pi(f)(1+o(1))
\end{aligned}
$$

Here, the second to last equation follows from the fact that for any $(\hat{f}, \mathcal{L}(Y, U \mid X)) \in \mathcal{E}_{\text {pub }}(b)$, it holds that $\left(\hat{f}, \mathbb{P}_{f}^{Y \mid X, U=u}\right) \in \mathcal{E}_{\text {priv }}(b)$. By (S.30), the private coin lower bound also holds in the public coin case and the result follows.
D. Proof of the lower bounds in Theorems 7.1 and 7.2. Let $f^{L}$ and $\tilde{X}_{L^{\prime}: L}^{j}$ as defined in (29) and (30), respectively. Let $T=\left(T, K, \mathbb{P}^{U}\right)$ be a given distributed testing protocol (with $U$ degenerate in the case it is a private coin protocol) and fix $\alpha \in(0,1)$. For given $s_{\text {min }}<s_{\text {max }}$, consider for $s \in\left[s_{\min }, s_{\max }\right]$ the map $s \mapsto \rho_{s}$.

Recall that for $\Psi_{L}$ as defined in (S.20) and any distribution $\pi_{L}$ on $\mathbb{R}^{\nu(L)}, \pi_{L} \circ \Psi_{L}^{-1}$ defines a probability measure on the Borel sigma algebra of $L_{2}[0,1]$. Define the mixture of the above probability measures by

$$
\begin{equation*}
\Pi=\frac{1}{\left|\mathcal{C}_{0}\right|} \sum_{L \in \mathcal{C}_{0}} \pi_{L} \circ \Psi_{L}^{-1} \tag{S.31}
\end{equation*}
$$

where $\mathcal{C}_{0} \subseteq \mathcal{C}$. There exists a grid of points $\mathcal{S} \subset\left[s_{\min }, s_{\max }\right]$ such that the map $s \mapsto L_{s}$ is a one-to-one map from $\mathcal{S}$ to $\mathcal{C}$. Let $L \mapsto s_{L}$ denote its inverse.

By the same steps as in (S.1),

$$
\begin{equation*}
\sup _{f \in H_{H_{\alpha}}^{s_{L}^{L}, R} \rho_{L}} \mathbb{P}_{f}^{Y}(T=0) \geqslant \mathbb{P}_{\pi_{L}}^{Y}(T=0)-\pi_{L} \circ \Psi_{L}^{-1}\left(f \notin H_{c_{\alpha} \rho_{s_{L}}}^{s_{L}, R}\right) \tag{S.32}
\end{equation*}
$$

for all $L \in \mathcal{C}$. Using the above display, we can bound the risk in the adaptive setting from below:

$$
\begin{align*}
& \sup _{s \in\left[s_{\min }, s_{\max }\right]} \mathcal{R}\left(H_{c_{\alpha} \rho_{s}}^{s, R}, T\right)
\end{aligned} \begin{aligned}
& |\mathcal{C}| \\
& \text { S.33) }  \tag{S.33}\\
&
\end{align*}
$$

Taking $\pi_{L}$ as in the proof of Theorem 6.1, then by the same reasoning as in proof the proof of Theorem 6.1 that the third term in the above display can be made arbitrarily small per choice of $c_{\alpha}$ for $\rho_{s}$ satisfying (25)-(26). For the first two terms, define

$$
\mathcal{L}_{\pi_{L}}^{Y \mid u}:=\int \frac{d \mathbb{P}_{f}^{Y \mid U=u}}{d \mathbb{P}_{0}^{Y \mid U=u}} d \pi_{L}(f)
$$

and note that

$$
\begin{aligned}
\mathbb{P}_{0}^{Y}(T=1)+\mathbb{P}_{\Pi}^{Y}(T=0) & =\frac{1}{\left|\mathcal{C}_{0}\right|} \sum_{L \in \mathcal{C}_{0}} \int \mathbb{P}_{0}^{Y \mid U=u}\left(T+\mathcal{L}_{\pi_{L}}^{Y \mid u}(1-T)\right) d \mathbb{P}^{U}(u) \\
& \geqslant \frac{1}{\left|\mathcal{C}_{0}\right|} \sum_{L \in \mathcal{C}_{0}} \int \mathbb{E}_{0}^{Y \mid U=u}\left(\gamma T+\mathcal{L}_{\pi_{L}}^{Y \mid u}(1-T)\right) \mathbb{1}\left\{\mathcal{L}_{\pi_{L}}^{Y \mid u}>\gamma\right\} d \mathbb{P}^{U}(u) \\
& \geqslant \gamma \frac{1}{\left|\mathcal{C}_{0}\right|} \sum_{L \in \mathcal{C}_{0}} \int \mathbb{P}_{0}^{Y \mid U=u}\left(\mathcal{L}_{\pi_{L}}^{Y \mid u}>\gamma\right) d \mathbb{P}^{U}(u),
\end{aligned}
$$

where the conditioning follows from the Markov chain structure (2) and the inequality holds for $0<\gamma<1$. We can conclude that it suffices to show that for all $\varepsilon>0$,

$$
\begin{equation*}
\frac{1}{\left|\mathcal{C}_{0}\right|} \sum_{L \in \mathcal{C}_{0}} \mathbb{P}_{0}^{(Y, U)}\left(\left|\mathcal{L}_{\pi_{L}}^{Y \mid U}-1\right|>\varepsilon\right) \tag{S.34}
\end{equation*}
$$

can be made arbitrarily small per small enough choice of $c_{\alpha}$ in order obtain the required lower bound in (S.33). Using $\mathbb{P}_{0}^{(Y, U)}=d \mathbb{P}^{U} d \mathbb{P}_{0}^{Y \mid U}$, conditioning on the $\mathbb{P}_{0}^{Y \mid U}$-variance of $\mathcal{L}_{\Pi}^{Y \mid u}$ with Chebyshev's inequality and $\mathbb{E}_{0}^{Y \mid U=u} \mathcal{L}_{\Pi}^{Y \mid u}=1$ lead to

$$
\frac{1}{\left|\mathcal{C}_{0}\right|} \sum_{L \in \mathcal{C}_{0}} \mathbb{P}_{0}^{(Y, U)}\left(\left(\mathcal{L}_{\pi_{L}}^{Y \mid U}-1\right)^{2}>\varepsilon^{2}\right) \leqslant \frac{1}{\left|\mathcal{C}_{0}\right|} \sum_{L \in \mathcal{C}_{0}} \mathbb{P}^{U}\left(\mathbb{E}^{Y \mid U}\left(\mathcal{L}_{\pi_{L}}^{Y \mid U}\right)^{2}>1+\zeta\right)+\frac{\zeta}{\varepsilon^{2}}
$$

for all $\varepsilon>0$ and $\zeta>0$. Noting that $\mathbb{E}^{Y \mid U=u}\left(\mathcal{L}_{\pi_{L}}^{Y \mid U=u}\right)^{2} \geqslant 1$, sufficiently bounding (S.34) follows from Markov's inequality and showing

$$
\begin{equation*}
\frac{1}{\left|\mathcal{C}_{0}\right|} \sum_{L \in \mathcal{C}_{0}} \int \log \left(\mathbb{E}^{Y \mid U=u}\left(\mathcal{L}_{\pi_{L}}^{Y \mid U=u}\right)^{2}\right) d \mathbb{P}^{U}(u) \lesssim c_{\alpha} . \tag{S.35}
\end{equation*}
$$

Noting that $\mathbb{E}^{Y \mid U=u}\left(\mathcal{L}_{\pi_{L}}^{Y \mid U=u}\right)^{2}=D_{\chi^{2}}\left(\mathbb{P}_{0, K}^{Y \mid U=u} ; \mathbb{P}_{\pi_{L}, K}^{Y \mid U=u}\right)+1$, we can apply the argument of the proof of Theorem 3.1 (foregoing the bound of (40)) for bounding the Chi-square divergence and we obtain that for some fixed $C>0$,

$$
\log \left(\mathbb{E}^{Y \mid U=u}\left(\mathcal{L}_{\pi_{L}}^{Y \mid U=u}\right)^{2}\right) \leqslant \begin{cases}C c_{\alpha} \frac{n^{4} \rho_{s_{L}}^{4}}{m^{2}{ }^{2} L} \operatorname{Tr}\left(\Xi_{L, u}\right)^{2}+A_{L, u}, & \text { if } U \text { is degenerate, },  \tag{S.36}\\ C c_{\alpha} \frac{n^{3} \rho_{s_{L}}}{m^{2} 2^{2 L}} \operatorname{Tr}\left(\Xi_{L, u}\right)+A_{L, u}, & \text { otherwise, }\end{cases}
$$

where

$$
A_{L, u}=\sum_{j=1}^{m} \log \left(\mathbb{E}_{0}^{Y^{j} \mid U=u}\left(\mathbb{E}_{0}\left[\left.\int \frac{d \mathbb{P}_{f}^{\tilde{X}^{j}}}{d \mathbb{P}_{0}^{\tilde{X}^{j}}}\left(\tilde{X}_{L}^{j}\right) d \pi_{L}(f) \right\rvert\, Y^{j}, U=u\right]^{2}\right)\right)
$$

and $\Xi_{L, u}=\sum_{j=1}^{m} \Xi_{L, u}^{j}$ with $\Xi_{L, u}^{j}=\mathbb{E}_{0} \mathbb{E}_{0}\left[\tilde{X}_{L}^{j} \mid Y^{j}, U=u\right] \mathbb{E}_{0}\left[\tilde{X}_{L}^{j} \mid Y^{j}, U=u\right]^{\top}$. Via a data processing argument (Lemma E. 5 in the supplement),

$$
\frac{1}{\left|\mathcal{C}_{0}\right|} \sum_{L \in \mathcal{C}_{0}} \int A_{L, u} d \mathbb{P}^{U}(u) \lesssim \max _{L \in \mathcal{C}_{0}} \frac{c_{\alpha} n^{2} \rho_{s_{L}}^{4}\left(b \wedge\left|\mathcal{C}_{0}\right|\right)}{m 2^{L}\left|\mathcal{C}_{0}\right|} .
$$

When $U$ is degenerate, Lemma E. 3 implies that there exists a choice for $\mathcal{C}_{0} \subset \mathcal{C}_{0}$ such that for all $L \in \mathcal{C}_{0}$,

$$
\operatorname{Tr}\left(\Xi_{L, u}\right)^{2} \lesssim\left(\frac{b}{|\mathcal{C}|} \wedge 2^{L}\right)^{2} \frac{m^{4}}{n^{2}}
$$

When $U$ is not degenerate, Lemma E. 4 implies that taking $\mathcal{C}_{0}=\mathcal{C}$,

$$
\frac{1}{|\mathcal{C}|} \sum_{L \in \mathcal{C}} \frac{n^{3} \rho_{s_{L}}^{4}}{m^{2} 2^{2 L}} \operatorname{Tr}\left(\Xi_{L, u}\right) \lesssim \max _{L \in \mathcal{C}} \frac{n^{2} \rho_{s_{L}}^{4}}{2^{2 L}}\left(\frac{b}{|\mathcal{C}|} \wedge 2^{L}\right)
$$

Combining the above with the fact that $s \mapsto L_{s}=\left\lfloor s^{-1} \log \left(1 / \rho_{s}\right)\right\rfloor \vee 1$ maps a grid $\mathcal{S} \subset$ [ $s_{\text {min }}, s_{\text {max }}$ ] one-to-one to $\mathcal{C}_{0}$ with inverse map $L \mapsto s_{L}$ on $\mathcal{C}_{0}$, we obtain

$$
\frac{1}{\left|\mathcal{C}_{0}\right|} \sum_{L \in \mathcal{C}_{0}} \int \log \left(\mathbb{E}^{Y \mid U=u}\left(\mathcal{L}_{\pi_{L}}^{Y \mid U=u}\right)^{2}\right) d \mathbb{P}^{U}(u) \lesssim c_{\alpha} \cdot\left\{\begin{array}{l}
\max _{L \in \mathcal{C}} \frac{n^{2} \rho_{s_{L}}^{4}\left(\frac{b}{\log (n)} \wedge 2^{L}\right)^{2}}{2^{3 L}} \bigvee \frac{n^{2} \rho_{s_{L}}^{4}(b \wedge \log (n))}{m 2^{L} \log (n)}, \\
\max _{L \in \mathcal{C}} \frac{n^{2} \rho_{s_{L}}^{4}\left(\frac{b}{\log (n)} \wedge 2^{L}\right)}{2^{2 L}} \bigvee \frac{n^{2} \rho_{s_{L}}^{4}(b \wedge \log (n))}{m 2^{L} \log (n)},
\end{array}\right.
$$

where the first case corresponds to a degenerate $U$, the latter to the general (public coin) case. The conditions (25)-(26) for $\rho_{s_{L}}$ yield (S.35), which in turn finishes the proof.
E. Lemmas concerning the adaptation upper and lower bounds. The following lemma controls the Type I error of the adaptive tests defined in Section 7.

Lemma E.1. Consider for $L \in \mathbb{N}$ and a nonnegative positive integer sequence $K_{n}$,

$$
S_{n}(L):=\frac{1}{\sqrt{K_{n}}} \sum_{i=1}^{K_{n}} \zeta_{i, L}
$$

where $\left(\zeta_{1, L}, \ldots, \zeta_{K_{n}, L}\right)$ independent random variables with mean 0 and unit variance.
Assume that the random variables satisfy Cramér's condition, i.e. for some $\epsilon>0$ and all $t \in(-\epsilon, \epsilon), i=1, \ldots, K_{n}$ and $L \in \mathcal{C}$, for some set $\mathcal{C} \subset \mathbb{N}$ satisfying $|\mathcal{C}|=\log (n)$,

$$
\mathbb{E} e^{t \zeta_{i, L}}<\infty
$$

Then for $K_{n} \gg(\log \log n)^{6}$, it holds that

$$
\operatorname{Pr}\left(\max _{L \in \mathcal{C}}\left|S_{n}(L)\right| \geqslant c \sqrt{\log \log (n)}\right) \rightarrow 0
$$

for all $c>\sqrt{2}$ as $n \rightarrow \infty$.
If the random variables are iid Rademacher or are of the form

$$
\zeta_{i, L}=\frac{1}{4 Q}\left[\left(\sum_{q=1}^{Q} R_{q L}\right)^{2}-Q\right]
$$

with $R=\left(R_{1 L}, \ldots, R_{Q L}\right)$ independent Rademacher random variables and $Q \in \mathbb{N}$, the statement holds for any sequence $K_{n}$ as $n \rightarrow \infty$.

Proof. By using union bounds,

$$
\begin{gathered}
\operatorname{Pr}\left(\max _{L \in \mathcal{C}} S_{n}(L) \geqslant c \sqrt{\log \log (n)}\right) \leqslant \sum_{L \in \mathcal{C}} \operatorname{Pr}\left(\left|S_{n}(L)\right| \geqslant c \sqrt{\log \log (n)}\right) \leqslant \\
\sum_{L \in \mathcal{C}}\left[\operatorname{Pr}\left(S_{n}(L) \geqslant c \sqrt{\log \log (n)}\right)+\operatorname{Pr}\left(-S_{n}(L) \geqslant c \sqrt{\log \log (n)}\right)\right]
\end{gathered}
$$

The proof follows by showing that $S_{n}(L)$ and $-S_{n}(L)$ are or tend to sub-Gaussian variables with sub-Gaussianity constant less than or equal to 1 , since this allows for bounding the above display by

$$
2 \sum_{L \in \mathcal{C}} e^{-\frac{c^{2}}{2} \log \log (n)} \lesssim \frac{1}{(\log (n))^{c^{2} / 2-1}}
$$

and the result follows.
For the first statement, by Cramér's theorem (see e.g. Theorem 7 in Section 8.2 of [28]),

$$
\frac{\operatorname{Pr}\left(S_{n}(L) \geqslant c \sqrt{\log \log (n)}\right)}{1-\Phi(c \sqrt{\log \log (n)})}=\exp \left(O(1) \cdot \frac{(\log \log n)^{3}}{\sqrt{K_{n}}}\right)\left(1+O\left(\frac{\log \log n}{\sqrt{K_{n}}}\right)\right) \rightarrow 1
$$

Note that the above statement holds for $-S_{n}(L)$ also. The statement now follows by using $1-\Phi(x) \leqslant e^{-x^{2} / 2}$.

For the second statement, note that by symmetry of the Rademacher distribution, it suffices to consider only $S_{n}(L)$. In case the $\zeta_{i, L}$ 's are iid Rademacher, note that a Chernoff bound yields

$$
\begin{equation*}
\operatorname{Pr}\left(S_{n}(L) \geqslant c \sqrt{\log \log (n)}\right) \leqslant \inf _{t>0} e^{\frac{t^{2}}{2}-c t \sqrt{\log \log (n)}}=e^{-\frac{c^{2}}{2} \log \log (n)} \tag{S.37}
\end{equation*}
$$

Similarly, for the sum of Rademacher random variables, we have

$$
\begin{aligned}
\mathbb{E} \exp \left(\frac{t}{\sqrt{K_{n}}} \zeta_{i, L}\right) & =\mathbb{E} \exp \left(\frac{t}{4 Q \sqrt{K_{n}}}\left[\sum_{q \neq q^{\prime}}^{Q} R_{q L} R_{q^{\prime} L}\right]\right) \\
& \leqslant \mathbb{E} \exp \left(\frac{t}{Q \sqrt{K_{n}}}\left[\sum_{q \neq q^{\prime}}^{Q} R_{q L} R_{q^{\prime} L}^{\prime}\right]\right),
\end{aligned}
$$

where the inequality follows from e.g. Theorem 6.1 .1 in [40] with $R^{\prime}=\left(R_{1 L}^{\prime}, \ldots, R_{Q L}^{\prime}\right)$ independent of $R$. The latter implies that $\left(R_{q L} R_{q^{\prime} L}^{\prime}\right)_{\left(q, q^{\prime}\right) \in\{1, \ldots, Q\}^{2}}$ itself is a vector of independent Rademacher random variables, and consequently the above display is further bounded by

$$
\exp \left(\frac{t^{2} Q(Q-1)}{2 K_{n} Q^{2}}\right) \leqslant \exp \left(\frac{t^{2}}{2 K_{n}}\right)
$$

The proof of the last statement now follows via Chernoff bound as in (S.37).

The next lemma controls the Type 2 error of the adaptive test in the high-budget case under public coin protocol.

LEmMA E.2. Consider $S_{I I}\left(L_{s}\right)$ as in (33) in the paper. It holds that

$$
\mathbb{E}_{f} \mathbb{1}\left\{S_{I I}\left(L_{s}\right)<2 \sqrt{\log \log n}\right\} \leqslant \alpha / 2
$$

whenever $f \in H_{C_{\alpha} \rho_{s}}^{s, R}$ with $\rho^{2} \geqslant C_{0} \sqrt{\log \log (n)} \frac{2^{L_{s}}}{n \sqrt{\frac{b}{\log (n)} \wedge 2^{L_{s}}}}$ for $C_{0}$ large enough, depending only on $R$.

Proof. The proof is similar in spirit to that of the risk bound in the finite dimensional, non-adaptive, public coin setting given in Lemma A.7.

We show below that the event

$$
\left.A=\left\{\frac{m^{\prime}-1}{2 \sqrt{b^{\prime}}} \sum_{i=1}^{b^{\prime}}\left(Y_{\mathrm{II}}^{j}(L)\right)_{i}-1 / 2\right)^{2} \geqslant 2 \sqrt{\log \log n}\right\}
$$

occurs with $\mathbb{P}_{f}$-probability greater than $1-\alpha / 4$. Since on $A$ the condition of Lemma A. 4 is satisfied with $c_{\alpha, n}=2 \sqrt{\log \log n}$ and consequently, by the conclusion of Lemma A.4, $\mathbb{E}_{f} \mathbb{1}\left\{S_{\mathrm{II}}\left(L_{s}\right)<2 \sqrt{\log \log n}\right\}$ is bounded by $\alpha / 2$.

Following the proof of Lemma A. 7 (with $d=\nu_{L_{s}}$, considering the $\nu_{L_{s}}$ dimensional vector $f^{\nu_{L_{s}}}$, and taking $N_{\alpha}=2 \sqrt{\log \log n}$ ), and noting that for $C_{0}^{2}>4 R^{2}$

$$
\left\|\tilde{f}^{L_{s}}\right\|_{2}^{2} \geqslant\|f\|_{2}^{2} / 2-R^{2} 2^{-2 L_{s} s} \gtrsim \frac{C_{0} 2^{L_{s}} \sqrt{\log \log (n)}}{2 n \sqrt{\frac{b}{\log (n)} \wedge 2^{L_{s}}}} \gtrsim \frac{C_{0} 2^{L_{s}} \sqrt{\log \log (n)}}{n \sqrt{b^{\prime} \frac{m^{\prime}}{m}}}
$$

we get that

$$
\begin{equation*}
\mathbb{E}_{f} \mathbb{1}_{A^{c}} \leqslant \operatorname{Pr}\left(\frac{m^{\prime}-1}{24 \sqrt{b^{\prime}}} \sum_{i=1}^{b^{\prime}} \min \left\{\frac{C_{0} \sqrt{\log \log n} 2^{L_{s}} Z_{i}^{2}}{2 m^{\prime} \sqrt{b^{\prime}}\|Z\|_{2}^{2}}, 1\right\} \leqslant 2 \sqrt{\log \log n}\right) \tag{S.38}
\end{equation*}
$$

Considering the intersection with the event $\left\{\|Z\|_{2}^{2} \leqslant k 2^{L_{s}}\right\}$ for some large enough $k>0$, and noting that by Lemma A.12,

$$
\operatorname{Pr}\left(\max _{1 \leqslant i \leqslant b^{\prime}} Z_{i}^{2} \geqslant \frac{2 m^{\prime} \sqrt{b^{\prime}} k}{C_{0} \sqrt{\log \log n}}\right) \leqslant 2 b^{\prime} \exp \left(-\frac{m^{\prime} \sqrt{b^{\prime}} k}{2 C_{0} \sqrt{\log \log n}}\right)=o(1)
$$

the right hand side of (S.38) is further bounded by

$$
\operatorname{Pr}\left(\sum_{i=1}^{b^{\prime}} Z_{i}^{2} \leqslant \frac{96 b^{\prime} m^{\prime} k}{C_{0}\left(m^{\prime}-1\right)}\right)+o(1)+\alpha / 8 \leqslant \alpha / 4
$$

where the last inequality holds for large enough choices $m^{\prime}:=\frac{m(b \wedge \log (n))}{\log (n)}, b^{\prime}:=\frac{m b}{m^{\prime}|\mathcal{C}|} \wedge \nu_{L}$ and large enough choice of $C_{0}$ (depending on $k$ ), see e.g. (S.12) in the proof of Lemma A.7, which finishes the proof of our statement.

Next we provide the lemmas for the lower bound. From now on in this section, we consider the setting of Section 7. That is, let $\tilde{X}_{L}^{j}, \tilde{X}_{1: L}^{j}$ denote the wavelet coefficients of $X^{j}$ as in (30). Define in addition the matrices

$$
\begin{aligned}
\Xi_{L, u}^{j} & =\mathbb{E}_{0} \mathbb{E}_{0}\left[\tilde{X}_{L}^{j} \mid Y^{j}, U=u\right] \mathbb{E}_{0}\left[\tilde{X}_{L}^{j} \mid Y^{j}, U=u\right]^{\top} \\
\Xi_{L^{\prime}: L, u}^{j} & =\mathbb{E}_{0} \mathbb{E}_{0}\left[\tilde{X}_{L^{\prime}: L}^{j} \mid Y^{j}, U=u\right] \mathbb{E}_{0}\left[\tilde{X}_{L^{\prime}: L}^{j} \mid Y^{j}, U=u\right]^{\top},
\end{aligned}
$$

$\Xi_{L, u}:=\sum_{j=1}^{m} \Xi_{L, u}^{j}$ and $\Xi_{u}=\sum_{j=1}^{m} \Xi_{L_{\min }: L_{\max }, u}^{j}$. The lemma below allows for extending the data processing inequality of Lemma A. 3 to the adaptive private coin case, in which extra demands are placed on the communication budget in terms of the budget needing to cover the coordinates corresponding to each resolution level.

LEmmA E.3. Suppose $Y^{j}$ takes values in a space with cardinality at most $2^{b} \in \mathbb{N}$, for $j=1, \ldots, m$ and let $\mathcal{C}=\left\{L_{\min }, \ldots, \mathbb{L}_{\max }\right\}$, for some $L_{\min }<L_{\max } \in \mathbb{N}$. There exists $\mathcal{C}_{0} \subset \mathcal{C}$ such that

$$
\operatorname{Tr}\left(\Xi_{L, u}\right) \lesssim\left(\frac{b}{|\mathcal{C}|} \wedge 2^{L}\right) \frac{m^{2}}{n}
$$

for all $L \in \mathcal{C}_{0}$.

Proof. Define $\Delta_{L}=\operatorname{Tr}\left(\Xi_{L, u}\right)$ and let $\ell:\left\{1, \ldots, L_{\text {max }}-L_{\text {min }}+1\right\} \rightarrow \mathcal{C}$ a map that respects the ordering of the $\Delta_{L}$ 's in the sense that

$$
\Delta_{\ell(i)} \leqslant \Delta_{\ell(k)} \text { if } i \leqslant k
$$

Let $\mathcal{C}_{0}$ denote the first $\left\lfloor\frac{L_{\max }-L_{\min }+1}{2}\right\rfloor$ elements of the collection $\left\{\Delta_{\ell 1}, \Delta_{\ell(2)}, \ldots, \Delta_{\ell\left(L_{\max }-L_{\min }+1\right)}\right\}$. For all $L^{\circ} \in \mathcal{C}$,

$$
\operatorname{Tr}\left(\Xi_{L^{\circ}, u}\right) \leqslant \frac{2}{|\mathcal{C}|} \sum_{L \in \mathcal{C} \backslash \mathcal{C}_{0}} \operatorname{Tr}\left(\Xi_{L, u}\right)
$$

By definition of the trace of a matrix, $\sum_{L} \operatorname{Tr}\left(\Xi_{L, u}\right)=\operatorname{Tr}\left(\Xi_{L_{\min }: L_{\text {max }}, u}\right)$. By Lemma A.3,

$$
\operatorname{Tr}\left(\Xi_{L_{\min }: L_{\max }, u}\right)=\sum_{j=1}^{m} \operatorname{Tr}\left(\Xi_{L_{\min }: L_{\max }, u}^{j}\right) \leqslant \frac{2 \log (2) m^{2} b}{n}
$$

Combining the above two displays, we obtain that

$$
\operatorname{Tr}\left(\Xi_{L^{\circ}, u}\right) \lesssim \frac{m^{2} b}{n(|\mathcal{C}|}
$$

By an application of Lemma A. 2 and a straightforward computation as in the proof of Lemma A.3,

$$
\begin{equation*}
\operatorname{Tr}\left(\Xi_{L^{\circ}, u}\right) \leqslant \frac{m^{2}}{n} 2^{L^{\circ}} \tag{S.39}
\end{equation*}
$$

Combining the two bounds for $\operatorname{Tr}\left(\Xi_{L^{\circ}, u}\right)$ gives the result.
The next lemma applies to the adaptive public coin setting. The bound below is slightly more relaxed than the previous one, which relates to the private coin setting. The reason for this is the fact that in the public coin setting, the hyperprior cannot be chosen in an adversarial way because the public coin draw.

LEMmA E.4. With the notation as in the proof of Theorem 7.1, it holds that

$$
\frac{1}{|\mathcal{C}|} \sum_{L \in \mathcal{C}} \frac{n^{3} \rho_{s_{L}}^{4}}{m^{2} 2^{2 L}} \operatorname{Tr}\left(\Xi_{L, u}\right) \lesssim \max _{L \in \mathcal{C}} \frac{n^{2} \rho_{s_{L}}^{4}}{2^{2 L}}\left(\frac{b}{|\mathcal{C}|} \wedge 2^{L}\right)
$$

Proof. Similarly to the proof of Lemma E.3, we note that by the linearity of the trace,

$$
\sum_{L \in \mathcal{C}} \operatorname{Tr}\left(\Xi_{L, u}\right)=\operatorname{Tr}\left(\Xi_{u}\right)
$$

where $\Xi_{u}=\sum_{j=1}^{m} \Xi_{L_{\min }: L_{\max }, u}^{j}$. Lemma A. 3 yields $\operatorname{Tr}\left(\Xi_{u}\right) \leqslant 2 \log (2) \frac{b m^{2}}{n}$. Otherwise, applying Lemma A. 2 yields $\operatorname{Tr}\left(\Xi_{L, u}\right) \leqslant \frac{2^{L} m^{2}}{n}$. Combining these two inequalities yields the result:

$$
\begin{aligned}
\frac{1}{|\mathcal{C}|} \sum_{L \in \mathcal{C}} \frac{n^{2} \rho_{s_{L}}^{4}}{2^{2 L}} \operatorname{Tr}\left(\Xi_{L, u}\right) & \leqslant \frac{1}{|\mathcal{C}|} \sum_{L \in \mathcal{C}} \frac{n^{2} \rho_{s_{L}}^{4}}{2^{2 L}}\left(\frac{n}{m^{2}} \operatorname{Tr}\left(\Xi_{L, u}\right) \bigwedge 2^{L}\right) \\
& \leqslant \max _{L^{*}} \frac{n^{2} \rho_{s_{L}^{*}}^{4}}{2^{2 L^{*}}}\left(\frac{n}{m^{2}} \frac{1}{|\mathcal{C}|} \sum_{L \in \mathcal{C}} \operatorname{Tr}\left(\Xi_{L, u}\right) \bigwedge 2^{L^{*}}\right) \\
& \leqslant \max _{L^{*}} \frac{n^{2} \rho_{s_{L}^{*}}^{4}}{2^{2 L^{*}}}\left(\frac{b}{|\mathcal{C}|} \bigwedge 2^{L^{*}}\right)
\end{aligned}
$$

Whereas in the nonadaptive setting of Theorem 3.1 and Theorem 6.1 the local "chi-square" based terms need no special data processing treatment, it does in the adaptive case. For each of the $\log (n)$ resolution levels $L$, information on the norm of $\tilde{X}_{L}^{j}$ is communicated. Using $b=\log (n)$ to this without loss (compared to Theorem 6.1) turns out to be fundamental, as is the content of the lemma below. The proof of the lemma is based on exploiting the fact that even though $2^{-L / 2}\left(\left\|\sqrt{n / m} \tilde{X}_{L}^{j}\right\|_{2}^{2}-2^{L}\right)$ is sub-exponential, the fact that it tends to a subGaussian random variable can be exploited whenever the communication budget is small enough.

Lemma E.5. Let $\pi_{L}$ as in the proof of Theorem 7.1, with $\rho_{s}=\rho_{s_{L}}$ satisfying (26) or (25). Furthermore, let

$$
A_{L, u}=\sum_{j=1}^{m} \log \left(\mathbb{E}_{0}^{Y^{j} \mid U=u}\left(\mathbb{E}_{0}\left[\left.\int \frac{d \mathbb{P}_{f}^{\tilde{X}^{j}}}{d \mathbb{P}_{0}^{\tilde{X}^{j}}}\left(\tilde{X}_{L}^{j}\right) d \pi_{L}(f) \right\rvert\, Y^{j}, U=u\right]^{2}\right)\right)
$$

Then for arbitrary $\mathcal{C} \subset \mathbb{N}$,

$$
\frac{1}{|\mathcal{C}|} \sum_{L \in \mathcal{C}} \int A_{L, u} d \mathbb{P}^{U}(u) \lesssim \max _{L \in \mathcal{C}} \frac{c_{\alpha} n^{2} \rho_{s_{L}}^{4}(b \wedge|\mathcal{C}|)}{m 2^{L}|\mathcal{C}|} .
$$

Proof. Recalling the notation from Section B, we shall write $\mathscr{L}_{\pi_{L}}\left(\tilde{X}_{L}^{j}\right)=\int \mathscr{L}_{f}\left(\tilde{X}_{L}^{j}\right) d \pi_{L}(f)$ with

$$
\mathscr{L}_{f}\left(\tilde{X}_{L}^{j}\right):=\frac{d \mathbb{P}_{f}^{\tilde{X}^{j}}}{d \mathbb{P}_{0}^{\tilde{X}^{j}}}\left(\tilde{X}_{L}^{j}\right)=e^{\frac{n}{m} f^{\top} \tilde{X}_{L}^{j}-\frac{n}{2 m}\|f\|_{2}^{2}}
$$

Note that, using $\log (x) \leqslant x-1, \mathbb{E}_{0} \mathscr{L}_{\pi_{L}}\left(\tilde{X}_{L}^{j}\right)=1$ and the fact that by the law of total probability

$$
\mathbb{E}_{0}^{Y^{j} \mid U=u} \mathbb{E}_{0}\left[\mathscr{L}_{\pi_{L}}\left(\tilde{X}_{L}^{j}\right) \mid Y^{j}, U=u\right]=1
$$

we obtain that

$$
\begin{equation*}
A_{L, u} \leqslant \sum_{j=1}^{m} \mathbb{E}_{0}^{Y^{j} \mid U=u}\left(\mathbb{E}_{0}\left[\mathscr{L}_{\pi_{L}}\left(\tilde{X}_{L}^{j}\right)-1 \mid Y^{j}, U=u\right]^{2}\right) \tag{S.40}
\end{equation*}
$$

We work out the case where $\pi=N\left(0, \epsilon_{s}^{2} I_{2^{L}}\right)$, the case where $\pi=N\left(0, \epsilon_{s}^{2} \Gamma\right)$ with $\|\Gamma\|=1$ follows similarly with additional bookkeeping. Since $f \sim N\left(0, \epsilon_{s}^{2} I_{2 L}\right)$ with $\epsilon_{s}=c_{\alpha}^{1 / 4} \rho_{s} / 2^{L / 2}$,

$$
\begin{equation*}
\mathscr{L}_{\pi_{L}}\left(\tilde{X}_{L}^{j}\right)=\prod_{i=0}^{2^{L}-1} \int \frac{e^{\frac{n}{m} f_{i} \tilde{X}_{L i}^{j}-\frac{1}{2}\left(\frac{n}{m}+\epsilon_{s}^{-2}\right) f_{i}^{2}}}{\sqrt{2 \pi \epsilon_{s}^{2}}} d f_{i}=\frac{e^{\frac{n}{m} \epsilon_{s}^{2} \frac{\sqrt{\frac{n}{m}}}{2\left(1+\frac{n}{m}\left\|_{2}^{\prime}\right\|_{2}^{2}\right.}}}{\left(1+\frac{n}{m} \epsilon_{s}^{2}\right)^{2^{L} / 2}} \tag{S.41}
\end{equation*}
$$

where the last equality follows by the substitution $u=f_{i} \sqrt{1+\frac{n}{m} \epsilon_{s}^{2}}$ and completing the square. Taking the logarithm and using that $\frac{(1+x) \log (1+x)}{x}>1$ for $x>0$, we find

$$
\begin{equation*}
V_{L}^{j}:=\frac{n}{m} \epsilon_{s}^{2} \frac{\left\|\sqrt{\frac{n}{m}} \tilde{X}_{L}^{j}\right\|_{2}^{2}}{2\left(1+\frac{n}{m} \epsilon_{s}^{2}\right)}-2^{L-1} \log \left(1+\frac{n}{m} \epsilon_{s}^{2}\right) \leqslant \frac{\frac{n}{m} \epsilon_{s}^{2}}{2}\left(\left\|\sqrt{\frac{n}{m}} \tilde{X}_{L}^{j}\right\|_{2}^{2}-2^{L}\right) \tag{S.42}
\end{equation*}
$$

Therefore, using (S.41), Taylor expanding, $(a+b)^{2} \leqslant 2 a^{2}+2 b^{2}$ and (S.42), we can upper bound (S.40) by

$$
\begin{equation*}
2 \sum_{j=1}^{m} \mathbb{E}_{0}^{Y^{j} \mid U=u}\left(\mathbb{E}_{0}\left[V_{L}^{j} \mid Y^{j}, U=u\right]^{2}\right)+2 \sum_{j=1}^{m} \mathbb{E}_{0}^{Y^{j} \mid U=u}\left(D^{j}\right)^{2}, \tag{S.43}
\end{equation*}
$$

with

$$
D^{j}=\mathbb{E}_{0}\left[\left.\sum_{k=2}^{\infty} \frac{n^{k} \epsilon_{s}^{2 k}}{2^{k} m^{k} k!}\left|\left\|\sqrt{\frac{n}{m}} \tilde{X}_{L}^{j}\right\|_{2}^{2}-2^{L}\right|^{k} \right\rvert\, Y^{j}, U=u\right] .
$$

We deal with the two terms in (S.43) separately. Since conditional expectation contracts the $L_{2}$-norm,

$$
\sum_{j=1}^{m} \mathbb{E}_{0}^{Y j \mid U=u}\left(D^{j}\right)^{2} \lesssim m \cdot \sum_{k=2}^{\infty} \sum_{i=2}^{\infty} \frac{n^{k} c_{\alpha}^{k / 2} \rho_{s}^{2 k}}{2^{k} m^{k} 2^{k L_{s} / 2} k!} \frac{n^{i} c_{\alpha}^{i / 2} \rho_{s}^{2 i}}{2^{i} m^{i} 2^{L_{s} / 2} t!} \mathbb{E} W^{i+k}
$$

where $W \stackrel{d}{=}\left(\left\|\sqrt{\frac{n}{m}} \tilde{X}_{L}^{j}\right\|_{2}^{2}-2^{L}\right)$. Furthermore, since $\left\|\sqrt{\frac{n}{m}} X_{L}^{j}\right\|_{2}^{2} \sim \chi_{2^{L}}^{2}$ is sub-exponential, $\mathbb{E} W^{i+k} \leqslant C^{k+i}(i+k)^{i+k}$, where $C>0$ is a constant (see e.g. Proposition 2.7.1 in [40]). Then in view of $(i+k)^{i+k} \leqslant 2^{i+k} i!k$ !, we the above display is $O\left(\frac{c_{\alpha}^{2} n^{4} \rho_{s}^{8}}{m^{3} 2^{2} L_{s}}\right)$ whenever $\frac{c_{c}^{2} n^{4} \rho_{s}^{8}}{C^{2} m^{4} 2^{2 L L}}<1$. This is certainly the case when $\rho_{s}^{2} \lesssim\left(\frac{\sqrt{m \log (n)}}{n \sqrt{b \wedge \log (n)}}\right)^{\frac{2 s}{2 s+1 / 2}}$ and $m b \gtrsim \log (n)$, which yields that

$$
\sum_{j=1}^{m} \mathbb{E}_{0}^{Y_{j}^{j} \mid U=u}\left(D^{j}\right)^{2} \lesssim \frac{c_{\alpha}^{2} n^{2} \rho_{s}^{4}}{m 2^{L_{s} / 2}} \cdot O\left(\frac{\log (n)}{m(b \wedge \log (n))}\right)
$$

It remained to deal with the first term in (S.43), where we proceed by a data processing argument. When $b \geqslant \log (n)$,

$$
2 \sum_{j=1}^{m} \mathbb{E}_{0}^{Y^{j} \mid U=u}\left(\mathbb{E}_{0}\left[V_{L}^{j} \mid Y^{j}, U=u\right]^{2}\right) \leqslant 2 \sum_{j=1}^{m} \mathbb{E}_{0}^{\tilde{X}^{j}}\left(V_{L}^{j}\right)^{2} \leqslant \frac{c_{\alpha} n^{2} \rho_{s}^{4}}{m 2^{L_{s}}},
$$

in which case the result follows.
We continue with the case where $b<\log (n)$, which implies $\left|\mathcal{Y}^{j}\right| \leqslant 2^{\log (n)}$. We bound the average of the first terms in (S.43) over $\mathcal{C}$, by

$$
\begin{array}{r}
\frac{1}{|\mathcal{C}|} \sum_{L \in \mathcal{C} \mathcal{j}=1}^{m} \frac{n^{2} \rho_{s}^{4}}{m^{2} 2^{L_{s}}} \mathbb{E}_{0}^{Y^{j} \mid U=u}\left(\mathbb{E}_{0}\left[G_{L}^{j} \mid Y^{j}, U=u\right]^{2}\right) \leqslant  \tag{S.44}\\
\max _{L \in \mathcal{C}} \frac{n^{2} \rho_{s_{L}}^{4}}{m^{2} 2^{L}|\mathcal{C}|} \sum_{j=1}^{m} \mathbb{E}_{0}^{Y^{j} \mid U=u} \operatorname{Tr}\left(M^{j}\left(Y^{j}\right)\right),
\end{array}
$$

where $M^{j}(y)=\mathbb{E}_{0}\left[G_{\mathcal{C}}^{j} \mid Y^{j}=y, U=u\right] \mathbb{E}_{0}\left[G_{\mathcal{C}}^{j} \mid Y^{j}=y, U=u\right]^{\top}, G_{\mathcal{C}}^{j}=\left(G_{L}^{j}\right)_{L \in \mathcal{C}}$, and $G_{L}^{j}=\left(\frac{n \rho_{s}^{2}}{m 2^{L_{s} / 2}}\right)^{-1} V_{L}^{j}$. We show below that for all $v=\left(v_{L}\right)_{L \in \mathcal{C}}$ of unit norm

$$
\begin{equation*}
\mathbb{E}_{0}^{Y^{j} \mid U=u}\left\langle v_{\mathcal{C}}, G_{\mathcal{C}}^{j}\right\rangle^{2} \leqslant b, \tag{S.45}
\end{equation*}
$$

which by taking $v=G_{\mathcal{C}}^{j} /\left\|G_{\mathcal{C}}^{j}\right\|_{2}$ yields that (S.44) is $O\left(\max _{s} \frac{n^{2} \rho_{s}^{4}}{m 2^{L} s|C|} b\right)$ as required.
Therefore, it remained to verify (S.45). For any $\lambda \in \mathbb{R}$, independence and (S.42) yield

$$
\mathbb{E}_{0}^{X^{j}} e^{\lambda v^{\top} G_{\mathcal{C}}^{j}} \leqslant \prod_{L \in \mathcal{C}} \mathbb{E}_{0}^{X^{j}} e^{\frac{\lambda}{2^{2 / s} / 2} v_{L} \sum_{i=0}^{L^{L}-1}\left(\tilde{X}_{L i}^{2}-1\right)} .
$$

When $\frac{|\lambda|}{2 \cdot 2^{L_{s} / 2}} v_{L} \leqslant \frac{1}{4}$, the latter can be further bounded by

$$
\prod_{L \in \mathcal{C}} \exp \left(\lambda^{2} v_{L}^{2}\right)=\exp \left(\lambda^{2}\right),
$$

see e.g. Lemma 12 in [34]. In view of $0 \leqslant K\left(y \mid X^{j}, u\right) \leqslant 1$ and the previously shown subexponential behaviour of $\left\langle v_{\mathcal{C}}, G_{\mathcal{C}}^{j}\right\rangle$, we get that

$$
\begin{aligned}
& \mathbb{P}^{Y^{j} \mid U=u}(y) \mathbb{E}_{0}\left[\left\langle v_{\mathcal{C}}, G_{\mathcal{C}}^{j}\right\rangle \mid Y^{j}=y, U=u\right] \\
& \quad=\mathbb{E}_{0}^{X^{j}}\left\langle v_{\mathcal{C}}, G_{\mathcal{C}}^{j}\right\rangle K\left(y \mid X^{j}, u\right)=\mathbb{E}_{0}^{X^{j}} \int_{0}^{\infty} \mathbb{1}\left\{\left|\left\langle v_{\mathcal{C}}, G_{\mathcal{C}}^{j}\right\rangle\right|>t\right\} K\left(y \mid X^{j}, u\right) d t \\
& \quad \leqslant \int_{0}^{\infty} \min \left\{\mathbb{P}_{0}^{X^{j}}\left(\left|\left\langle v_{\mathcal{C}}, G_{\mathcal{C}}^{j}\right\rangle\right|>t\right), \mathbb{P}^{Y^{j} \mid U=u}(y)\right\} d t \leqslant e^{-t_{0}}+t_{0} \mathbb{P}^{Y^{j} \mid U=u}(y) .
\end{aligned}
$$

Taking $t_{0}=-\log \left(\mathbb{P}^{Y^{j} \mid U=u}(y)\right)$ yields

$$
\begin{equation*}
\mathbb{E}_{0}\left[\left\langle v_{\mathcal{C}}, G_{\mathcal{C}}^{j}\right\rangle \mid Y^{j}=y, U=u\right] \leqslant-2 \log \left(\mathbb{P}^{Y^{j} \mid U=u}(y)\right) . \tag{S.46}
\end{equation*}
$$

Furthermore, for $\lambda_{y} \in \mathbb{R}$ and $y$ satisfying

$$
\begin{equation*}
-2^{L_{s} / 2+2} \leqslant \lambda_{y}=\mathbb{E}_{0}\left[\left\langle v_{\mathcal{C}}, G_{\mathcal{C}}^{j}\right\rangle \mid Y^{j}=y, U=u\right] \leqslant 2^{L_{s} / 2+2} \tag{S.47}
\end{equation*}
$$

the argument of Lemma A. 3 yields

$$
\begin{equation*}
\mathbb{E}_{0}\left[\left\langle v_{\mathcal{C}}, G_{\mathcal{C}}^{j}\right\rangle \mid Y^{j}=y, U=u\right]^{2} \leqslant-\log \left(\mathbb{P}^{Y^{j} \mid U=u}(y)\right) \tag{S.48}
\end{equation*}
$$

Note, that if (S.47) does not hold, then in view of (S.46), $-\log \left(\mathbb{P}^{Y^{j} \mid U=u}(y)\right) \geqslant 2^{L_{s} / 2+1}$.
Let us write $p_{y}=\mathbb{P}^{Y^{j} \mid U=u}(y)$ and define $\mathcal{Y}_{*}^{j}=\left\{y \in \mathcal{Y}^{j}: \log \left(1 / p_{y}\right) \leqslant 2^{L_{s} / 2+2}\right\}$. Since $x \mapsto$ $x \log ^{2}(1 / x)$ is increasing on $\left(0, e^{-2}\right)$, it holds that $p_{y} \log ^{2}\left(1 / p_{y}\right) \leqslant e^{-2^{L_{s} / 2+2}+\left(L_{s}+4\right) \log (2)}$ for $y \in\left(\mathcal{Y}_{*}^{j}\right)^{c}$. Then, in view of (S.46) and (S.48) we get that

$$
\begin{aligned}
\sum_{y \in \mathcal{Y}^{j}} p_{y} \mathbb{E}_{0}\left[\left\langle v_{\mathcal{C}}, G_{\mathcal{C}}^{j}\right\rangle \mid Y^{j}=y, U=u\right]^{2} & \leqslant \sum_{y \in \mathcal{Y}_{*}^{j}} p_{y} \log \left(1 / p_{y}\right)+4 \sum_{y \in\left(\mathcal{Y}_{*}^{j}\right)^{c}} p_{y} \log ^{2}\left(1 / p_{y}\right) \\
& \lesssim \log \left|\mathcal{Y}^{j}\right|+e^{-2^{L_{s} / 2+2}+\left(L_{s}+4\right) \log (2)} \lesssim b
\end{aligned}
$$

concluding the proof of (S.45) and hence the lemma.
F. Proof of Lemma 10.1. We start by introducing some short hand notations for convenience. Write, for $x \in \mathbb{R}^{v k}, v \in\{1, m\}$,

$$
\phi_{v}(x)=\mathbb{E}^{H} \frac{p_{H}^{v}}{p_{0}^{v}}(x)=\mathbb{E}^{H} e^{H^{\top}}\left(\sum_{j=1}^{v} \Lambda^{-1} x^{j}\right)-\frac{v}{2}\left\|\Lambda^{-1 / 2} H\right\|_{2}^{2},
$$

with $\phi_{m}(x) p_{0}^{m}(x)=\mathbb{E}^{H} p_{H}^{m}(x), x=\left(x^{1}, \ldots, x^{m}\right)$, and $\Pi_{j=1}^{m} \phi_{1}\left(x^{j}\right)=\Pi_{j=1}^{m} \mathbb{E}^{H} p_{H}\left(x^{j}\right)$. Let $P_{0}^{m}$ denote the measure corresponding to the Lebesgue density $p_{0}^{m}$. Furthermore, recall that

$$
\begin{gathered}
\mathcal{Q} \equiv \mathcal{Q}(M, \Sigma):=\left\{q \in L_{1}\left(\mathbb{R}^{m k}, P_{0}^{m}\right): q \geqslant 0, \frac{q}{\int q(x) d P_{0}^{m}(x)} \leqslant M P_{0}^{m}-\right.\text { a.e., } \\
\left.\int x q(x) d P_{0}^{m}(x)=0, \text { and } \frac{\int x x^{\top} q(x) d P_{0}^{m}(x)}{\int q(x) d P_{0}^{m}(x)}=\Sigma\right\},
\end{gathered}
$$

where $L_{1}\left(\mathbb{R}^{m k}, P_{0}^{m}\right)=\left\{f: \mathbb{R}^{m k} \mapsto \mathbb{R}\right.$, such that $\left.\int f d P_{0}^{m}(x)<\infty\right\}$.

Let $\lambda \equiv \lambda_{m k}$ denote the Lebesgue measure on $\mathbb{R}^{m k}$, define for $r \in L_{1}\left(\mathbb{R}^{m k}, \lambda\right)$ nonnegative,

$$
\begin{equation*}
F(r):=\frac{\int \phi_{m}(x) r(x) d x}{\int \prod_{j=1}^{m} \phi_{1}\left(x^{j}\right) r(x) d x} \in[0, \infty] \tag{S.49}
\end{equation*}
$$

and set $G(q):=F\left(q p_{0}^{m}\right)$. Since $G(c q)=G(q)$ for any constant $c \in \mathbb{R}$, it suffices to show that

$$
\bar{G}=\sup _{q \in \mathcal{Q}} G(q) \leqslant \frac{\int \phi_{m}(x) d N(0, \Sigma)(x)}{\int \prod_{j=1}^{m} \phi_{1}\left(x^{j}\right) d N(0, \Sigma)(x)} .
$$

We will proceed through the following steps.

1. First, we show that the supremum $\bar{G}$ is finite and attained in $\mathcal{Q}$, i.e. by the Banach-Alaoglu theorem there exists $q \in \mathcal{Q}$ such that $G(q)=\bar{G}$.
2. We will then consider $\mathcal{Q}_{2}$, the class of all $Q \in L_{1}\left(\mathbb{R}^{2 k m}, \lambda\right)$ such that $x_{1} \mapsto Q\left(x_{1}, x_{2}\right)$ is in $\mathcal{Q}$ for $P_{0}^{m}$-almost every $x_{2} \in\left\{x_{1} \mapsto Q\left(x_{1}, x_{2}\right) \not \equiv 0\right\}$ and $x_{2} \mapsto Q\left(x_{1}, x_{2}\right)$ is in $\mathcal{Q}$ for $P_{0}^{m}$-almost every $x_{1} \in\left\{x_{2} \mapsto Q\left(x_{1}, x_{2}\right) \not \equiv 0\right\}$. It holds that

$$
G_{2}(Q):=\frac{\int \phi_{m}\left(x_{1}\right) \phi_{m}\left(x_{2}\right) p_{0}^{m}\left(x_{1}\right) p_{0}^{m}\left(x_{2}\right) Q\left(x_{1}, x_{2}\right) d\left(x_{1}, x_{2}\right)}{\int \prod_{j=1}^{m} \phi_{1}\left(x_{1}^{j}\right) \phi_{1}\left(x_{2}^{j}\right) p_{0}^{m}\left(x_{1}\right) p_{0}^{m}\left(x_{2}\right) Q\left(x_{1}, x_{2}\right) d\left(x_{1}, x_{2}\right)}
$$

satisfies $\sup _{Q \in \mathcal{Q}_{2}} G_{2}(Q)=\bar{G}^{2}$.
3. Next, we show that $\left(x_{1}, x_{2}\right) \mapsto q\left(\frac{x_{1}-x_{2}}{\sqrt{2}}\right) q\left(\frac{x_{1}+x_{2}}{\sqrt{2}}\right)$ is a maximizer of $G_{2}$ whenever $q \in \mathcal{Q}$ is a maximizer of $G$. This is a consequence of the conjugacy between the observation and the distribution of the parameter $H$.
4. Then it will be shown that for any maximizer $Q$ of $G_{2}, x_{1} \mapsto Q\left(x_{1}, x_{2}\right)$ maximizes $G$ for $P_{0}^{m}$-almost every $x_{2}$.
5. Combining the above steps, we obtain that for any maximizer $q$, an appropriately rescaled convolution of $q$ with itself is also a maximizer, i.e.

$$
F\left(\sqrt{2}\left(q p_{0}^{m}\right) *\left(q p_{0}^{m}\right)(\sqrt{2} \cdot)\right)=\bar{G},
$$

where $*$ denotes convolution.
6. By repeated application of Step 5 and the central limit theorem, the result follows.

Step 1. For $q \in \mathcal{Q}$, define the normalizing constant as $C_{q}:=\left(\int q d P_{0}^{m}\right)^{-1}$. As linear combinations and products of nonnegative convex functions are convex, the mapping

$$
x \mapsto \prod_{j=1}^{m} \mathbb{E}^{H} e^{H^{\top} \Lambda^{-1} x^{j}-\frac{1}{2}\left\|\Lambda^{-1 / 2} H\right\|_{2}^{2}}
$$

is convex. Then Jensen's inequality gives

$$
\frac{\int \mathbb{E}^{H} e^{H^{\top}\left(\sum_{j=1}^{m} \Lambda^{-1} x^{j}\right)-\frac{1}{2}\left\|\Lambda^{-1 / 2} H\right\|_{2}^{2}} q(x) d P_{0}^{m}(x)}{\int \prod_{j=1}^{m} \mathbb{E}^{H} e^{H^{\top} \Lambda^{-1} x^{j}-\frac{1}{2}\left\|\Lambda^{-1 / 2} H\right\|_{2}^{2}} q(x) d P_{0}^{m}(x)} \leqslant \frac{C_{q} \int \mathbb{E}^{H} e^{H^{\top}\left(\Lambda^{-1} \sum_{j=1}^{m} x^{j}\right)-\frac{1}{2}\left\|\Lambda^{-1 / 2} H\right\|_{2}^{2}} q(x) d P_{0}^{m}(x)}{\prod_{j=1}^{m} \mathbb{E}^{H} e^{C_{q} \int H^{\top} \Lambda^{-1} x^{j} q(x) d P_{0}^{m}(x)-\frac{1}{2}\left\|\Lambda^{-1 / 2} H\right\|_{2}^{2}}} .
$$

Since $X=\left(X_{1}, \ldots, X_{m}\right) \sim q d P_{0}^{m}$ has mean 0 , the denominator on the lhs is equal to $\left(\mathbb{E}^{H} e^{-\frac{1}{2} \| \Lambda^{-1 / 2}} H \|_{2}^{2}\right)^{m}>0$. This means that the denominator in the above display is bounded away from 0 over $q$. Since $q C_{q} \leqslant M$ a.e., the numerator is bounded above by $M \int \mathbb{E}^{H} p_{H}^{m}(x) d x=M$. We can conclude that the supremum of (S.49) over $q p_{0}^{m}, q \in \mathcal{Q}$ is finite. It is easy to construct a $q^{*} \in \mathcal{Q}$ such that $G\left(q^{*}\right)>0$, so we can conclude that $0<\bar{G}<\infty$.

Let $q_{t}$ be a maximizing sequence for $G$, rescale $q_{t}$ such that $\int q_{t} P_{0}^{m}=1$ and note that $q_{t} \in \mathcal{Q}$ and $q_{t}$ is contained in the $L_{\infty}\left(\mathbb{R}^{m k}\right)$ ball of radius $M$. Since $L_{\infty}\left(\mathbb{R}^{m k}\right)$ is the dual of $L_{1}\left(\mathbb{R}^{m k}, \lambda\right)$, by the Banach-Alaoglu theorem the $L_{\infty}\left(\mathbb{R}^{m k}\right)$ ball of radius $M$ is weak-*compact. Therefore, there exists a subsequence, again denoted by $q_{t}$, along which $q_{t} \xrightarrow{\text { wk-* }}$ $q$ for some $q$ in the $L_{\infty}\left(\mathbb{R}^{m k}\right)$ ball of radius $M$. Since $x=\left(x^{1}, \ldots, x^{m}\right) \mapsto \phi_{m}(x)$ is in $L_{1}\left(\mathbb{R}^{m k}, P_{0}^{m}\right)$, the weak-*-convergence implies that

$$
\int \phi_{m}(x) q_{t}(x) d P_{0}^{m}(x) \rightarrow \int \phi_{m}(x) q(x) d P_{0}^{m}(x) .
$$

Similarly,

$$
\int \Pi_{j=1}^{m} \phi_{1}\left(x^{j}\right) q_{t}(x) d P_{0}^{m}(x) \rightarrow \int \Pi_{j=1}^{m} \phi_{1}\left(x^{j}\right) q(x) d P_{0}^{m}(x) \in(0, \infty),
$$

where the boundedness away from 0 has been concluded earlier on in the proof. We have now obtained that

$$
\begin{equation*}
\bar{G}=\lim _{t \rightarrow \infty} \frac{\int \phi_{m}(x) q_{t}(x) d P_{0}^{m}(x)}{\int \Pi_{j=1}^{m} \phi_{1}\left(x^{j}\right) q_{t}(x) d P_{0}^{m}(x)}=\frac{\int \phi_{m}(x) q(x) d P_{0}^{m}(x)}{\int \Pi_{j=1}^{m} \phi_{1}\left(x^{j}\right) q(x) d P_{0}^{m}(x)} . \tag{S.50}
\end{equation*}
$$

Since $q_{t} \in \mathcal{Q}$, we have

$$
\int x q_{t}(x) d P_{0}^{m}(x)=0 \text { and } \int x x^{\top} q_{t}(x) d P_{0}^{m}(x)=\Sigma \text { for all } t .
$$

As $x \mapsto 1, x \mapsto x$ and $x \mapsto x x^{\top}$ are all $P_{0}^{m}$ integrable, the weak-*-convergence yields that $\int q(x) d P_{0}^{m}(x)=1, \int x q(x) d P_{0}^{m}(x)=0$ and $\Sigma=\int x x^{\top} q(x) d P_{0}^{m}(x)$. Since we have that $\int \zeta(x) q_{t}(x) d P_{0}^{m}(x) \rightarrow \int \zeta(x) q_{t}(x) d P_{0}^{m}(x)$ for every continuous and bounded function $\zeta$ : $\mathbb{R}^{m k} \rightarrow \mathbb{R}^{m k}$, the Portmanteau lemma yields that $\int_{B} q d P_{0}^{m} \geqslant 0$ for all open sets $B$ so $q \geqslant 0$ almost everywhere. We conclude that $G(q)=\bar{G}$ and $q \in \mathcal{Q}$.

Step 2. Let $Q \in \mathcal{Q}_{2}$ be given. By definition, the marginals $x_{1} \mapsto Q\left(x_{1}, x_{2}\right), x_{2} \mapsto$ $Q\left(x_{1}, x_{2}\right)$ are in $\mathcal{Q} P_{0}^{m}$-a.e. and $\mathbb{E}^{H} p_{H}(x) d x=\phi_{m}(x) p_{0}^{m}(x) d x$ is equivalent to the Lebesgue measure, hence

$$
\begin{aligned}
G_{2}(Q) & =\int \phi_{m}(x) p_{0}^{m}\left(x_{1}\right) \int \phi_{m}(x) p_{0}^{m}\left(x_{2}\right) Q\left(x_{1}, x_{2}\right) d x_{2} d x_{1} \\
& \leqslant \bar{G} \int \phi_{m}(x) p_{0}^{m}\left(x_{1}\right) \int \Pi_{j=1}^{m} \phi_{1}\left(x_{2}^{j}\right) p_{0}^{m}\left(x_{2}\right) Q\left(x_{1}, x_{2}\right) d x_{2} d x_{1} \\
& \leqslant \bar{G}^{2} \int \Pi_{j=1}^{m} \phi_{1}\left(x_{2}^{j}\right) p_{0}^{m}\left(x_{2}\right) \int \Pi_{j=1}^{m} \phi_{1}\left(x_{1}^{j}\right) p_{0}^{m}\left(x_{1}\right) Q\left(x_{1}, x_{2}\right) d x_{1} d x_{2}
\end{aligned}
$$

Let $q \in \mathcal{Q}$ be a maximizer of $G$. Then, the above steps hold with equality for $Q\left(x_{1}, x_{2}\right):=$ $q\left(x_{1}\right) q\left(x_{2}\right)$. For almost every $x_{1} \in\{q \neq 0\} \equiv\left\{x_{2} \mapsto Q\left(x_{1}, x_{2}\right) \not \equiv 0\right\}$,

$$
\frac{Q\left(x_{1}, x_{2}\right)}{\int Q\left(x_{1}, x_{2}\right) d P_{0}^{m}\left(x_{2}\right)}=\frac{q\left(x_{2}\right)}{\int q\left(x_{2}\right) d P_{0}^{m}\left(x_{2}\right)} \leqslant M .
$$

By similar calculations, the rescaled marginal has the correct mean and covariance. By symmetry, we conclude that the marginals of $\left(x_{1}, x_{2}\right) \mapsto q\left(x_{1}\right) q\left(x_{2}\right)$ belong to $\mathcal{Q}$ and it is a maximizer of $G_{2}$ over $\mathcal{Q}_{2}$.

Step 3. Consider a maximizer $q \in \mathcal{Q}$ of $G$. By a change of variables $w_{1}=\left(x_{1}-x_{2}\right) / \sqrt{2}$ and $w_{2}=\left(x_{1}+x_{2}\right) / \sqrt{2}$,

$$
\begin{aligned}
& \int \phi_{m}\left(x_{1}\right) \phi_{m}\left(x_{2}\right) q\left(\frac{x_{1}-x_{2}}{\sqrt{2}}\right) q\left(\frac{x_{1}+x_{2}}{\sqrt{2}}\right) p_{0}^{m}\left(x_{1}\right) p_{0}^{m}\left(x_{2}\right) d\left(x_{1}, x_{2}\right)= \\
& \int \phi_{m}\left(\frac{w_{1}+w_{2}}{\sqrt{2}}\right) \phi_{m}\left(\frac{w_{1}-w_{2}}{\sqrt{2}}\right) q\left(w_{1}\right) q\left(w_{2}\right) p_{0}^{m}\left(\frac{w_{1}-w_{2}}{\sqrt{2}}\right) p_{0}^{m}\left(\frac{w_{1}+w_{2}}{\sqrt{2}}\right) d\left(w_{1}, w_{2}\right) .
\end{aligned}
$$

Since $p_{0}^{m}$ is a Gaussian density, $p_{0}^{m}\left(\frac{w_{1}-w_{2}}{\sqrt{2}}\right) p_{0}^{m}\left(\frac{w_{1}+w_{2}}{\sqrt{2}}\right)=p_{0}^{m}\left(w_{1}\right) p_{0}^{m}\left(w_{2}\right)$. This follows from direct computation, but it characterizes Gaussian functions in general, see e.g. Theorem 1 in [15]. Likewise, for $H^{\prime}$ an independent copy of the centered Gaussian random vector $H, \frac{H-H^{\prime}}{\sqrt{2}}$ and $\frac{H+H^{\prime}}{\sqrt{2}}$ are independent and furthermore equal in distribution to $H$. Therefore,

$$
\begin{aligned}
& \phi_{m}\left(\frac{w_{1}+w_{2}}{\sqrt{2}}\right) \phi_{m}\left(\frac{w_{1}-w_{2}}{\sqrt{2}}\right) \\
&=\mathbb{E}^{\left(H, H^{\prime}\right)} e^{H^{\top} \Lambda^{-1} \sum_{j=1}^{m} \frac{w_{1}^{j}+w_{2}^{j}}{\sqrt{2}}+\left(H^{\prime}\right)^{\top} \Lambda^{-1} \sum_{j=1}^{m} \frac{w_{1}^{j}-w_{2}^{j}}{\sqrt{2}}-\frac{m}{2}\left\|\Lambda^{-1 / 2} H\right\|_{2}^{2}-\frac{m}{2}\left\|\Lambda^{-1 / 2} H^{\prime}\right\|_{2}^{2}} \\
&=\mathbb{E}^{\left(H, H^{\prime}\right)} e^{\left(\frac{H+H^{\prime}}{\sqrt{2}}\right)^{\top} \Lambda^{-1} \sum_{j=1}^{m} w_{1}^{j}-\frac{m}{2}\left\|\Lambda^{-1 / 2} \frac{H+H^{\prime}}{\sqrt{2}}\right\|_{2}^{2}+\left(\frac{H-H^{\prime}}{\sqrt{2}}\right)^{\top} \Lambda^{-1} \sum_{j=1}^{m} w_{2}^{j}-\frac{m}{2}\left\|\Lambda^{-1 / 2} \frac{H-H^{\prime}}{\sqrt{2}}\right\|_{2}^{2}} \\
& \quad=\phi_{m}\left(w_{1}\right) \phi_{m}\left(w_{2}\right) .
\end{aligned}
$$

Since $\left(x_{1}, x_{2}\right) \mapsto q\left(x_{1}\right) q\left(x_{2}\right)$ was established to be a maximizer of $G_{2}$ in the second step, the above establishes that $\left(x_{1}, x_{2}\right) \mapsto q\left(\frac{x_{1}-x_{2}}{\sqrt{2}}\right) q\left(\frac{x_{1}+x_{2}}{\sqrt{2}}\right)$ is a maximizer of $G_{2}$ also.

Step 4. Next, we will show that for a maximizer $Q \in \mathcal{Q}_{2}$ of $G_{2}, x \mapsto Q(x, w)$ is in $\mathcal{Q}$ and is a maximizer of $G$ for almost every $w$. We prove this by contradiction. Take an arbitrary measurable set $A \subset \mathbb{R}^{m k}$ s.t. $\lambda(A)>0$. Note that Gaussian measures are equivalent to the Lebesgue measure, so both $\mathbb{E}^{H} P_{H}^{m}(A)$ and $\prod_{j=1}^{m} \mathbb{E}^{H} P_{H}^{1}(A)$ are bounded away from zero. Suppose that for $Q \in \mathcal{Q}_{2}$ a maximizer of $G_{2}$ it holds that

$$
\begin{align*}
& \int_{A} \phi_{m}(w) \int \phi_{m}(x) Q(x, w) d P_{0}^{m}(x) d P_{0}^{m}(w) \\
& \quad<\bar{G} \int_{A} \phi_{m}(w) \int \Pi_{j=1}^{m} \phi_{1}\left(x^{j}\right) Q(x, w) d P_{0}^{m}(x) d P_{0}^{m}(w) . \tag{S.51}
\end{align*}
$$

Since the marginal $w \mapsto Q(x, w)$ is in $\mathcal{Q}$ for almost every $x \in\{w \mapsto Q(x, w) \not \equiv 0\}$,

$$
\begin{aligned}
& \bar{G}^{2} \int \Pi_{j=1}^{m} \phi_{1}\left(w^{j}\right) \Pi_{j=1}^{m} \phi_{1}\left(x^{j}\right) Q(x, w)\left(d P_{0}^{m} \times P_{0}^{m}\right)(x, w) \\
& \quad \geqslant \bar{G} \int \Pi_{j=1}^{m} \phi_{1}\left(x^{j}\right) \int \phi_{m}(w) Q(x, w) d P_{0}^{m}(w) d P_{0}^{m}(x) .
\end{aligned}
$$

Likewise, $x \mapsto Q(x, w)$ is in $\mathcal{Q}$ for almost every $u \in A^{c} \cap\{x \mapsto Q(x, w) \not \equiv 0\}$, so

$$
\begin{array}{r}
\bar{G} \int \Pi_{j=1}^{m} \phi_{1}\left(x^{j}\right) \int_{A^{c}} \phi_{m}(w) Q(x, w) d P_{0}^{m}(w) d P_{0}^{m}(x) \\
\quad \geqslant \int_{A^{c}} \phi_{m}(w) \int \phi_{m}(x) Q(x, w) d P_{0}^{m}(x) d P_{0}^{m}(w) .
\end{array}
$$

Together with (S.51) and the second to last display, we obtain that

$$
\begin{gathered}
\bar{G}^{2} \int \Pi_{j=1}^{m} \phi_{1}\left(w^{j}\right) \Pi_{j=1}^{m} \phi_{1}\left(x^{j}\right) Q(x, w)\left(d P_{0}^{m} \times P_{0}^{m}\right)(x, w) \\
\quad>\iint \phi_{m}(x) \phi_{m}(w) Q(x, w) d P_{0}^{m}(w) d P_{0}^{m}(x),
\end{gathered}
$$

which contradicts with $Q$ maximizing $G_{2}$.
Step 5. Let $q \in \mathcal{Q}$ be a maximizer of $G$ over $\mathcal{Q}$, where $q$ is normalized such that $\int q d P_{0}^{m}=$ 1. Define $q_{2}$ as

$$
q_{2}(x):=\int q\left(\frac{x-w}{\sqrt{2}}\right) q\left(\frac{x+w}{\sqrt{2}}\right) d P_{0}^{m}(u) .
$$

The map $x \mapsto q\left(\frac{x-w}{\sqrt{2}}\right) q\left(\frac{x+w}{\sqrt{2}}\right):=Q(x, w)$ is in $\mathcal{Q}$ for almost all $w$ s.t. $Q(x, w) \not \equiv 0$ and as a consequence of the previous step, it is a maximizer of $G$ for such $w$. Hence, $q_{2}(x)$ is a maximizer of $G$ :

$$
\begin{aligned}
\int \phi_{m}(x) q_{2}(x) d P_{0}^{m}(x) & =\iint \phi_{m}(x) q\left(\frac{x-w}{\sqrt{2}}\right) q\left(\frac{x+w}{\sqrt{2}}\right) d P_{0}^{m}(x) d P_{0}^{m}(w) \\
& =\bar{G} \int \Pi_{j=1}^{m} \phi_{1}\left(x^{j}\right) q_{2}(x) d P_{0}^{m}(x) .
\end{aligned}
$$

Let $h \in L_{1}\left(\mathbb{R}^{m k}, p_{0}^{m}\right)$. Using again that $p_{0}^{m}\left(\frac{w_{1}-w_{2}}{\sqrt{2}}\right) p_{0}^{m}\left(\frac{w_{1}+w_{2}}{\sqrt{2}}\right)=p_{0}^{m}\left(w_{1}\right) p_{0}^{m}\left(w_{2}\right)$ and applying a change of variable $w=\sqrt{2} w-x$, we get

$$
\begin{aligned}
\int h(x) q_{2}(x) p_{0}^{m}(x) d x & =\iint h(x) q\left(\frac{x-w}{\sqrt{2}}\right) q\left(\frac{x+w}{\sqrt{2}}\right) p_{0}^{m}\left(\frac{x-w}{\sqrt{2}}\right) p_{0}^{m}\left(\frac{x+w}{\sqrt{2}}\right) d x d w \\
& =\iint h(x) q(\sqrt{2} x-w) q(w) p_{0}^{m}(\sqrt{2} x-w) p_{0}^{m}(w) d x \sqrt{2} d w \\
& =\int h(x) \sqrt{2}\left(q p_{0}^{m}\right) *\left(q p_{0}^{m}\right)(\sqrt{2} x) d x
\end{aligned}
$$

where $f * g$ denotes convolution. Therefore, $q p_{0}^{m}$ being a probability density with mean 0 and covariance $\Sigma$ implies that $q_{2} p_{0}^{m}$ is too. So, $q_{2} \in \mathcal{Q}$ and maximizes $G$.

Step 6. Consider now $q_{4} \in \mathcal{Q}$ defined by $q_{4}(x):=\int q_{2}\left(\frac{x-w}{\sqrt{2}}\right) q_{2}\left(\frac{x+w}{\sqrt{2}}\right) d P_{0}^{m}(w)$. Since $q_{2} \in \mathcal{Q}$ is a maximizer, the above steps imply that $G\left(q_{4}\right)=\bar{G}$ and by a similar computation as above,

$$
q_{4}(x) p_{0}^{m}(x)=\sqrt{4} \stackrel{4}{*}\left(q p_{0}^{m}\right)(\sqrt{4} x),
$$

where ${ }^{4} r$ denotes $r * r * r * r$. Repeating the above steps, we obtain a maximizer $q_{2^{N}} \in \mathcal{Q}$ of $G$ for $N \in \mathbb{N}$ which satisfies

$$
\begin{aligned}
r_{2^{N}}(x):=q_{2^{N}}(x) p_{0}^{m}(x) & =\int q_{2^{N-1}}\left(\frac{x-w}{\sqrt{2}}\right) q_{2^{N-1}}\left(\frac{x+w}{\sqrt{2}}\right) p_{0}^{m}(x) p_{0}^{m}(w) d x d w \\
& =\sqrt{2} \int q_{2^{N-1}}(\sqrt{2} x-w) p_{0}^{m}(\sqrt{2} x-w) q_{2^{N-1}}(w) d P_{0}^{m}(w) \\
& =\sqrt{2}\left(q_{2^{N-1}} p_{0}^{m}\right) *\left(q_{2^{N-1}} p_{0}^{m}\right)(\sqrt{2} x) .
\end{aligned}
$$

We conclude that

$$
r_{2^{N}}(x)=2^{N / 2} \stackrel{2^{N}}{*}\left(q p_{0}^{m}\right)\left(2^{N / 2} x\right)
$$

and

$$
\frac{\int \phi_{m}(x) r_{2^{N}}(x) d x}{\int \Pi_{j=1}^{m} \phi_{1}\left(x^{j}\right) r_{2^{N}}(x) d x}=G\left(q_{2^{N}}\right)=\bar{G}
$$

for all $N \in \mathbb{N}$. Let $r=q p_{0}^{m}$. The characteristic function of $r_{2^{N}}$ equals, for $s \in \mathbb{R}^{m k}$,

$$
\begin{aligned}
\mathcal{F} r_{2^{N}}(s) & :=\int e^{-i s^{\top} x} r_{2^{N}}(x) d x=\int e^{-i \frac{s^{\top}}{2^{N / 2}} x} * r(x) d x=\left(\int e^{-i \frac{s^{\top}}{2^{N / 2}} x} r(x) d x\right)^{2^{N}} * r\left(\frac{s}{2^{N}}\right. \\
& =\left(\int\left(1-i \frac{s}{2^{N / 2}} x-\frac{\left(s^{\top} x\right)^{2}}{2^{N+1}}+O\left(\frac{\left(s^{\top} x\right)^{3}}{2^{3 N / 2}}\right)\right) r(x) d x\right)^{2^{N}} .
\end{aligned}
$$

Since $r$ has mean 0 , covariance $\Sigma$ and bounded third moment (by the boundedness of $q$ and $p_{0}^{m} d \lambda$ possessing a third moment), $\mathcal{F} r_{2^{N}}(s) \rightarrow e^{-\frac{1}{2} s^{\top} \Sigma s}$. Consequently, $r_{2^{N}} d \lambda$ converges weakly to a Gaussian distribution with mean 0 and covariance $\Sigma$. In particular, $\int \phi r_{2^{N}} d \lambda \rightarrow$ $\int \phi d N(0, \Sigma)$ for all $\phi \in C^{\infty}\left(\mathbb{R}^{m k}\right)$, so

$$
\bar{G}=\lim _{N \rightarrow \infty} \frac{\int \phi_{m}(x) r_{2^{N}}(x) d x}{\int \prod_{j=1}^{m} \phi_{1}\left(x^{j}\right) r_{2^{N}}(x) d x}=\frac{\int \phi_{m}(x) d N(0, \Sigma)(x)}{\int \Pi_{j=1}^{m} \phi_{1}\left(x^{j}\right) d N(0, \Sigma)(x)},
$$

which finishes the proof.
G. Definitions and notations for wavelets. In this section we briefly introduce wavelets and collect some properties used in the article. For a more detailed and elaborate introduction of wavelets we refer to [23, 20].

In our work we consider the Cohen, Daubechies and Vial construction of compactly supported, orthonormal, $N$-regular wavelet basis of $L_{2}[0,1]$, see for instance [17]. First for any $N \in \mathbb{N}$ one can follow Daubechies' construction of the father $\phi($.$) and mother \psi($.$) wavelets$ with $N$ vanishing moments and bounded support on $[0,2 N-1]$ and $[-N+1, N]$, respectively, see for instance [18]. The basis functions are then obtained as

$$
\left\{\phi_{j_{0} m}, \psi_{j k}: m \in\left\{0, \ldots, 2^{j_{0}}-1\right\}, \quad j>j_{0}, \quad k \in\left\{0, \ldots, 2^{j}-1\right\}\right\},
$$

with $\psi_{j k}(x)=2^{j / 2} \psi\left(2^{j} x-k\right)$, for $k \in\left[N-1,2^{j}-N\right]$, and $\phi_{j_{0} k}(x)=2^{j_{0}} \phi\left(2^{j_{0}} x-m\right)$, for $m \in\left[0,2^{j_{0}}-2 N\right]$, while for other values of $k$ and $m$, the basis functions are specially constructed, to form a basis with the required smoothness property. For notational convenience we take $j_{0}=0$ and denote the father wavelet by $\psi_{00}$. Then the function $f \in L_{2}[0,1]$ can be represented in the form

$$
f=\sum_{j=j_{0}}^{\infty} \sum_{k=0}^{2^{j}-1} f_{j k} \psi_{j k}
$$

with $f_{j k}=\left\langle f, \psi_{j k}\right\rangle$. Note that in view of the orthonormality of the wavelet basis the $L_{2}$-norm of the function $f$ is equal to

$$
\|f\|_{2}^{2}=\sum_{j=j_{0}}^{\infty} \sum_{k=0}^{2^{j}-1} f_{j k}^{2} .
$$

Next we give an equivalent definition of Sobolev spaces using wavelets. Let us define the norm for $s \in(0, N)$ as

$$
\|f\|_{\mathcal{H}^{s}}^{2}=\sum_{j \geqslant j_{0}} 2^{2 j s} \sum_{k=0}^{2^{j}-1} f_{j k}^{2} .
$$

Then the Sobolev space $\mathcal{H}^{s}([0,1])$ and Sobolev ball $\mathcal{H}^{s, R}([0,1])$ of radius $R>0$ are defined as

$$
\mathcal{H}^{s}=\left\{f \in L_{2}[0,1]:\|f\|_{\mathcal{H}^{s}}<\infty\right\}, \quad \text { and } \quad \mathcal{H}^{s, R}([0,1])=\left\{f \in L_{2}[0,1]:\|f\|_{\mathcal{H}^{s}}<R\right\}
$$

respectively. The above definition of the Sobolev space and norm is equivalent to the classical one based on the weak derivatives of the function.


[^0]:    MSC2020 subject classifications: Primary 62G10, 62F30; secondary 62F03.
    Keywords and phrases: Distributed methods, Nonparametric, Hypothesis testing, Minimax optimal.

